ARMA Process

Bruno Pereira, Economist and Data Scientist

Contents

```r
library(forecast)
library(ggplot2)
library(tidyverse)
library(gridExtra)
```

1 First-Order Autoregressive Process (AR(1))

Let us consider the following process:

$$y_t = \beta_0 + \beta_1 y_{t-1} + \varepsilon_t \tag{1}$$

Where:

- y_t is the value of the series at time t;
- β_0 is a constant (intercept);
- β_1 is the autoregressive coefficient;
- $\varepsilon_t \sim \mathcal{N}(0, \sigma^2)$ is white noise (random shocks), that is, noise with zero mean, constant variance σ^2, and no autocorrelation.

The trajectory of the AR(1) process depends on the value of the parameter β_1:

- If $|\beta_1| \geq 1$, the shocks ε_t accumulate over time, causing the series to exhibit a trend — it may become explosive or follow a random walk. In these cases, the process is **non-stationary**.

- If $|\beta_1| < 1$, the effects of past shocks gradually dissipate, which characterizes a **stationary process**.

Assuming the process is stationary, we can apply the **lag operator** L, defined as $Ly_t = y_{t-1}$, to equation (1) to obtain the compact form of the first-order autoregressive process:

$$y_t - \beta_1 y_{t-1} = \beta_0 + \varepsilon_t$$

Applying the operator L:

$$(1 - \beta_1 L) y_t = \beta_0 + \varepsilon_t \tag{2}$$

This model is called a **first-order autoregressive (AR(1))** process because the current value of the series y_t depends only on a single time lag, that is, the past value y_{t-1} — known as **lag 1**.

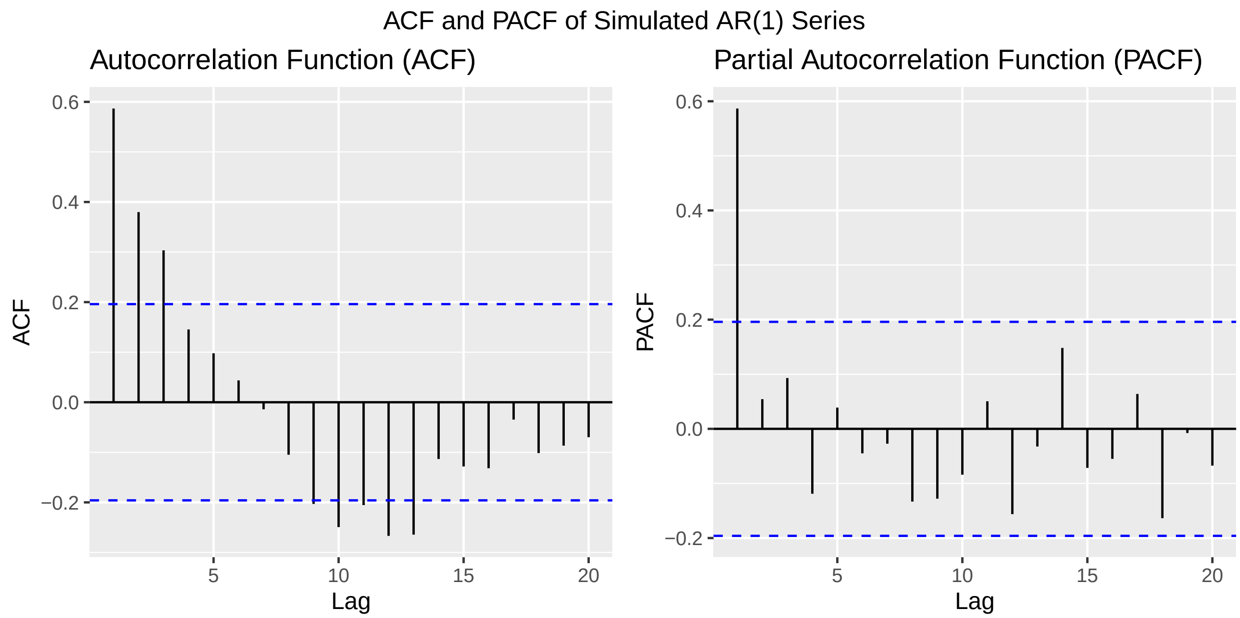
```r
# Simulando AR(1) com beta_1 = 0.6
set.seed(123)

serie <- arima.sim(model = list(ar = 0.6), n = 100)

g1 <- ggAcf(serie, type = "correlation") +
  ggtitle("Autocorrelation Function (ACF)")
```

```r
g2 <- ggPacf(serie) +
  ggtitle("Partial Autocorrelation Function (PACF)")

grid.arrange(g1, g2, ncol = 2, top = "ACF and PACF of Simulated AR(1) Series")
```



1.1 Stationarity and the Mean of the AR(1) Process

Our goal is to understand the stationarity conditions for a first-order autoregressive (AR(1)) process and show how it can be rewritten as an infinite weighted sum of past shocks. In other words, we want to show how the present depends on all past shocks, but with exponentially decaying weights.

Assuming **weak stationarity**, that is:

- $\mathbb{E}(y_t) = \mu$ (constant mean),
- $\text{Var}(y_t) = \sigma_y^2$ (constant variance),
- $\text{Cov}(y_t, y_{t-k}) = \gamma_k$ (autocovariance depends only on the lag k, not on time t),

we will compute the mean of the AR(1) process.

1.1.1 AR(1) Process Equation

The equation of the first-order autoregressive process is:

$$y_t = \beta_0 + \beta_1 y_{t-1} + \varepsilon_t \tag{1}$$

Taking the expectation on both sides:

$$\mathbb{E}(y_t) = \beta_0 + \beta_1 \mathbb{E}(y_{t-1})$$

If the process is stationary, then $\mathbb{E}(y_t) = \mathbb{E}(y_{t-1}) = \mu$. Substituting:

$$\mu = \beta_0 + \beta_1 \mu$$

Rearranging the terms:

$$\mu(1 - \beta_1) = \beta_0 \quad \Rightarrow \quad \mu = \frac{\beta_0}{1 - \beta_1}$$

This is the mean of the AR(1) process, provided that $|\beta_1| < 1$. This means the mean exists only if $\beta_1 \neq 1$.

If $\beta_1 = 1$, the denominator becomes zero and μ is undefined — the process is **non-stationary** (for example, a **random walk**).

Moreover, the mean will be zero **if and only if** $\beta_0 = 0$, because:

$$\mu = \frac{\beta_0}{1 - \beta_1} = 0 \quad \Rightarrow \quad \beta_0 = 0$$

1.2 Mean-Centered Representation of the Process

We know that

$$\beta_0 = (1 - \beta_1)\mu,$$

so we can substitute this into equation (1):

$$y_t = (1 - \beta_1)\mu + \beta_1 y_{t-1} + \varepsilon_t.$$

Subtracting μ from both sides, we get:

$$y_t - \mu = \beta_1 (y_{t-1} - \mu) + \varepsilon_t.$$

This is the **mean-centered model**, meaning it describes fluctuations around the mean μ.

1.3 Infinite Expansion: MA(∞) Representation

The equation above,

$$y_t - \mu = \beta_1(y_{t-1} - \mu) + \varepsilon_t,$$

can be solved by recursive substitution, expanding the lags in terms of past shocks ε_t.

We begin with:

$$y_t - \mu = \beta_1(y_{t-1} - \mu) + \varepsilon_t.$$

We substitute $y_{t-1} - \mu$ using the same equation lagged by one period:

$$y_{t-1} - \mu = \beta_1(y_{t-2} - \mu) + \varepsilon_{t-1}.$$

Plugging it back in:

$$\begin{aligned} y_t - \mu &= \beta_1 \left[\beta_1(y_{t-2} - \mu) + \varepsilon_{t-1}\right] + \varepsilon_t \\ &= \beta_1^2(y_{t-2} - \mu) + \beta_1\varepsilon_{t-1} + \varepsilon_t. \end{aligned}$$

Now we substitute $y_{t-2} - \mu$ by:

$$y_{t-2} - \mu = \beta_1(y_{t-3} - \mu) + \varepsilon_{t-2},$$

and obtain:

$$\begin{aligned} y_t - \mu &= \beta_1^2 \left[\beta_1(y_{t-3} - \mu) + \varepsilon_{t-2}\right] + \beta_1\varepsilon_{t-1} + \varepsilon_t \\ &= \beta_1^3(y_{t-3} - \mu) + \beta_1^2\varepsilon_{t-2} + \beta_1\varepsilon_{t-1} + \varepsilon_t. \end{aligned}$$

After three substitutions, we have:

$$y_t - \mu = \varepsilon_t + \beta_1\varepsilon_{t-1} + \beta_1^2\varepsilon_{t-2} + \beta_1^3(y_{t-3} - \mu).$$

At each step, the lagged term $y_{t-k} - \mu$ gets an additional coefficient β_1, while a new shock ε_{t-k} is added with weight β_1^k.

With infinite substitutions, the term

$$\beta_1^k(y_{t-k} - \mu)$$

tends to zero if $|\beta_1| < 1$, because

$$\lim_{k\to\infty} \beta_1^k(y_{t-k} - \mu) = 0.$$

Assuming this holds, we obtain the infinite representation:

$$y_t - \mu = \sum_{i=0}^{\infty} \beta_1^i \varepsilon_{t-i}.$$

Or, reintroducing the mean μ,

$$y_t = \mu + \sum_{i=0}^{\infty} \beta_1^i \varepsilon_{t-i}.$$

This is the **MA(∞) representation** of the AR(1) process — that is, an infinite weighted sum of past shocks with geometrically decaying weights.

1.4 Variance of y_t in an AR(1)

We have the process:

$$y_t = \mu + \beta_1(y_{t-1} - \mu) + \varepsilon_t$$

Or, centering:

$$y_t - \mu = \beta_1(y_{t-1} - \mu) + \varepsilon_t$$

Now, we want the variance of y_t. Taking variance on both sides:

$$\text{Var}(y_t - \mu) = \text{Var}\big(\beta_1(y_{t-1} - \mu) + \varepsilon_t\big)$$

We know that ε_t is independent of y_{t-1}, so the variance adds up:

$$\text{Var}(y_t) = \beta_1^2 \text{Var}(y_{t-1}) + \sigma_\varepsilon^2$$

Since we are assuming stationarity, then:

$$\text{Var}(y_t) = \text{Var}(y_{t-1}) = \sigma_y^2,$$

therefore:

$$\sigma_y^2 = \beta_1^2 \sigma_y^2 + \sigma_\varepsilon^2$$

Rearranging:

$$\sigma_y^2(1 - \beta_1^2) = \sigma_\varepsilon^2 \quad \Rightarrow \quad \sigma_y^2 = \frac{\sigma_\varepsilon^2}{1 - \beta_1^2}$$

This variance only exists if $|\beta_1| < 1$ — this is one of the stationarity conditions.

1.5 2. Autocorrelation Function (ACF)

Now we want the covariance between y_t and y_{t-k}, denoted by:

$$\gamma_k = \text{Cov}(y_t, y_{t-k}).$$

From the model:

$$y_t - \mu = \beta_1(y_{t-1} - \mu) + \varepsilon_t,$$

multiplying both sides by $y_{t-k} - \mu$ and taking expectations:

$$\text{Cov}(y_t, y_{t-k}) = \beta_1 \text{Cov}(y_{t-1}, y_{t-k}) + \text{Cov}(\varepsilon_t, y_{t-k}).$$

Since ε_t is independent of y_{t-k} (for $k \geq 1$), we have:

$$\gamma_k = \beta_1 \gamma_{k-1}.$$

That is, the autocovariance of order k is a multiple of the one of order $k-1$. Repeating:

$$\gamma_k = \beta_1 \gamma_{k-1} = \beta_1^2 \gamma_{k-2} = \cdots = \beta_1^k \gamma_0.$$

1.6 3. Autocorrelation Function (ACF) and Exponential Decay

The autocorrelation ρ_k is the autocovariance γ_k normalized by the variance:

$$\rho_k = \frac{\gamma_k}{\gamma_0} \quad \Rightarrow \quad \rho_k = \beta_1^k.$$

This means that:

- For $\beta_1 > 0$, the decay is smooth and exponential.
- For $\beta_1 < 0$, the decay alternates (zigzag pattern), since the signs of β_1^k change with k.

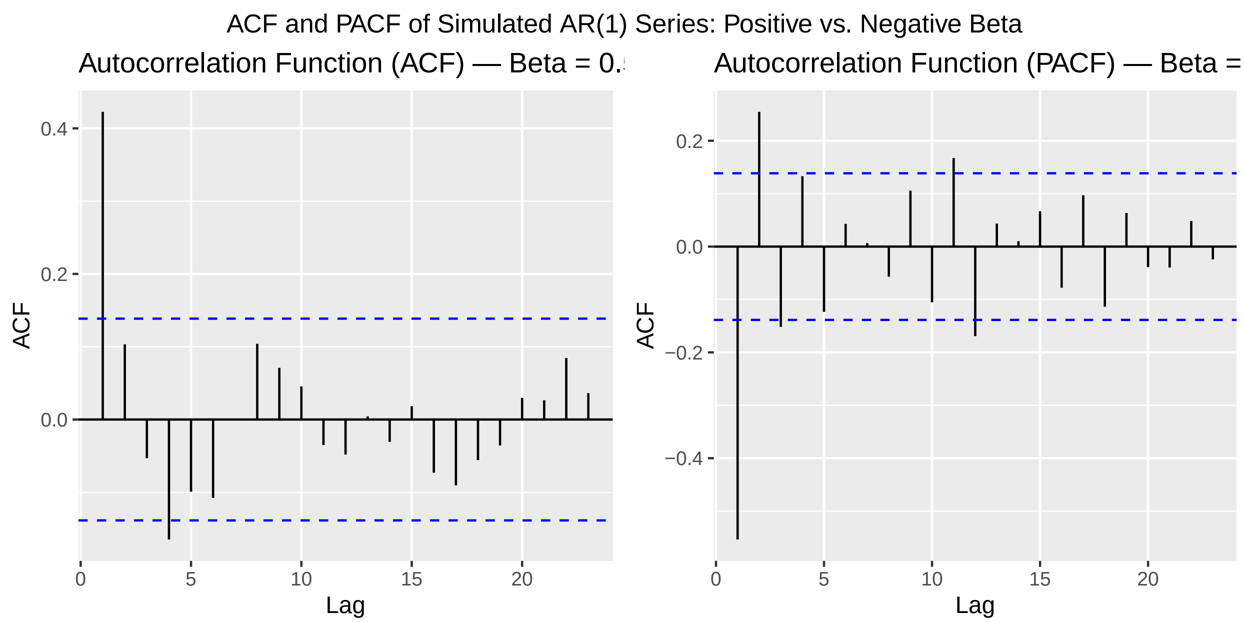
```r
# Simulate AR(1) processes with positive and negative beta coefficients
beta_positive <- arima.sim(model = list(ar = 0.5), n = 200)
beta_negative <- arima.sim(model = list(ar = -0.5), n = 200)

# Plot ACF for positive beta
p1 <- ggAcf(beta_positive) +
  ggtitle("Autocorrelation Function (ACF) - Beta = 0.5")

# Plot PACF for negative beta
p2 <- ggAcf(beta_negative) +
```

```
  ggtitle("Autocorrelation Function (PACF) - Beta = -0.5")

# Arrange plots side by side with a main title
grid.arrange(p1, p2, ncol = 2, top = "ACF and PACF of Simulated AR(1) Series: Positive vs. Negative Beta
```



1.6.1 Generalization: AR(p) Process

An AR(p) process is written as:

$$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \cdots + \beta_p y_{t-p} + \varepsilon_t$$

Or, in compact form using the lag operator L:

$$(1 - \beta_1 L - \beta_2 L^2 - \cdots - \beta_p L^p) y_t = \beta_0 + \varepsilon_t$$

2 Moving Average (MA) Processes of Order 1

A moving average process of order 1 (MA(1)) is defined by the equation:

$$y_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}$$

where:

- μ is the mean of the series.
- ε_t is white noise with zero mean, constant variance σ^2, and no autocorrelation.
- θ is a parameter that measures the effect of the lagged shock (delayed by one period).

The properties of the MA(1) process — specifically the moments (mean and autocovariances) — are:

Mean:

$$E(y_t) = \mu$$

Autocovariance at lag 0 (variance):

$$\gamma_0 = \text{Var}(y_t) = (1 + \theta^2)\sigma^2$$

Autocovariance at lag 1:

$$\gamma_1 = \text{Cov}(y_t, y_{t-1}) = \theta\sigma^2$$

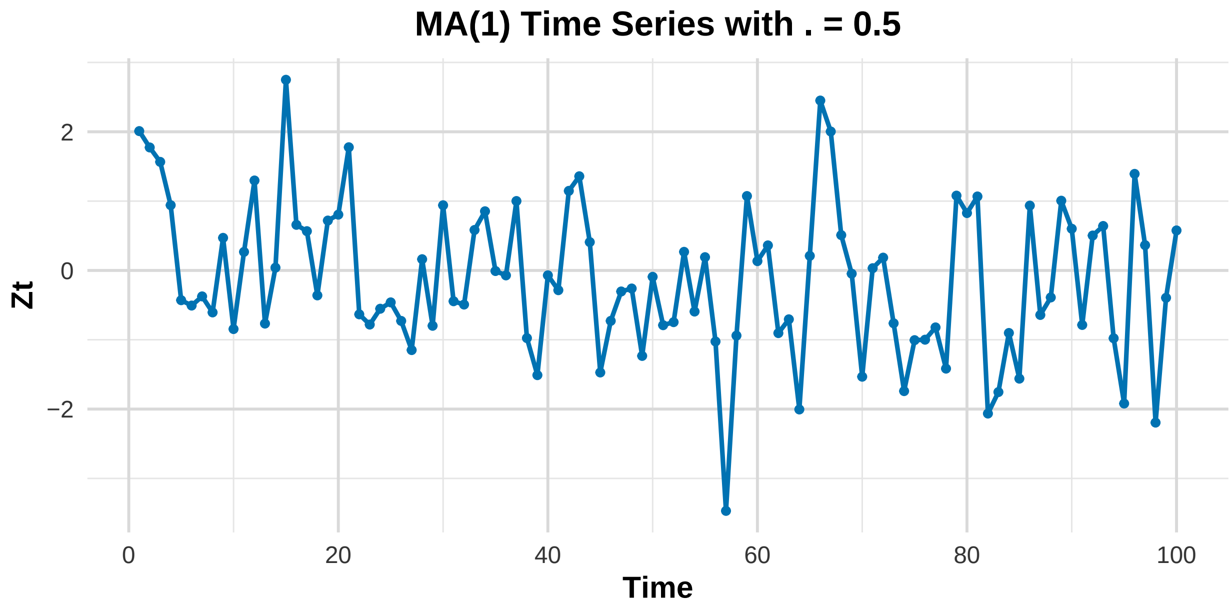
Autocovariances for lags greater than 1:

$$\gamma_k = 0, \quad \text{for } k > 1$$

Below we simulate an MA(1) process with $\theta = 0.5$.

```r
# Simulation of the MA(1) series
Zt <- arima.sim(model = list(ma = 0.5), n = 100)

# plot
autoplot(Zt) +
  ggtitle("MA(1) Time Series with   = 0.5") +
  xlab("Time") +
  ylab("Zt") +
  theme_minimal(base_size = 14) +
  theme(
    plot.title = element_text(hjust = 0.5, face = "bold", size = 16),
    axis.title = element_text(face = "bold"),
    panel.grid.major = element_line(color = "gray85"),
    panel.grid.minor = element_line(color = "gray90"),
    axis.text = element_text(color = "gray20")
  ) +
  geom_line(color = "#0072B2", size = 1) +
  geom_point(color = "#0072B2", size = 1.5)
```



3 Generalization of the MA(1) to MA(q)

The MA(1) process depends on the current shock ε_t and the shock from the previous period ε_{t-1} multiplied by a parameter θ:

$$y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$$

Now, we can generalize this process to consider up to q lagged shocks, resulting in the MA(q) process:

$$y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \cdots + \theta_q\varepsilon_{t-q}$$

where

- μ is the mean of the series.
- ε_t are white noise shocks at different time points.
- The coefficients $\theta_1, \theta_2, \ldots, \theta_q$ measure the impact of past shocks on the current value.

4 What are ARMA(p, q) processes?

So far, you have seen two main types of processes:

- AR(p) (autoregressive process of order p): where the current value depends on its own past values $y_{t-1}, y_{t-2}, \cdots, y_{t-p}$.
- MA(q) (moving average process of order q): where the current value depends on past shocks $\varepsilon_{t-1}, \varepsilon_{t-2}, \cdots, \varepsilon_{t-q}$.

The ARMA(p, q) model combines these two, allowing the current value to depend both on past values of the series and on past shocks:

$$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \cdots + \beta_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

where

- The terms with β_i correspond to the autoregressive part.
- The terms with θ_j correspond to the moving average part.
- ε_t remains the contemporaneous white noise.

5 Practical example in R: Simulating an ARMA(1,1)

The following code generates a simulated time series from an ARMA process with:

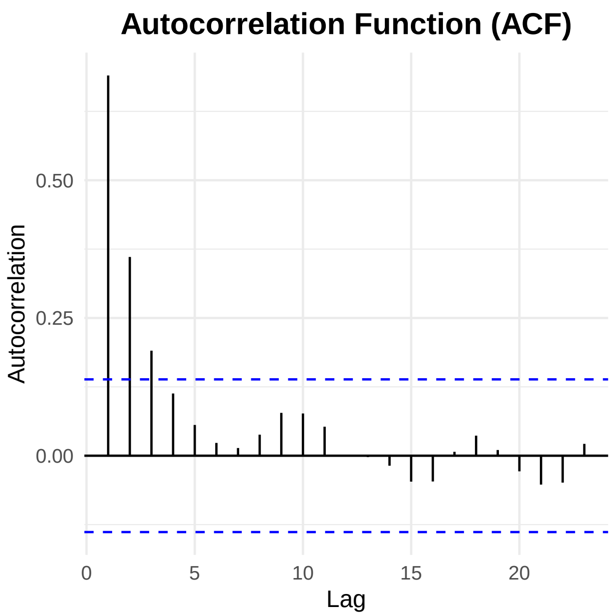
- $p = 1$ (one autoregressive term),
- $q = 1$ (one moving average term),
- $\phi = 0.5$ (AR coefficient),
- $\theta = 0.3$ (MA coefficient),
- 200 observations.

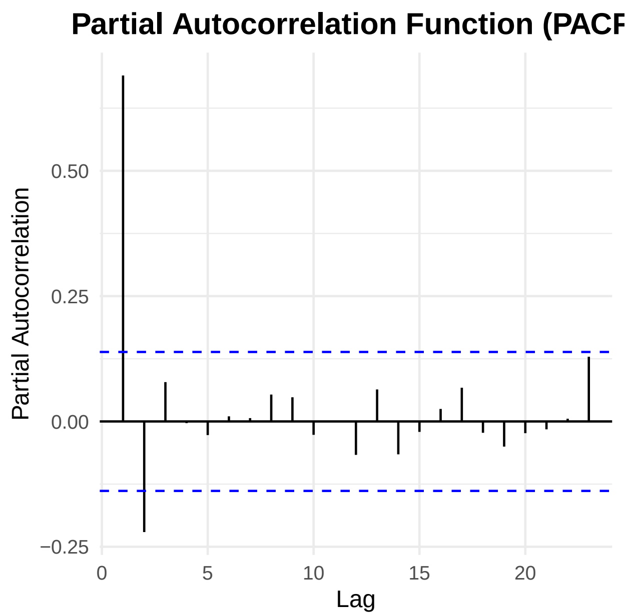
```r
# Simulation of the ARMA(1,1) model order = c(p, d, q)
arma <- arima.sim(list(order = c(1,0,1), ar = 0.5, ma = 0.3), n = 200)

# Autocorrelation Function (ACF)
g1 <- ggAcf(arma) +
  ggtitle("Autocorrelation Function (ACF)") +
  xlab("Lag") + ylab("Autocorrelation") +
  theme_minimal() +
  theme(plot.title = element_text(hjust = 0.5, face = "bold", size = 14))

# Partial Autocorrelation Function (PACF)
g2 <- ggPacf(arma) +
  ggtitle("Partial Autocorrelation Function (PACF)") +
  xlab("Lag") + ylab("Partial Autocorrelation") +
  theme_minimal() +
  theme(plot.title = element_text(hjust = 0.5, face = "bold", size = 14))

#
grid.arrange(g1, g2, ncol = 2)
```


Autocorrelation Function (ACF)
Autocorrelation
0.50
0.25
0.00
0
5
10
15
20
Lag


Partial Autocorrelation Function (PACF
Partial Autocorrelation
0.50
0.25
0.00
−0.25
0
5
10
15
20
Lag