$u, v \in W^{k,p}(\Omega)$

i. $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$

$u \in W^{k,p}(\Omega) \Rightarrow u \in \mathcal{L}^p(\Omega)$ e $D^\alpha u \in \mathcal{L}^p(\Omega)$ $\forall \alpha$ com $|\alpha| \leq k$

suponha que $u \notin W^{k-|\alpha|,p}(\Omega)$ então $\cancel{u \notin \mathcal{L}^p(\Omega)}$ ou $\exists \beta$ com $|\beta| \leq k-|\alpha|$ tq $D^\beta(D^\alpha u) \in \mathcal{L}^p(\Omega)$

$$\Downarrow$$

$$|\alpha| + |\beta| \leq k$$

seja $\gamma = \alpha + \beta \Rightarrow |\gamma| = |\alpha| + |\beta| \Rightarrow D^\gamma u \notin \mathcal{L}^p(\Omega)$ com $|\gamma| \leq k$ $\rightarrow\!\times\!\leftarrow$
pois $D^\gamma u \in \mathcal{L}^p(\Omega)$ $\forall \gamma$ com $|\gamma| \leq k$ $(u \in W^{k,p}(\Omega))$

ii. $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$ $\quad \phi \in C_c^\infty(\Omega)$ então aqui $D^\cdot$ é a derivada parcial

$$\int_\Omega D^\alpha u \cdot D^\beta \phi \, dx = (-1)^{|\alpha|} \int_\Omega u \, D^\alpha(D^\beta \phi) = (-1)^{|\alpha|} \int_\Omega u \, D^{\alpha+\beta} \phi = \underbrace{(-1)^{|\alpha|}(-1)^{|\alpha+\beta|}}_{(-1)^{|\alpha|+|\alpha|+|\beta|}} \int_\Omega \phi \, D^{\alpha+\beta} u = (-1)^{|\beta|} \int_\Omega \phi \, D^{\alpha+\beta} u$$

derivada fraca

$\forall \phi \in C_c^\infty(\Omega)$

$$\cancel{(-1)^{2|\alpha|}}^{\,1} (-1)^{|\beta|}$$

$\Rightarrow D^\alpha(D^\beta u) = D^{\alpha+\beta} u$ no sentido fraco

o caso $D^\beta(D^\alpha u) = D^{\alpha+\beta} u$ é análogo

iii. $\forall \gamma, \lambda \in \mathbb{R}$, $\gamma u + \lambda v \in W^{k,p}(\Omega)$ e

$$D^\alpha(\gamma u + \lambda v) = \gamma D^\alpha u + \lambda D^\alpha v$$

$$\int_\Omega (\gamma u + \lambda v) D^\alpha \phi \, dx = \int_\Omega \gamma u \, D^\alpha \phi + \int_\Omega \lambda v \, D^\alpha \phi$$

$$= \gamma \int_\Omega u \, D^\alpha \phi + \lambda \int_\Omega v \, D^\alpha \phi$$

$$= \gamma (-1)^{|\alpha|} \int_\Omega \phi \, D^\alpha u + \lambda (-1)^{|\alpha|} \int_\Omega \phi \, D^\alpha v$$

$$= (-1)^{|\alpha|} \int_\Omega \phi \, \gamma D^\alpha u + (-1)^{|\alpha|} \int_\Omega \phi \, \lambda D^\alpha v$$

$$= (-1)^{|\alpha|} \int_\Omega \phi \, \underbrace{(\gamma D^\alpha u + \lambda D^\alpha v)}_{D^\alpha(\gamma u + \lambda v)}$$

deixar claro que está em $\mathcal{L}^1(\Omega)$ (com operações).

iv. se Ω_0 é um aberto de Ω então $u \in W^{k,p}(\Omega_0)$

$u \in W^{k,p}(\Omega) \Rightarrow u \in \mathcal{L}^p(\Omega)$ e $D^\alpha u \in \mathcal{L}^p(\Omega)$ $\forall \alpha$

como $\Omega_0 \subseteq \Omega$

$$\int_{\Omega_0} |u|\,dx \leq \int_{\Omega} \overbrace{|u|}^{\in \mathcal{L}^p(\Omega)}\,dx < \infty$$

$$\int_{\Omega_0} |D^\alpha u|\,dx \leq \int_{\Omega} \overbrace{|D^\alpha u|}^{\in \mathcal{L}^p(\Omega)\ \forall \alpha}\,dx < \infty$$

v. se $\eta \in C_c^\infty(\Omega)$ então $\eta u \in W^{k,p}(\Omega)$ e

$$D^\alpha(\eta u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \eta\, D^{\alpha-\beta} u$$

onde

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!\,(\alpha-\beta)!}, \quad \alpha! = \alpha_1! \cdots \alpha_n!$$

e $\beta \leq \alpha$ significa $\beta_j \leq \alpha_j \ \forall j$

utilizando indução sobre $|\alpha|$
suponha que $|\alpha| = 1$, dada uma função $\phi \in C_c^\infty(\Omega)$
temos

$$\int_\Omega \eta u D^\alpha \phi\, dx = \int_\Omega u D^\alpha(\eta\phi) - u(D^\alpha \eta)\phi\, dx$$

$\downarrow$ regra do produto usual

$$u(\phi D^\alpha \eta + \eta D^\alpha \phi)$$
$$u\phi D^\alpha \eta + u\eta D^\alpha \phi \nearrow \cancel{u\phi D^\alpha \eta} + u\eta D^\alpha \phi - \cancel{u(D^\alpha \eta)\phi}$$

$$= \int_\Omega u D^\alpha \underbrace{(\eta\phi)}_{\in C_c^\infty(\Omega)} - \int_\Omega u\phi D^\alpha \eta$$

derivada fraca

igual

$$= -\int_\Omega \eta\phi D^\alpha u - \int_\Omega u\phi D^\alpha \eta$$

$$= -\int_\Omega \underbrace{(\eta D^\alpha u + u D^\alpha \eta)}_{D^\alpha(\eta u) \text{ com } |\alpha| = 1} \phi\, dx$$

! mostrar que se supp ϕ é compacto então supp$(D^\alpha \phi)$ também é compacto

isso mostra que $D^\alpha(\eta u) = \eta D^\alpha u + u D^\alpha \eta$

agora seja $l < k$ e suponha que

$$D^\alpha(\eta u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \eta\, D^{\alpha-\beta} u \quad *$$

para todo α com $|\alpha| \leq l$ e toda função $\eta \in C_c^\infty(\Omega)$
seja α um MI com $|\alpha| = l+1$. então $\alpha = \beta + \gamma$ com $|\beta| = l$
e $|\gamma| = 1$, daí

$$\int_\Omega \eta u D^\alpha \phi\, dx \overset{\alpha = \beta+\gamma}{=} \int_\Omega \eta u D^{\beta+\gamma} \overset{\text{ii.}}{=} \int_\Omega \eta u D^\beta (D^\gamma \phi)\, dx \overset{\text{der fraca}}{=} (-1)^{|\beta|} \int_\Omega D^\beta(\eta u) D^\gamma \phi\, dx \overset{*\ D^\beta(\eta u)}{=} \int_\Omega \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} D^\sigma \eta\, D^{\beta-\sigma} u\, D^\gamma \phi$$

pela hipótese de indução em $D^{\beta-\sigma}u \in W^{k-|\beta|+|\sigma|,p}(\Omega)$ e $D^\sigma\phi \in C_c^\infty(\Omega)$, $|\gamma|=1$ temos

$$(-1)^{|\beta|}\int_\Omega \sum_{\sigma\le\beta}\binom{\beta}{\sigma} D^\sigma\eta\, D^{\beta-\sigma}u\, D^\gamma\phi = (-1)^{|\beta|+|\gamma|}\int_\Omega \sum_{\sigma\le\beta}\binom{\beta}{\sigma} \underbrace{D^\gamma(D^\sigma\eta\, D^{\beta-\sigma}u)}\phi\, dx$$

vamos utilizar a hipótese de indução denovo para calcular D^γ do produto. $|\gamma|=1$

$$=(-1)^{|\alpha|}\int_\Omega \sum_{\sigma\le\beta}\binom{\beta}{\sigma}\Big(\underbrace{D^{\beta-\sigma}u}_{D^{\beta-\sigma}=D^{\alpha-\rho}}\ \underbrace{D^\gamma(D^\sigma\eta)}_{D^{\gamma+\sigma}\eta=D^\rho\eta} + D^\sigma\eta\,\underbrace{D^\gamma(D^{\beta-\sigma}u)}_{D^{\gamma+\beta-\sigma}u=D^{\alpha-\sigma}u}\Big)\phi\, dx$$

$\rho=\sigma+\gamma \Rightarrow \rho=\sigma+\alpha-\beta \Rightarrow \beta-\sigma=\alpha-\rho$

$$=(-1)^{|\alpha|}\int_\Omega \sum_{\sigma\le\beta}\binom{\beta}{\sigma}\left[D^{\alpha-\rho}u\, D^\rho\eta + D^{\alpha-\sigma}u\, D^\sigma\eta\right]\phi\, dx$$

$$=(-1)^{|\alpha|}\int_\Omega\left[\sum_{\sigma\le\beta}\binom{\beta}{\sigma}D^\rho\eta\, D^{\alpha-\rho}u + \sum_{\sigma\le\beta}\binom{\beta}{\sigma}D^\sigma\eta\, D^{\alpha-\sigma}u\right]\phi\, dx$$

$0\le\sigma\le\beta$

$\gamma\le\sigma+\gamma\le\beta+\gamma$

$\gamma\le\rho\le\alpha$

$$\sum_{\sigma\le\beta}\binom{\beta}{\sigma}D^\rho\eta\, D^{\alpha-\rho}u + \sum_{\sigma\le\beta}\binom{\beta}{\sigma}D^\sigma\eta\, D^{\alpha-\sigma}u$$

$$=\underbrace{\sum_{\gamma\le\rho\le\alpha}\binom{\beta}{\rho-\gamma}D^\rho\eta\, D^{\alpha-\rho}u} + \underbrace{\sum_{\sigma\le\beta}\binom{\beta}{\sigma}D^\sigma\eta\, D^{\alpha-\sigma}u}$$

$$=\sum_{\gamma\le\rho\le\beta}X_\rho + \sum_{\boxed{\beta<\rho\le\alpha}}X_\rho + \sum_{\boxed{0\le\sigma<\gamma}}X_\sigma + \sum_{\gamma\le\sigma\le\beta}X_\sigma$$

$\rho=\alpha \qquad \sigma=0$

$X_\rho = \binom{\beta}{\beta}u\,D^\alpha\eta \qquad X_\sigma = \binom{\beta}{0}\eta\, D^\alpha u$

$\rho=\alpha \quad \binom{\beta}{\beta} = \binom{\alpha}{\alpha} \qquad \sigma=0 \quad \binom{\beta}{0}=\binom{\alpha}{0}$

$\rho\le\gamma \Rightarrow \rho_i\le\gamma_i \quad \forall i$

$\Rightarrow \rho>\gamma \Rightarrow \exists i$ tq $\rho_i>\gamma_i$

$0\le\rho<\gamma$

$0\le\rho_1<1$ e $0\le\rho_2<0$ $\Rightarrow \rho=0$

$\vdots$

n

$$\binom{\alpha}{0}\eta D^\alpha u + \sum_{\gamma\le\sigma\le\beta}\binom{\beta}{\sigma}D^\sigma\eta\, D^{\alpha-\sigma}u + \sum_{\gamma\le\rho\le\beta}\binom{\beta}{\rho-\gamma}D^\rho\eta\, D^{\alpha-\rho}u + \binom{\alpha}{\alpha}u\,D^\alpha\eta$$

ao menos de uma mudança de variaveis podemos juntar os somatórios e utilizando

$$\binom{\beta}{\sigma}+\binom{\beta}{\rho-\gamma}=\binom{\alpha}{\sigma}$$

que é válido pois $\alpha=\beta+\gamma$ $(\beta=\alpha-\gamma)$ e $\rho=\sigma+\gamma$, obtemos

$$\binom{\alpha}{0}\eta D^\alpha u + \sum_{\gamma\le\sigma\le\beta}\binom{\alpha}{\sigma}D^\sigma\eta\, D^{\alpha-\sigma} + \binom{\alpha}{\alpha}u\,D^\alpha\eta$$

$$=\sum_{0\le\sigma\le\alpha}\binom{\alpha}{\sigma}D^\sigma\eta\, D^{\alpha-\sigma}u$$

ou seja

$$\int_\Omega \eta u \, D^\alpha \phi \, dx = \int_\Omega \Bigg[\overbrace{\sum_{0 \leq \sigma \leq \alpha} \binom{\alpha}{\sigma} D^\sigma \eta \, D^{\alpha - \sigma} u}^{D^\alpha(\eta u)} \Bigg] \phi \, dx$$

isto nos diz que a derivada fraca de ηu é

$$D^\alpha(\eta u) = \sum_{0 \leq \sigma \leq \alpha} \binom{\alpha}{\sigma} D^\sigma \eta \, D^{\alpha - \sigma} u$$

por fim, note que

$$\int_\Omega |D^\alpha(\eta u)|^p dx = \int_\Omega \Big| \sum_{\sigma \leq \alpha} \binom{\alpha}{\sigma} D^\sigma \eta \, D^{\alpha-\sigma} u \Big|$$

$$\leq \sum \binom{\alpha}{\sigma} \int_\Omega |D^\sigma \eta| |D^{\alpha-\sigma} u|$$

$$\leq c \sum \binom{\alpha}{\sigma} \int_\Omega |D^{\alpha-\sigma} u| < \infty$$

$\Rightarrow D^\alpha(\eta u) \in L^p(\Omega)$.