Mathematical Morphology

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Context

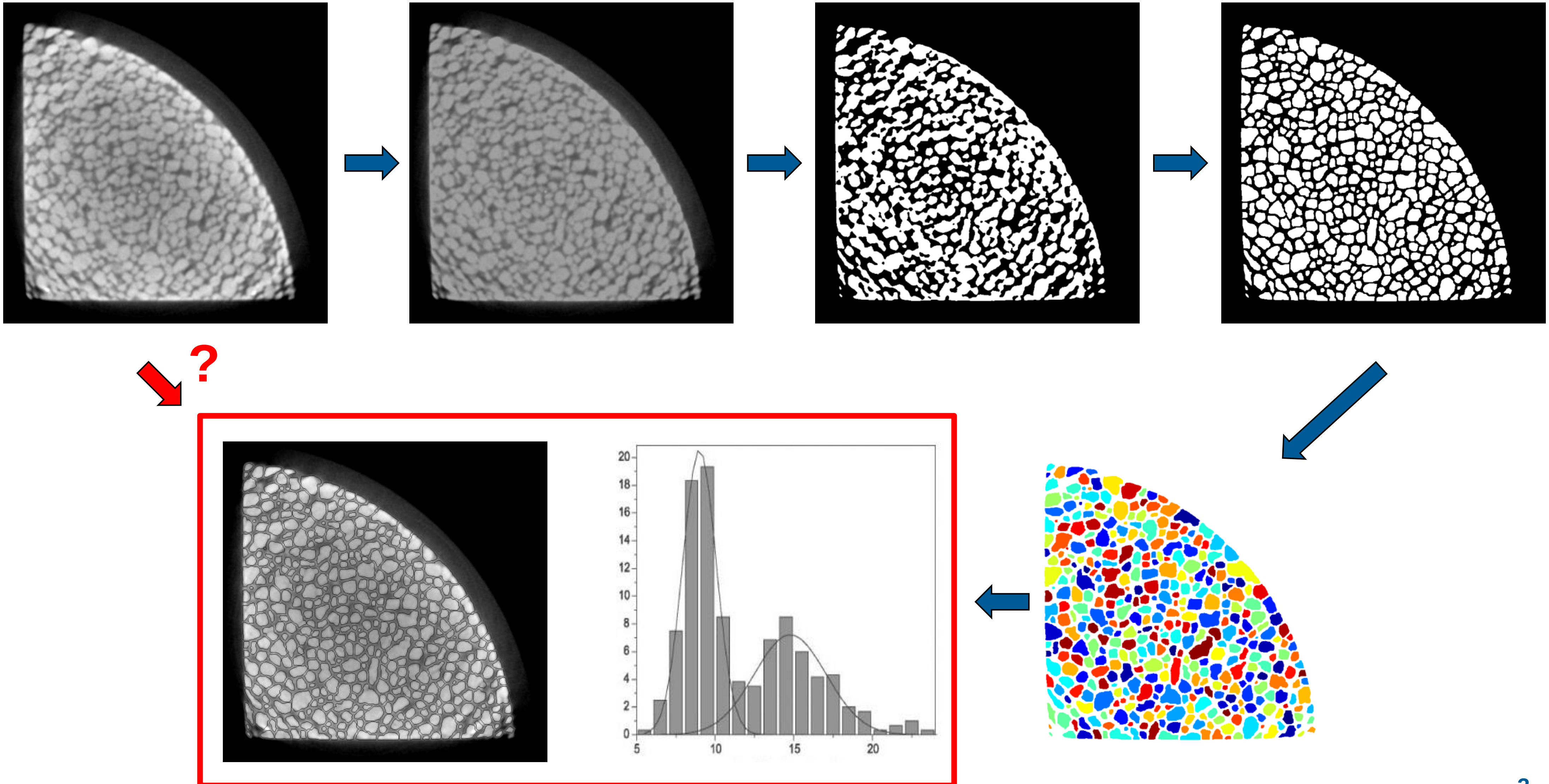
Image Processing & Analysis

- **What for?**
  - Restoration (removing noise/blur)
  - Enhancement (highlighting specific features, improving contrast)
  - Segmentation (extracting objects of interest)
  - Compression (reducing memory size)
  - Registration (putting images in spatial correspondence)
  - Classification (discriminating/clustering objects)
  - Recognition (recognizing objects)
  - Quantification (characterizing objects, texture)
  - …

- **Mathematical Morphology is useful for several tasks!**

Context

Example: Image Characterization (Grain Size Distribution)

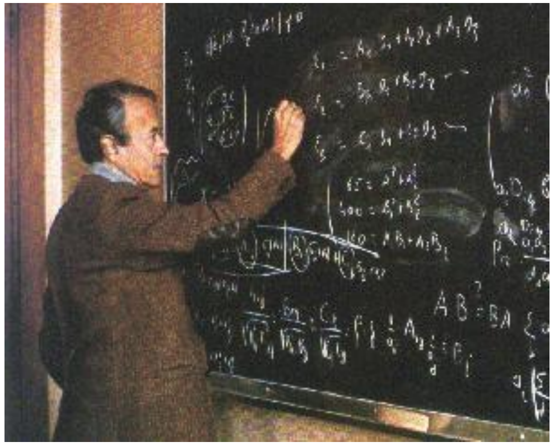


Introduction

History

- **Morphology**
  - The word “morphology” stems from the Greek words ‘morphé ‘and ‘logos’, meaning “the study of forms”
  - It is used in a large number of scientific disciplines including biology and geography

- **Origin**
  - Invented in the early 1960s by Georges Matheron and Jean Serra
  - Study of porous media (geostatistics), automatic analysis of images occurring in mineralogy and petrography



G. Matheron (1930-2000)

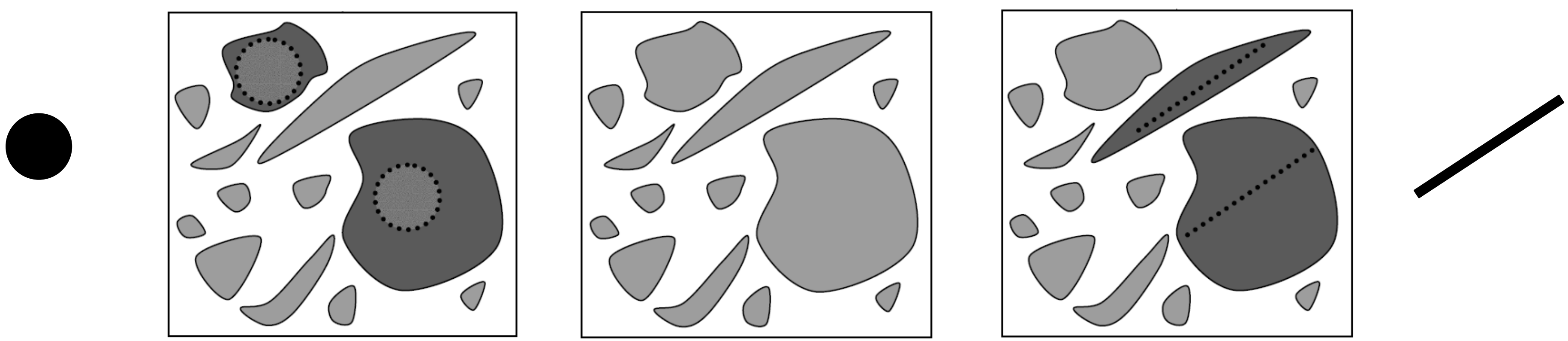
J. Serra (1940-~)

Introduction

Principle of Mathematical Morphology (MM)

- **Study the Shape of Objects (Images)**
  - Geometry
  - Topology
  - Neighborhood information

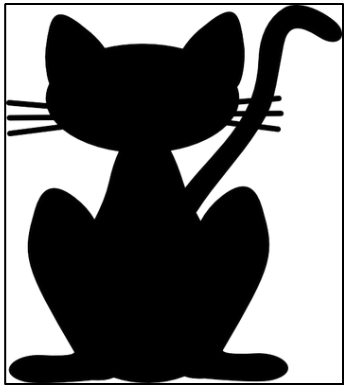
- **Key Idea**
  - Extracting knowledge from the relation of an image and a simple, small probe (called the structuring element), which is a predefined shape.
  - It is checked in each pixel, how does this shape match or miss local shapes in the image.

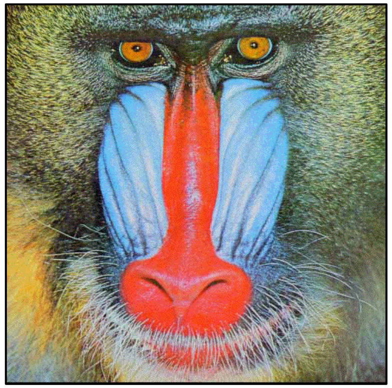


Introduction

Morphological Image Processing

- **Tools for**
  - Filtering
  - Segmentation
  - Measurements
  - Texture analysis
  - Shape recognition
  - Scene interpretation
- **Mathematical Bases**
  - Set theory
  - Lattice algebra
- **Application in Various Domains**
  - Material sciences
  - Process engineering
  - Biomedical imaging
  - …
- **Nature of MM**
  - Set-Valued (binary images)
  - Functional (gray level images)
  - Multivariate (color/multispectral images)
  - Tensorial (tensor images)

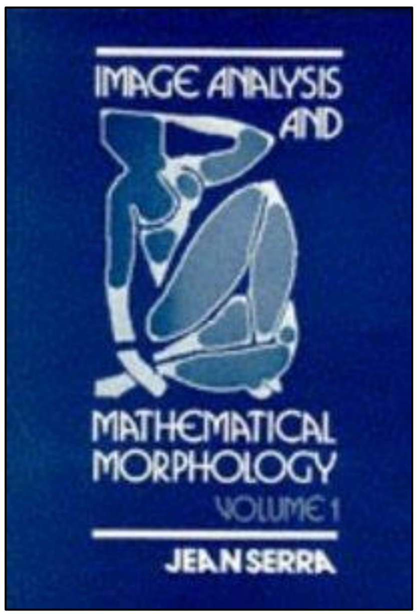


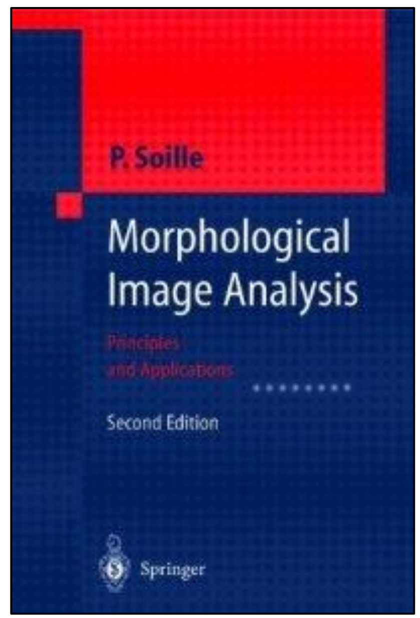


Introduction

A Few References

- **J. Serra**, *Image Analysis and Mathematical Morphology, Volume I*, Academic Press, New-York, 1982.
- **J. Serra** (Ed.), *Image Analysis and Mathematical Morphology, Part II: Theoretical Advances*, Academic Press, London, 1988.
- **P. Soille**, *Morphological Image Analysis, 2nd Edition*, Springer-Verlag, Berlin, 2004.





Binary Mathematical Morphology

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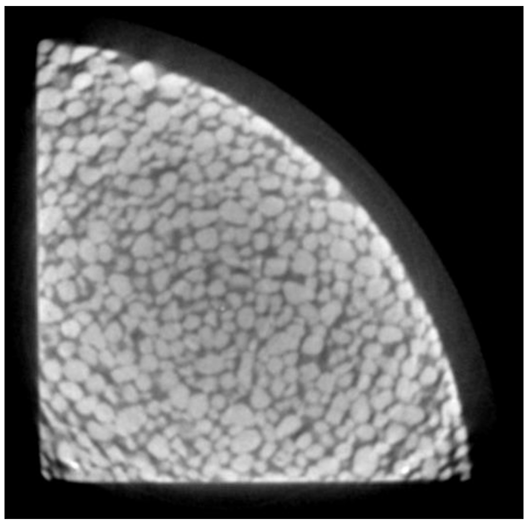
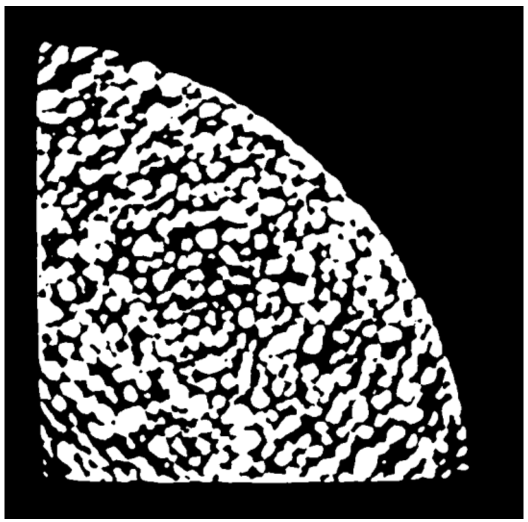
Binary Morphology

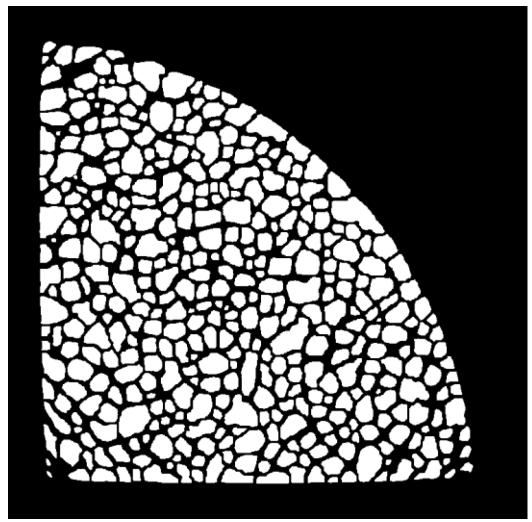
- **Mainly used for**
  - Object segmentation/recognition
    - Binary images often suffer from noise
    - Binary regions also suffer from noise (holes, cracks, protusions…)
  - Object measurements



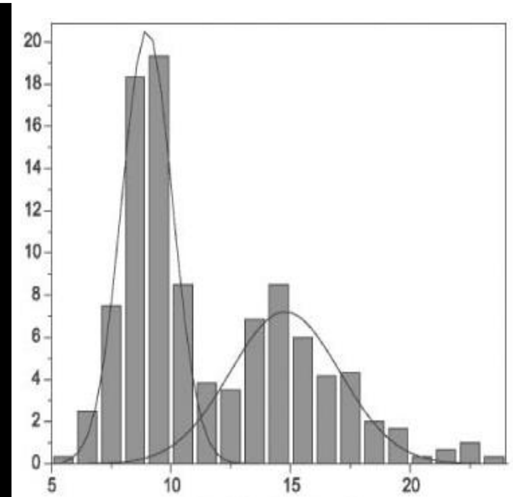
Original image
of sand grains
(X-ray tomography)



Thresholded image



Segmented image



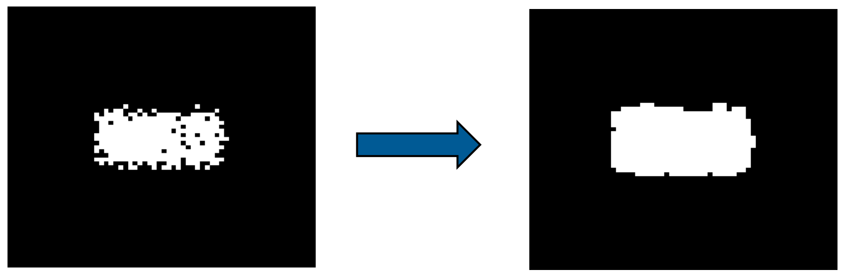
Quantification

Binary Mathematical Morphology

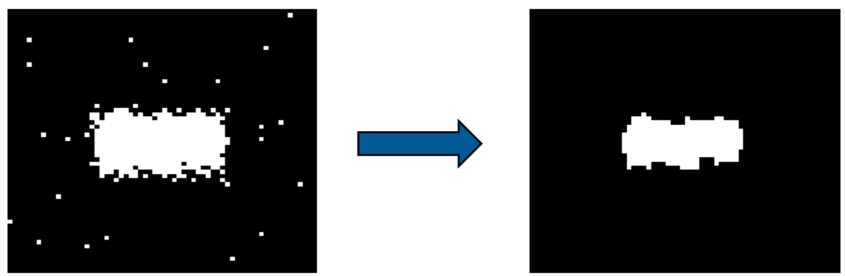
Erosion, Dilation

Erosion and Dilation

- **Primary Operations of MM**
- **More Complicated Operators can be designed by means of combining Dilations and Erosions**
- **Dilation**
  - Fills holes
  - Smoothes object boundaries
  - Objects become slightly larger



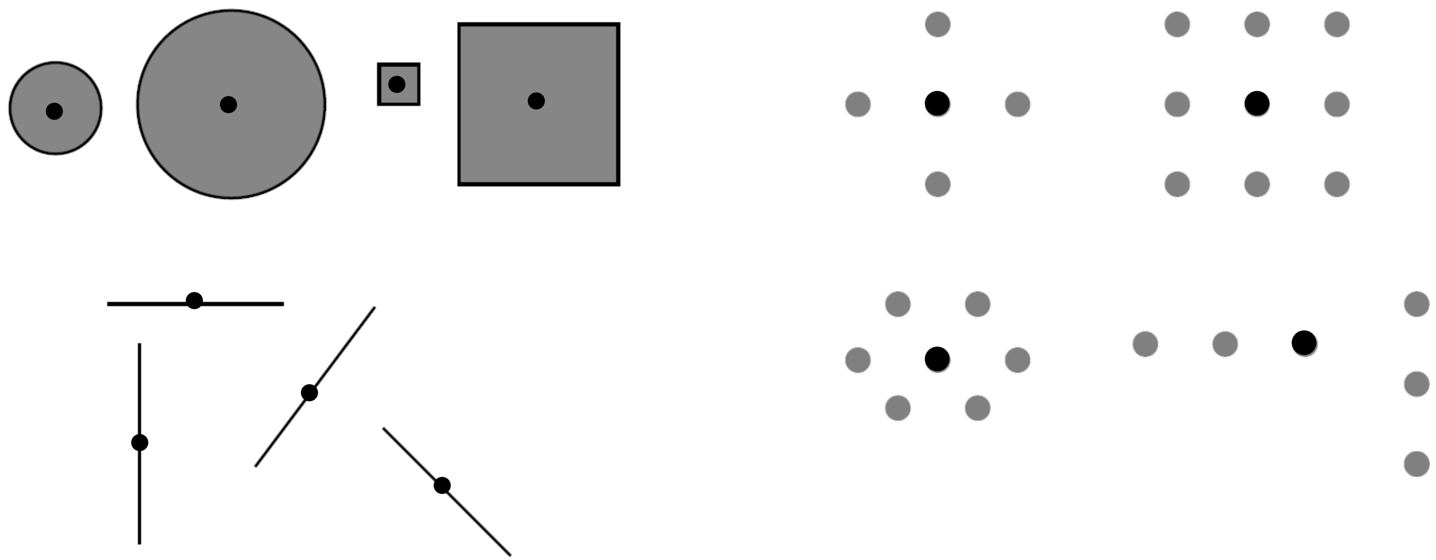
- **Erosion**
  - Removes isolated noisy pixels
  - Smoothes object boundaries
  - Objects become slightly smaller



- **Use of a Structuring Element (SE)**

Structuring Element (SE)

- **Various Shape, Size**
- **Origin**
  - Not necessarily inside the SE
- **Continuous and Discrete**



Erosion, Dilation

Erosion

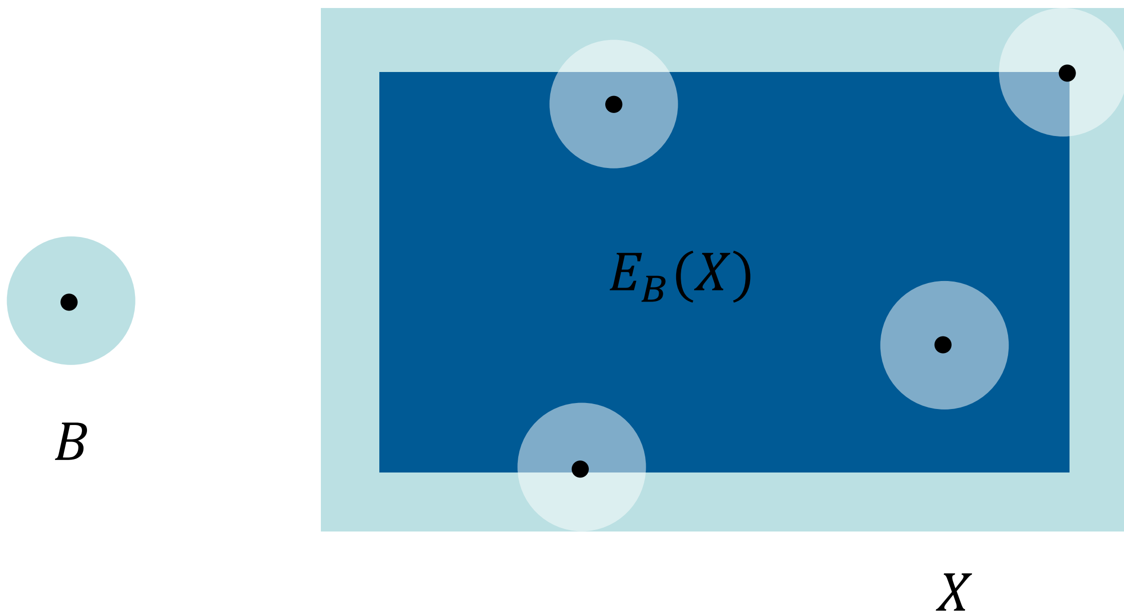
- **Does the SE Match the Set?**

- **Definition**

$$E_B(X) = \{x;\ B_x \subseteq X\}$$

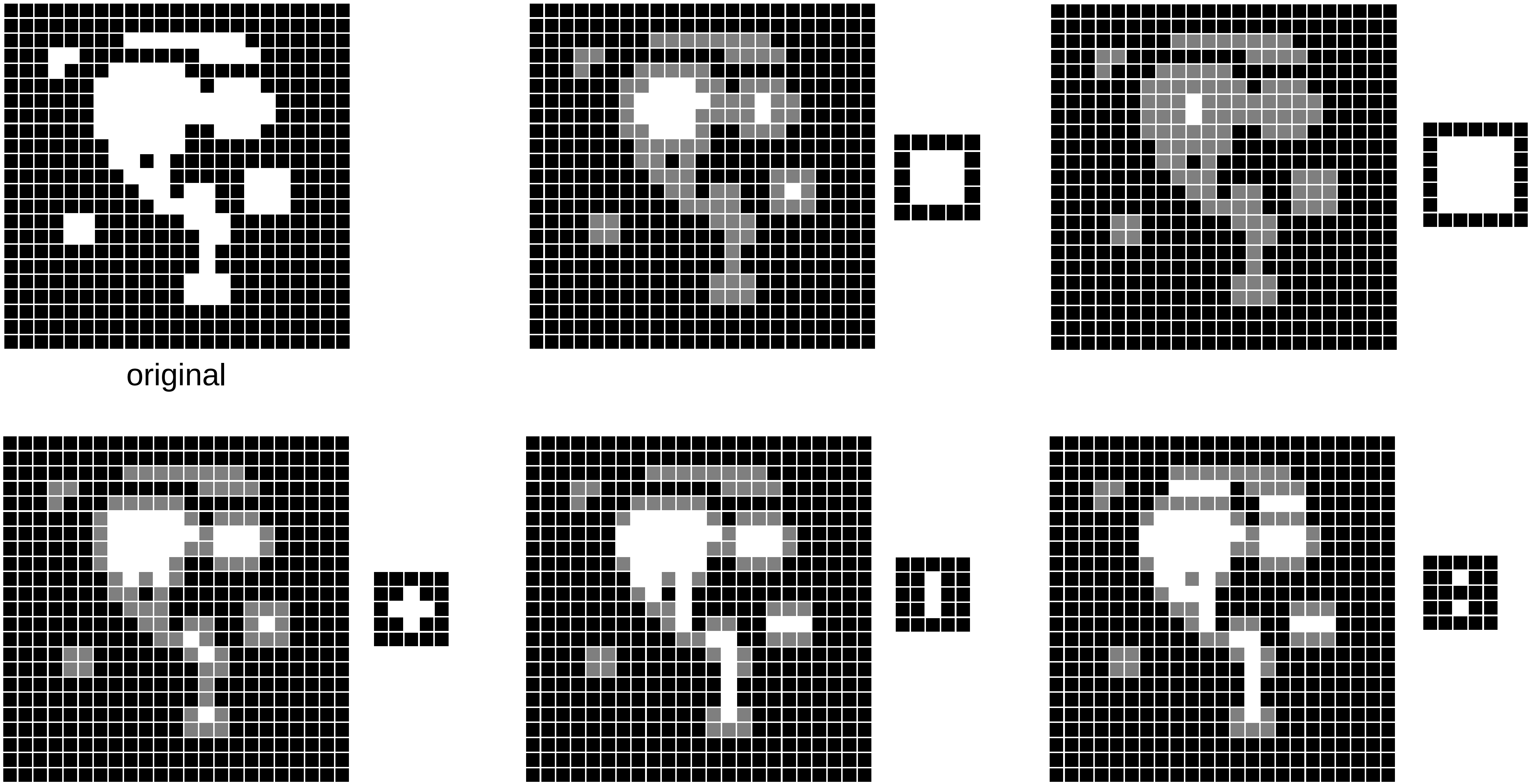
where $B_x = \{b + x; x \in X\}$

- **Illustration**



Erosion, Dilation

Erosion with Various Structuring Elements



original

Erosion, Dilation

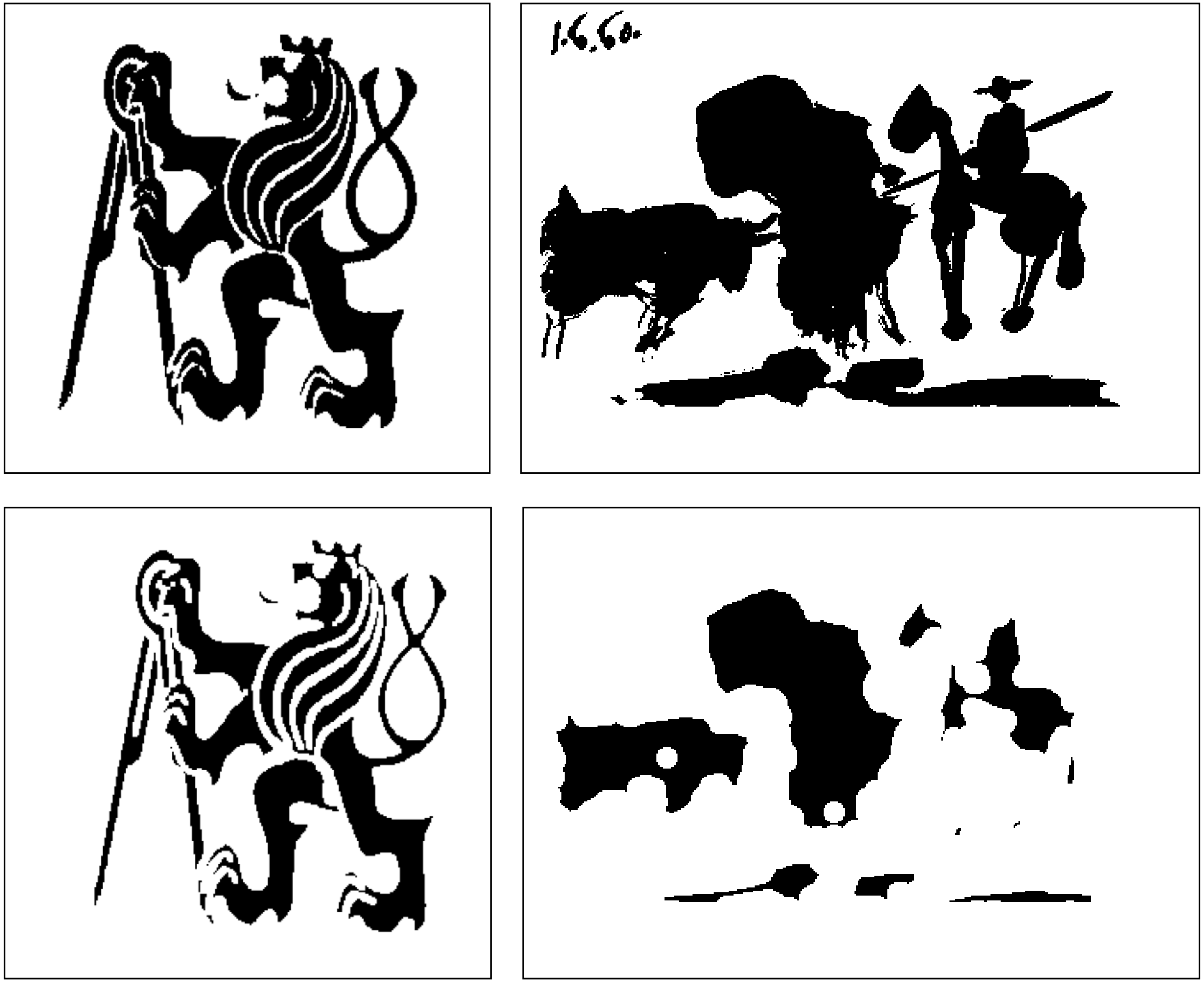
Properties of Erosion

- **Anti-Extensivity:** $E_B(X) \subseteq \mathrm{X}$ if $0 \in B$
- **Increasing:** $X \subseteq Y \Rightarrow E_B(X) \subseteq E_B(Y)$
- **Decreasing with respect to the SE:** $B \subseteq B' \Rightarrow E_{B'}(X) \subseteq E_B(X)$
- **Commutativity with Intersection, not with Union:** $E_B(X \cap Y) = E_B(X) \cap E_B(Y)$
  $E_B(X) \cup E_B(Y) \subseteq E_B(X \cup Y)$
- **Structuring Element Decomposition:** $E_{B \cup B'}(X) = E_B(X) \cap E_{B'}(X)$
- **Iterativity:** $E_{B'}(E_B(X)) = E_{B \oplus B'}(X)$
- **Translation Invariance:** $E_B(X_z) = (E_B(X))_z$
- **Compatibility with Scales:** $\lambda E_B(X) = E_{\lambda B}(\lambda X)$

Erosion, Dilation

Erosion

- **Illustration**
  - Foreground is black
  - Background is white



Erosion, Dilation

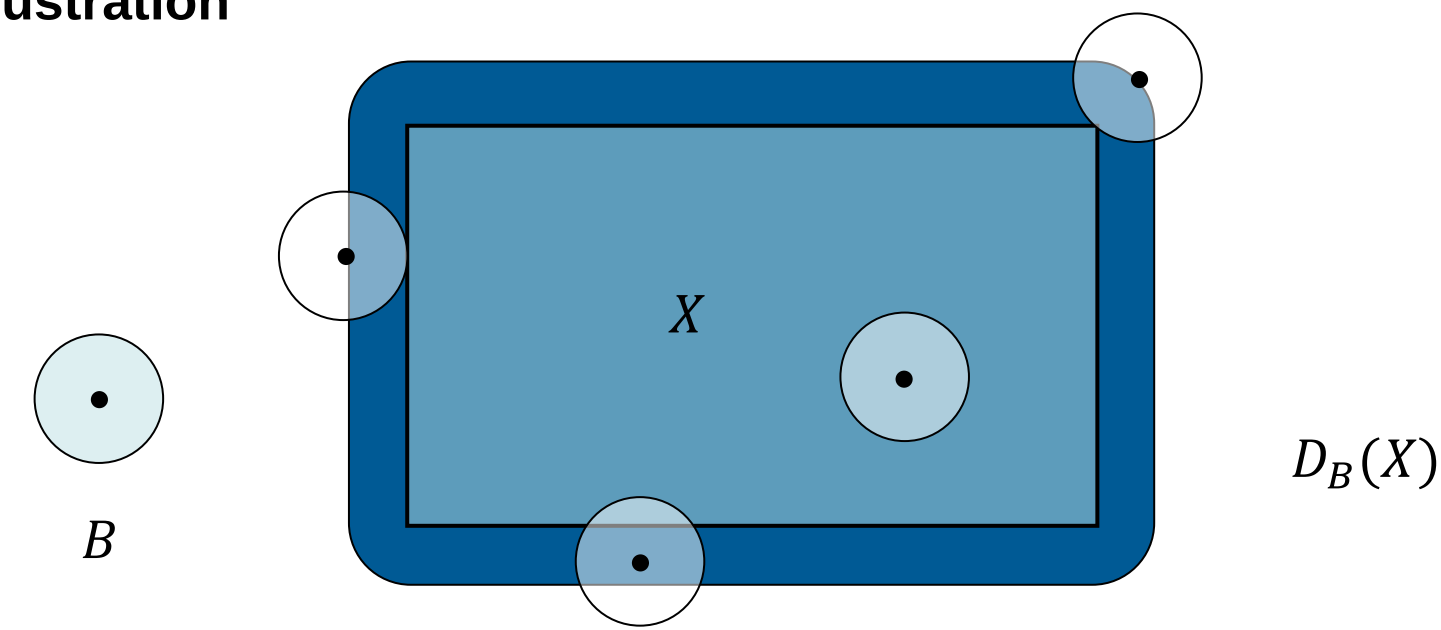
Dilation

- **Definition**

$$D_B(X) = \{x;\ B_x \cap X \neq \emptyset\}$$

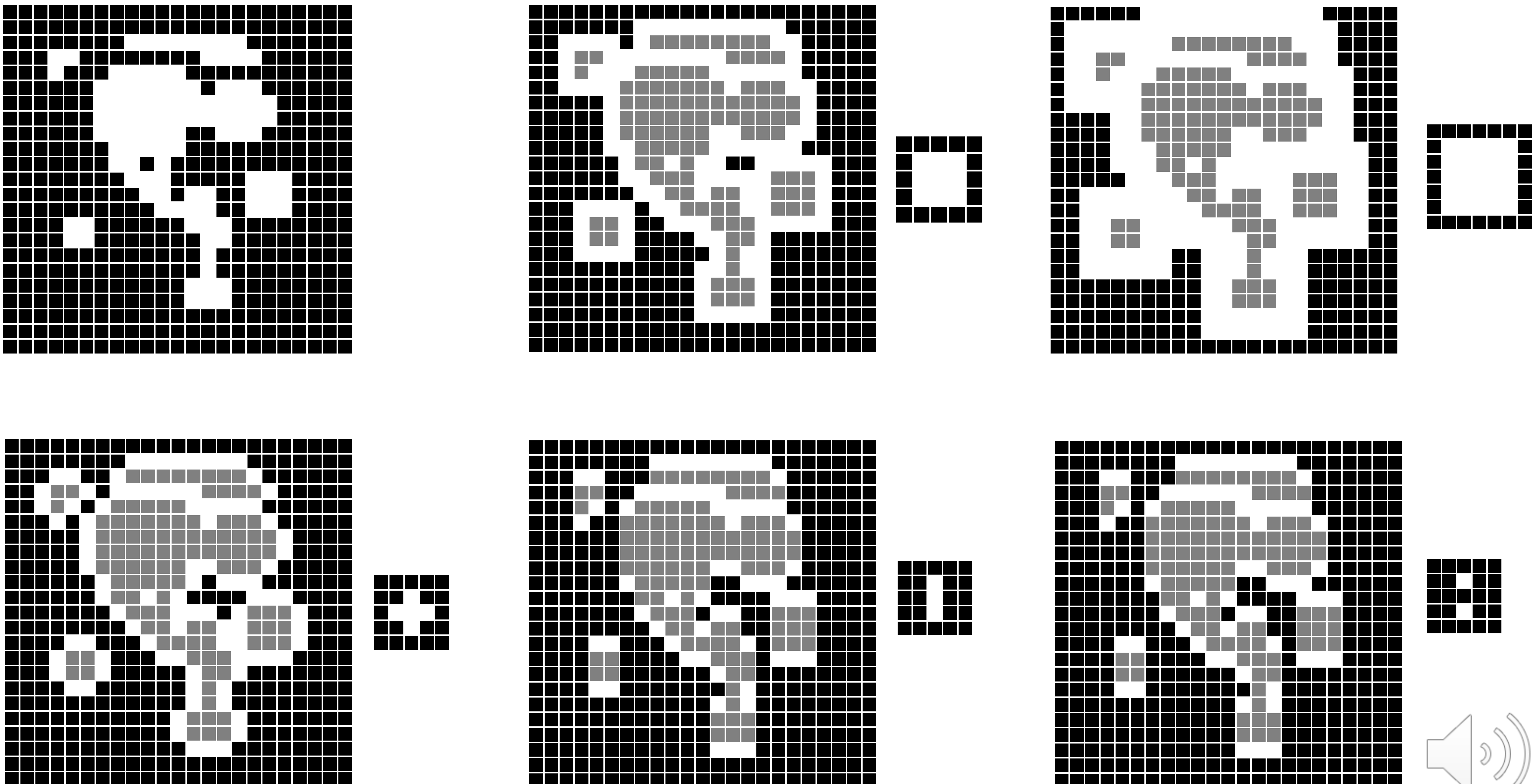
where $B_x = \{b + x; x \in X\}$

- **Illustration**



Erosion, Dilation

Dilation with Various Structuring Elements



Erosion, Dilation

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Erosion, Dilation

Dilation

- **Illustration**
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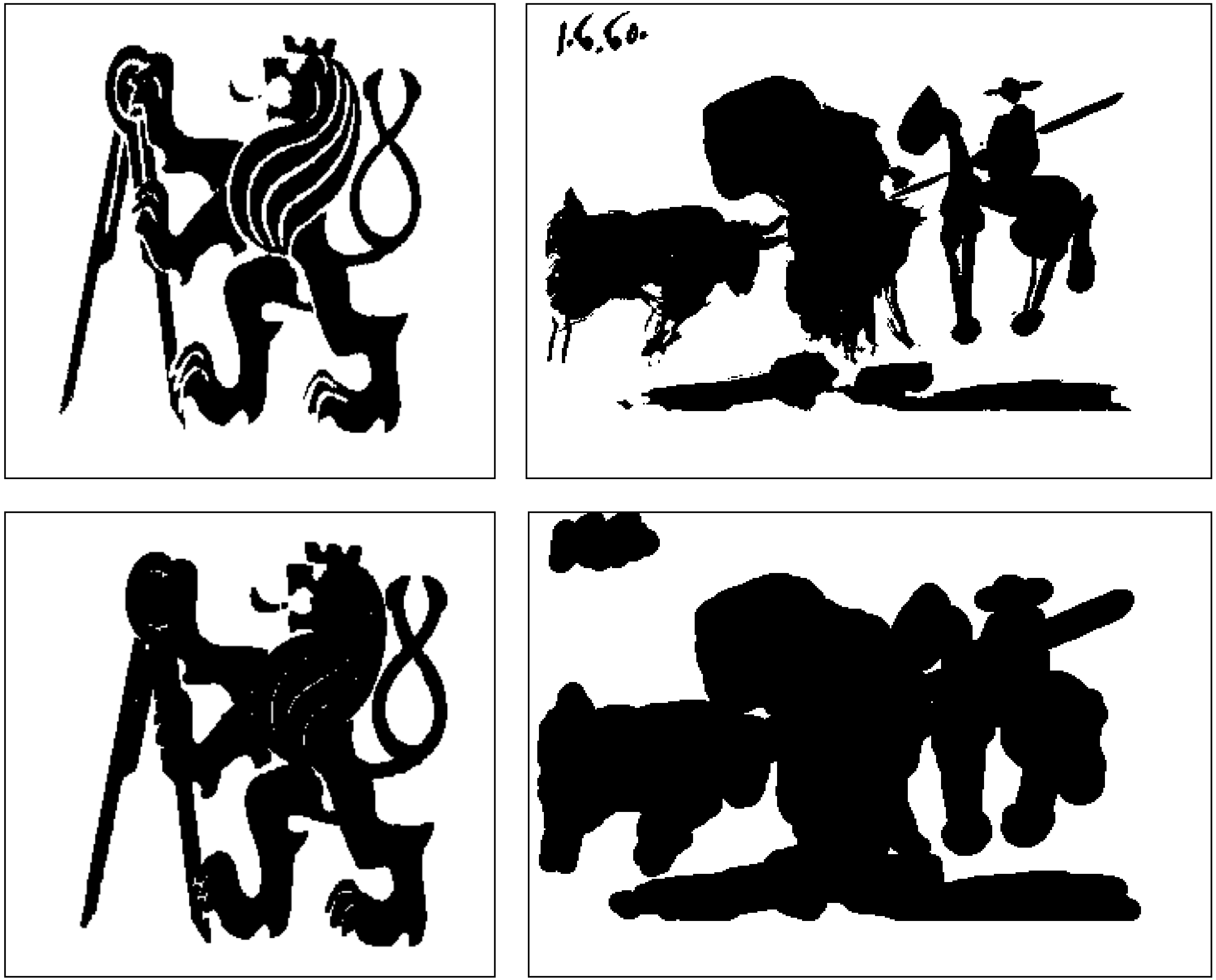
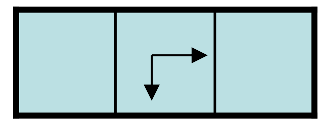


Erosion, Dilation

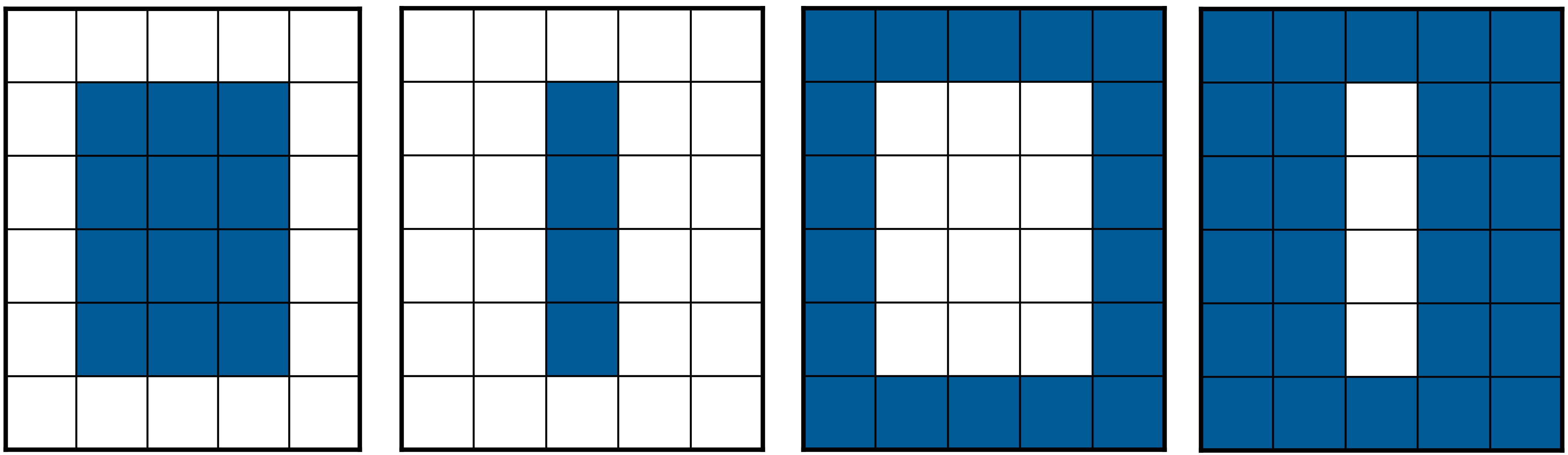
Duality Relationship between Erosion and Dilation

- **Duality with respect to the Complementation**

$$(E_B(X))^c = D_B(X^c)$$



B

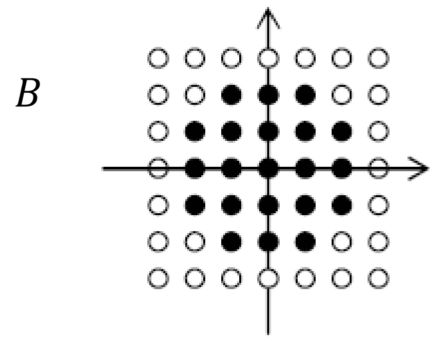


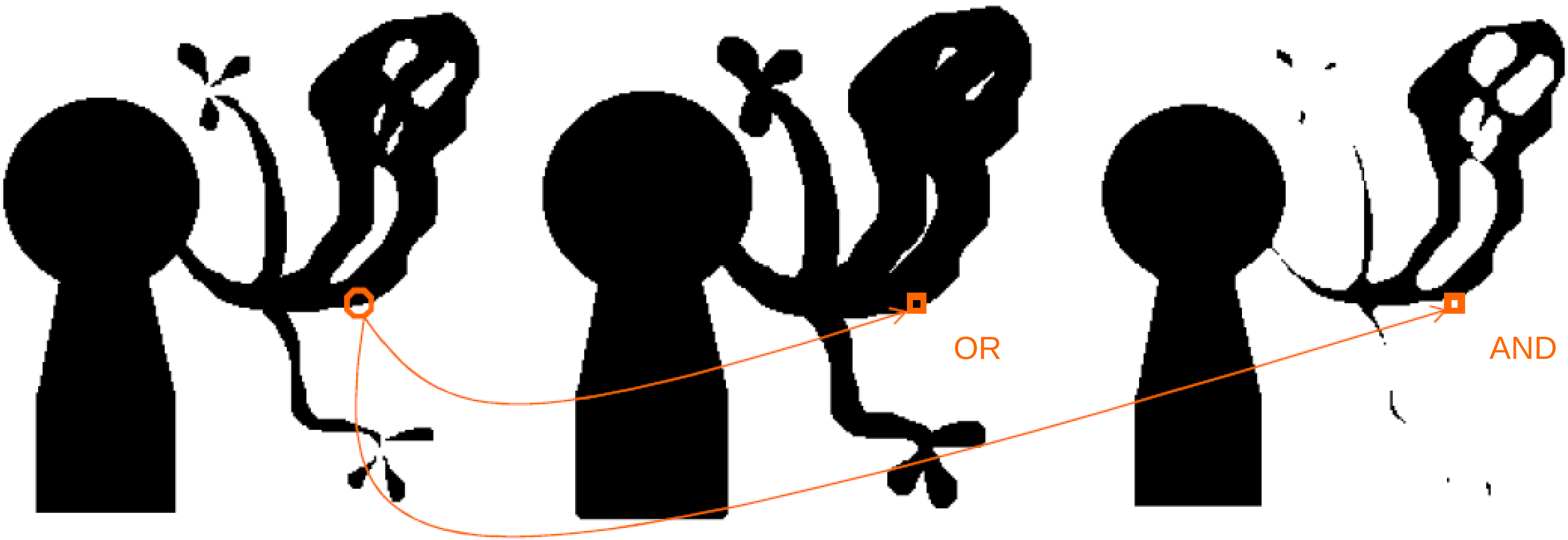
X $\qquad E_B(X)$ $\qquad X^c$ $\qquad D_B(X^c)$

Erosion, Dilation

Application to Digital Images

- **Logical Operations**



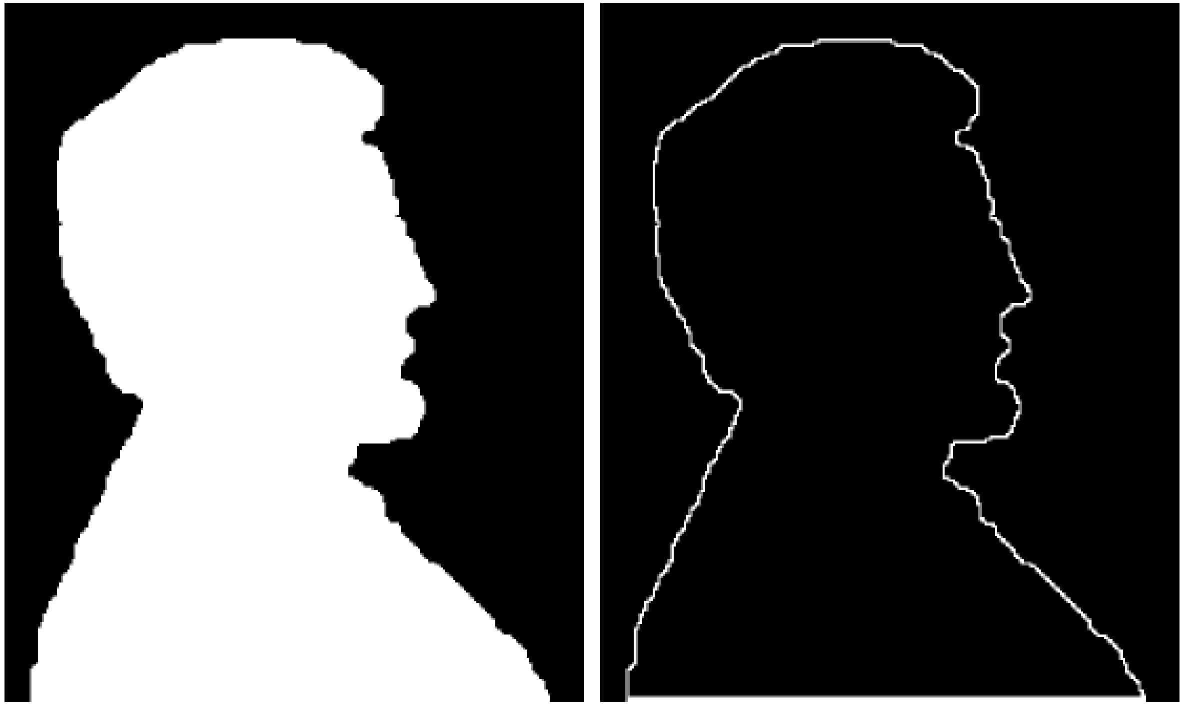


original image X dilated image $D_B(X)$ eroded image $E_B(X)$

Application to Boundary Extraction

- **Morphological Gradient**
  - For a small SE size

$$\partial_B(X) = D_B(X) \backslash E_B(X)$$



Erosion, Dilation

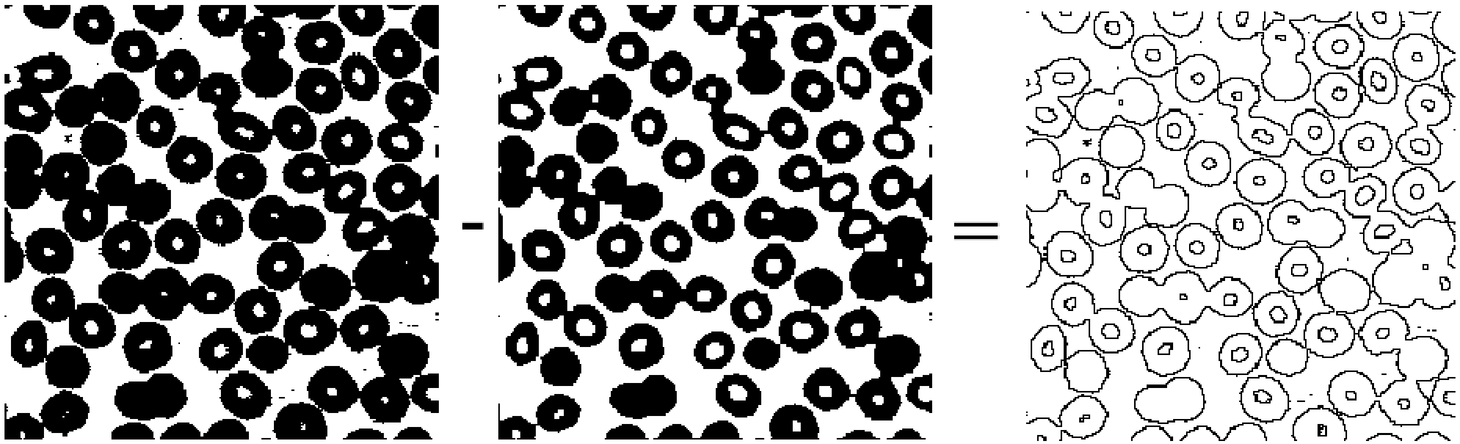
Other Definition of Gradients

- **External Boundary**

$$\partial_B^{ext}(X) = D_B(X) \backslash X$$

- **Internal Boundary**

$$\partial_B^{int}(X) = X \backslash \mathrm{E}_B(X)$$



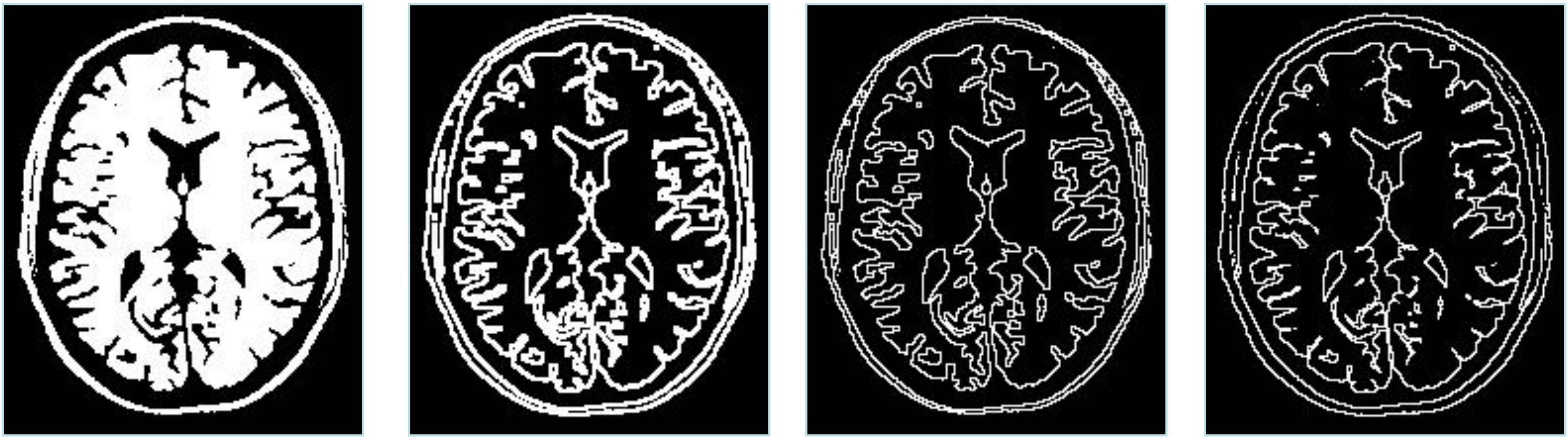
- **Boundary**

$$\partial_B(X) = \partial_B^{int}(X) \cup \partial_B^{ext}(X)$$

Erosion, Dilation

Boundary Extraction

- **Comparison**
  - Brain image

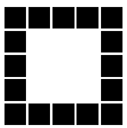


X $\qquad$ $\partial_B(X)$ $\qquad$ $\partial_B^{int}(X)$ $\qquad$ $\partial_B^{ext}(X)$

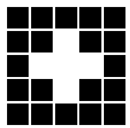
Erosion, Dilation

Effect of the SE Topology

- **Example at the Fine Scale**

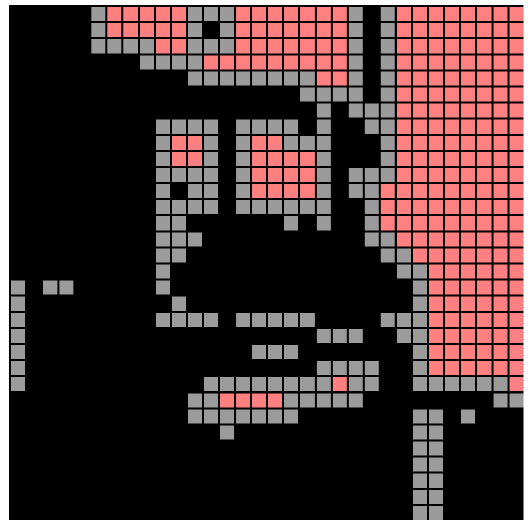


8-connected SE



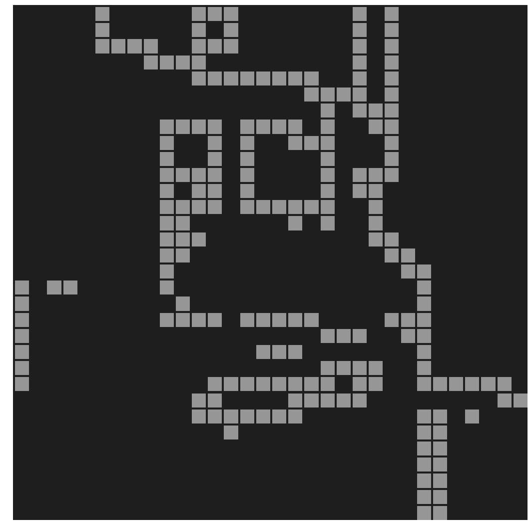
4-connected SE

8-connected SE

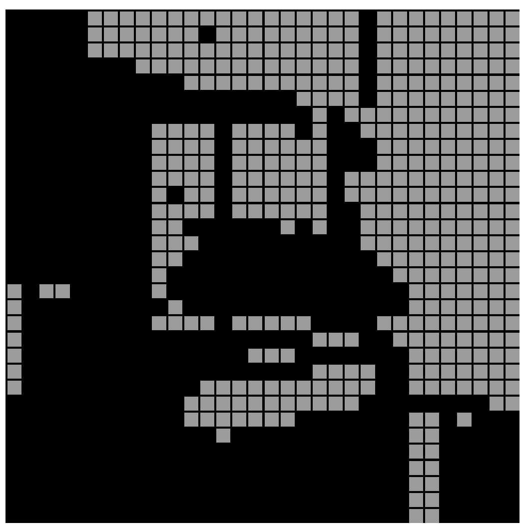


$E_B(X)$

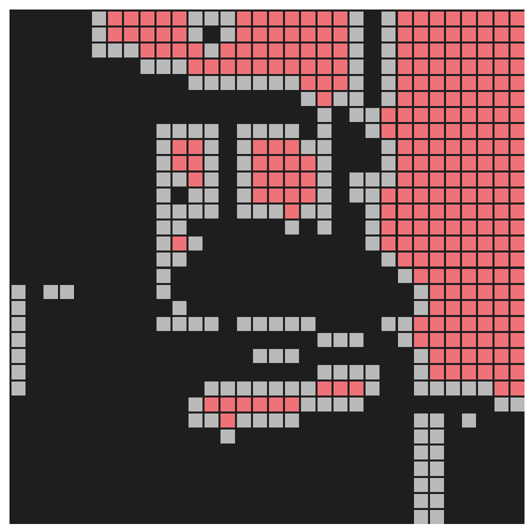
4-connected boundary



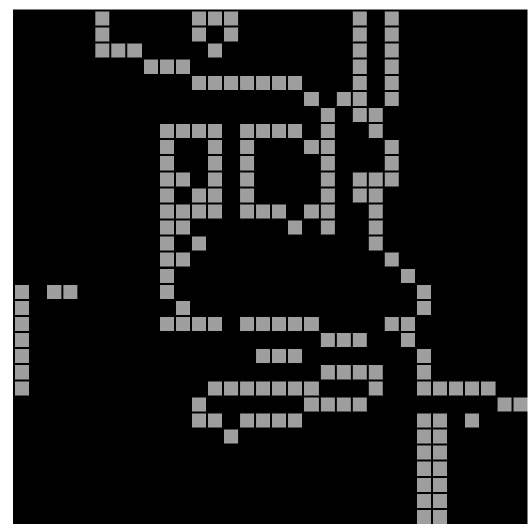
$\partial_B^{int}(X)$



X



4-connected SE

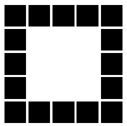


8-connected boundary

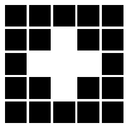
Erosion, Dilation

Effect of the SE Topology

- **Example at the Fine Scale**

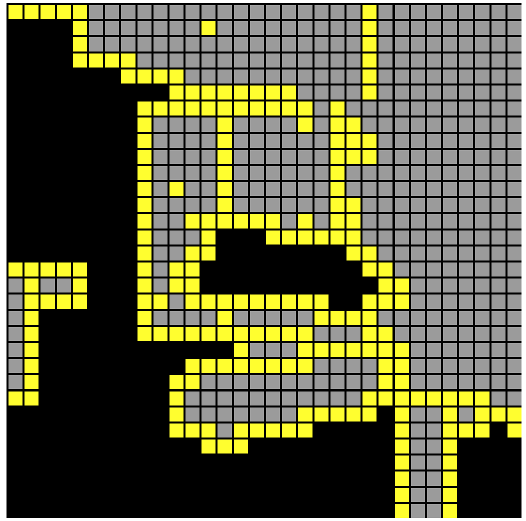


8-connected SE



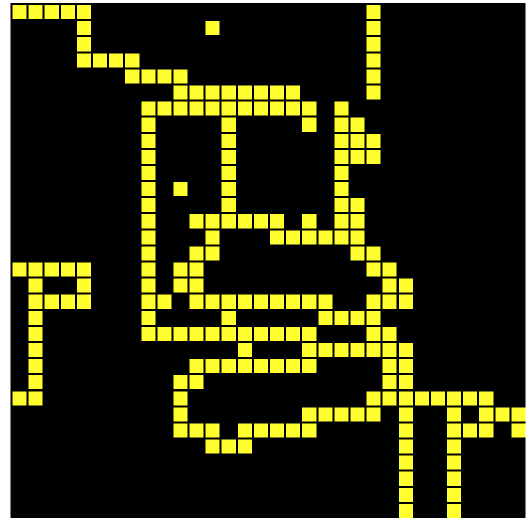
4-connected SE

8-connected SE

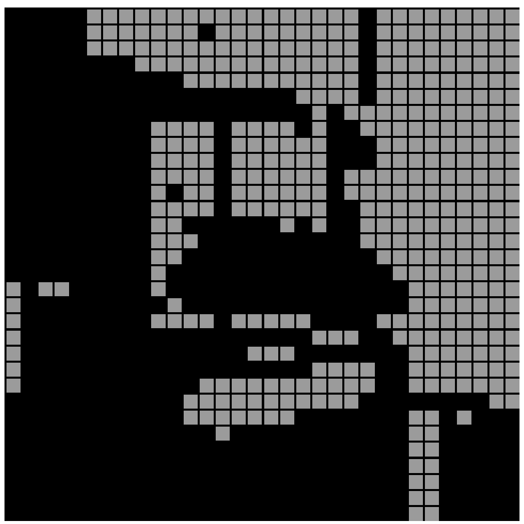


$D_B(X)$

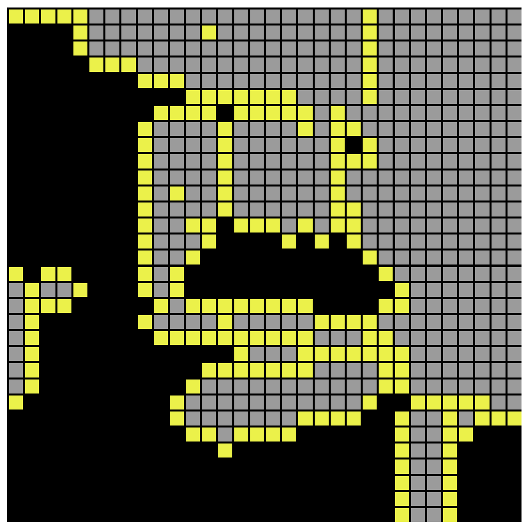
4-connected boundary



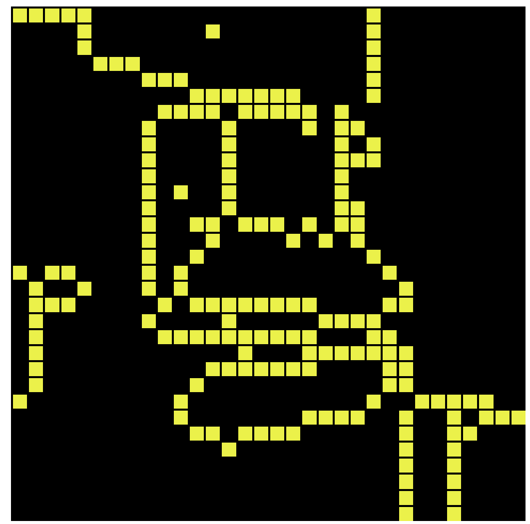
$\partial_B^{ext}(X)$



X



4-connected SE



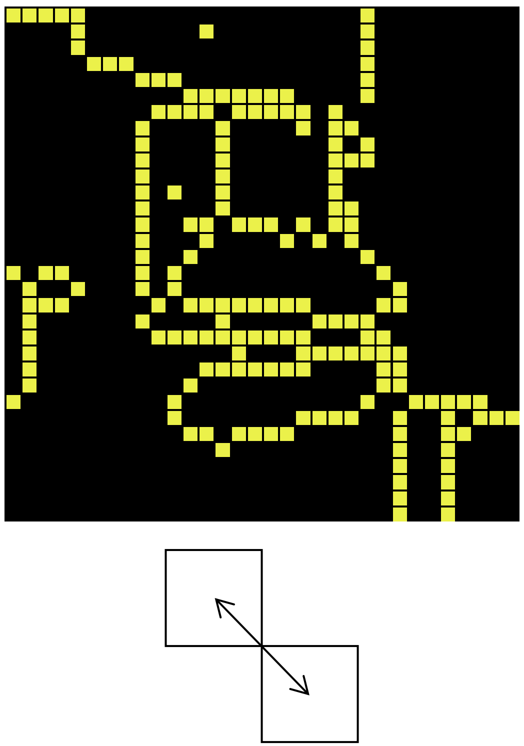
8-connected boundary

Erosion, Dilation

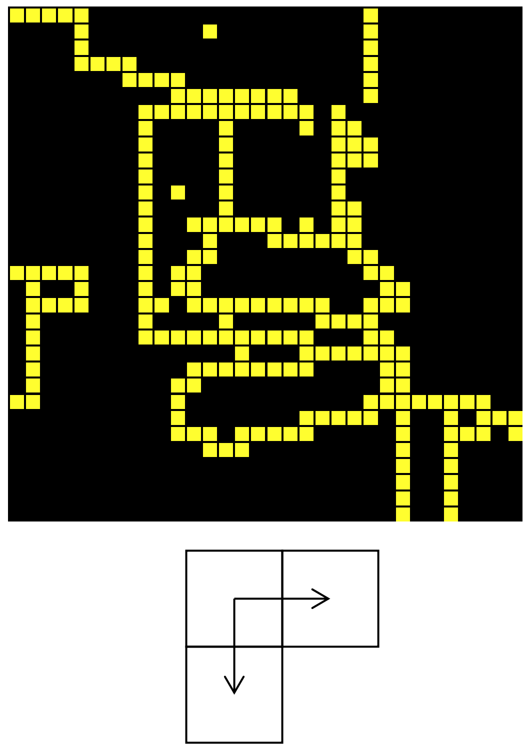
Boundary Measurement

- Interior Boundaries are disjoint from Exterior Boundaries

- Topology is important for the Perimeter Computation



$\sqrt{2}$



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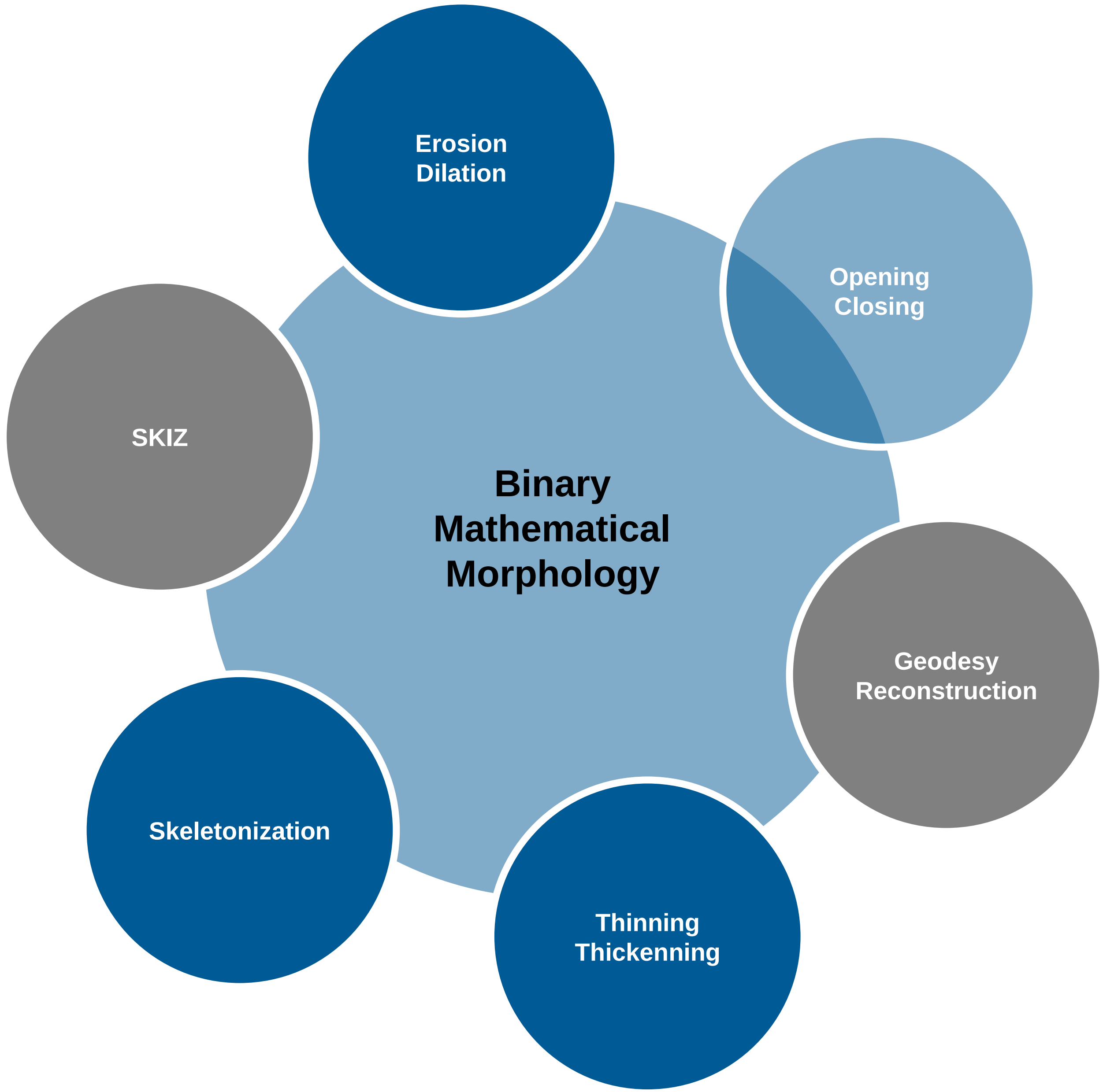
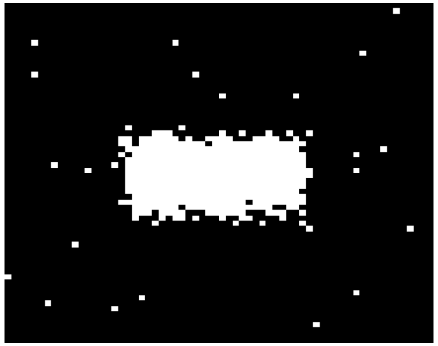
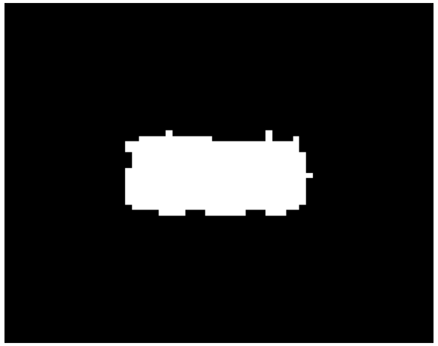
Binary Mathematical Morphology



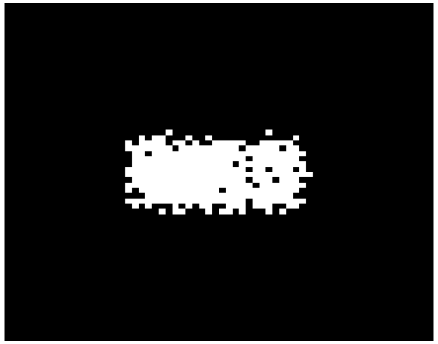
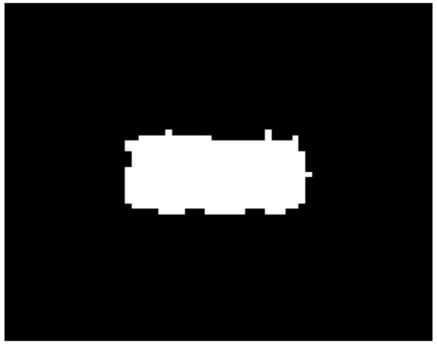
Opening, Closing

Opening and Closing

- **Opening**
  - Removes isolated noisy pixels
  - Smoothes object boundaries
  - Eliminates narrow protrusions
  - Does not significantly change the object’s size

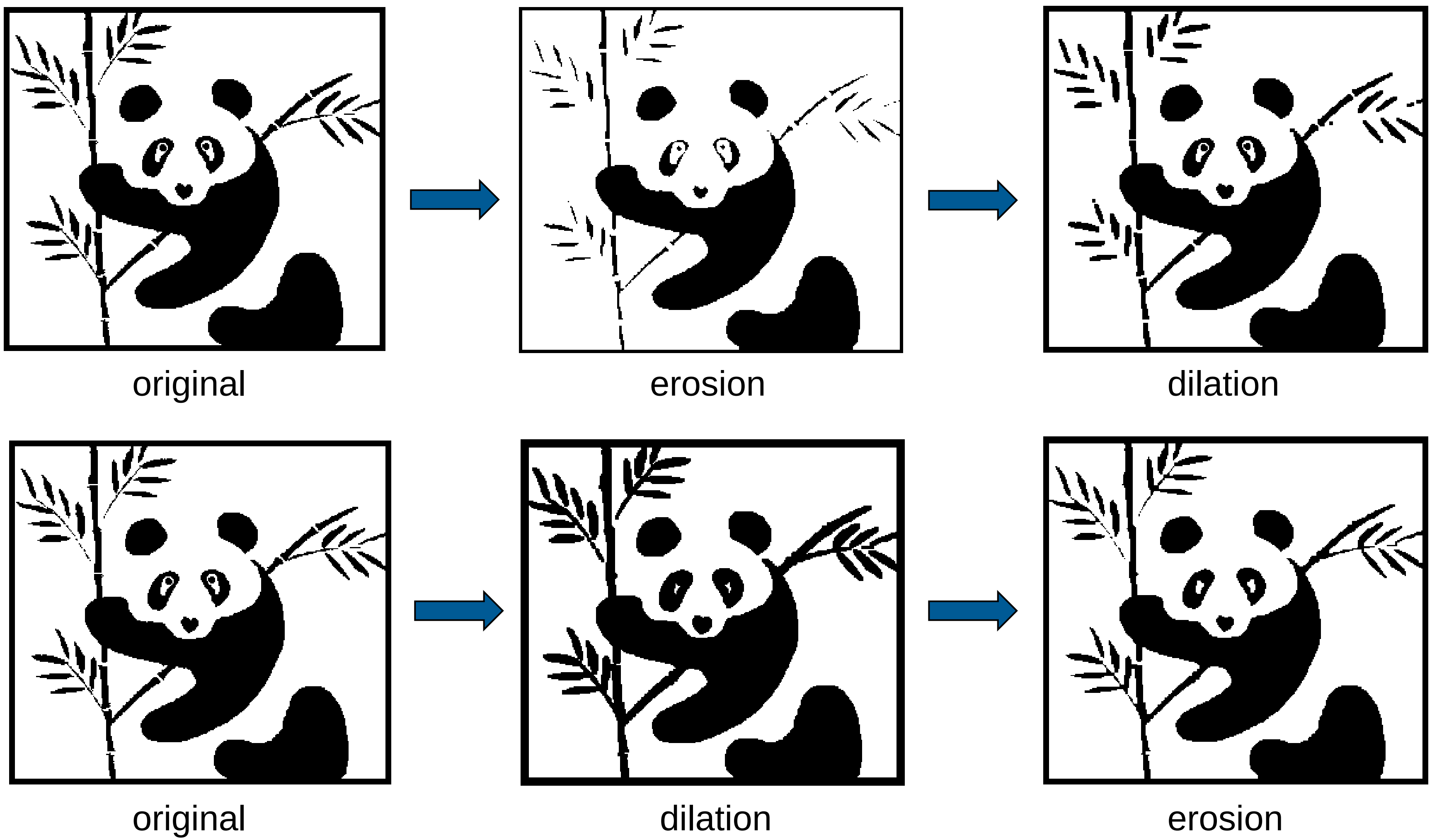



- **Closing**
  - Eliminates small holes
  - Smoothes object boundaries
  - Fills narrow gaps in the contour
  - Does not significantly change the object’s size

Opening and Closing

- **Composition of Dilation and Erosion**

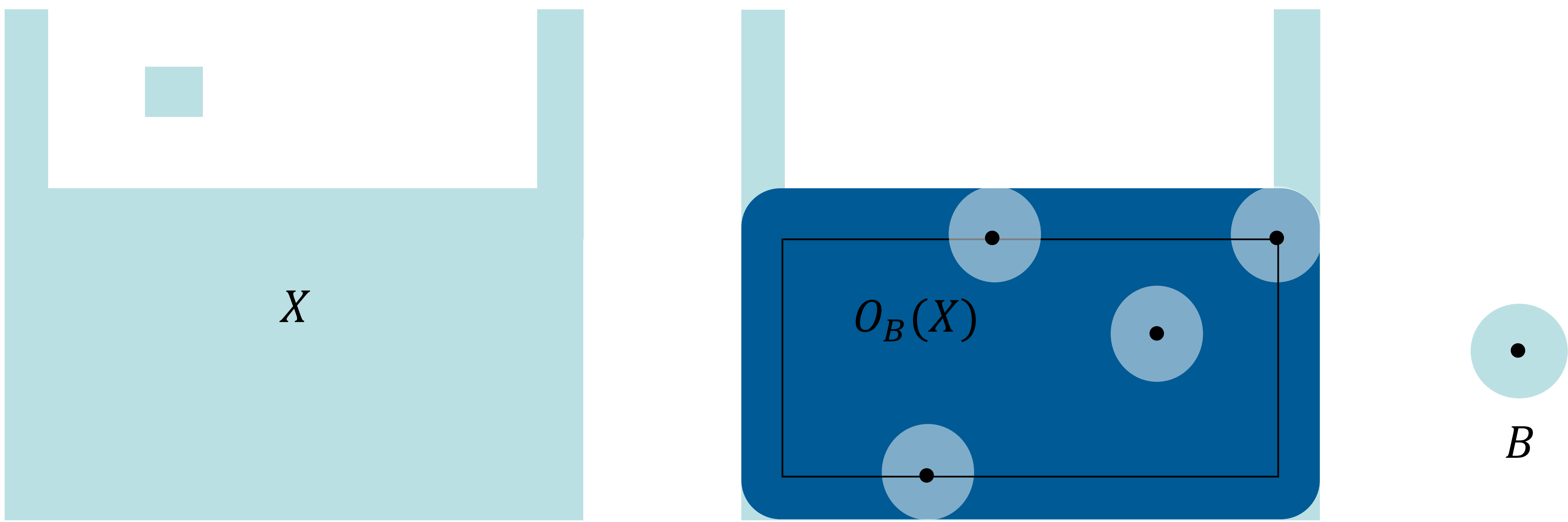


Opening

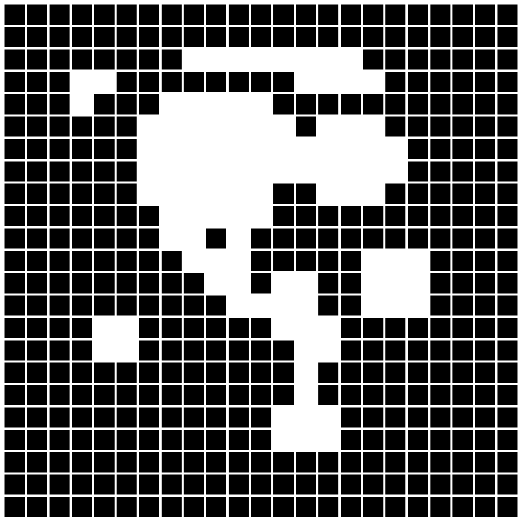
- **Definition**

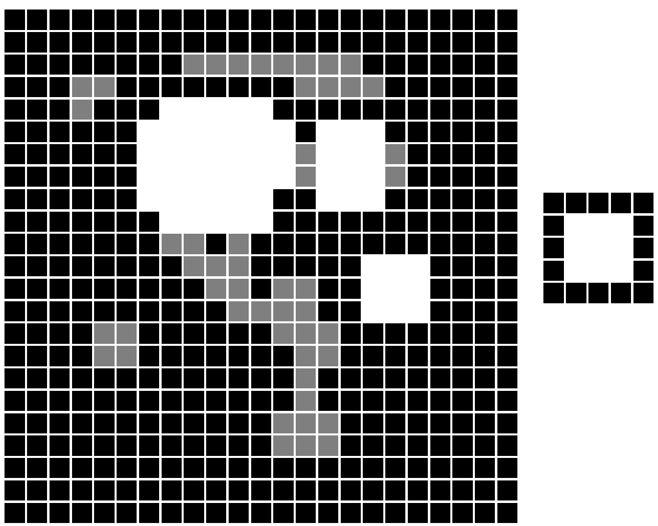
$$O_B(X) = D_{\check{B}}(E_B(X)) = \left(X \ominus \check{B}\right) \oplus B = \bigcup_{B_x \subseteq X} B_x$$

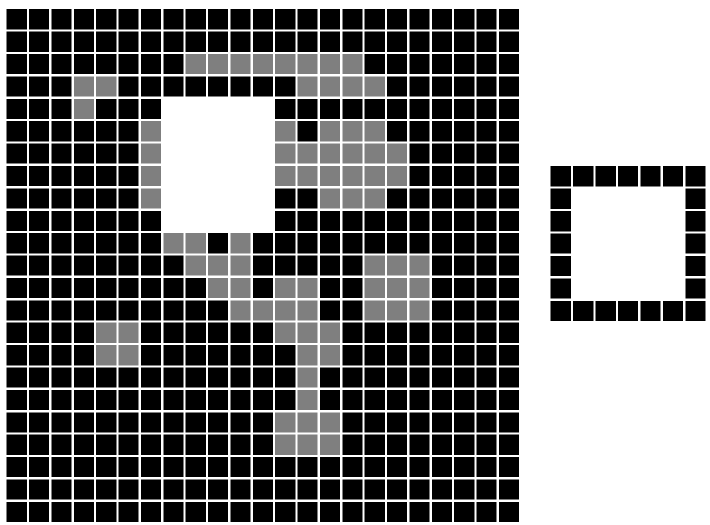
- **Illustration**

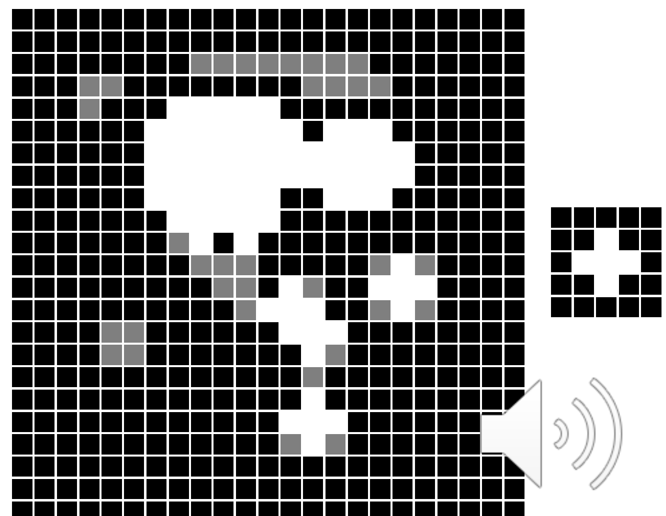


Opening with Other Structuring Elements









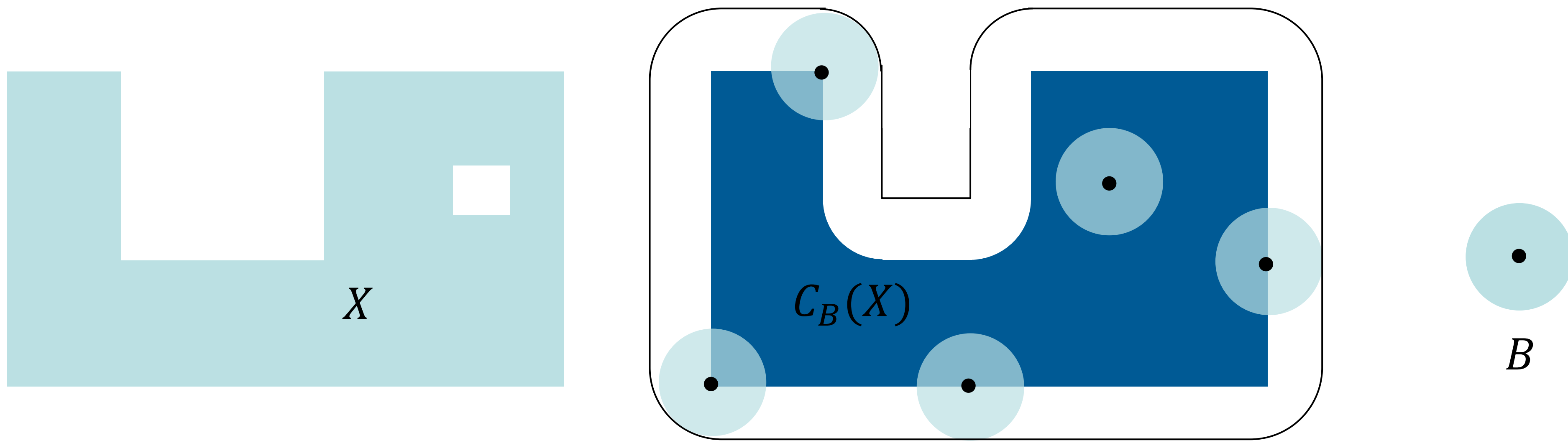
Opening, Closing

Closing

- **Definition**

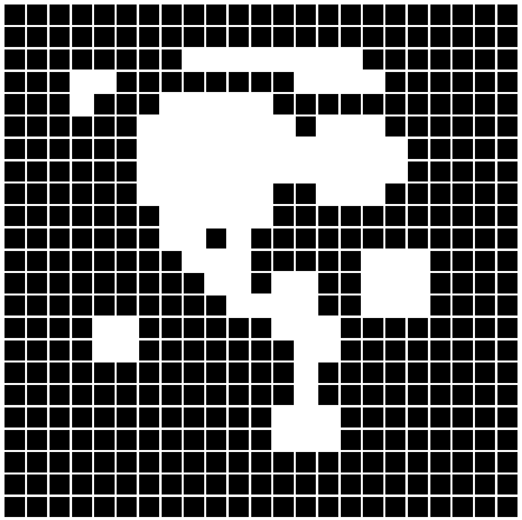
$$C_B(X) = E_{\check{B}}(D_B(X))$$

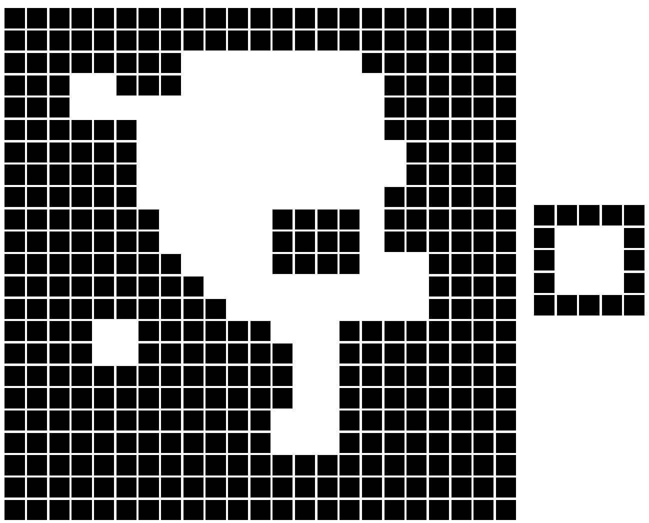
- **Illustration**

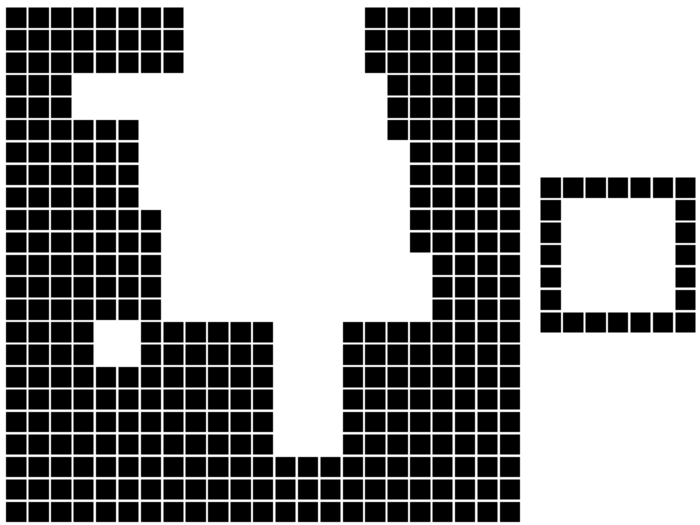


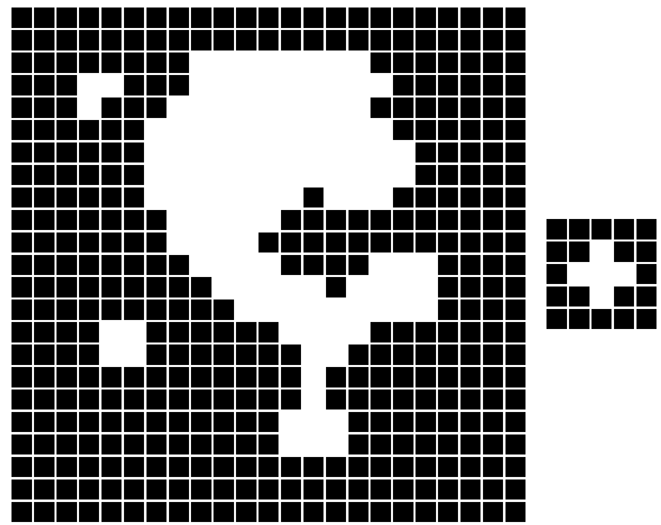
Opening, Closing

Closing with Other Structuring Elements









Opening, Closing

Main Properties

- **Extensivity, Anti-Extensivity**

  $$O_B(X) \subseteq X \subseteq C_B(X)$$

- **Increasing**

  $$X \subseteq Y \Rightarrow O_B(X) \subseteq O_B(Y)$$
  $$X \subseteq Y \Rightarrow C_B(X) \subseteq C_B(Y)$$

- **Idempotence**

  $$O_B(O_B(X)) = O_B(X)$$
  $$C_B(C_B(X)) = C_B(X)$$

- **Duality with respect to the Complementation**

  $$O_B(X) = (C_B(X^c))^c$$

- **Duality with respect to the Adjunction**

  $$C_{\check{B}}(X) \subseteq Y \Leftrightarrow X \subseteq O_B(Y)$$

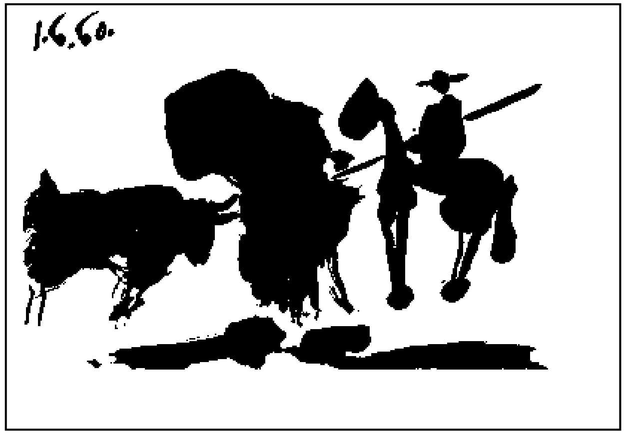
- **Translation Invariance**
- **Compatibility with Scales**

- **Decreasing, Increasing with respect to the SE**

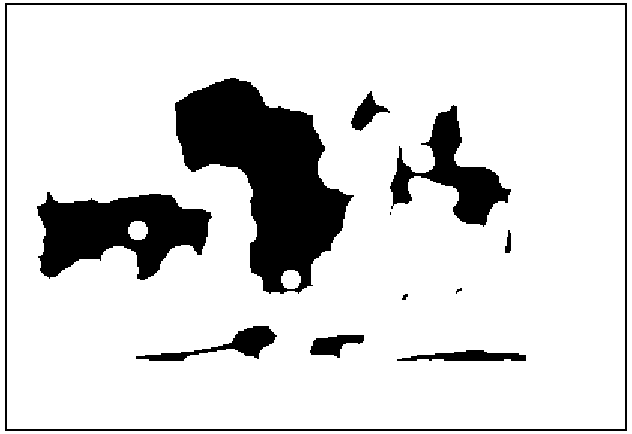
  $$B \subseteq B' \Rightarrow O_{B'}(X) \subseteq O_B(X)$$
  $$B \subseteq B' \Rightarrow C_B(X) \subseteq C_{B'}(X)$$

Erosion, Dilation, Opening, Closing

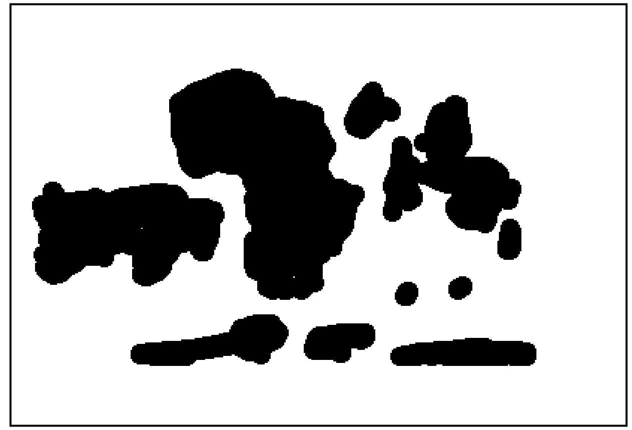
- **Comparison**
  - Foreground is black, background is white



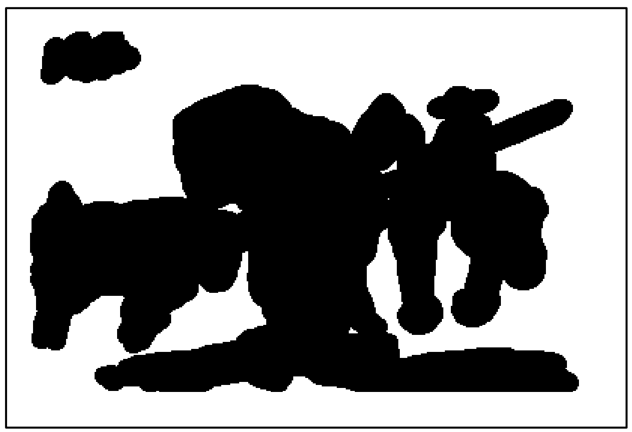
original



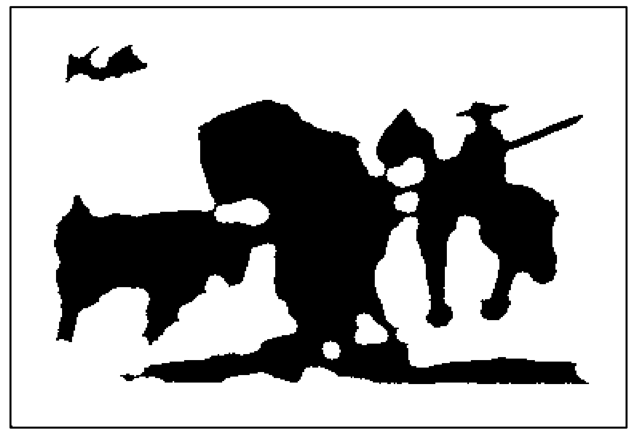
erosion



opening



dilation

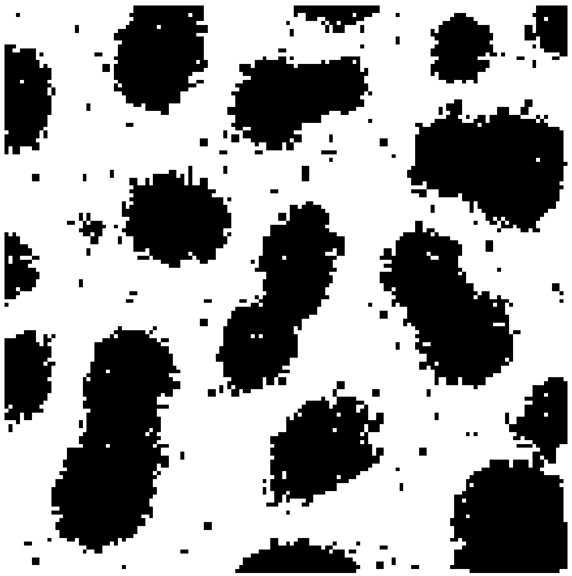


closing

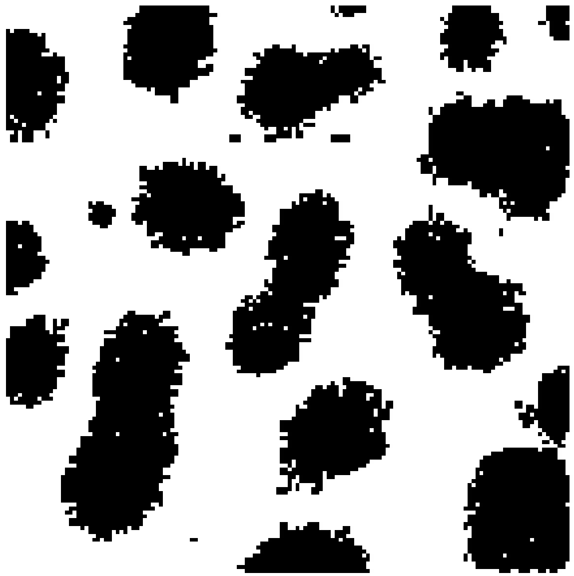
Opening, Closing

Application to Image Denoising

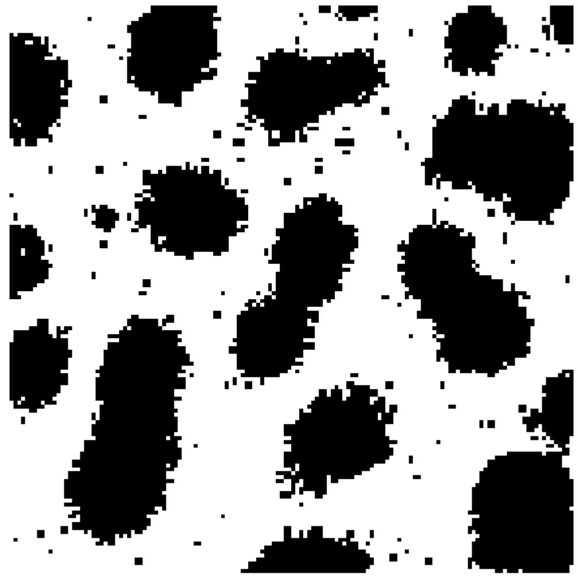
- **Salt and Pepper Noise Reduction**
  - Using closing or opening



original



closing
(pepper removed)

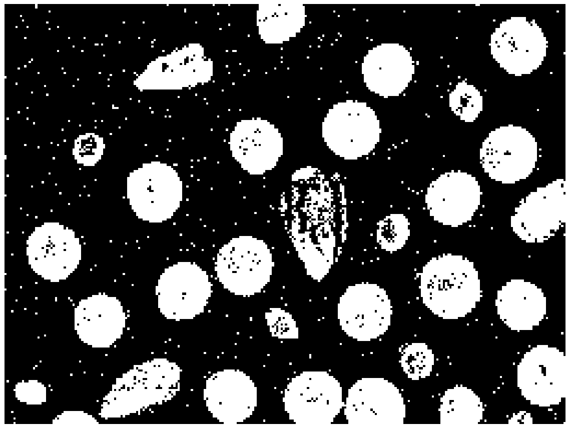


opening
(salt removed)

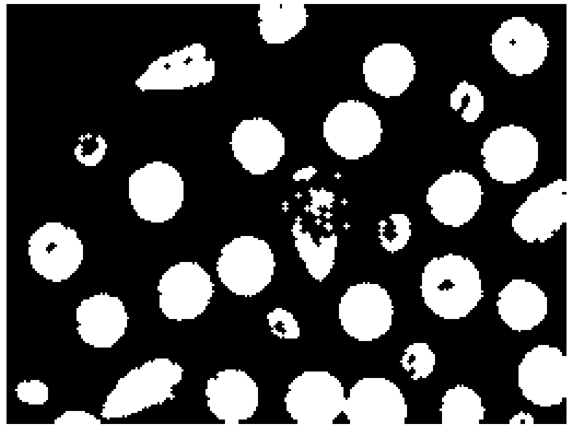
Opening, Closing

Application to Image Denoising

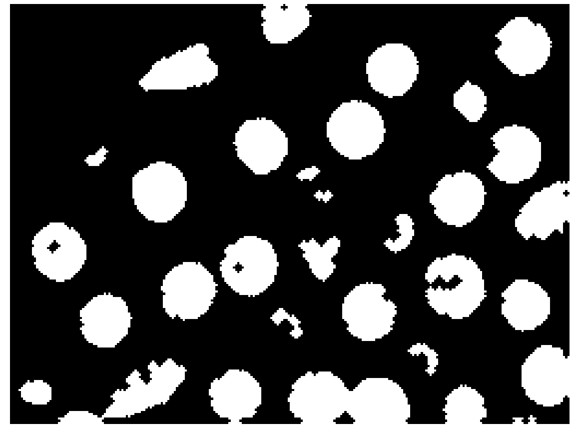
- **Salt and Pepper Noise Reduction**
  - Using alternate filters



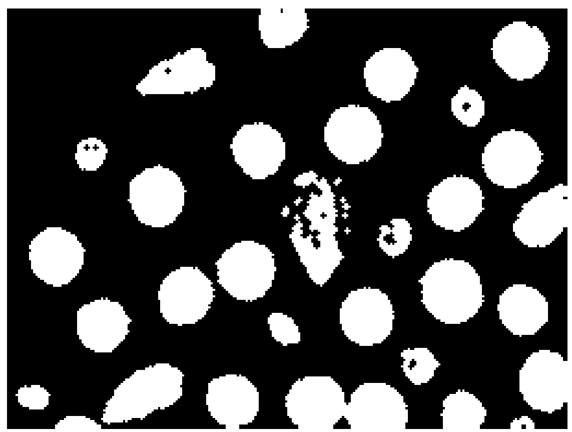
original



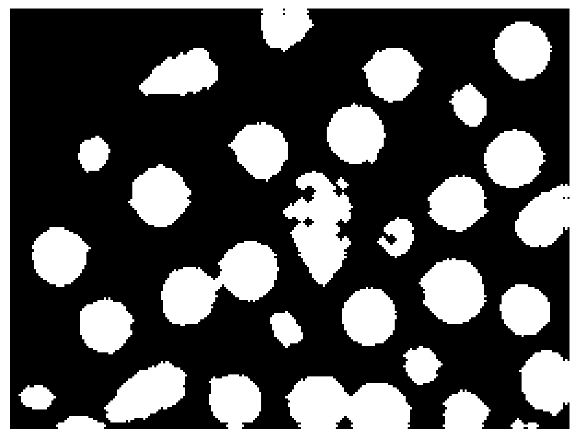
opening + closing (3x3 SE)



opening + closing (5x5 SE)



closing + opening (3x3 SE)

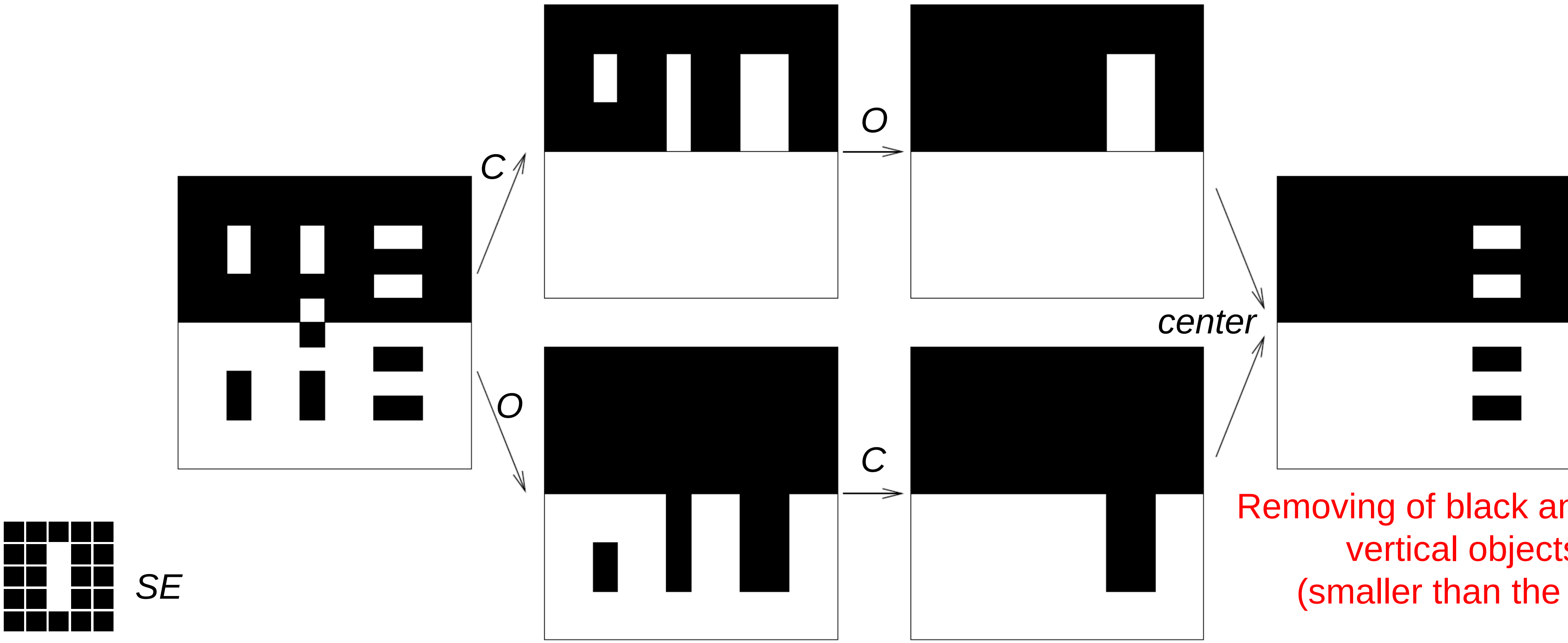


closing + opening (5x5 SE)

Opening, Closing

Morphological Center

- **Acting on Black and White Areas**
  - Auto-dual operator
  - Kind of median morphological operator
- **Illustration**
  - $Median[f, O_B(C_B(f)), C_B(O_B(f))]$



Removing of black and white vertical objects (smaller than the SE)

Opening, Closing

Granulometry

- **Aim**
  - Study of the size characteristics of sets
  - In physics: use of sieves ψ_λ of increasing meshes $\lambda > 0$

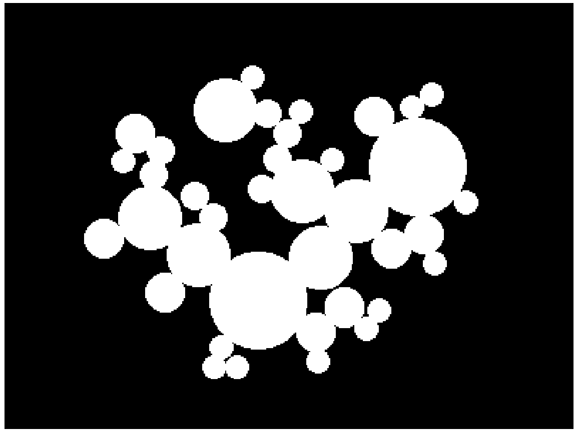
- **Granulometry by Openings**
  - Physical sieving refers to openings
  - The size of the SE plays the role of the sieve meshes
  - Decreasing with respect to the parameter: $\lambda \geq \mu > 0 \Rightarrow O_{\lambda B} \leq O_{\mu B}$
  - Semi-group property: $O_{\lambda B} O_{\mu B} = O_{\mu B} O_{\lambda B} = O_{\sup(\lambda,\mu) B}$

- **Anti-Granulometry by Closings**
  - Increasing with respect to the parameter: $\lambda \geq \mu > 0 \Rightarrow C_{\lambda B} \geq C_{\mu B}$
  - Semi-group property: $C_{\lambda B} C_{\mu B} = C_{\mu B} C_{\lambda B} = C_{\sup(\lambda,\mu) B}$

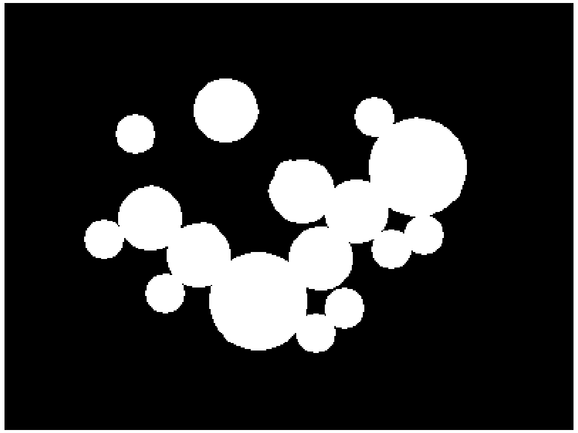
Opening, Closing

Granulometry

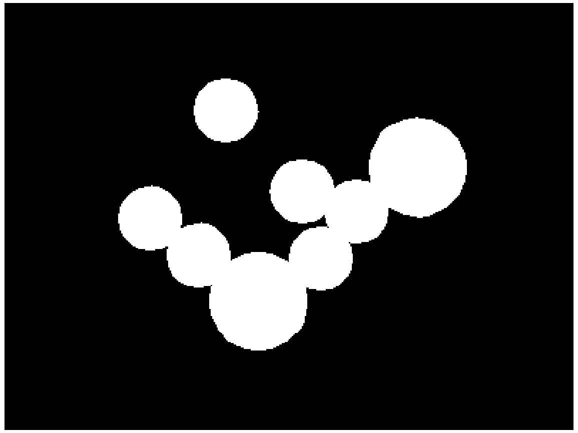
- **Illustration**



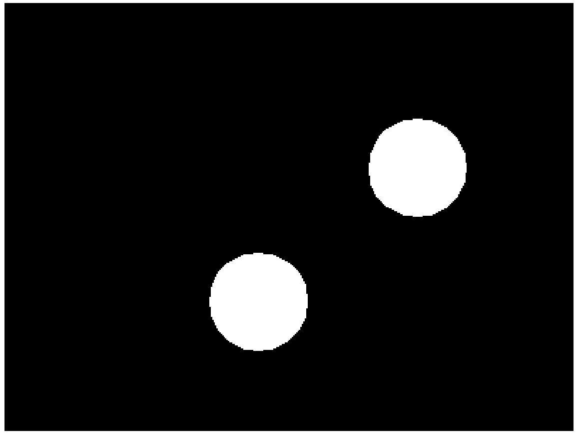
original



opening of size 10



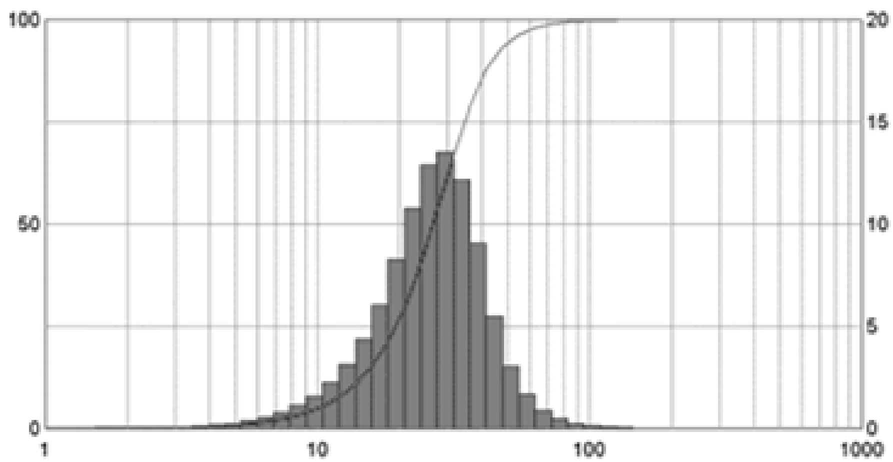
opening of size 15



opening of size 25

Application to Particle Size Distribution

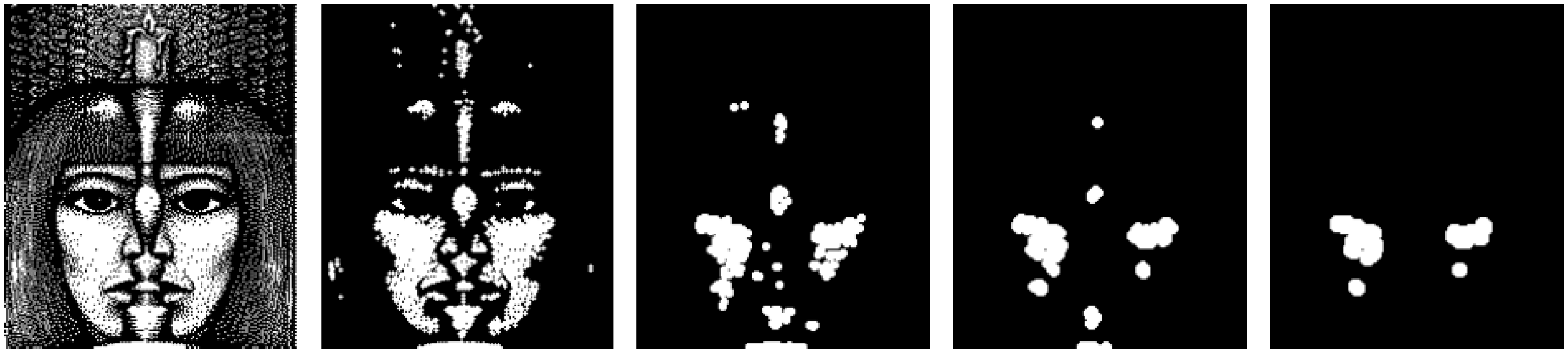
- **Size Measurement**
  - Cumulative Distribution function: $F_X(\lambda) = 1 - \frac{\mu(O_{\lambda B}(X))}{\mu(X)}$
  - Density function / Pattern spectrum: $f_X(\lambda) = F'_X(\lambda)$



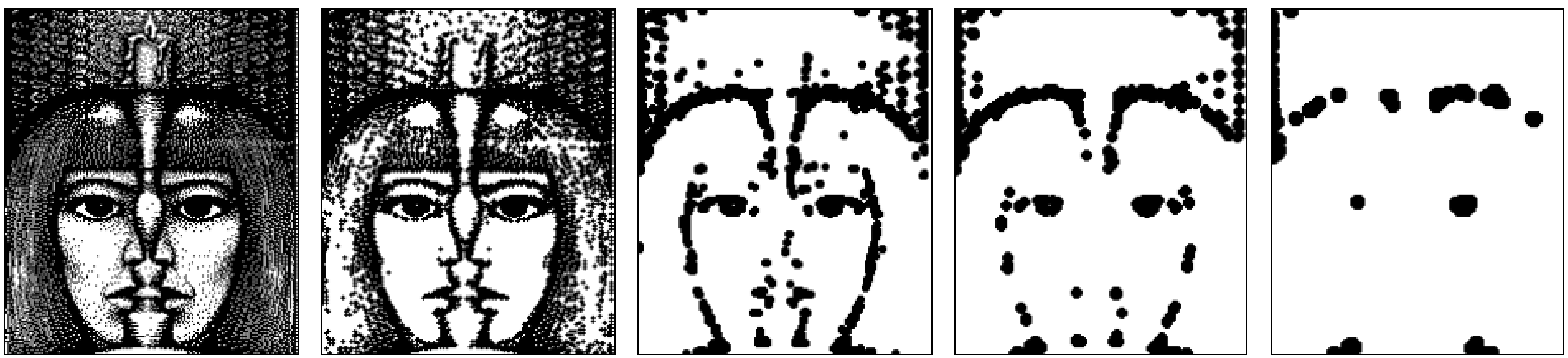
Opening, Closing

Application to Particle Size Distribution

- **Granulometry**



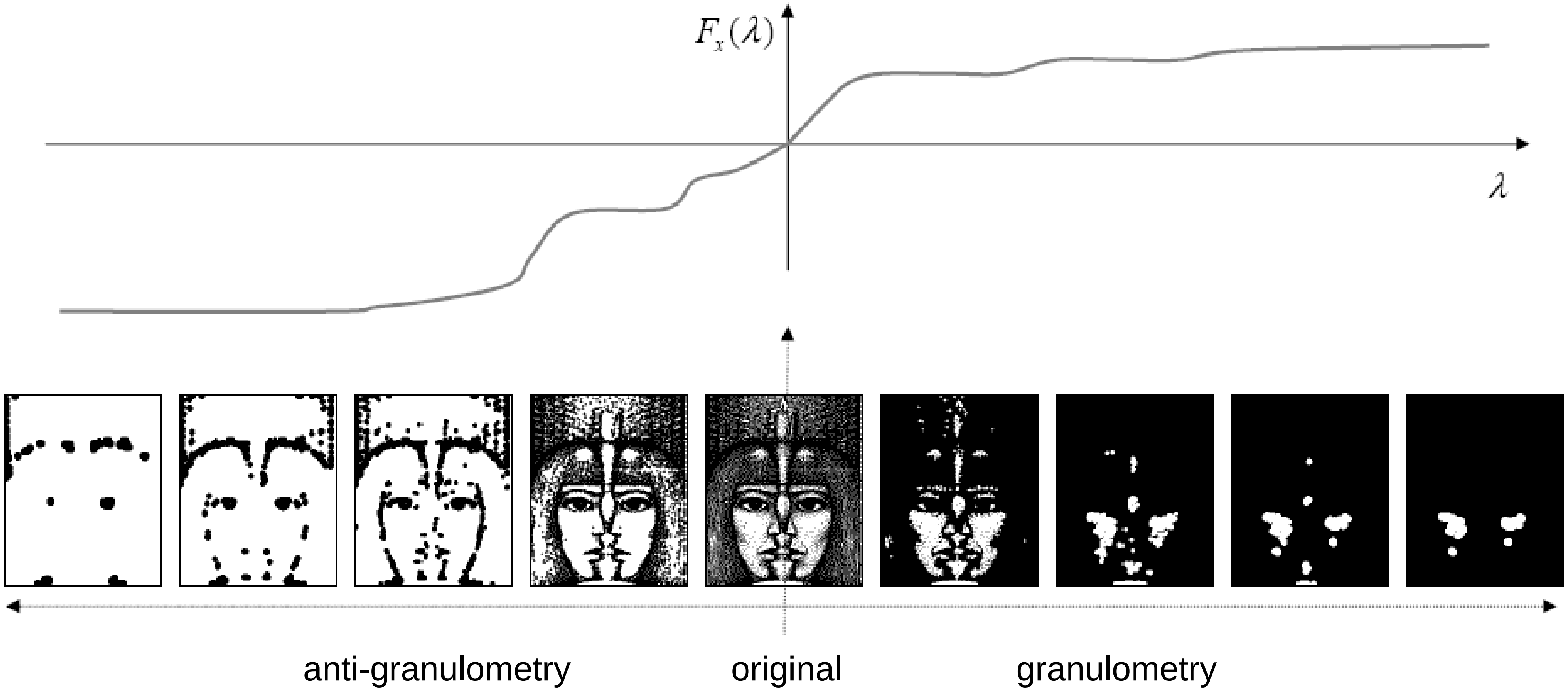
- **Anti-Granulometry**



Opening, Closing

Application to Particle Size Distribution

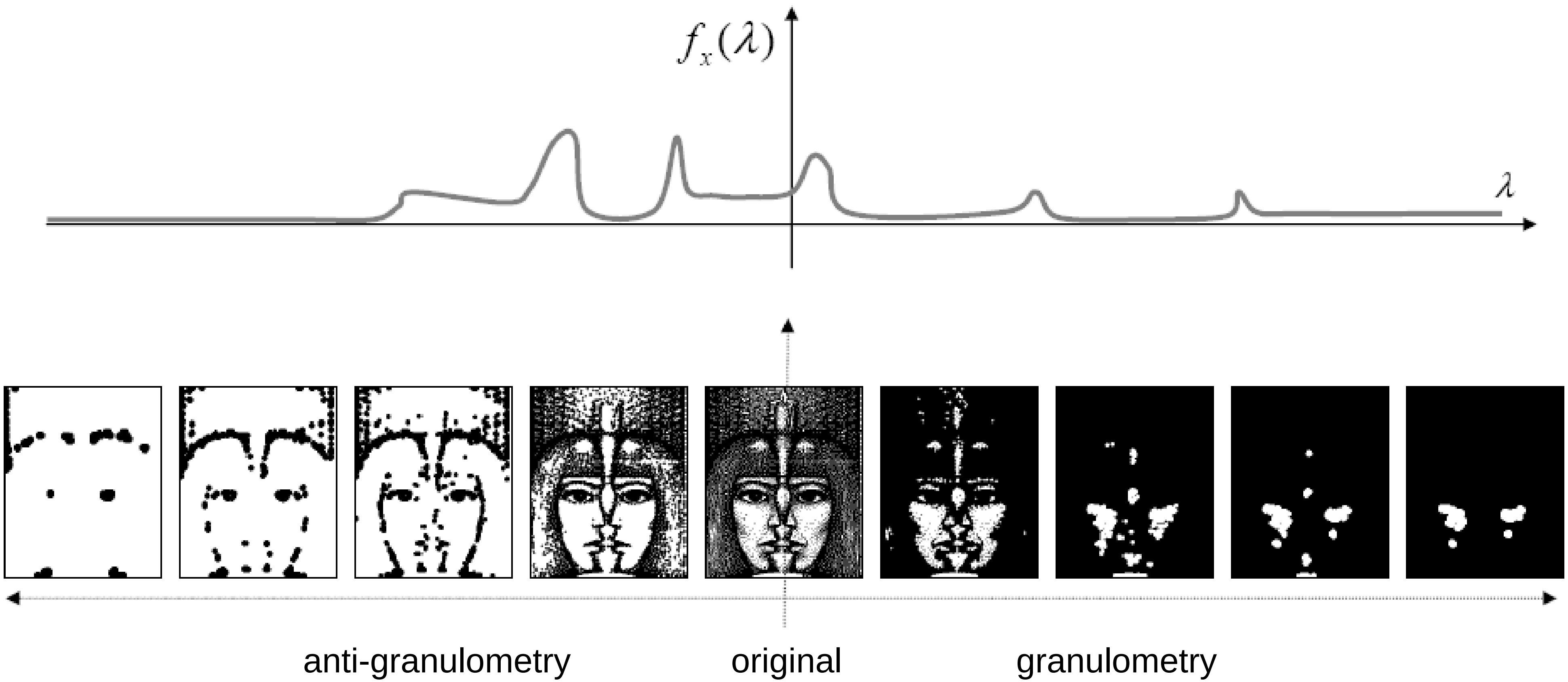
- **Cumulative Distribution Function**



Opening, Closing

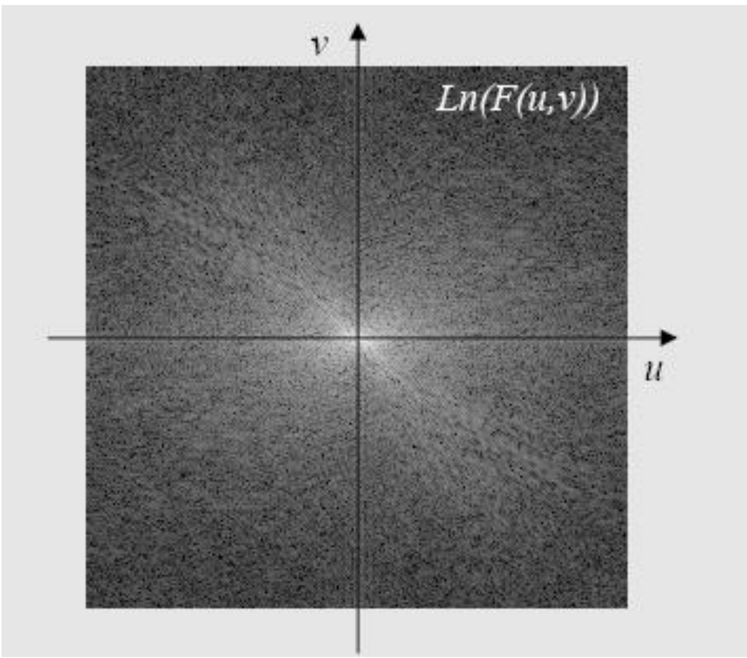
Application to Particle Size Distribution

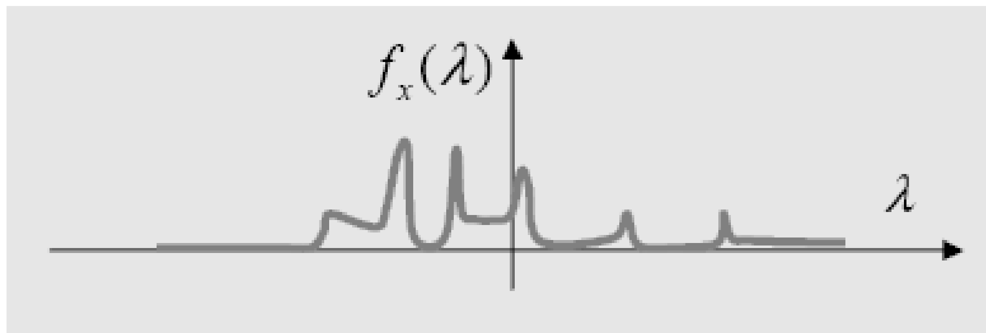
- **Pattern Spectrum**



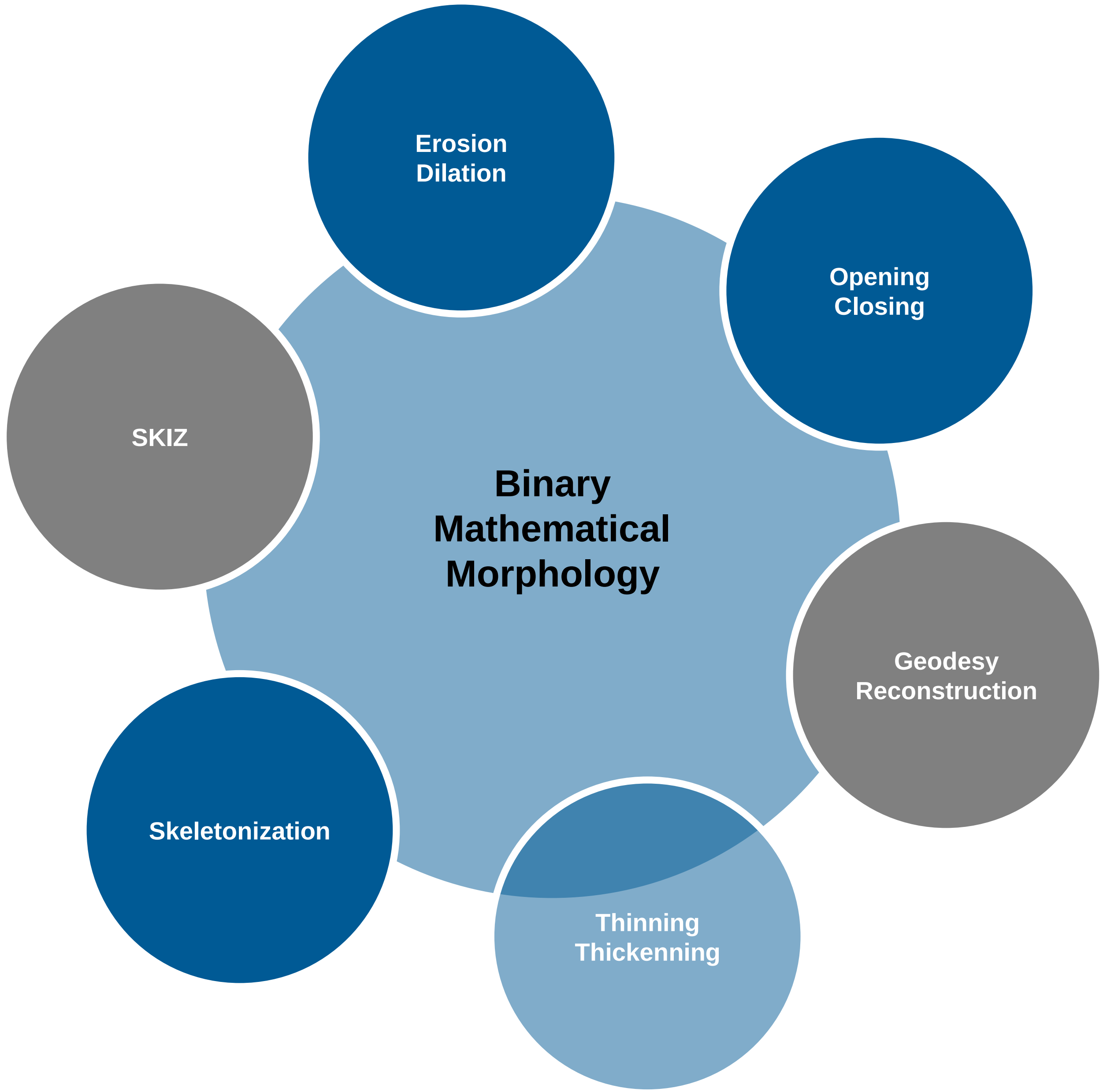
Fourier Analysis vs. Granulometric Analysis

- **Fourier Analysis**
  - Components = complex sinusoids
- **Granulometric Analysis**
  - Components = black and white discs





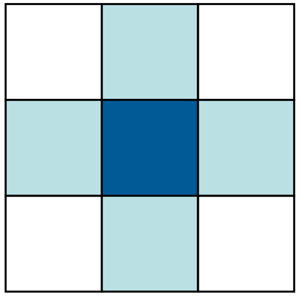
Binary Mathematical Morphology



Thinning, Thickening

Hit-or-Miss Transform

- **Bi-Phase Structuring Element**
  - $B = (B_1, B_2)$ with $B_1 \cap B_2 = \emptyset$



B

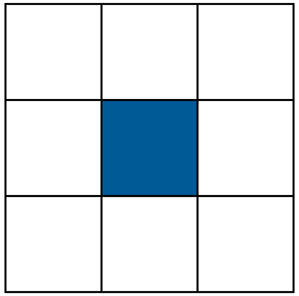
- **Hit or Miss Principle**
  - Does B_1 fit the object?

  while simultaneously
  - Does B_2 miss the object, i.e. fit the background?

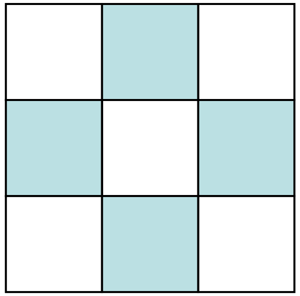
$$\{x;\ (B_1)_x \subseteq X, (B_2)_x \subseteq X^c\}$$

- **Definition**

$$X \otimes B = E_{B_1}(X) \cap E_{B_2}(X^c)$$



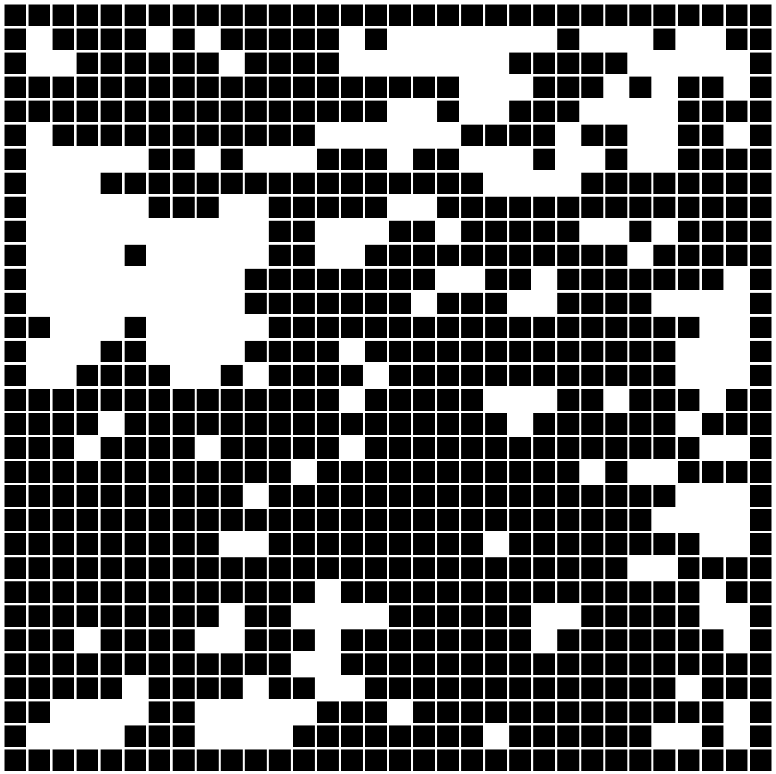
B_1: ‘hit’ part



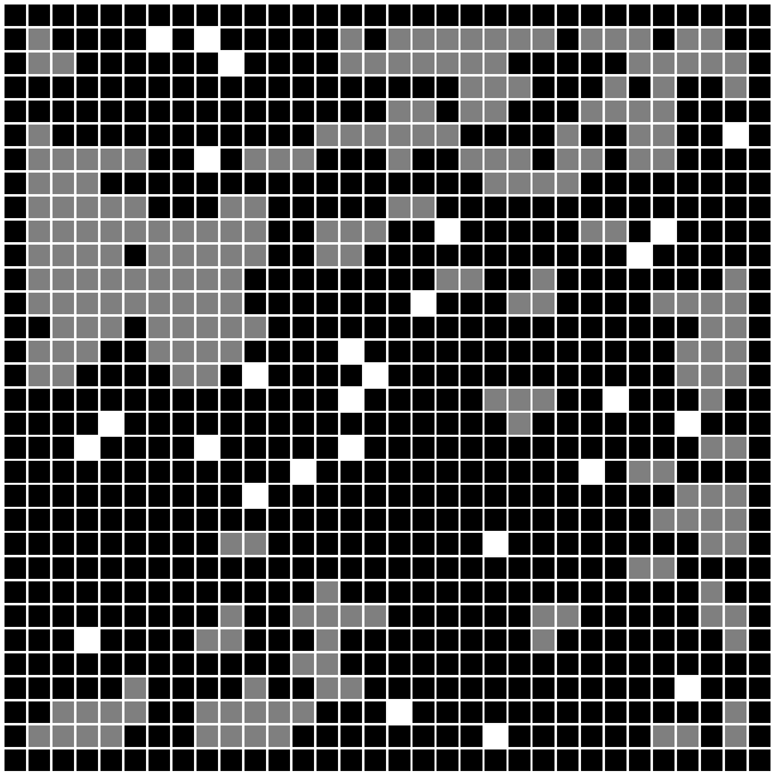
B_2: ‘miss’ part

Hit-or-Miss Transform

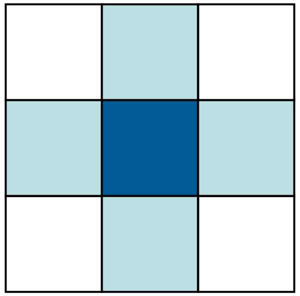
- **Illustration**
  - Isolated points in 4-connectivity



original



hit-or-miss



B

Thinning, Thickening

Definitions

- **Thinning**

$$X \circ B = X \setminus (X \otimes B)$$

- **Thickening**

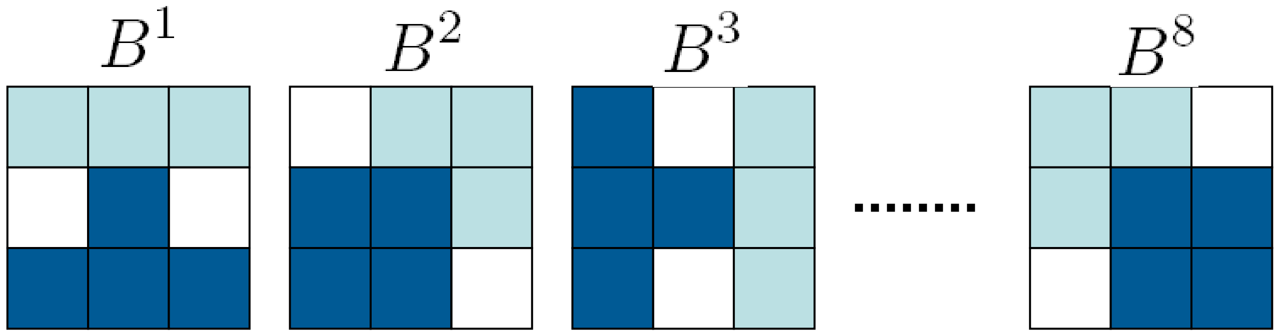
$$X \odot B = X \cup (X \otimes B)$$

- **Depending on the SE (or series of SE), very different results can be achieved**
  - Pruning
  - Skeletonization
  - Convex hull
  - …

Thinning, Thickening

Illustration

- **Isotropic Processing: Using Specific Series of SE**



original

thinning

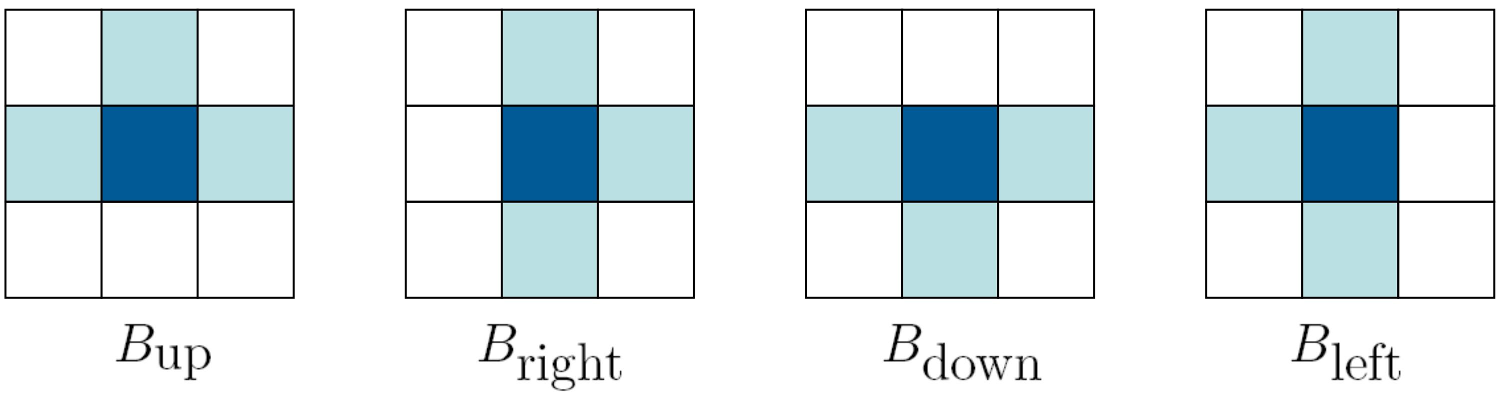
thickening

Thinning, Thickening

Pruning

- **Remove End Points by a Sequence of Thinnings**

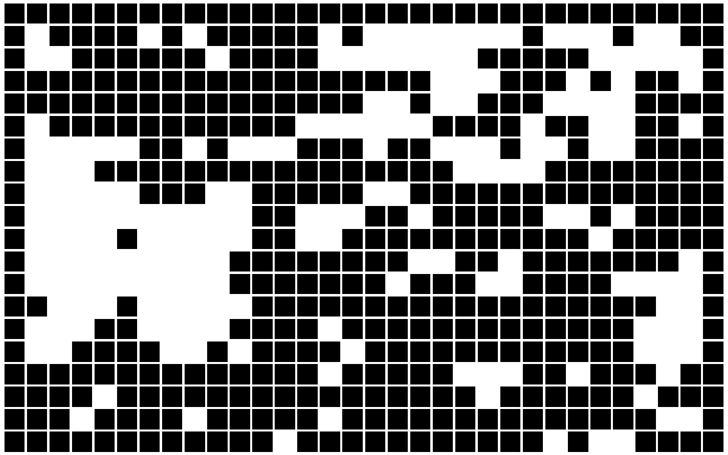
$$\textcolor{red}{\text{1 iteration}} = (((X \circ B_{\text{up}}) \circ B_{\text{right}}) \circ B_{\text{down}}) \circ B_{\text{left}}$$



Thinning, Thickening

Pruning

- **Illustration**



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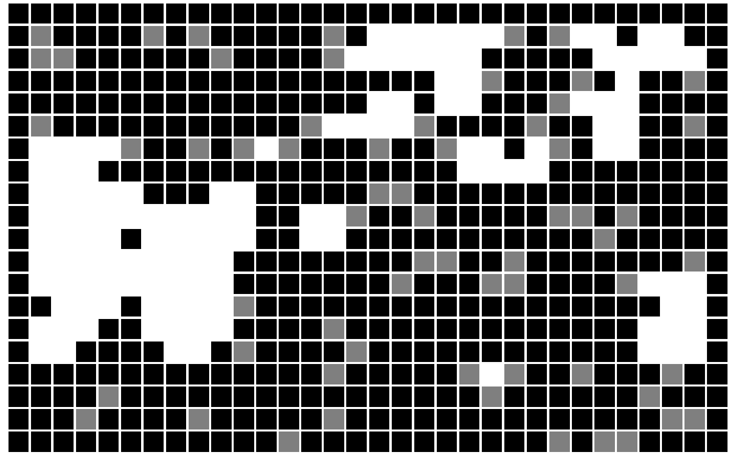
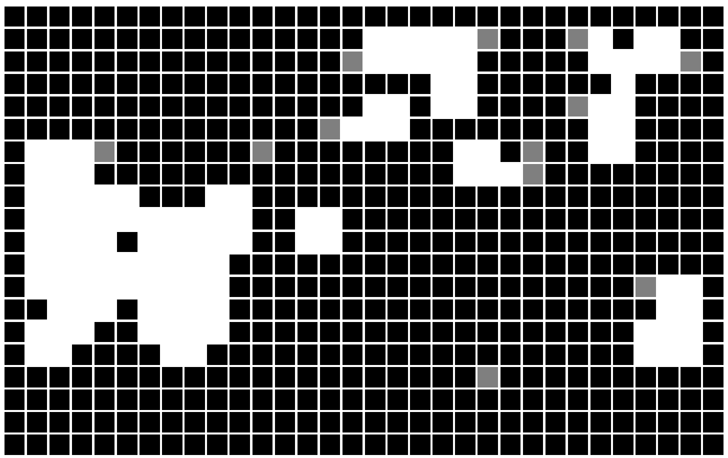
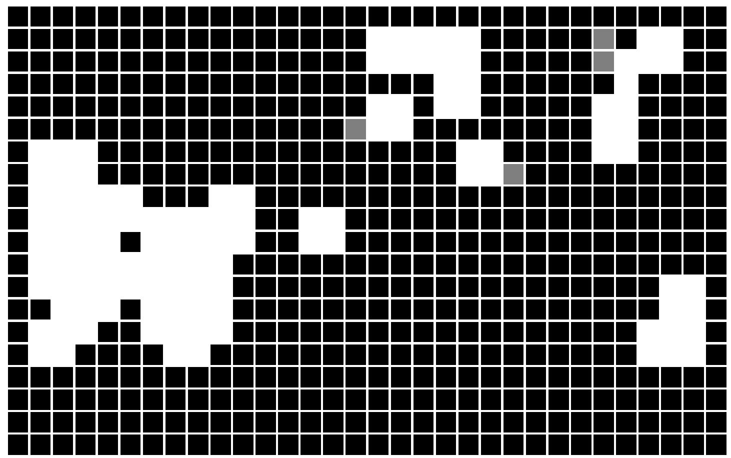


1st iteration



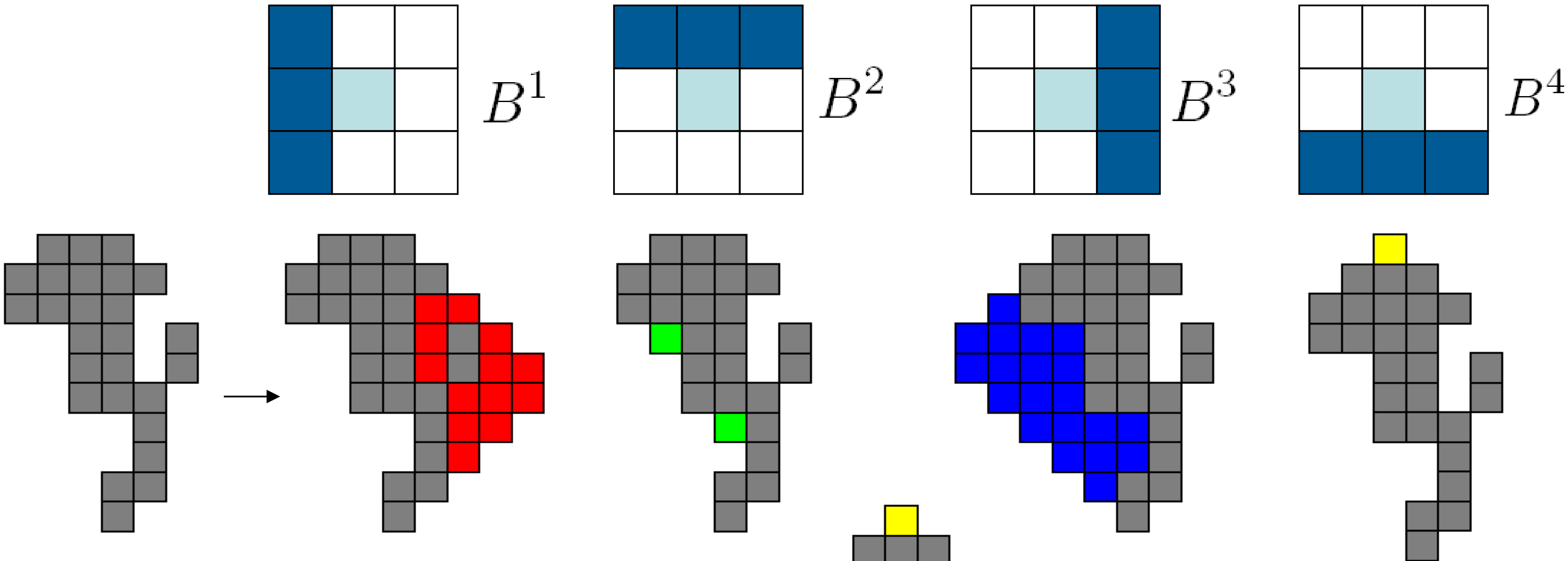
2nd iteration



3rd iteration: idempotence

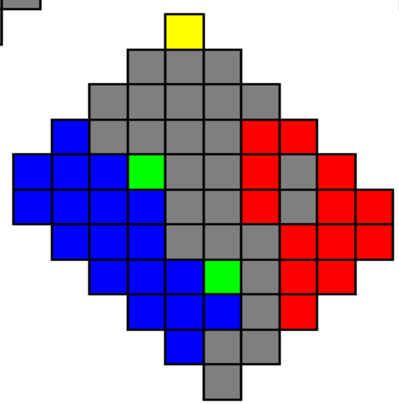
Convex Hull

- **Smallest Convex Set Containing the Object**
  - Union of thickenings, each up to idempotence



Original shaper

Thickening with the first mask

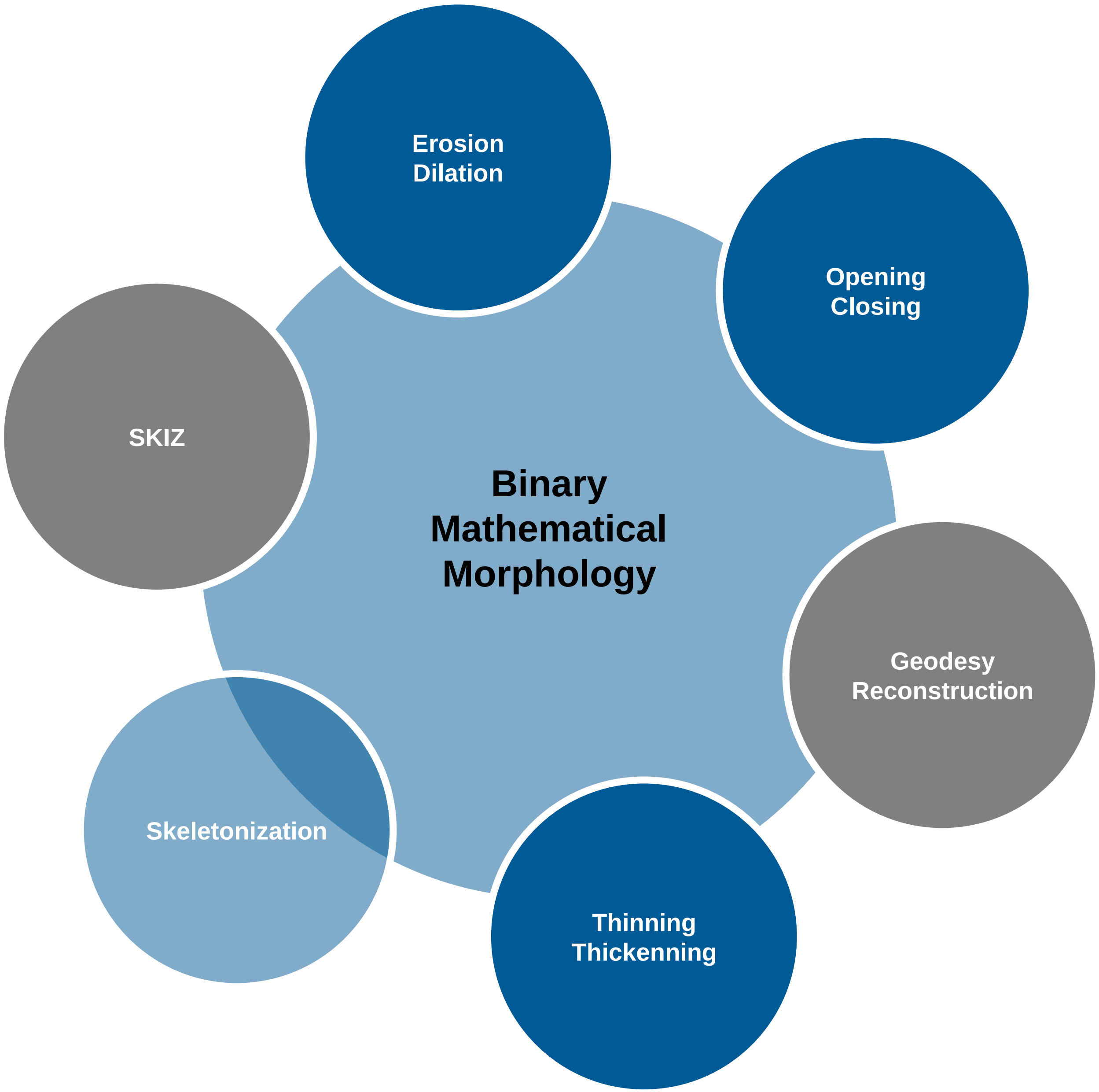


Union of four thickenings

Convex Hull

- **Illustration**

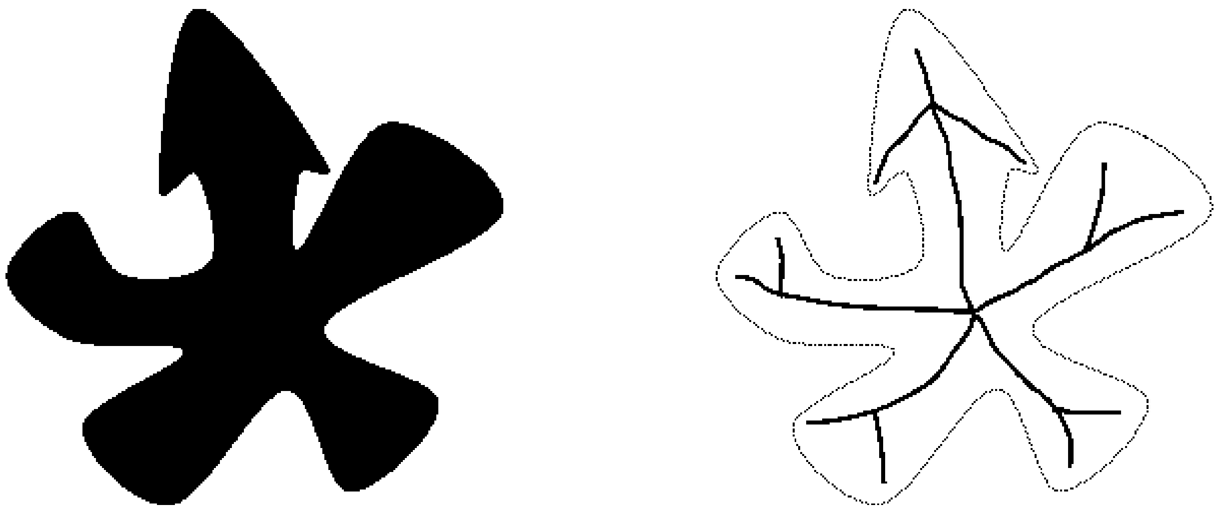
Binary Mathematical Morphology



Skeletonization

Aim

- **Compact Representation of Objects by Minimal Geometrical Information**

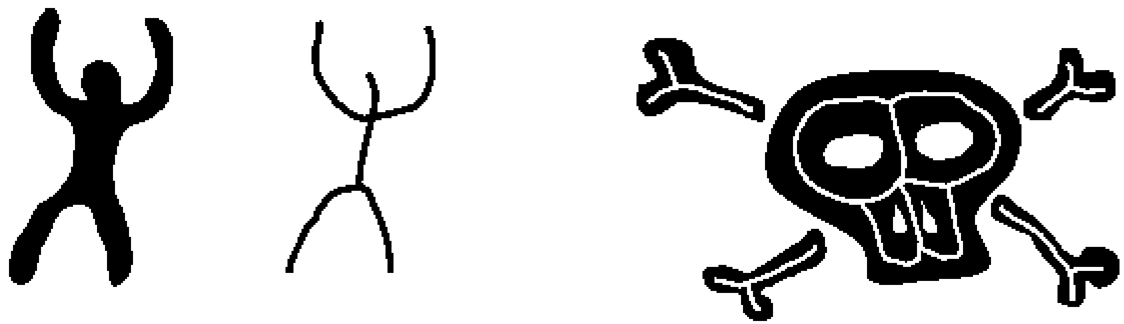


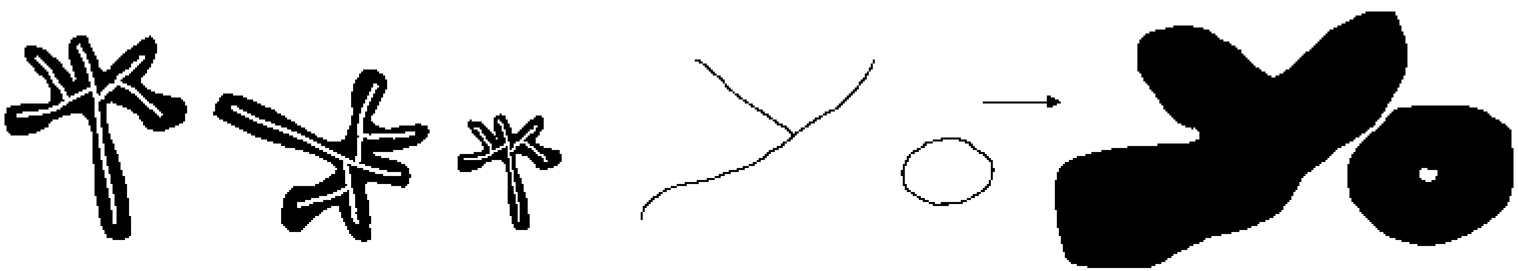
- **Different Approximations/Definition**
  - Skeleton by iterative thinnings
  - Skeleton by maximal balls

Skeletonization

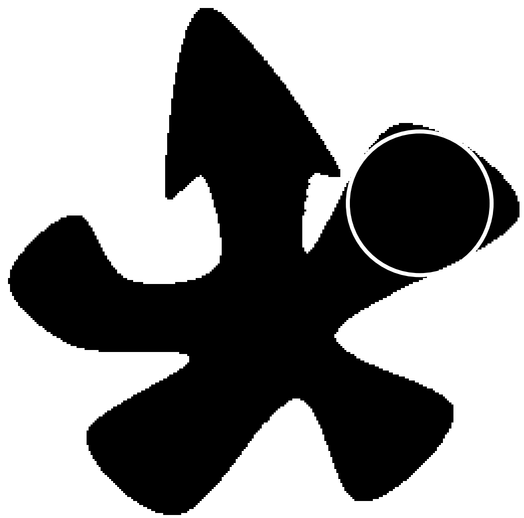
Requirements

- Thin Lines
- Geometry Preservation
- Topology Preservation
- Affine Invariance
- Invertibility
- Continuity





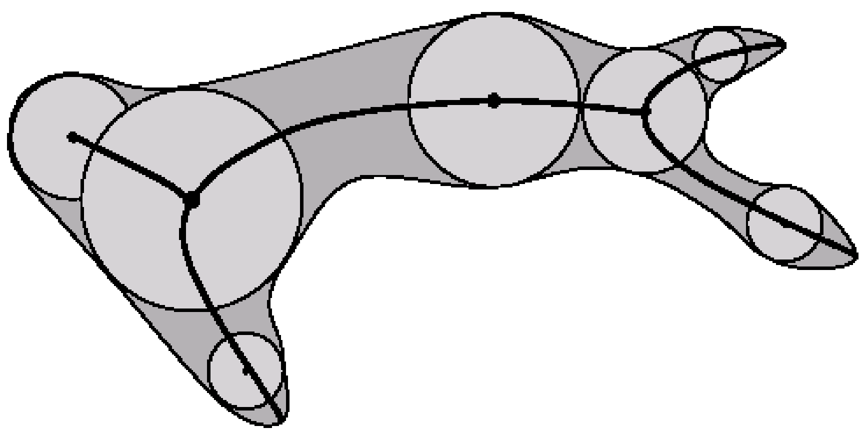
Skeleton by Maximal Balls

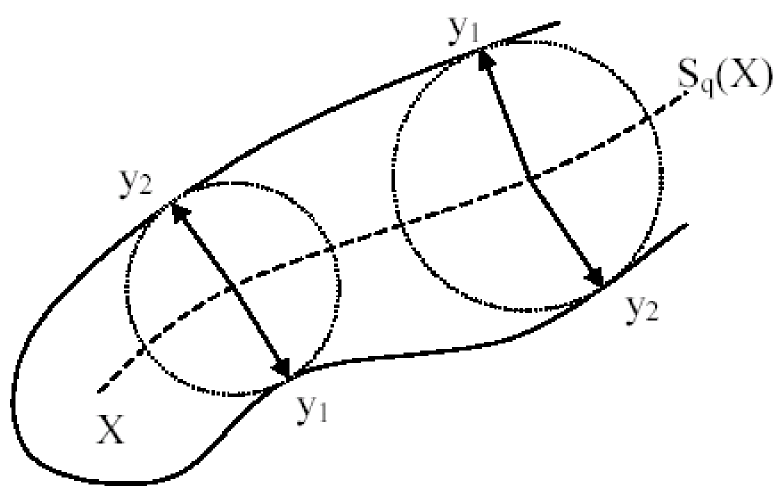


- **Maximal Ball *B***

  $$B \subseteq B' \subseteq X \Rightarrow B = B'$$

- **Skeleton**
  - Set of all centers of maximal balls
  - $s \in S(X) \Leftrightarrow (\exists y_1, y_2 \in \partial X, y_1 \neq y_2 \mid d(s, \partial X) = d(s, y_1) = d(s, y_2))$





Skeletonization

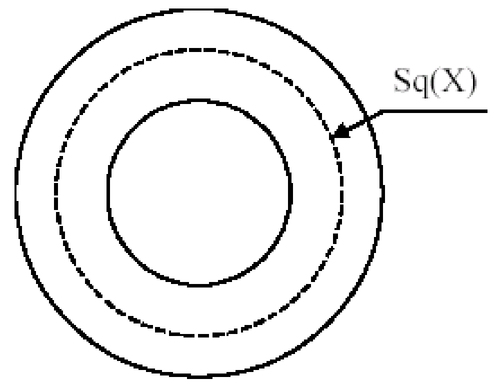
Skeleton by Maximal Balls

- **Definition in the Continuous Case**

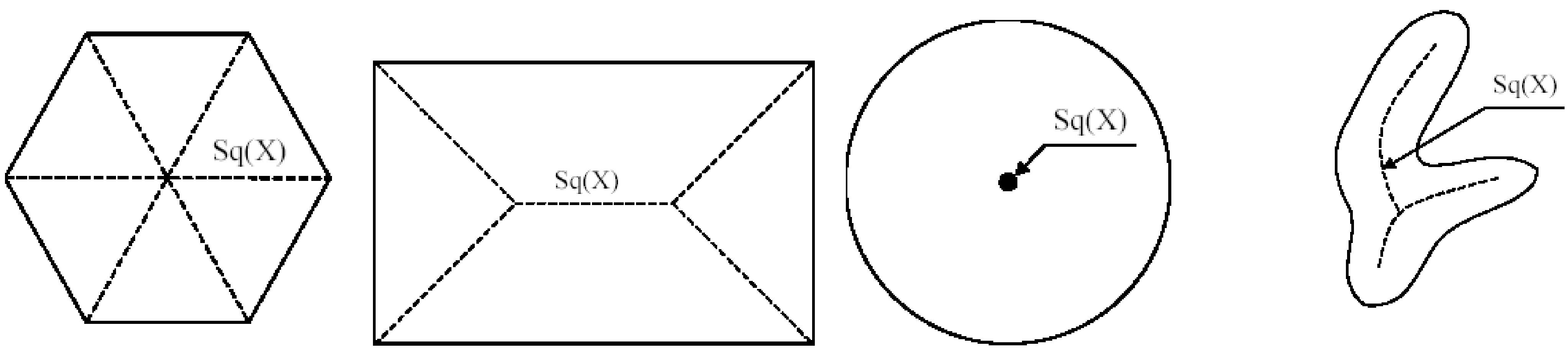
$$S(X) = \bigcup_{\lambda>0} \bigcap_{\mu>0} \{E_{B(\lambda)}(X) \setminus O_{B(\mu)}(E_{B(\lambda)}(X))\}$$

- **Definition in the Digital Case**

$$S(X) = \bigcup_{i\in\mathbb{N}} S_i(X) = \bigcup_{i\in\mathbb{N}} \{E_{B(i)}(X) \setminus O_{B(1)}(E_{B(i)}(X))\}$$



- **Illustration**

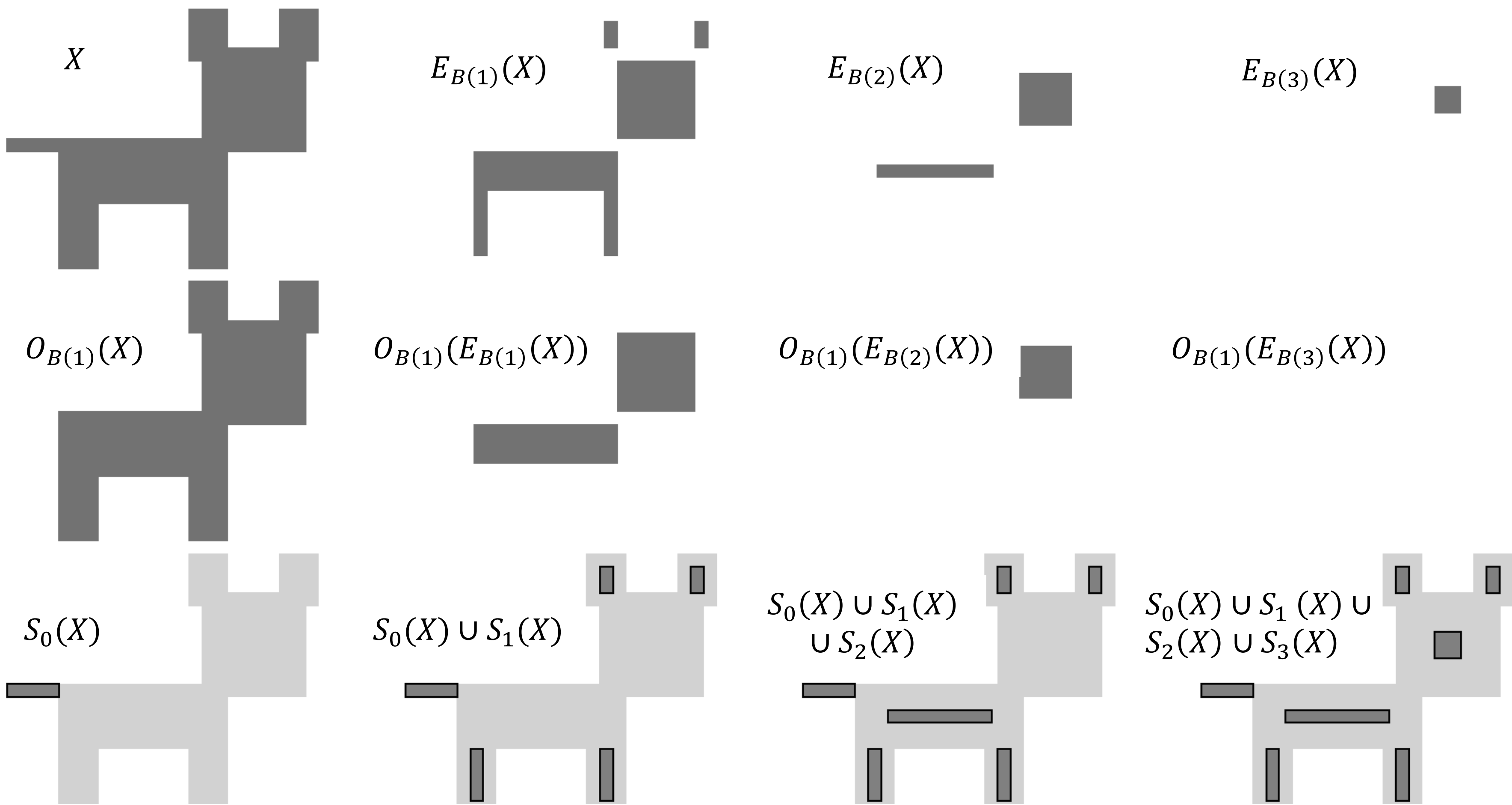


Skeletonization

Skeleton by Maximal Balls

$$S(X) = \bigcup_{i\in\mathbb{N}} S_i(X) = \bigcup_{i\in\mathbb{N}} \{E_{B(i)}(X) \backslash O_{B(1)}(E_{B(i)}(X))\}$$

- **Illustration in the Digital Case**



Properties

- **Anti-Extensivity**

  $$S(X) \subseteq X$$

- **Idempotence**

  $$S\big(S(X)\big) = S(X)$$

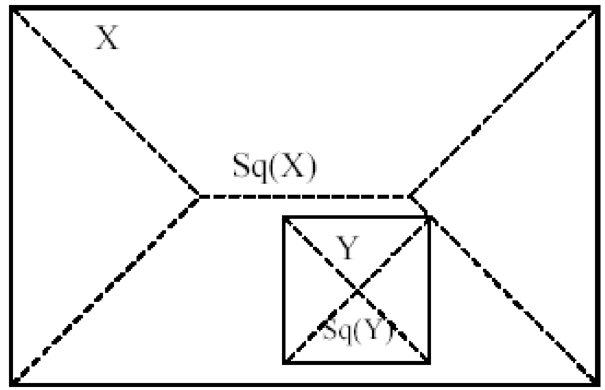
- **Invertibility**

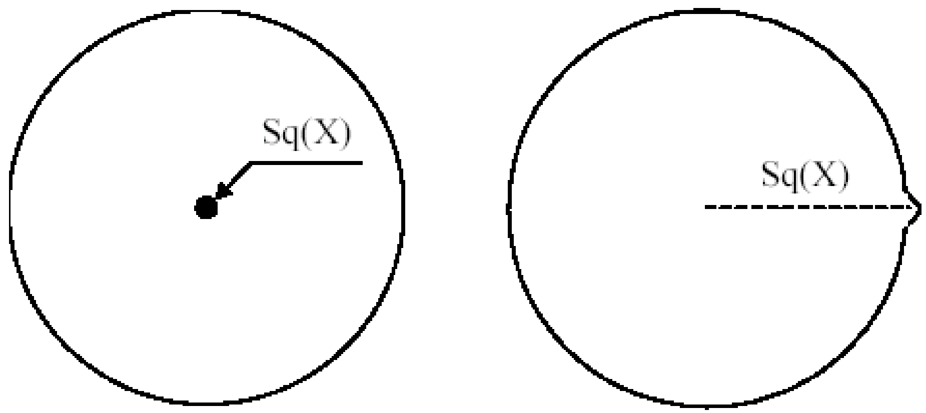
  $$X = \bigcup_i D_{B(i)}(S_i(X))$$

- **No Increasing, No Decreasing**

- **No Continuity**

- **Connectivity only in Continuous Case**

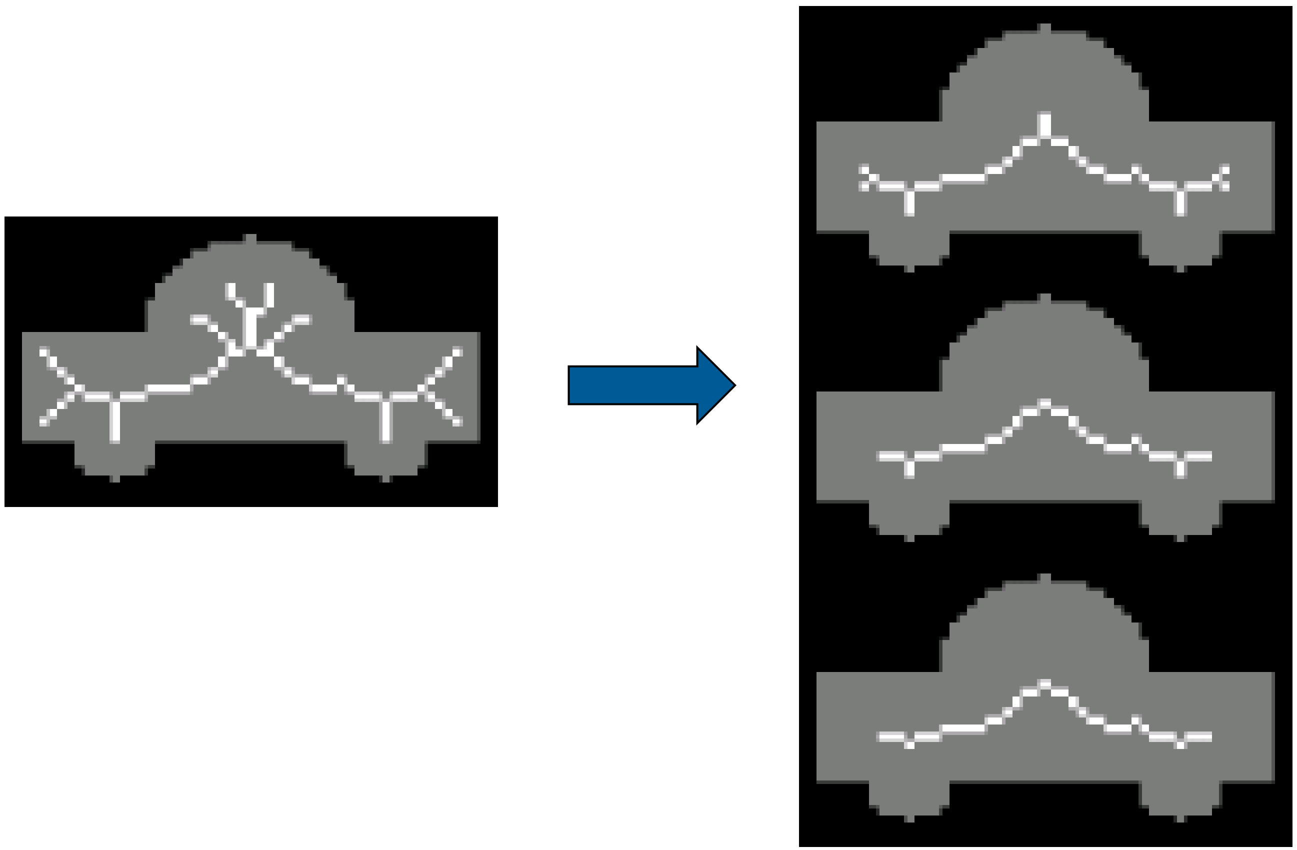




Skeletonization

Needs of Pruning

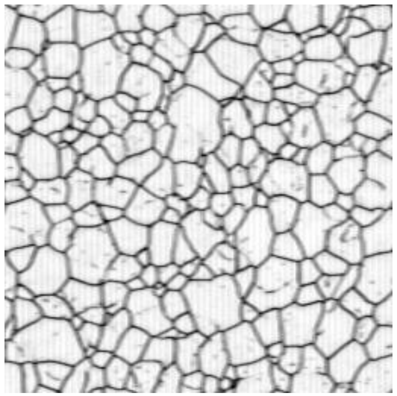
- Iterative Removing of Extremal Points



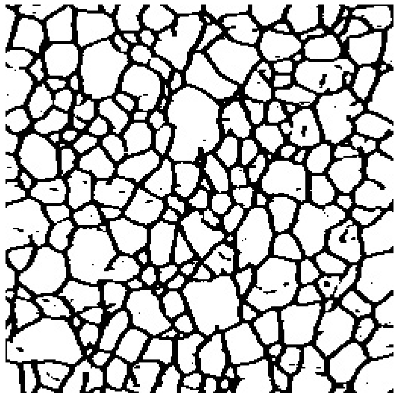
Skeletonization

Illustration

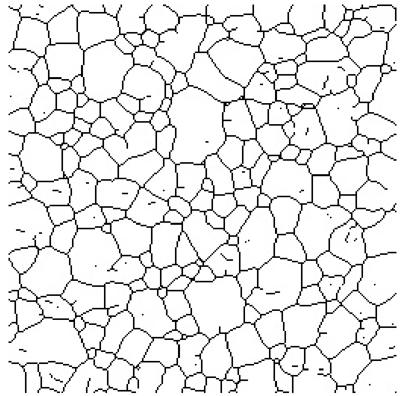
- **Metallurgic Grains and Textile Fibers**



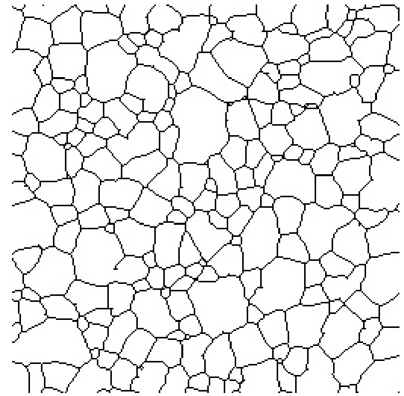
original



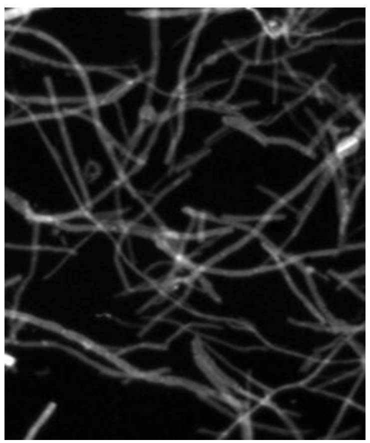
thresholding



skeletonization

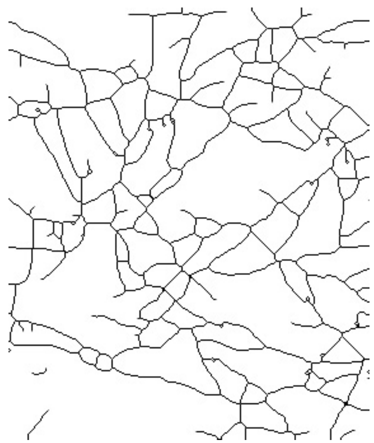


pruning

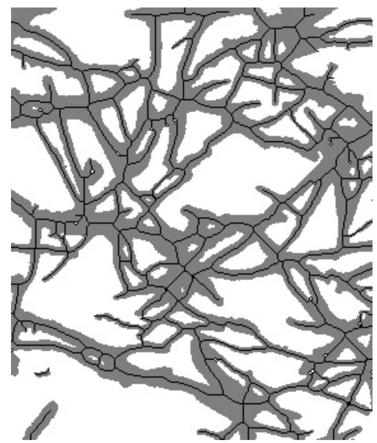


original

thresholding



skeletonization

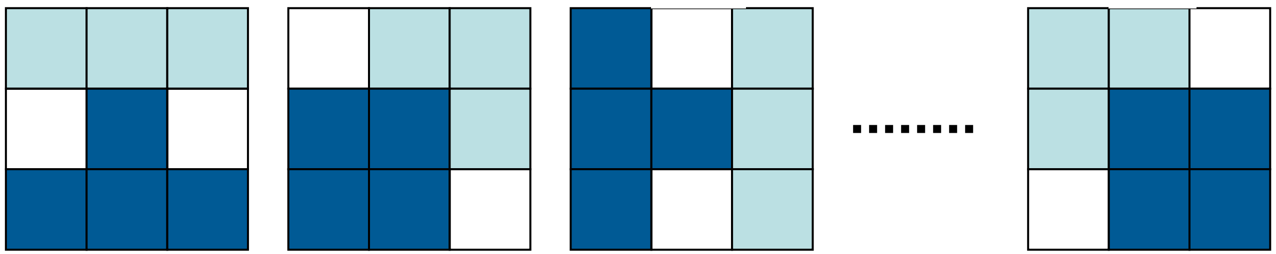


superimposition

Homotopic Skeleton

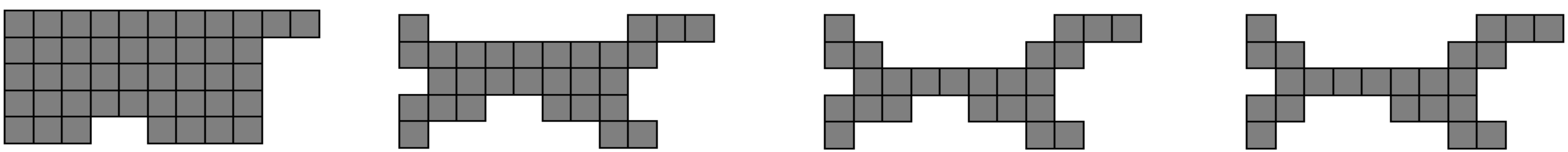
- **If Connectivity is of Importance!**
- **Isotropic Processing: Using Specific Series of SE**

B^1 B^2 B^3 …….. B^8



- **Iterative Thinnings**

X $(\ldots(X \bigcirc B^1)\ldots \bigcirc B^8)$



original first iteration second iteration reaches idempotence

Skeletonization

Properties

- **Anti-Extensivity**

  $$S(X) \subseteq X$$

- **Idempotence**

  $$S\big(S(X)\big) = S(X)$$

- **No Invertibility**

- **No Increasing, No Decreasing**

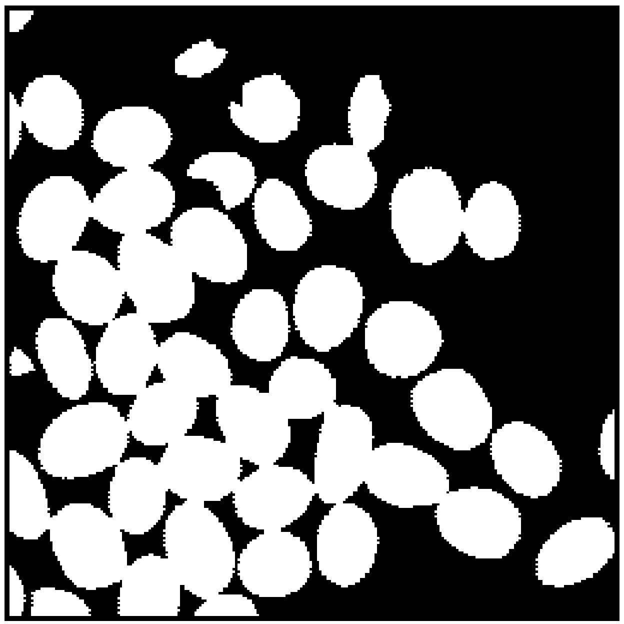
- **No Continuity**

- **Connectivity**

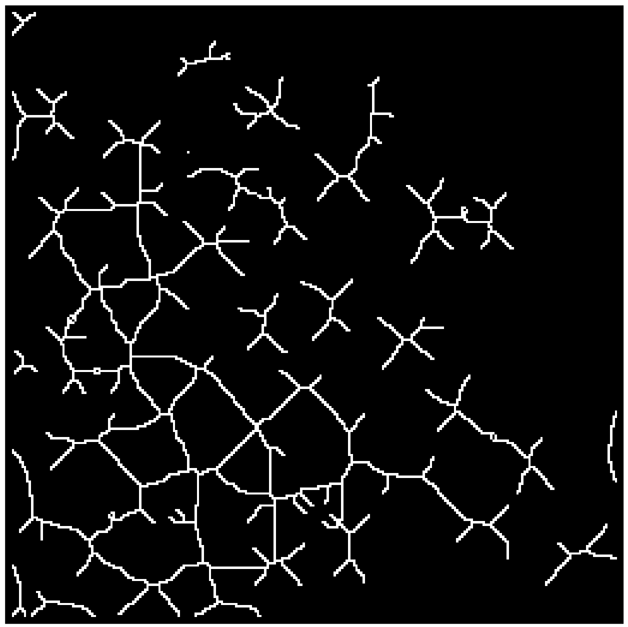
Skeletonization

Homotopic Skeleton

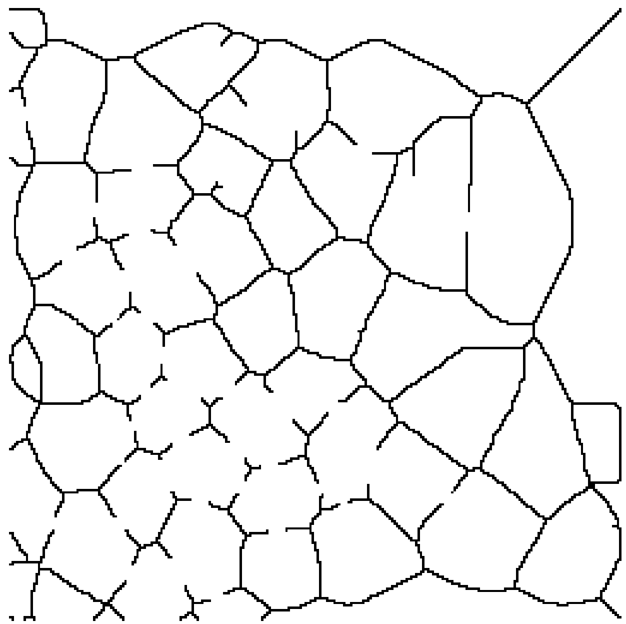
- **Illustration**



original



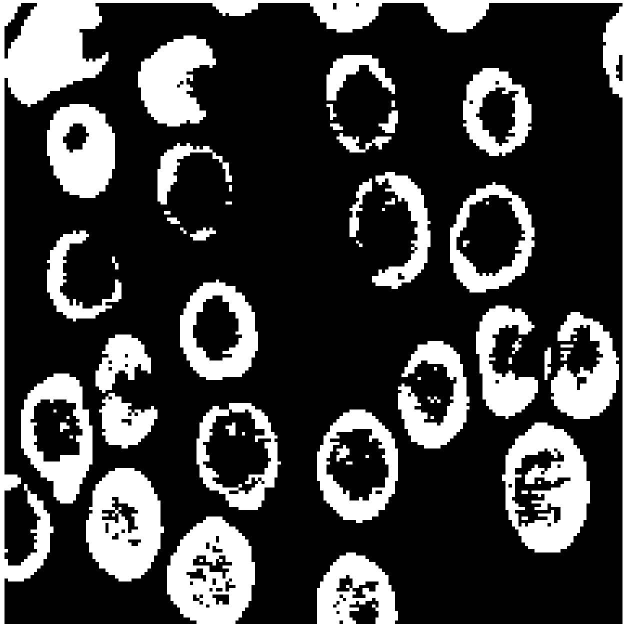
skeleton of the foreground



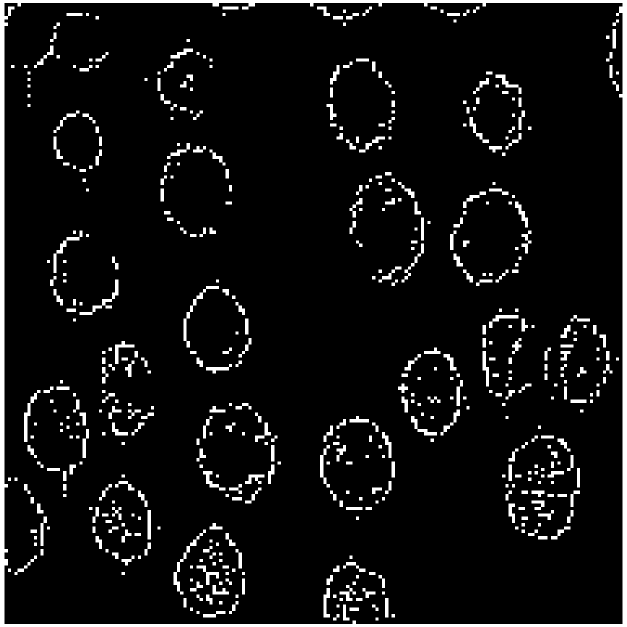
skeleton of the background

Homotopic vs. Maximal Ball Skeleton

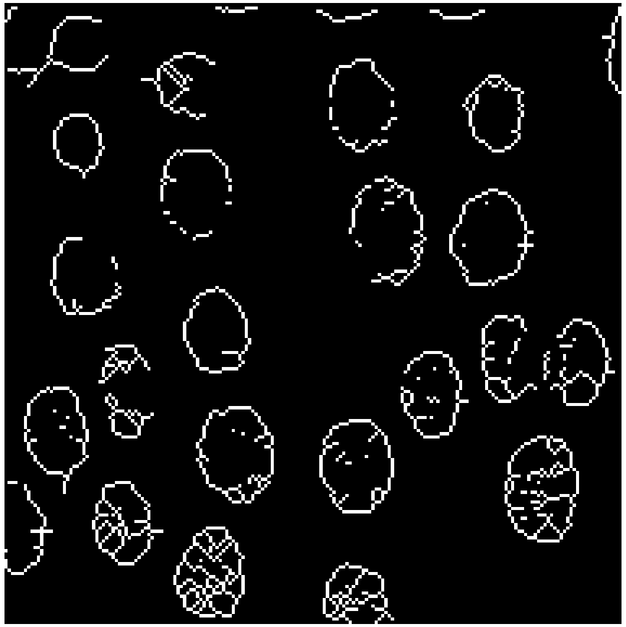
- **Comparison**



original



skeleton by maximal balls



homotopic skeleton