Voronoi diagram and Delaunay triangulation

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Plan

Section 1

Introduction

Section 2

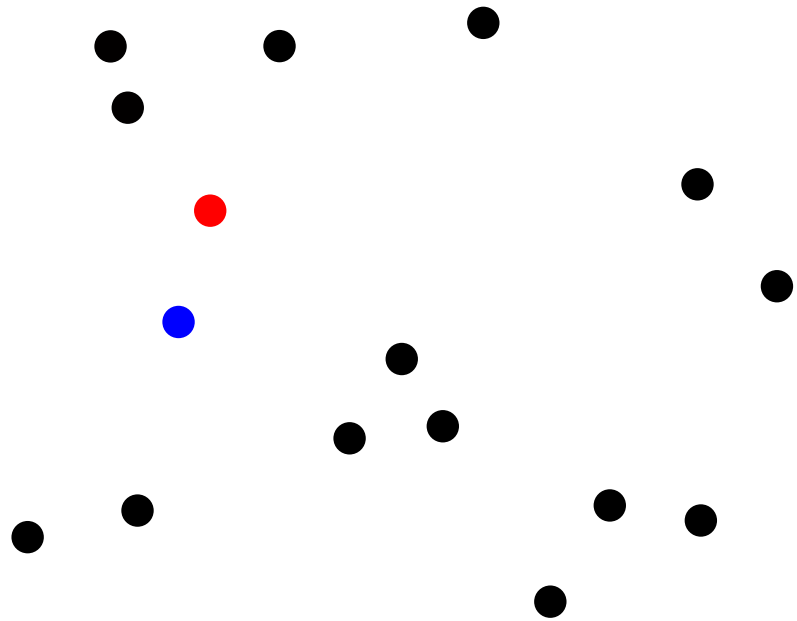
Nearest Neighbors

Nearest Neighbors search

Problem

Search for the (k) closest point(s) to a query point $q \in M$ in a set $S \subset M$.

- M is a not necessarily a metric space.
- Usual metrics are used: L_1, L_2, L_∞.
- $M = \mathbb{R}^d$ is common



Methods [Bhatia and Vandana, 2010]

Exact methods

- Linear search (naive algorithm)
- Space partitioning
  - KD tree
  - Ball tree
  - R tree

Approximate methods

return a points at most c times the distance from the query to its nearest points.

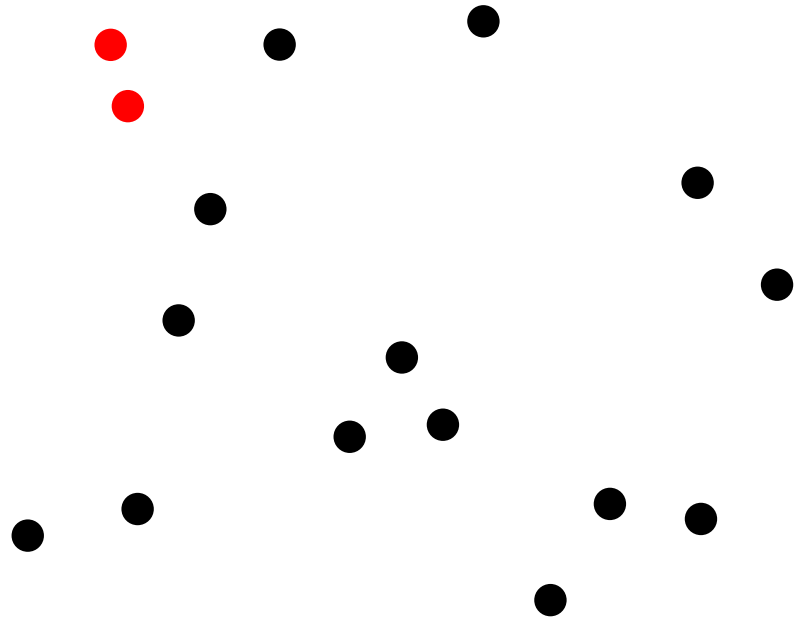
Particular case: closest pair of points

- $S \subset M$ set of points
- Minimum for all points $q \in S$
- search for closest point $p_q \in S$, $p_q \neq q$

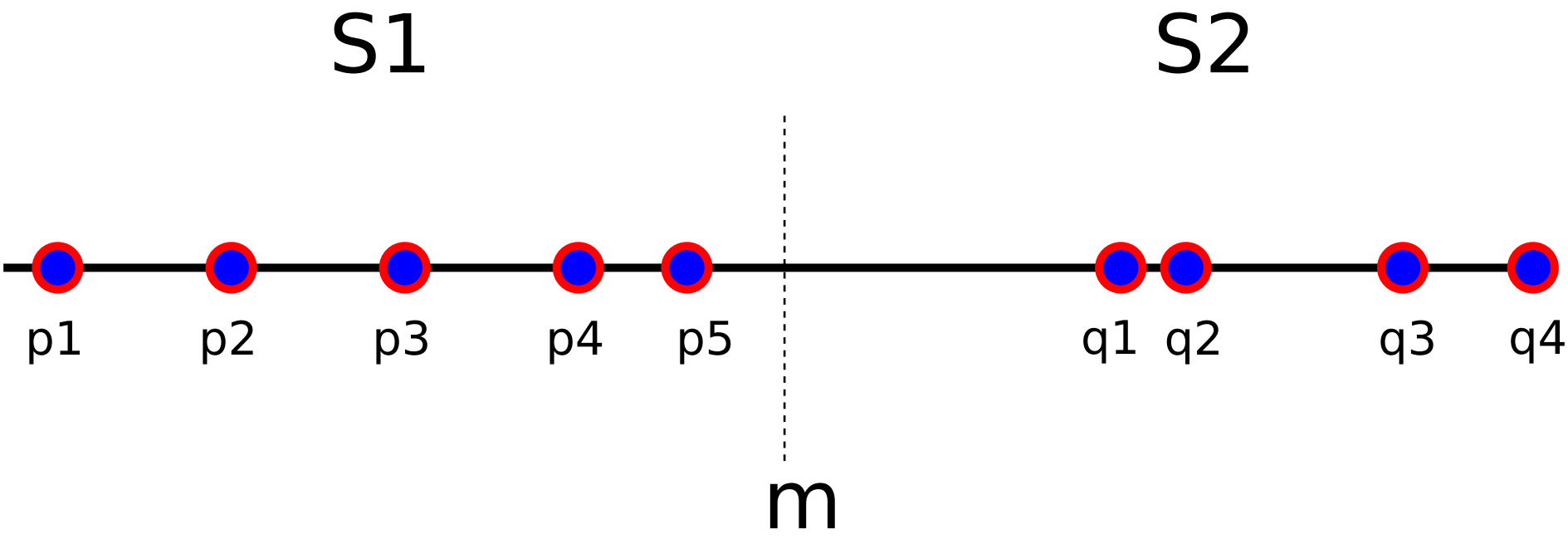
Closest pair of points

Exhaustive search

- n points in $\mathbb{R}^d$
- All pairs of points: $O(dn^2)$
- One query point: $O(nd)$
- NB: d is sometimes missed in complexity notations (constant)

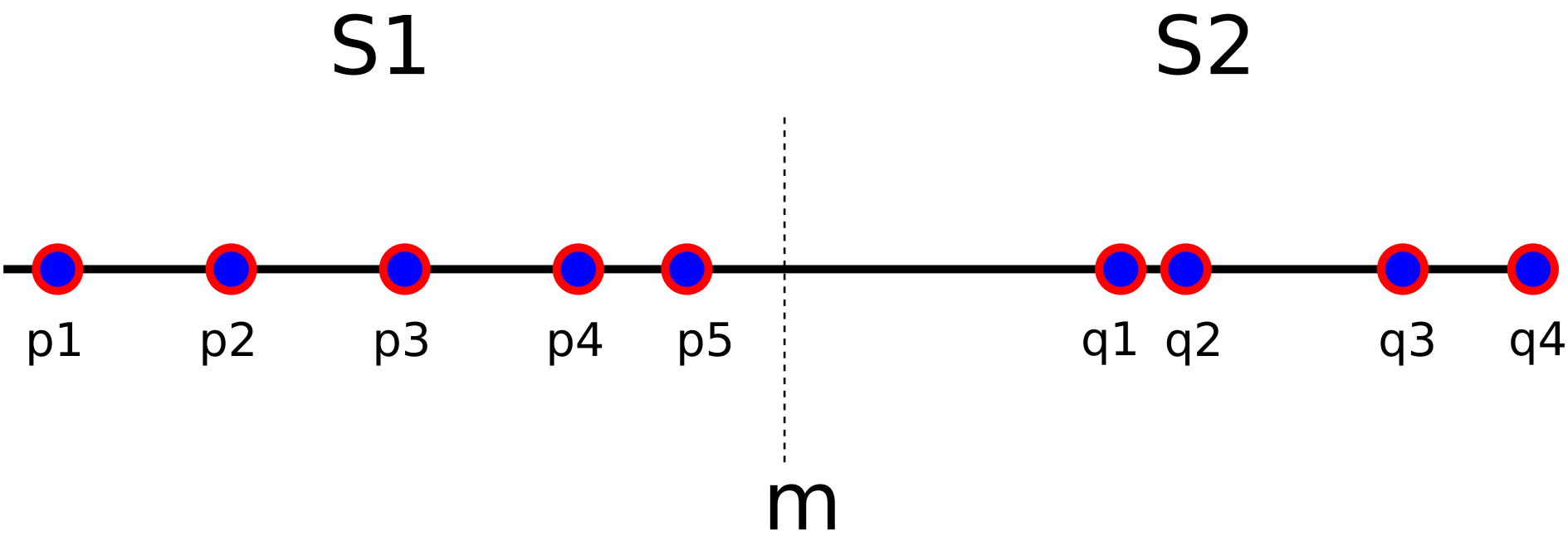


Closest pair of points: 1D problem



- Divide and conquer
  $O(n \log n)$
- Based on sorting

Closest pair of points: 1D problem



- Separate into 2 sets S_L and S_R by value m
- $p_i \in S_L$, $q_j \in S_R$
- $\forall i, j, p_i < q_j$

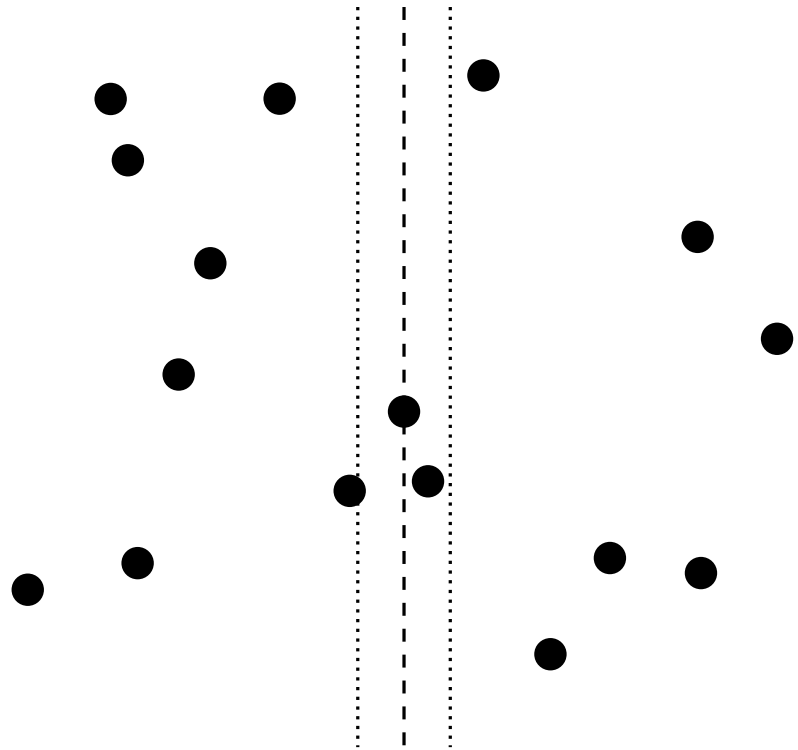
- Recursively compute closest pairs (p_k, p_{k+1}), (q_l, q_{l+1})
- $\delta = \min\{p_{k+1} - p_k, q_{l+1} - q_l\}$
- Merge: either (p_k, p_{k+1}), (q_l, q_{l+1}) or (p_m, q_n)

Closest pair of points: 2D problem [Cormen et al., 2009]

Algorithm

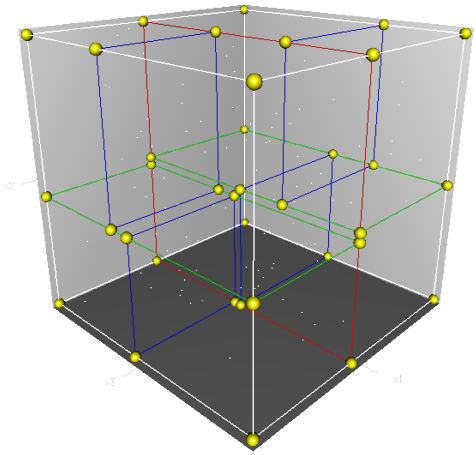
- 2 sets S_L and S_R
- $\delta = \min(\delta_L, \delta_R)$
- Combine: 8 points can lie in a $\delta \times 2\delta$ rectangle

- Complexity: $T(n) = 2T(n/2) + O(n)$
- $O(n \log n)$
- $O(n \log^{d-1} n)$ in $\mathbb{R}^d$



Space partitionning

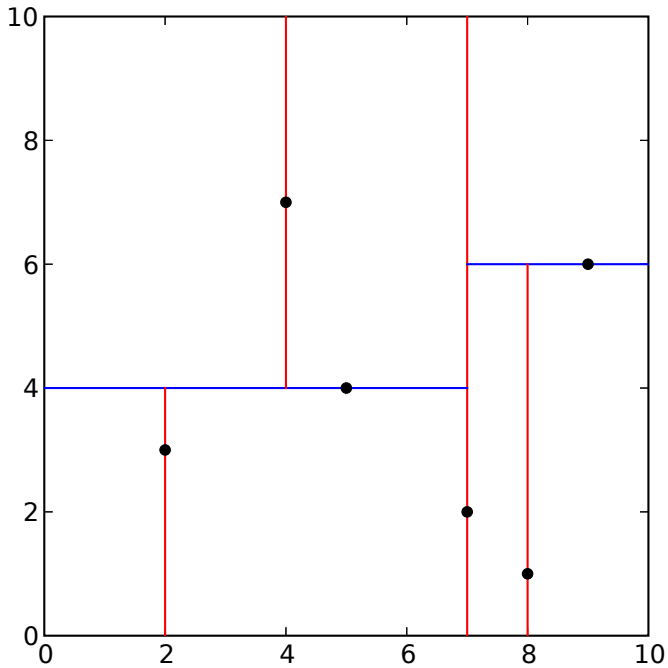
- Divides a space into non-overlapping regions
- Construction of a data structure
  - trees
  - graphs

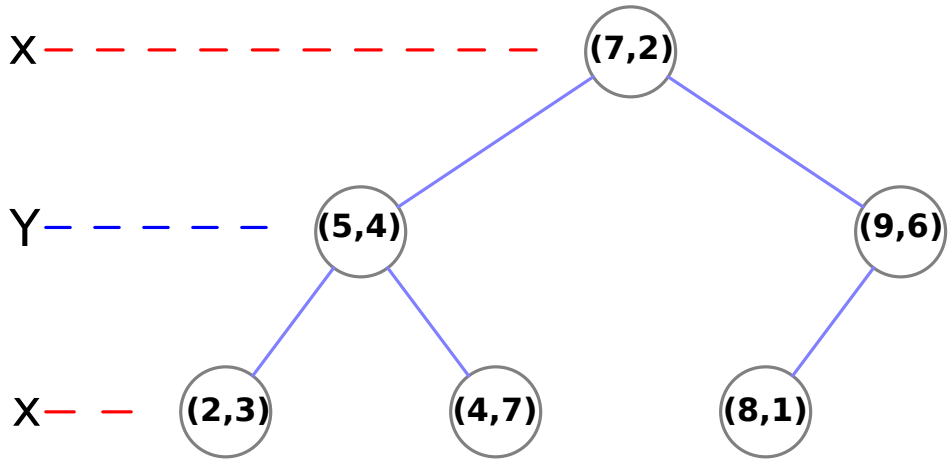


- BSP trees (Binary Space Partitioning)
- k-dtrees;
- Quadtrees;
- Octrees;
- Bins;
- R-trees;
- Bounding volume hierarchies.

k-d tree

- Partitions the space by hyperplanes
- in $\mathbb{R}^d$, performs partitionning axis by axis
- median value is used for balancing the tree





k-d tree of the points
$(2,3),(5,4),(9,6),(4,7),(8,1),(7,2)$

k-d tree algorithms

Median value

- Sorting in $O(n \log n)$
- ...would be inefficient
- Median approximation: sort a subsample

Complexity

	Average	**Worst**
Space	$O(n)$	$O(n)$
Search	$O(n \log n)$	$O(n)$
Insert	$O(n \log n)$	$O(n)$
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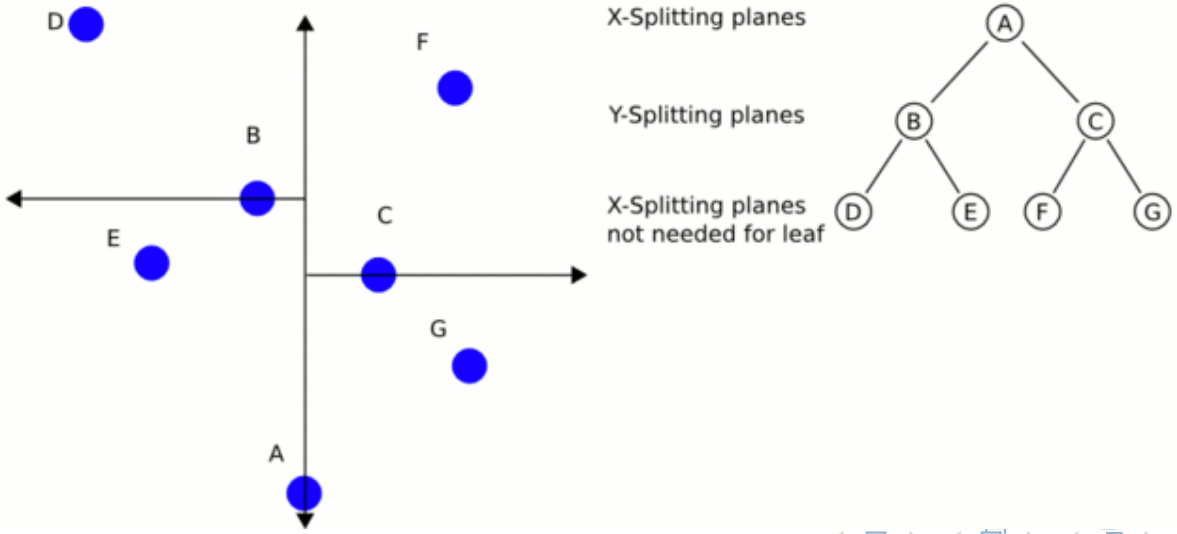

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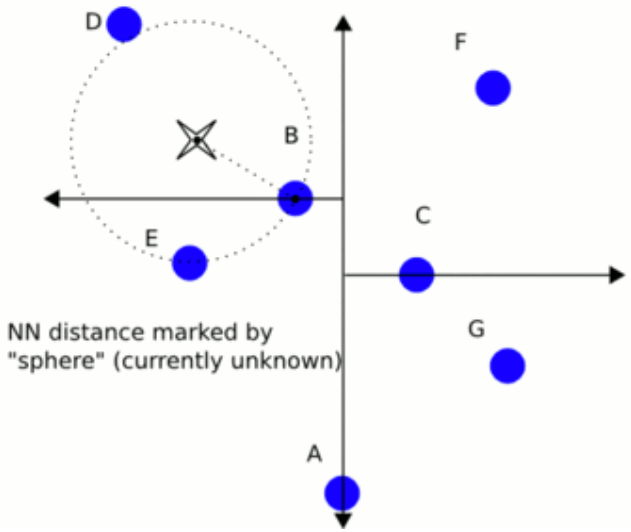

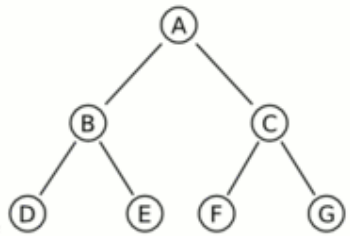

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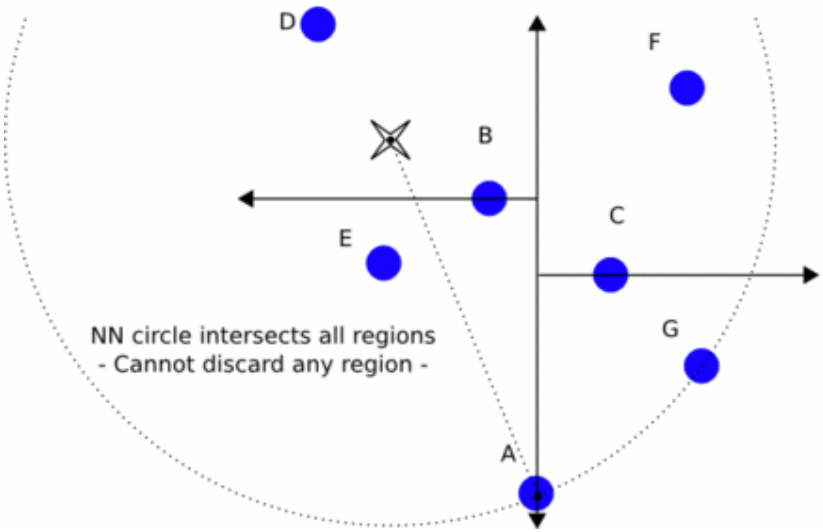

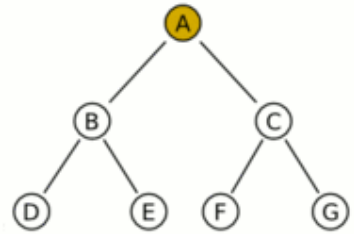

Start at A, then proceed in depth-first search (maintain a stack of parent-nodes if using a singly-linked tree). Set best estimate to A's distance Then examine left child node

k-d tree algorithms

Median value

- Sorting in $O(n \log n)$
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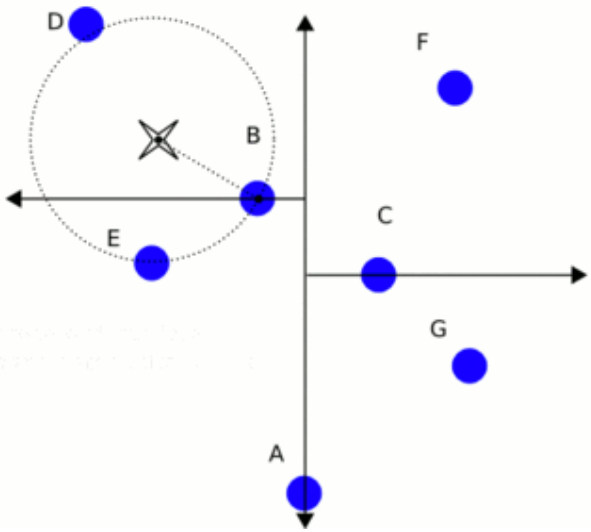

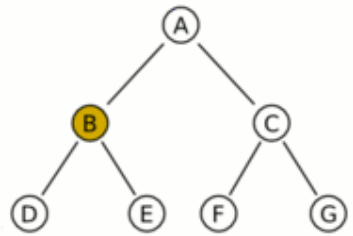

Calculate B's distance and
compare against best estimate
- It is smaller distance, so update
best estimate. Examine children (left then right)

k-d tree algorithms

Median value

- Sorting in $O(n \log n)$
- ...would be inefficient
- Median approximation: sort a subsample

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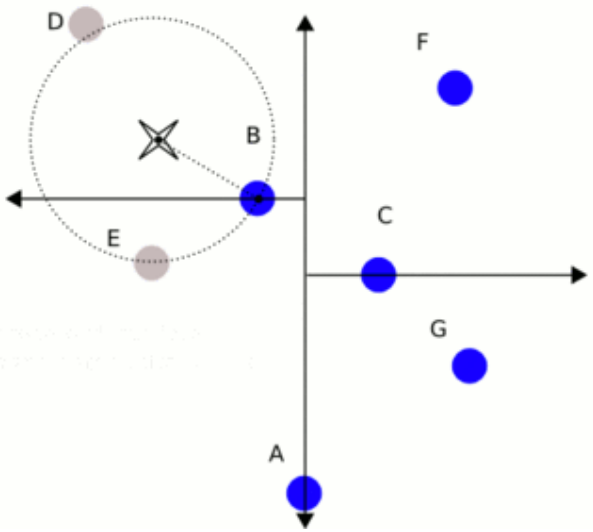

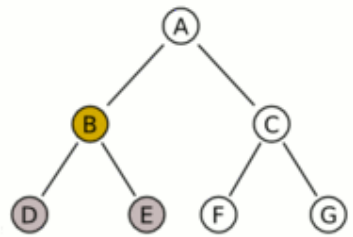

D & E Discarded as B
(already visited) is closer.
B is the best estimate for B's sub-branch
Proceed back to parent node

k-d tree algorithms

Median value

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- ...would be inefficient
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Complexity

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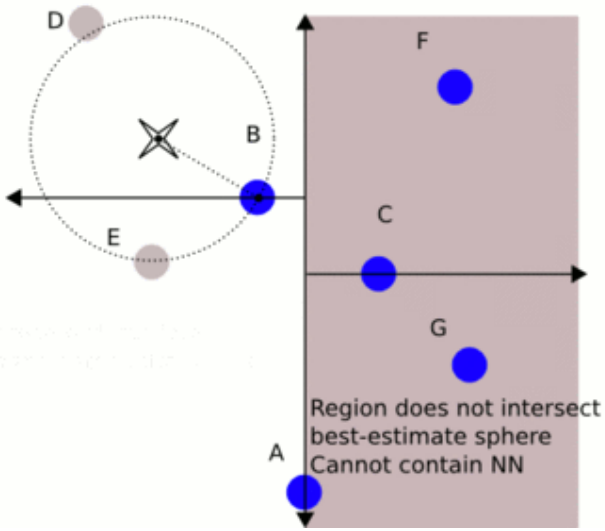
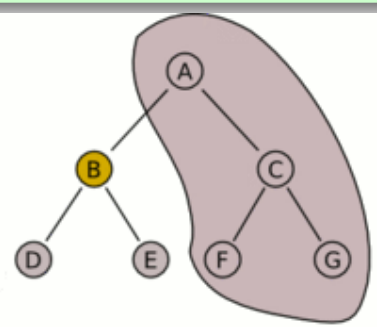




A's children have all been searched,
B is the best estimate for entire tree

k-d tree: improvements

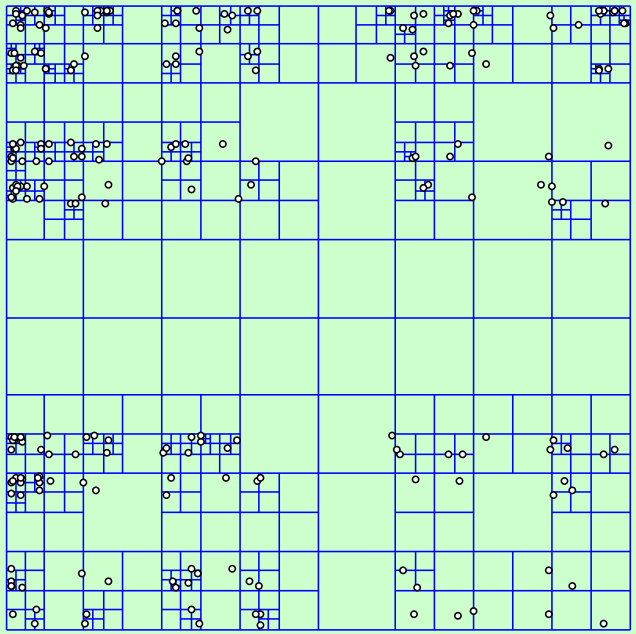
- Extension to k nearest neighbors
- Approximation: stop search after
  - some times
  - a number of branches
  - see best-bin-first search algorithm

Other trees

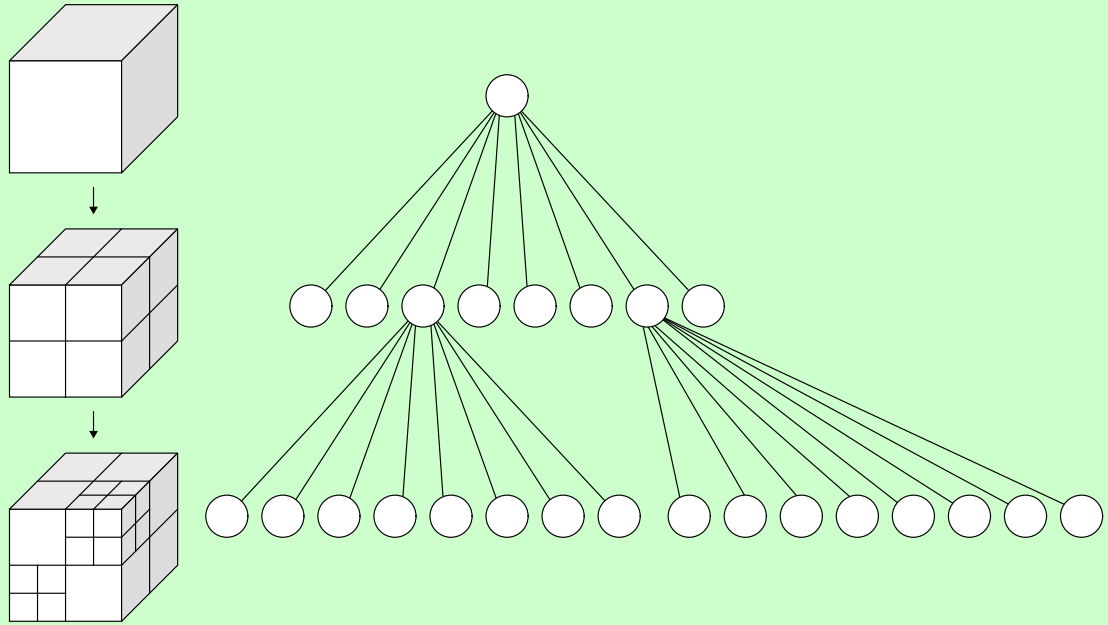
- quadtree: split into 4 children (2D)
- octree: split into 8 children (3D)
- Ball tree, a multi-dimensional space partitioning useful for nearest neighbor search
- R-tree and bounding interval hierarchy, for objects rather than points, with overlapping regions
- Vantage-point tree, a variant of a k-d tree that uses hyperspheres

Quadtree and Octree

Quadtree



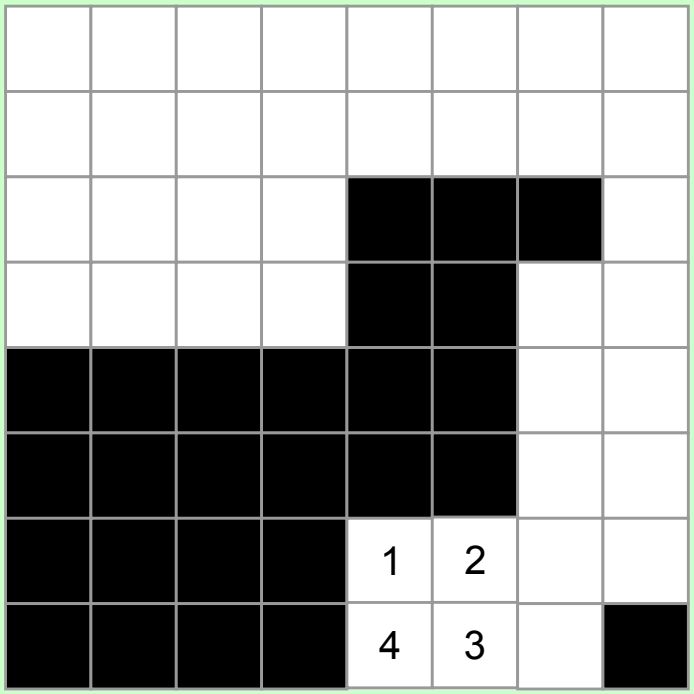
Octree

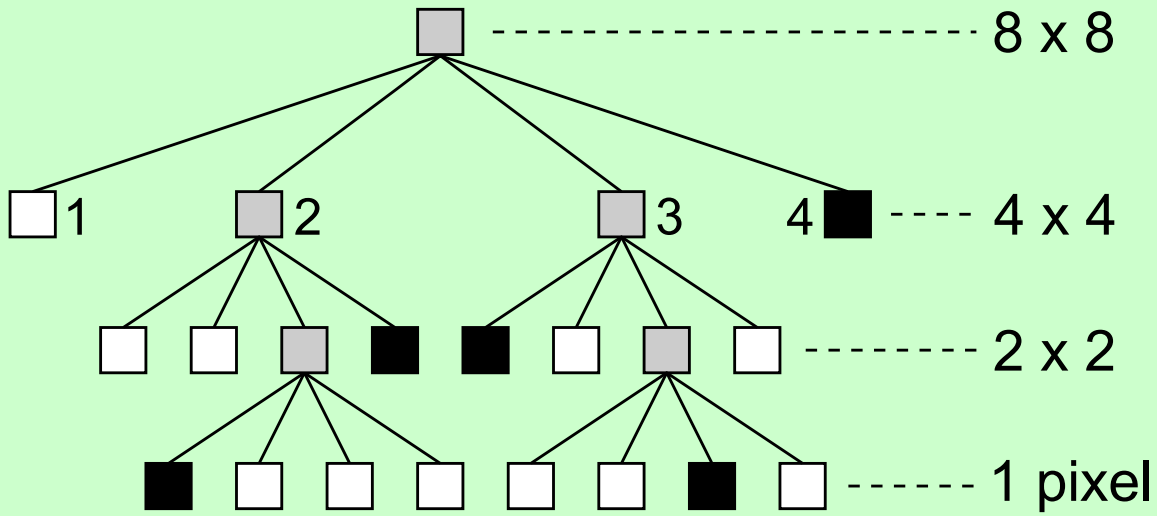


Quadtree usage

Bitmap

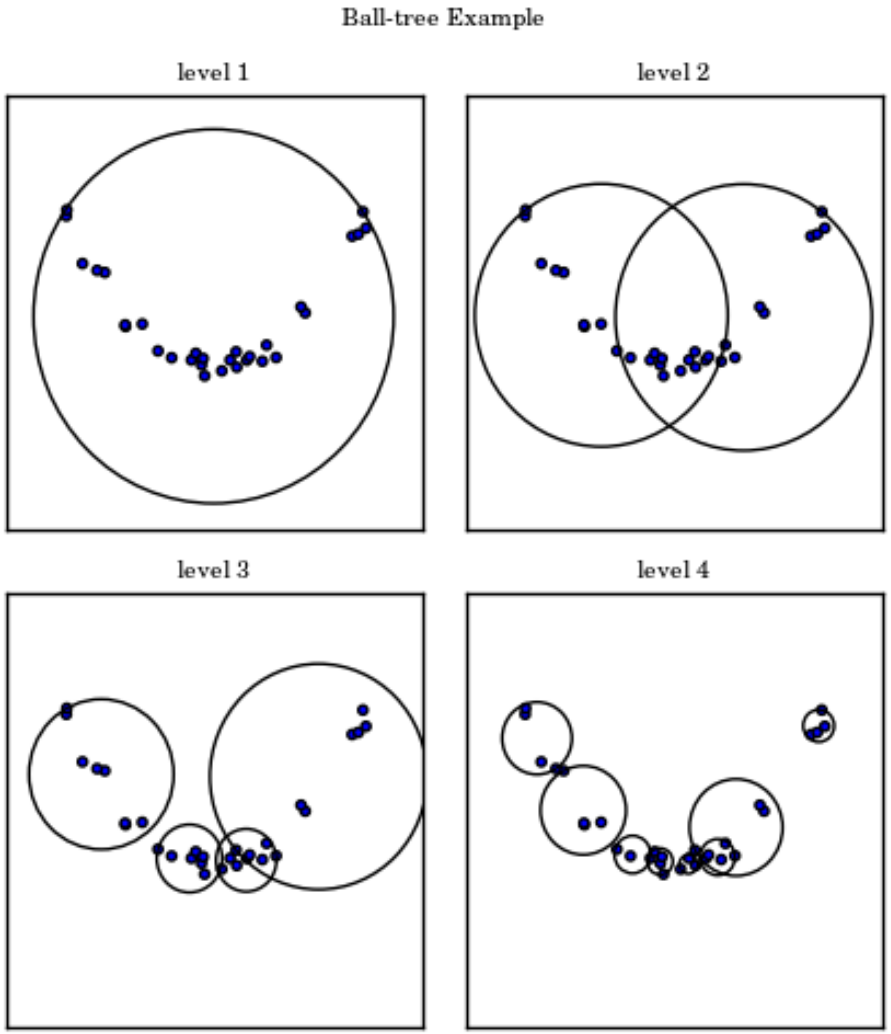
- compression
- segmentation: split/merge algorithm





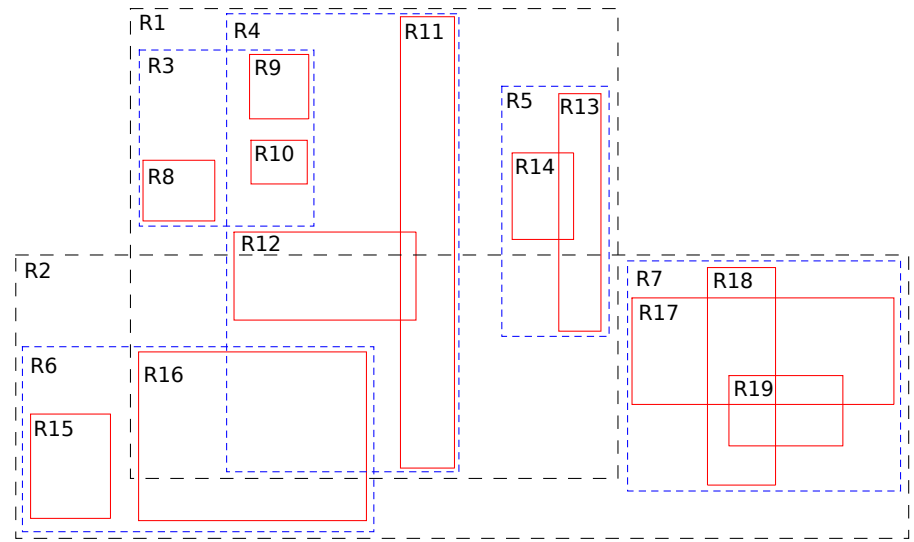
Ball tree (M-tree)

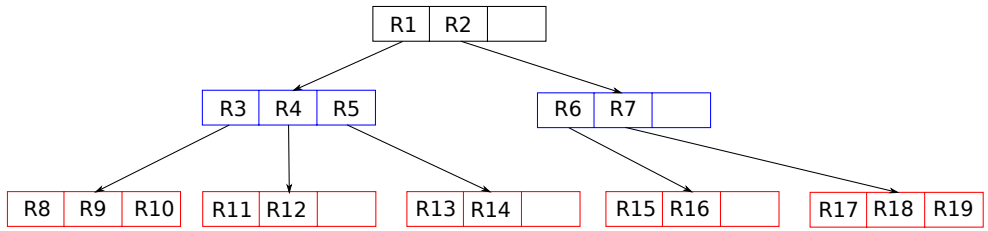
- M for Metric
- Use hyperspheres instead of hyperplanes



R tree

- groups objects
- rectangle bounding boxes





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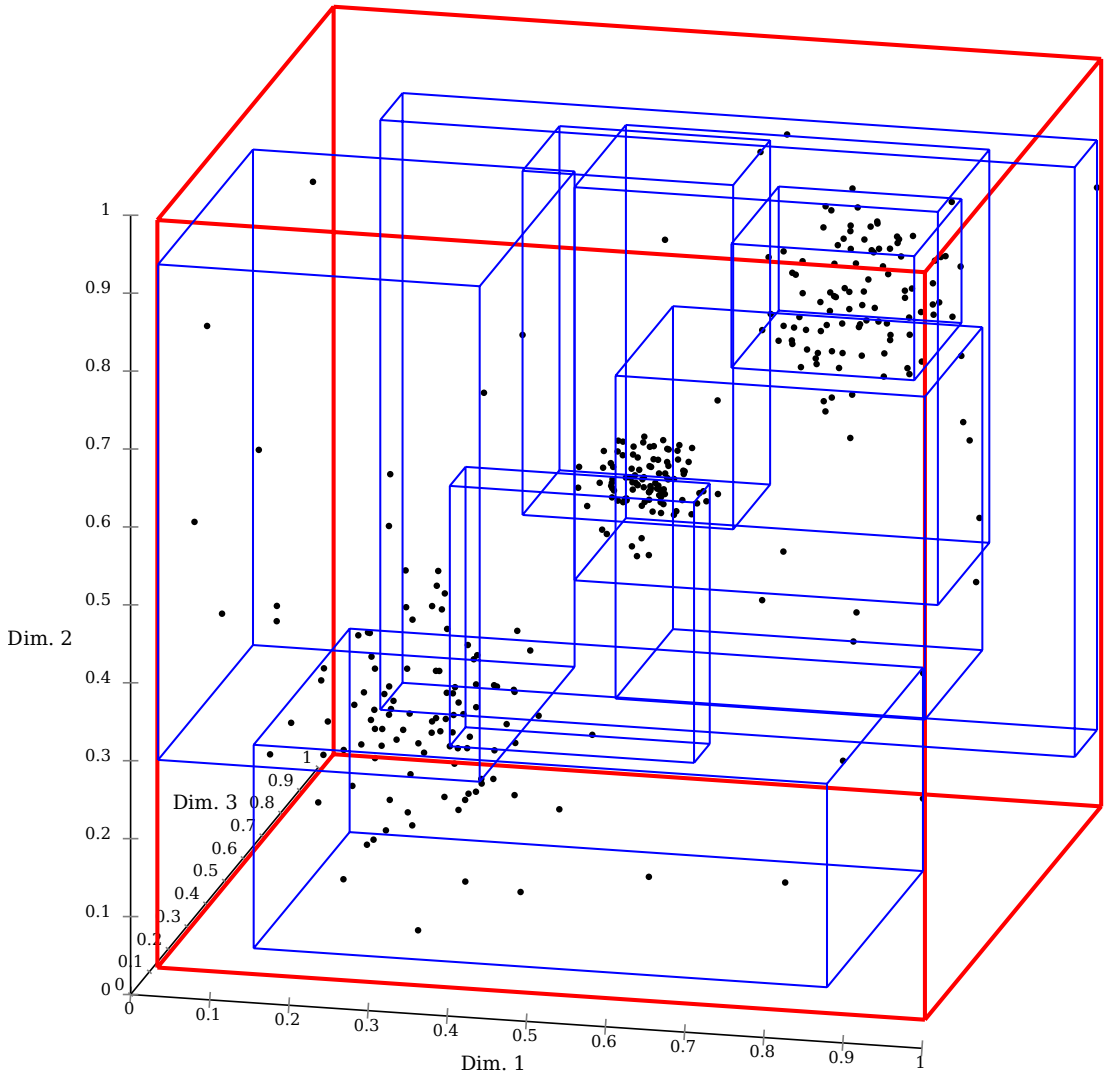


Vantage-Point tree (VP)

- partitions the space with spheres
- given a point (VP)
- and distances
- may be faster than k-d for high dimensions

Section 3

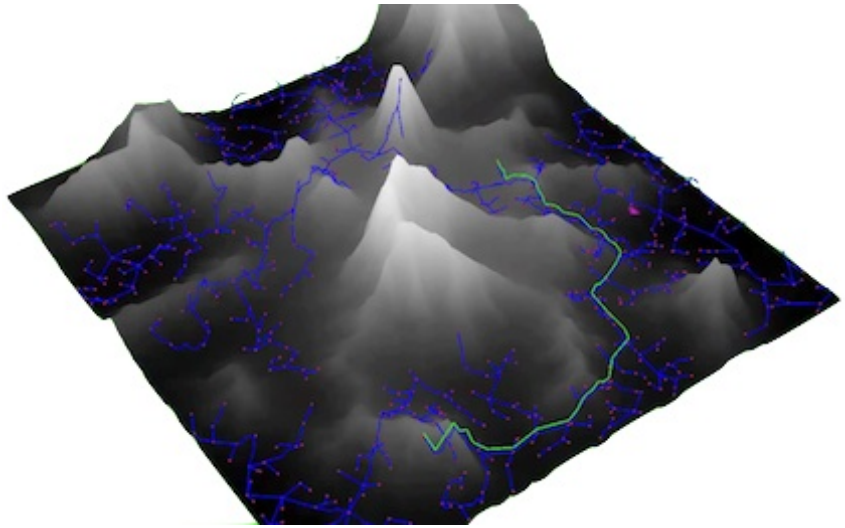
Voronoi Diagram

References

- [Devadoss and O'Rourke, 2011]
- [de Berg, 1997]

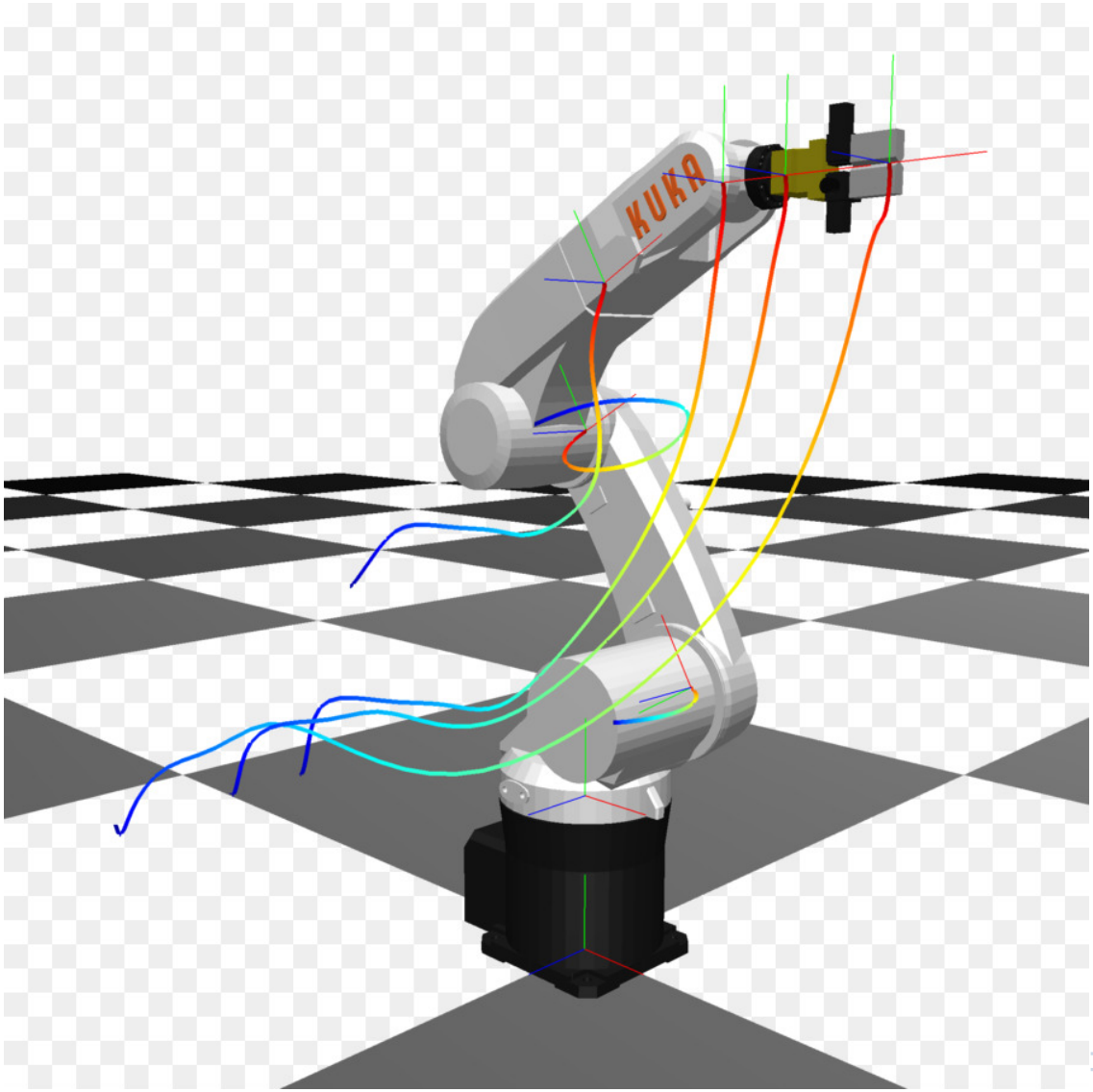
Illustrations and application domains

Motion planning



Illustrations and application domains

Motion planning: robotics

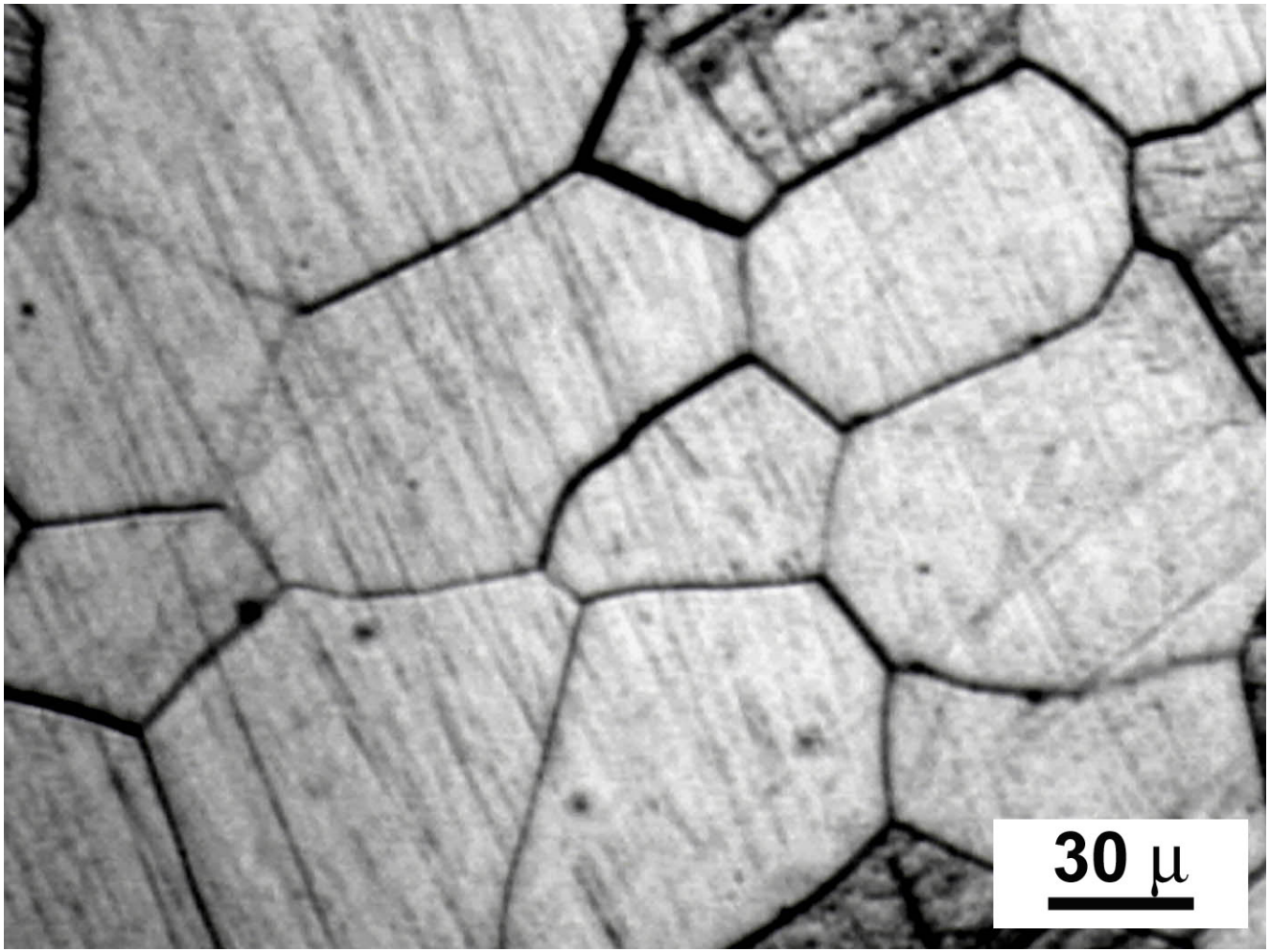


Illustrations and application domains

Post-office problem: delivery routes

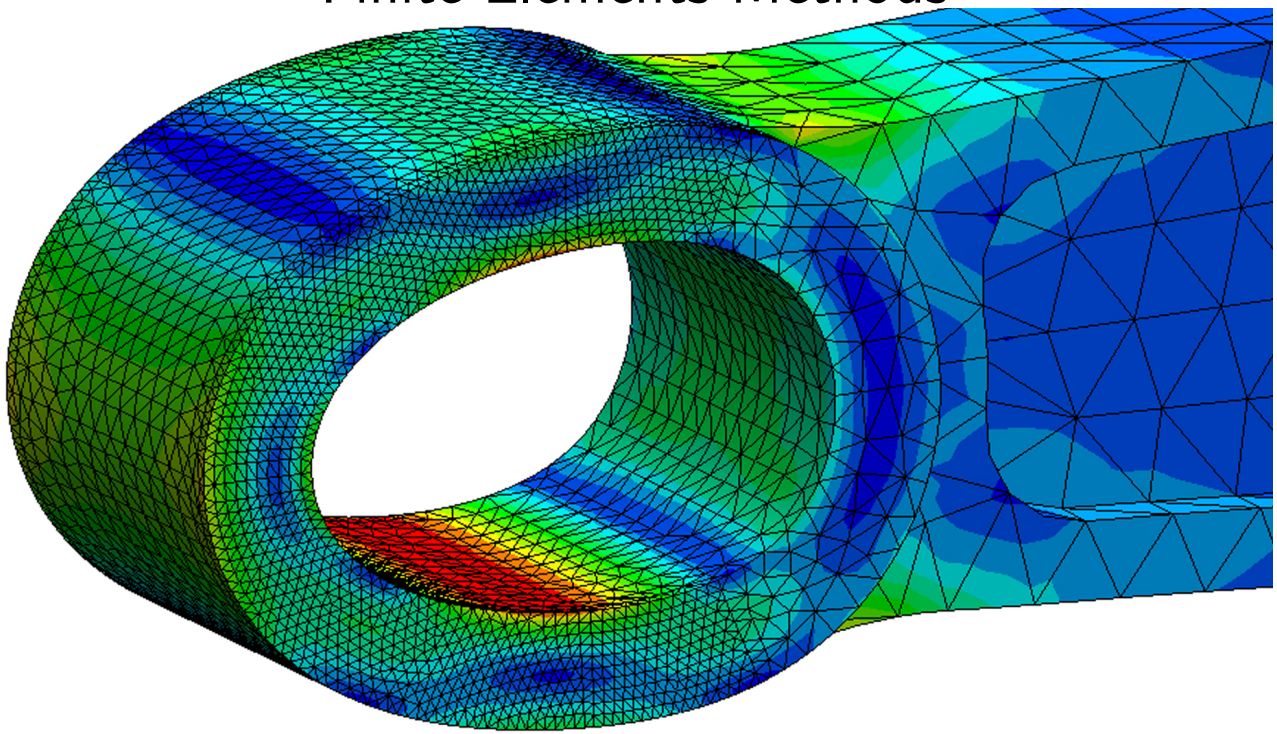
Illustrations and application domains

cristallography



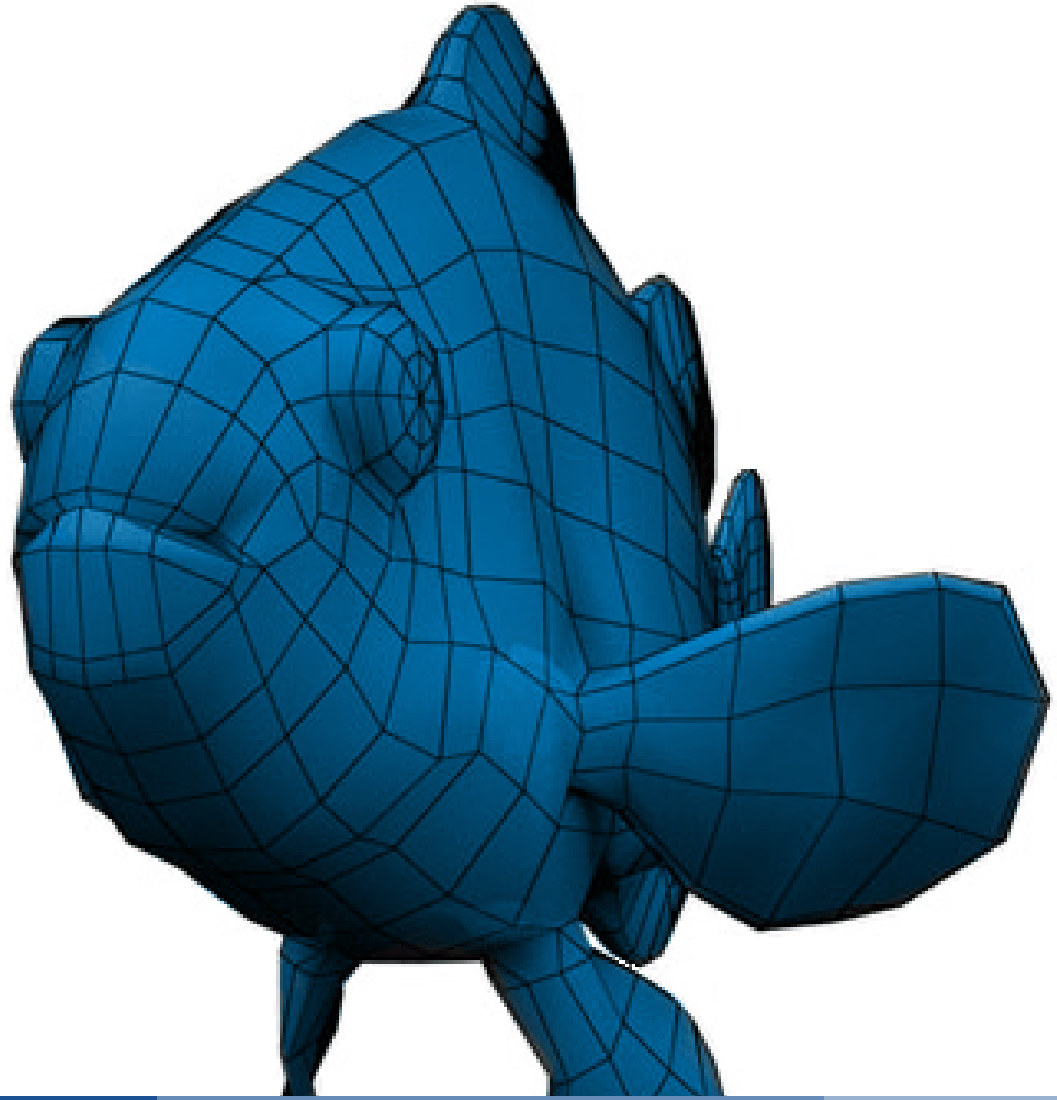
Illustrations and application domains

Finite Elements Methods



Illustrations and application domains

Meshing for video games and animations



Illustrations and application domains

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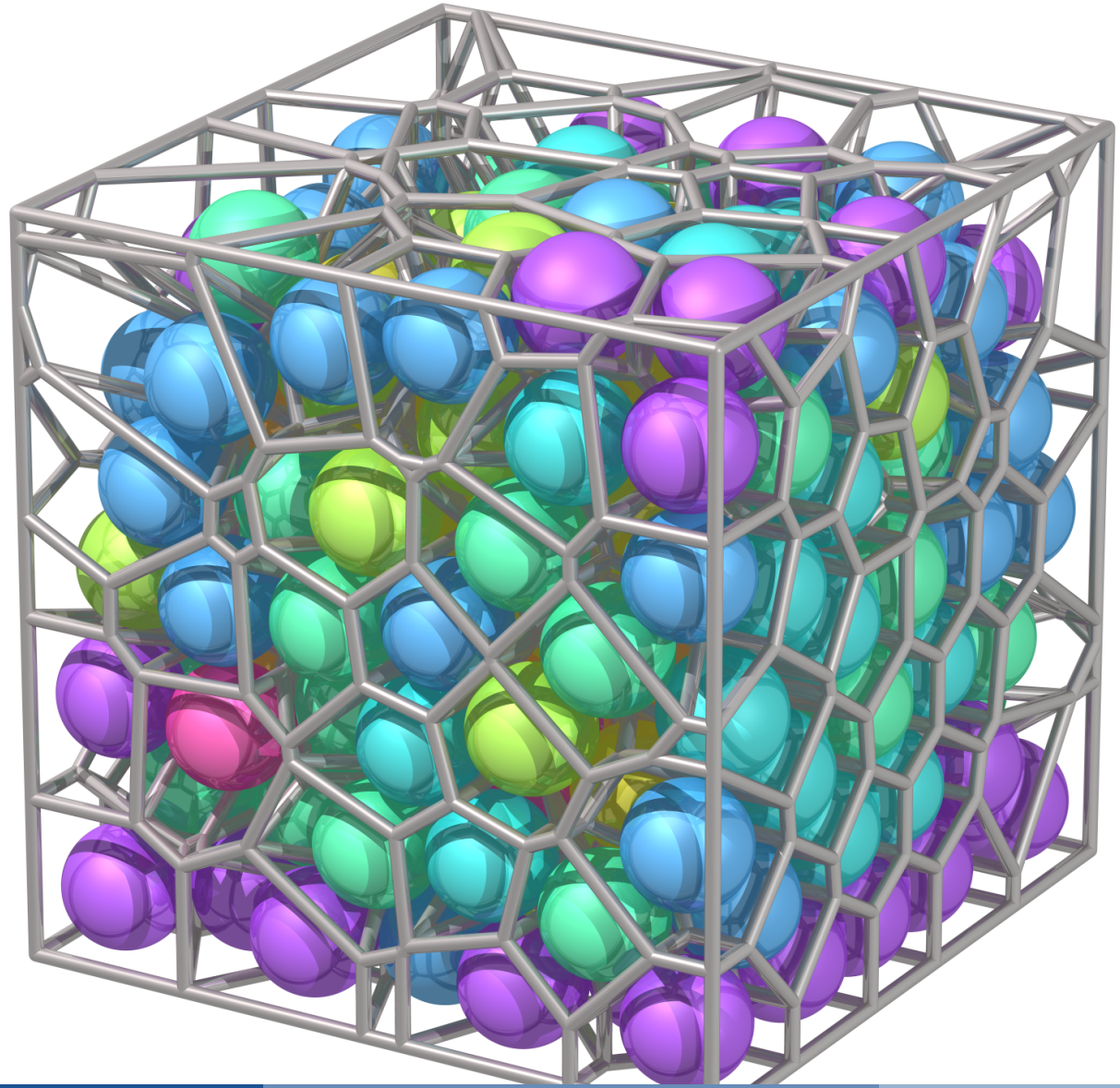


Illustrations and application domains

Plan your trip to Venice

Voronoi diagram

- X is a metric space, with distance d.
- points (aka sites) $P_k \in X$, $k \in K$
- $S = \{P_k, k \in K\}$
- $d(x, A) = \inf\{d(x, a) \mid a \in A\}$ denotes the distance between the point x and the set A.

Definition (Voronoi region)

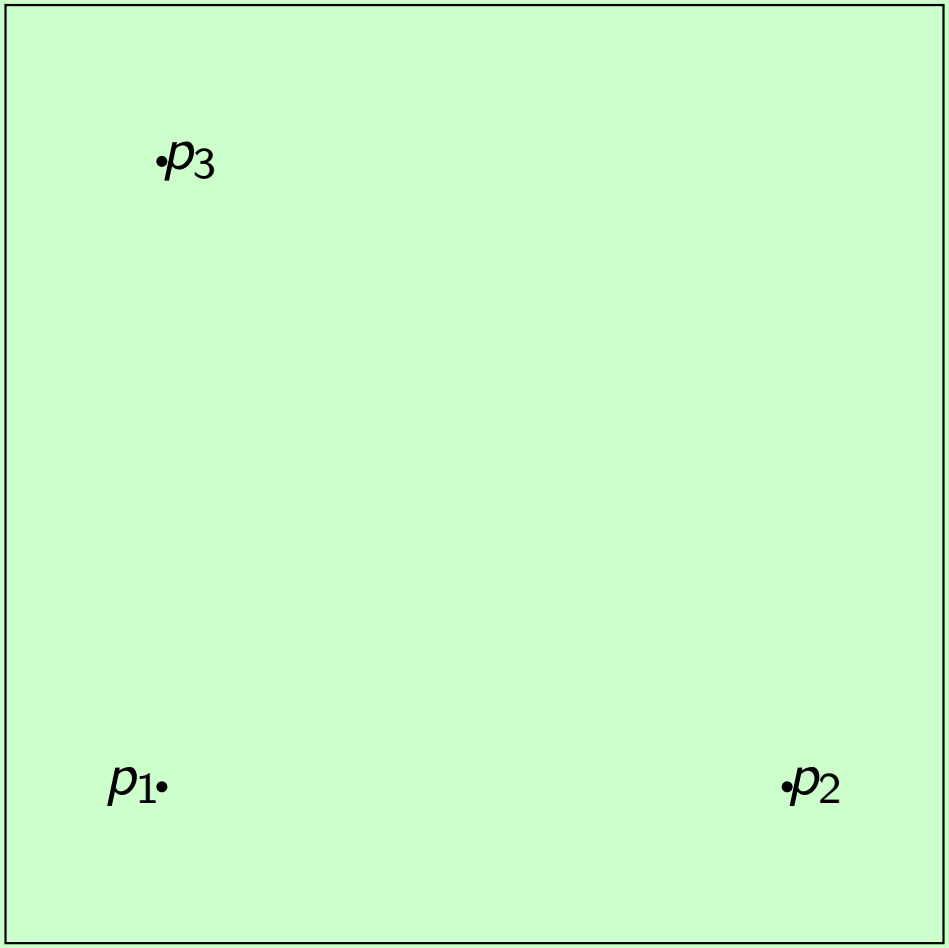
$$\mathrm{Vor}_S(P_k) = R_k = \{x \in X \mid d(x, P_k) \leq d(x, P_j)\ \forall j \neq k\}$$

Definition (Voronoi diagram)

Set of all regions

$$\mathrm{Vor} = (R_k)_{k \in K}$$

Illustration



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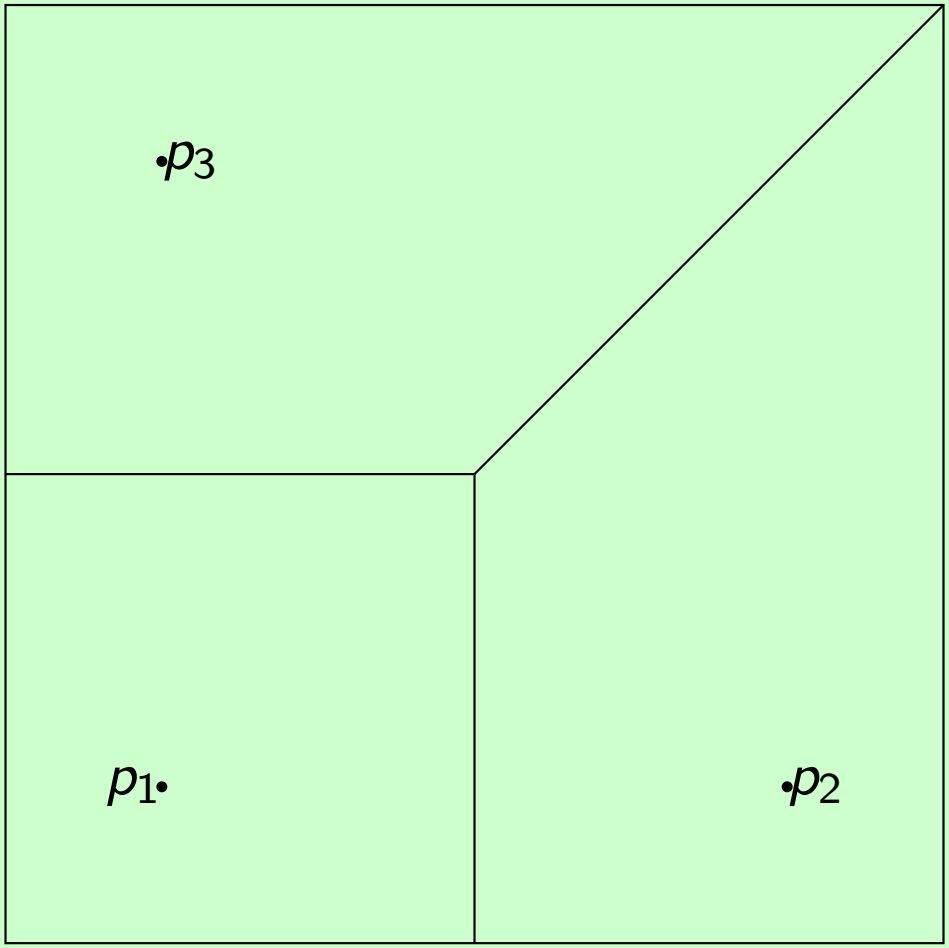

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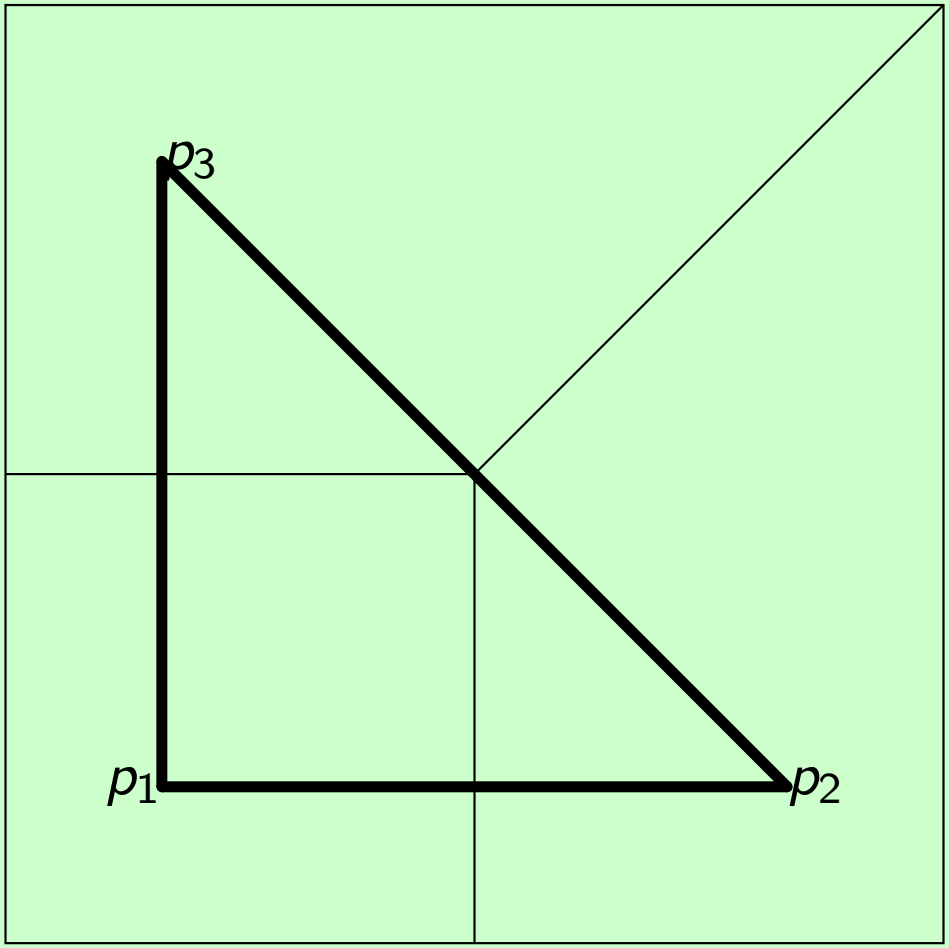

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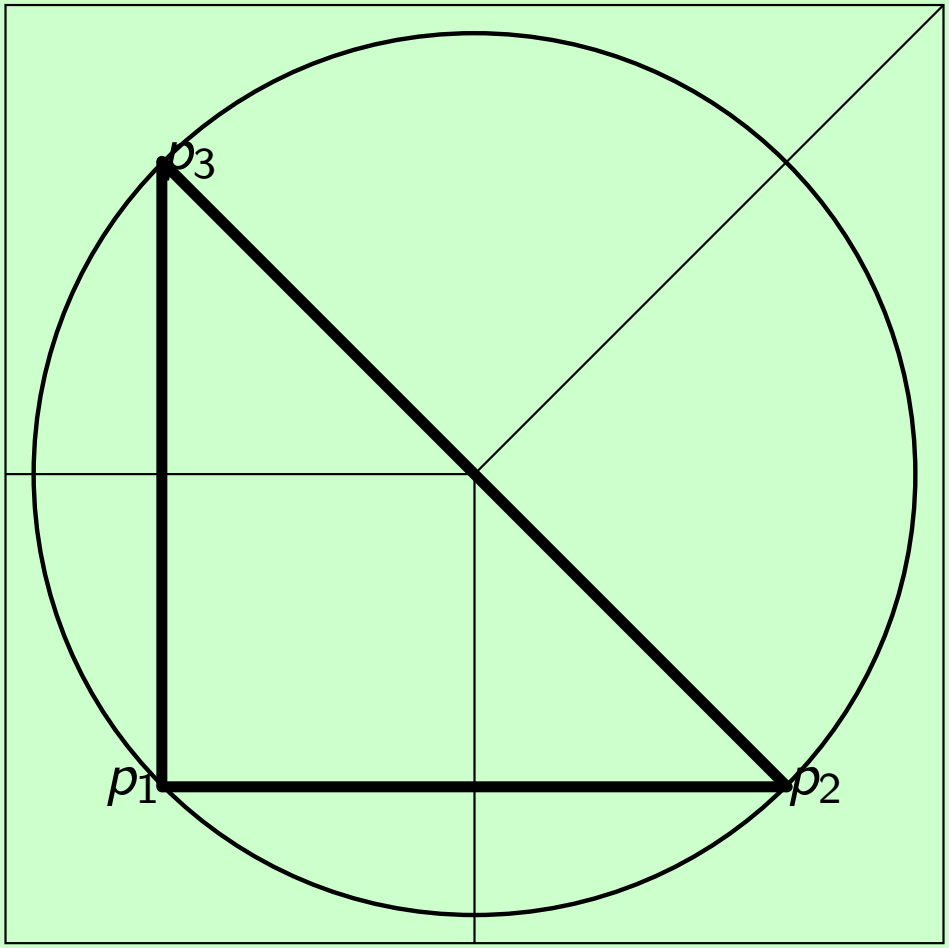
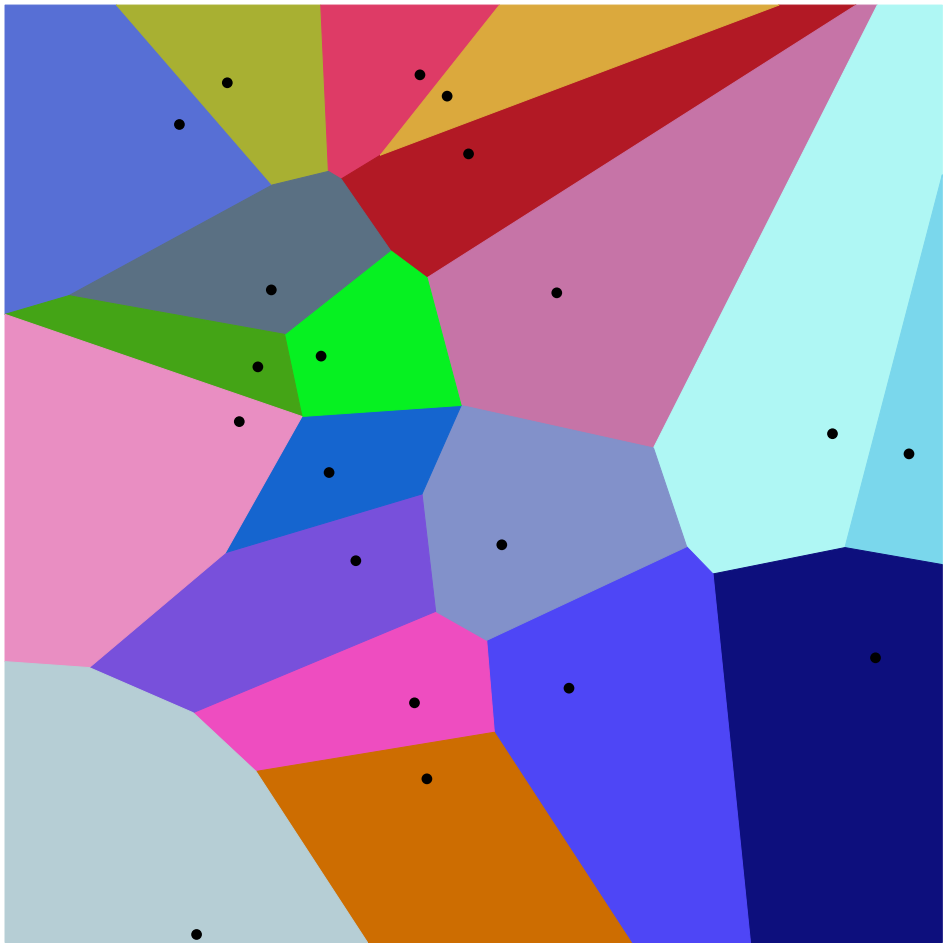


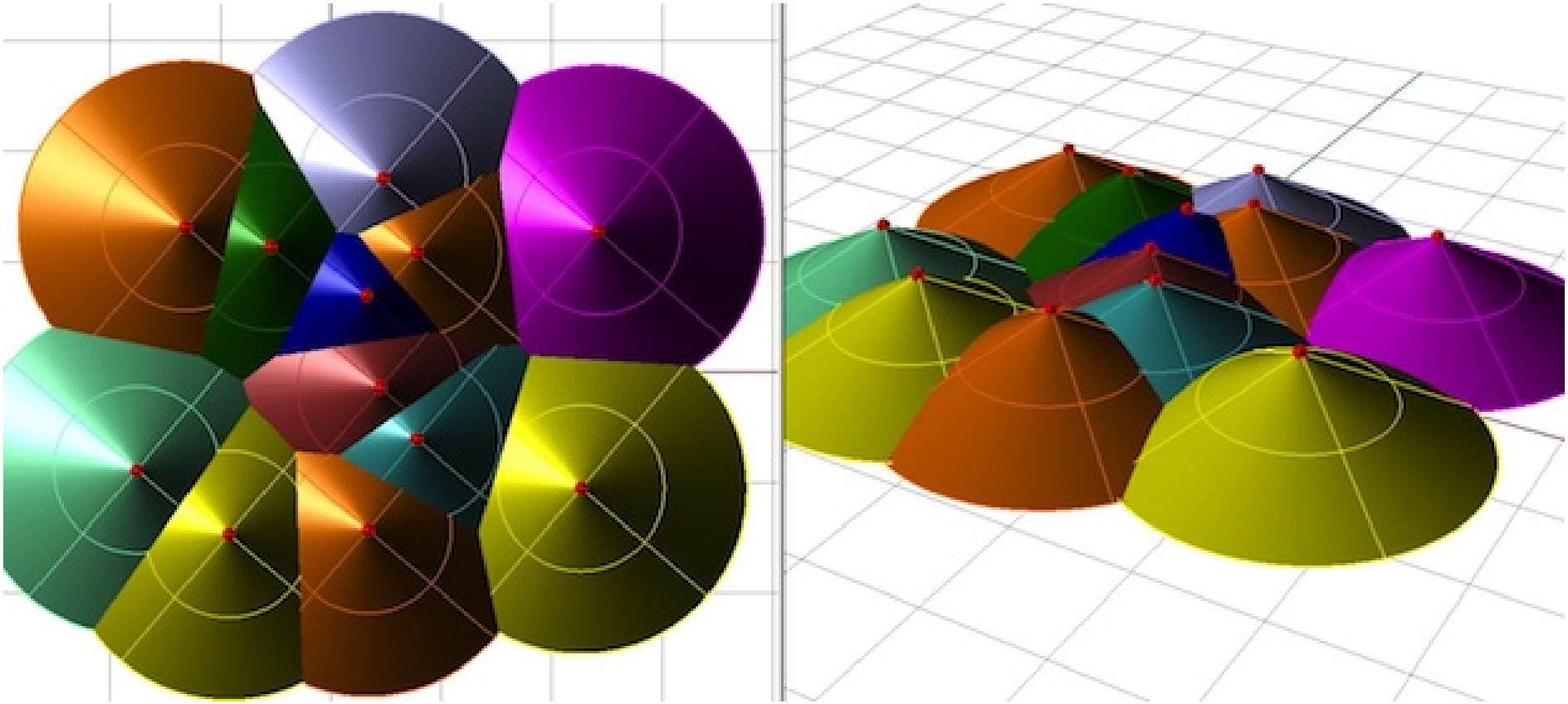
2D examples

Euclidean distance



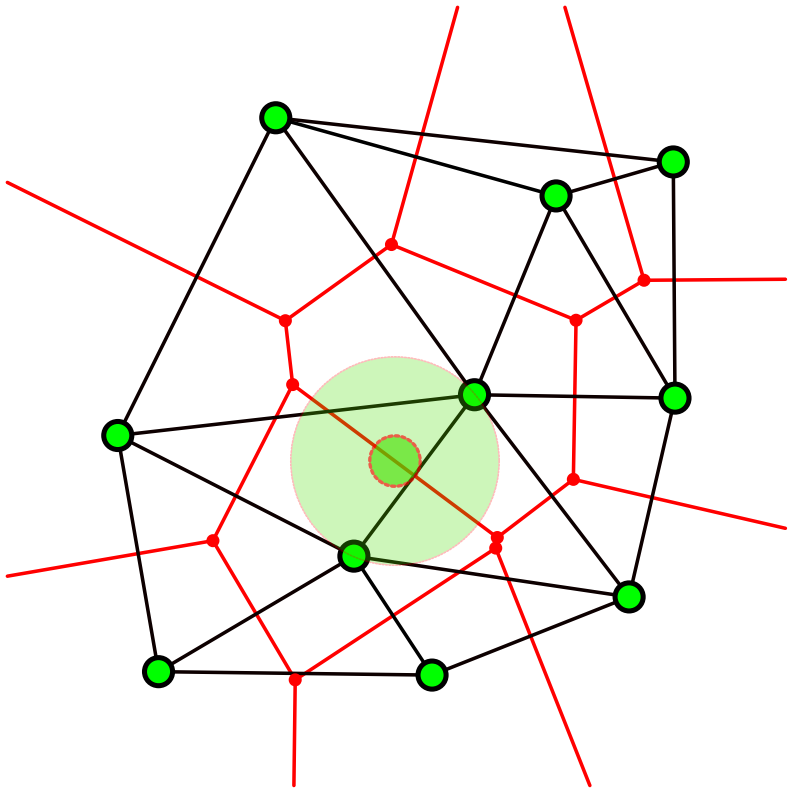
Manhattan distance

Intersection of cones



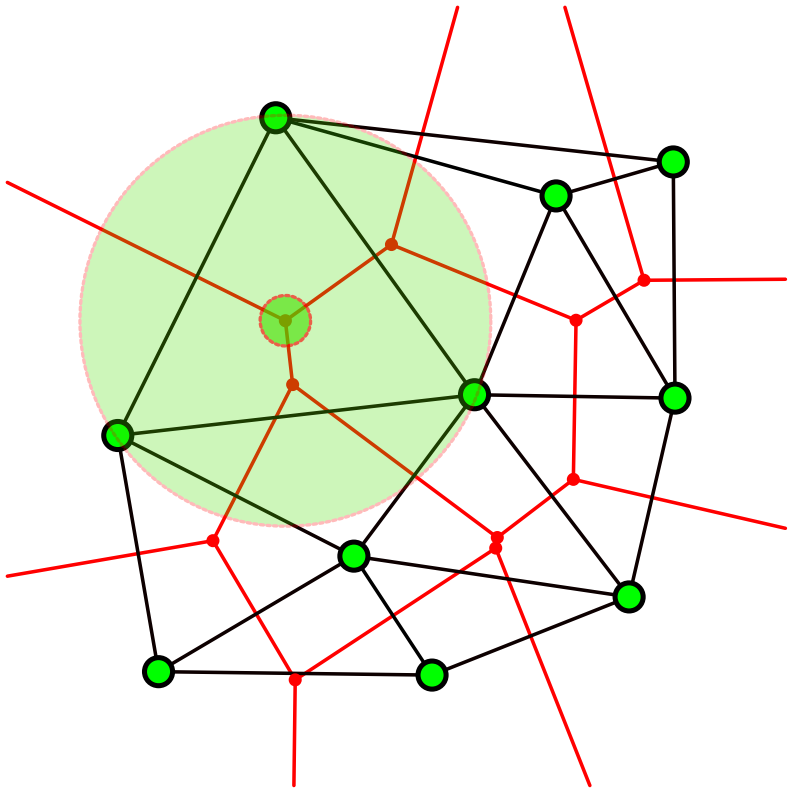
Some properties

- A point in an edge is at equal distance of 2 points in S



Some properties

- A point in an edge is at equal distance of 2 points in S
- A point in a vertex is at equal distance of 3 points in S



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- A point in an edge is at equal distance of 2 points in S
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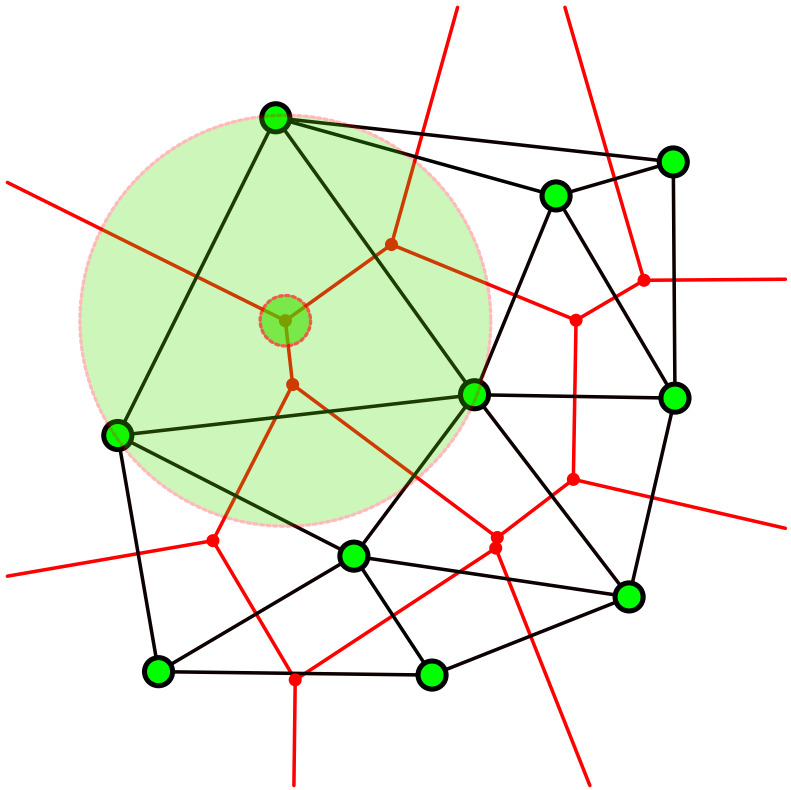
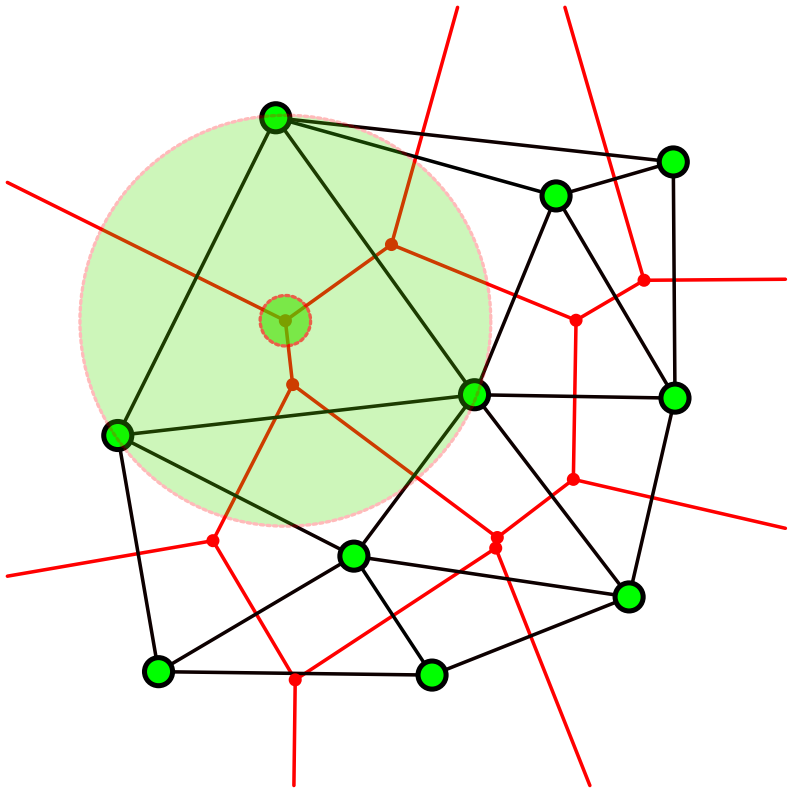
- The closest pair of points corresponds to two adjacent cells in the Voronoi diagram.



Some properties

- A point in an edge is at equal distance of 2 points in S
- A point in a vertex is at equal distance of 3 points in S
- The closest pair of points corresponds to two adjacent cells in the Voronoi diagram.
- Generally (Euclidean distance, unique points...) two points are adjacent on the convex hull if and only if their Voronoi cells share an infinitely long side.



Euler Number

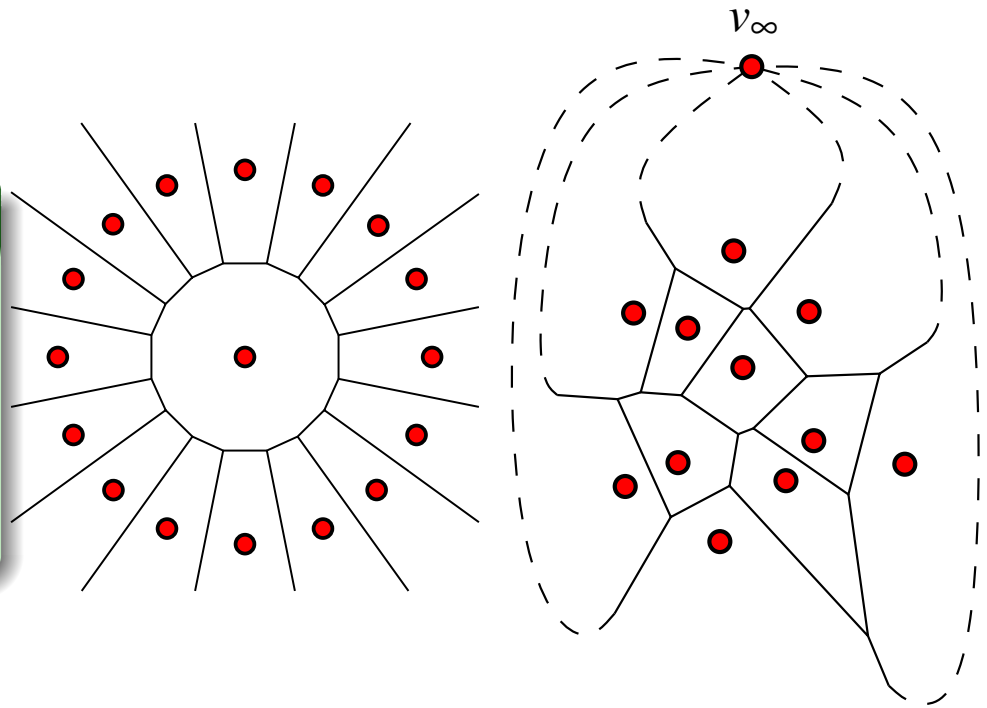
Theorem

For $n \geq 3$, the number of vertices in the Voronoi diagram of n points in the plane is at most $2n - 5$, and the number of edges is at most $3n - 6$.

Element of proof

consider a vertex at infinity so that Euler theorem can apply.

$$(n_v + 1) - n_e + n = 2$$



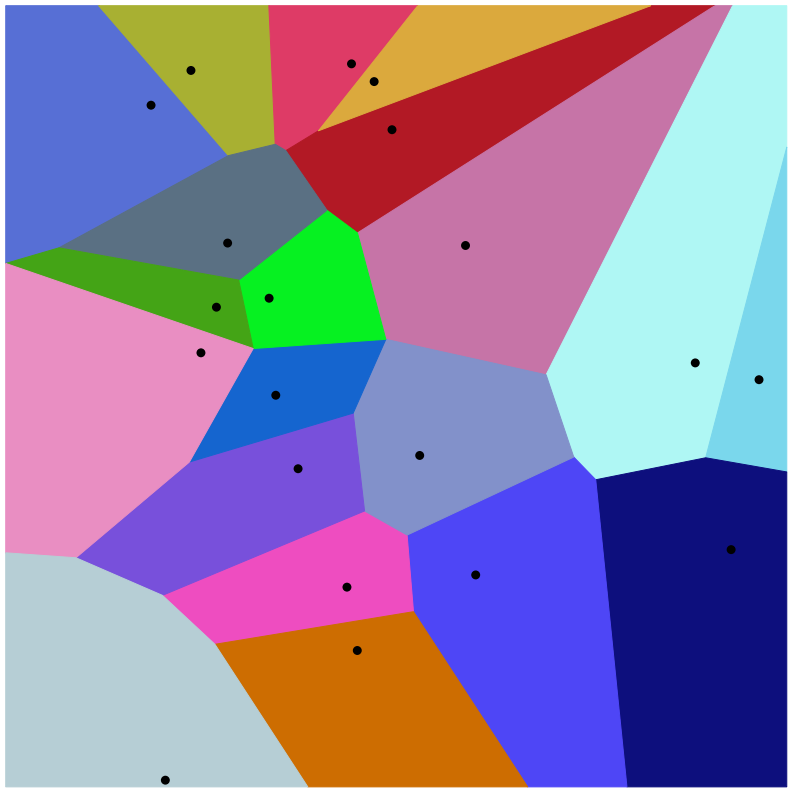
Algorithm from affine half-plane

Definition (Affine (bisector) hyperplane)

$$\mathrm{H}(p,q) = \{x \in \mathbb{R}^2 / d(x,p) \leq d(x,q)\}$$

$$\mathrm{Vor}_S(p) = \bigcap_{q \in \mathrm{S} \setminus \{p\}} \mathrm{H}(p,q)$$

- For a point, construct all half-planes
- Compute the intersection
- $O(n^2)$

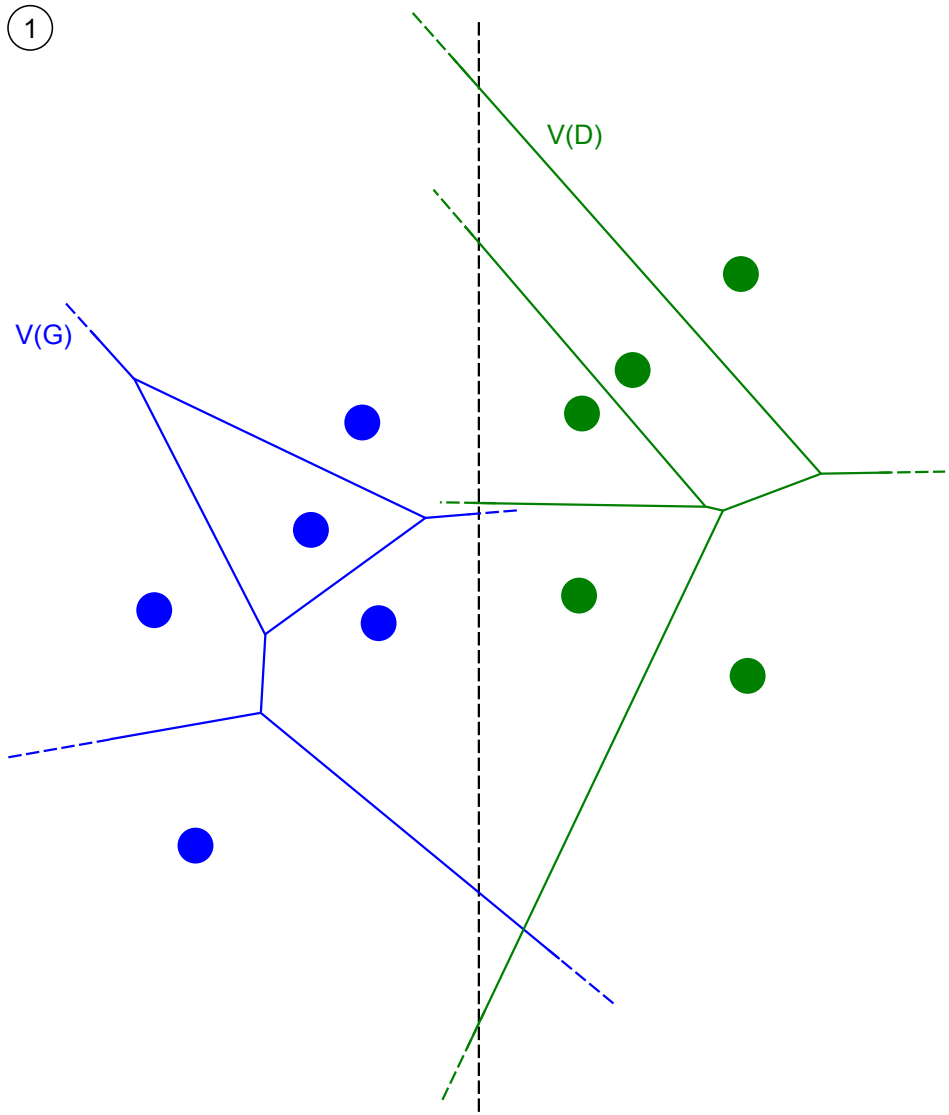


Shamos et Hoey, 1975

Divide and conquer algorithm

- Separate in 2 sets
- Compute Voronoi Diagram
- Compute convex hull
- Merge diagrams

$O(n \log n)$, but too hard to code!



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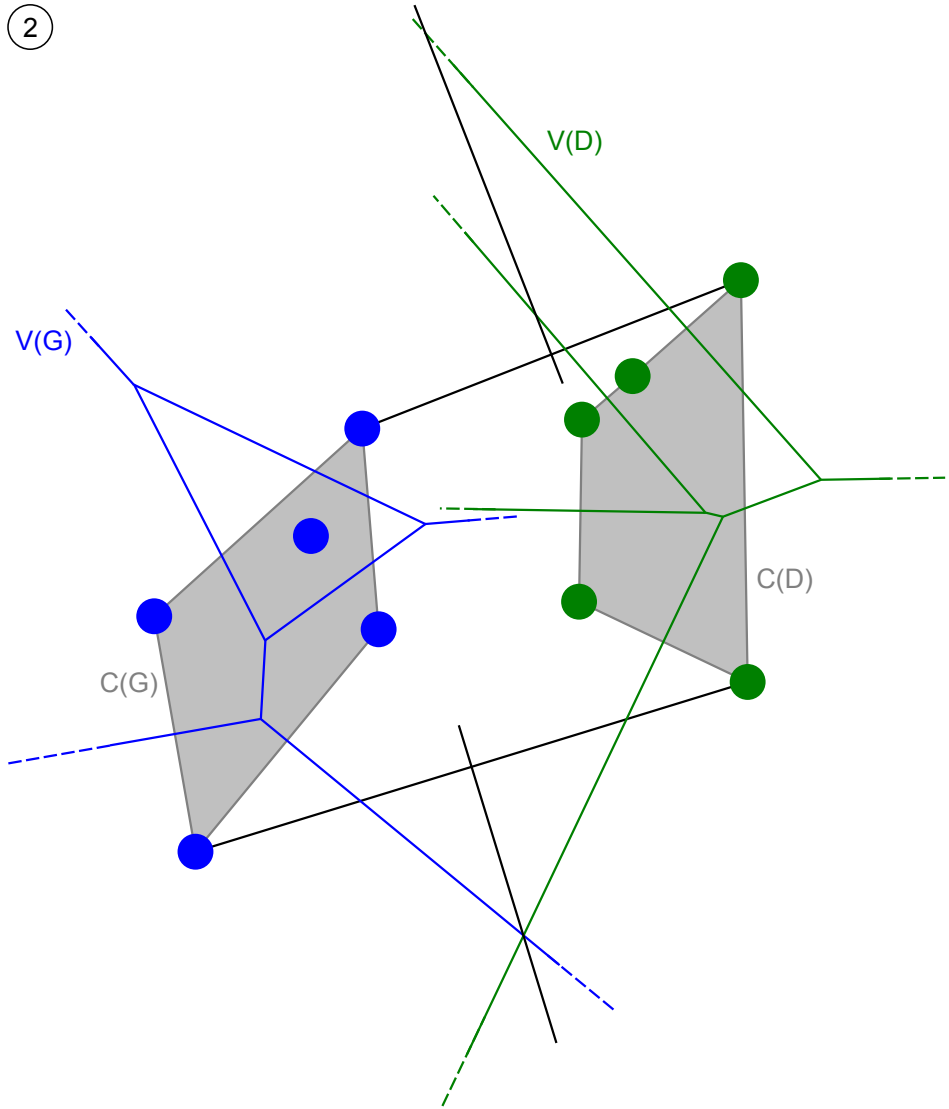

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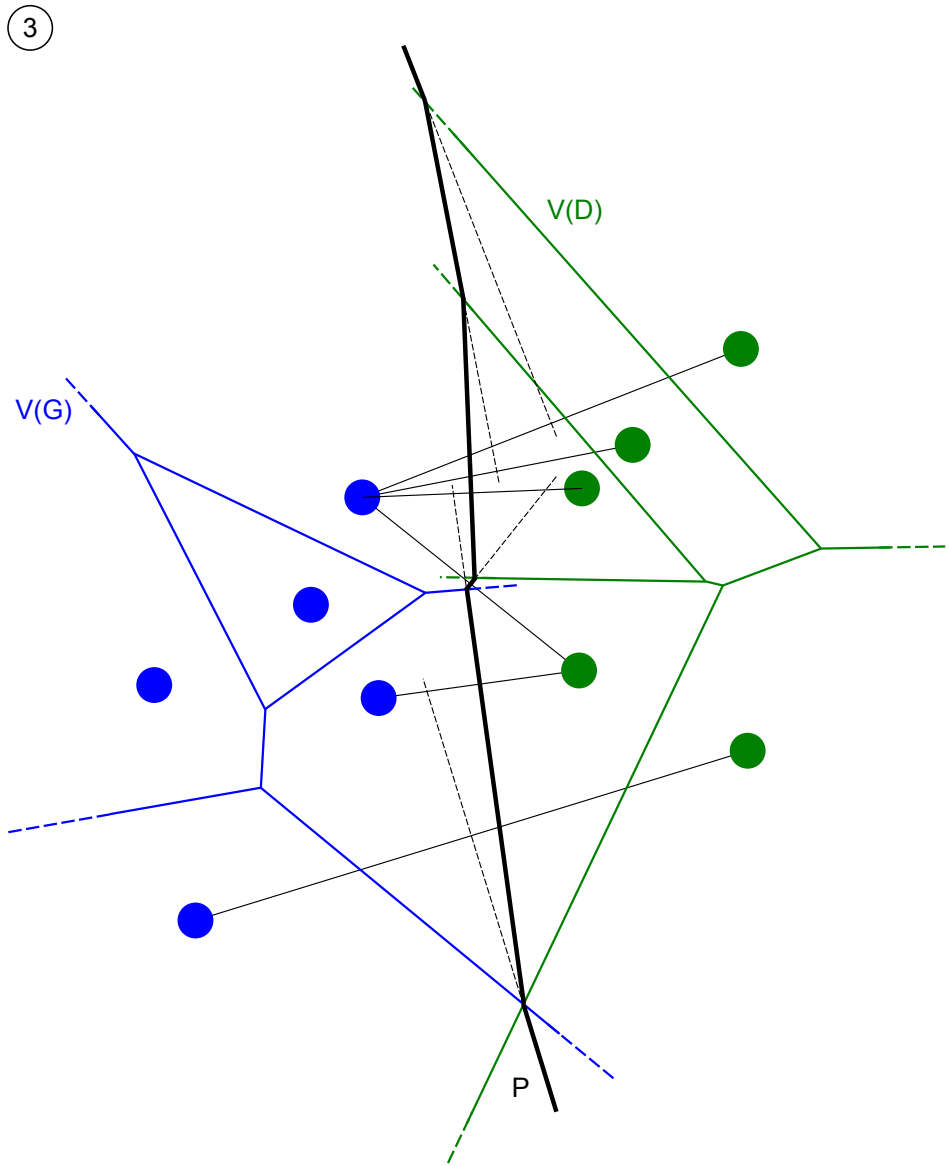

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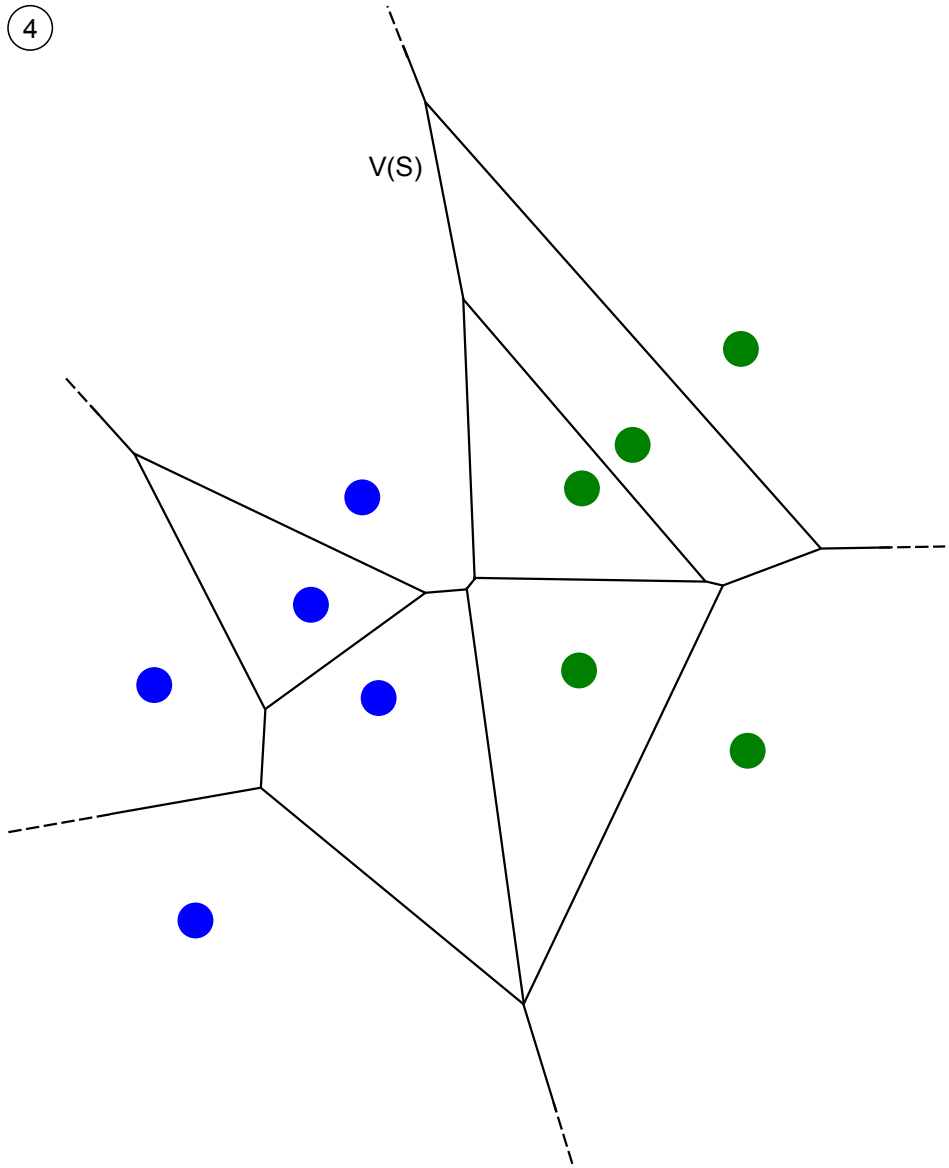
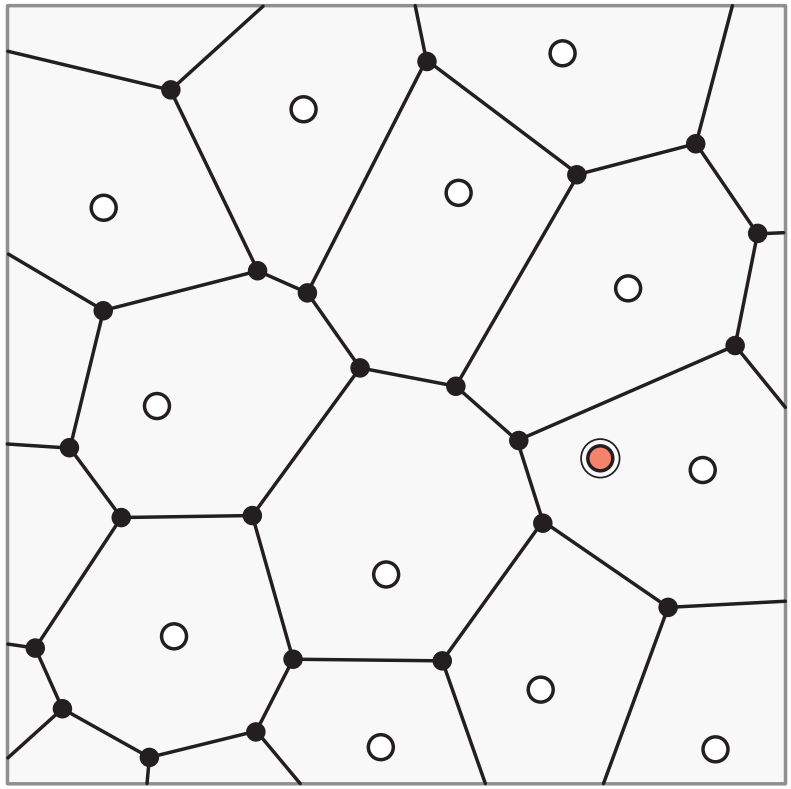


Green et Sibson, 1978

- Incremental algorithm
- Key idea: a new point will modify the diagram locally
- $O(n^2)$

- Find the cell $\mathrm{Vor}_S(p_n)$ containing the new point q
- Compute bisector of $[q; p_n]$
- Find intersections with the cell
- Iterate from cells touching this intersection



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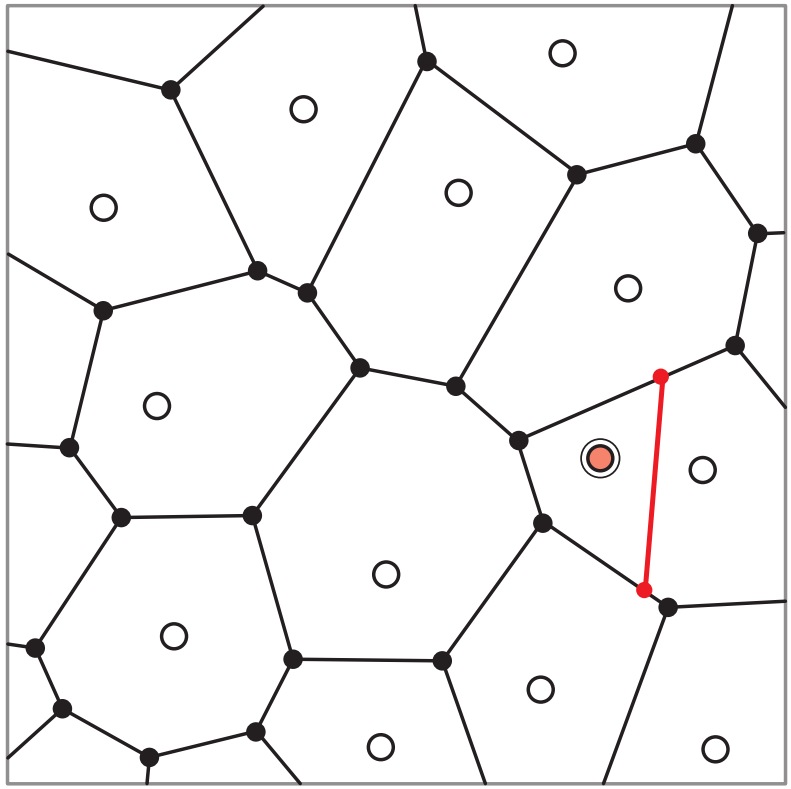

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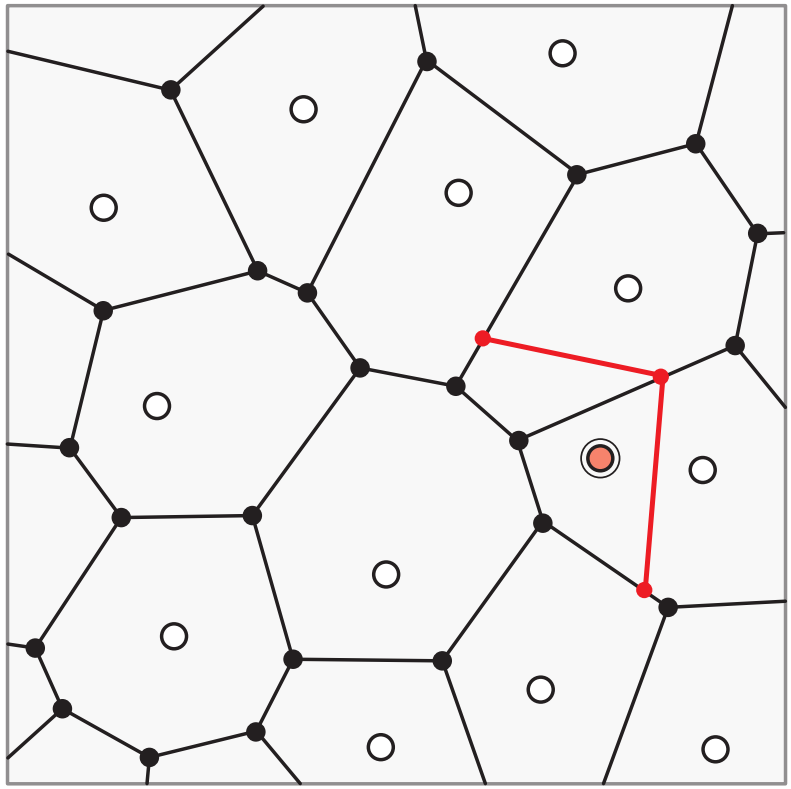

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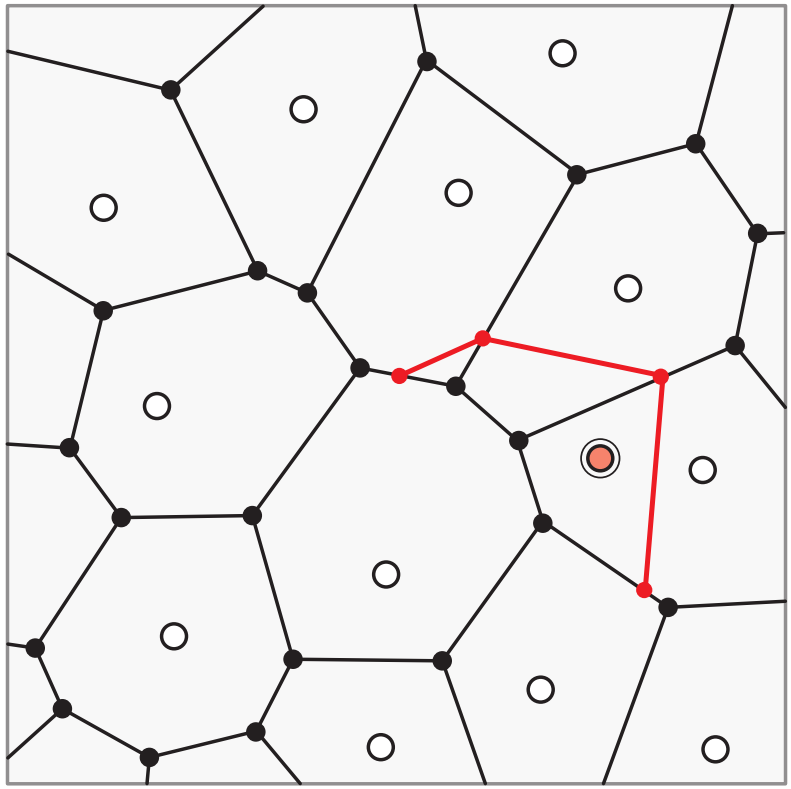

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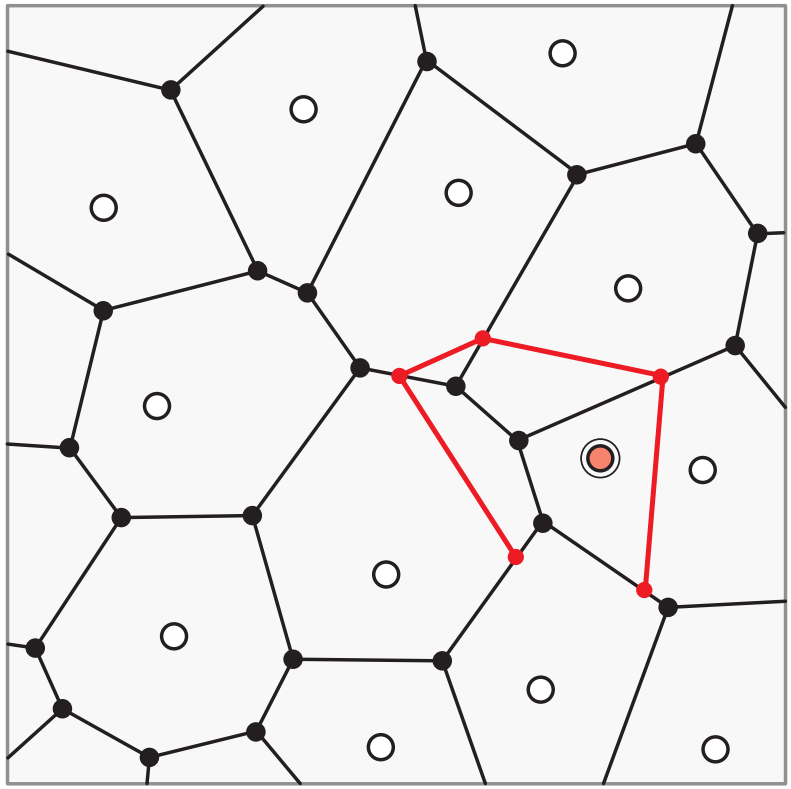

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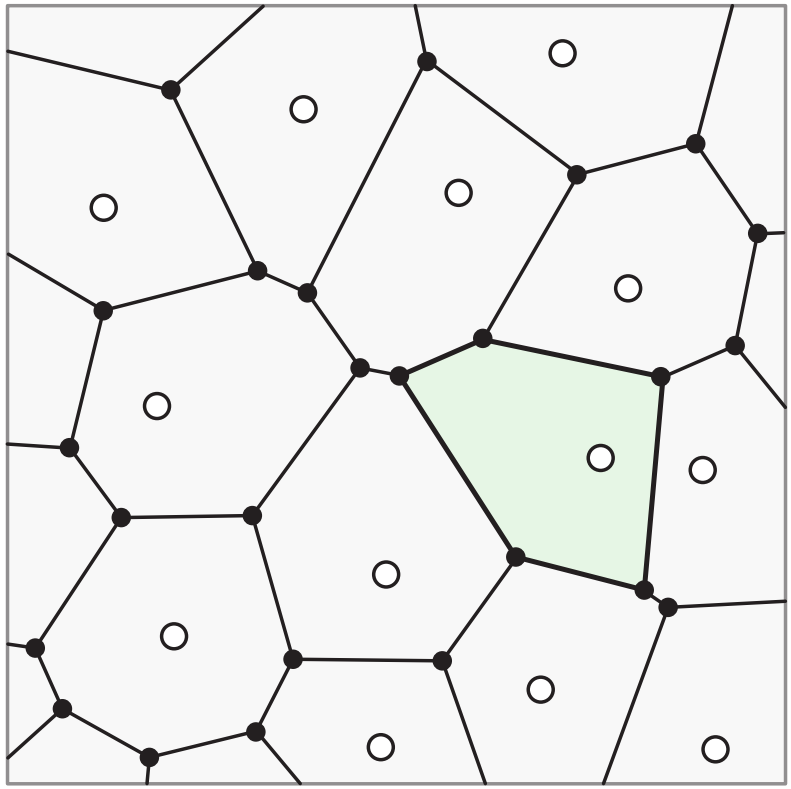
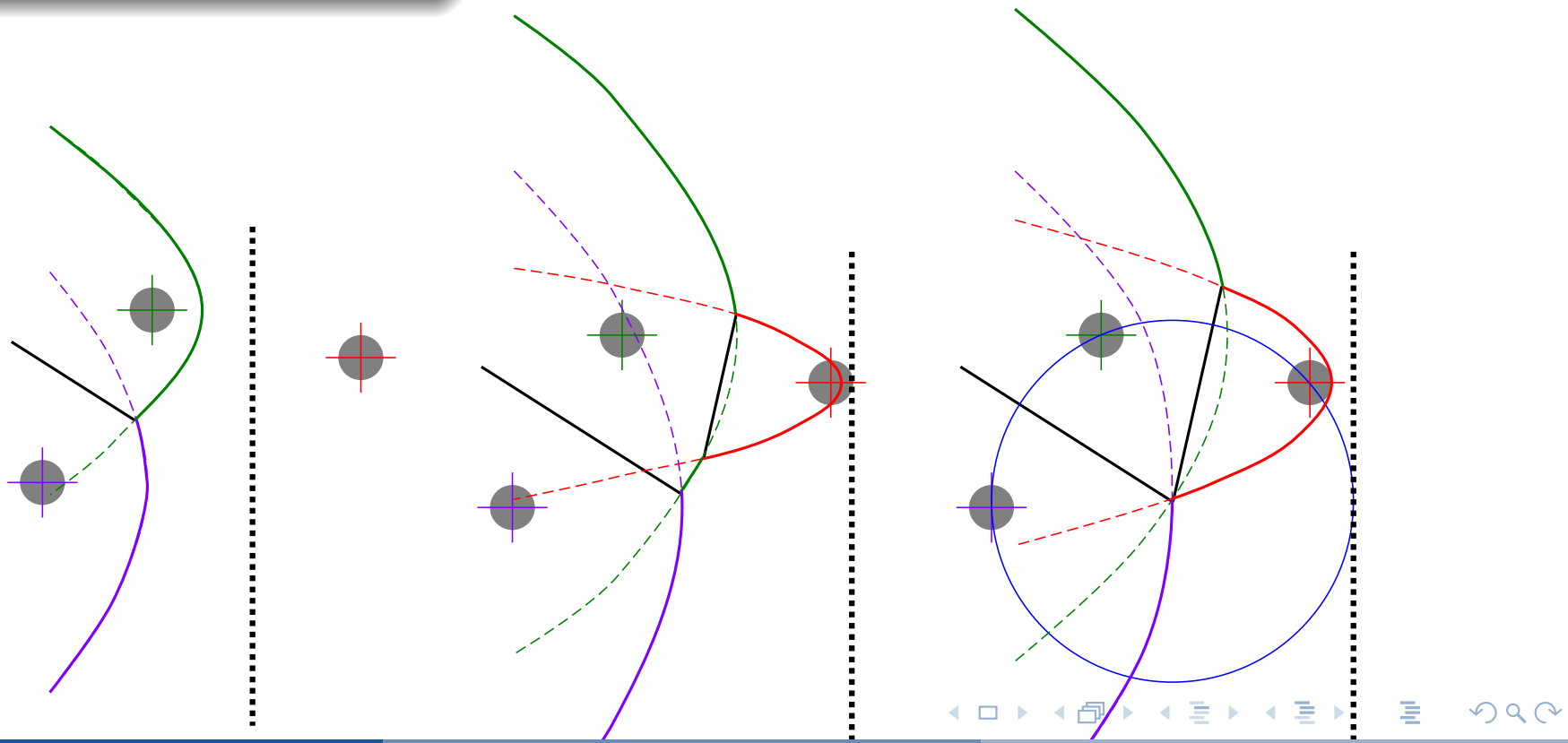


Fortune's algorithm, 1987

- Sweep line $\mathcal{L}$
- Beach curve $\mathcal{B}$
- Compute intersections

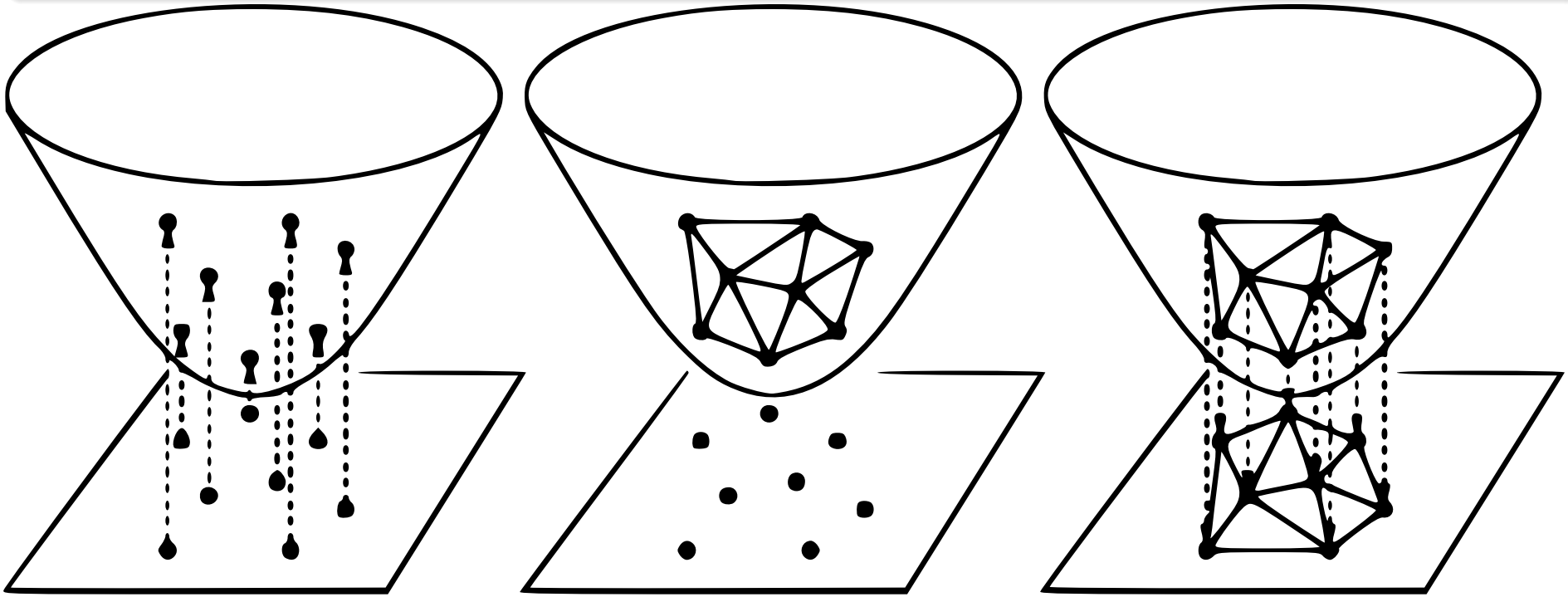
Optimal algorithm $O(n \log n)$

- 1D: equivalent to sorting points (x_i)
- Consider Vor_S, with $S = (x_i, x_i^2)$



For 2D points: Convex Hull in 3D

- $S = \{(x_i, y_i)\}$
- Projection on a paraboloid: $S_{3D} = \{(x_i, y_i, x_i^2 + y_i^2)\}$
- Compute convex hull of S_{3D}
- Projection on the plane



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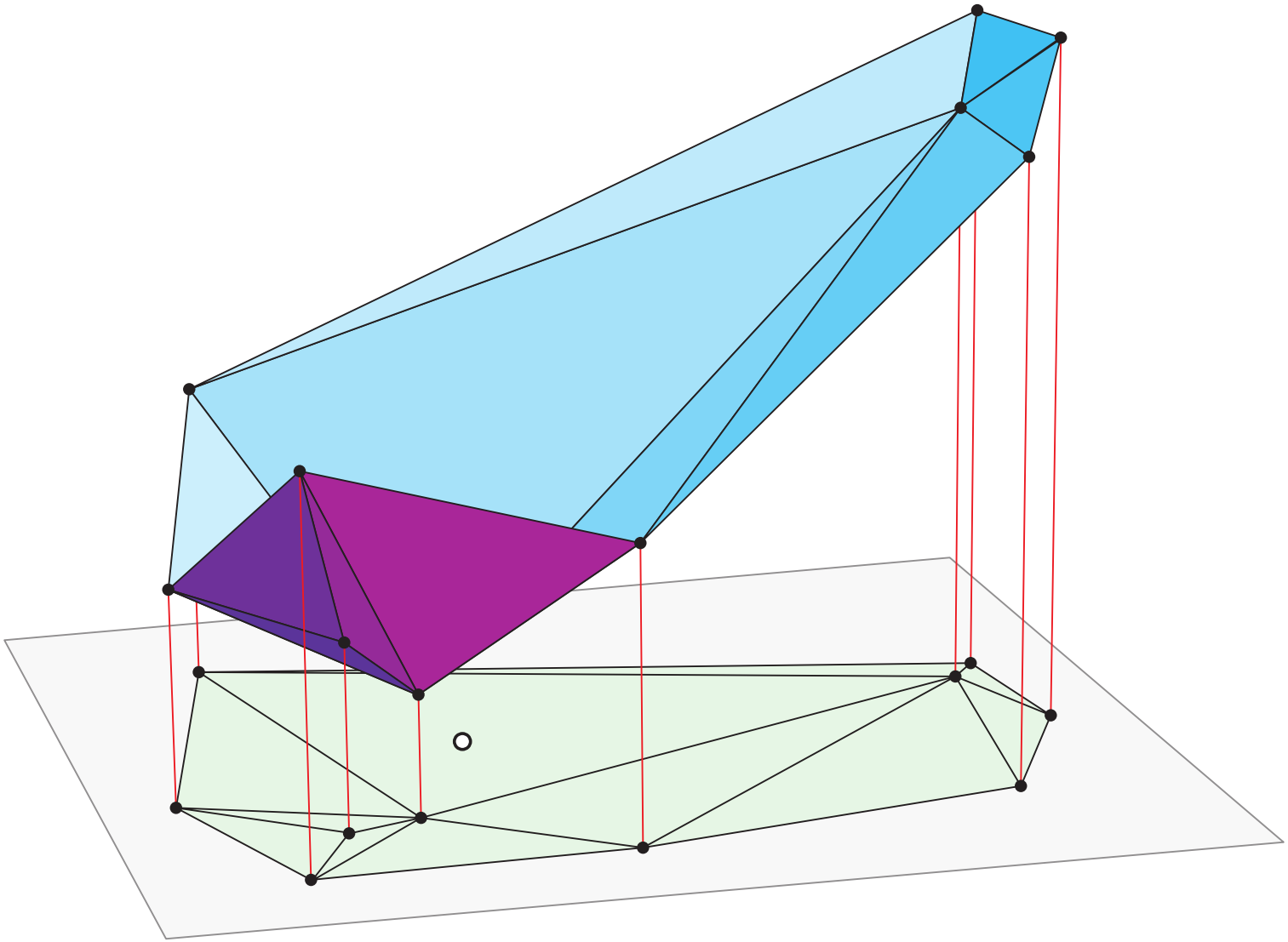


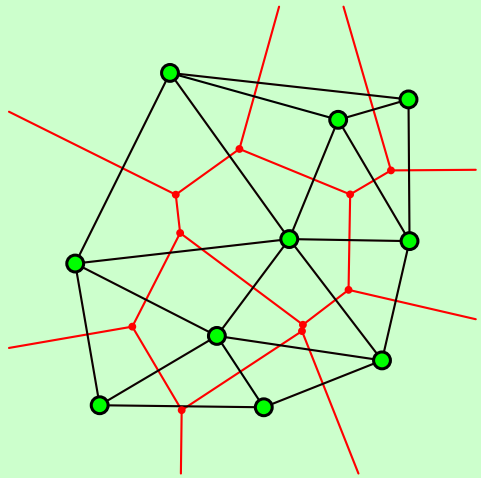
Delaunay triangulation

$$\mathrm{Vor}_S(p) = \bigcap_{q \in \mathrm{S} \setminus \{p\}} \mathrm{H}(p, q)$$

Definition (Delaunay triangulation)

$$\mathrm{DEL}(\mathrm{S}) = \{(p, q) \in \mathrm{S}^2 \ / \ \mathrm{Vor}_\mathrm{S}(p) \cap \mathrm{Vor}_\mathrm{S}(q) \neq \varnothing\}$$

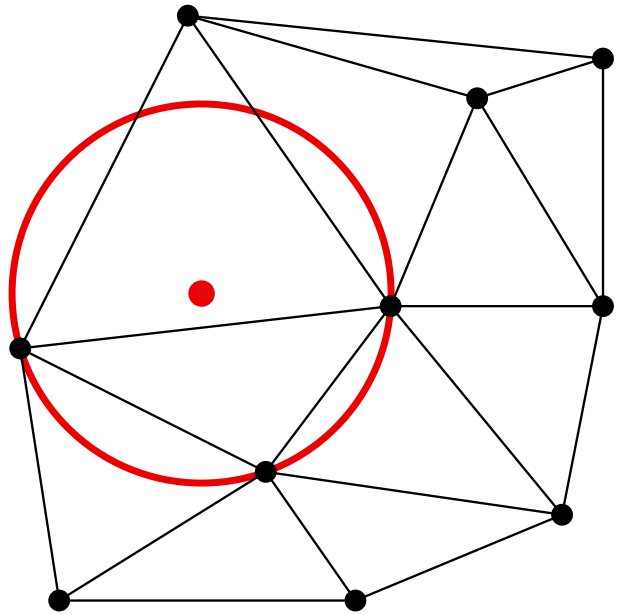
- Red: Voronoi graph
- Black: Delaunay triangulation
- Green: sites



Property: circumscribed circle

Property

A circle circumscribing any Delaunay triangle does not contain any other input points in its interior



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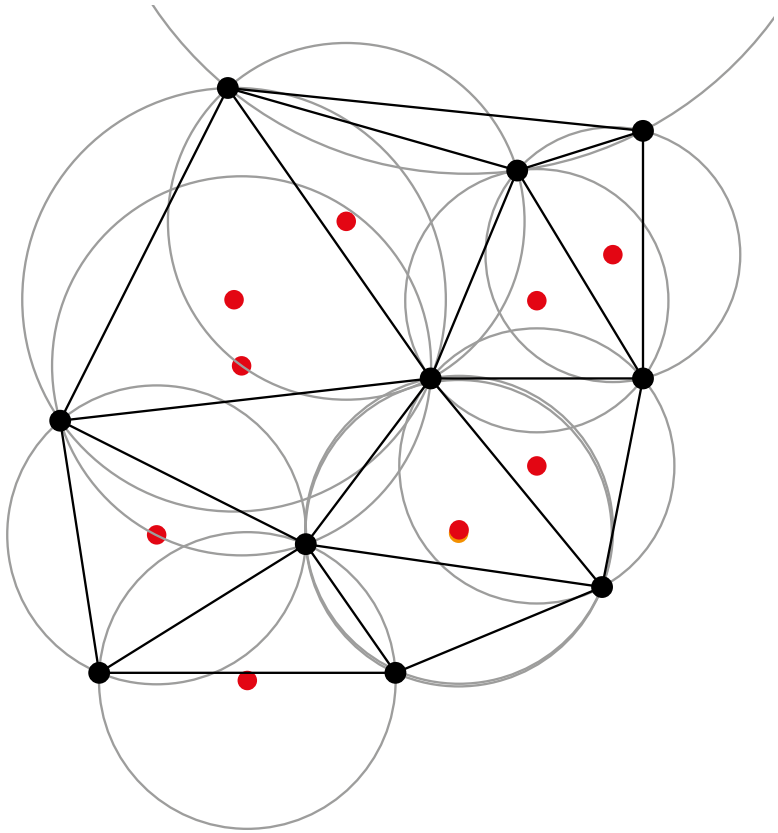


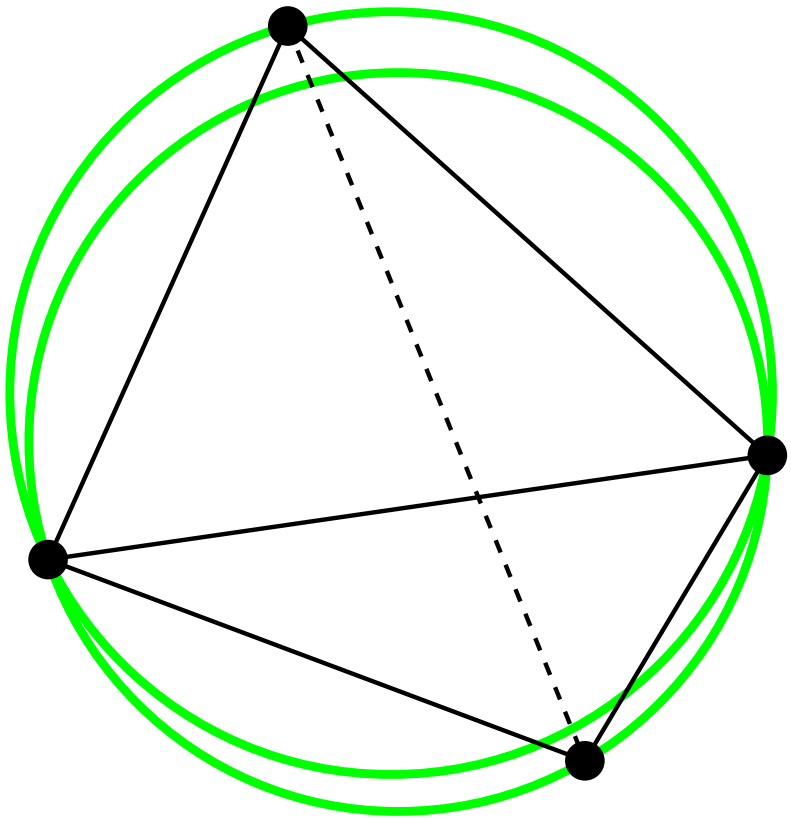
Flip algorithm

Construction algorithm

- Start by an arbitrary triangulation
- For all edges, test the 2 circles
- Flip edges if the property is not respected

Robust and fast method for in-circle testing

- use determinant of matrix
- $\Omega(n^2)$ operations



Power/Laguerre diagram

Definition (Power function)

- $S = \{(P_k, w_k)\}$
- Power of a point: $\mathrm{pow}(y, (P_k, w_i)) = d(y, P_k)^2 - w_k$
- Sphere of radius $r_k = \sqrt{w_k}$

Definition (Laguerre diagram)

$$\mathrm{Lags}(P_k) = \{x \in X \mid \mathrm{pow}(x, (P_k, w_k) \leq \mathrm{pow}(x, (P_j, w_k)) \; \forall j \neq k\}$$

- Laguerre cell can be empty
- It may not contain its seed
- It may contain other seeds

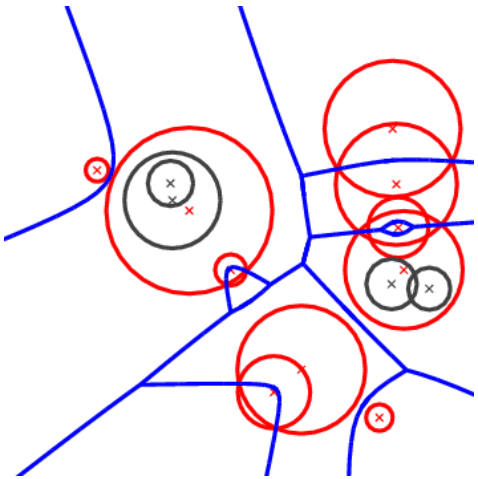
Apollonius diagram

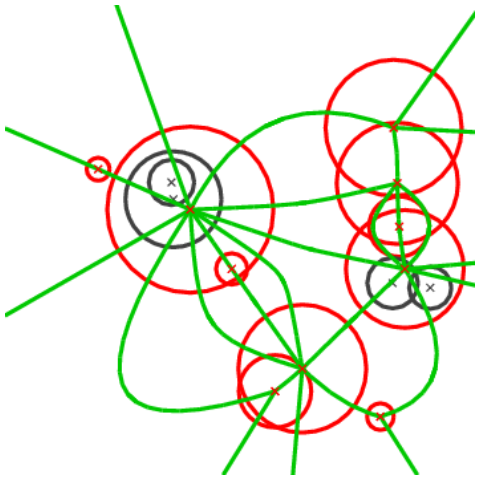
Definition (Apollonius diagram)

$$\mathrm{Apos}(P_k) = \{x \in X \mid \delta(x,(P_k,w_k) \leq \delta(x,(P_j,w_k))\ \forall j \neq k\}$$

Definition (Apollonius distance)

$$\delta(x,(P_k,w_k)) = d(x,P_k) - w_k$$



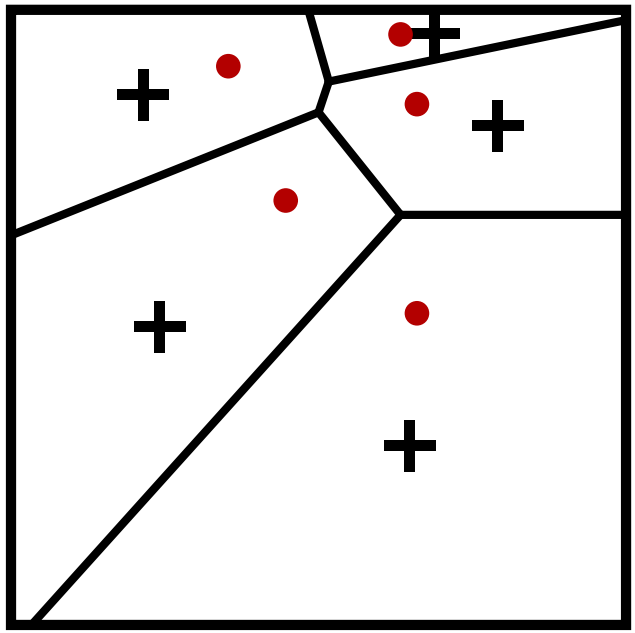


Centroidal Voronoi Tessellation

Lloyd's algorithm

Make several iterations of these steps:

- Compute the Voronoi diagram
- Compute the center of each cell
- Move each seed to its center



- It may take a long time to convergence
- How to handle infinite cells?
- Converges to hexagonal cells

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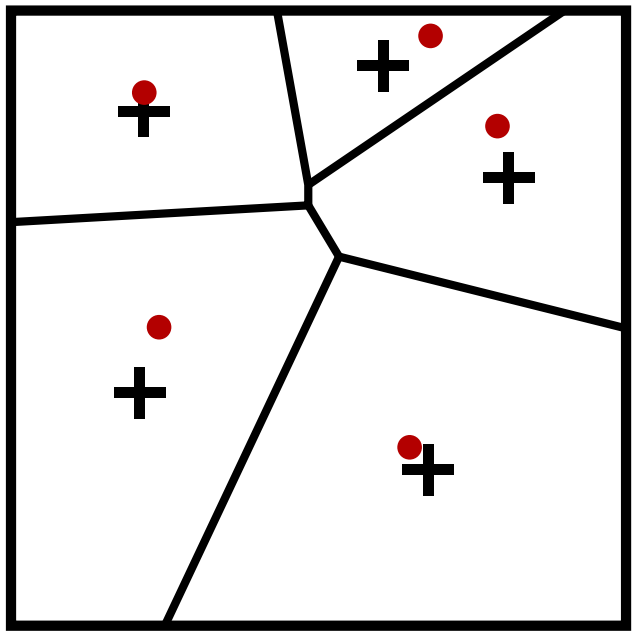

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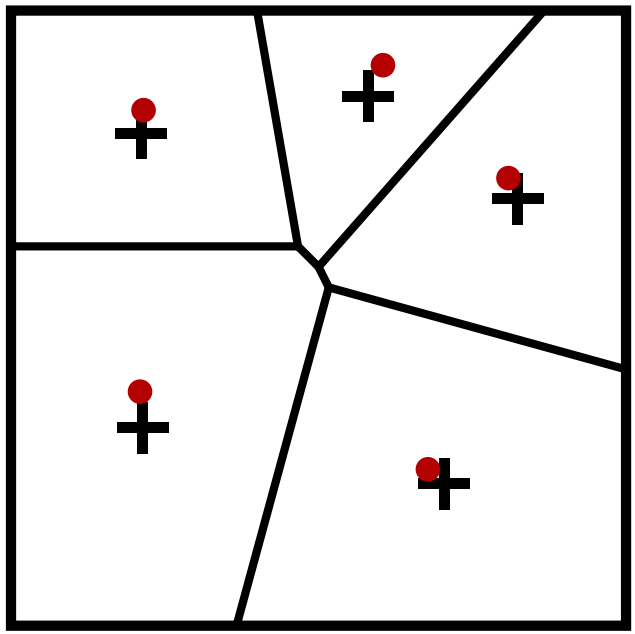

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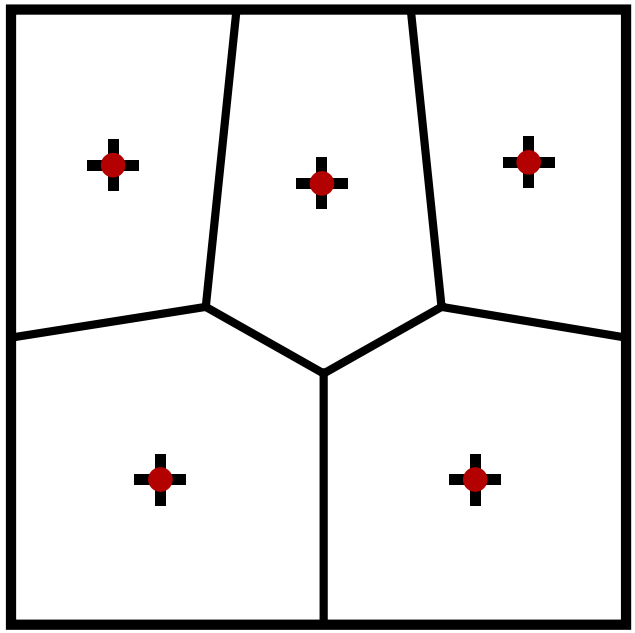
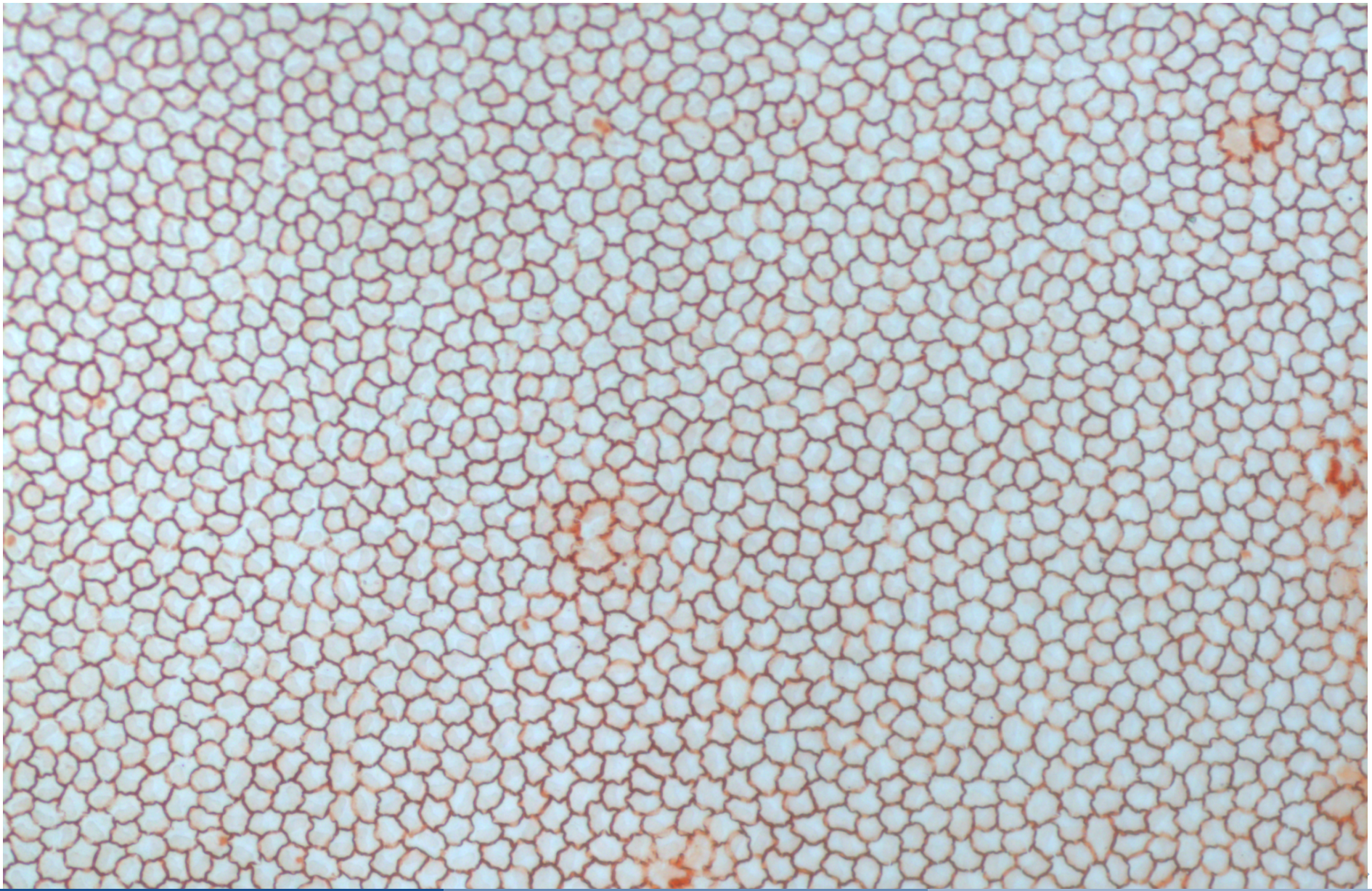


Application: simulation of corneal endothelium cells

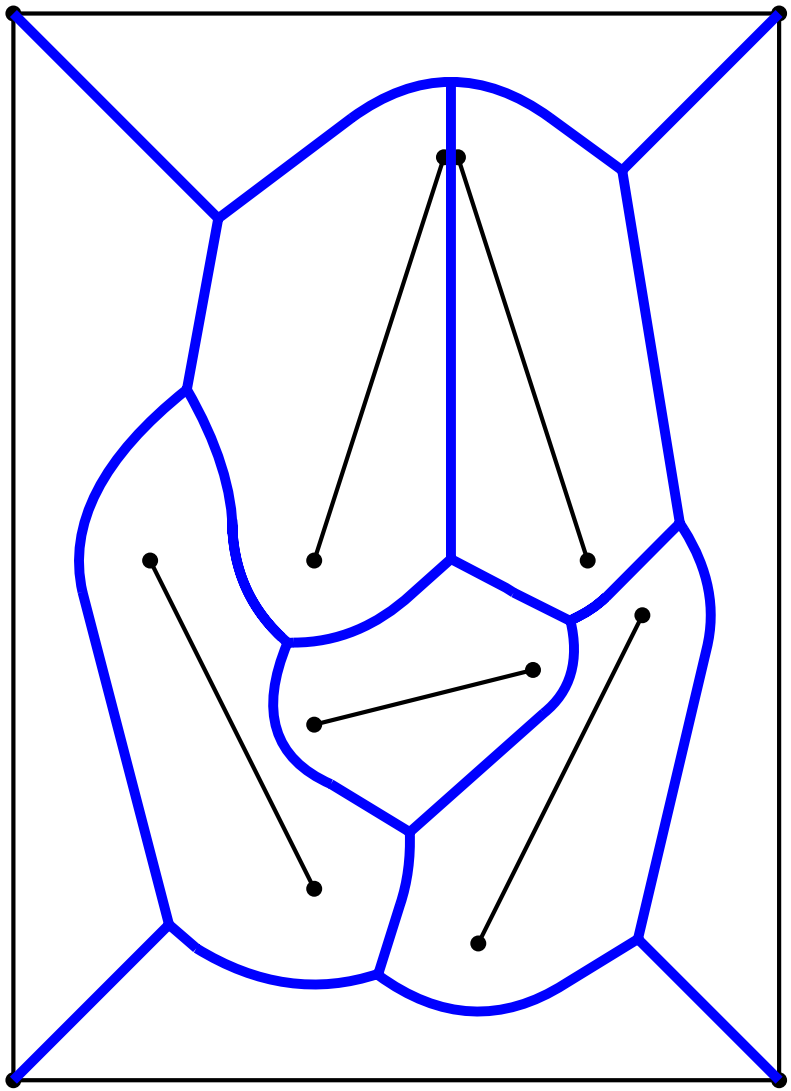


Voronoi diagram of objects

- Voronoi diagram of segments
- Voronoi diagram of objects
- Medial axis

Path planning

Medial axis is the location with the largest area around it.

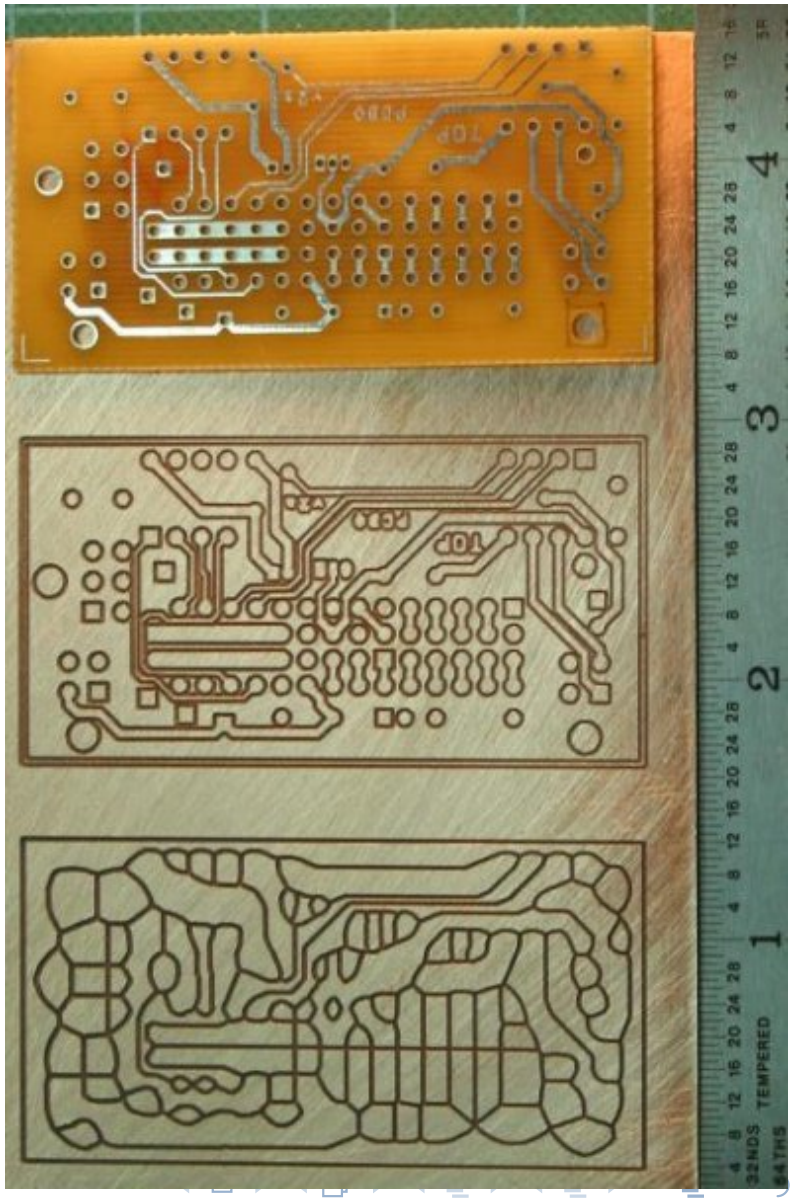


Example: PCB Printed Circuit Board conception

The same printed circuit board manufactured three different ways:

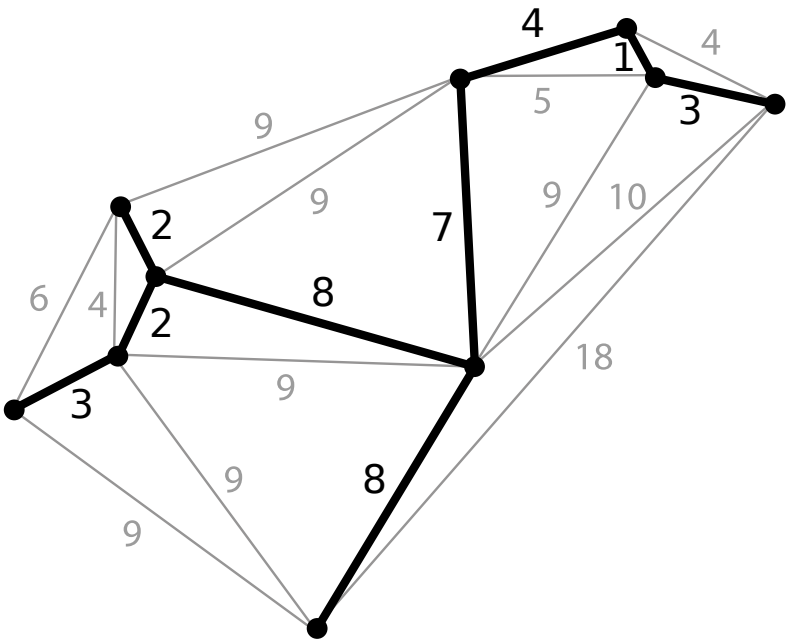
- by traditional photochemical process (top);
- by mechanical etch with standard outline toolpaths (middle);
- by mechanical etch with Voronoi toolpaths (bottom).

See [Vona and Rus, 2005].



Minimum spanning tree

- Planar graph
- No cycle
- Each vertex is connected
- Minimum weight
- Not necessarily unique



Greedy algorithms [Cormen et al., 2009]

n_v vertices, n_e edges.

- Borůvka, 1926: $O(n_e \log n_v)$
- Prim $O(n_e \log n_v)$ or $O(n_e + n_v \log n_v)$ (depends of data structure)
- Kruskal $O(n_e \log n_e)$
- Chazelle $O(n_e \alpha(n_e, n_v))$, α is the inverse of Ackerman function

$O(n_e \log n_v)$ is equivalent to $O(n_e \log n_e)$

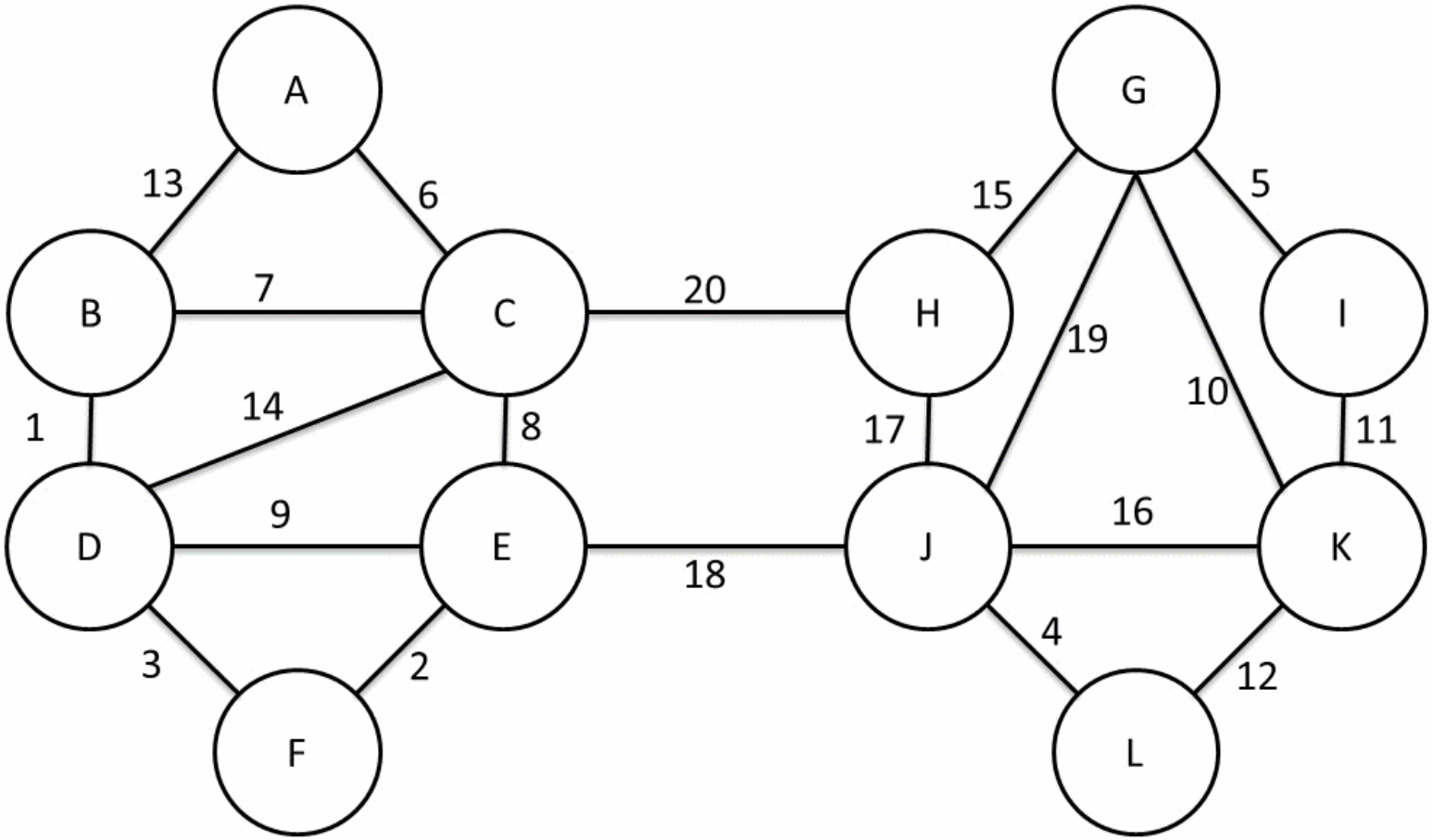
Boruvka algorithm, 1926

Warning

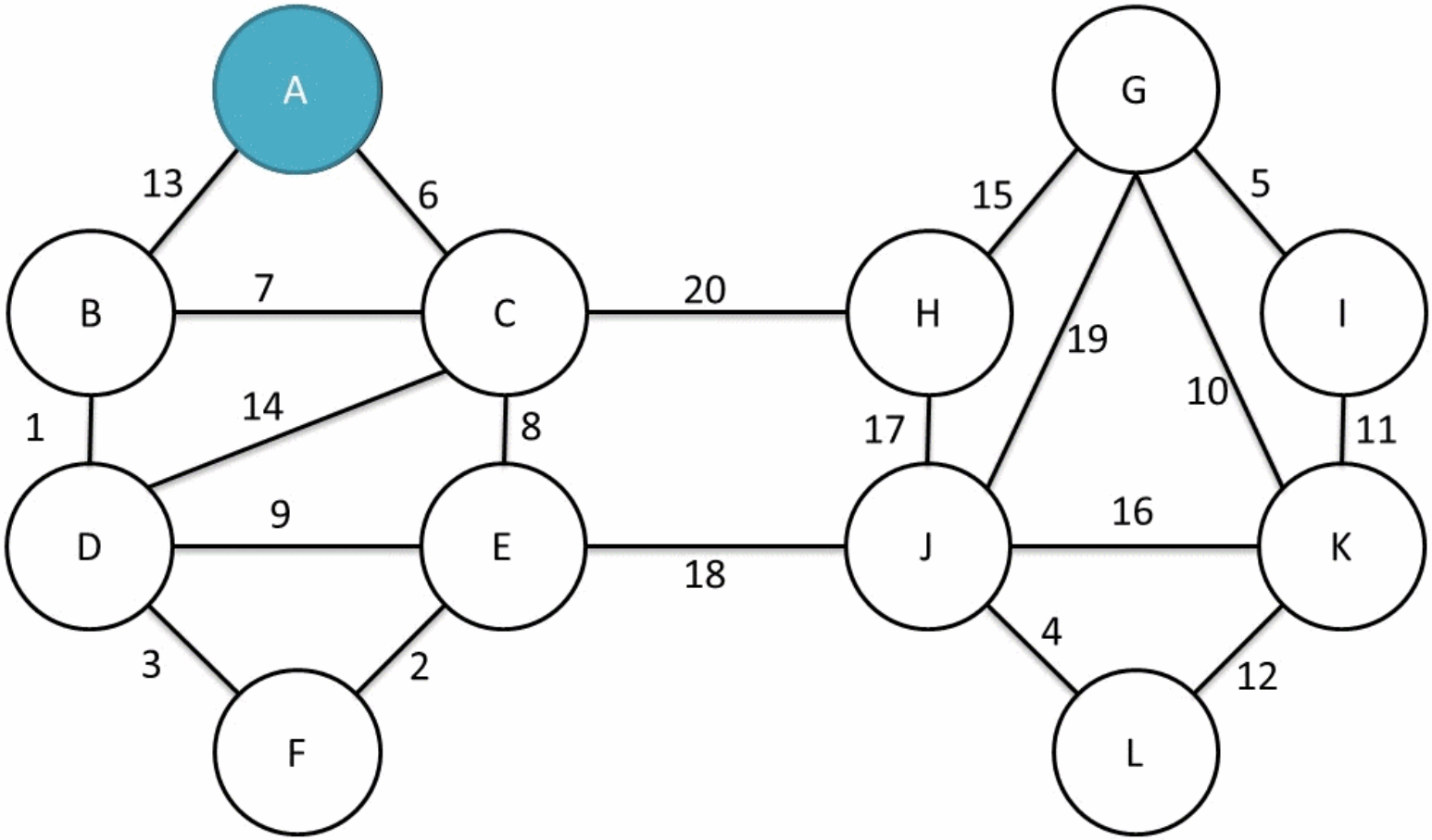
all edge weights are distinct,

Otakar Boruvka: an efficient electricity network for Moravia.

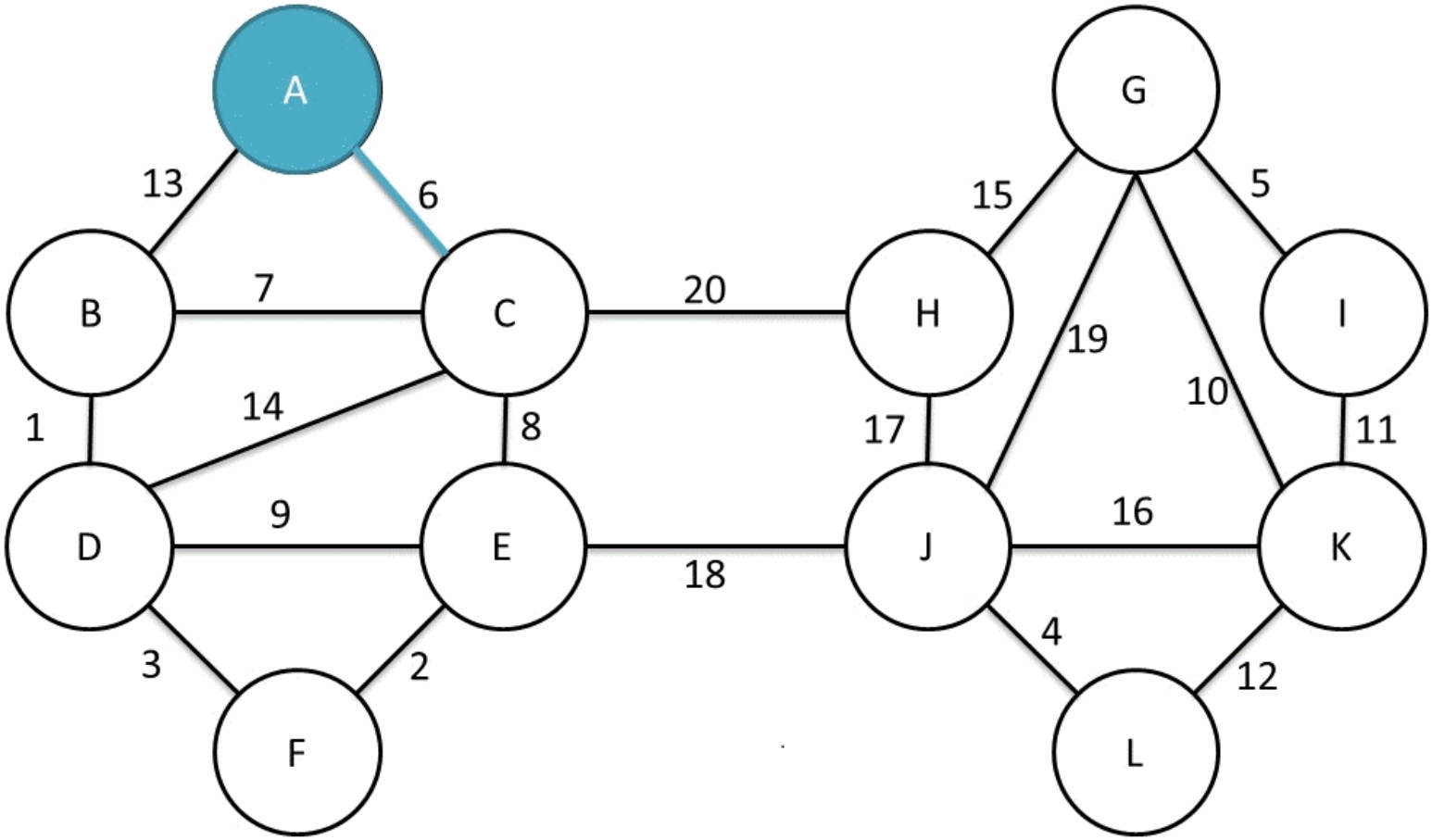
Boruvka algorithm, 1926



Boruvka algorithm, 1926



Boruvka algorithm, 1926



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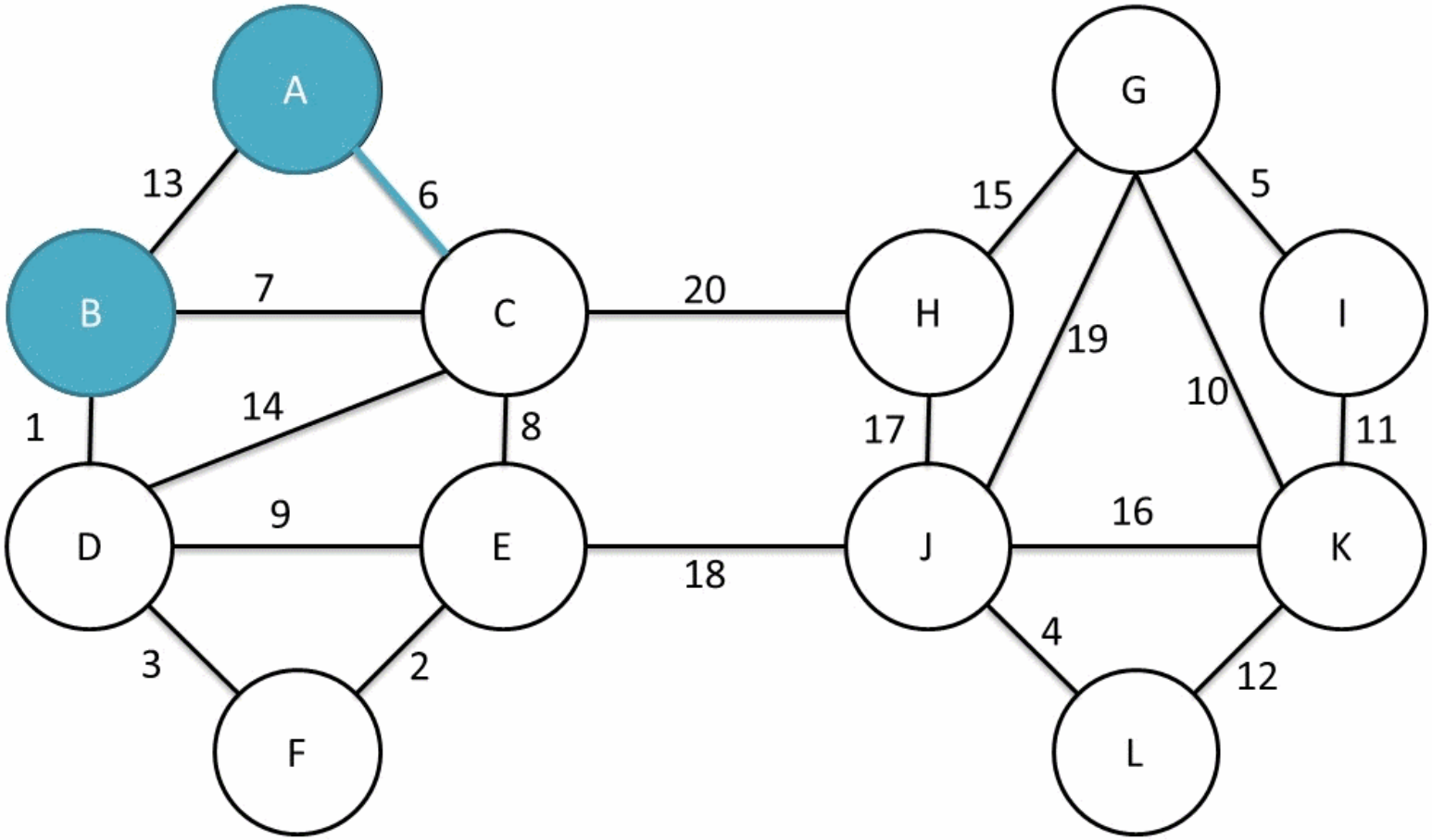

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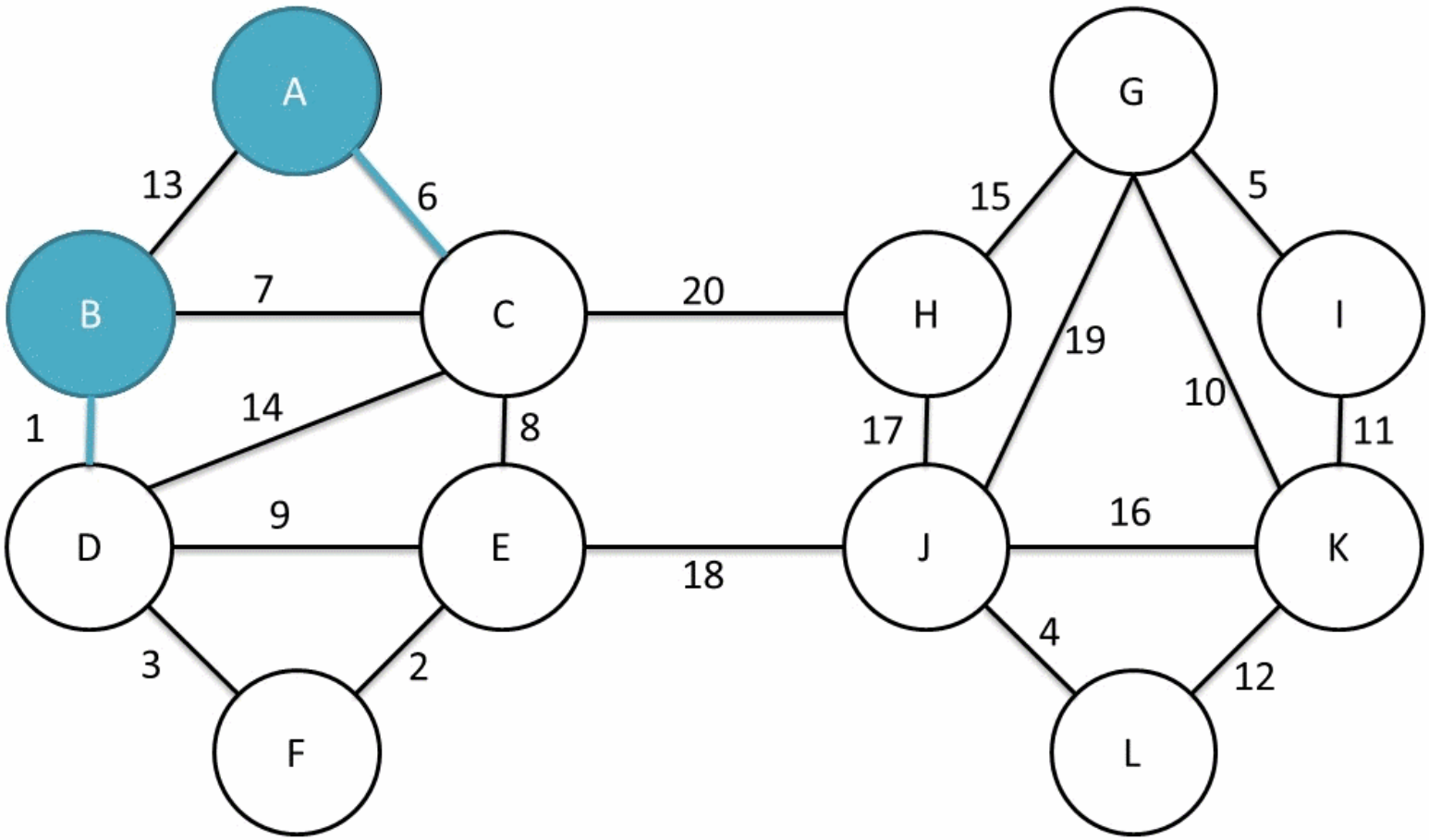

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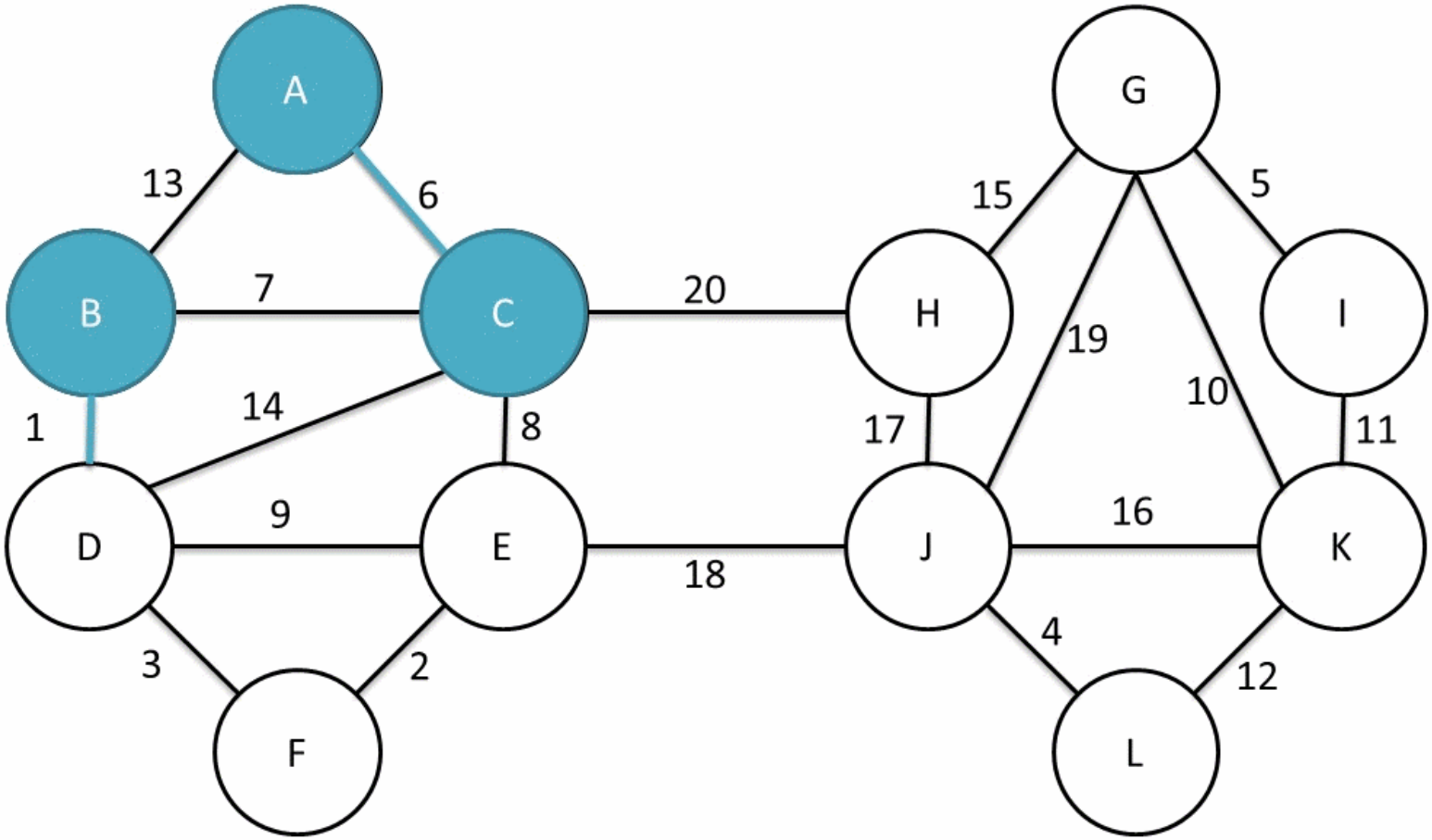

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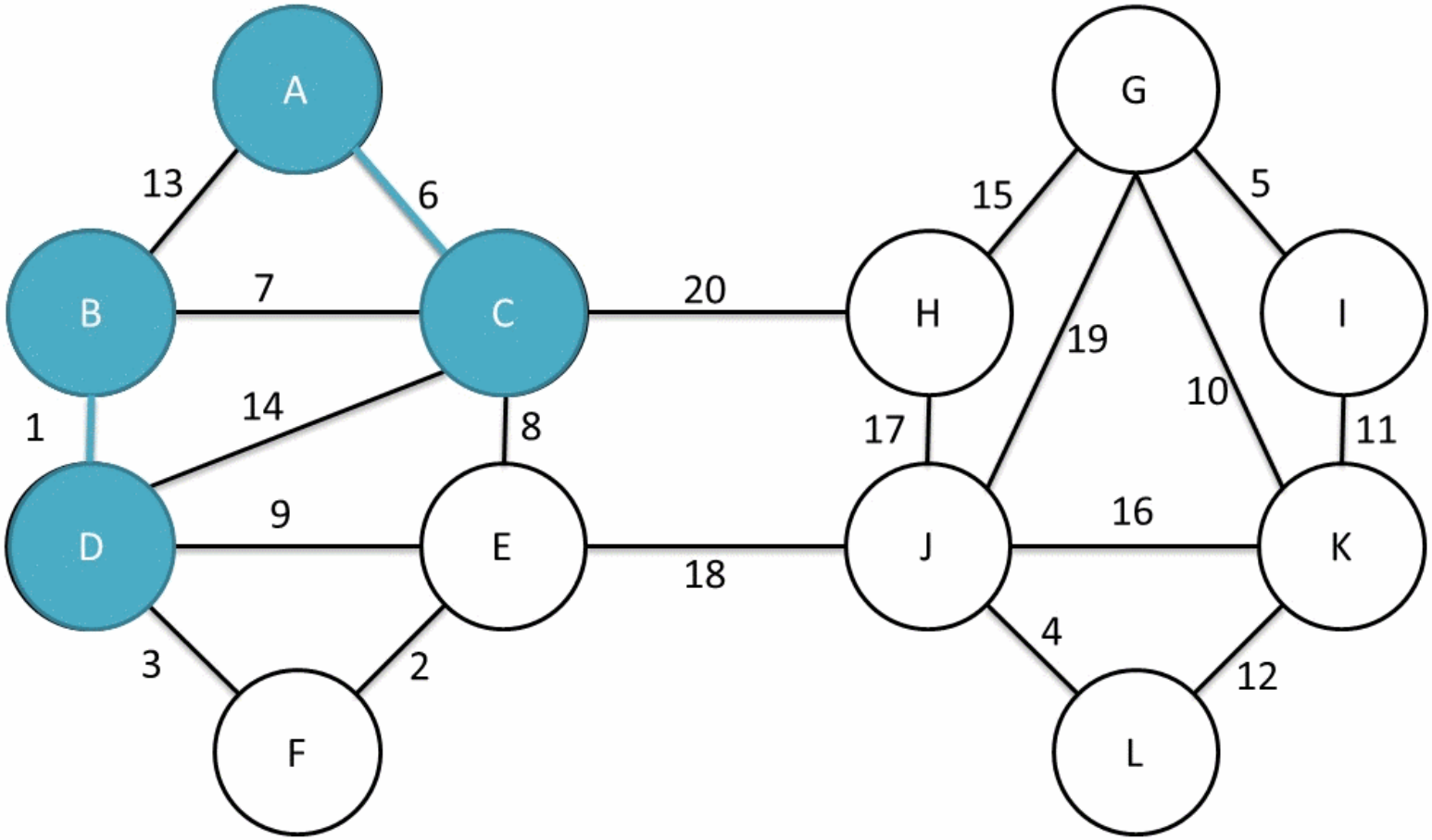
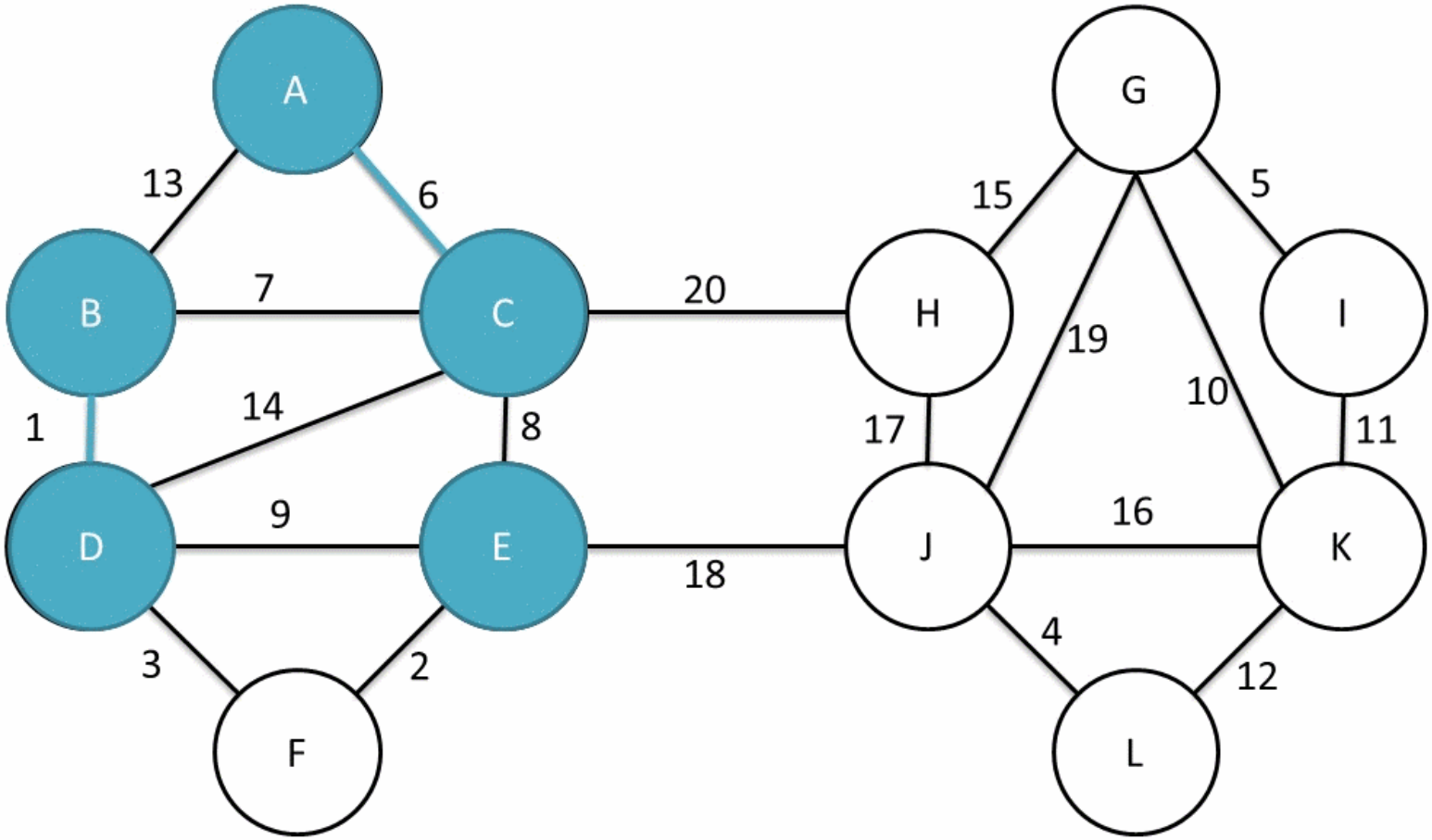


Boruvka algorithm, 1926



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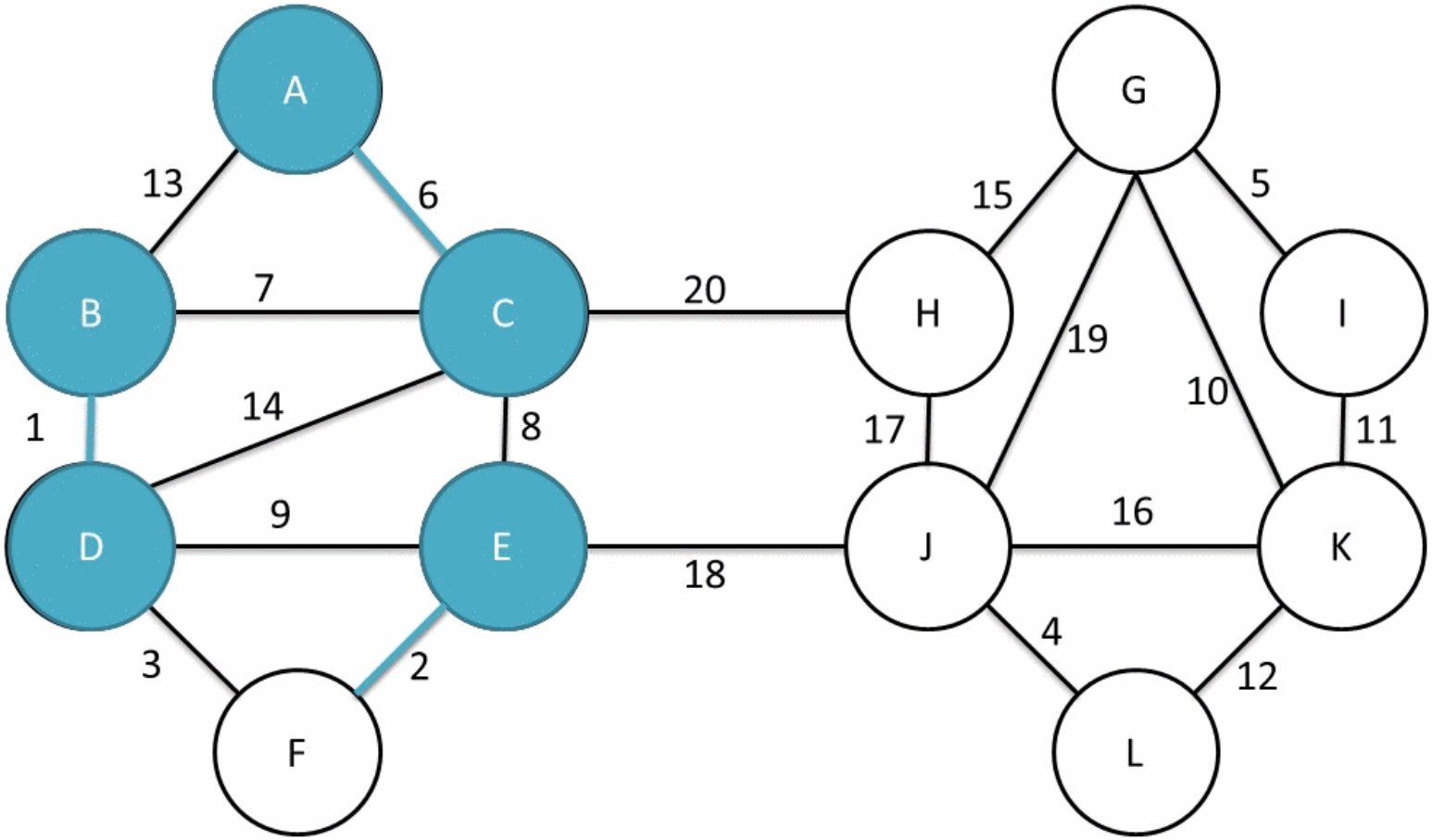

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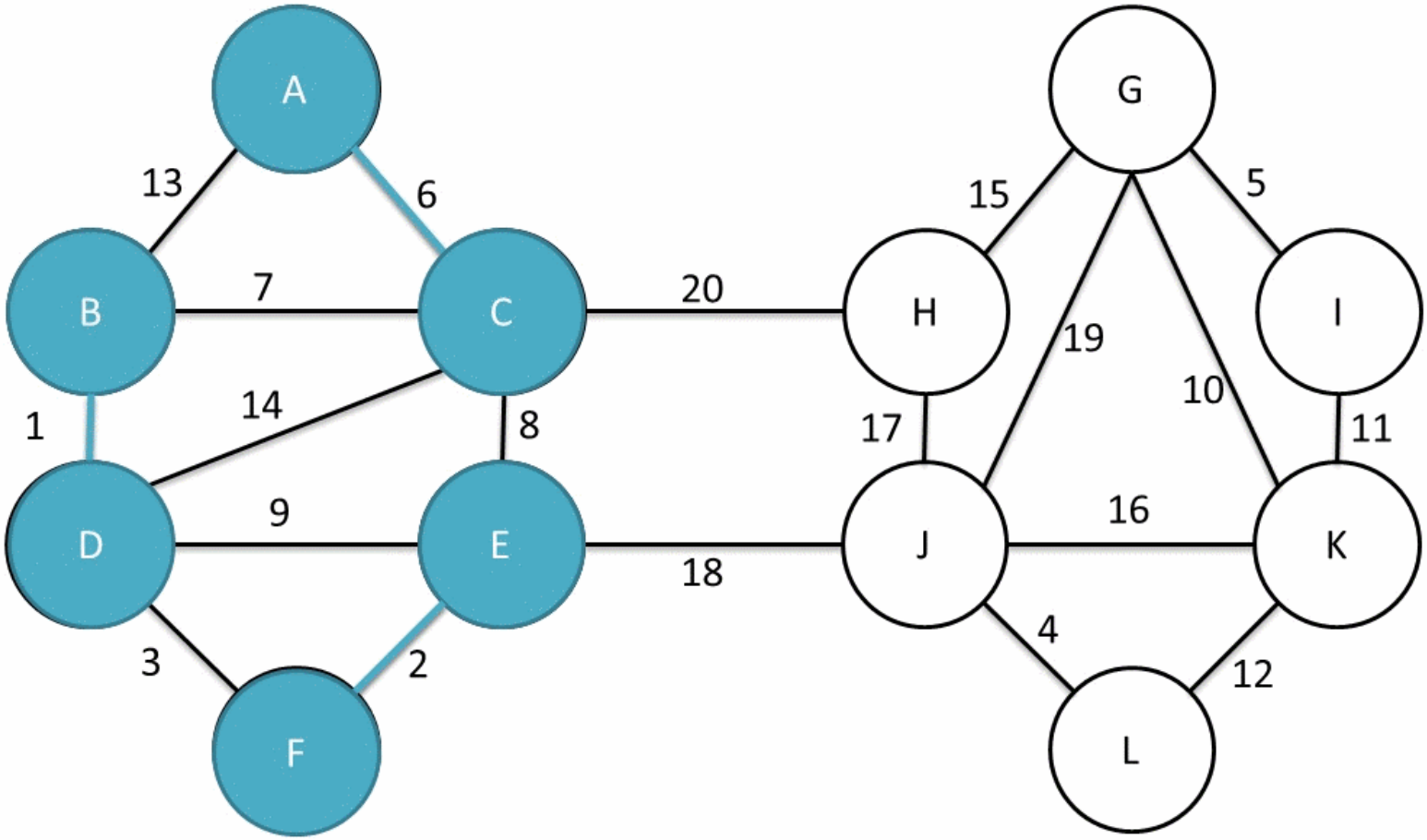
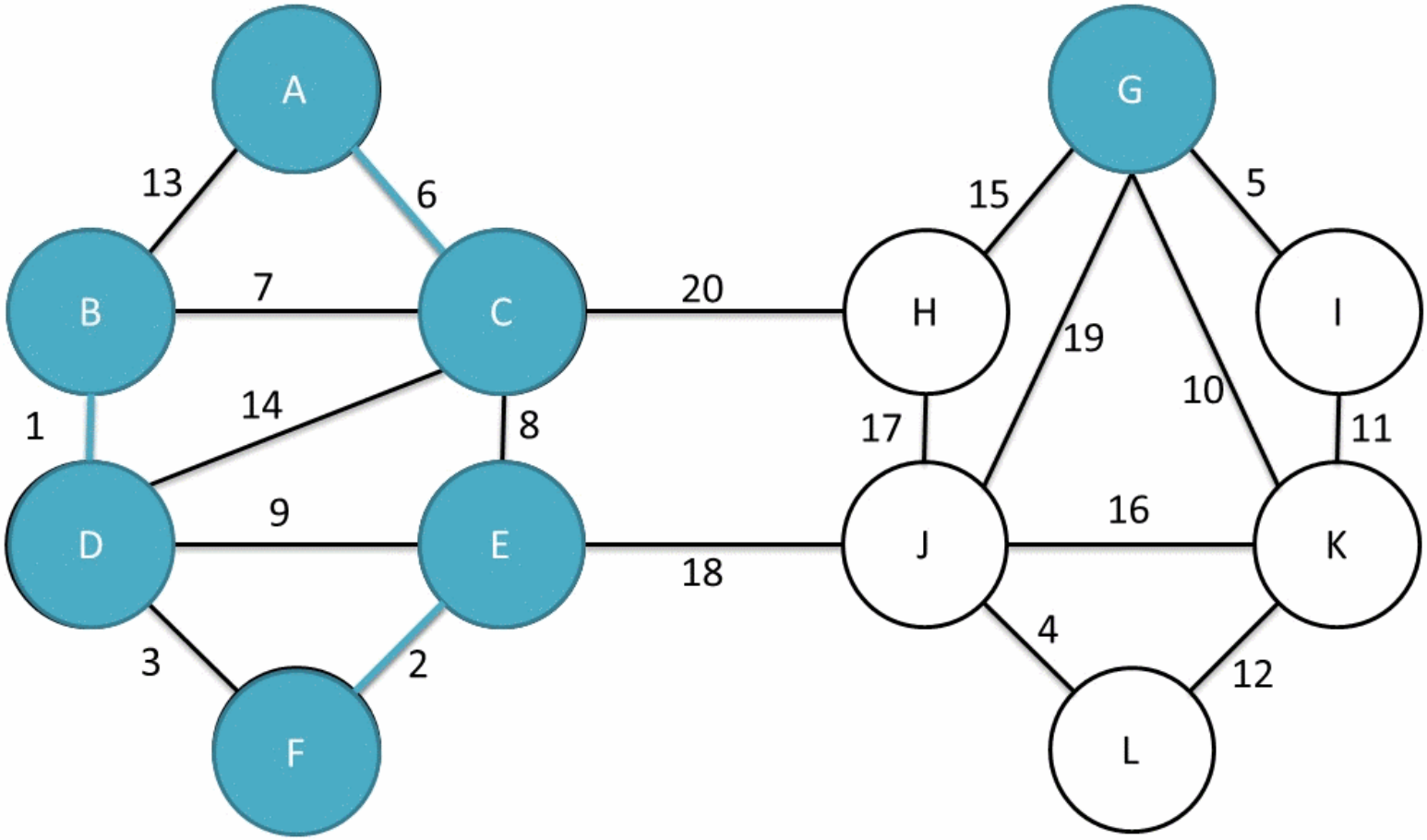


Boruvka algorithm, 1926



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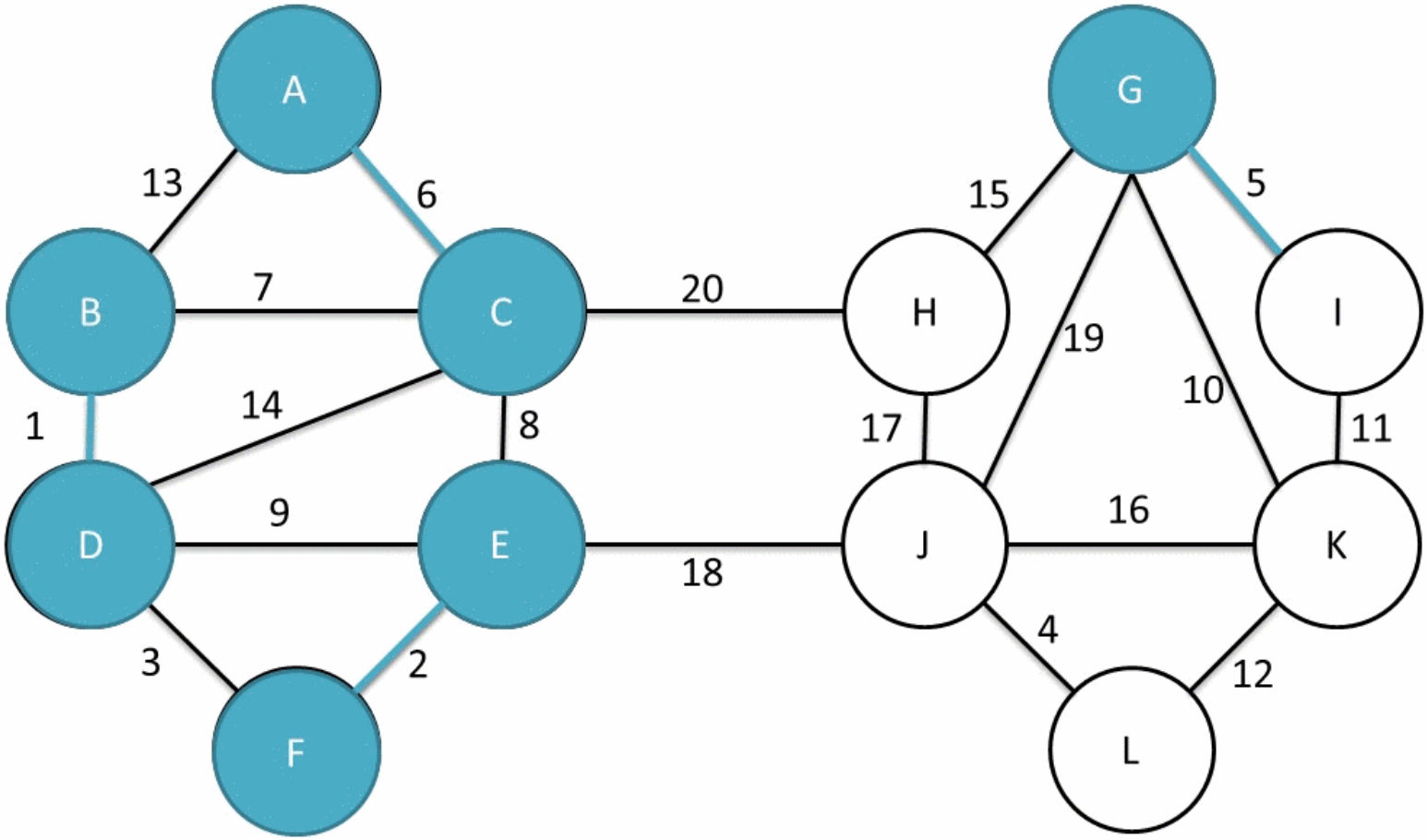
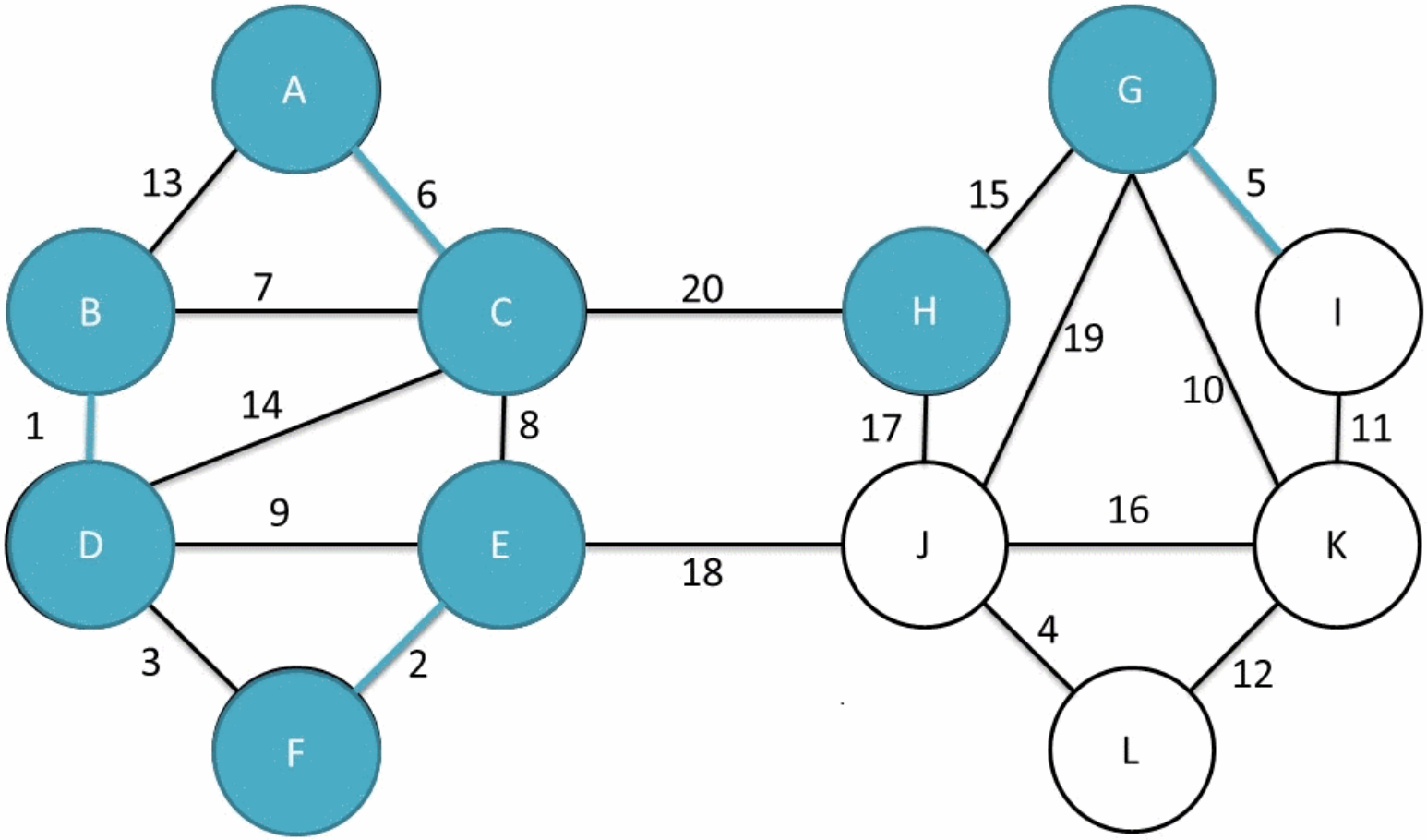
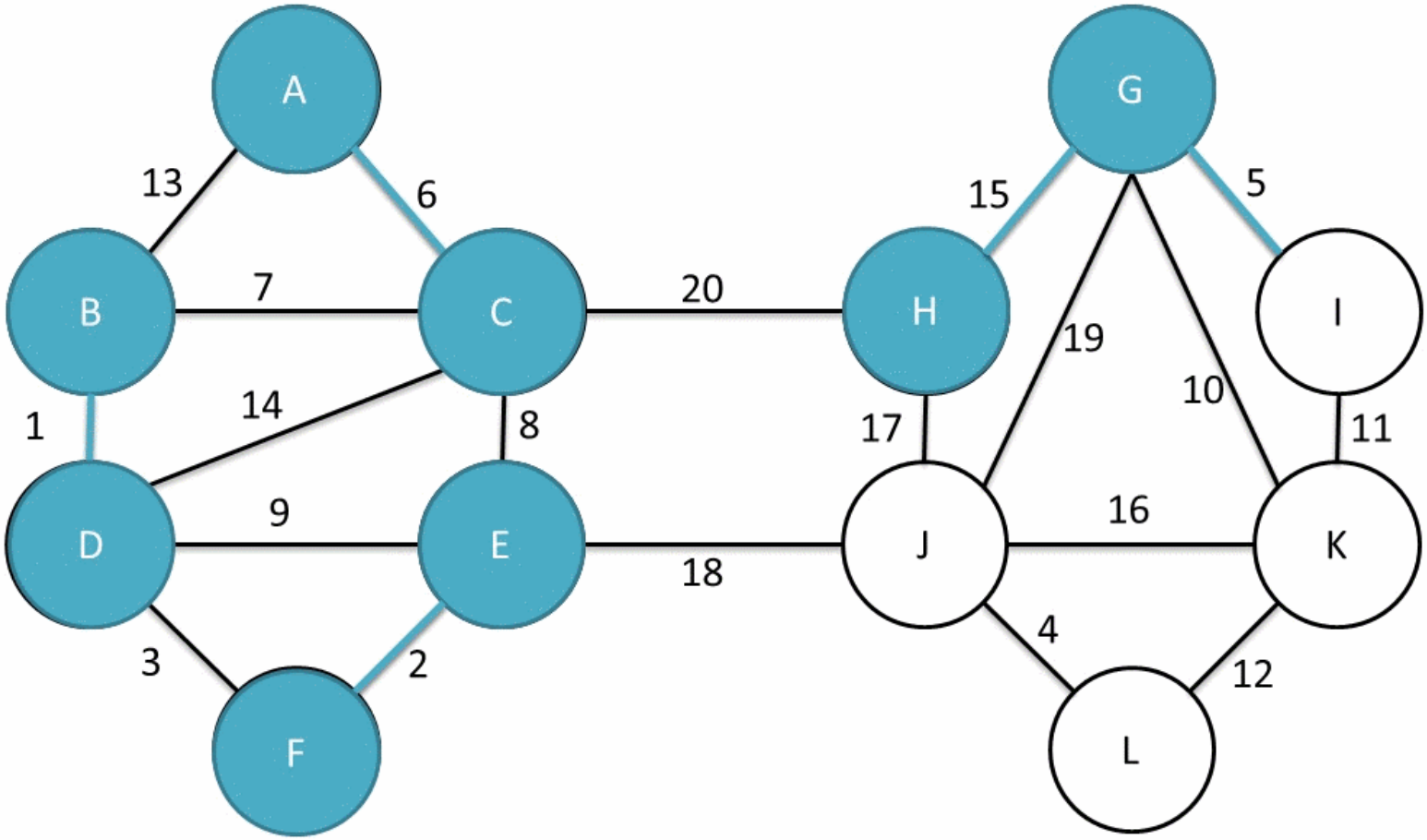


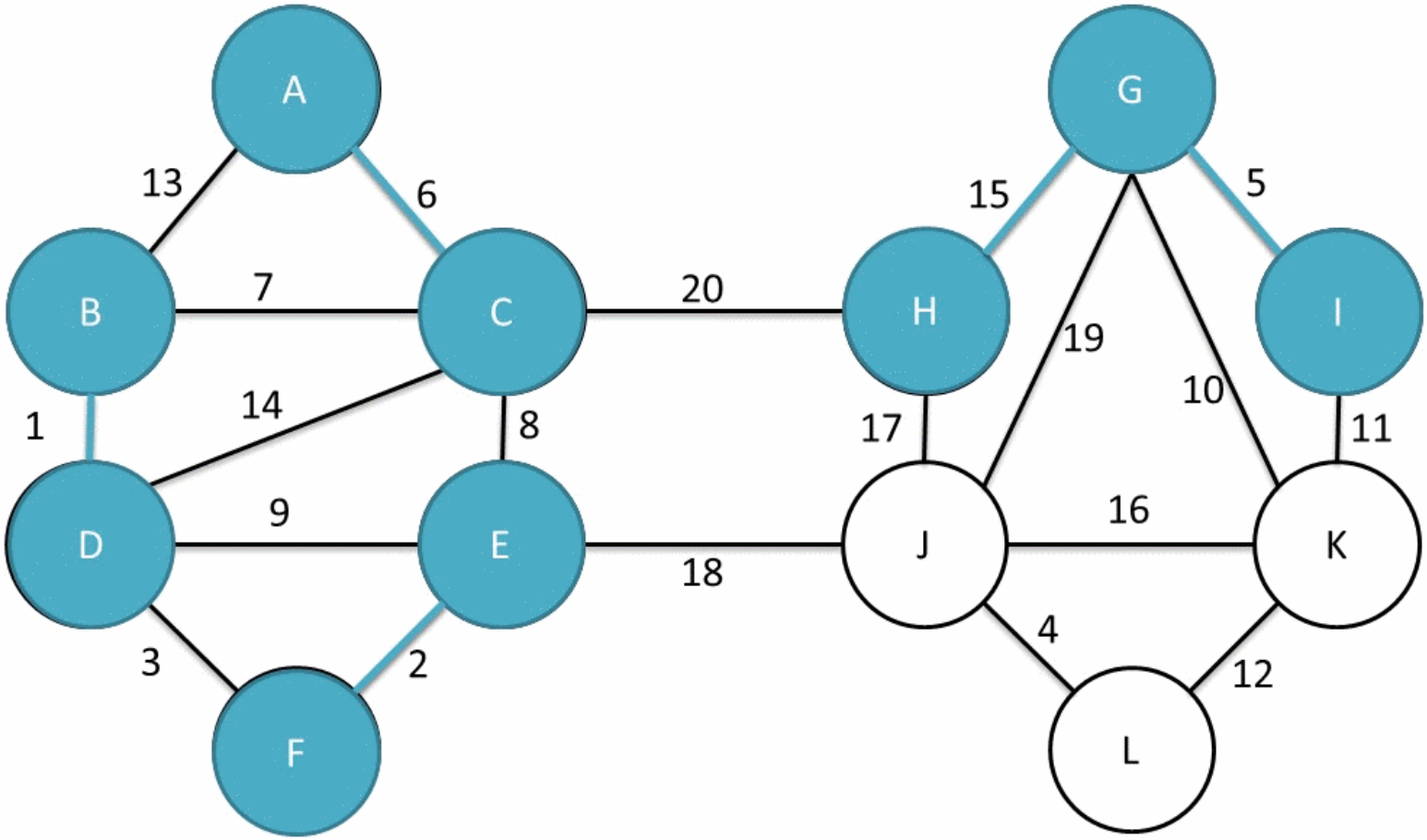
Boruvka algorithm, 1926



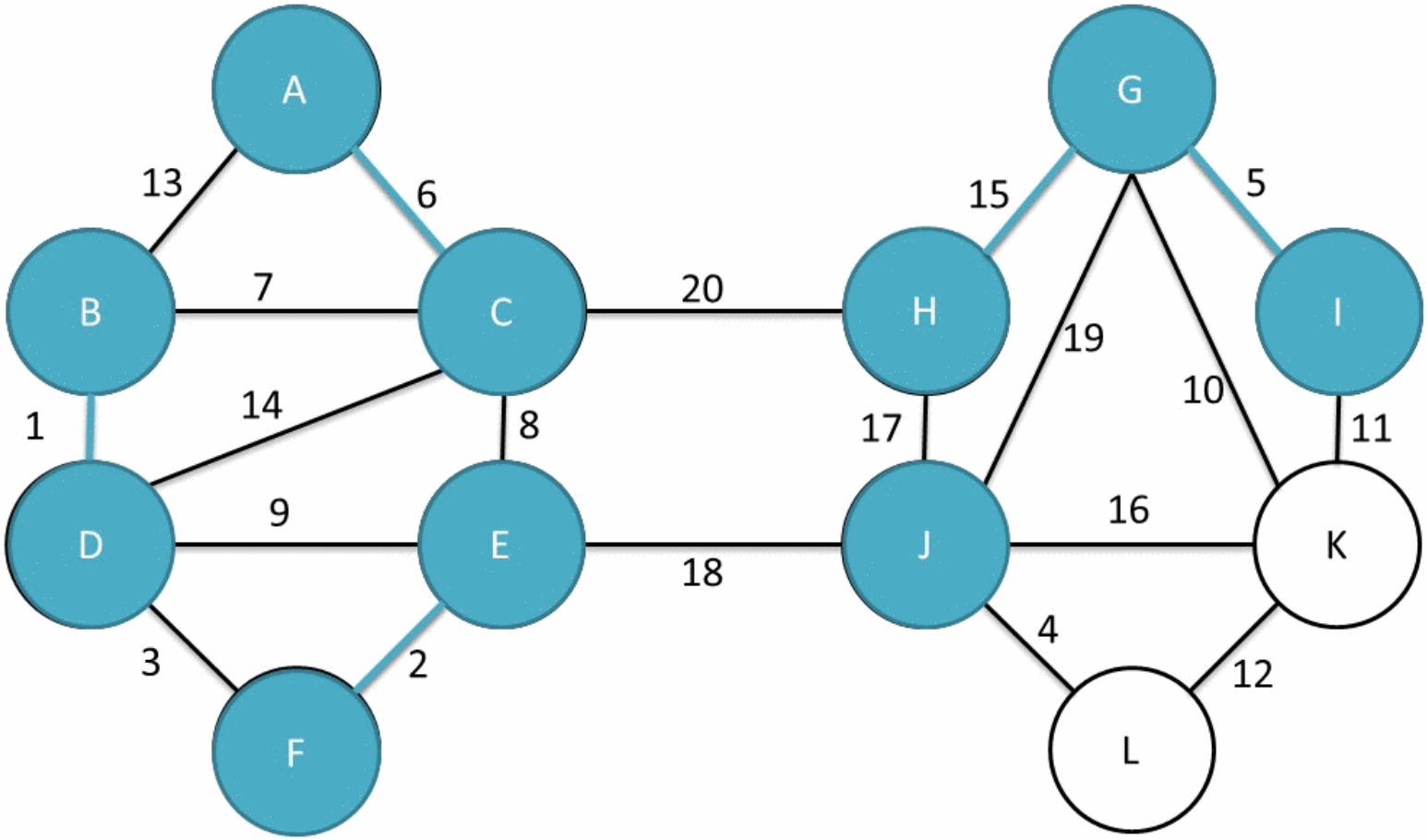
Boruvka algorithm, 1926



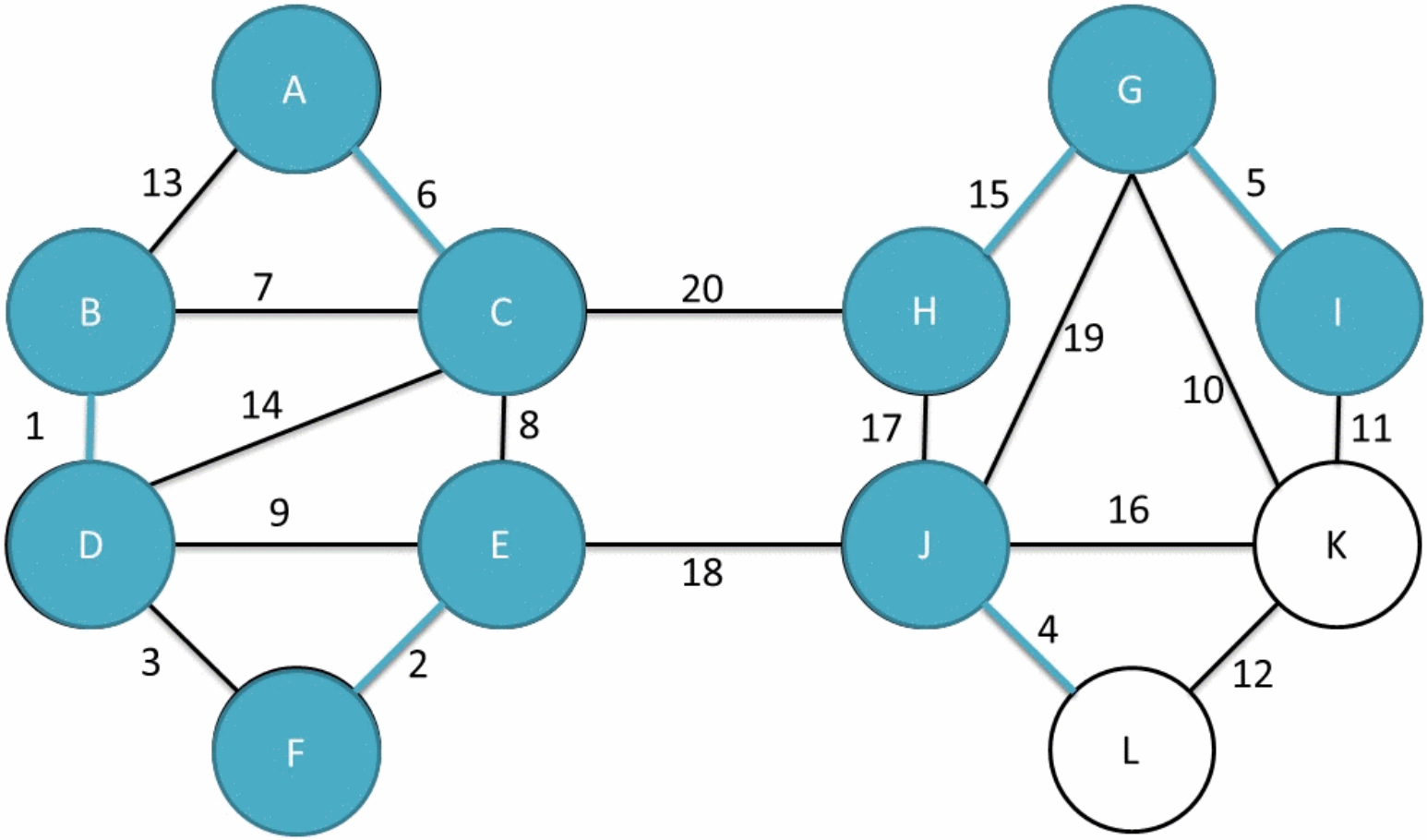
Boruvka algorithm, 1926



Boruvka algorithm, 1926



Boruvka algorithm, 1926



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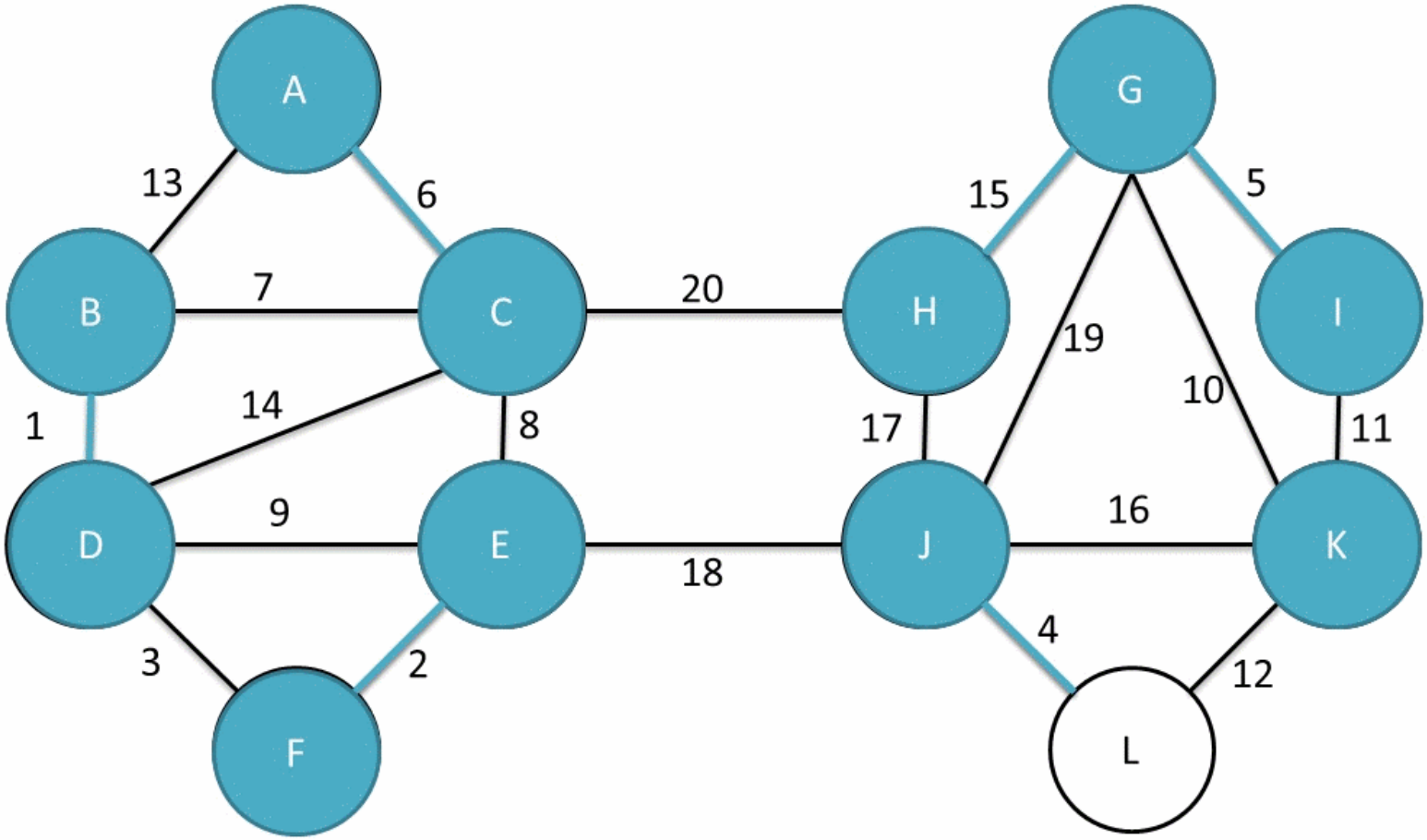

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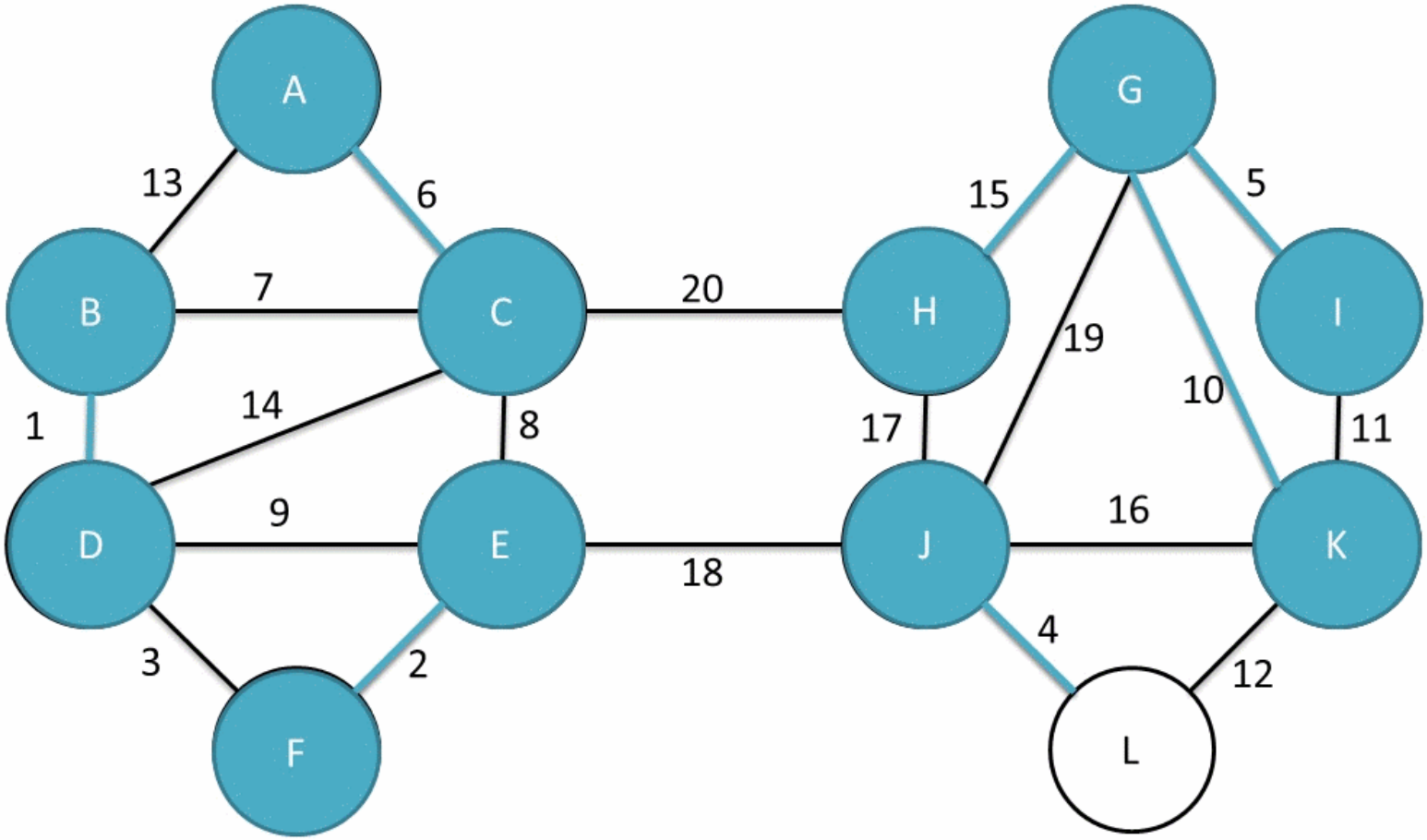
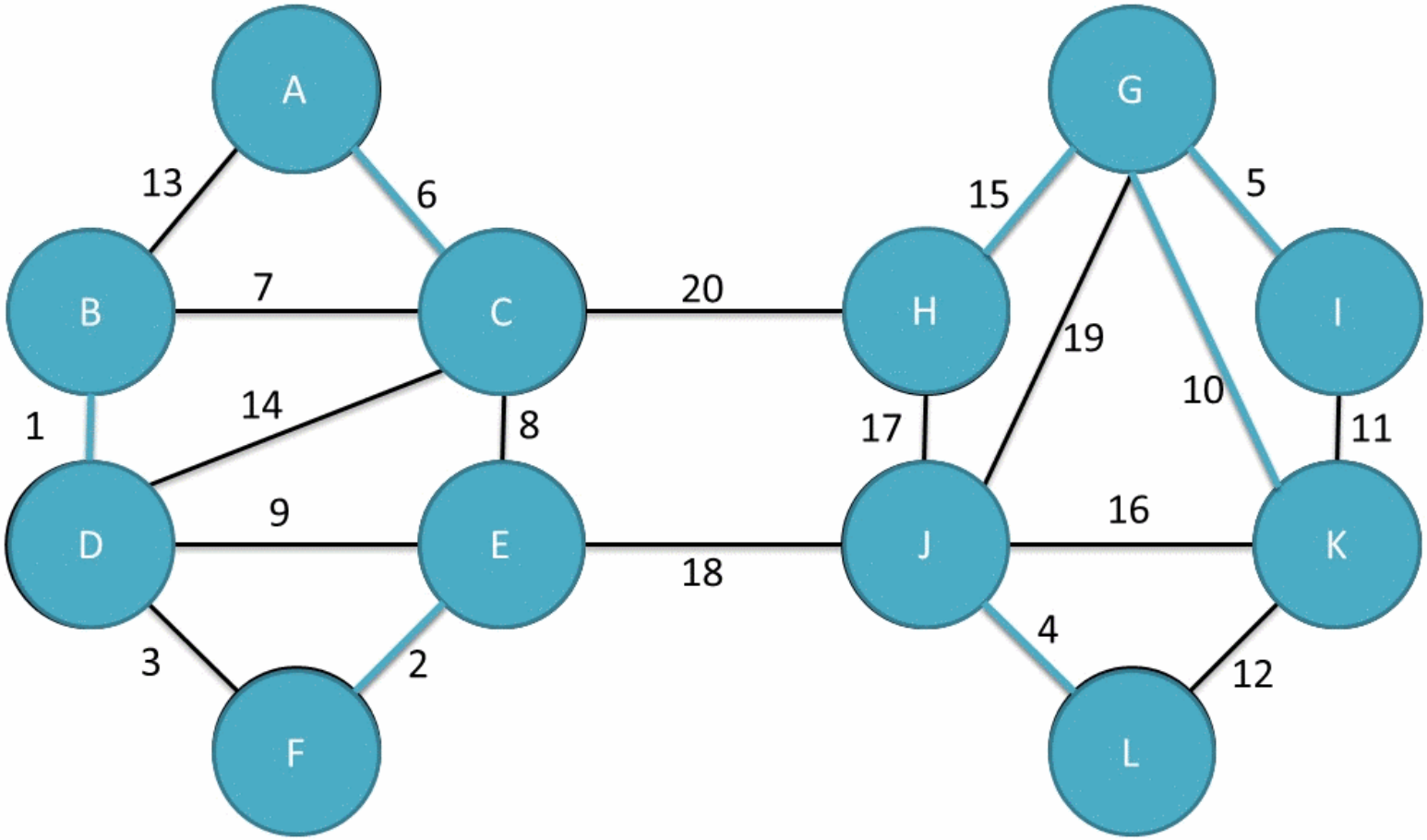
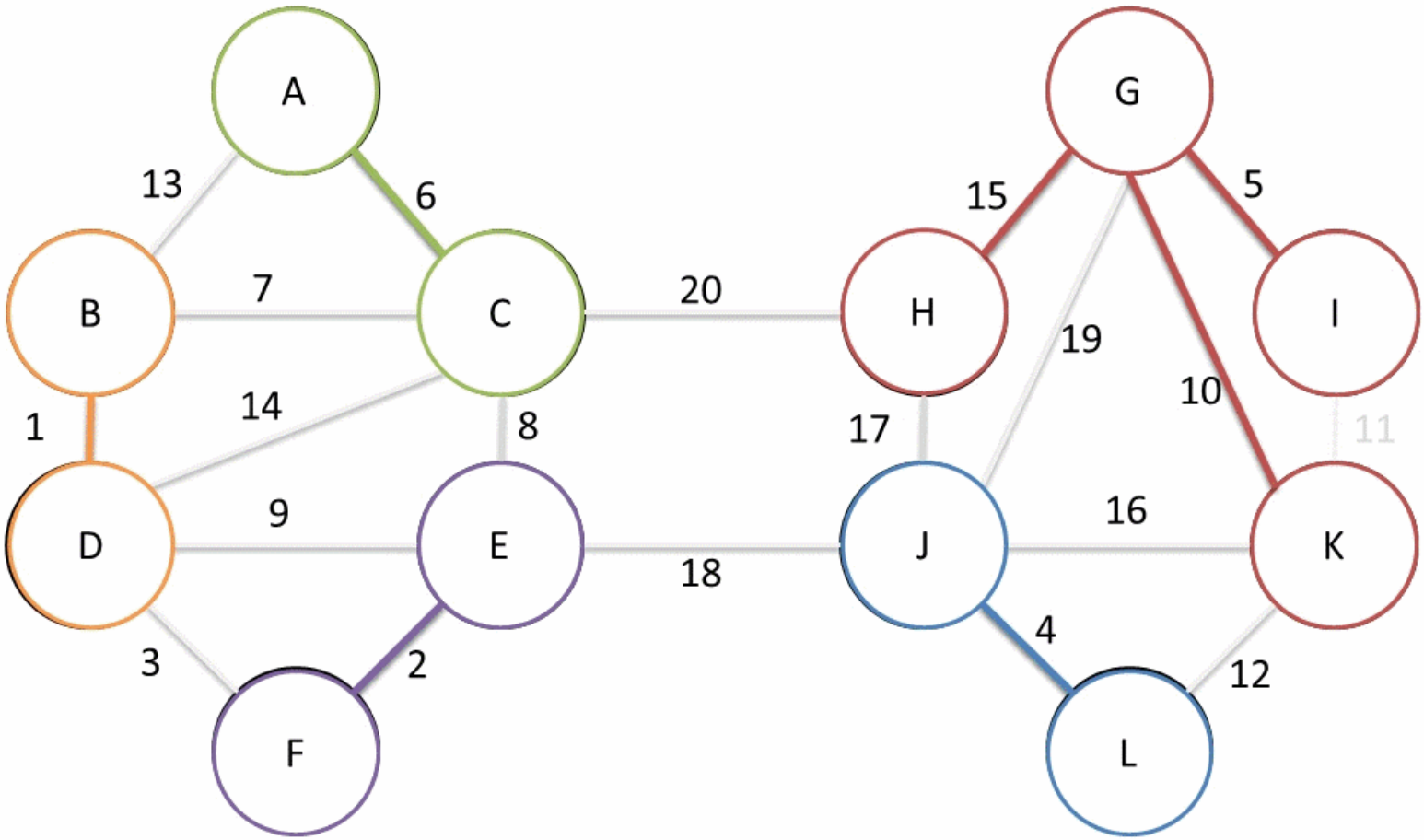


Boruvka algorithm, 1926



Boruvka algorithm, 1926



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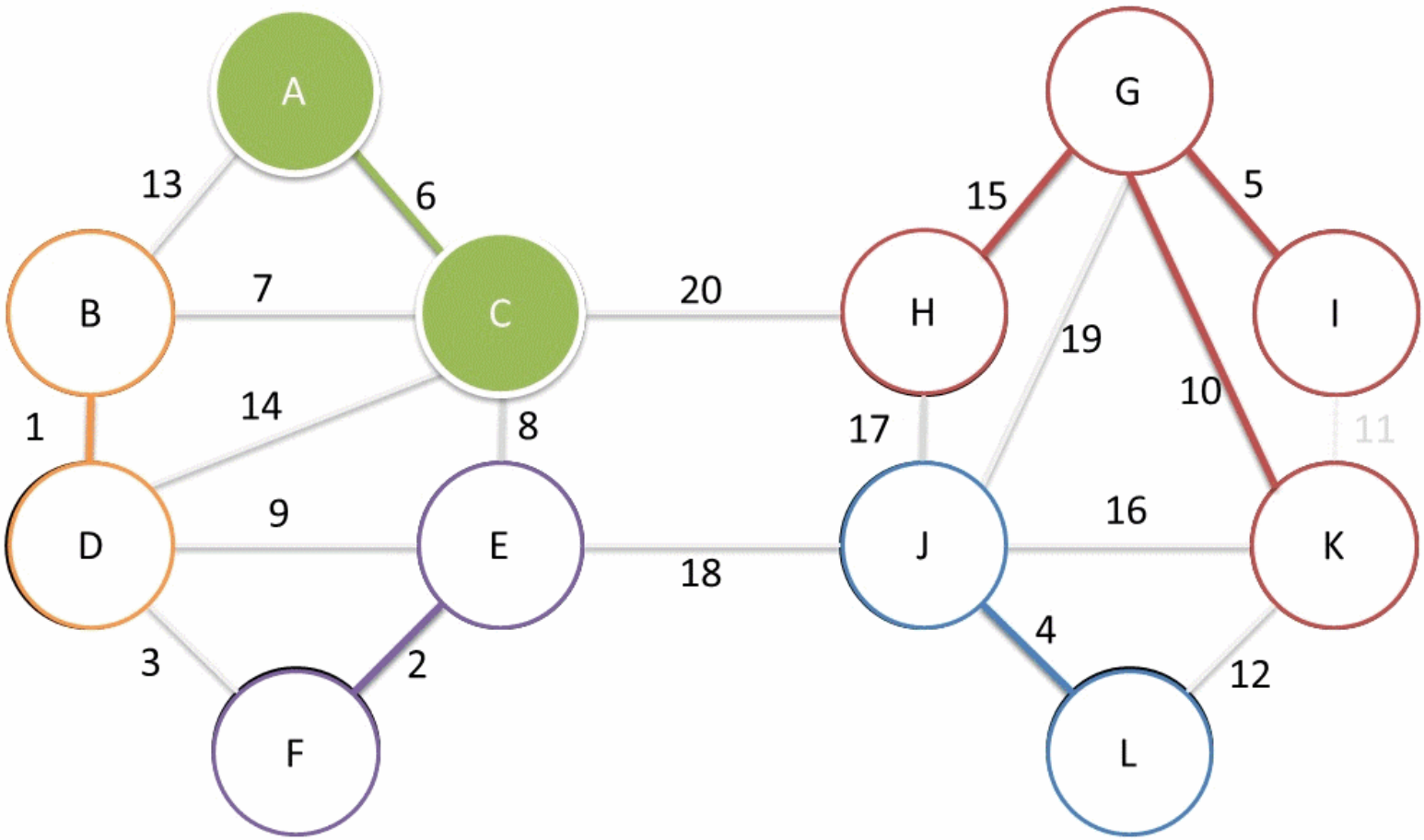

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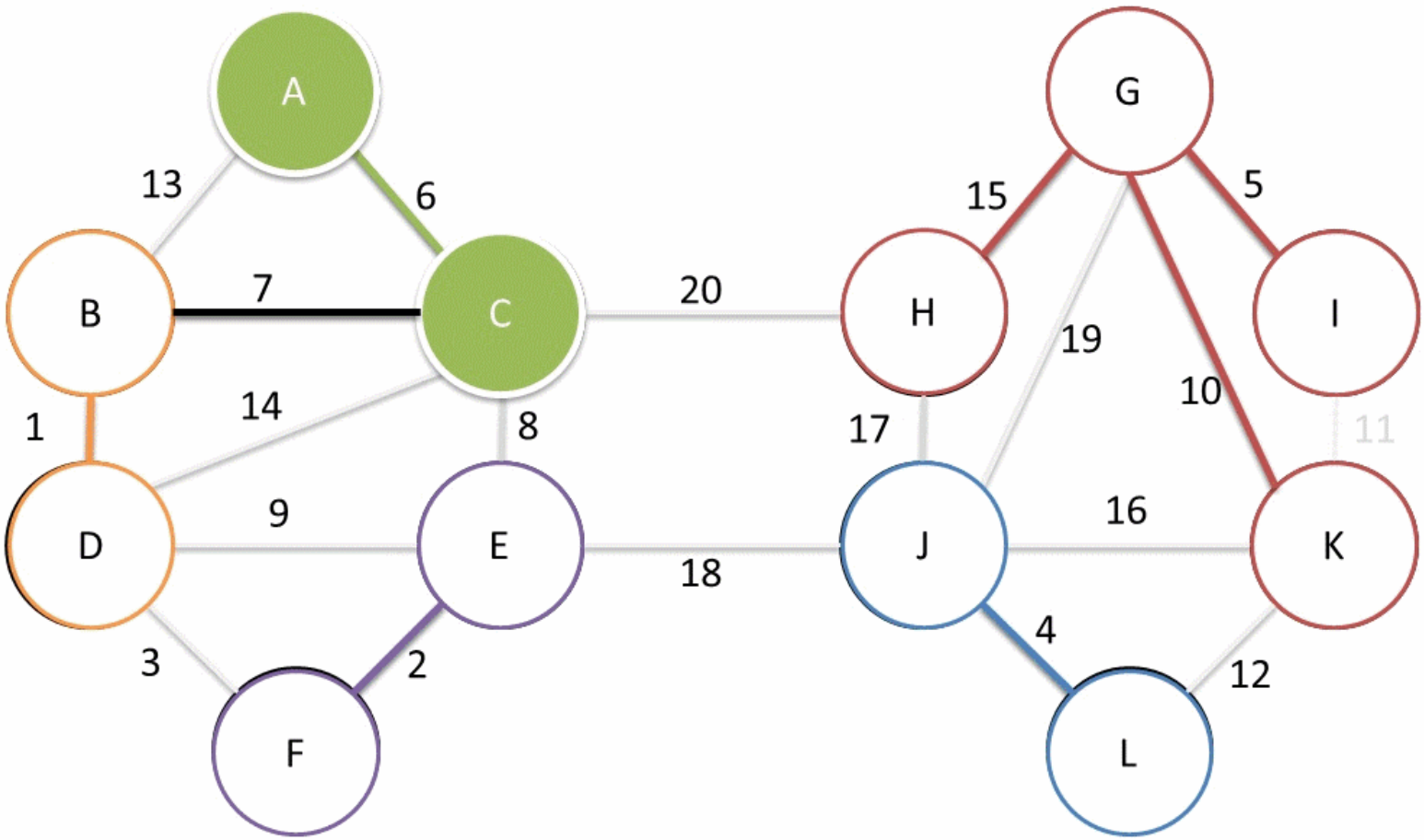

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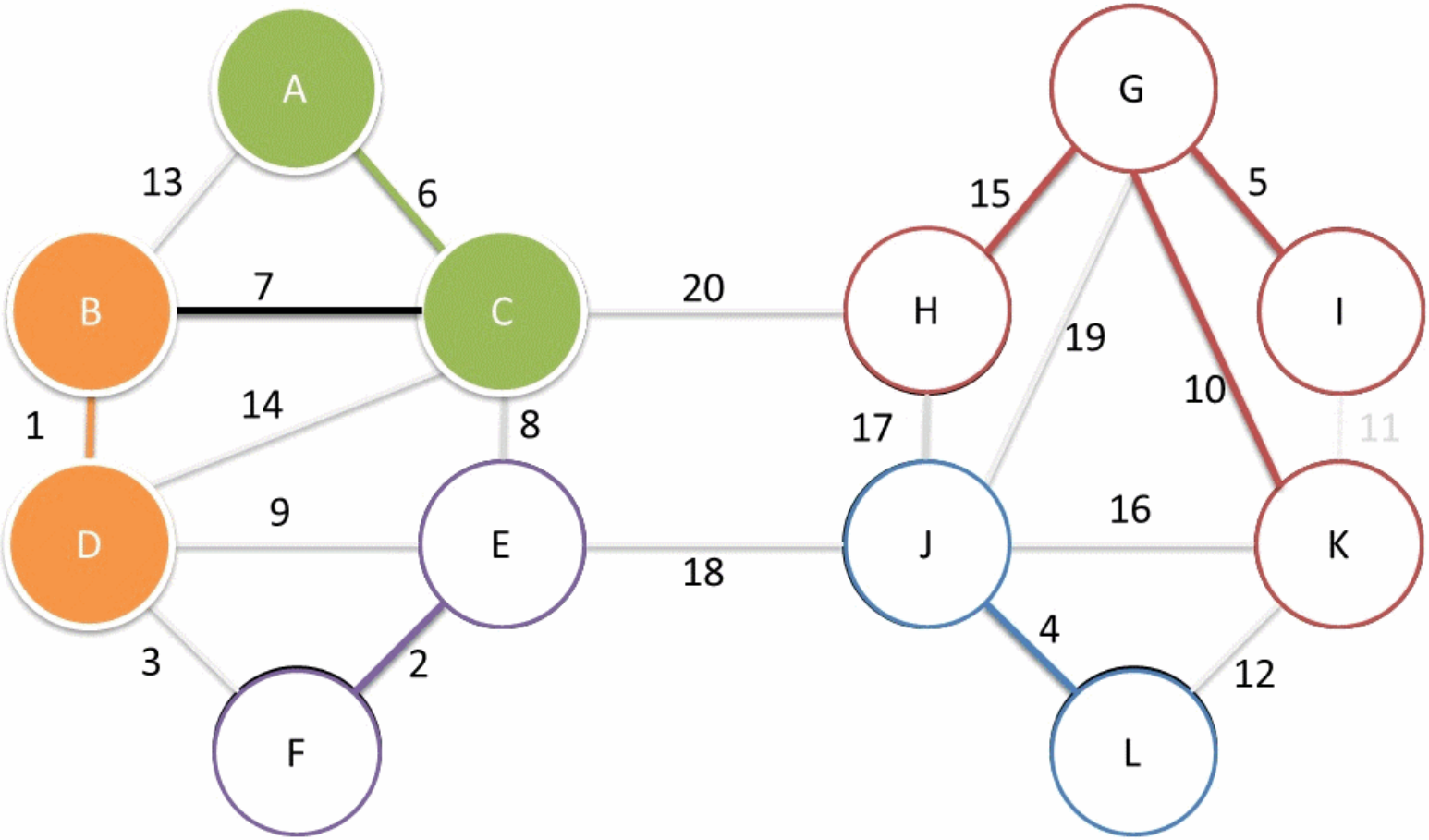

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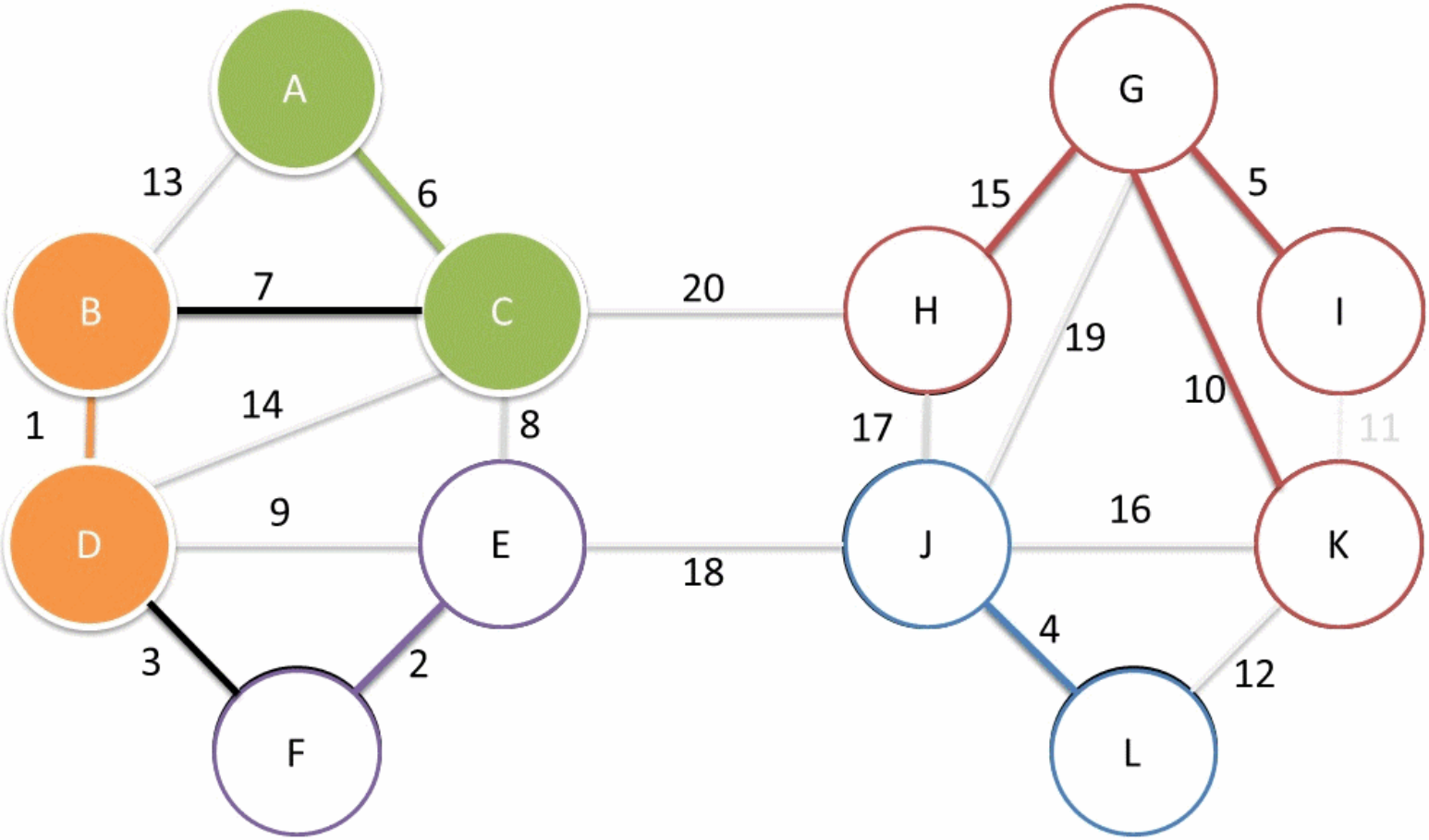

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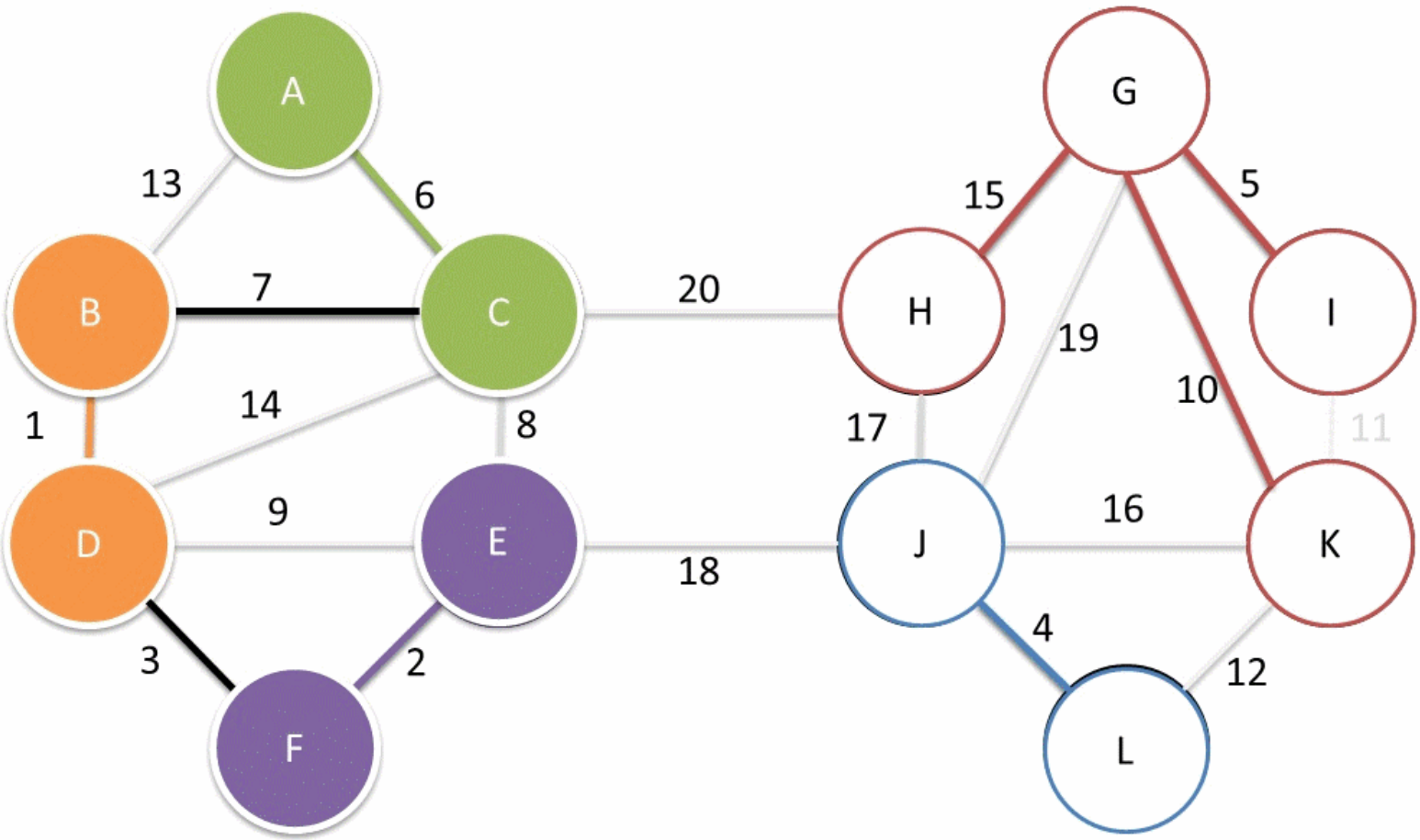

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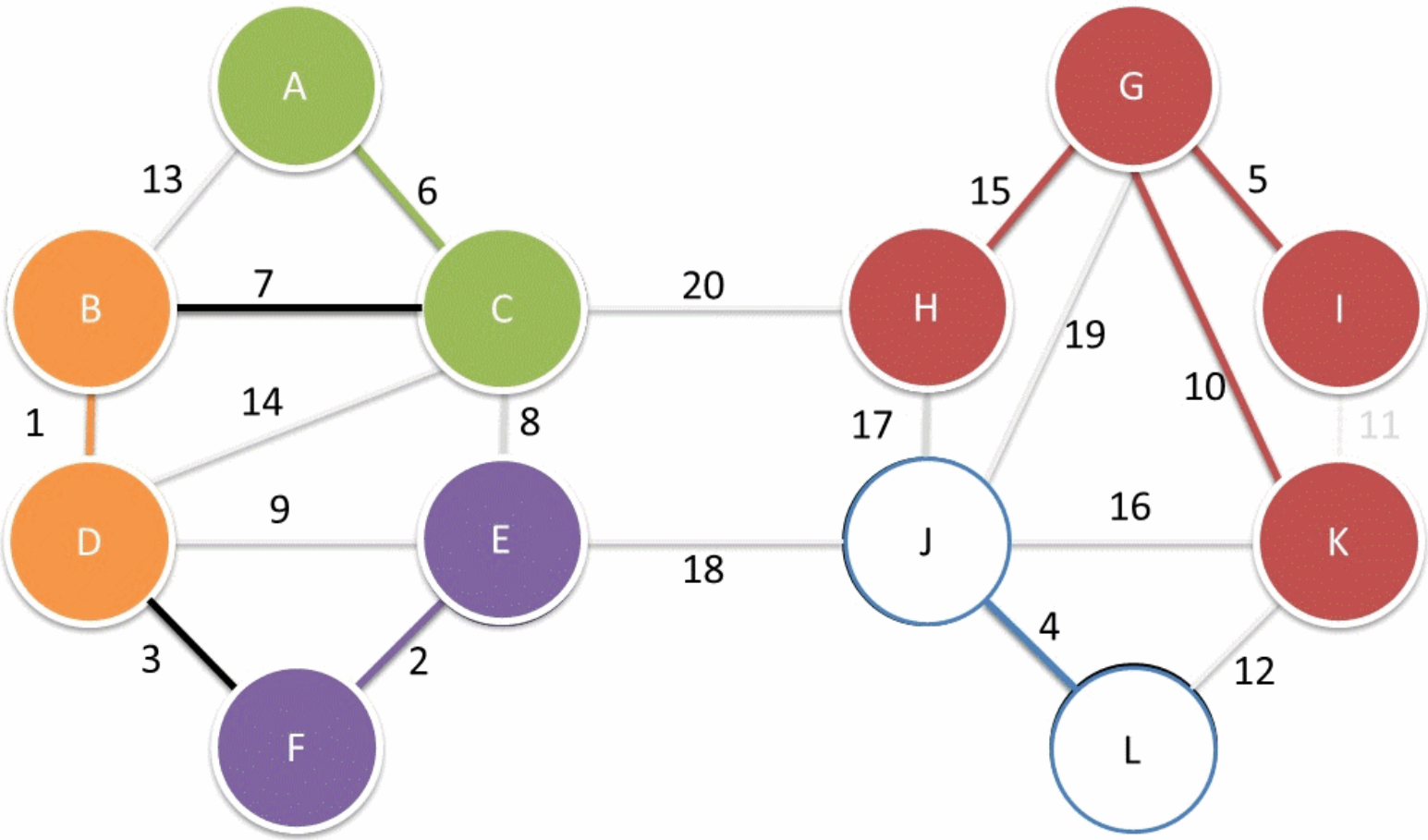

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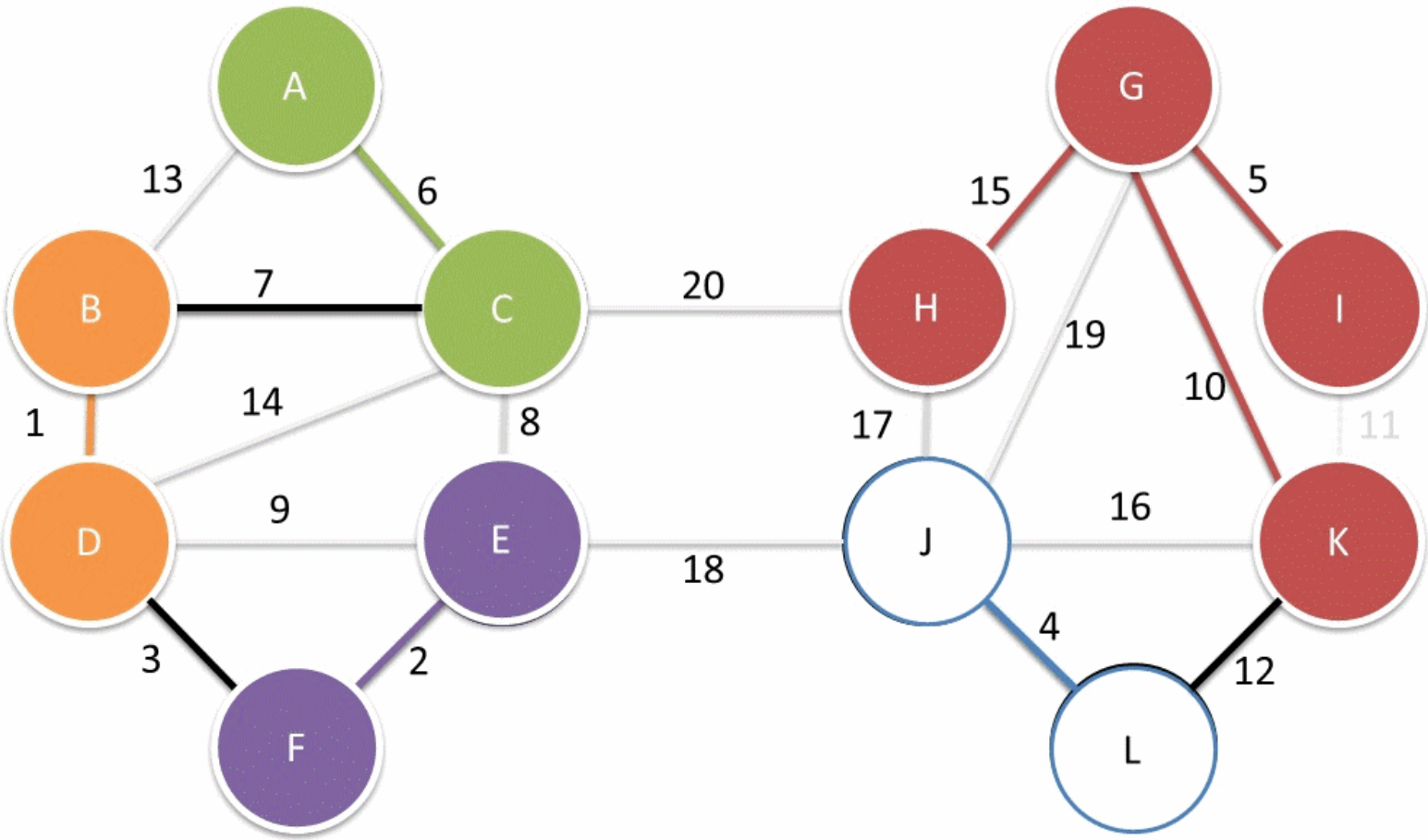

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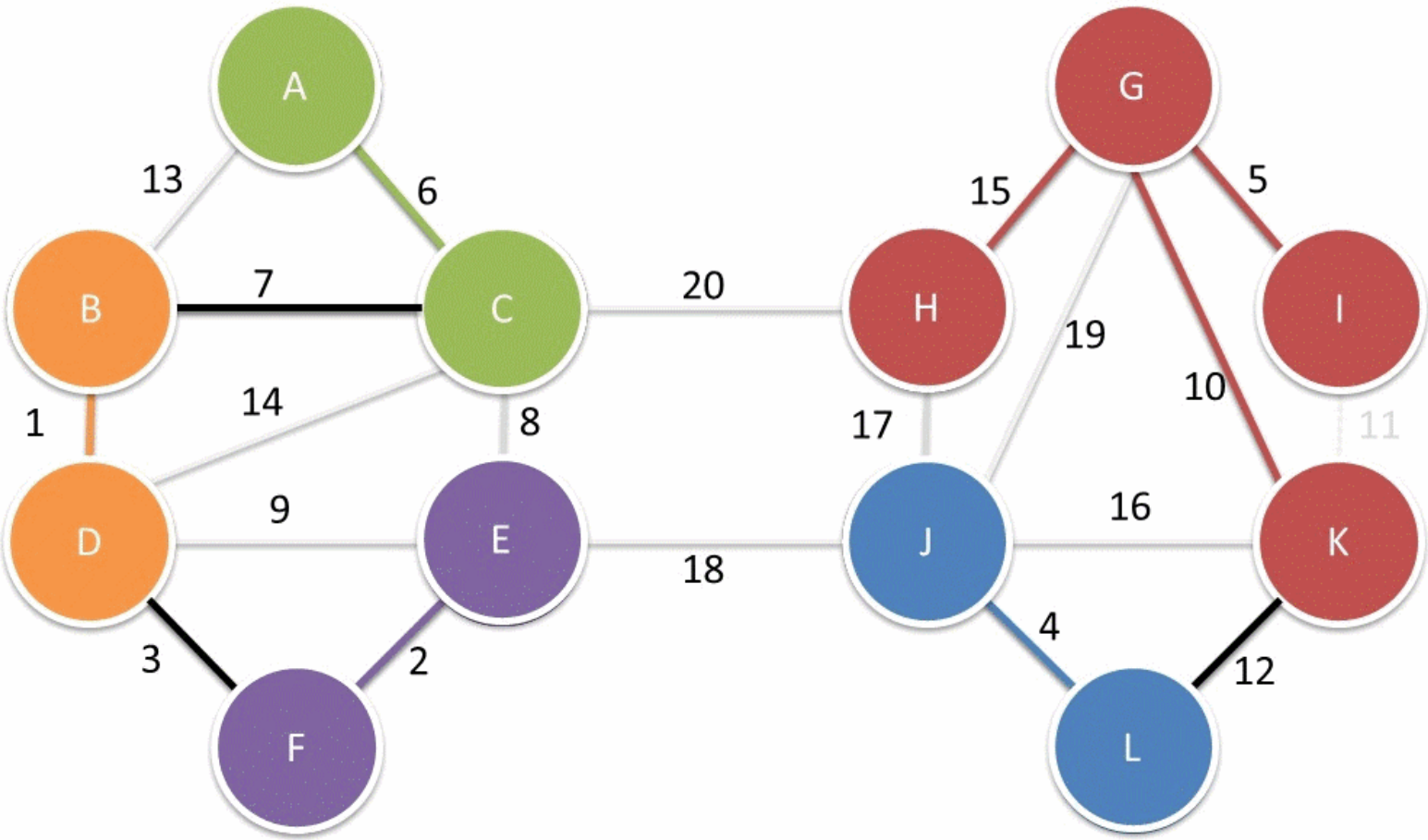

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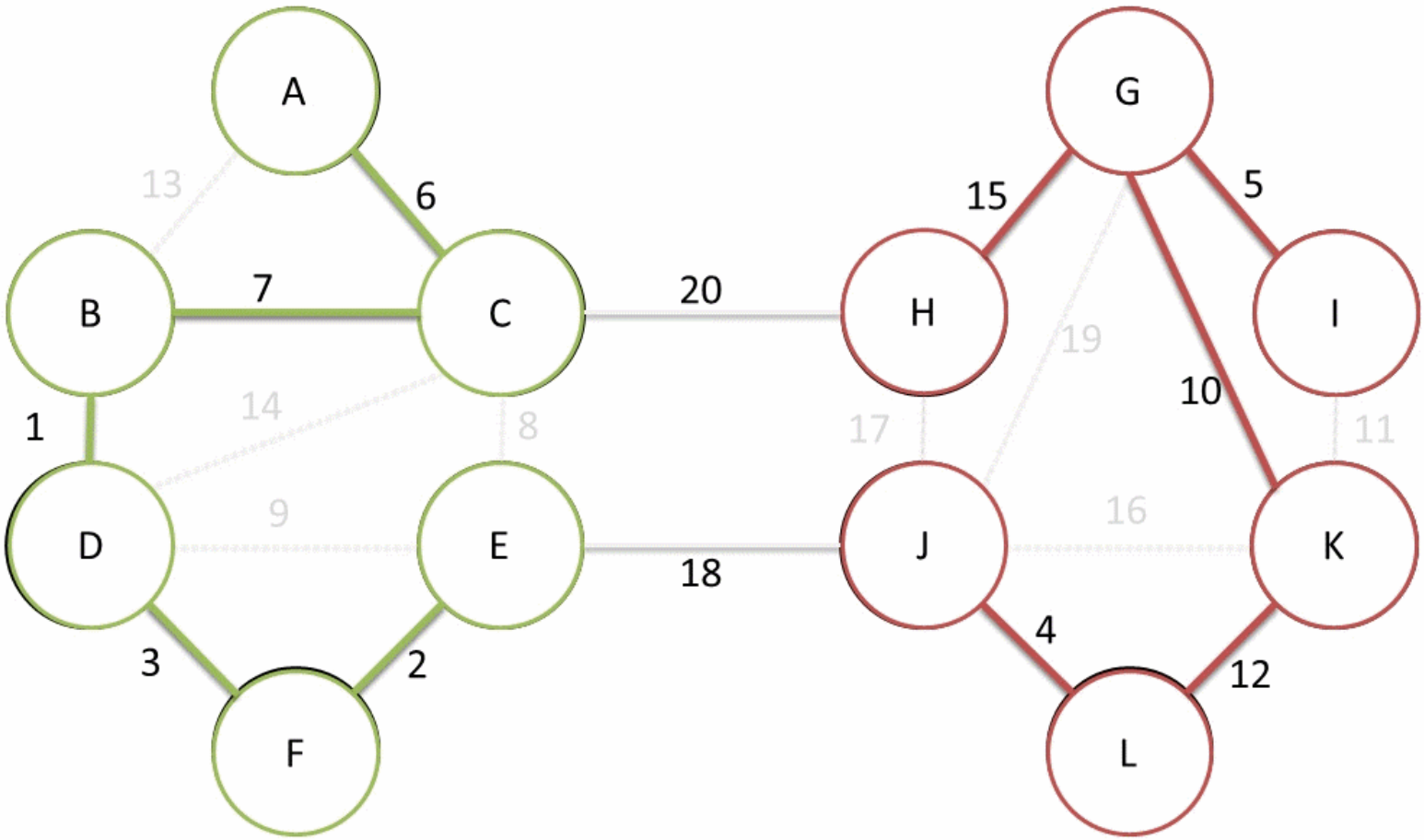

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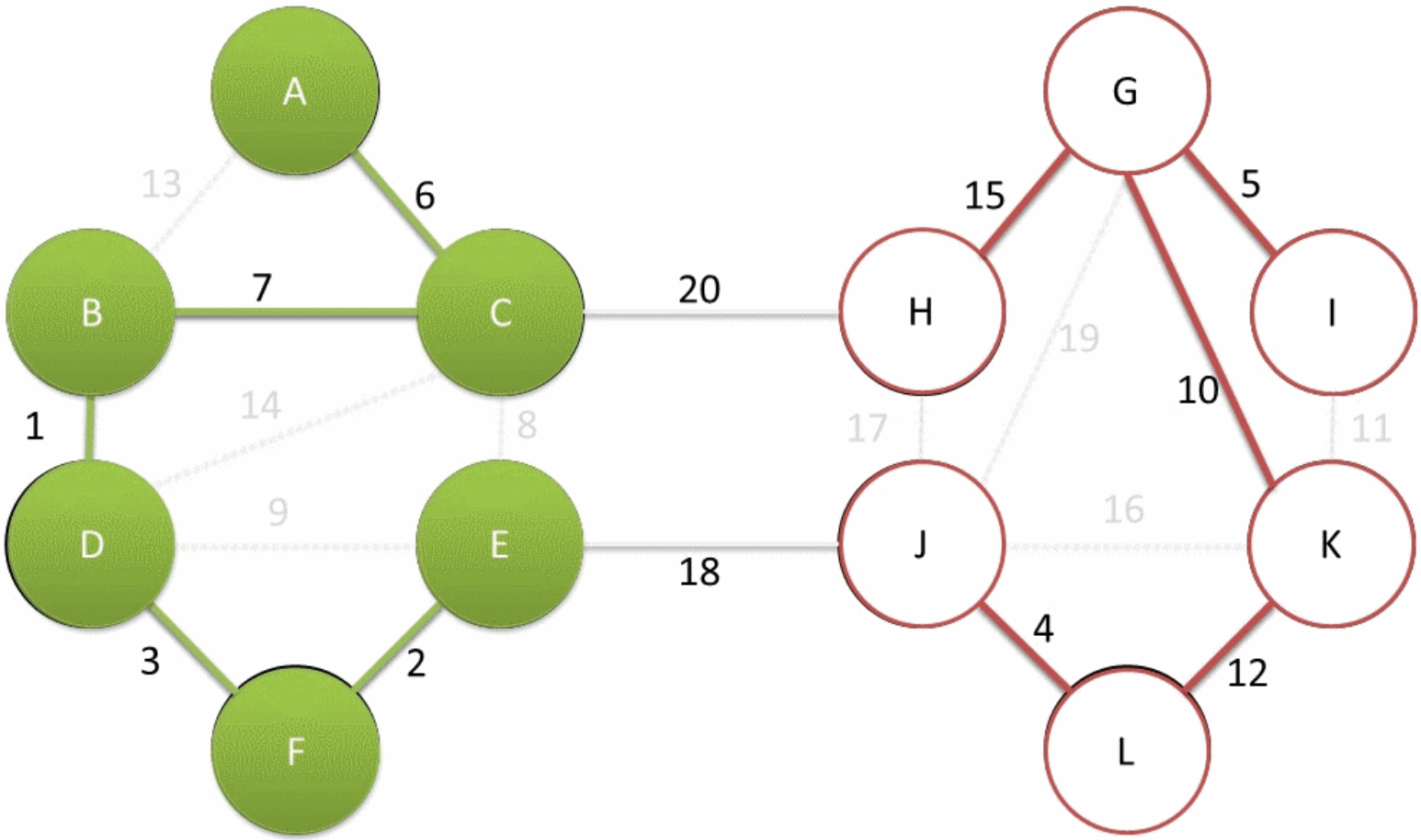

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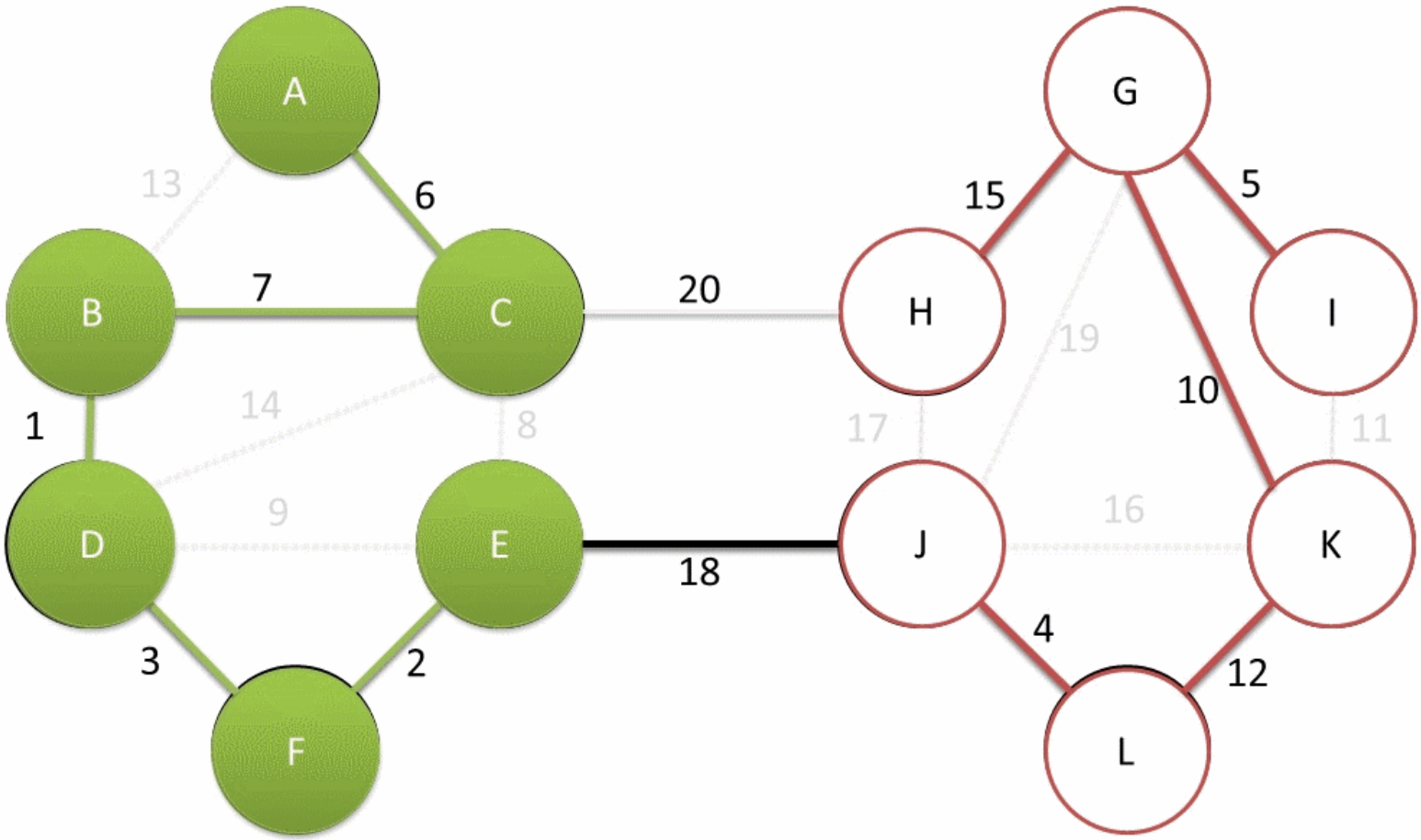

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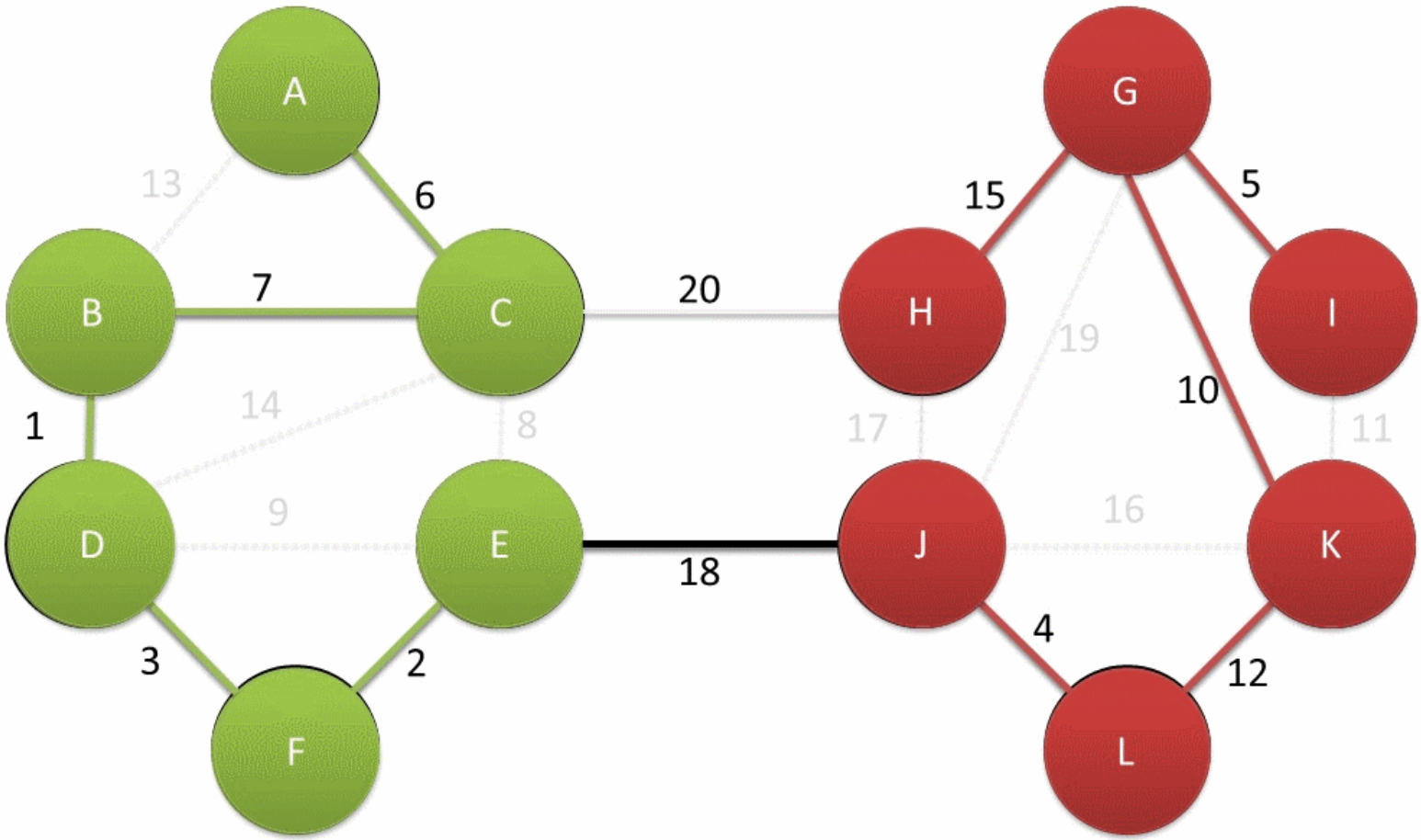

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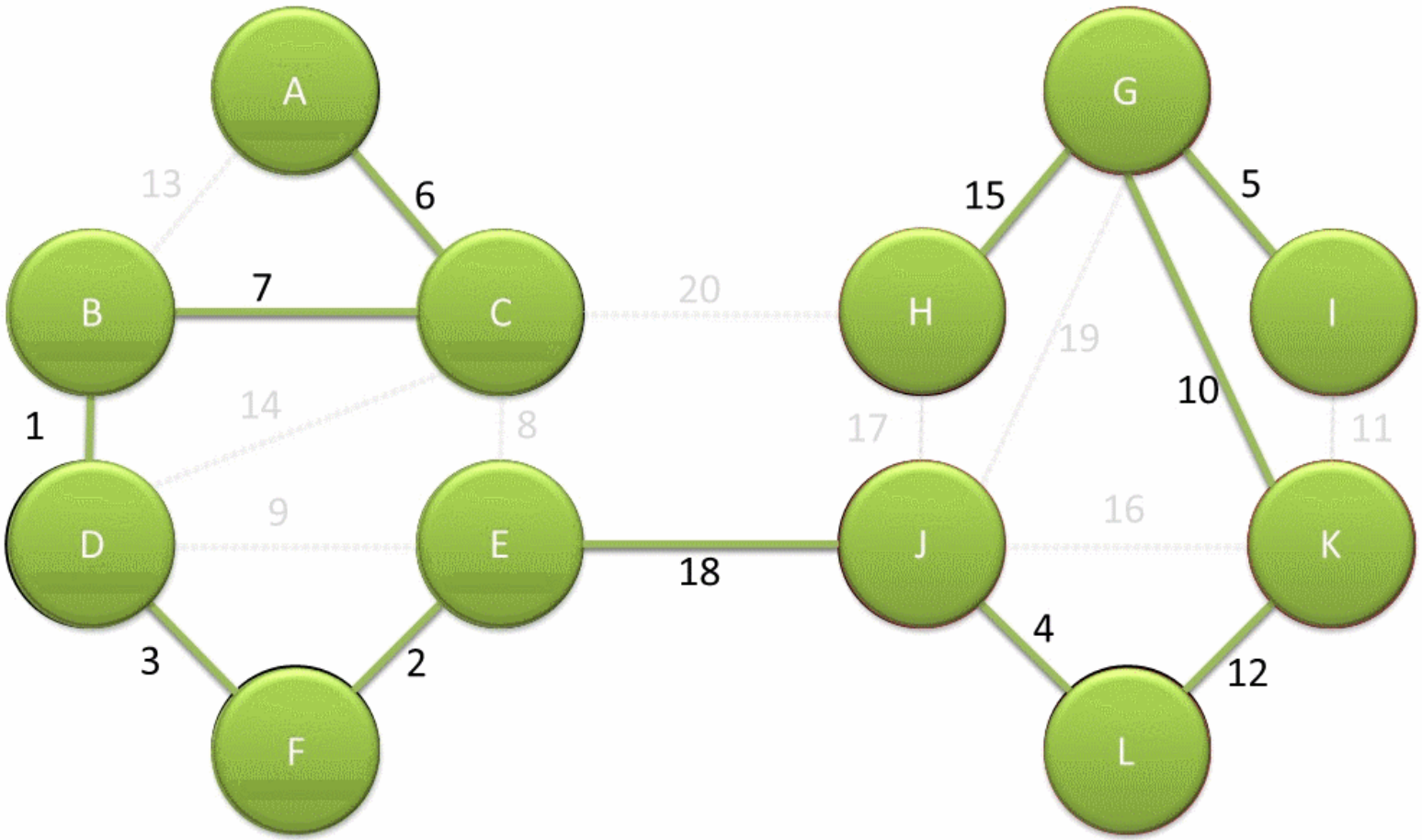

Kruskal's algorithm

- Sort the edges by weight
- Add all edges in this order so that they do not form a cycle

5	5
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7	7
8	8
9	9
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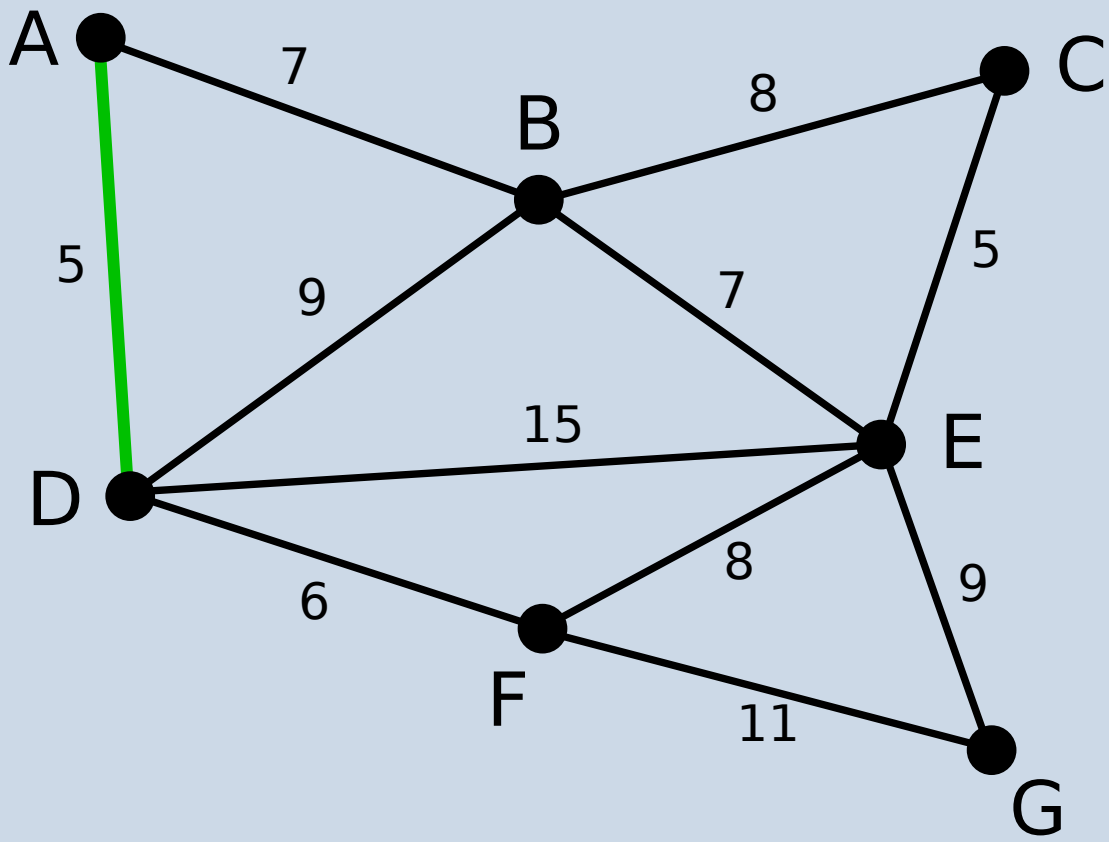

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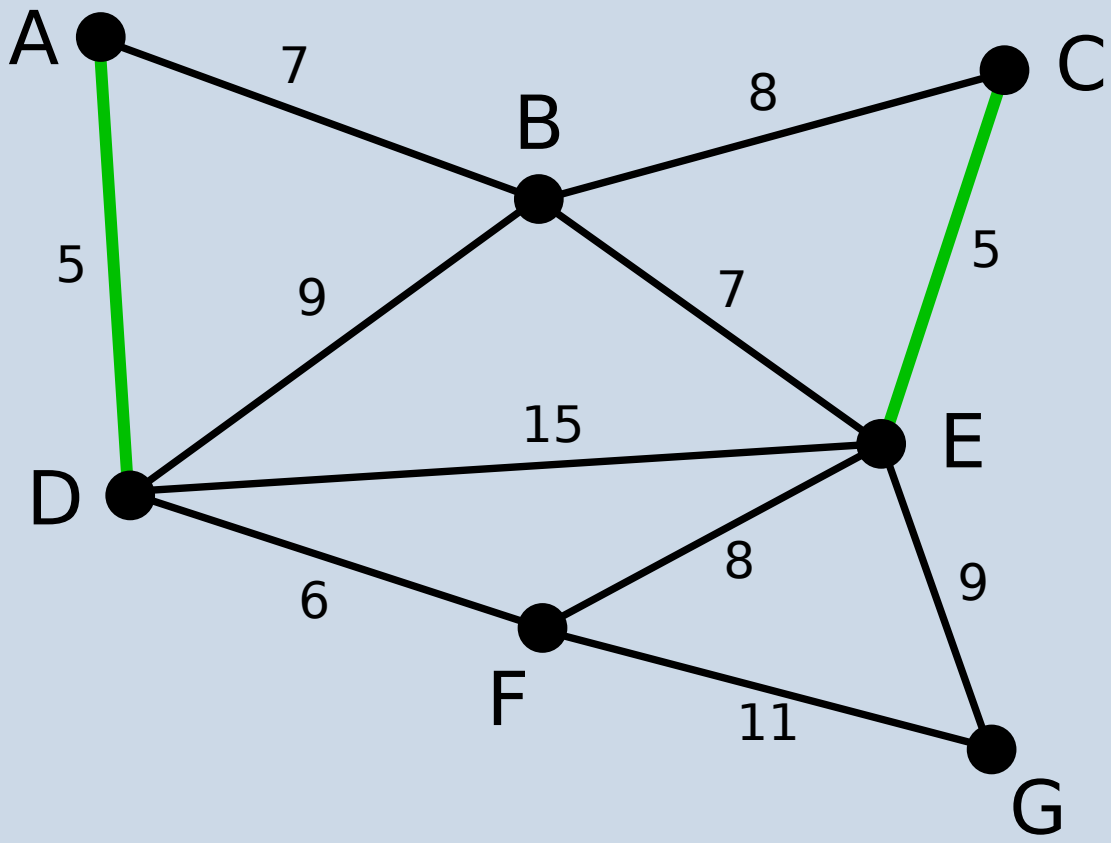

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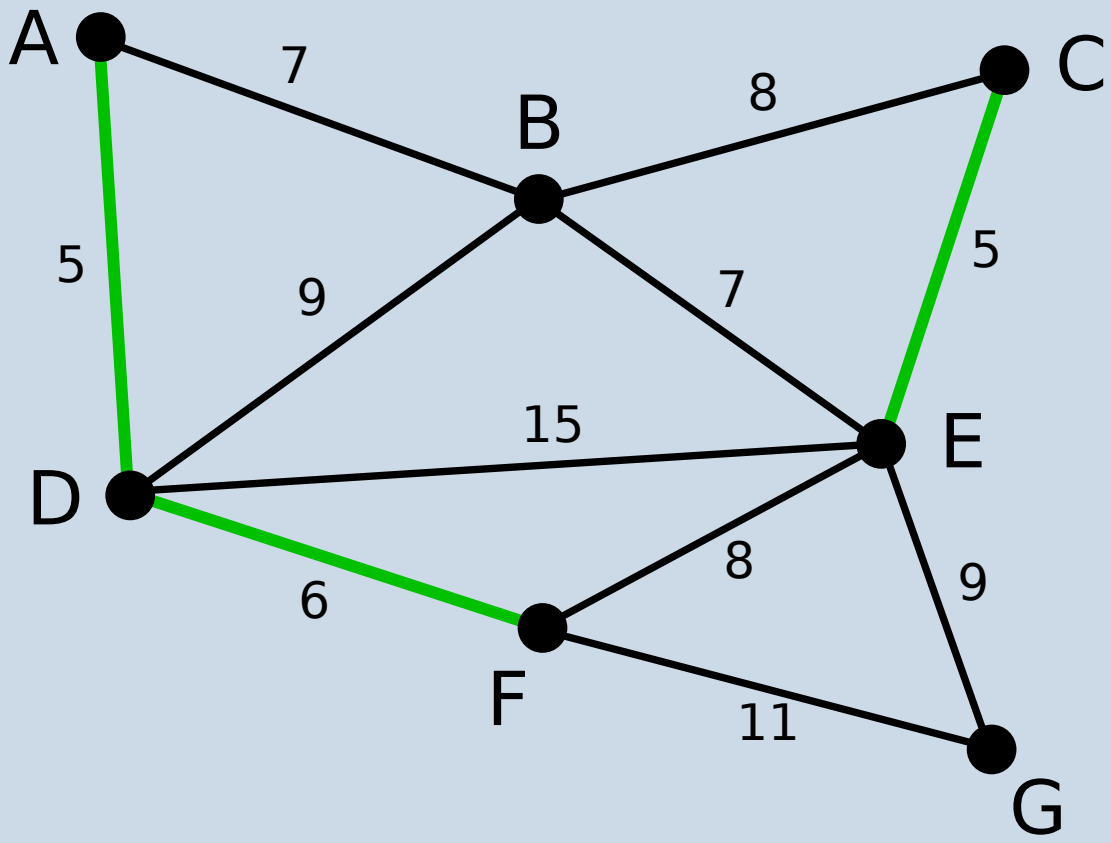

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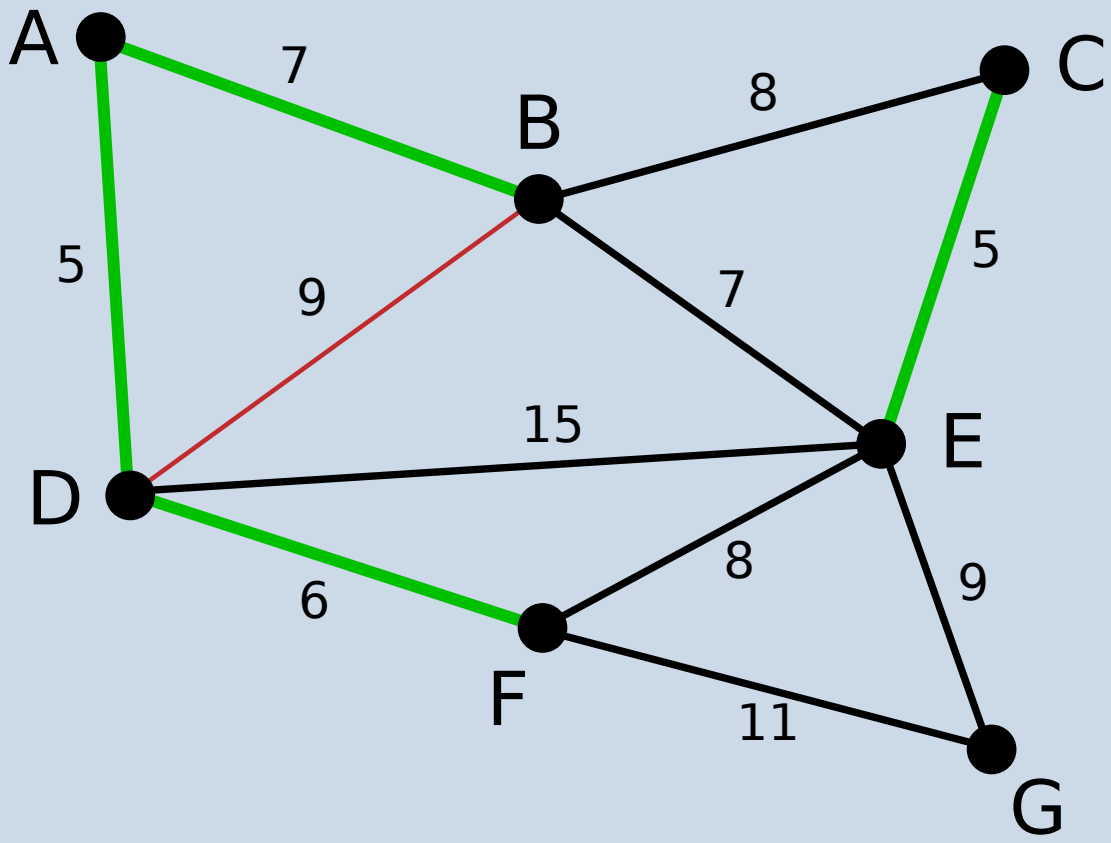

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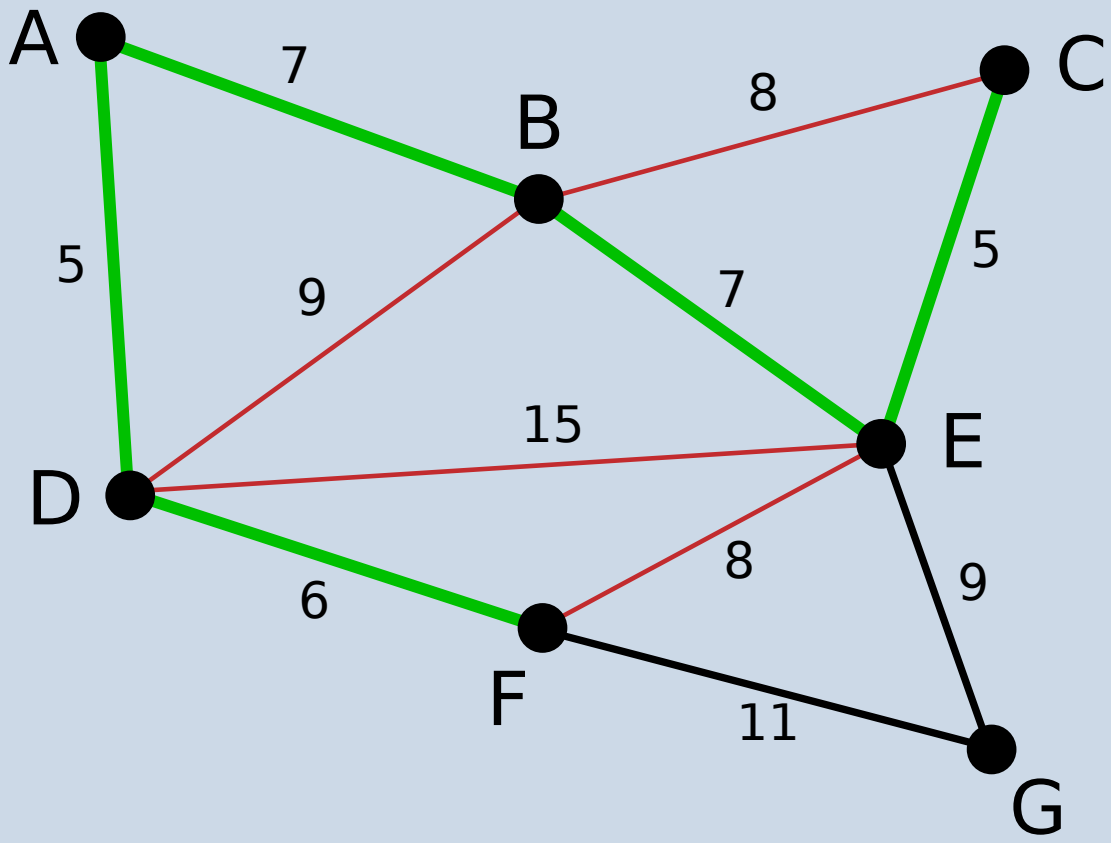

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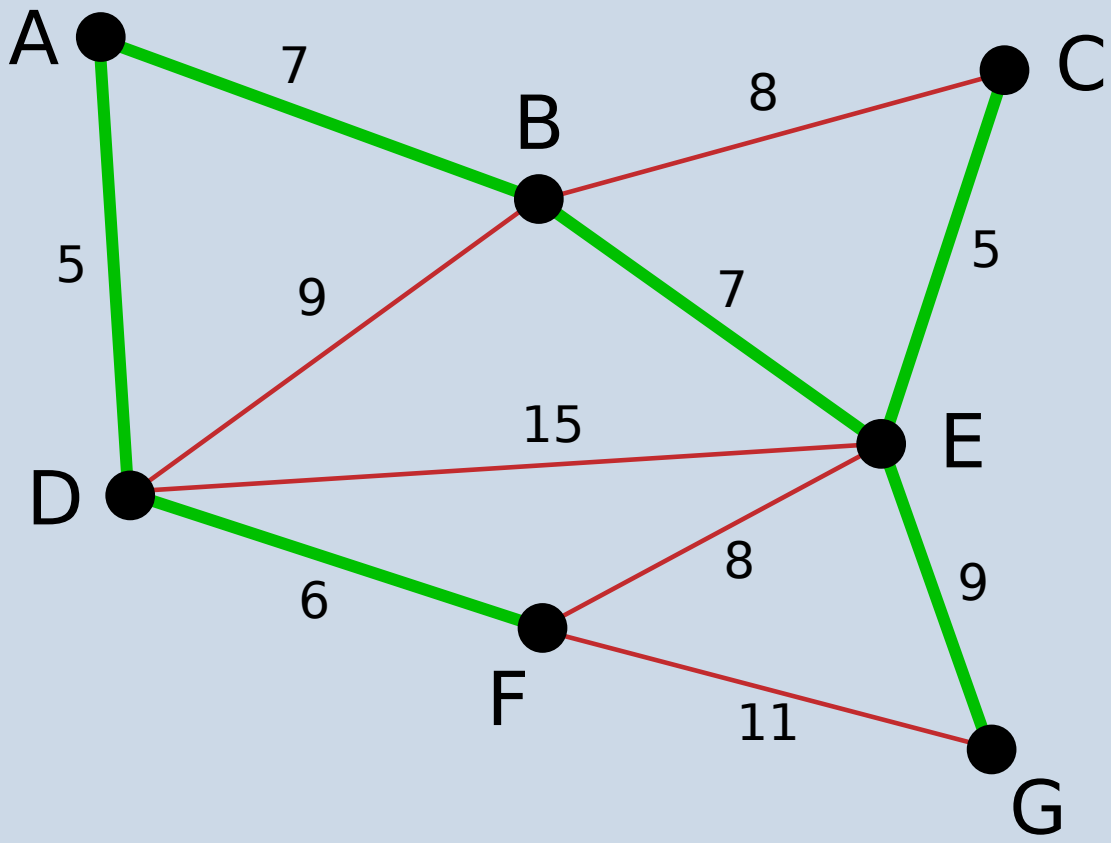
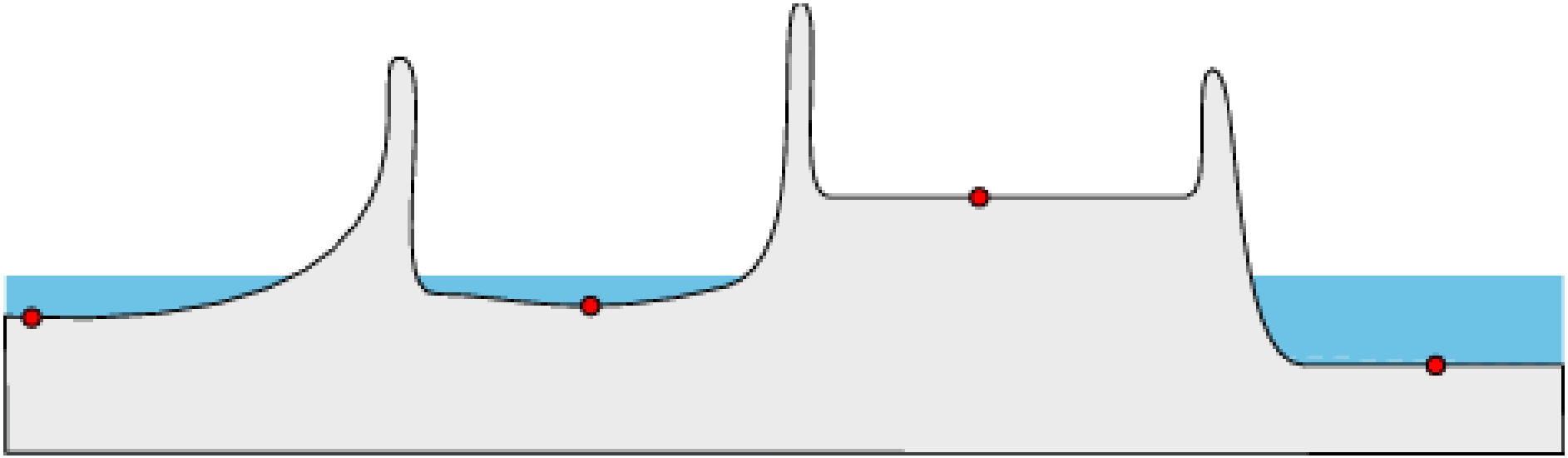


Section 4

Watershed

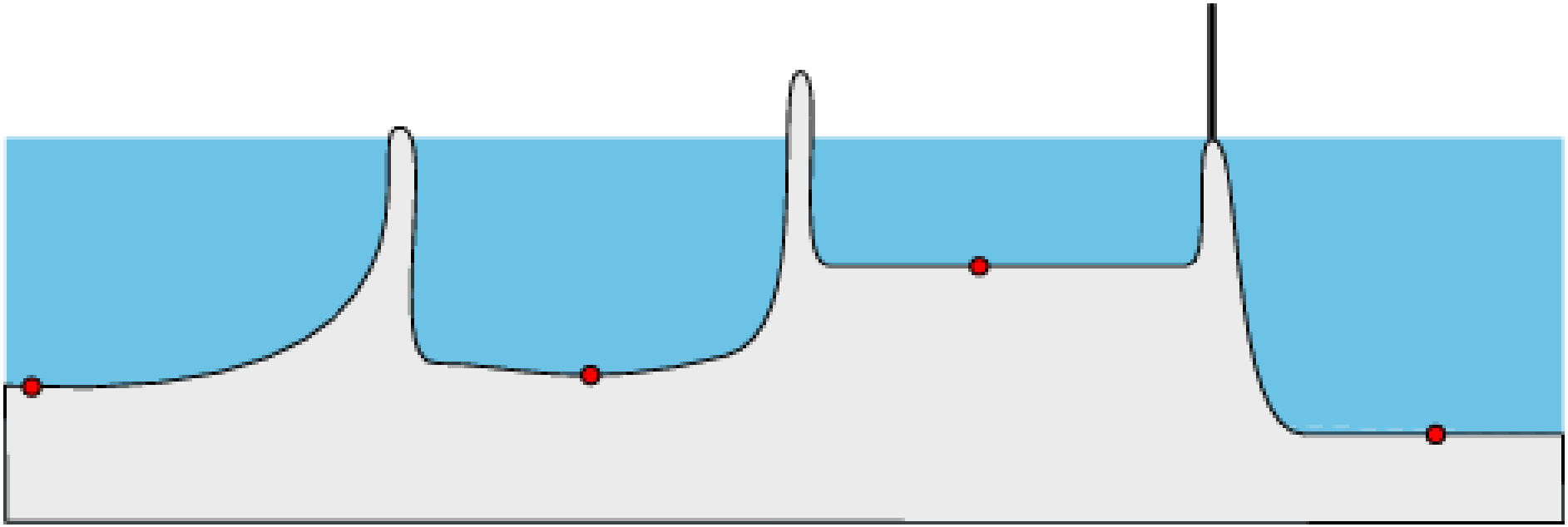
Watershed, catchment basin

Image processing tool: mathematical morphology



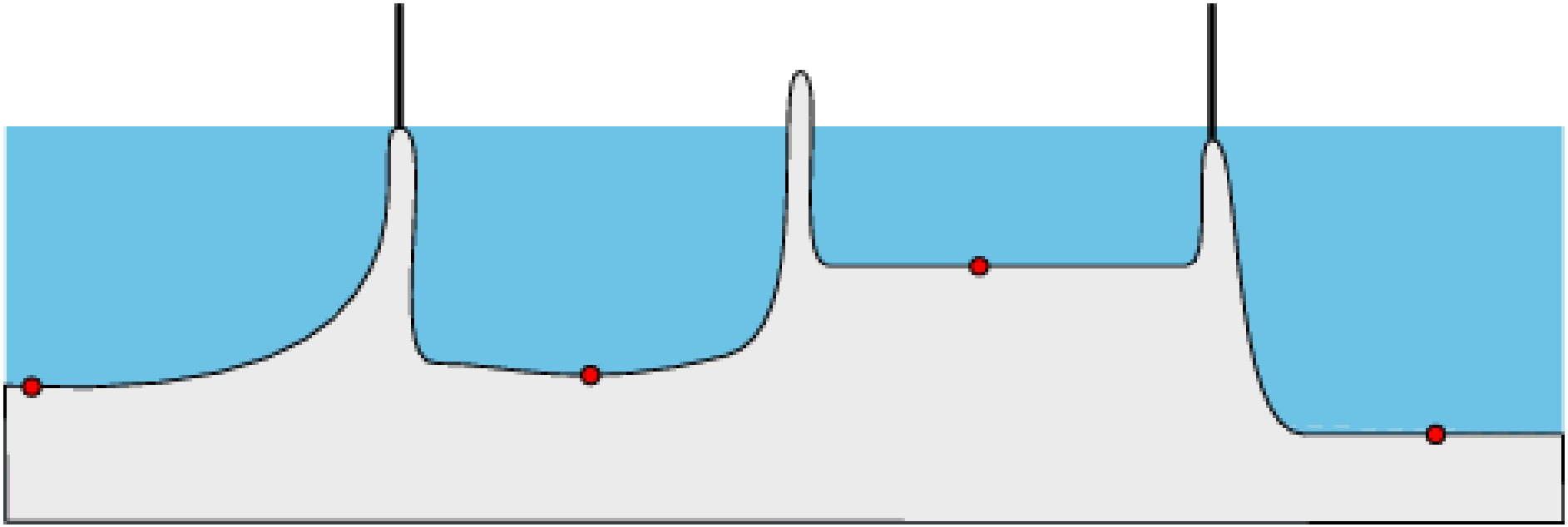
Watershed, catchment basin

Image processing tool: mathematical morphology



Watershed, catchment basin

Image processing tool: mathematical morphology



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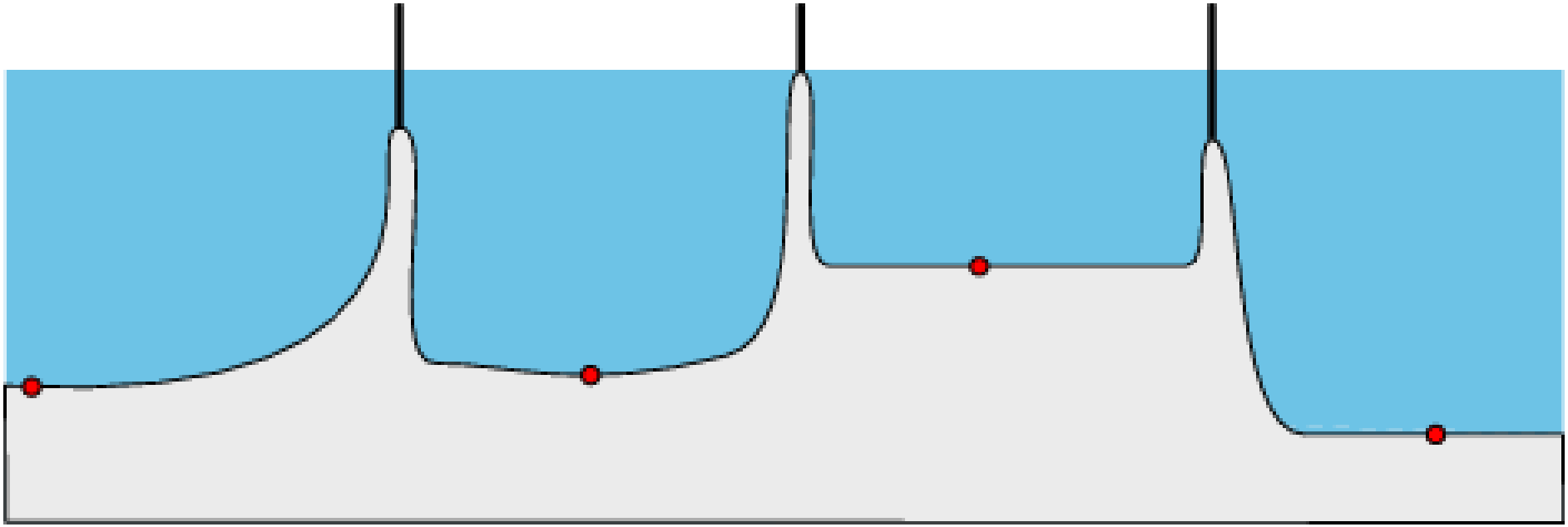

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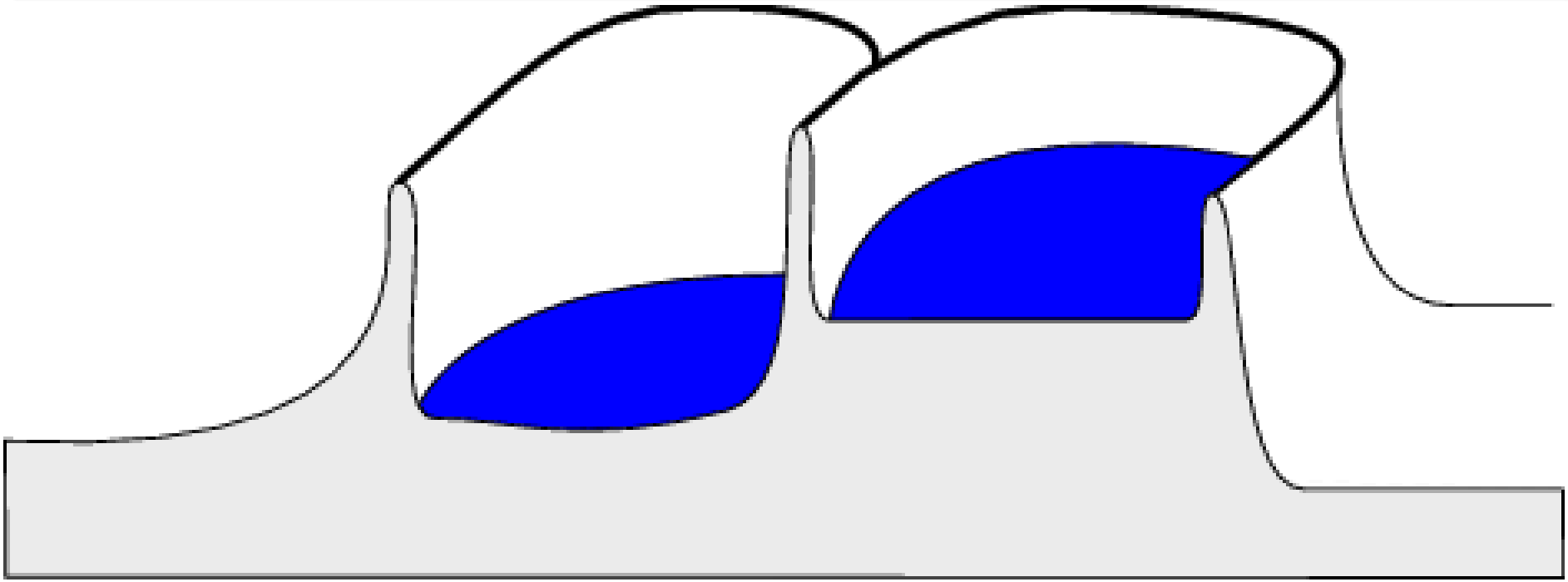

Geodesic distance [Roerdink and Meijster, 2001]

Definition (Geodesic distance between 2 points)

- $A \subseteq \mathbb{R}^n$ or $\mathbb{Z}^n$.
- $a, b \in A$.

Geodesic distance $d_A(a, b)$ between a and b on A is the path of minimal length on A that links a tp b (continuous case, **min** is **inf**).

Definition (Point to set geodesic distance)

$B \subset A$,

$$d_A(a, B) = \min_{b \in B}(d_A(a, b))$$

Geodesic influence zone

Definition (Influence zone)

- $B \subset A$, partitionned into k connected components, B_i, $i \in [1:k]$

The influence zone iz_A of $B_i \in A$ is defined by:

$$iz_A(B_i) = \{p \in A | \forall j \in [1:k] \setminus \{i\} : d_A(p, B_i) < d_A(p, B_j)\}$$

This is the definition of a **Voronoi** diagram.

SKIZ

Definition

L'union des zones d'influence est définie par: The union of influence zones is defined by:

$$IZ_A(B) = \bigcup_i iz_A(B_i)$$

Complementary set is called the SKIZ (skeleton by influence zone):

$$SKIZ_A(B) = A \setminus IZ_A(B)$$

Practically, the SKIZ connects all the points that are equidistant from at least 2 connected compoments B_i

Definition in continuous case

Definition (Topographic distance)

Not a geodesic distance. See it as a positive elevation change.

- $D \subset \mathbb{R}^n$ connected (spatial support).
- f is a function $\mathcal{C}^2(D)$ (Morse function, twice differentiable almost everywhere).
- 2 point $p, q \in D$
- γ is a **path** (smooth curve) included in D, $\gamma(0) = p$, $\gamma(1) = q$.

$$T_f(p, q) = \inf_{\gamma} \int_{\gamma} \|\nabla f_{|\gamma}(s))\| \mathrm{d}s \tag{1}$$

Watershed transform

Definition (Catchment basin CB)

Let $f \in \mathcal{C}(D)$, and $\{m_k\}_{k \in I}$ its minima.

$$CB(m_i) = \{ \quad x \in D | \forall j \in I \setminus \{i\} : \\ f(m_i) + T_f(x, m_i) < f(m_j) + T_f(x, m_j)\}$$

The watershed is the set of dams:

Definition (Watershed)

$$Wshed(f) = D \cap \left(\bigcup_i CB(m_i) \right)^c$$

Comments

- If f is a metric, the topographic distance becomes a **geodesic distance**.
  The watershed is then the **SKIZ** (by construction, slope is constant).

Comments

- If f is a metric, the topographic distance becomes a **geodesic distance**.
  The watershed is then the **SKIZ** (by construction, slope is constant).
- Differentiability constraints are too high on f.

Comments

- If f is a metric, the topographic distance becomes a **geodesic distance**.
  The watershed is then the **SKIZ** (by construction, slope is constant).
- Differentiability constraints are too high on f.
- Minima of a general function are numerous, thus leading to **oversegmentation**.
  Watershed constrained by markers.


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