Homework

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1. An investor wishes to invest his savings in a set $S = \{1, \dots, 7\}$ of market stocks. Using 0-1 variables formulate the following constraints in integer (linear) programming.

   (a) the portfolio cannot invest in all stocks;

   (b) at least one stock must be selected;

   (c) stock 1 cannot be selected if stock 3 is selected;

   (d) stock 4 can be selected only if stock 2 can be selected.

   (e) either stocks 1 and 5 are selected simultaneously, or none of them;

   (f) at least one stock of the set $\{1, 2, 3\}$ or at least two stocks of the set $\{2, 4, 5, 6\}$ mus be selected.

   Remark: your formulation should be efficient.

2. Suppose that $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is the vector of decision variables. How would you represent the following constraints in integer linear programming?

$$\|x\|_0 \leq k$$
$$\|x\|_1 \leq k$$
$$\|x\|_\infty \leq k$$

   where k is a given constant. The 0-norm $\|x\|_0$ is the number of nonzero vector entries. For instance, $\|(-1.2, 0, 0, 1, -2)\| = 3$.

3. A set of n jobs must be processed in a machine that can handle one job at a time. Task j needs p_j hours to be processed.

   A directed and acyclic graph $G = (V, E)$, with $V = \{1, \dots, n\}$, establishes a partial order for job processing in the machine. That is, if there exists a path $\delta_{i,j}$ from i to j in G, then job i must be processed before job j.

   Given nonnegative weights $w_j, j = 1, \dots, n$, in which order should we process the jobs in order to minimize the weighted sum of the start processing time of all jobs, while respecting the precedence order? For the modeling task that follows, s_j is the instant that job j starts to be processed.

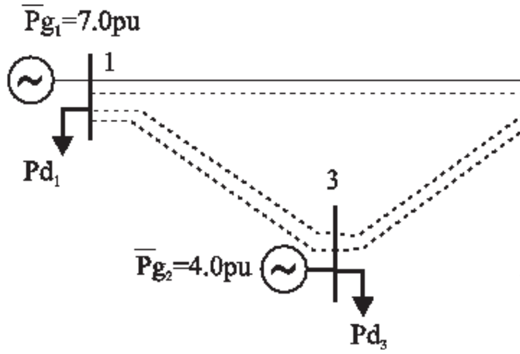
   Tasks:

   (a) Formulate the problem in mixed-integer linear programming using discrete and continuous variables.

(b) Model the problem in AMPL and solve the instance given below, in which $V = \{1, \ldots, 12\}$. Present the results.

j	p_j	w_j	Arcs (j, i)
1	3	5	(1,3)
2	2	3	
3	6	7	(3,12), (3,7)
4	2	6	
5	5	1	
6	4	2	(6,7)
7	4	8	
8	3	4	(8,6)
9	10	7	
10	1	1	(10,12)
11	8	6	
12	7	2	

4. In this exercise, you will have the opportunity to approximate a nonlinear problem as a MILP by means of piecewise-linear models. The power system has 3 buses as depicted in Fig. 1. Buses 1 and 3 have generation units, whereas bus 2 is a power consumer. This figures also gives the maximum power generation ($\overline{Pg}_i$) and the power consumption ($\overline{Pd}_i$) of each bus i, under low and high demand. The properties of the transmission lines appear in Table 1, in pu using a 100 MVA basis, are indicated in the figure. The parameters are the resistance $r_{i,j}$, the reactance $x_{i,j}$ and the number of lines installed between buses i and j.



Load (pu)	Pd_1	Pd_2	Pd_3
High	1.0	3.5	5.0
Low	1.0	2.0	3.0

Figura 1: Illustrative 3-bus Power System

Tabela 1: Line Parameters of the 3-bus System

Line	$r_{i,j}$ (pu)	$x_{i,j}$ (pu)	$n_{i,j}$
1-2	0.030	0.23	2
1-3	0.035	0.25	1
2-3	0.025	0.20	1

A simplified model is adopted for the transmission network, in which:

- Lines and transformers are represented by their series impedances in per unit:

$$z_{i,j} = r_{i,j} + jx_{i,j} \tag{1}$$

where $r_{i,j}$ is the resistance and $x_{i,j}$ is the reactance of line (i,j).
- Voltage magnitudes are fixed at 1.0 pu.
- Reactive power balance is supposed to be satisfied.

With these assumptions, active power flows are expressed as

$$\text{p.flow } i \to j : P_{i,j} = g_{i,j} - (g_{i,j}\cos\theta_{i,j} + b_{i,j}\sin\theta_{i,j}) \tag{2a}$$
$$\text{p.flow } j \to i : P_{j,i} = g_{i,j} - (g_{i,j}\cos\theta_{i,j} - b_{i,j}\sin\theta_{i,j}) \tag{2b}$$

where g_{ij} and b_{ij} are, respectively, the series conductance and series susceptance of line (i,j), $\theta_{i,j} = (\theta_i - \theta_j)$ and θ_i is the voltage angle of bus i. Conductance and susceptance are calculated as follows

$$g_{i,j} = \frac{r_{i,j}}{(r_{i,j}^2 + x_{i,j}^2)}$$
$$b_{i,j} = -\frac{x_{i,j}}{(r_{i,j}^2 + x_{i,j}^2)}$$

The power injected into bus i is defined as

$$P_i = \sum_{j\in N_i} n_{i,j}P_{i,j} = \sum_{j\in N_i} n_{i,j}\left[g_{i,j} - (g_{i,j}\cos\theta_{i,j} + b_{i,j}\sin\theta_{i,j})\right] \tag{3}$$

where N_i is the set of neighboring buses of bus i. To ensure energy conservation, the following equations must also be satisfied

$$Pg_i = \overline{Pd}_i + P_i \tag{4}$$

Aiming to minimize the power loss in transmission, the power-flow optimization problem could be solved:

$$\min \sum_{i\in N} |P_i| \tag{5a}$$

$$\text{s.t. :} \begin{cases} Pg_i = \overline{Pd}_i + P_i, \\ P_i = \sum_{j\in N_i} n_{i,j}P_{i,j}, \\ 0 \le Pg_i \le \overline{Pg}_i, \\ \theta_i \in [-\frac{\pi}{2}, \frac{\pi}{2}],\ i \in N \end{cases} \tag{5b}$$

$$\begin{cases} P_{i,j} = g_{i,j} - (g_{i,j}\cos\theta_{i,j} + b_{i,j}\sin\theta_{i,j}), \\ \theta_{i,j} = \theta_i - \theta_j, \\ \theta_{i,j} \in [-\pi, \pi],\ i \in N,\ j \in N_i \end{cases} \tag{5c}$$

Tasks:

- Reformulate the power-flow optimization problem in MILP using the following piecewise-linear models: CC and SOS2.

- Implement the models in AMPL, choosing a suitable number of breakpoints to induce a good approximation of the power flow equations. You may plot the piecewise linear approximations for $\sin\theta_{i,j}$ and $\cos\theta_{i,j}$ in order to show the degree of approximation.
- Solve the problem for the low and high power demand cases. Present and illustrate the solutions.