Alpha-Shapes

Johan DEBAYLE, Yann GAVET, Jean-Charles PINOLI

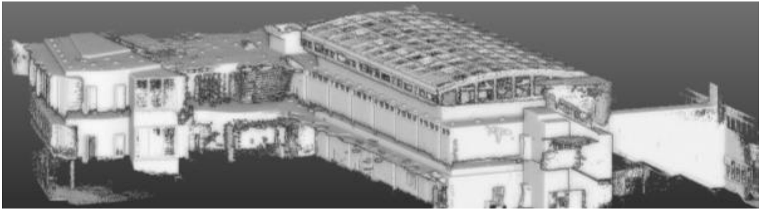
Ecole Nationale Supérieure des Mines de Saint-Etienne
Laboratoire Georges Friedel, UMR CNRS 5307

Section 1

Context

Point clouds

- Acquisition: hardware, sensors (e.g. LiDAR)
- Structuring: classification, segmentation, reconstruction, mesh generation

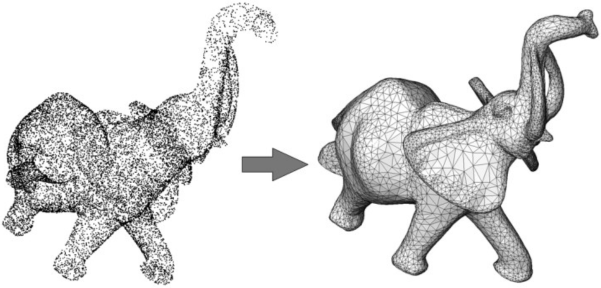


Where do we use point clouds?

- Generation / updating of 3D city models
- Forest mapping / vegetation analytics
- Monitoring infrastructures
- Volume computations (mining, landslides, etc.)
- Heritage documentation and valorization
- Building Information Modeling (BIM)
- Flood modeling
- Change detection
- Tunnel inspection
- Monitoring coastal erosion
- ...

Shape reconstruction from point clouds

- Alpha-shapes (Edelsbrunner et al.)
- Improved methods
  - Crust, Power Crust (Amenta et al.)
  - Cocone, Tight Cocone (Dey et al.)
  - Natural Neighbors (Boissonnat, Cazals)
  - LDA alpha-shapes (Maillot et al.)



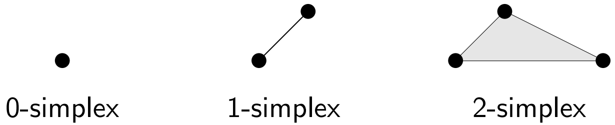
Outline

Section 2

Simplicial Complexes

Simplices

- Generalization of the notion of a triangle to arbitrary dimensions.
- A k-**simplex** $\sigma_T = conv(T)$ is the convex hull of an affinely independent point set $T \subseteq \mathbb{R}^n$, $\#T = k+1; 0 \leq k \leq n$.



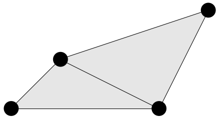
k is the dimension of the simplex σ_T.

- The convex hull of any nonempty subset of the $k+1$ points that define a k-simplex is called a **face** of the simplex.

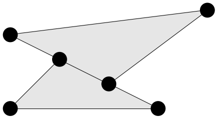
Simplicial complexes

A **simplicial complex** K is a finite collection of simplices with the following two properties:

1. $\sigma_V \in K$ and $U \subset V \Rightarrow \sigma_U \in K$ (face)
2. $\sigma_U, \sigma_V \in K \Rightarrow \sigma_U \cap \sigma_V$ is either empty or a face of both



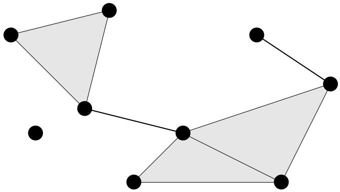
simplicial complex



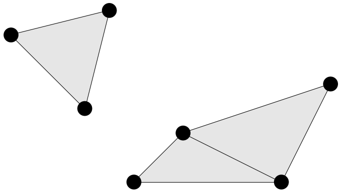
not simplicial complex

Solid simplicial complexes

The **solid simplicial complex** of K and denoted $\bar{K}$ has no isolated simplices, i.e., k-simplices that are not faces of a simplex with greater dimension. $\bar{K}$ is a subsimplex of K.



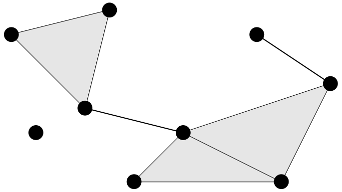
simplicial complex K



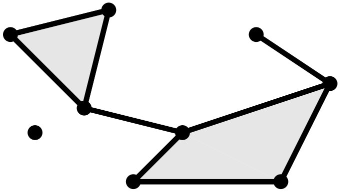
solid simplicial complex $\bar{K}$

Simplicial complex shapes

The **simplicial complex shape** of K is the underlying space $|K| \subseteq \mathbb{R}^n$ corresponding to the union of the simplices of K (polygonal shape).

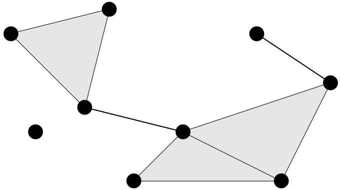


simplicial complex K

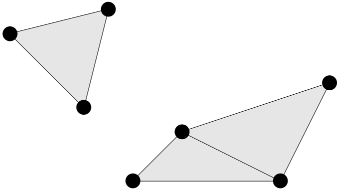


simplicial complex shape $|K|$

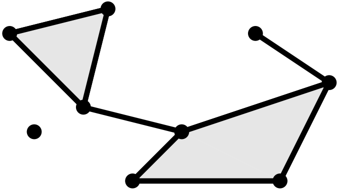
Synthesis



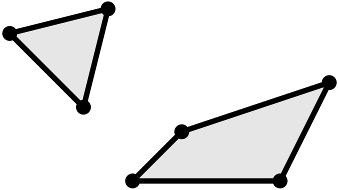
simplicial complex K



solid simplicial complex $\overline{K}$



simplicial complex shape $|K|$

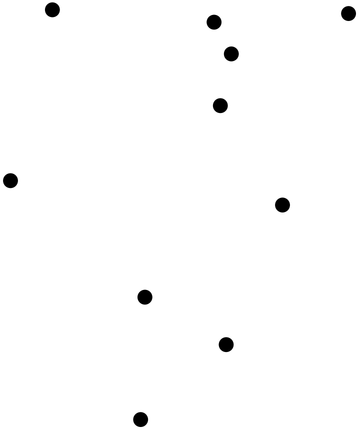


solid simplicial complex shape $|\overline{K}|$

Section 3

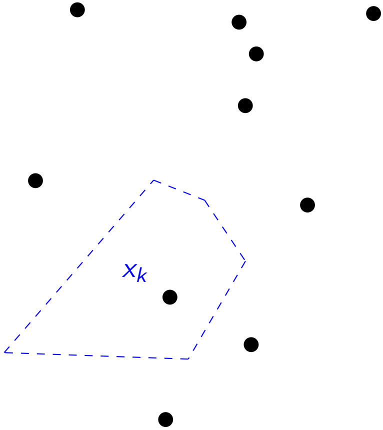
Delaunay triagulation and Voronoi diagram

A set of points S



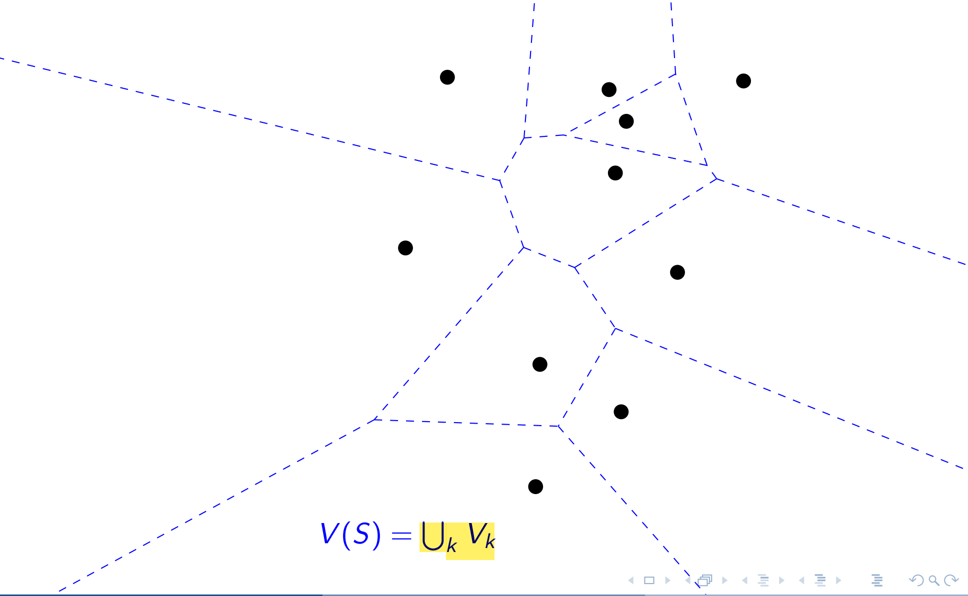
$$S = \{x_k \in \mathbb{R}^2\}_k$$

A Voronoi cell V_k of a point $x_k \in S$



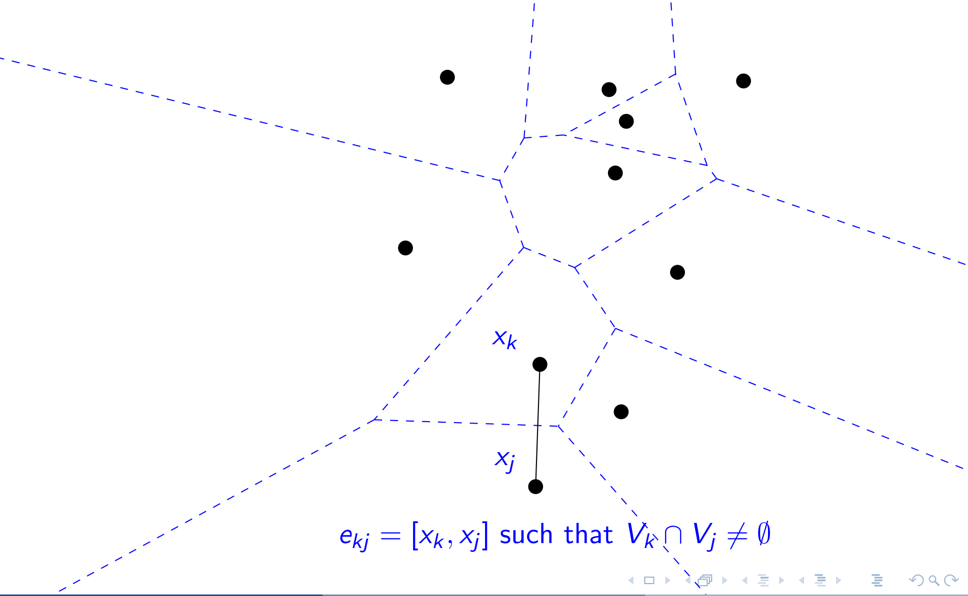
$$V_k = \{x \in \mathbb{R}^2; ||x - x_k|| \leq ||x - x_j|| \text{ for all } j \neq k, x_j \in S\}$$

The Voronoi diagram $V(S)$



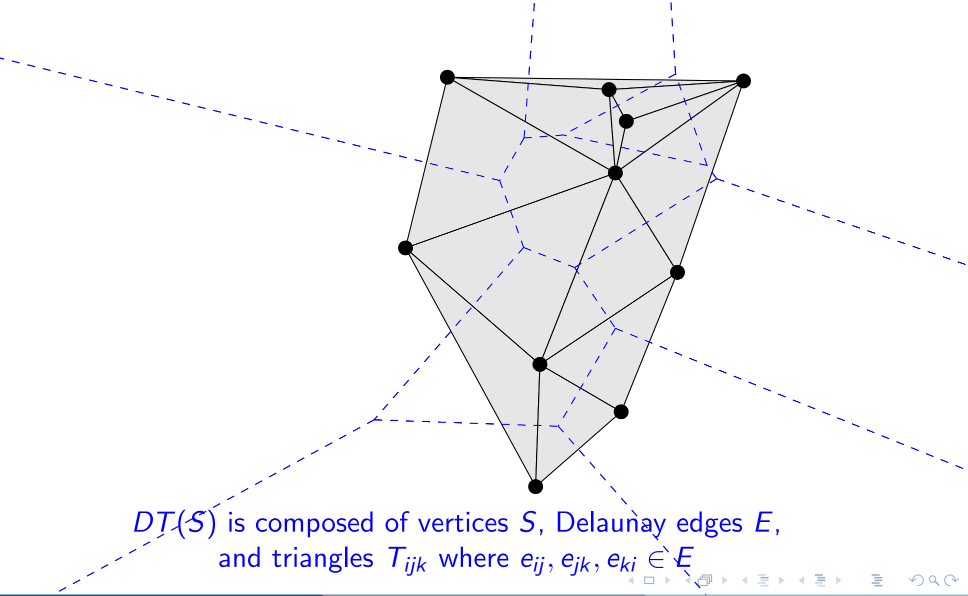
$V(S) = \bigcup_k V_k$

A Delaunay edge between two points



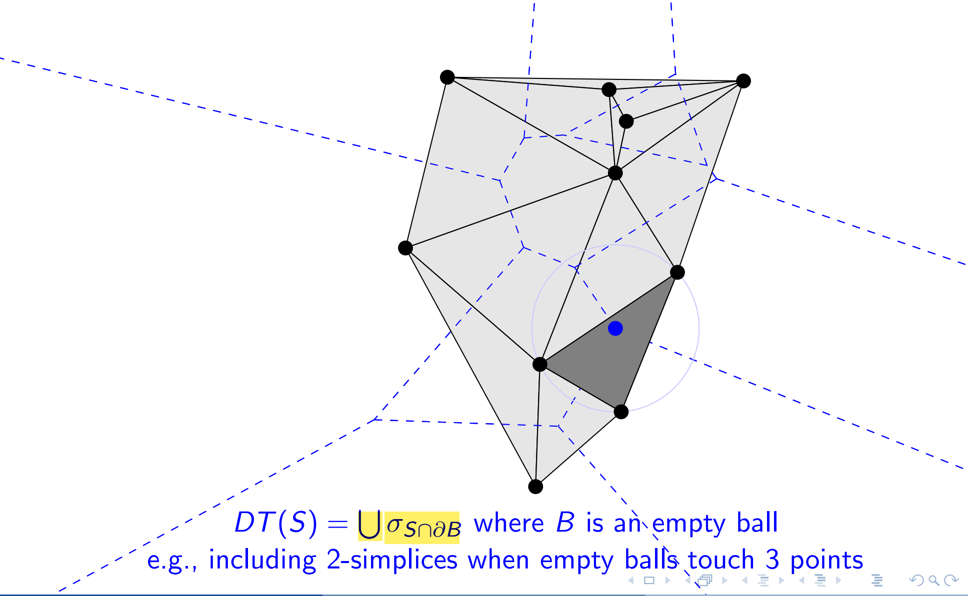
$e_{kj} = [x_k, x_j]$ such that $V_k \cap V_j \neq \emptyset$

The Delaunay triangulation $DT(S)$



$DT(S)$ is composed of vertices S, Delaunay edges E, and triangles T_{ijk} where $e_{ij}, e_{jk}, e_{ki} \in E$

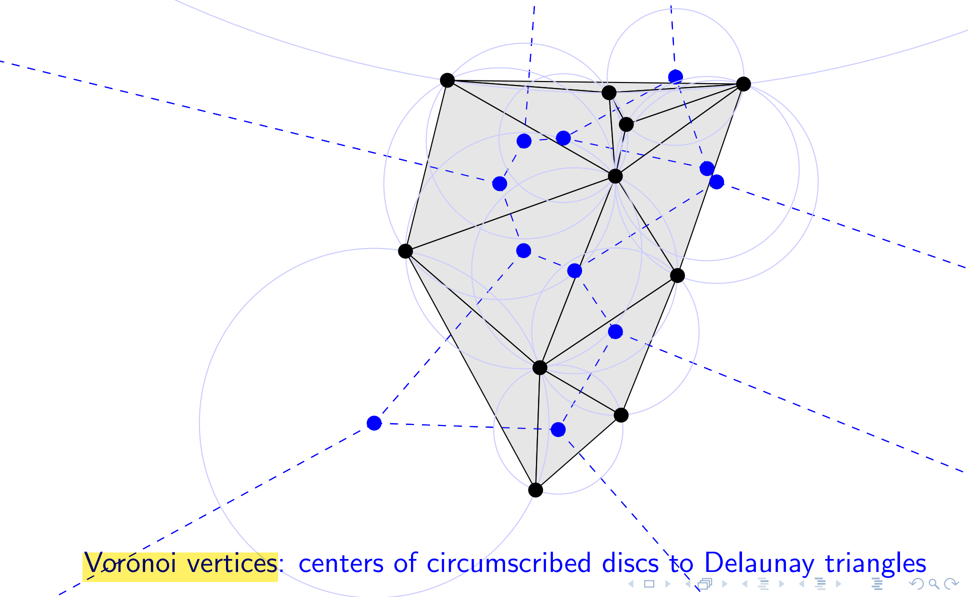
The Delaunay trianglulation is a simplicial complex!



$DT(S) = \bigcup \sigma_{S \cap \partial B}$ where B is an empty ball

e.g., including 2-simplices when empty balls touch 3 points

Duality: Voronoi diagram / Delaunay triangulation

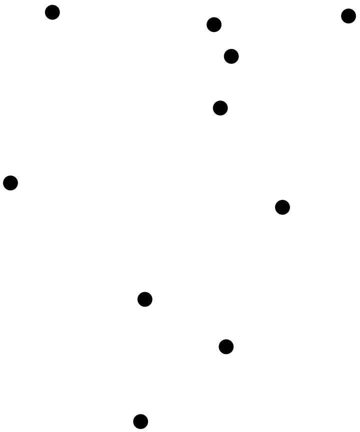


Voronoi vertices: centers of circumscribed discs to Delaunay triangles

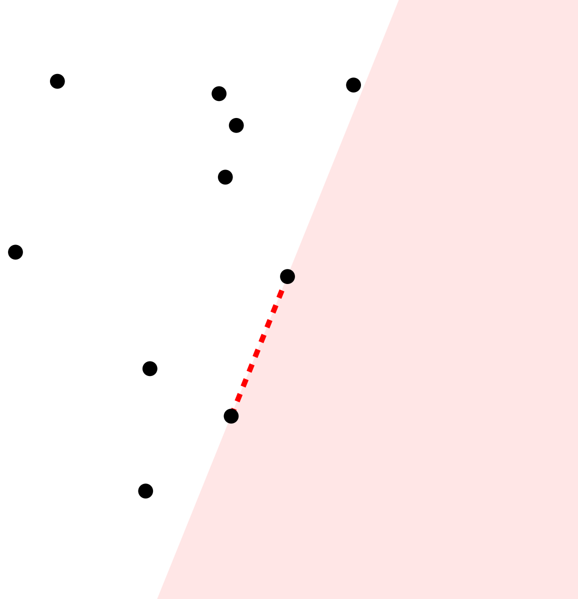
Section 4

Convex hull

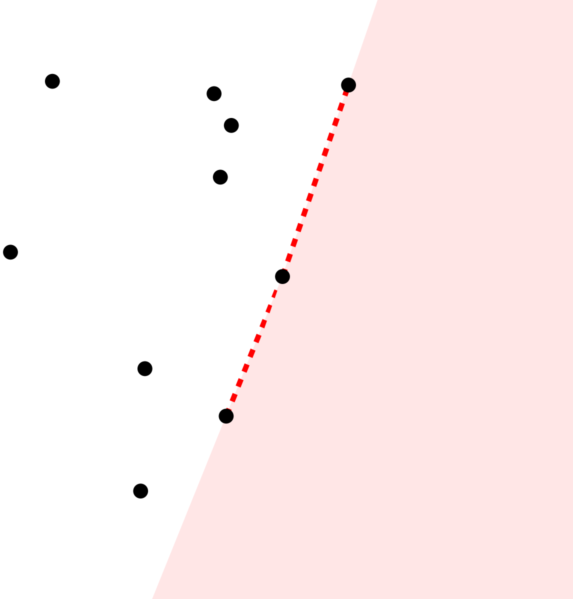
From a set of points. . .



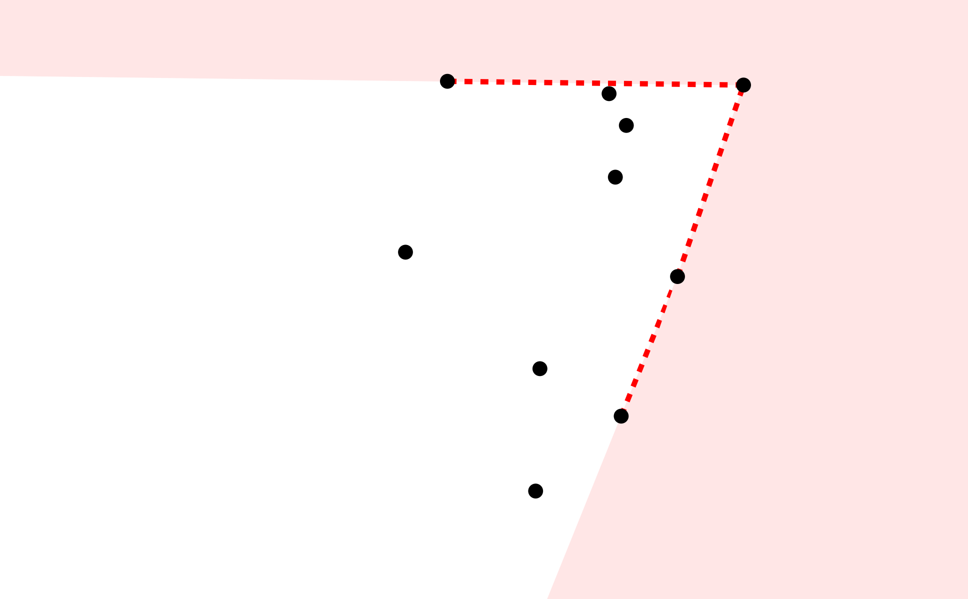
. . . to all empty halfplanes . . .



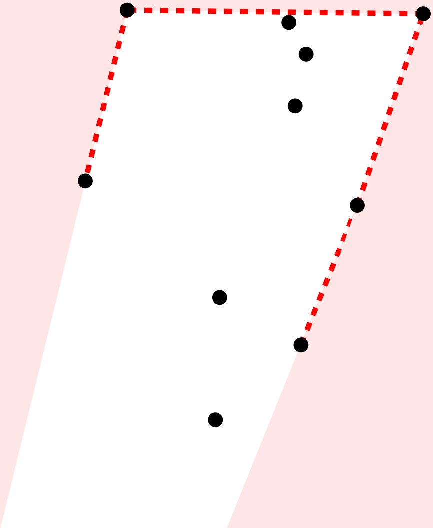
. . . to all empty halfplanes . . .



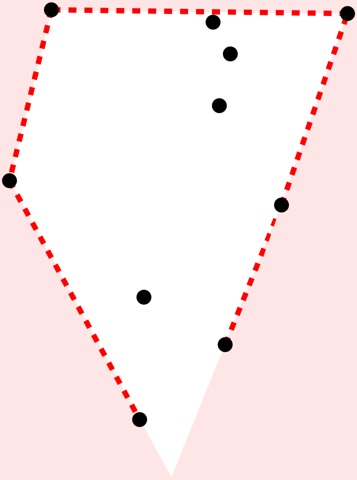
... to all empty halfplanes ...



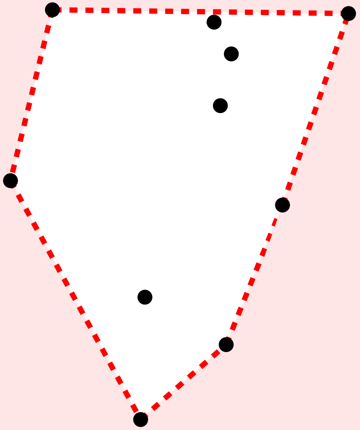
. . . to all empty halfplanes . . .



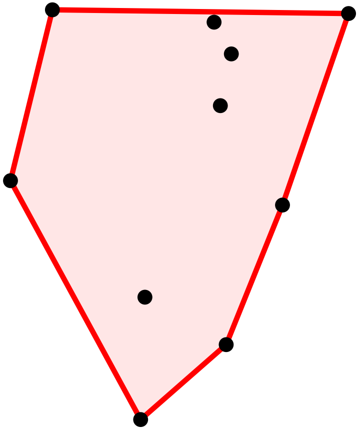
. . . to all empty halfplanes . . .



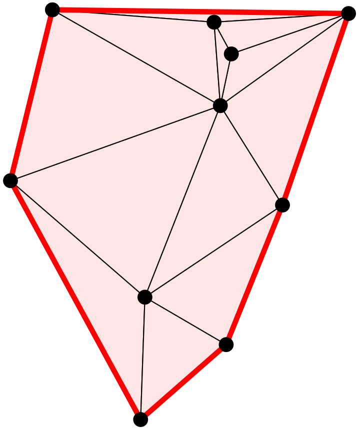
. . . to all empty halfplanes . . .



... to the convex hull (by complementation)



. . . to the simplicial complex $DT(S)$



Convex hull: simplicial complex shape $|DT(S)| = |\overline{DT(S)}|$

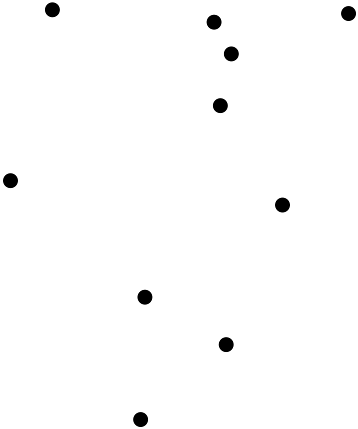
Section 5

Alpha-shapes

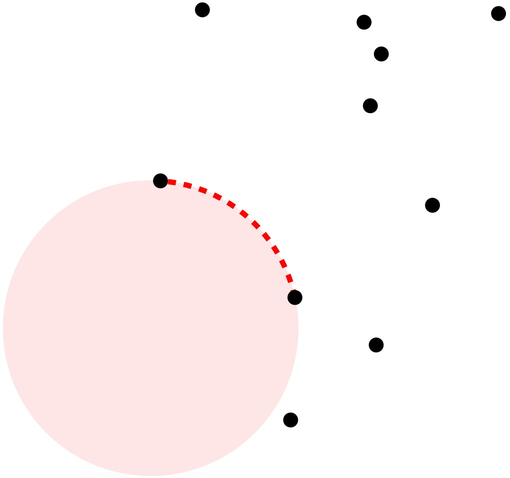
Generalization of the convex hull

- Reconstruction of convex shapes from point clouds: too restrictive!
- Need more flexibility regarding convexity
- Notions of α-complexes and α-shapes
- Subcomplexes of the Delaunay triangulation
- Empty halfplanes $\rightarrow$ empty discs (of radius α)

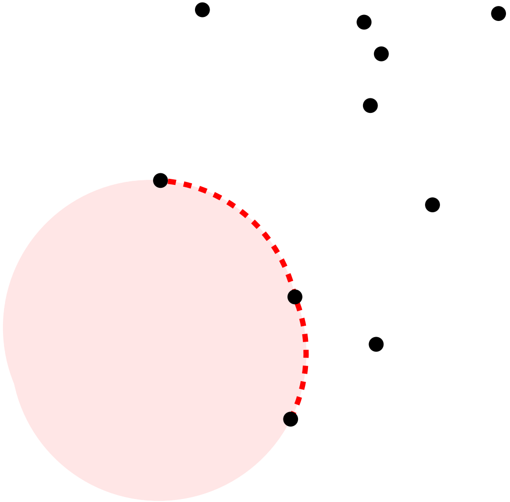
From a set of points. . .



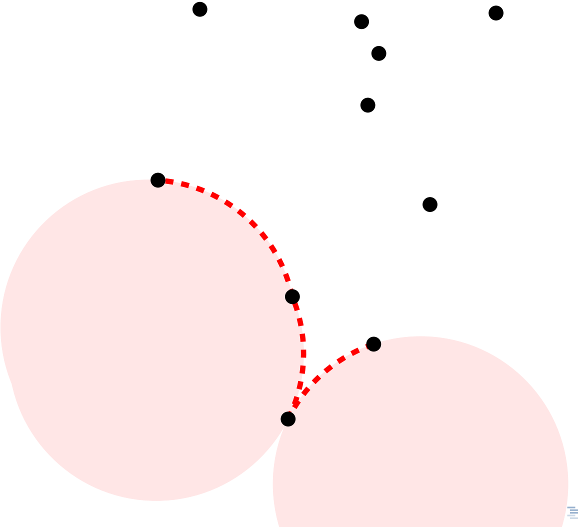
. . . to all empty α-disks . . .



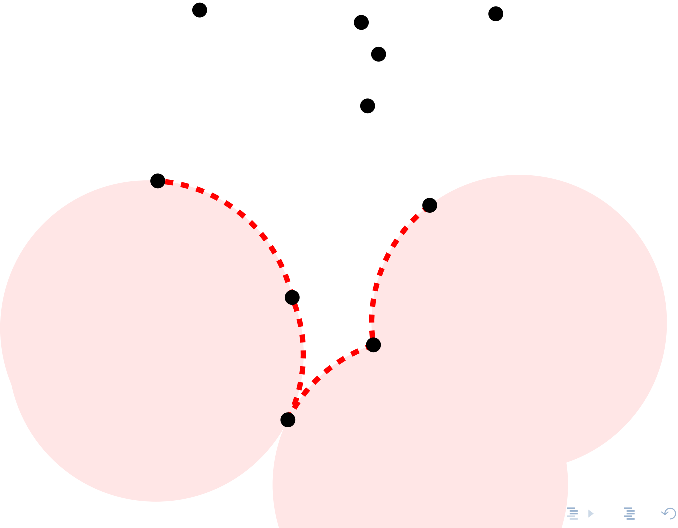
. . . to all empty α-disks . . .



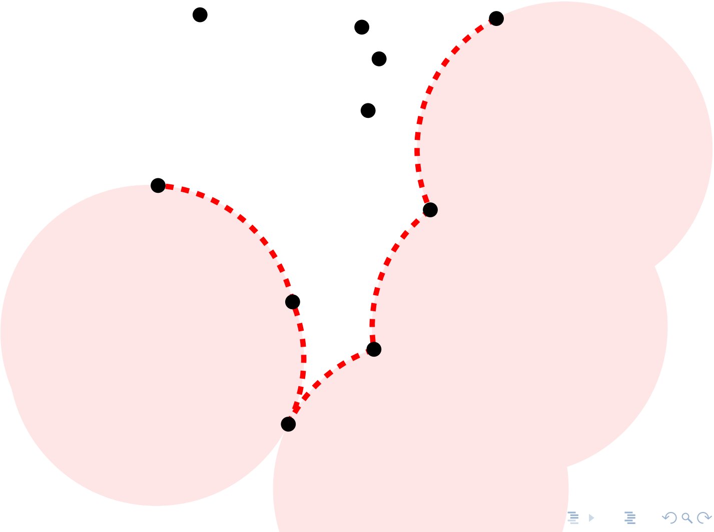
. . . to all empty α-disks . . .



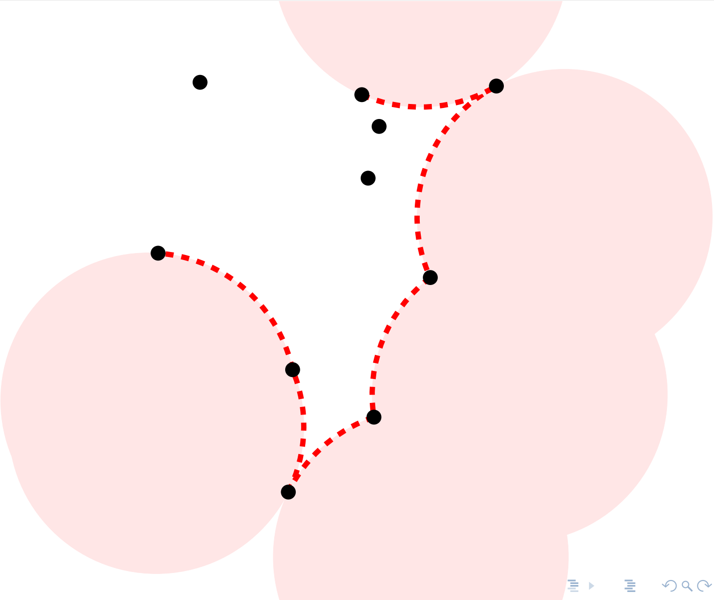
. . . to all empty α-disks . . .



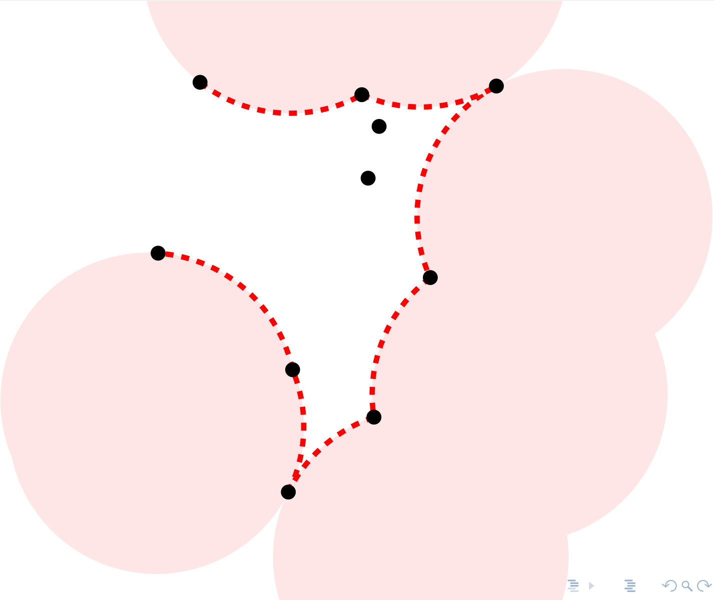
. . . to all empty α-disks . . .



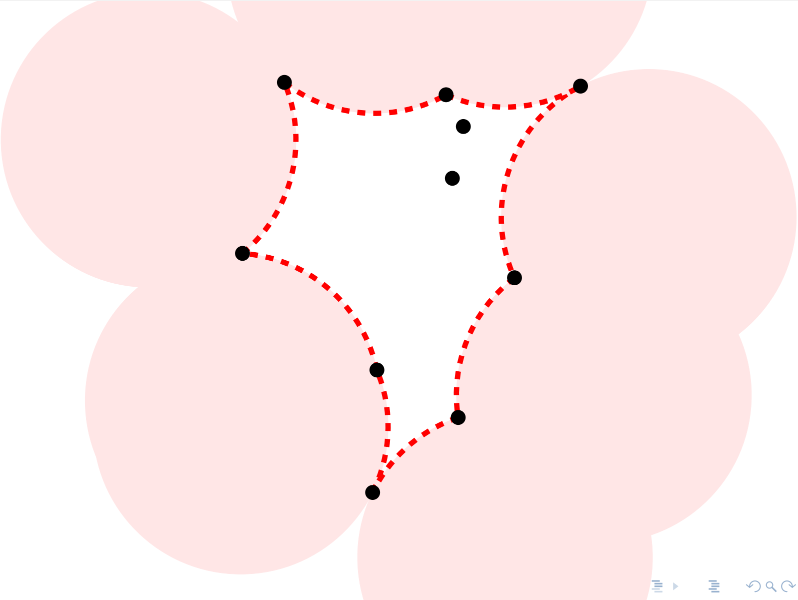
. . . to all empty α-disks . . .



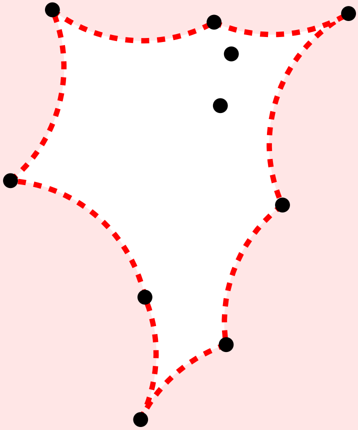
. . . to all empty α-disks . . .



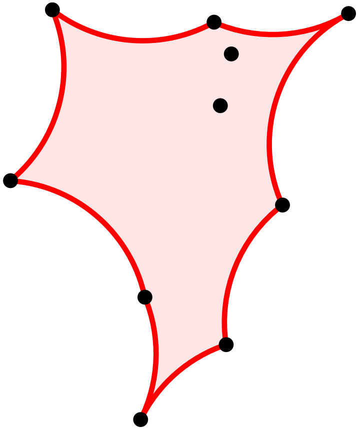
. . . to all empty α-disks . . .



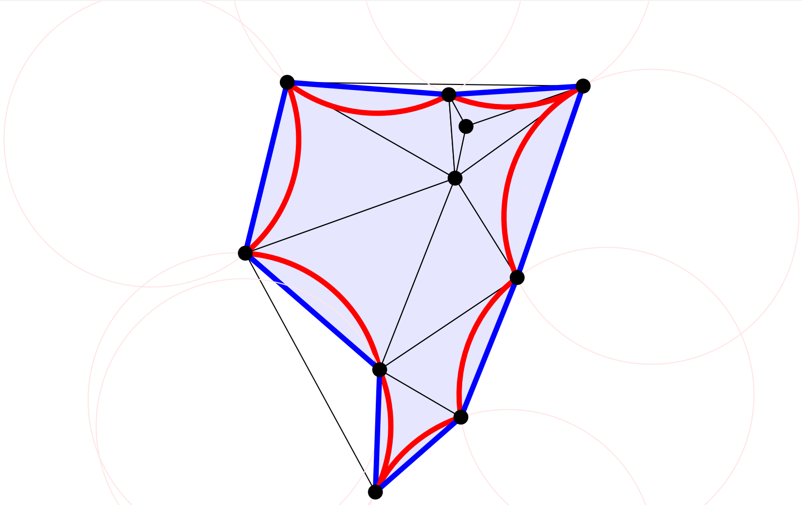
... to all empty α-disks ...



. . . to the α-hull



. . . to the α-complex



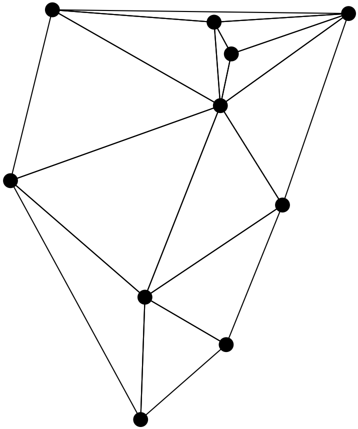
$$C_\alpha(S) = \bigcup \sigma_{S \cap \partial B_{\leq\alpha}}$$

where $B_{\leq\alpha}$ is an empty disc of radius $\leq \alpha$

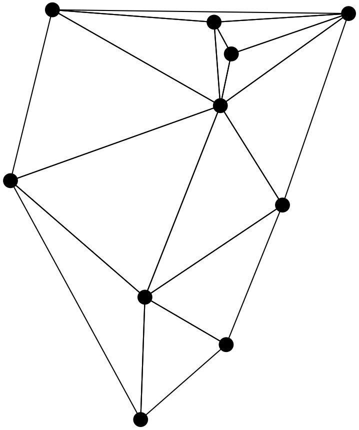
. . . to the α-shape

- α-complex: $C_\alpha(S)$
- α-shape: $S_\alpha(S) = |C_\alpha(S)|$
- Solid α-shape: $\overline{S_\alpha}(S) = |\overline{C_\alpha(S)}|$

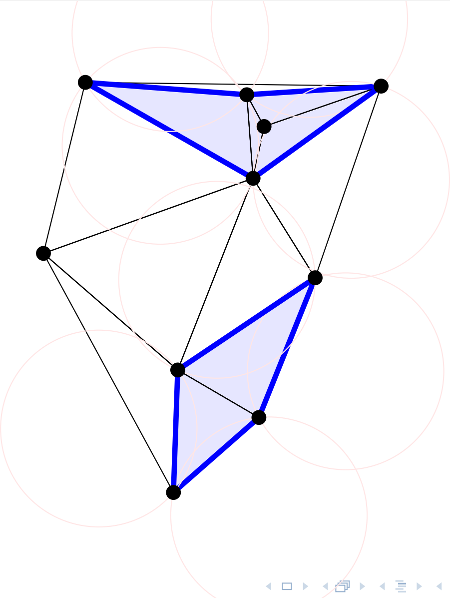
$\overline{S_0}(S)$: empty set



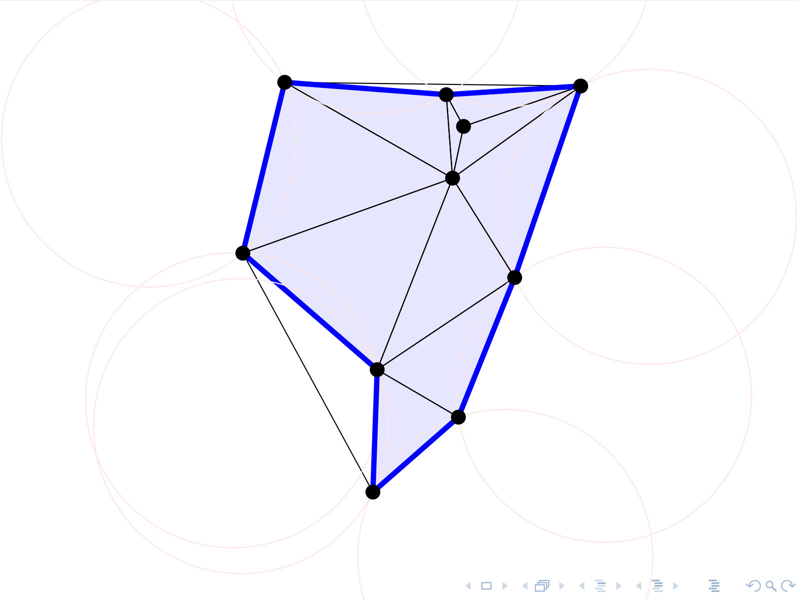
$\overline{S_{0.1}}(S)$



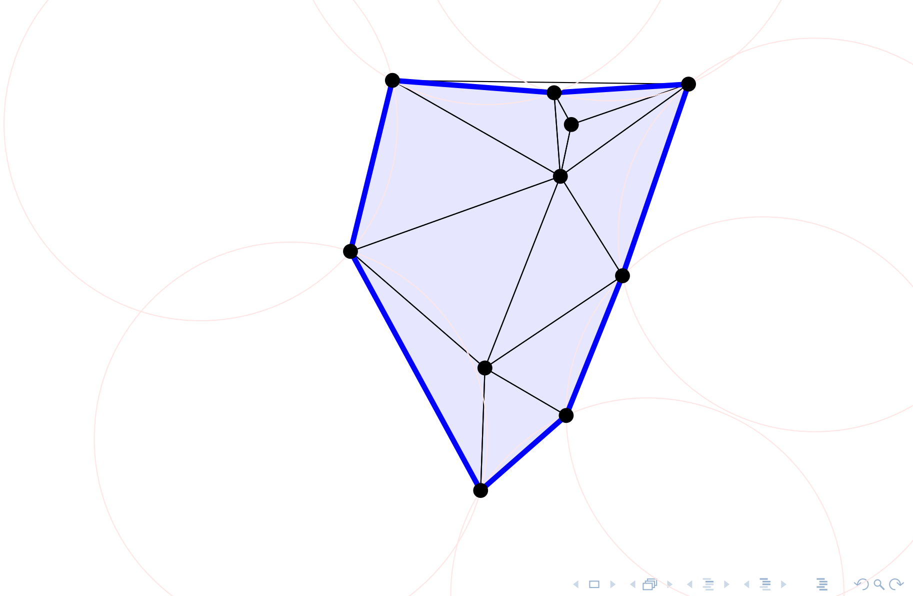
$\overline{S_{0.2}}(S)$



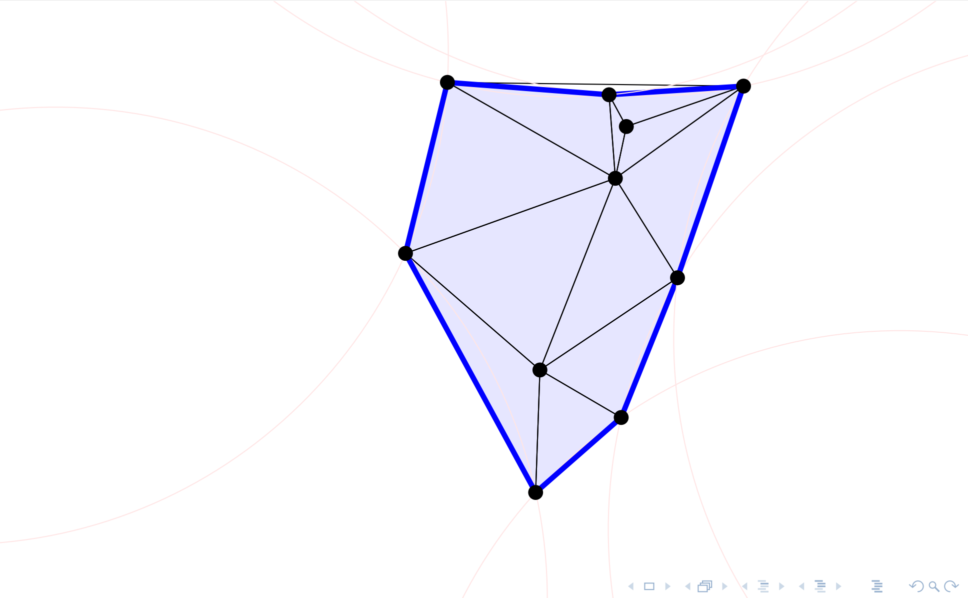
$\overline{S_{0.3}}(S)$



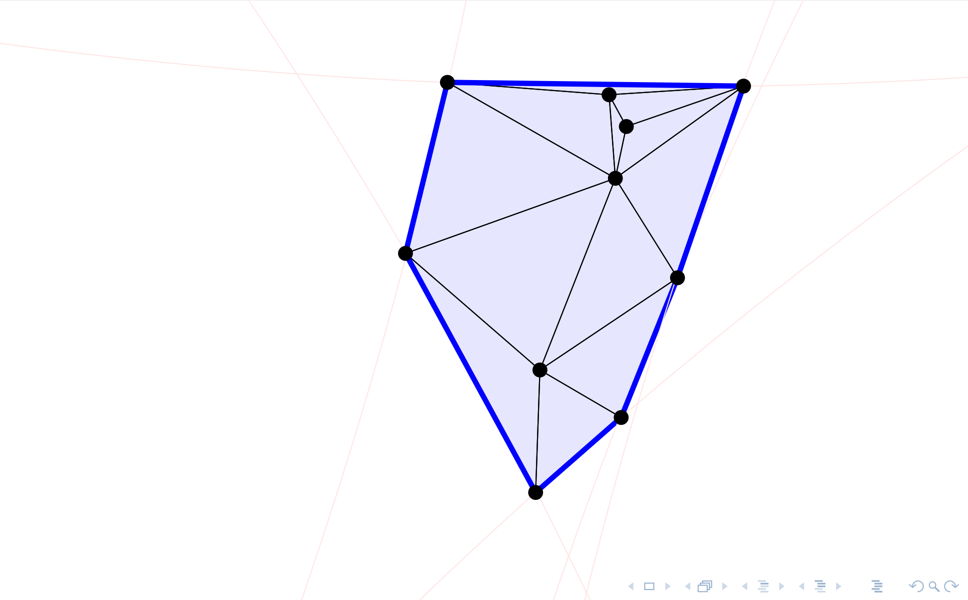
$\overline{S_{0.4}}(S)$



$\overline{S_1}(S)$



$\overline{S_{10}}(S) = \overline{S_\infty}(S)$: convex hull



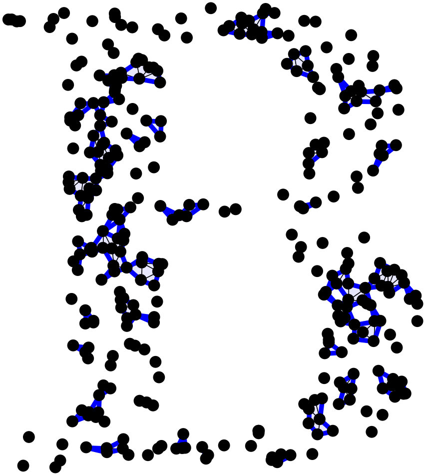
Section 6

Illustrations

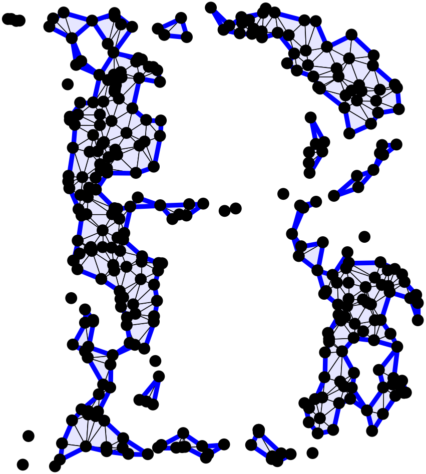
$\overline{S_0}(S)$: empty set

$\overline{S_{0.01}}(S)$

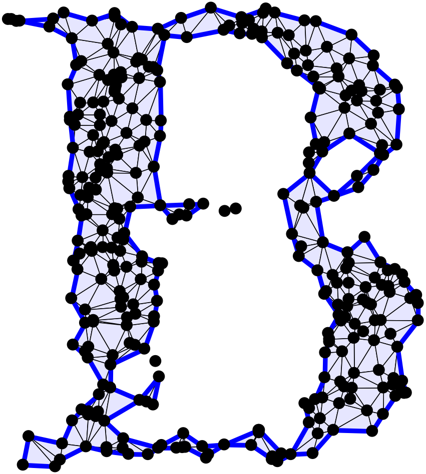
$\overline{S_{0.02}}(S)$



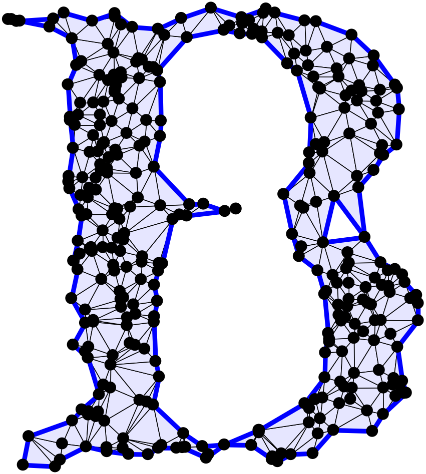
$\overline{S_{0.03}}(S)$



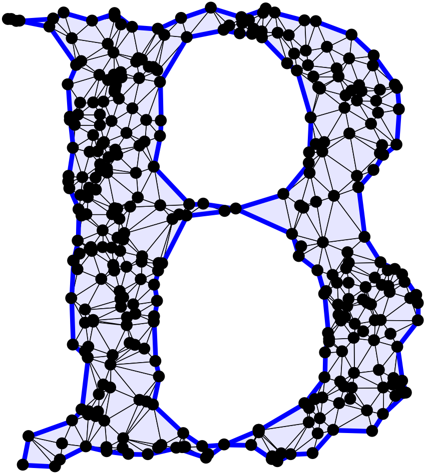
$\overline{S_{0.04}}(S)$



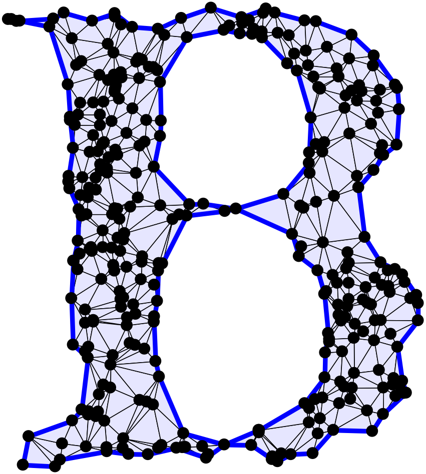
$\overline{S_{0.05}}(S)$



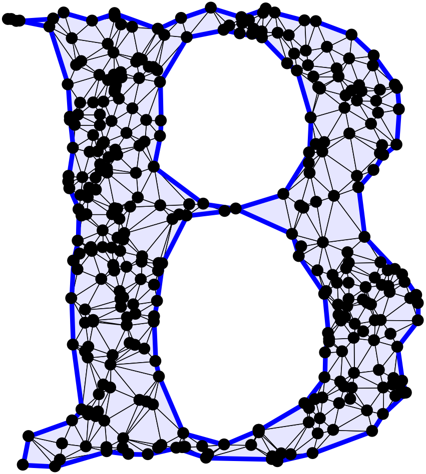
$\overline{S_{0.06}}(S)$



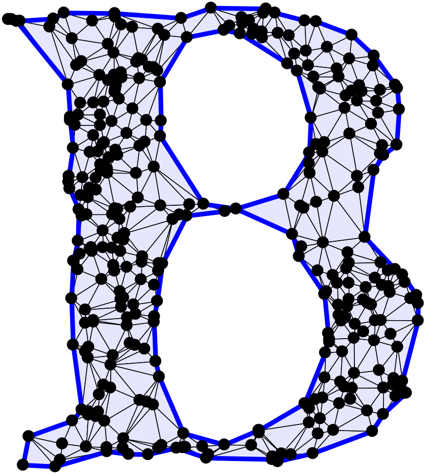
$\overline{S_{0.07}}(S)$



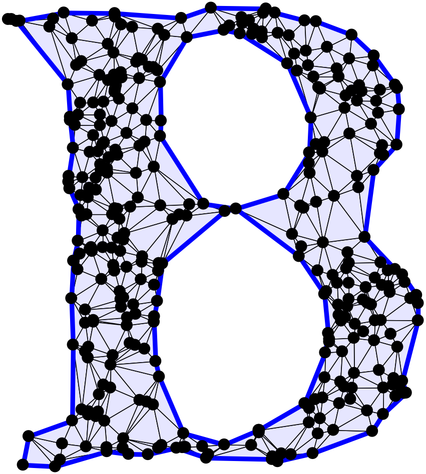
$\overline{S_{0.08}}(S)$



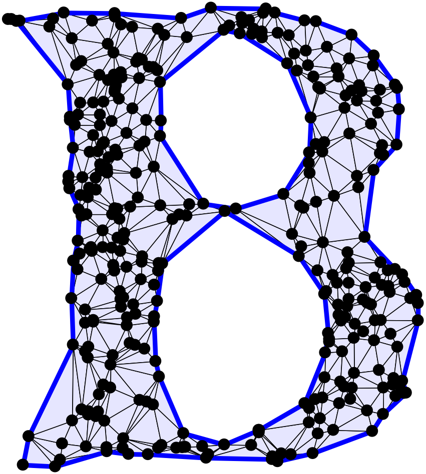
$\overline{S_{0.09}}(S)$



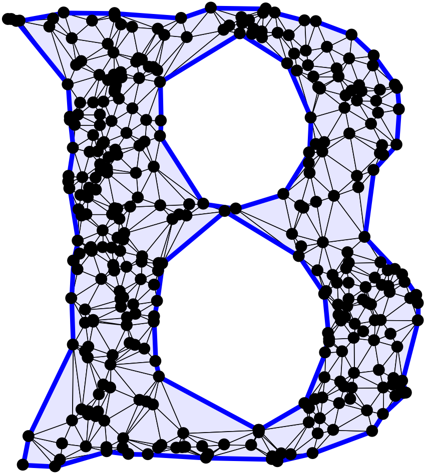
$\overline{S_{0.1}}(S)$



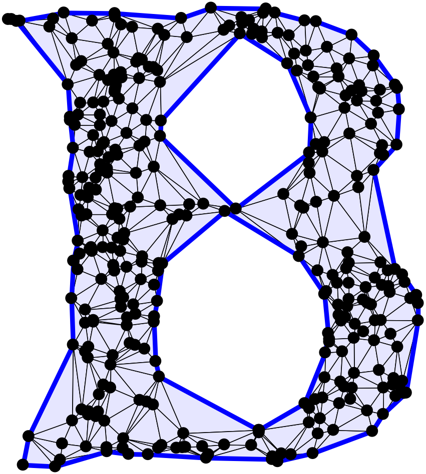
$\overline{S_{0.11}}(S)$



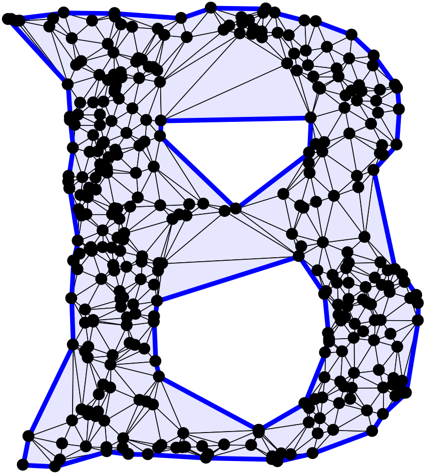
$\overline{S_{0.12}}(S)$



$\overline{S_{0.13}}(S)$



$\overline{S_{0.14}}(S)$



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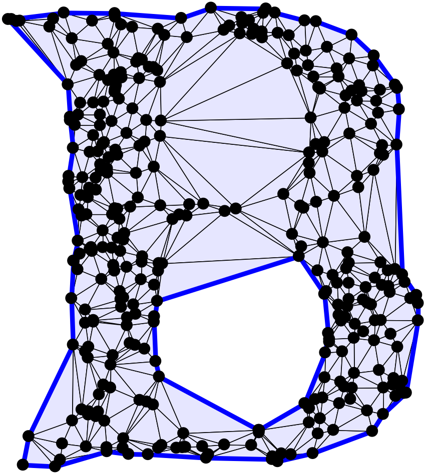

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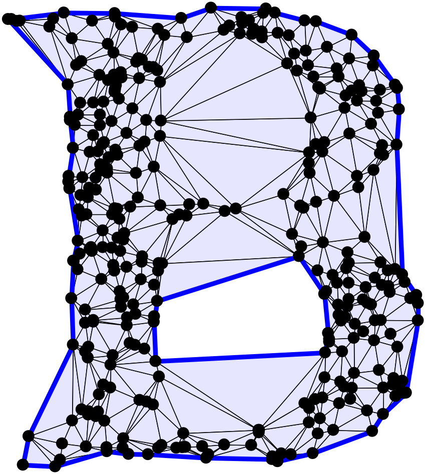

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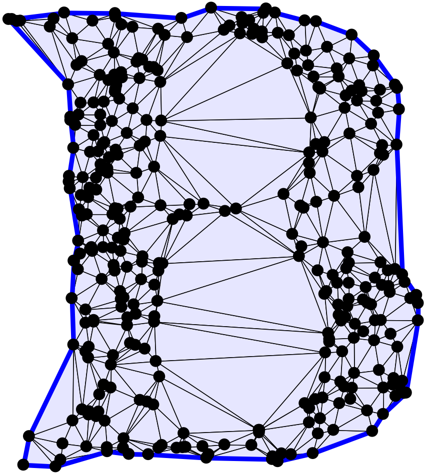

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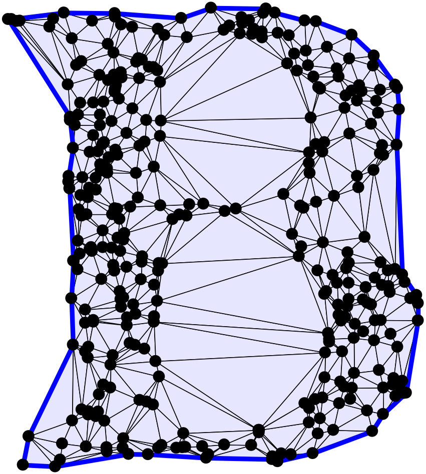

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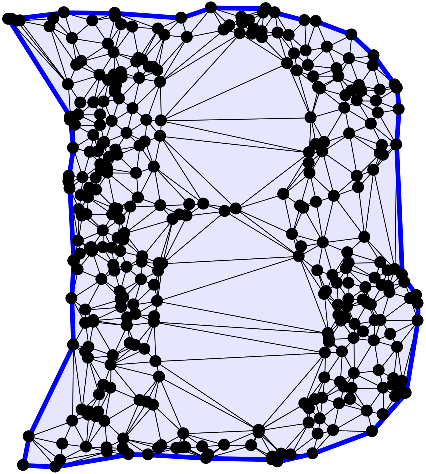

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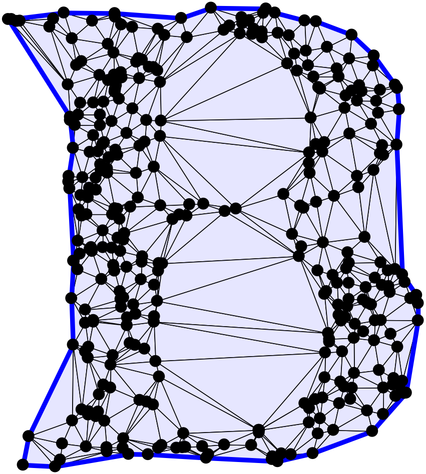

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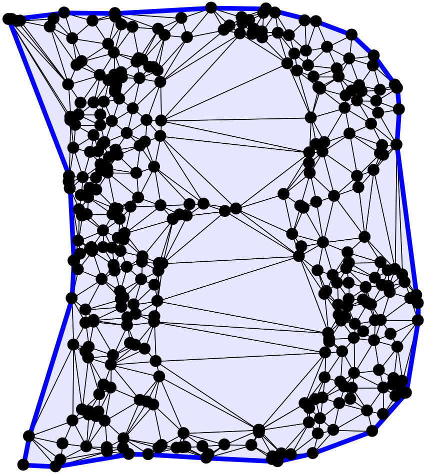

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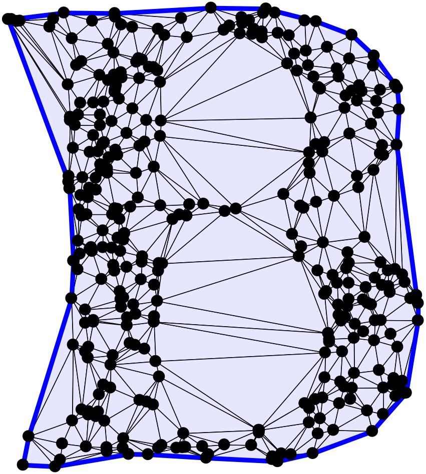

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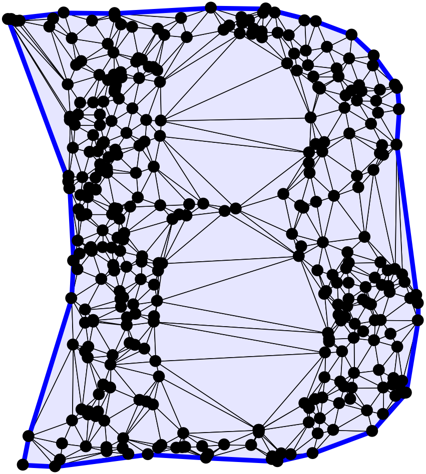

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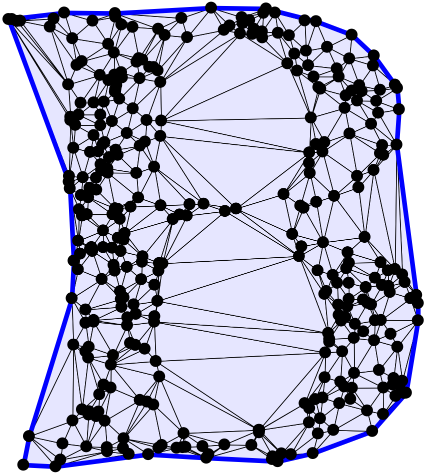

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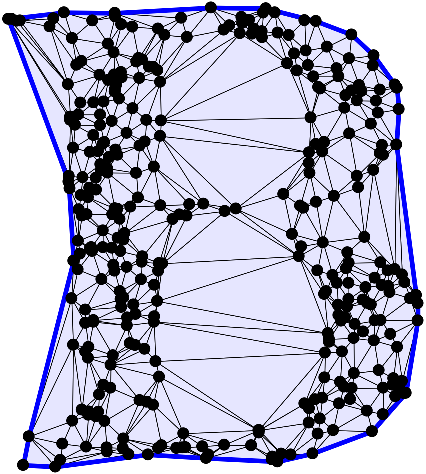
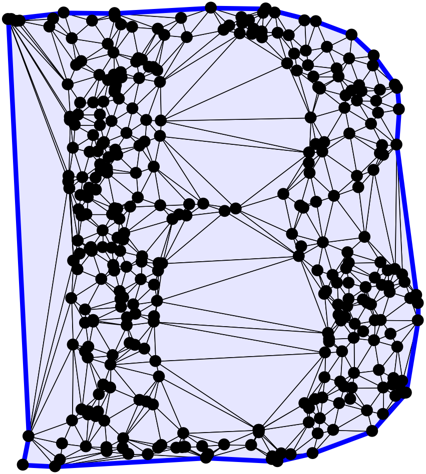
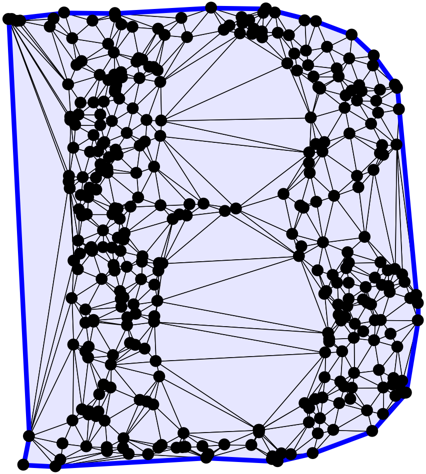


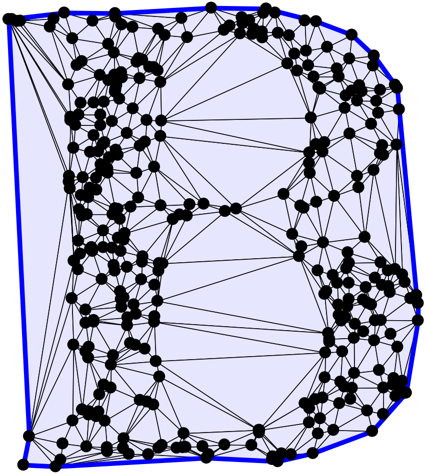
$\overline{S_{0.8}(S)}$



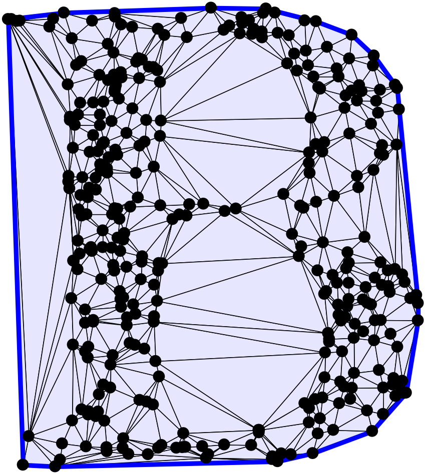
$\overline{S_{0.9}}(S)$



$\overline{S_1}(S)$

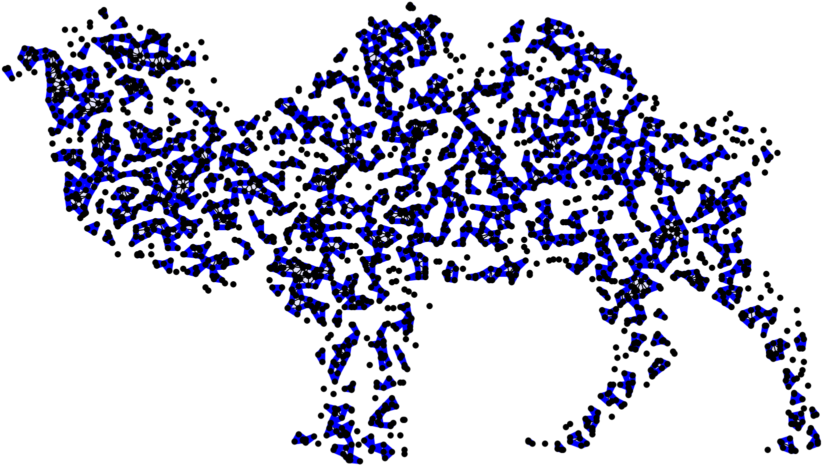


$\overline{S_{10}}(S) = \overline{S_\infty}(S)$: convex hull



$\overline{S_0}(S)$: empty set

$\overline{S_{0.01}}(S)$



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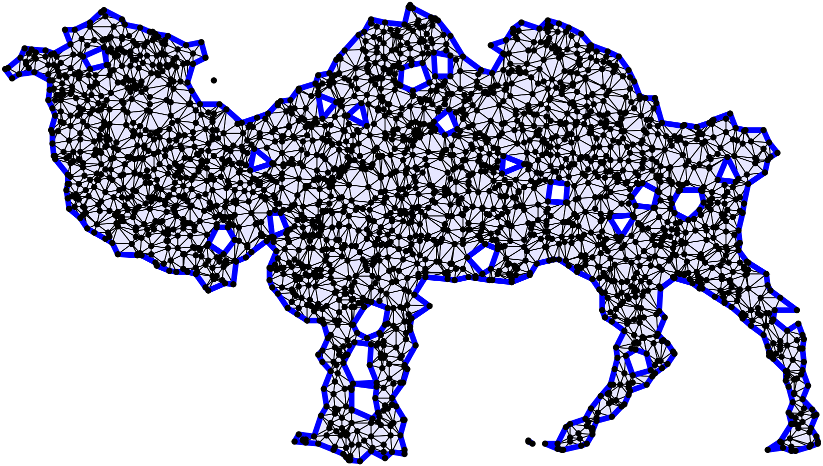

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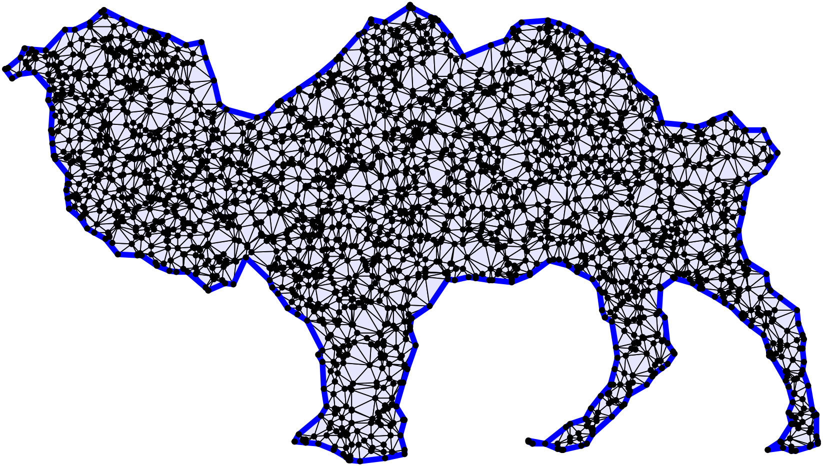
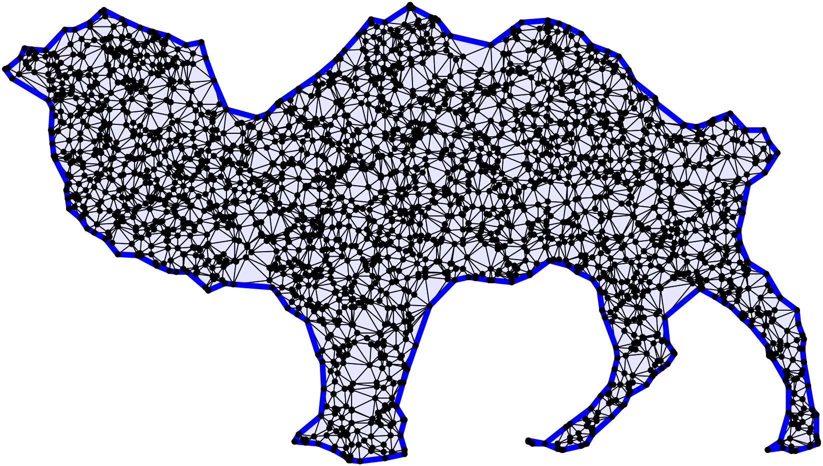
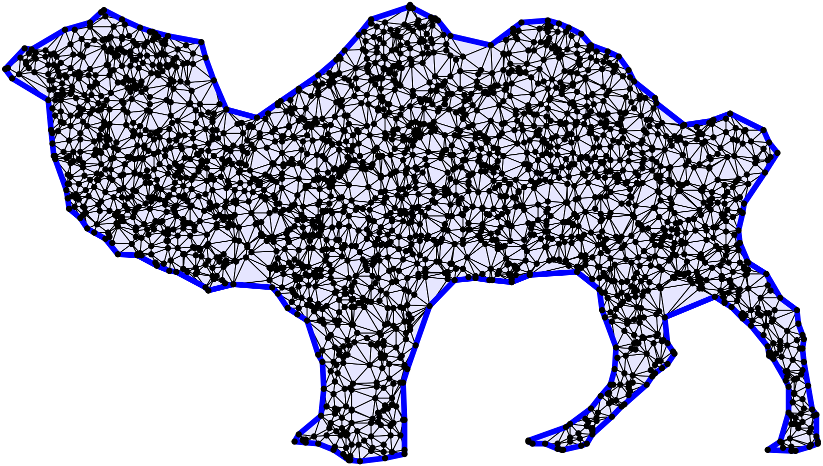


$\overline{S_{0.04}}(S)$



$\overline{S_{0.05}}(S)$



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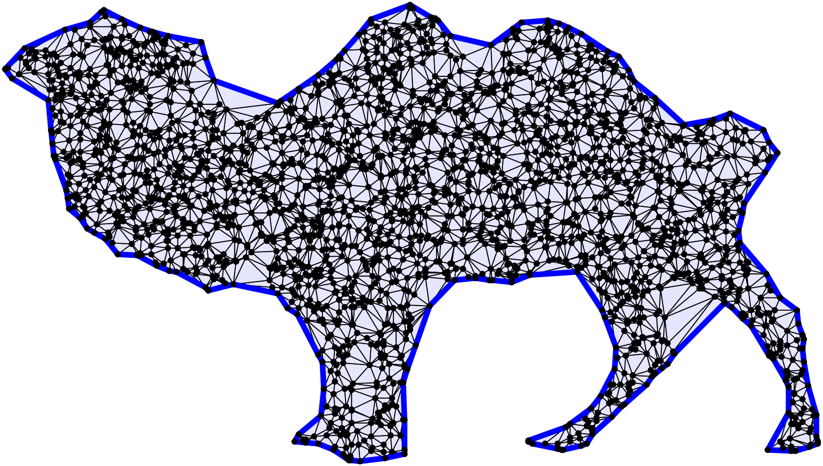

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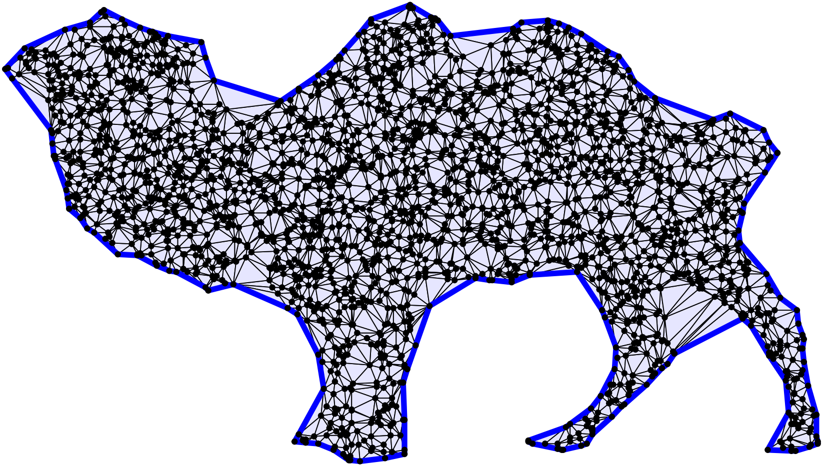
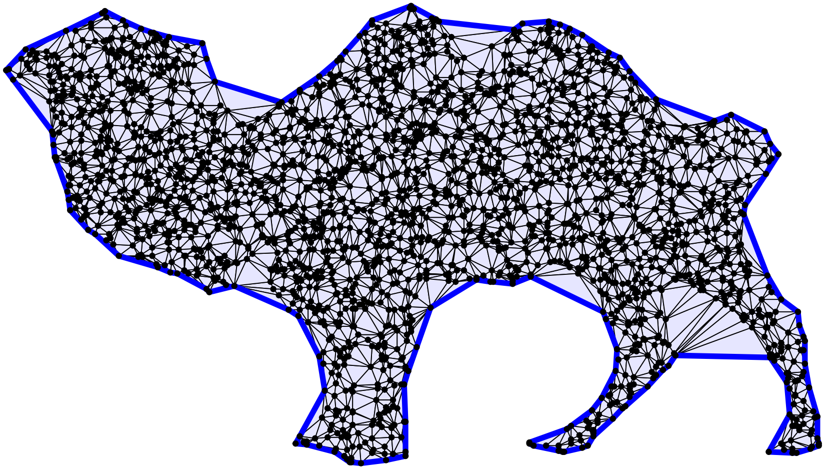
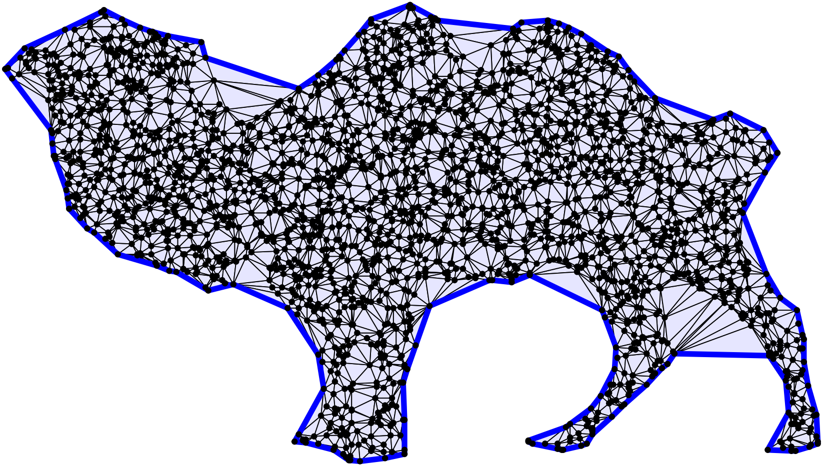


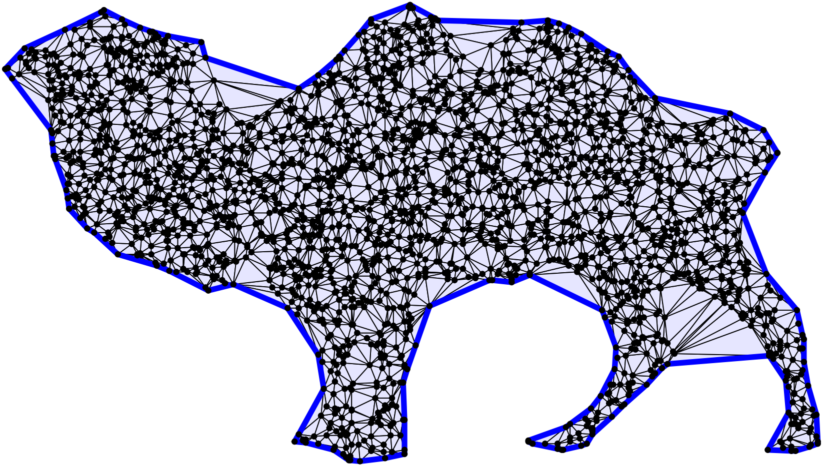
$\overline{S_{0.08}}(S)$



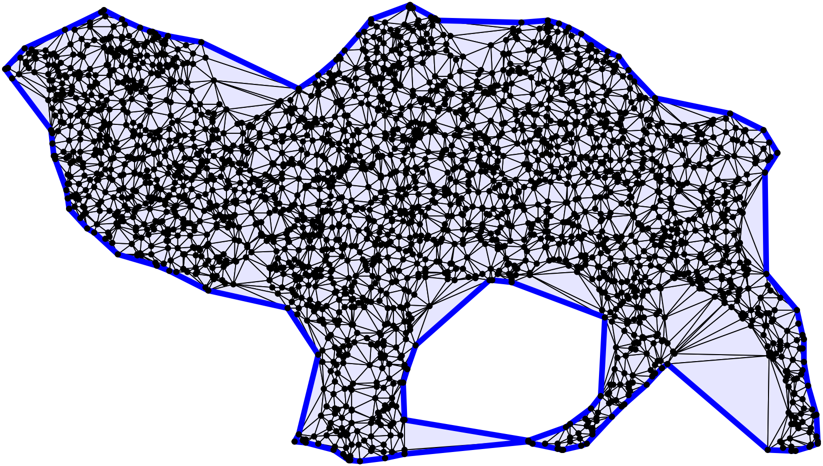
$\overline{S_{0.09}}(S)$



$\overline{S_{0.1}}(S)$



$\overline{S_{0.11}}(S)$



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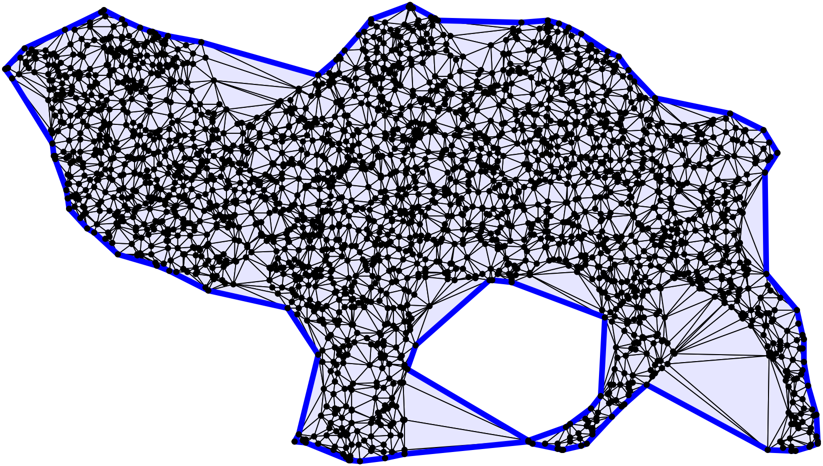

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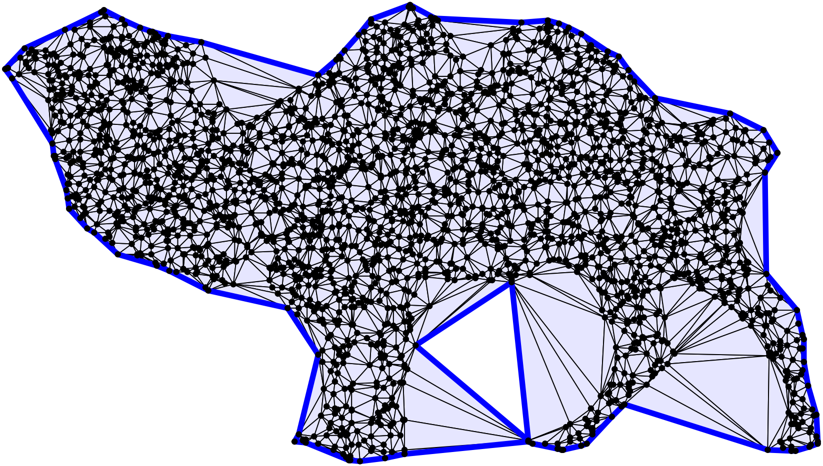

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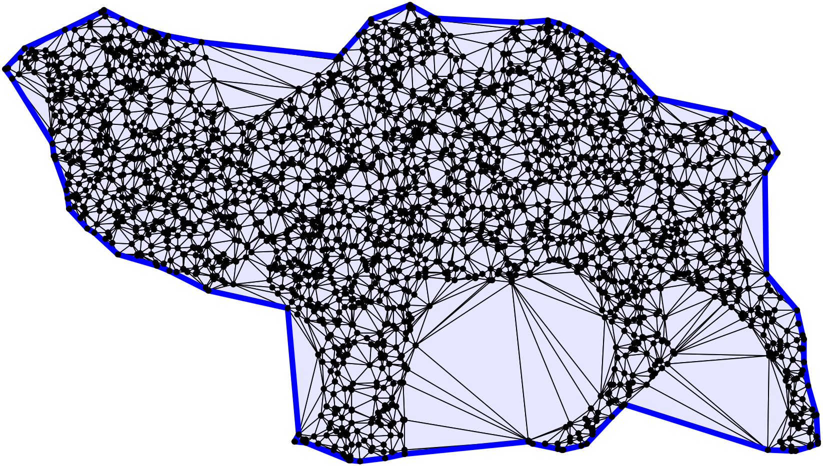

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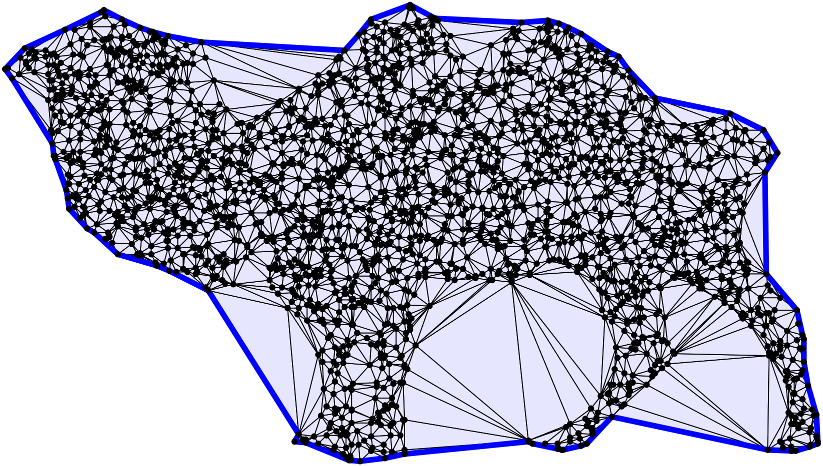

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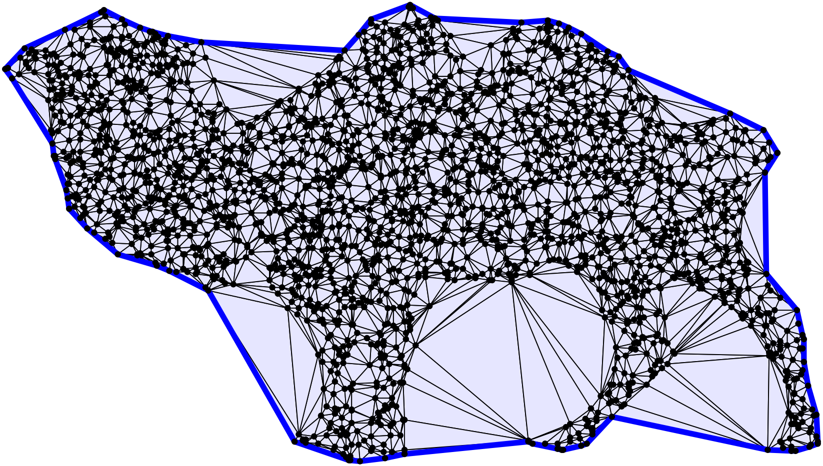
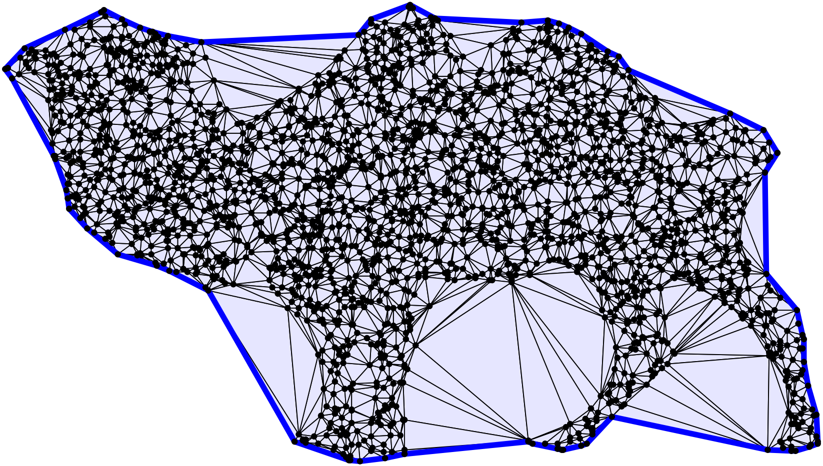


$\overline{S_{0.17}}(S)$



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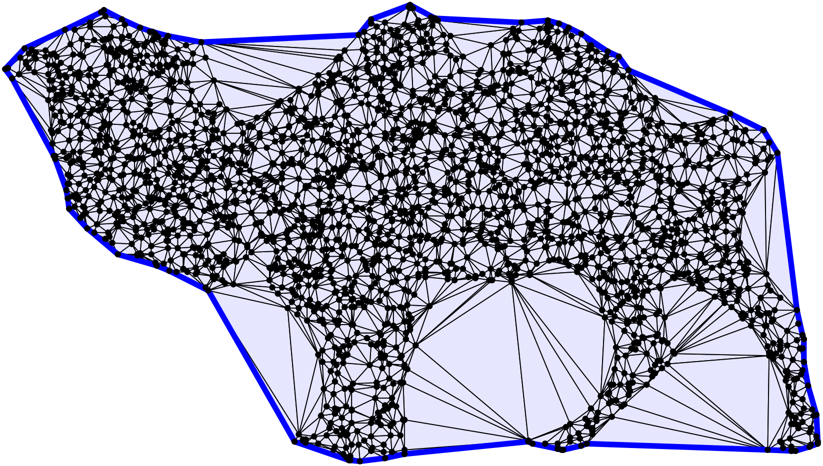

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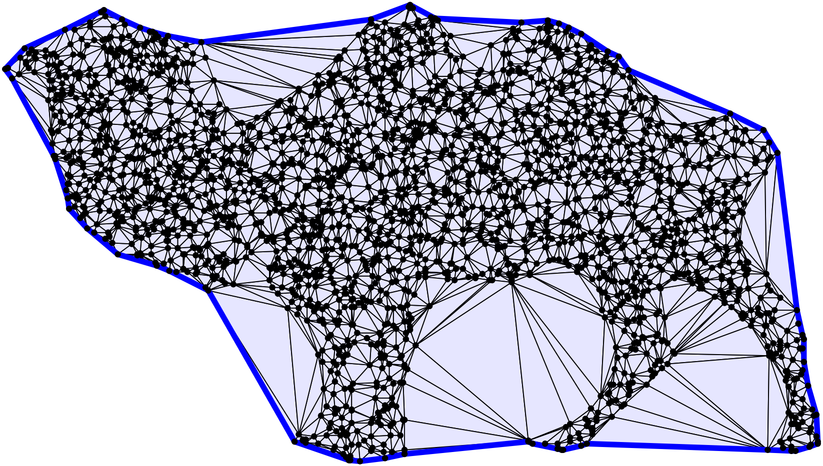
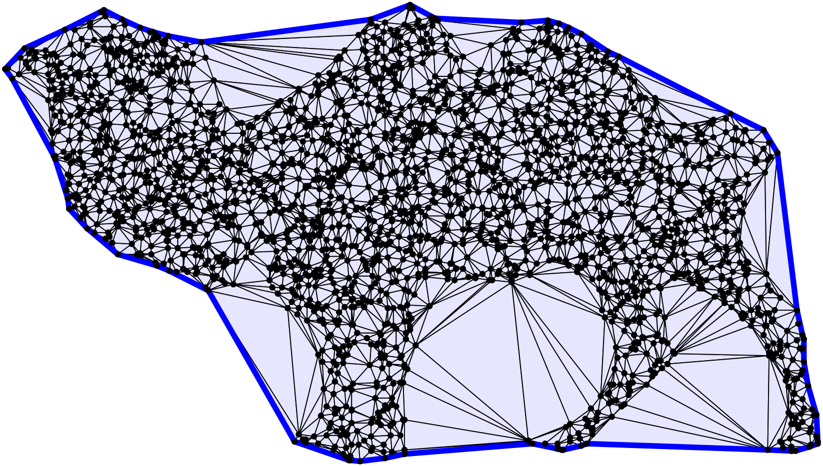


$\overline{S_{0.2}}(S)$



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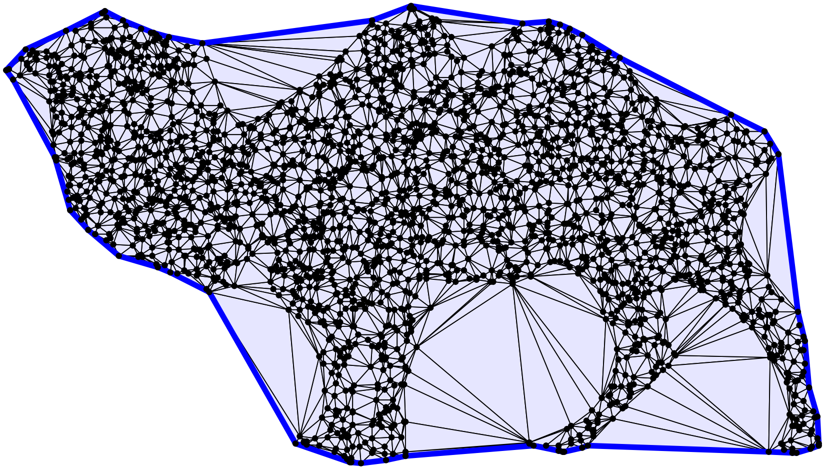

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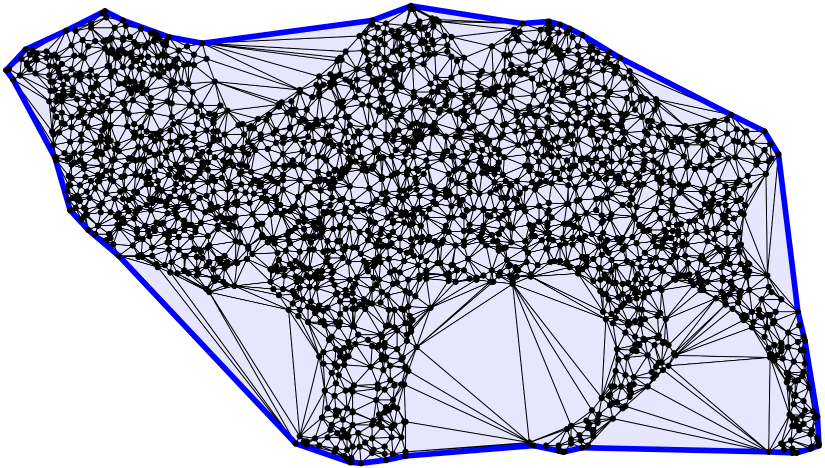

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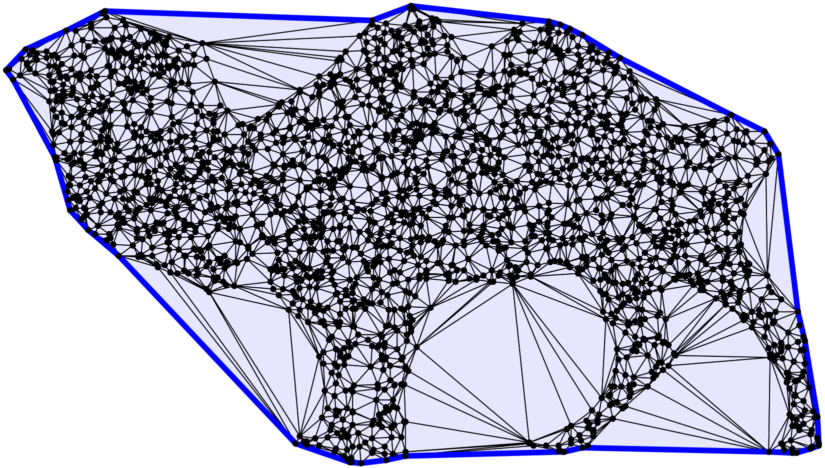

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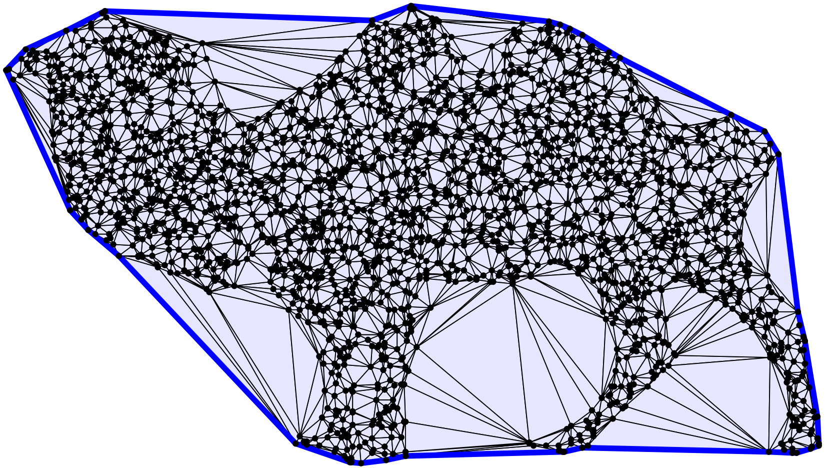

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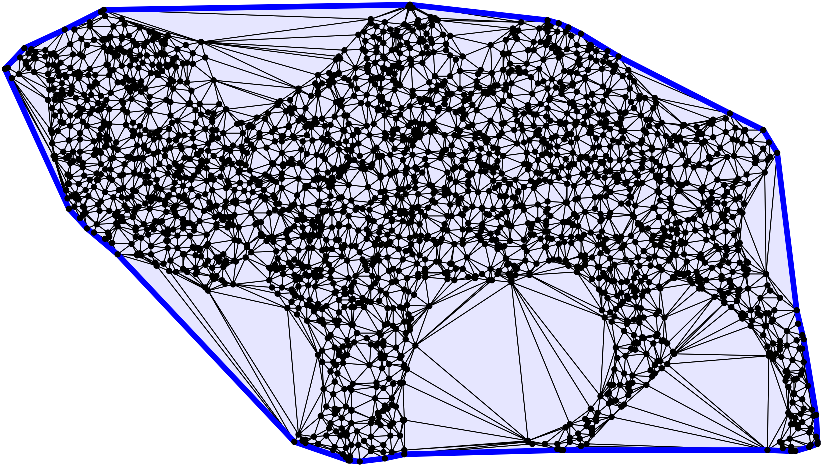

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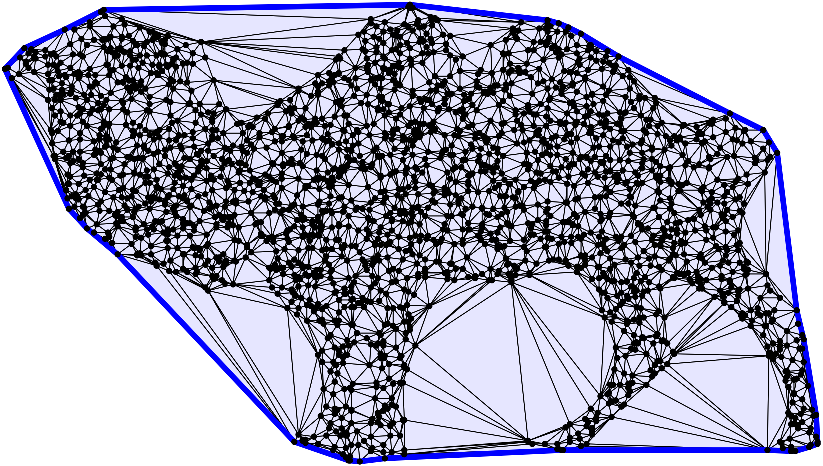
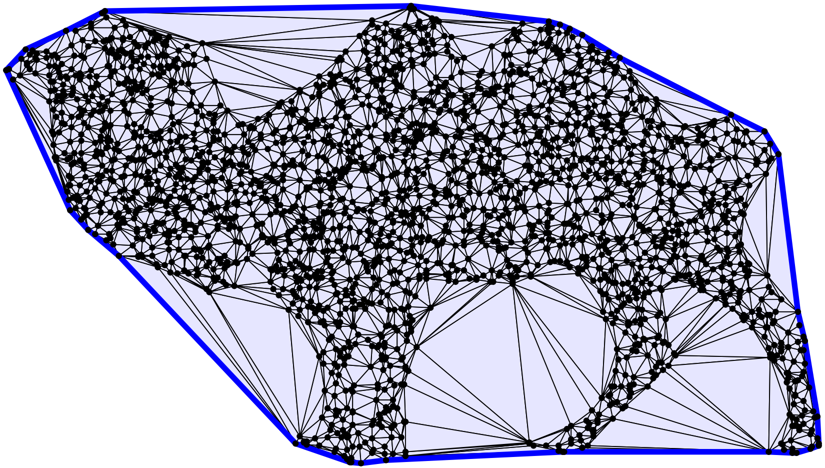
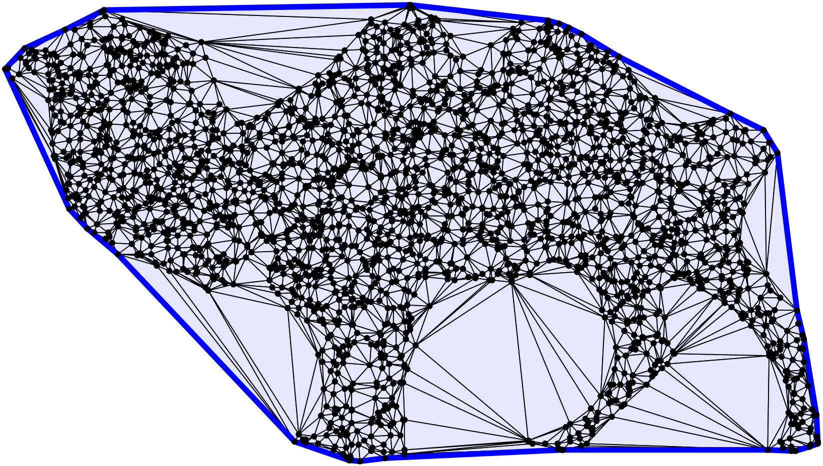


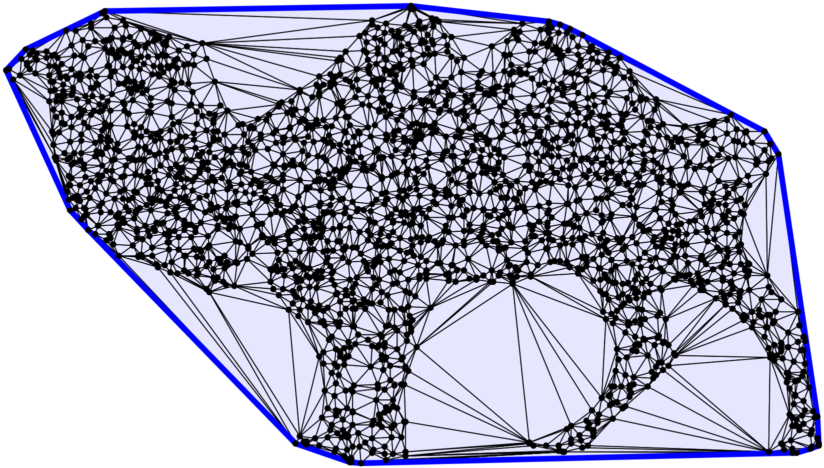
$\overline{S_{0.9}}(S)$



$\overline{S_1}(S)$



$\overline{S_{10}}(S) = \overline{S_\infty}(S)$: convex hull

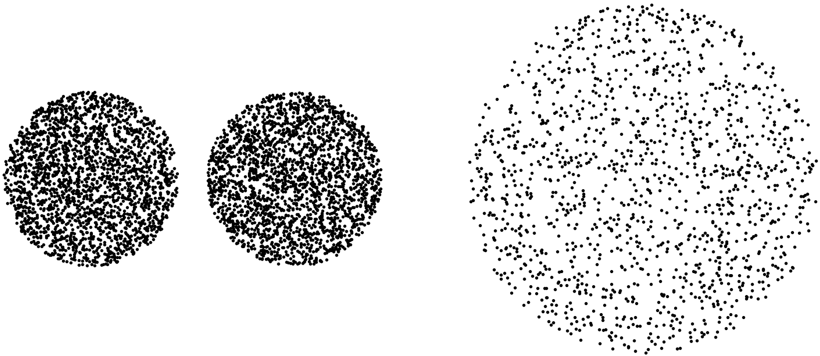


Section 7

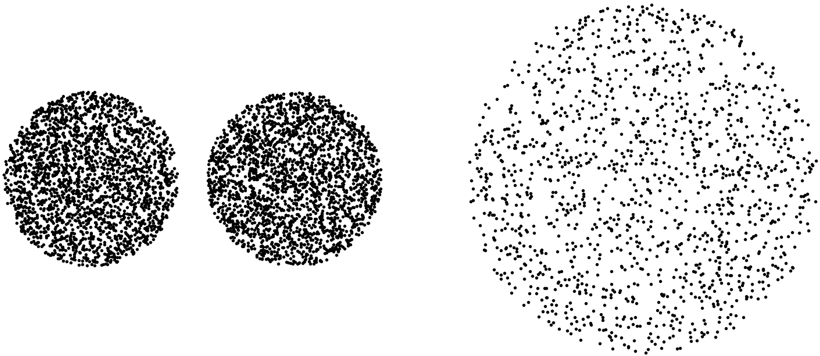
LDA alpha-shapes

Inhomogeneous point clouds?

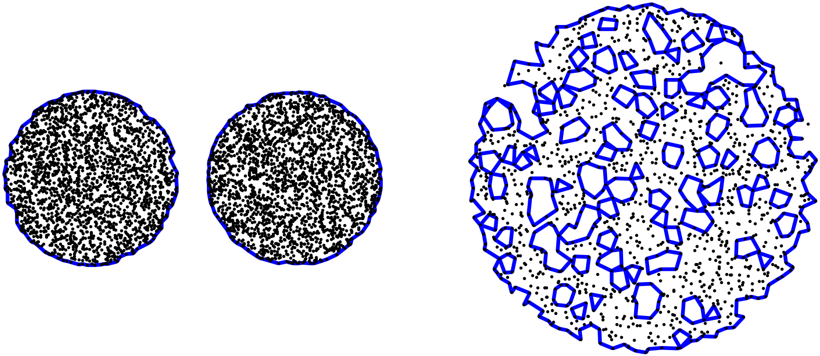
Inhomogeneous point clouds?



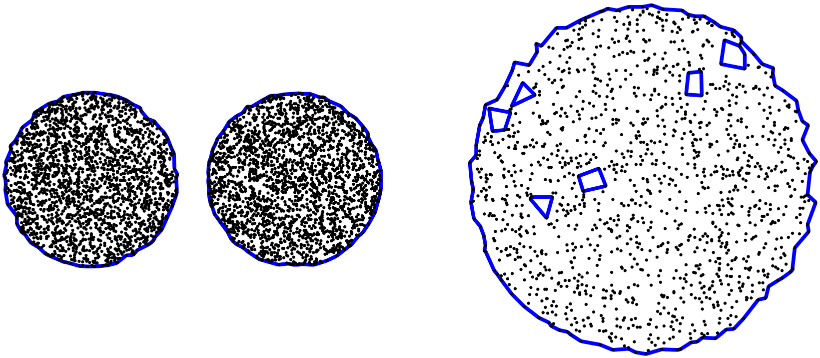
$\overline{S_0}(S)$: empty set



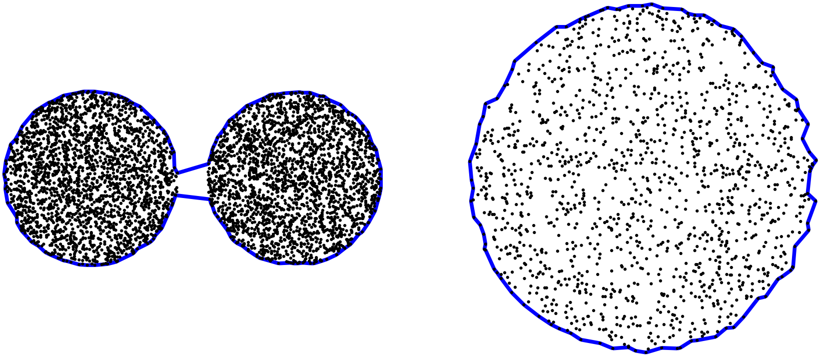
$\overline{S_{0.01}}(S)$



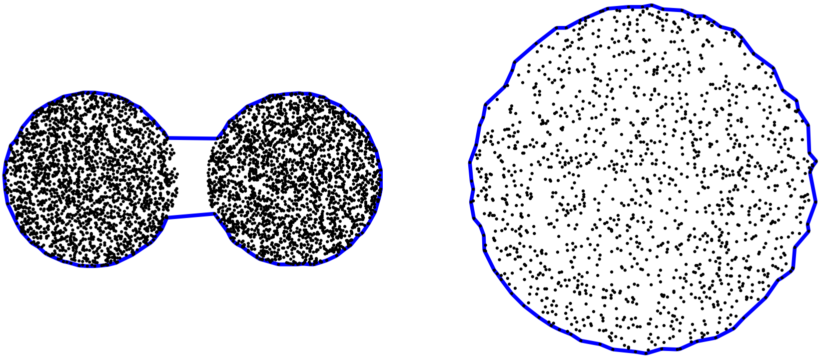
$\overline{S_{0.015}}(S)$



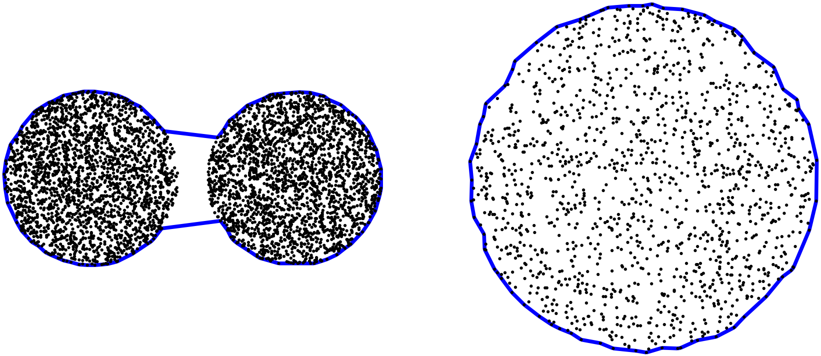
$\overline{S_{0.02}}(S)$



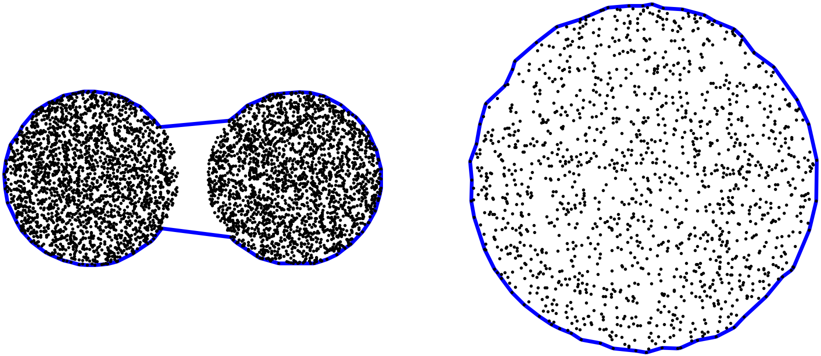
$\overline{S_{0.03}}(S)$



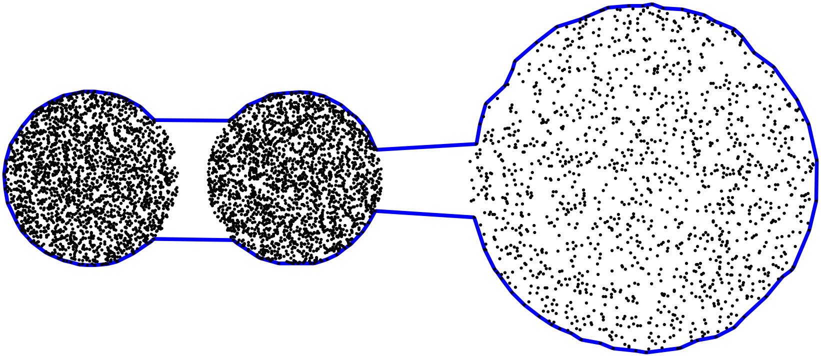
$\overline{S_{0.04}}(S)$



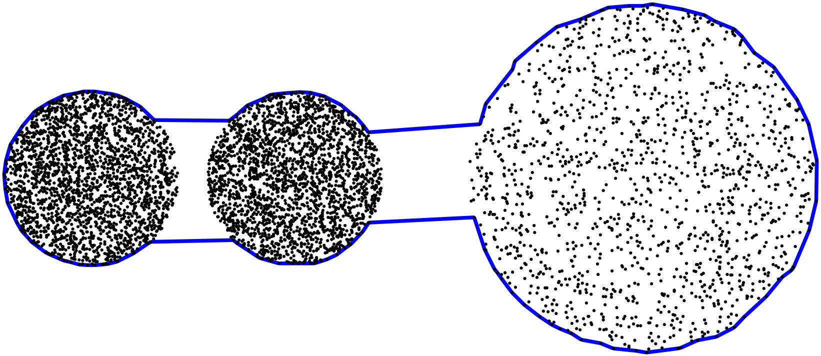
$\overline{S_{0.05}}(S)$



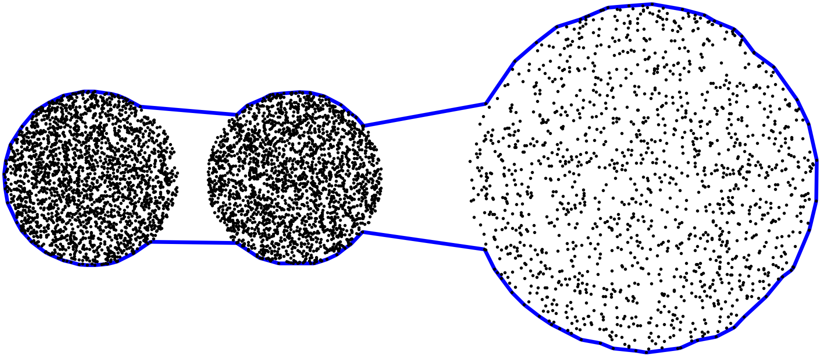
$\overline{S_{0.06}}(S)$



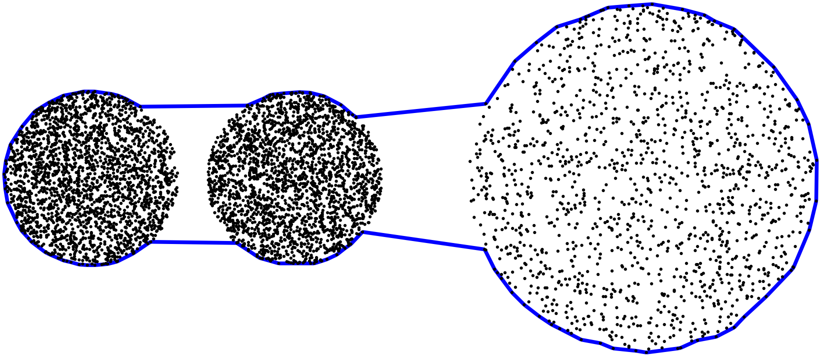
$\overline{S_{0.07}}(S)$



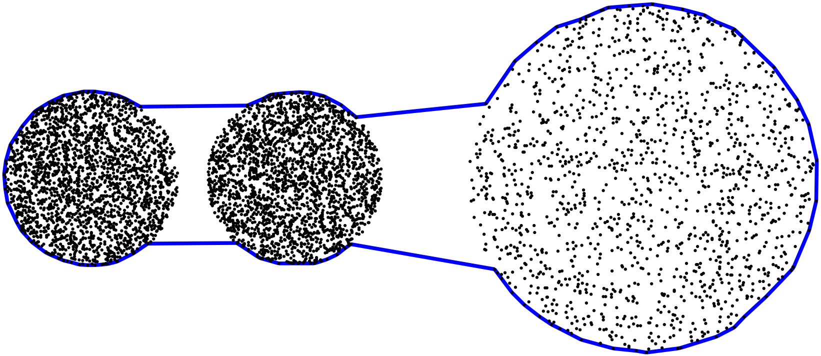
$\overline{S_{0.08}}(S)$



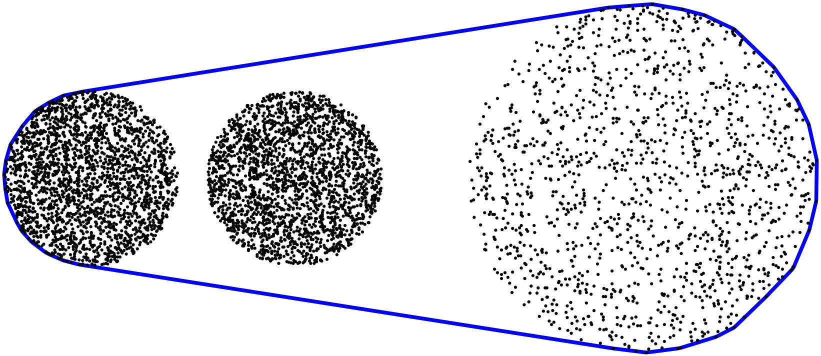
$\overline{S_{0.09}}(S)$



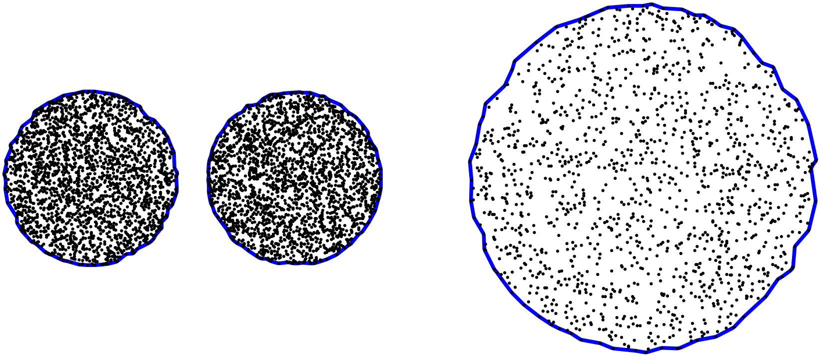
$\overline{S_{0.1}}(S)$



$\overline{S_{10}}(S) = \overline{S_\infty}(S)$: convex hull



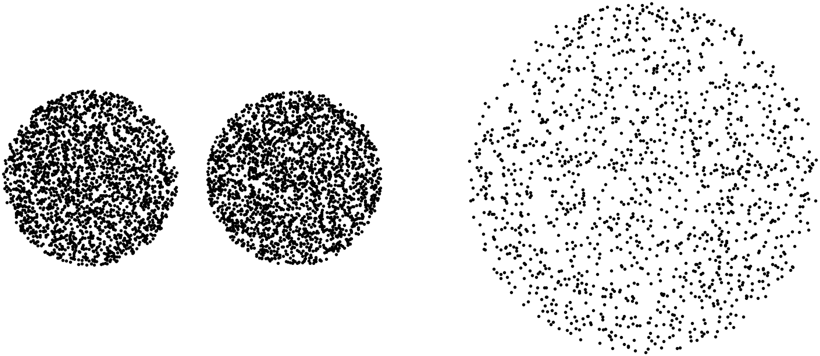
$\overline{S_5^{LDA}}(S)$



Intuitive idea

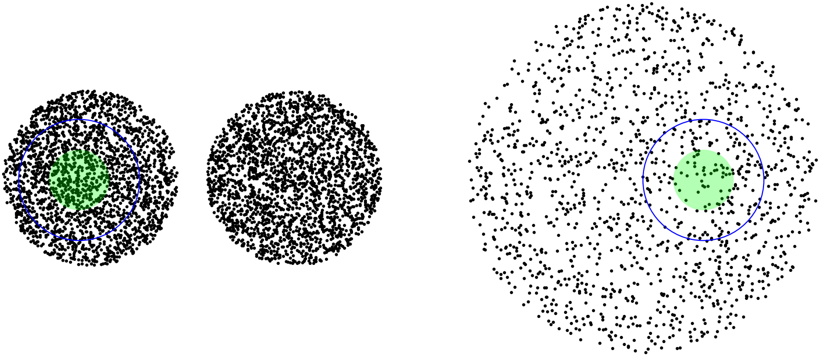
- Hypothesis
  - The sampling points are inside the shape
  - The sampling is not necessarily uniform
- Which zones are outside the shape?
- Which zones are inside the shape?

Intuitive idea



Intuitive idea

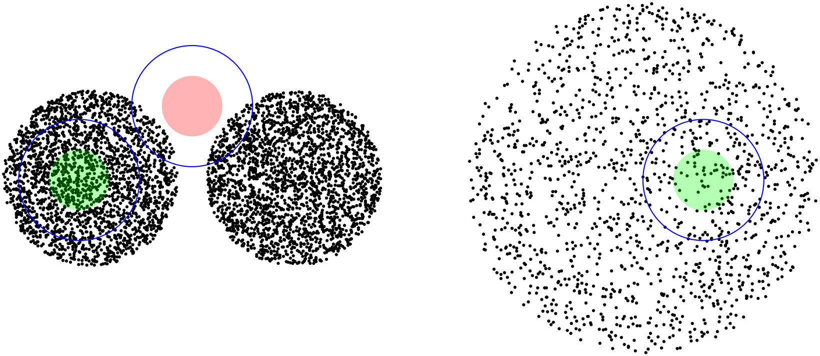
neighborhood region



probably inside the shape (the density in the neigborhood is similar)

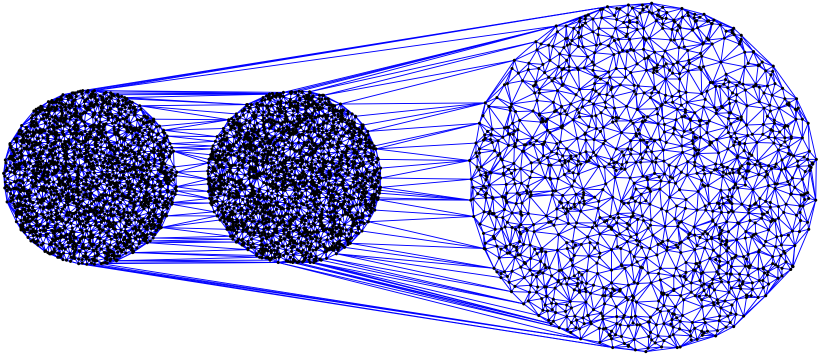
Intuitive idea

neighborhood region



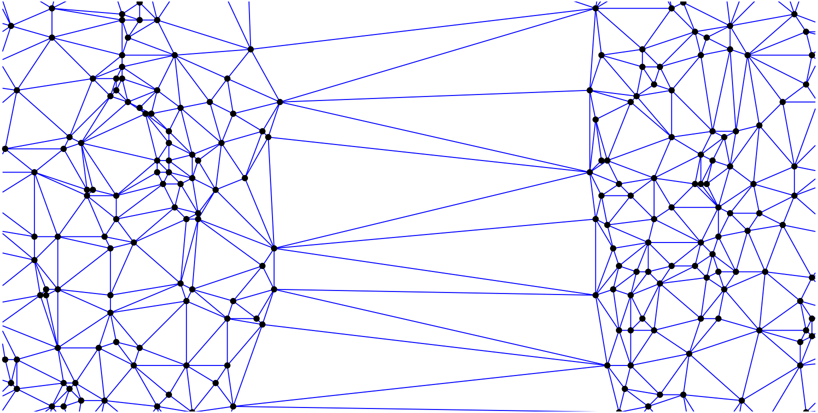
probably inside the shape (the density in the neigborhood is similar)
probably outside the shape (the density in the neigborhood is higher)

Intuitive idea



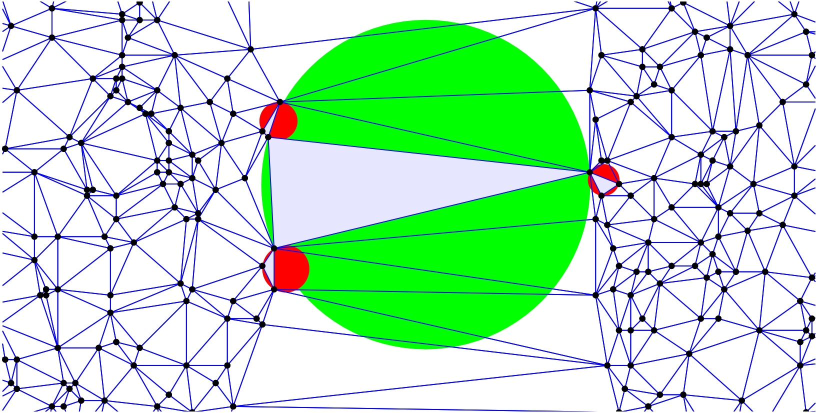
the Delaunay triangulation gives a good idea of this density variation!

Intuitive idea



zoom in

Intuitive idea



the green empty disk probably corresponds to a hole

LDA-α shape / LDA-α hull

- For a vertex a, one can associate the minimal circumradius r_a of all the triangles with vertex a.
- A triangle T_{abc} is LDA-α removed if its circumradius r_{abc} is α-greater than the minimal circumradius r_a, r_b and r_c, i.e.:

$$\frac{r_{abc}}{\max(r_a, r_b, r_c)} > \alpha$$

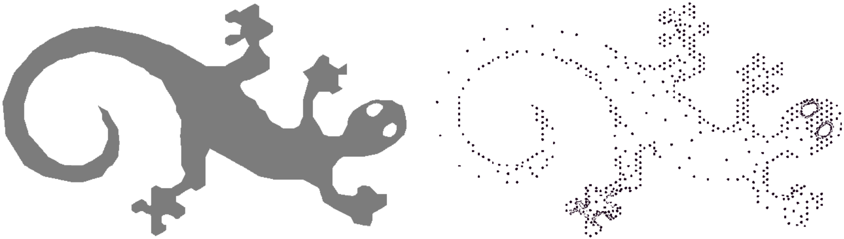
- The LDA-α shape is composed of all the triangles that are not LDA-α removed.
- The LDA-α hull is composed of the free boundary edges of the LDA-α shape.

Section 8

Application to shape reconstruction

Related published research work

- Collaboration: MINES Saint-Etienne and University of Haute-Alsace
- Conference: ICIAR 2011, Burnaby, Canada
- Authors: Benoît PRESLES, Johan DEBAYLE, Yvan MAILLOT, Jean-Charles PINOLI
- Title: Automatic recognition of 2D shapes from a set of points



What is the perceived shape of a set of points?

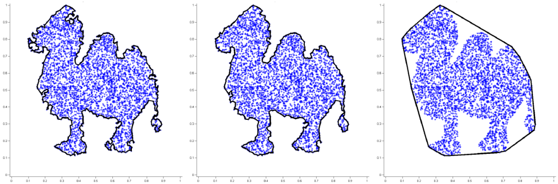
- Highly subjective
- Spatial arrangment of the points + several cognitive factors
- Candidates: LDA alpha-shapes

What is the perceived shape of a set of points?

- Hypothesis
  - All the points should be interior to the reconstructed shape
  - The reconstructed shape is composed of only one connected component
- In this way, only a subset $\{\alpha_1, ..., \alpha_n = \infty\}$, defined by the LDA alpha-shapes, are candidates
- How to define criteria for quantitatively evaluating the perceived shape (i.e how to automatically select alpha)?
  - The shape should be close to the set of points
  - The shape should be regular (a tortuous hull is not desirable)

What is the perceived shape of a set of points?

- According to the previous citeria, the perceived shape has:
  - Minimum area (close to the points)
  - Minimum perimeter (regular shape)
- Nevertheless, these two objectives are controversial:
  - The first hull (α_1) has minimum area and maximal perimeter
  - The last (convex) hull ($\alpha_n = \infty$) has maximum area and minimal perimeter
- Consequence: The "reasonable" hull achieves a compromise between reducing the area and the perimeter



left: $A = 0.35, P = 6.45$; middle: $A = 0.36, P = 5.17$; right: $A = 0.50, P = 2.66$

The proposed criterion

- Normalized area and perimeter (between 0 and 1)

$$\overline{A}(\alpha) = \frac{A(\alpha) - A(\alpha_n)}{A(\alpha_1) - A(\alpha_n)}, \quad \overline{P}(\alpha) = \frac{P(\alpha) - P(\alpha_1)}{A(\alpha_n) - A(\alpha_1)}$$

- The criterion to be minimized, depending on the α value, is defined with the $L1$-norm of the vector $(\overline{A}(\alpha), \overline{P}(\alpha))$.
- Nevertheless, the norm has to be weighted according to the convexity C of the expected shape:
  - If the perceived shape is convex: only the perimeter needs to be minimzed
  - If the perceived shape is tortuous holding several concavities: only the area has to be minimized

The proposed criterion

- The proposed criterion:

$$APC(\alpha) = C^6\overline{P}(\alpha) + (1 - C^6)\overline{A}(\alpha)$$

- where C denotes the convexity of the expected shape, calculated as:

$$C = \frac{1}{n}\sum_{i=1}^{n}\frac{A(\alpha_i)}{A(\alpha_n)}$$

  This approximation has been validated on real samples where the real hull is known.

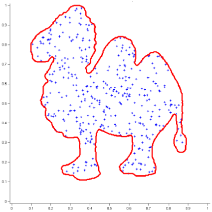
- Finally:

$$\alpha_{opt} = \arg\min_{\alpha} APC(\alpha)$$

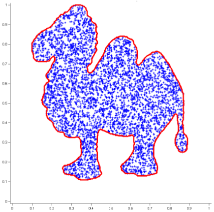
Relevance of the proposed criterion

- Experiments on the KIMIA image database
  - For each binary image, the boundary points of the set are extracted and normalized into the unit square
  - From these contour points, the polygonal hull is retrieved giving the reference shape S
  - This reference shape S is sampled with random points using either an homogeneous or an hetereogeneous law (the probability is higher near the medial axis of the complementary of the shape)
  - In addition, different point densities are used: low (1000 points per unit area), medium (10000 points per unit area) and high (20000 points per unit area)

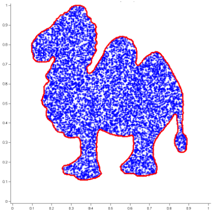
Relevance of the proposed criterion



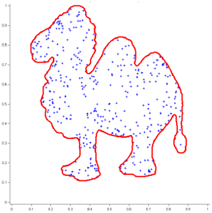
(a) low sampling with a uniform law



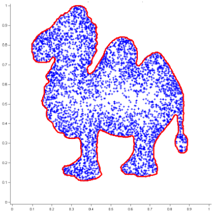
(b) medium sampling with a uniform law



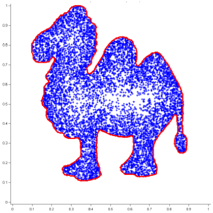
(c) high sampling with a uniform law



(d) low sampling with an heterogeneous law



(e) medium sampling with an heterogeneous law



(f) high sampling with an heterogeneous law

Relevance of the proposed criterion

- The optimal alpha value α_{opt} is compared with the $\tilde{\alpha}$ for which the LDA $\tilde{\alpha}$-shape best approximates the reference shape S (among all the alpha values α_1 to α_n):

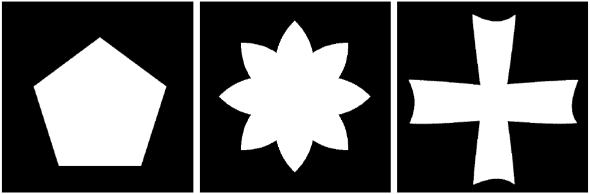
$$\tilde{\alpha} = \arg\min_{\alpha} ASD(LDA(\alpha), S)$$

  where ASD is the area of the symmetric difference.
- Finally, the error between the two shapes $LDA(\alpha_{opt})$ and $LDA(\tilde{\alpha})$ is measured for each image I:

$$E(I) = \frac{ASD(LDA(\alpha_{opt}), LDA(\tilde{\alpha}))}{A(\tilde{\alpha})}$$

Relevance of the proposed criterion

- Evaluation on three images of different convexity



(a) I_1 (b) I_2 (c) I_3

- Results

	uniform random sampling			heterogeneous random sampling		
	low	medium	high	low	medium	high
$E(I_1)$ $(C = 0.9)$	$\mu = 8.47$ $\sigma = 4.55$	$\mu = 2.65$ $\sigma = 0.67$	$\mu = 1.03$ $\sigma = 0.17$	$\mu = 7.62$ $\sigma = 5.32$	$\mu = 2.22$ $\sigma = 0.53$	$\mu = 0.87$ $\sigma = 0.14$
$E(I_2)$ $(C = 0.7)$	$\mu = 13.57$ $\sigma = 8.02$	$\mu = 3.61$ $\sigma = 2.45$	$\mu = 0.59$ $\sigma = 0.46$	$\mu = 12.21$ $\sigma = 8.89$	$\mu = 2.55$ $\sigma = 1.79$	$\mu = 0.48$ $\sigma = 0.39$
$E(I_3)$ $(C = 0.5)$	$\mu = 23.75$ $\sigma = 9.30$	$\mu = 10.12$ $\sigma = 4.44$	$\mu = 3.38$ $\sigma = 1.29$	$\mu = 22.79$ $\sigma = 10.61$	$\mu = 8.21$ $\sigma = 4.09$	$\mu = 2.51$ $\sigma = 0.98$

Experiments with quantitative evaluation

- Evaluation on 1400 images from the KIMIA database
- Performance evaluation:

$$E(I) = \frac{ASD(LDA(\alpha_{opt}), S)}{A(S)}$$

- Results

	uniform random sampling			heterogeneous random sampling		
	low	medium	high	low	medium	high
KIMIA database	$\mu = 32.89$ $\sigma = 21.94$	$\mu = 15.90$ $\sigma = 14.15$	$\mu = 6.92$ $\sigma = 7.33$	$\mu = 31.33$ $\sigma = 22.57$	$\mu = 14.30$ $\sigma = 15.61$	$\mu = 5.95$ $\sigma = 6.72$