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G.P. Galdi

An Introduction to the Mathematical Theory of the Navier-Stokes Equations

Steady-State Problems

Second Edition

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Steady-State Problems

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G.P. Galdi
Department of Mechanical Engineering
and Materials Science
University of Pittsburgh
O'Hara Street 3700
15261 Pittsburgh Pennsylvania
USA
galdi@pitt.edu

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I dedicate this work to four generations of loving people:
My parents, Gino and Elena;
My sister, Marisa;
My children, Adriana, Giovanni, Elena, Lisa,
Francesca and Sierra;
My grandchildren, Chiara, Davide, Sofia and Mirko.

Preface

This book is the new edition of the original two-volume book, published in 1994, and of its corrected second printing published in 1998. The first volume dealt with linearized problems, while the second one was dedicated to fully nonlinear problems.

The current edition is new in many significant ways.

In the first place, the two volumes have been merged into one. This seems to me a more appropriate choice that on the one hand, provides a natural logical unity to the subject, and on the other hand, furnishes a better presentation of the topics. In fact, nonlinear problems cannot be addressed properly without a careful analysis of their suitable linear counterparts, and, conversely, linearized problems can find full justification only as approximations of the complete nonlinear model.

In the second place, I have added two entirely new chapters (Chapter VIII and Chapter XI). The motivation for this addition comes from the increasing effort that mathematicians, especially over the past decade, have devoted to problems describing the interaction of a viscous liquid with rigid bodies. For this reason, I dedicated the above chapters to a systematic and updated analysis of a fundamental question of liquid–solid interaction, namely, the steady flow of a viscous liquid around a body that is allowed to translate and to rotate. In the years 2003 through 2010, over fifty relevant mathematical papers, directly or indirectly dedicated to this subject, have been published. Therefore, I deem it very useful for the interested researcher to have a place where all significant basic results are collected and treated in an organized and detailed fashion.

Furthermore, several important new contributions to the field that were published after 1998 have led me to update numerous sections extensively, as well as to add other new ones, not to mention the corresponding substantial increase in the number of bibliographic items. Among the above contributions, I would like to point out especially those dedicated to the regularity of solutions to the nonlinear problem in arbitrary dimension, as well as those concerning the asymptotic behavior in two-dimensional exterior domains.

Another new feature of this edition is that most of the proofs given in the previous editions, not only for the main results but also for those on the periphery, have been clarified either through a simplification or else through an extended treatment. For completeness, I also have included the proofs of several basic theorems that were not provided in the previous editions.

Finally, again for the reader's sake, I have included an introductory section collecting the basic properties of Banach spaces and related results that are very often referenced in the text.

Last, but not least, I take this opportunity to convey my warmest thanks to all my colleagues from whose collaboration I have benefited enormously in writing this new edition, and in particular to Josef Bemelmans, Paul Deuring, Reinhard Farwig, Mads Kyed, Anne Robertson (who is not "just" a collaborator), Ana Silvestre, Christian Simader, and Hermann Sohr.

In conclusion, I hope that the interested scientist will enjoy this book and derive great benefit from reading it just as I did while writing it.

Pittsburgh, Beechwood Blvd
April 2011 *G.P. Galdi*

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I

Steady-State Solutions of the Navier–Stokes Equations: Statement of the Problem and Open Questions

O muse o alto ingegno or m'aiutate.
O mente che scrivesti ciò ch'io vidi
qui si parrà la tua nobilitate.

DANTE, Inferno II, vv. 7-9

Introduction

Let us consider a viscous fluid of constant density (in short: a *viscous liquid*) $\mathcal{L}$ moving within a fixed region Ω of three-dimensional space $\mathbb{R}^3$. We shall assume that the generic motion of $\mathcal{L}$, with respect to an *inertial frame of reference*, is described by the following system of equations:

$$
\begin{aligned}
\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\right) &= \mu\Delta\boldsymbol{v} - \nabla\pi - \rho\boldsymbol{f}(x,t)\,,\\
\nabla\cdot\boldsymbol{v} &= 0\,,
\end{aligned}
\tag{I.0.1}
$$

where t is the time, $x = (x_1, x_2, x_3)$ is a point of Ω, ρ is the constant density of $\mathcal{L}$, $\boldsymbol{v} = \boldsymbol{v}(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ and $\pi = \pi(x,t)$ are the Eulerian velocity and pressure fields, respectively, and the positive constant μ is the shear viscosity coefficient. Moreover,

$$
\boldsymbol{v}\cdot\nabla\boldsymbol{v} \equiv \sum_{i=1}^{3} v_i \frac{\partial \boldsymbol{v}}{\partial x_i}
$$

is the *convective term*, and

$$
\Delta \equiv \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}
$$

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is the *Laplace* operator, while

$$\nabla \equiv \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$$

is the *gradient* operator, and

$$\nabla \cdot \boldsymbol{v} \equiv \sum_{1=1}^{3} \frac{\partial v_i}{\partial x_i}$$

is the *divergence* of $\boldsymbol{v}$. Finally, $-\boldsymbol{f}$ is the external force per unit mass (*body force*) acting on $\mathcal{L}$.

Equation $(\mathrm{I.0.1})_1$ expresses the balance of linear momentum (*Newton's law*), while $(\mathrm{I.0.1})_2$ ensures that the velocity field is *solenoidal* and represents the equation of conservation of mass (*incompressibility condition*). Notice that in contrast to the compressible scheme, here the pressure π is *not* a thermodynamic variable; rather it represents the "reaction force" that must act on $\mathcal{L}$ in order to leave any material volume unchanged.

System (I.0.1) was proposed for the first time by the French engineer C.L.M.H. Navier in 1822, cf. Navier (1827, p. 414), on the basis of a suitable molecular model.[1] However, it was only later, by the efforts of Poisson (1831), de Saint Venant (1843), and mainly by the clarifying work of Stokes (1845), that equations (I.0.1) found a completely satisfactory justification on the basis of the continuum mechanics approach.[2] Nowadays, equations (I.0.1) are usually referred to as *Navier–Stokes equations.*[3] In the language of modern rational mechanics we may say that the underlying constitutive assumption on the liquid $\mathcal{L}$ that leads to (I.0.1) is that the dynamical part of the *Cauchy stress tensor* $\boldsymbol{T}$ is linearly related to the *stretching tensor* $\boldsymbol{D}$, namely,

$$\boldsymbol{T} = -\pi \boldsymbol{I} + 2\mu \boldsymbol{D}, \tag{I.0.2}$$

[1] In this regard, we wish to quote the following comment of Truesdell (1953, p. 455):

> "Such models were not new, having occurred in philosophical or qualitative speculations for millennia past. Navier's magnificent achievement was to put these notions into sufficiently concrete form that he could derive equations of motion for them."

[2] A detailed account of the history of the Navier–Stokes equations can be found in the beautiful paper of Darrigol (2002).

[3] Some authors, showing questionable semantic taste, often speak of "incompressible" and "compressible" Navier-Stokes equations. The latter definition is given to the generalization of (I.0.1), that takes into account the variation of the density in space and time. Besides the awkward nomenclature, it should be observed, for the sake of precision, that C.L.M.H. Navier never obtained such a generalization and that it was derived for the first time by S.D. Poisson in 1829 (Poisson 1831), and later clarified on a sound and clear phenomenological basis by G.G. Stokes (Stokes 1845).

where $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{D} = \{D_{ij}\}$ with

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{I.0.3}$$

Liquids satisfying the constitutive equation (I.0.2) are also called *Newtonian liquids.*[4]

In several mathematical questions related to the unique solvability of problem (I.0.1), two-dimensional solutions describing the *plane motions* of $\mathcal{L}$ deserve separate attention. For these solutions the fields $\boldsymbol{v}$ and p depend only on x_1, x_2 (say), and t, and, moreover, $v_3 \equiv 0$. Consequently, the relevant (spatial) region of flow Ω becomes a subset of the plane $\mathbb{R}^2$.

Till a few decades ago (early 1930s), there was unanimous opinion that the Navier–Stokes equations were useful (in agreement with the experiments, that is) only at "low"-velocity regimes. It is also thanks to the efforts of outstanding mathematicians such as Jean Leray, Eberhard Hopf, Olga Ladyzhenskaya, and Robert Finn that they are nowadays regarded as the universal foundation of fluid mechanics, no matter how complicated and unpredictable the behavior of its solutions may be. In fact, when we struggle to give an answer to a "simply" formulated problem, we never know whether it is because of lack of sufficient mathematical knowledge or because of some hidden physical phenomenon. In any case, there is a firm belief in the mathematical community that these equations hide many mysteries and secrets that are still far beyond our current reach.

When dealing with equations (I.0.1), the primary goal is to study such significant and basic properties of solutions $\boldsymbol{v}, \pi$ as existence, uniqueness, and regularity, and asymptotic behavior in space (when Ω is unbounded) and in time.[5] Of course, these properties may be rather different according to whether

[4] Roughly speaking, the relation (I.0.2) states that the shear stress in a viscous liquid produces a gradient of velocity that is proportional to the stress. In Newton's words: "Resistentiam, quæ oritur ex defectu lubricitatis partium Fluidi, cæteris paribus, proportionalem esse velocitati, qua partes Fluidi separantur ab invicem" ("The resistance, arising from the want of lubricity in the parts of a fluid is, *cæteris paribus*, proportional to the velocity with which the parts of the fluid are separated from each other"); see Newton (1686, Book 2, Sect. IX, p. 373).

[5] We wish to observe that with a view to solving the above-mentioned questions, the Navier–Stokes equations can be considered the mathematical prototype of more complicated models that can be used to take into account other than purely mechanical phenomena of the liquid, such as thermal conduction in the Boussinesq approximation or electrical conduction in the nonrelativistic (incompressible) magnetohydrodynamic scheme. In fact, for these more general systems, one can prove results that are qualitatively analogous to those achieved for the simpler system (I.0.1) and that present for their proof approximately the same degree of difficulty one encounters for (I.0.1). However, the situation becomes completely different, and, in principle, more complicated, if the liquid is modeled

we are interested in *steady* or *unsteady* flows of $\mathcal{L}$, that is, according to whether the velocity and pressure fields depend or do not depend explicitly on time.

Throughout this book we shall consider steady motions, for which system (I.0.1), with $\partial \boldsymbol{v}/\partial t \equiv 0$ and $\boldsymbol{f} = \boldsymbol{f}(x)$, takes the form

$$\begin{aligned} \nu \Delta \boldsymbol{v} &= \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla p + \boldsymbol{f}\,, \\ \nabla \cdot \boldsymbol{v} &= 0\,, \end{aligned} \tag{I.0.4}$$

where $p = \pi/\rho$, and $\nu = \mu/\rho$ is the kinematic viscosity coefficient. We shall continue to call p "the pressure" (or "the pressure field") of the liquid $\mathcal{L}$.

In order to perform our study, however, we need to add to (I.0.4) appropriate supplementary conditions that may depend on the physics of the problem we want to address, which, in turn, can be broadly characterized in terms of the type of region Ω where the flow occurs. To this end, we shall distinguish the following cases:

(i) Ω is bounded;
(ii) Ω is the complement of a bounded region (in short: Ω is an *exterior* region).

In both circumstances Ω has a bounded boundary. However, it is also of great relevance to study flow in regions with an unbounded boundary, such as infinite tubes or pipes. Therefore, we shall also consider the following situation:

(iii) Ω has an unbounded boundary.

In all three cases (i), (ii), and (iii), the associated mathematical problems present several difficulties and basic open questions. The aim of the following sections is, for each case, to formulate these problems, to outline the related difficulties, and to point out those fundamental questions that still have no answer.

I.1 Flow in Bounded Regions

Usually, for flow in bounded regions, the driving mechanism is due either to the movement of part of the boundary, as in the flow between two rotating, concentric spheres, or to the injection and removal of liquid through the permeable part of the boundary, as in the case of a flow in a region with a finite number of "sources" and "sinks." In such situations, we append to (I.0.4) the following condition:

$$\boldsymbol{v}(y) = \boldsymbol{v}_*(y), \quad y \in \partial\Omega, \tag{I.1.1}$$

by a constitutive equation different from (I.0.2). These latter liquids are called *non-Newtonian*. We refer the reader to the Notes at the end of this chapter, for the basic mathematical literature related to certain relevant non-Newtonian liquid models.

that describes the situation in which the velocity field is prescribed at the bounding walls $\partial\Omega$ of the region Ω.[6]

It is worth noticing that in particular, condition (I.1.1), often referred to as a *no-slip* boundary condition, requires that the particles of the liquid "adhere" to the boundary $\partial\Omega$ in the case that $\partial\Omega$ is motionless, rigid, and impermeable ($\boldsymbol{v}_* \equiv \mathbf{0}$). This fact appears to be verified within a very good degree of experimental accuracy.[7]

Because of the solenoidality condition $(\text{I.0.4})_2$, and in view of Gauss theorem, it turns out that the field $\boldsymbol{v}_*$ must satisfy the *compatibility condition*:

$$\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} = 0 \tag{I.1.2}$$

with $\boldsymbol{n}$ unit outer normal to $\partial\Omega$. From the physical viewpoint, relation (I.1.2) requires that the *total mass flux* through the boundary must vanish. If $\partial\Omega$ is constituted by the union of $m \geq 1$ closed non-intersecting surfaces $\Gamma_1, \ldots, \Gamma_m$, condition (I.1.2) becomes

$$\sum_{i=1}^{m} \int_{\Gamma_i} \boldsymbol{v}_* \cdot \boldsymbol{n}_i \equiv \sum_{i=1}^{m} \Phi_i = 0 \tag{I.1.3}$$

with $\boldsymbol{n}_i$ unit outer normal to Γ_i.

The problem of proving or disproving existence of solutions to (I.0.4), (I.1.1), and (I.1.2) under no restrictions on $\boldsymbol{v}_*$ (other than (I.1.2) and, of course, certain regularity requirements) and on the number m of surfaces Γ_i is among the most outstanding and still open questions, not only in the steady-state case but in the whole mathematical theory of Navier–Stokes equations.

---

[6] Instead of (I.1.1), one may consider different boundary conditions. A popular alternative is the so-called *stress-free* (or *pure slip*) *boundary condition*, which consists in prescribing the normal component of the velocity and the tangential component of the stress vector at the boundary. This type of condition is often used in stability problems, since it considerably simplifies the calculations (Chandrasekhar 1981). Throughout this book *we shall always assume condition* (I.1.1) *at the bounding walls*. We refer the reader to the literature quoted in the Notes at the end of this chapter for other work related to the Navier-Stokes equations with boundary conditions different from (I.1.1).

[7] The adherence condition is quite reasonable, at least for liquids filling rigid vessels under ordinary conditions (Perucca 1963, Vol. I, p. 451). However, such a condition need not be satisfied in different situations. For instance, assuming adherence at the contact line Γ of a free surface of a liquid with a rigid plane wall steadily moving along itself would lead to an infinite dissipation rate of the liquid near Γ (Pukhnacev & Solonnikov 1982, pp. 961–962). Furthermore, the adherence condition is not expected to hold for fluids other than liquids. In particular, experimental evidence shows that in high-altitude aerodynamics an adherence condition is no longer true (Serrin 1959b, §64); see also Bateman, Dryden, & Murnaghan (1932; §§1.2, 1.7, 3.2), and Truesdell (1952, §79).

Actually, so far, one can prove its solvability only when the fluxes Φ_i through Γ_i satisfy, in addition to (I.1.3), a condition of the type

$$\sum_{i=1}^{m} |\Phi_i| < c\nu, \tag{I.1.4}$$

where c depends only on Ω. Notice that in view of (I.1.2) and (I.1.3), inequality (I.1.4) is *automatically satisfied if* $m = 1$; otherwise, it becomes an extra requirement.

It is not clear whether the above restriction is really needed. This is a typical example, which we mentioned previously, where we do not know whether the fact that we are unable to give an answer is due to a lack of a sufficient mathematical knowledge or to a "hidden" physical phenomenon that we are not able to see. The mathematician often experiences similar situations while studying the Navier–Stokes equations, and probably, it is just for this reason that these equations are so particularly fascinating.

Referring the reader to Chapter IX for a detailed analysis of the question, here we may wish to briefly explain why it may be difficult to avoid restriction (I.1.4). The resolution of (I.0.4), (I.1.1), and (I.1.2) relies, usually, on some approximating procedure whose convergence requires a uniform *a priori* bound on the approximating solutions. If $\boldsymbol{v}_* \equiv 0$, following the ideas of Leray (1933, 1936), this bound can be simply obtained by the following *formal* computation. We dot-multiply through both sides of $(\text{I.0.4})_1$ by $\boldsymbol{v}$ and use the known identity

$$\nabla \cdot (\varphi\, \boldsymbol{w}) = \boldsymbol{w} \cdot \nabla\varphi + \varphi \nabla \cdot \boldsymbol{w}$$

together with $(\text{I.0.4})_2$ to obtain[8]

$$-\nu \nabla \boldsymbol{v} : \nabla \boldsymbol{v} + \nabla \cdot \left(\frac{\nu}{2} \nabla v^2 - p\, \boldsymbol{v} - \frac{1}{2} v^2\, \boldsymbol{v} \right) = \boldsymbol{f} \cdot \boldsymbol{v}.$$

Integrating this expression over Ω and taking into account $\boldsymbol{v} = \boldsymbol{v}_* \equiv 0$ at $\partial\Omega$,

[8] For $\boldsymbol{A} = \{A_{ij}\}$ and $\boldsymbol{B} = \{B_{ij}\}$ second-order tensors in the n-dimensional Euclidean space $\mathbb{R}^n$ we set, as is customary,

$$\boldsymbol{A} : \boldsymbol{B} \equiv \sum_{i,j=1}^{n} A_{ij} B_{ij}, \quad |\boldsymbol{A}| \equiv \left(\sum_{i,j=1}^{n} A_{ij} A_{ij} \right)^{1/2}.$$

Moreover, if $\boldsymbol{a}$ is a vector of $\mathbb{R}^n$, by the symbols

$$\boldsymbol{a} \cdot \boldsymbol{A} \quad \text{and} \quad \boldsymbol{A} \cdot \boldsymbol{a}$$

we mean vectors with components

$$\sum_{i=1}^{n} a_i A_{ij} \quad \text{and} \quad \sum_{i=1}^{n} a_j A_{ij},$$

respectively.

we deduce[9]

$$\nu \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{v} = -\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,. \tag{I.1.5}$$

By the well-known inequality (see (II.5.1)[10])

$$\int_\Omega v^2 \leq \gamma \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{v}, \tag{I.1.6}$$

where γ depends on Ω, from (I.1.5) and the Schwarz inequality we obtain

$$\int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{v} \leq \frac{\gamma}{\nu} \int_\Omega f^2 \,, \tag{I.1.7}$$

which (formally) furnishes the a priori bound for $\boldsymbol{v}$ when $\boldsymbol{v}_* \equiv 0$. When $\boldsymbol{v}_* \not\equiv 0$, again following the ideas of Leray (1933) and Hopf (1941, 1957), one writes [11]

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{V}, \tag{I.1.8}$$

where $\boldsymbol{V}$ is a (sufficiently smooth) solenoidal vector field in Ω that equals $\boldsymbol{v}_*$ at $\partial\Omega$. Placing (I.1.8) into $(\text{I.0.4})_1$ and proceeding as before, we arrive at the following identity:

$$\nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u} = -\int_\Omega \big(\boldsymbol{u} \cdot \nabla \boldsymbol{V} \cdot \boldsymbol{u} + \boldsymbol{V} \cdot \nabla \boldsymbol{V} \cdot \boldsymbol{u} + \nu \nabla \boldsymbol{V} : \nabla \boldsymbol{u} + \boldsymbol{f} \cdot \boldsymbol{u} \big), \tag{I.1.9}$$

which generalizes (I.1.6) to the case $\boldsymbol{v}_* \not\equiv 0$. By use of the Schwarz inequality and (I.1.6), it is easily seen that the last three terms on the right-hand side of (I.1.9) can be increased, for instance, as follows:

$$\int_\Omega |\boldsymbol{V} \cdot \nabla \boldsymbol{V} \cdot \boldsymbol{u}| \leq \sqrt{\gamma} \sup_\Omega |\boldsymbol{V}| \left(\int_\Omega \nabla \boldsymbol{V} : \nabla \boldsymbol{V} \right)^{1/2} \left(\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u} \right)^{1/2} ,$$

$$\nu \int_\Omega |\nabla \boldsymbol{V} : \nabla \boldsymbol{u}| \leq \nu \left(\int_\Omega \nabla \boldsymbol{V} : \nabla \boldsymbol{V} \right)^{1/2} \left(\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u} \right)^{1/2} ,$$

$$\int_\Omega |\boldsymbol{f} \cdot \boldsymbol{u}| \leq \gamma \left(\int_\Omega f^2 \right)^{1/2} \left(\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u} \right)^{1/2} .$$

Placing these inequalities back into (I.1.9), we obtain

$$\nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u} \leq -\int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{V} \cdot \boldsymbol{u} + C \left(\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u} \right)^{1/2} , \tag{I.1.10}$$

[9] Unless their use clarifies the context, the infinitesimal volume and surface elements in the integrals will generally be omitted.

[10] That is, the first display in Section 5 of Chapter II.

[11] For another approach, again due to Leray, we refer the reader to the Notes for Chapter IX.

with C depending only on Ω, $\boldsymbol{V}$ (i.e., $\boldsymbol{v}_*$), and $\boldsymbol{f}$. Therefore, in order to obtain a bound depending only on the data for the quantity

$$\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u},$$

or, what amounts to the same thing, for

$$\int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{v},$$

it is sufficient to prove the following *one-sided* inequality:

$$-\int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{V} \cdot \boldsymbol{u} \le k \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u}, \tag{I.1.11}$$

with some constant k (depending on $\boldsymbol{V}$ and Ω) such that

$$k < \nu, \tag{I.1.12}$$

and for all smooth solenoidal vector fields $\boldsymbol{u}$ vanishing at $\partial\Omega$. Of course, if we do not want to impose restrictions on ν, we should be able to prove that for *any* $k > 0$ there exists a solenoidal field $\boldsymbol{V} = \boldsymbol{V}(x; k)$ assuming the value $\boldsymbol{v}_*$ at $\partial\Omega$ and satisfying (I.1.11) for all the above specified vectors $\boldsymbol{u}$. However, Takeshita (1993) and, more recently, Heywood (2010) showed by means of counterexamples that in general, the latter property does not hold if $\partial\Omega$ is composed by more than one surface Γ_i, and consequently, if one follows in such a case the method of Leray–Hopf, one *must* impose some restrictions on ν or, equivalently, on $\boldsymbol{v}_*$. To date, the best one can show, in general,[12] is that (I.1.11) and (I.1.12) are certainly satisfied, provided the fluxes Φ_i satisfy the restriction (I.1.4). As already observed, (I.1.4) becomes redundant if the number of surfaces Γ_i reduces to one.[13]

I.2 Flow in Exterior Regions

The most significant physical problem that motivates this type of study is the motion of a rigid body $\mathcal{B}$ through a viscous liquid. Such a problem originates

[12] In some special cases of plane flow, one can remove the extra condition (I.1.4); see the Notes to Chapter IX.

[13] It is interesting to observe that for a certain class of non-Newtonian liquids, where the shear viscosity coefficient μ depends in a monotonically increasing way on the magnitude of the stretching tensor $|\boldsymbol{D}|$, the corresponding steady-state problem in the bounded domain Ω can be solved *under the sole compatibility condition* (I.1.2); see the Notes at the end of this chapter.

with the pioneering work of G.G. Stokes (1851) on the effect of internal friction on the movement of a pendulum in a liquid.

Usually, the influence on the motion of $\mathcal{B}$ of the boundary walls of the container of the liquid $\mathcal{L}$ can be safely disregarded, and this leads to the mathematical (simplifying, in principle) assumption that $\mathcal{L}$ fills the whole space region Ω complementary to $\mathcal{B}$, and that it is at rest at spatial infinity. Hence, Ω becomes an *exterior* region.

Another classical problem that can be formulated as an exterior problem is the *flow past an obstacle*, where the center of mass of $\mathcal{B}$ is held in place by appropriate forces and the liquid flows past $\mathcal{B}$ tending to a uniform velocity field at large distances from $\mathcal{B}$ (as in a *wind tunnel*).

In the above situations, it is convenient to describe the motion of the liquid from a frame of reference $\mathcal{S}$ attached to $\mathcal{B}$. This is because the region occupied by the liquid then becomes time-independent. However, since, in general, $\mathcal{B}$ may rotate, the frame $\mathcal{S}$ is no longer inertial, and consequently, we have to modify the original equations (I.0.1) accordingly, in order to take into account the fictitious forces. Assuming that the angular velocity, $\boldsymbol{\omega}$, of $\mathcal{B}$ with respect to the initial frame is *constant in time* (a case of particular interest in many applications), this amounts to adding, on the right-hand side of $(\text{I.0.4})_1$, the term $2\boldsymbol{\omega}\times\boldsymbol{v}$ representing the *Coriolis force* (per unit mass), while the centrifugal force (per unit mass), $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{x})$, can be formally absorbed in the pressure term. Equations (I.0.1) then modify to the following ones (see, e.g., Batchelor 1999, pp. 139–140, or Galdi 2002, §1):

$$\begin{aligned}\frac{\partial\boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}&=\nu\Delta\boldsymbol{v}-2\boldsymbol{\omega}\times\boldsymbol{v}-\nabla p-\boldsymbol{f}\,,\\ \nabla\cdot\boldsymbol{v}&=0\,,\end{aligned}\tag{I.2.1}$$

where now $\boldsymbol{v}$ has to be interpreted as the velocity of the particles of $\mathcal{L}$ *relative to* $\mathcal{S}$, and we have still denoted by p the original pressure field modified by the addition of the term $-\frac{1}{2}(\boldsymbol{\omega}\times\boldsymbol{x})^2$. The steady-state counterpart of (I.2.1) thus becomes

$$\begin{aligned}\nu\Delta\boldsymbol{v}&=\boldsymbol{v}\cdot\nabla\boldsymbol{v}+2\boldsymbol{\omega}\times\boldsymbol{v}-\nabla p+\boldsymbol{f}\,,\\ \nabla\cdot\boldsymbol{v}&=0\,.\end{aligned}\tag{I.2.2}$$

These equations must be endowed with a condition on $\boldsymbol{v}$ at the bounding walls $\partial\Omega$ and at infinity. For the former, we will adopt (I.1.1), whereas the latter will be taken to be of the form

$$\lim_{|x|\to\infty}(\boldsymbol{v}(x)+\boldsymbol{v}_\infty(x))=0\,,\tag{I.2.3}$$

with $\boldsymbol{v}_\infty=\boldsymbol{v}_0+\boldsymbol{\omega}\times\boldsymbol{x}$, and $\boldsymbol{v}_0$ a constant vector. A most significant situation described by the problem (I.2.2)–(I.2.3), is that of a rigid body moving with constant angular velocity $\boldsymbol{\omega}$ in a liquid that is quiescent at large distances. In such a case, the vector $\boldsymbol{v}_0$ can be viewed as the velocity of the center of mass of $\mathcal{B}$ with respect to $\mathcal{S}$, assumed to be a constant.[14,15]

The problem just described is formulated for three-dimensional motions. However, it also has a very significant counterpart in the case of *plane motions* of $\mathcal{L}$. Specifically, assuming that the relevant region of motion is located in the plane Π, we have that the vector $\boldsymbol{v}_0$ is prescribed parallel to Π, while $\boldsymbol{\omega}$ is orthogonal to it. A classic example of such flows is furnished by the motion of $\mathcal{L}$ past a fixed long and straight cylinder $\mathcal{C}$, of cross-section S, and axis a assuming that the velocity field $\boldsymbol{v}$ tends to a given constant at large distances from $\mathcal{C}$. In a region of flow sufficiently far from the two ends of $\mathcal{C}$ and including $\mathcal{C}$, one can assume that $\boldsymbol{v}$ is independent of the coordinate parallel to a and, moreover, that there is no motion in the direction of a. The plane Π is thus orthogonal to a, while Ω becomes the region of Π exterior to S.

The study of problem (I.2.2), (I.1.1), (I.2.3) and the description of the corresponding known results is tightly related to whether we consider three- or two-dimensional flow. Therefore, we shall analyze these two cases separately.

I.2.1 Three-Dimensional Flow

The cases $\boldsymbol{\omega} = \boldsymbol{0}$ and $\boldsymbol{\omega} \neq \boldsymbol{0}$ will be referred to as the *irrotational* and *rotational* cases, respectively.

Irrotational Case. In this situation, problem (I.2.2), (I.1.1), (I.2.3) reduces to

$$\left.\begin{aligned} &\nu\Delta\boldsymbol{v} = \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p + \boldsymbol{f} \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega\,,$$
$$\boldsymbol{v} = \boldsymbol{v}_* \ \text{ at } \partial\Omega\,, \tag{I.2.4}$$
$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = -\boldsymbol{v}_0\,.$$

As in the case of a bounded region of flow, the first fundamental contribution to the existence problem was given by Leray (1933), who showed that

[14] It is worth noticing that even though the velocity of the center of mass G is constant in $\mathcal{S}$, it can be time-dependent in an inertial frame $\mathcal{I}$. In fact, denoting by $\boldsymbol{\eta}$ the velocity of G with respect to $\mathcal{I}$, we find that $\boldsymbol{v}_0$ is constant in $\mathcal{S}$ if and only if $d\boldsymbol{\eta}/dt = \boldsymbol{\omega}\times\boldsymbol{\eta}$, that is, if and only if G moves, in $\mathcal{I}$, with constant speed along a circular helix whose axis is parallel to $\boldsymbol{\omega}$. However, the components of $\boldsymbol{\eta}$ and $\boldsymbol{v}_0$ along $\boldsymbol{\omega}$ must coincide, that is, $\boldsymbol{\eta}\cdot\boldsymbol{\omega} = \boldsymbol{v}_0\cdot\boldsymbol{\omega}$. Thus, in particular, the velocity of G will be constant also in $\mathcal{I}$ if and only if it is parallel to $\boldsymbol{\omega}$ (in which case $\boldsymbol{\eta} = \boldsymbol{v}_0$), and the helix degenerates into a straight line. On the other hand, if $\boldsymbol{\eta}$ is orthogonal to $\boldsymbol{\omega}$ (or, equivalently, $\boldsymbol{v}_0$ is orthogonal to $\boldsymbol{\omega}$), the helix will degenerate into a circle lying in a plane orthogonal to $\boldsymbol{\omega}$, which means that the motion of the body reduces to a constant rotation around an axis not necessarily passing through G. Further details can be found, e.g., in Galdi (2002, §1).

[15] For other interesting (unsolved) exterior problems, we refer the reader to the Notes at the end of this chapter.

an a priori estimate of the type (I.1.7) suffices to ensure the convergence of a suitable approximating procedure to a regular solution to (I.2.4);[16] see Chapter X. However, the outstanding problem Leray left out was the investigation of the asymptotic behavior at large distances of such solutions. This question is of primary importance since, as is easily realized, it is intimately related to the physical properties that any solution that deserves this name should possess. For example, solutions must satisfy the *energy equation*

$$2\nu\int_\Omega \boldsymbol{D}(\boldsymbol{v}):\boldsymbol{D}(\boldsymbol{v}) - \int_{\partial\Omega}\big[(\boldsymbol{v}_*+\boldsymbol{v}_0)\cdot\boldsymbol{T}(\boldsymbol{v},p) - \tfrac{1}{2}(\boldsymbol{v}_*+\boldsymbol{v}_0)^2\boldsymbol{v}_*\big]\cdot\boldsymbol{n} \\ + \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}+\boldsymbol{v}_0) = 0, \tag{I.2.5}$$

with $\boldsymbol{D}$ and $\boldsymbol{T}$ stretching and stress tensors (I.0.2), which represents the balance between the power of the work of external force, the power of the work done on the "bodies" represented by the connected components of $\mathbb{R}^3 - \Omega$, and the energy dissipated by the viscosity. Also, if $\boldsymbol{f}, \boldsymbol{v}_*$, and $\boldsymbol{v}_0$ are "sufficiently small" with respect to the viscosity ν,[17] the corresponding solutions are expected to be unique. In addition, in the case when $\boldsymbol{v}_0 \neq \boldsymbol{0}$, the flow must exhibit an infinite wake extending in the direction opposite to $\boldsymbol{v}_0$, and the order of convergence of $\boldsymbol{v}$ to $-\boldsymbol{v}_0$ has to be rather different according to whether it is calculated inside or outside the wake. Finally, in conformity with the boundary layer concept, the flow is expected to be potential outside a small neighborhood of the bodies and of the wake, which means that the vorticity should decay sufficiently fast at large distances and outside the wake.

If $\boldsymbol{v}_0 \neq \boldsymbol{0}$, the above questions found a definitive answer through the fundamental work of R. Finn, K.I. Babenko, and their co-workers during the years 1959–1973. In particular, using the results of Finn, Babenko has shown that the solution constructed by Leray admits an asymptotic development at infinity in which the dominant term is a solution to the corresponding *linearized* equations, which, for $\boldsymbol{v}_0 \neq 0$, are the *Oseen equations*; see also Galdi (1992b), Farwig & Sohr (1998). Therefore, Leray's solution behaves at infinity in such a way as to ensure the validity of all the above-mentioned properties; see Chapter X.

If $\boldsymbol{v}_0 = \boldsymbol{0}$, the picture is much less clear: the methods adopted by the above authors generally do not work. Nevertheless, by means of completely different tools, and following the approach given by Galdi (1992c), one can show that conclusions somewhat analogous to the case $\boldsymbol{v}_0 \neq \boldsymbol{0}$ can be drawn also when

[16] As a matter of fact, Leray proved $(\text{I.2.4})_4$ only for $\boldsymbol{v}_0 = \boldsymbol{0}$, while for $\boldsymbol{v}_0 \neq \boldsymbol{0}$ he proved only that $\boldsymbol{v}$ tends to $\boldsymbol{v}_0$ in a weaker sense. (As Leray himself noted, his proof for $\boldsymbol{v}_0 = \boldsymbol{0}$ fails if $\boldsymbol{v}_0 \neq \boldsymbol{0}$.) The validity of $(\text{I.2.4})_4$ when $\boldsymbol{v}_0 \neq \boldsymbol{0}$ was shown, independently, by Finn (1959a) and by Ladyzhenskaya (1969, Chapter 5 Theorem 8 and p. 206).

[17] More precisely, if a suitable nondimensional (Reynolds) number depending on $\boldsymbol{v}_*$, $\boldsymbol{f}$, ν, and Ω is "sufficiently small."

$\boldsymbol{v}_0 = \mathbf{0}$, provided, however, that $\boldsymbol{v}_*$ and $\boldsymbol{f}$ are not too large compared to ν.[18] Specifically, *under the stated restrictions on the data*, the solution constructed by Leray admits an asymptotic development at large spatial distances whose dominant term *behaves* like the solution of the corresponding linearized equations that, for $\boldsymbol{v}_0 = \mathbf{0}$, are the *Stokes equations*. However, as shown by Deuring & Galdi (2000), the dominant term in this expansion *cannot be* a solution to the corresponding Stokes equations. In fact, more recent results due to Korolev & Šverák (2007, 2011) establish that the leading term in the asymptotic expansion coincides with a suitable *exact, singular solution* of the full nonlinear problem obtained by Landau (1944); see Chapter X.

Thus, if $\boldsymbol{v}_0 = \mathbf{0}$, several basic questions remain open, among others the following:

(i) *Given $\boldsymbol{v}_*$ and $\boldsymbol{f}$, no matter how smooth but of unrestricted size, do there exist corresponding solutions satisfying the energy equation?*
(ii) *Do these solutions, if they exist at all, admit a suitable asymptotic expansion for large $|x|$ whose leading term is the corresponding Landau solution?*

It is quite clear that in order to answer these questions, one has to employ ideas and methods that go well beyond the simple perturbative analysis, by which the "small" data results referred above are obtained.

Another puzzling question that arises in the case $\boldsymbol{v}_0 = \mathbf{0}$ is the following *Liouville-like* problem. Consider

$$\left.\begin{aligned} &\nu\Delta\boldsymbol{v} = \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3 ,$$
$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = \mathbf{0}\,, \quad \int_{\mathbb{R}^3} \nabla\boldsymbol{v} : \nabla\boldsymbol{v} < \infty\,. \tag{I.2.6}$$

As will be shown later on, in Chapter X, all possible solutions $(\boldsymbol{v}, p)$ to (I.2.6) are infinitely differentiable, and moreover, all derivatives of $\boldsymbol{v}$ and ∇p tend to zero as $|x| \to \infty$. Of course, the null solution $\boldsymbol{v} \equiv \nabla p \equiv \mathbf{0}$ is a solution to (I.2.6). The question is this:

(iii) *Is the null solution the only (smooth) solution to* (I.2.6)*?*

In connection with this problem, it is worth emphasizing the following facts. In the first place, if the homogeneous condition at infinity is replaced by $\lim_{|x|\to\infty} \boldsymbol{v}(x) = \boldsymbol{v}_0$, for some *nonzero* $\boldsymbol{v}_0$, then one can show that the only solution is $\boldsymbol{v}(x) = \boldsymbol{v}_0$, $\nabla p(x) = \mathbf{0}$, for all $x \in \mathbb{R}^3$; see Chapter X. In the second place, the n-dimensional counterpart of this question, i.e., obtained by replacing, in (I.2.6), $\mathbb{R}^3$ with the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 2$ and $n \neq 3$, admits a *positive answer*, so that only the case $n = 3$ remains open; see Chapter X and Chapter XII.

[18] See footnote 17.

We shall finally describe another interesting question that might deserve an appropriate investigation. For the existence of Leray's solution it is not required that the *total* mass flux through $\partial\Omega$, i.e.,

$$\Phi \equiv \int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n}, \tag{I.2.7}$$

satisfy the vanishing condition (I.1.2). For, unlike the case of flow in a bounded region, (I.1.2) is no longer a priori a compatibility condition. This latter fact can be explained as follows. By the incompressibility condition $(\text{I.0.4})_2$, the flux Φ is equal to the flux Φ_R of $\boldsymbol{v}$ through the surface S_R of a ball of radius R surrounding the bodies $\mathcal{B}_i$. Since the behavior of the velocity field $\boldsymbol{v}$ at large distances is expected to be, in general, like $|x|^{-1}, \Phi_R$ need not vanish as R tends to infinity, and so Φ need not be zero a priori. However, to date, one is able to prove existence only if Φ is sufficiently small with respect to ν. Thus:

(iv) *Is it possible to prove existence for arbitrarily large values of Φ?*

Rotational Case. As probably expected, the rotational case presents further and new challenges, and, consequently, more unresolved issues, if compared to the irrotational case. To describe these latter, we begin by observing that, as shown in Chapter VIII and Chapter XI, after a suitable change of coordinates (*Mozzi–Chasles transformation*) problem (I.2.2), (I.1.1), and (I.2.3) with $\boldsymbol{\omega} \neq \boldsymbol{0}$ is formally equivalent to the following one:

$$\left.\begin{aligned} &\nu \Delta \boldsymbol{u} = (\boldsymbol{u} - \lambda\boldsymbol{\omega} - \boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{u} + \nabla p + \boldsymbol{f} \\ &\nabla \cdot \boldsymbol{u} = 0 \end{aligned}\right\} \ \text{in } \Omega\,,$$
$$\boldsymbol{u} = \boldsymbol{v}_* + \boldsymbol{v}_\infty \equiv \boldsymbol{u}_* \ \text{ at } \partial\Omega\,, \tag{I.2.8}$$
$$\lim_{|x|\to\infty} \boldsymbol{u}(x) = \boldsymbol{0}\,,$$

where $\boldsymbol{u} := \boldsymbol{v} + \boldsymbol{v}_\infty$, $\lambda := \boldsymbol{v}_0 \cdot \boldsymbol{\omega}/|\boldsymbol{\omega}|^2$, p is the "original" pressure field and Ω is a suitable exterior region.

The characteristic feature of problem (I.2.8) is that the term $\boldsymbol{\omega} \times \boldsymbol{x} \cdot \nabla \boldsymbol{u}$ has a coefficient that becomes *unbounded* at large distances. Therefore, the rotational case *cannot be viewed as a perturbation of the irrotational case.*

Despite this difficulty, one can show with relative ease that for data of *arbitrary* "size" (in appropriate function classes), problem (I.2.8) has at least one solution. As in the cases previously described, the proof of such a result, originally due to Leray (1933, Chapter III),[19] is based on the fact that (I.2.8) admits the following formal a priori estimate

$$\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u} \leq M\,, \tag{I.2.9}$$

[19] In his 1933 memoir, Leray proved $(\text{I.2.8})_4$ only in a certain weak sense. The first proof of $(\text{I.2.8})_4$ is due to Galdi (2002, Theorem 4.6).

where M is a positive constant depending on the data, Leray (1933, §17); see also Borchers (1992, Korollar 4.1) and Chapter XI. In analogy with the case $\boldsymbol{\omega} = \mathbf{0}$ (and the case of a bounded domain as well), solutions to (I.2.8) satisfying (I.2.9) will be called *Leray solutions*. Such solutions can be shown to possess as much regularity as allowed by the data, so that the next and crucial question is whether they exhibit all fundamental properties expected on physical grounds, already described in the previous section for the case $\boldsymbol{\omega} = \mathbf{0}$. To date, the answer to this question depends on whether $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq 0$ or $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0$ (namely, $\lambda \neq 0$ or $\lambda = 0$).

If $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq 0$, Galdi & Kyed (2011a) have proved that the velocity field (respectively, its gradient) of every Leray solution corresponding to $\boldsymbol{f}$ of bounded support is pointwise *bounded above*, at large distances, by a function that behaves like the solution (respectively, its gradient) to the (linearized) Oseen equations. In particular, it exhibits a wake region extending in the direction opposite to $\boldsymbol{\omega}$ if $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} > 0$, and along $\boldsymbol{\omega}$ otherwise, satisfies the energy equation and is unique, in the class of Leray solutions, for sufficiently "small" data. However, the investigation of Galdi & Kyed leaves unanswered the following important question:

(v) *Do Leray solutions admit an asymptotic expansion for large $|x|$ whose leading term is a solution to the Oseen equations?*

If $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0$, the physical properties of the solution originally constructed by Leray are known to hold, to date, only for "small" data. In particular Galdi & Kyed (2010) have shown that under these assumptions, the velocity field and its gradient are *bounded above*, at large distances, by the corresponding quantities of a function that behaves like a solution to the Stokes equations. One significant property that follows from this result is that Leray solutions satisfy the energy equation, but for "small" data only. Therefore, question (i) listed previously for the case $\boldsymbol{\omega} = \mathbf{0}$ continues to be *a significant open question in the case $\boldsymbol{\omega} \neq \mathbf{0}$ as well.* However, on the bright side, and in contrast to the case $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq 0$, one is able to furnish a detailed asymptotic expansion of the velocity field and of its derivatives for large $|x|$, but again, for small data. In fact, under these circumstances, Farwig & Hishida (2009) and Farwig, Galdi, & Kyed (2010) have shown that, as in the case $\boldsymbol{\omega} = \mathbf{0}$, the leading term of the expansion is a suitable Landau solution. Consequently, question (ii) listed previously for the case $\boldsymbol{\omega} = \mathbf{0}$ is *a significant open question also in the case $\boldsymbol{\omega} \neq \mathbf{0}$*.

I.2.2 Plane Flow

The results and open questions considered so far refer to three-dimensional solutions of the problem described by (I.0.4), (I.1.1), and (I.2.3). The two-dimensional solutions representing *plane motions* of $\mathcal{L}$, deserve separate consideration. As is well known, for these solutions the fields $\boldsymbol{v}$ and p depend only on x_1, x_2 (say) and, moreover, $v_3 \equiv 0$. Therefore, the relevant region for a description of the motion becomes a two-dimensional one. We shall restrict ourselves to the case $\boldsymbol{v}_\infty(x) = \boldsymbol{v}_0$, that is, $\boldsymbol{\omega} = \boldsymbol{0}$.[20]

Again, the first contribution to the resolution of the existence problem of plane flow in an exterior region is due to Leray (1933). Given $\boldsymbol{f}$ and $\boldsymbol{v}_*$, with $\boldsymbol{v}_*$ satisfying condition (I.1.2), he proved the existence of a regular pair $\boldsymbol{v}, p$ satisfying (I.0.4) and (I.1.1). However, in contrast to the three-dimensional case, Leray was unable to show whether the velocity field $\boldsymbol{v}$ satisfies condition (I.2.3). This is because the only information available on the behavior of $\boldsymbol{v}$ at large distances is the finiteness of the Dirichlet integral:

$$\int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{v} < \infty. \tag{I.2.10}$$

Condition (I.2.10) alone does not even ensure that $\boldsymbol{v}$ remains *bounded* at large spatial distances.[21] The question left open by Leray has been reconsidered, more than forty years later, by several authors (Gilbarg and Weinberger 1974, 1978; Amick 1986, 1988; Galdi 2004); see also Chapter XII. One of the main results found by these authors is that if $\boldsymbol{v}, p$ is the solution constructed by Leray, or if $\boldsymbol{v}, p$ is any (regular) solution to (I.0.4) corresponding to $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv 0$, with $\boldsymbol{v}$ satisfying (I.2.10), then there exists a vector $\widetilde{\boldsymbol{v}}$ to which $\boldsymbol{v}$ converges uniformly pointwise; also, the pressure field tends pointwise and uniformly at infinity to some constant. Notice that in general, no information is available about $\widetilde{\boldsymbol{v}}$. In principle, $\widetilde{\boldsymbol{v}}$ can be zero even though $\boldsymbol{v}_\infty \neq \boldsymbol{0}$. The fundamental question that still needs an answer is then

(vi) *Does the vector $\widetilde{\boldsymbol{v}}$ coincide with $\boldsymbol{v}_\infty$?*

A somewhat related question is

(vii) *If $\widetilde{\boldsymbol{v}} \neq 0$, does $\boldsymbol{v}$ tend pointwise to $\widetilde{\boldsymbol{v}}$?*[22]

Another question that naturally arises is that of the *order of decay* of $\boldsymbol{v}$ at infinity. Here one may expect that if $\boldsymbol{v}$ satisfies (I.2.10) and tends uniformly pointwise to some limit $\widetilde{\boldsymbol{v}}$, then $\boldsymbol{v}$ can be represented asymptotically by an expansion in "reasonable" functions of $r \equiv |x|$ with coefficients independent of r. However, if $\widetilde{\boldsymbol{v}} = \boldsymbol{0}$, not every such solution can be represented in this way,

[20] In this regard, it is worth noticing that to date, the case $\boldsymbol{\omega} \neq \boldsymbol{0}$ is virtually untouched. See also the Notes at the end of this chapter.

[21] Take, for instance, Ω the exterior of the unit circle, and

$$v(x) = \log^\alpha |x|, \quad 0 < \alpha < 1/2.$$

Clearly, v vanishes at $\partial\Omega$, has a finite Dirichlet integral, and becomes unbounded for large $|x|$.

[22] Concerning this question, it should be observed that under suitable conditions on the symmetry of the flow, it admits a positive answer; see Amick (1988) and Chapter XII. Unfortunately, the proof of the result reported in Galdi (2004, Theorem 3.4) is not correct.

for one can exhibit examples of solutions that satisfy (I.2.10) and decay more slowly than any negative power of r (see Hamel 1916 and cf. also Chapter XII). It is interesting to remark that for these solutions the flux Φ defined in (I.2.7) is *not* zero. Nevertheless, it should be observed that if $\boldsymbol{v}(x)$ tends uniformly pointwise to $\widetilde{\boldsymbol{v}} \neq \mathbf{0}$, and satisfies (I.2.10), Sazonov (1999) has shown, with the help of the results of Smith (1965), that $\boldsymbol{v}$ admits such an expansion, where the leading term is a solution to the (linearized) Oseen equations; see Chapter XII. This property can be considered the analogue of that established by Babenko for the three-dimensional case.

In view of what we said, the following question deserves attention:

(viii) *Is it possible to characterize the behavior, in a neighborhood of infinity, of a solution $\boldsymbol{v}$ satisfying* (I.2.10) *and tending uniformly to $\widetilde{\boldsymbol{v}} = 0$?*

The most important feature of Leray's solution is that it is *global* in the sense that it does not require restrictions on the size of the data. On the other hand, by what we have seen, one does not know, to date, whether such a solution satisfies condition (I.2.3).[23] Thus, we may wonder whether, using a different construction, we could prove existence in the full problem (I.0.4), (I.1.1), and (I.1.2) at least *in the small*, that is, by imposing suitable restrictions on the size of the data. Here again, we have to distinguish between the cases $\boldsymbol{v}_\infty \neq 0$ and $\boldsymbol{v}_\infty = 0$. In the former case, the answer is positive and is due to Finn and Smith (1967b) and Galdi (1993); see also Chapter XI. If, however, $\boldsymbol{v}_\infty = 0$, no result is available. So we are led to formulate the following questions:

(ix) *Does existence in problem* (I.0.4), (I.1.1), *and* (I.1.2) *with $\boldsymbol{v}_\infty = \mathbf{0}$ hold, even for small data $\boldsymbol{f}$ and $\boldsymbol{v}_*$?*

(x) *Can the solution of Finn and Smith be related to that of Leray with the same data?*[24]

My conjecture is that *for generic data, the answer to* (ix) *is negative.* This view is reinforced by the fact that the analogous problem for the *linear* case, obtained by formally suppressing the term $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$ in (I.0.4), admits an affirmative answer *if and only if* the data satisfy a finite number of compatibility conditions, or in other words, the space of the data has finite codimension; see Section 7 in Chapter V. (For an analogous situation in the three-simensional case, see Galdi 2009.)

Question (x) can be framed within the more general problem of the *uniqueness* of two-dimensional solutions. In this regard it should be pointed out that such a subject is still essentially obscure and that no significant result is available, with the exception of that of Finn and Smith (1967b) and Galdi (1993),

[23] For this reason, in dimension two, the Leray solution strictly speaking should not be called a "solution" to the steady-state exterior problem.

[24] See the previous footnote. It should be observed that the solution of Finn and Smith has a bounded Dirichlet integral.

which concern a somewhat restricted class of solutions; see Chapter XII.[25] The main reason for the lack of results is that the traditional methods usually employed to test the uniqueness of solutions of Navier–Stokes equations absolutely do not work in such a case. Perhaps the introduction of genuinely new tools to attack uniqueness will open new avenues to a better understanding of the entire problem of plane flow.

All the above considerations raise the question of whether the classical two-dimensional exterior problem with $\boldsymbol{v}_\infty \neq \mathbf{0}$ is indeed solvable for data of "arbitrary size." This question has been analyzed by Galdi (1999b). In that paper it is assumed that $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv \mathbf{0}$, $\boldsymbol{v}_\infty \neq \mathbf{0}$, and that $\mathbb{R}^3 - \Omega \neq \emptyset$ is sufficiently smooth, and symmetric around $\boldsymbol{v}_\infty$, and it is shown that if there is $\overline{v}$ such that (I.2.4), with the above data and $\boldsymbol{v}_0 \equiv \boldsymbol{v}_\infty$, has no solution for *all* $|\boldsymbol{v}_\infty| \geq \overline{v}$, then the homogeneous problem

$$
\begin{gathered}
\left.\begin{aligned}
&\Delta \boldsymbol{u} = \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \tau \\
&\nabla \cdot \boldsymbol{u} = 0
\end{aligned}\right\} \quad \text{in } \Omega\,, \\
\boldsymbol{u}|_{\partial\Omega} = 0\,, \\
\lim_{|x|\to\infty} \boldsymbol{u}(x) = \mathbf{0} \quad \text{uniformly pointwise}\,,
\end{gathered}
\tag{I.2.11}
$$

with $\boldsymbol{u}$ satisfying (I.2.9), must have at least one *nonzero* smooth solution $(\boldsymbol{u}, \tau)$ that is also symmetric (in a well defined sense) around $\boldsymbol{v}_\infty$. Although at first glance, this possibility seems to be easily ruled out, a mathematical proof showing that problem (I.2.11) admits only the zero solution is not yet available, and it is probably far from being obvious. We thus have the following:

(xi) *Is* $\boldsymbol{u} \equiv \nabla\tau \equiv \mathbf{0}$ *the only solution to* (I.2.11) *in the specified class?*

It is worth emphasizing that the possibility that (I.2.11) may admit a nonzero smooth solution is very questionable on physical grounds, and the occurrence of such a situation would cast serious doubt on the meaning of the two-dimensional assumption.

I.3 Flow in Regions with Unbounded Boundaries

Even though some basic issues were formulated quite long ago, a systematic study of flow in region with unbounded boundaries began only more recently, in the period 1976-1978, through the fundamental work of J.G. Heywood, C.J. Amick, O.A. Ladyzhenskaya, and V.A. Solonnikov. Therefore, it is not

[25] In fact, the uniqueness results given by these authors are of such a "local" type that it is not known whether the solution of Finn & Smith coincides with that of Galdi.

surprising that several basic questions are still far from being answered. Here, we wish to mention a few of them.

It might be surprising that we don't find any *direct* contribution of Jean Leray to the subject; however, one of the main questions still open seems to be in fact due to him (Ladyzhenskaya 1959b, p. 77, 1959c, p. 551). Let us describe this problem. Let Ω be a "distorted tube" of $\mathbb{R}^n$ $(n = 2, 3)$ with two semi-infinite cylindrical ends (strips, for $n = 2$),[26] i.e.,

$$\Omega = \bigcup_{i=0}^{2} \Omega_i, \tag{I.3.1}$$

where Ω_0 is a smooth bounded subset of Ω, while $\Omega_i, i = 1, 2$, are disjoint regions that, in possibly different coordinate systems (depending on Ω_i, $i = 1, 2$), reduce to straight cylinders (strips, for $n = 2$), that is,

$$\Omega_i = \{x \in \mathbb{R}^n : x_n > 0,\ x' \equiv (x_1, \dots, x_{n-1}) \in \Sigma_i\}, \tag{I.3.2}$$

with Σ_i bounded and simply connected regions in $\mathbb{R}^{n-1}$. Denoting by Σ any bounded intersection of Ω with a plane, which in Ω_i reduces to Σ_i, and by $\boldsymbol{n}$ a unit vector orthogonal to Σ, oriented from Ω_1 toward Ω_2 (say) owing to the incompressibility of the liquid and assuming that $\boldsymbol{v}$ vanishes at the boundary, we at once deduce that the flux Φ of $\boldsymbol{v}$ through Σ is a constant:

$$\Phi \equiv \int_{\Sigma} \boldsymbol{v} \cdot \boldsymbol{n} = \text{const.} \tag{I.3.3}$$

Therefore, a natural question that arises is that of establishing existence of a flow subject to a given flux. This condition alone, of course, may not be enough to determine the flow uniquely, and similarly to what we do for flows in exterior regions, we must prescribe a velocity field $\boldsymbol{v}_{\infty i}$ as $|x| \to \infty$ in the exits Ω_i. However, in contrast to the case of flows past bodies, $\boldsymbol{v}_{\infty i}$ need *not* be uniform, and in fact, if $\Phi \neq 0$, it is easily seen that $\boldsymbol{v}_{\infty i}$ *cannot* be uniform. Thus, one has to figure out how to prescribe $\boldsymbol{v}_{\infty i}$. What is most natural to expect is that the flow corresponding to a given flux Φ should tend, as $|x| \to \infty$ in each Ω_i, to the *Poiseuille solution of the Navier–Stokes equation in Ω_i corresponding to the flux Φ*, that is, to a pair $\left(\boldsymbol{v}_0^{(i)}, p_0^{(i)}\right)$, where

$$\boldsymbol{v}_0^{(i)} \equiv (0, \dots, v_0^{(i)}(x')), \qquad \nabla p_0^{(i)} \equiv (0, \dots, -C_i) \tag{I.3.4}$$

for some constants $C_i = C_i(\Phi)$, and

$$\begin{gathered} \sum_{j=1}^{n-1} \frac{\partial^2 v_0^{(i)}}{\partial x_j^2} = -C_i \quad \text{in } \Sigma_i\,, \\ v_0^{(i)} = 0 \quad \text{at } \partial\Sigma_i\,. \end{gathered} \tag{I.3.5}$$

[26] Entirely analogous considerations could be performed for the case of more than two "outlets" Ω_i.

If, for instance, $n = 3$ and the sections are circles of radius R_i, the solution to (I.3.4), (I.3.5) is the *Hagen–Poiseuille flow*

$$v_0^{(i)}(x') = \frac{1}{4} C_i R_i^2 (1 - |x'|^2 / R_i^2).$$

The problem of determining a motion in a region Ω with cylindrical “exits,” subject to a given flux Φ and tending in each “exit” to the Poiseuille solution corresponding to Φ, is known as *Leray's problem* (Ladyzhenskaya 1959b, p.77, 1959c, p. 551). This problem has been the object of deep research by Amick (1977, 1978), Horgan and Wheeler (1978), and Ladyzhenskaya and Solonnikov (1980). However, its solvability has been shown only for Φ sufficiently small. We are therefore led to the following basic question:

(i) *Is Leray's problem solvable for any value of the flux Φ?*

Despite the seemingly different natures of the two physical problems, due to the quite different shape of the regions of flow, from the mathematical point of view question (i) appears to be intimately related to the analogous problem in a bounded region, which we discussed in Section 1.

Similar questions can be formulated for regions having “outlets” to infinity whose cross sections are not necessarily bounded. So, assume that Ω is of the type (I.3.1), (I.3.2), where now, the sections are allowed to vary with x_n and become unbounded as x_n tends to infinity.[27] This time, the condition that the limiting velocity fields $\boldsymbol{v}_{\infty i}$ are zero is no longer in conflict with the conservation of flux (I.3.3), and we may try to solve problem (I.0.4), (I.1.1) (with $\boldsymbol{v}_* = 0$) under the condition of prescribed flux and vanishing velocity as $|x|$ tends to infinity in each “outlet” Ω_i.

Unlike flow in exterior regions, here the case of two-dimensional solutions presents results more complete than in the case of fully three-dimensional motions, thanks to the thorough investigation of Amick and Fraenkel (1980). Specifically, these authors prove the existence of solutions and pointwise asymptotic decay of the corresponding velocity fields under different assumptions on the “growth” of Σ_i and with a “small” flux if the sections have a certain rate of “growth.” However, two important issues are left out, that is, uniqueness of solutions and their order of decay at large distances. These two problems have been recently studied and solved for “small” flux by K. Pileckas in the particular case that each Ω_i is a body of revolution of type

$$\left\{ x \in \mathbb{R}^2 : \; x_2 > 0, \; |x_1| < g_i(x_2) \right\},$$

provided the (smooth) positive functions $g_i(x_2)$ satisfy suitable “growth” conditions as $x_2 \to \infty$. As expected, the decay rate of solutions is related to the inverse power of the functions g_i; see Pileckas (1996c, 1997); see also Nazarov & Pileckas (1998).

[27] We also assume that Σ cannot shrink to a point, that is, the measure $|\Sigma|$ of Σ is bounded below by a positive constant.

On the other hand, several fundamental questions remain open for three-dimensional flows. Actually, one can prove that if the sections Σ_i become unbounded at a sufficiently large rate, then a solution exists that, in the mean, converges to zero at large distances; otherwise, the problem admits only partial answers, and in some cases, it is completely unsolved. Let us briefly explain why. The leading idea is to try to obtain, as in previous instances, a bound on the Dirichlet integral D (say) of the velocity field $\boldsymbol{v}$. Now if the cross sections become unbounded, we may distinguish the following two possibilities:[28]

(a) $\displaystyle\int_0^\infty |\Sigma_i|^{-2} dx_3 < \infty, \quad i = 1, 2,$

(b) $\displaystyle\int_0^\infty |\Sigma_i|^{-2} dx_3 = \infty, \quad i = 1, 2.$

In case (a), using inequality (I.1.6) one can show that the condition of constant flux is compatible with the finiteness of D, and in fact, using more or less standard, tools one proves an a priori bound that allows us to obtain the existence of a solution to the problem for arbitrary flux Φ (Ladyzhenskaya and Solonnikov 1980). However, in general, one cannot prove a pointwise decay of $\boldsymbol{v}$ at large distances; rather, only a weaker behavior in the mean is achieved. We are thus led to formulate the following questions:

(ii) *In case (a), is it possible to prove the pointwise decay of solutions whose velocity field has a bounded Dirichlet integral?*
(iii) *Is it possible to relate the asymptotic behavior of such solutions to the rate of growth of cross sections Σ_i?*

There is one particular, though interesting, situation in which both questions (ii) and (iii) are positively answered, namely when each outlet Ω_i "degenerates" into a half-space (Borchers & Pileckas 1992, Chang 1992, 1993, Coscia & Patria 1992, Galdi & Sohr 1992); see also Chapter XIII. In such a case, Ω becomes the so-called *aperture domain* (see Heywood,1976):

$$\Omega = \{x \in \mathbb{R}^n : x_n \neq 0 \text{ or } x' \in S\},$$

with S a bounded region of the plane (the "aperture"). However, unlike the results of Ladyzhenskaya and Solonnikov, the results of all the preceding authors require that the flux Φ be sufficiently small. Therefore, we have the following question:

(iv) *Is it possible to show the known results for three-dimensional flow in the aperture domain for an arbitrary flux Φ?*

It should be remarked that the two-dimensional analogue of this problem appears to be difficult to treat, and all methods employed by the above authors

[28] Of course, we may assume that one section verifies (a) and the other verifies (b). The considerations to follow should then be modified accordingly which, however, would produce no conceptual difference.

do not apply there. Nevertheless, by different tools, one proves the existence of a solution that tends to zero at large distances uniformly pointwise for any Φ (Galdi, Padula, and Passerini 1995); however, the asymptotic structure of such solutions is completely characterized only for small Φ and, more importantly, when the aperture S is symmetric around x_2 (Galdi, Padula and Solonnikov 1996; cf. also Chapter XIII).[29] Therefore, we have the following question:

(v) *Is it possible to characterize the asymptotic structure of plane flow in an aperture domain when the aperture is not symmetric, even for a small flux Φ?*

Let us next consider case (b). Again using inequality (I.1.6), one shows that the nonzero flux condition becomes incompatible with the existence of solutions whose velocity field has a bounded Dirichlet integral. However, if[30]

$$G_i \equiv \int_0^\infty |\Sigma_i|^{-3q/2+1} dx_3 < \infty, \quad i = 1, 2, \quad \text{some } q > 2\,, \tag{I.3.6}$$

one can show that solutions may still exist with corresponding velocity field $\boldsymbol{v}$ satisfying the condition

$$\int_\Omega (\nabla \boldsymbol{v} : \nabla \boldsymbol{v})^{q/2} < \infty, \quad q > 2. \tag{I.3.7}$$

Therefore, a subspace $\mathcal{S}_q$ (say) of functions satisfying (I.3.7) seems to be a "most natural" space in which to set the existence problem. Whether this conjecture is true is yet to be ascertained in the general case. However, if Ω_i is a body of revolution defined by a smooth positive function g_i, K. Pileckas (1996c, 1997) has proved the existence of solutions in the class (I.3.7) for *arbitrary* flux Φ, provided g_i satisfies certain conditions at large distances more restrictive than those merely required by (b) and (I.3.6).[31] Corresponding decay estimates are also given.

A last possibility arises when the sections become unbounded in such a way that the integrals G_i defined in (I.3.6) are infinite for *every* value of $q > 1$. For such regions of flow it is not even clear in which space the problem has to be formulated. In this regard we should not overlook the approach of Ladyzhenskaya and Solonnikov (1980), who, in a rather large class of regions Ω with outlets Ω_i of unbounded cross section, prove the existence of a solution whose velocity field has a finite Dirichlet integral on every *bounded* portion of Ω. Growth estimates *from above* on such a quantity are then provided in terms of the growth of the cross sections of Ω_i. The question whether these solutions tend to zero at infinity is, however, left open.

[29] In fact, the asymptotes are given by suitable Jeffery–Hamel solutions; see Rosenhead (1940).

[30] Notice that since $|\Sigma| \geq \Sigma_0 > 0$, in case (b) the integrals G_i are infinite for any $q \leq 2$.

[31] In such a case, $|\Sigma_i| = \pi g_i^2(x_3)$.

Notes for Chapter I

Section I.1. The study of the properties of solutions to the Navier–Stokes equations has received substantial attention also under boundary conditions other than (I.1.1). A popular one is the following:

$$\boldsymbol{v} + \beta \boldsymbol{n} \cdot \boldsymbol{T}(\boldsymbol{v}, p) \times \boldsymbol{n} = \boldsymbol{b}_*, \quad \text{at } \partial\Omega\,, \tag{$*$}$$

where β is a constant, $\boldsymbol{n}$ is the outer unit normal at $\partial\Omega$, $\boldsymbol{T}(\boldsymbol{v}, p)$ is the Cauchy stress tensor (I.0.2), and $\boldsymbol{b}_*$ is a prescribed vector field. Condition $(*)$ was introduced for the first time by Navier (1827).[32] If $\beta = 0$, $(*)$ reduces to to (I.1.1), and $\boldsymbol{v}$ is totally prescribed (*no slip*), while if $1/\beta \to 0$, only $\boldsymbol{v} \cdot \boldsymbol{n}$ is prescribed, and we lose information on the tangential component $\boldsymbol{v}_\tau$ (*pure slip*). However, if $\beta \neq 0$ and finite, $(*)$ allows for $\boldsymbol{v}_\tau$ to be nonzero, by an amount that depends on the magnitude of the tangential stress at the boundary (*partial slip*).

The Navier condition $(*)$, with $1/\beta \neq 0$ or $\to 0$, has been employed in a wide range of problems. They include free surface problems (see, e.g., Solonnikov 1982, Maz'ja, Plamenevskii, & Stupyalis 1984), turbulence modeling (see, e.g., Parés 1992, Galdi & Layton 2000), and inviscid limits (see, e.g., Xiao & Xin 2007, Beirão da Veiga 2010).

Concerning the use of $(*)$ in steady-state studies, after the pioneering work of Solonnikov and Ščadilov (1973), where pure slip boundary conditions are used along with a linearized system of equations (Stokes equations), in the last few years there has been considerably increasing interest. Besides the papers of Beirão da Veiga (2004, 2005), which generalize and simplify the proof of the results of Solonnikov & Ščadilov, we refer the interested reader, for example, to Ebemeyer & Frehse (2001) for flow in bounded domains, Mucha (2003), Konieczny (2006), and Beirão da Veiga (2006) for flow in infinite channels and pipes, to Konieczny (2009) for flow in exterior domains, and to the literature cited therein.

It is conceptually interesting to notice that some of the basic problems left open for liquids modeled by the Navier–Stokes equations may find a complete and positive answer for liquid models whose constitutive assumptions differ from that given in (I.0.2). Such models are generically referred to as *non-Newtonian.*[33] Among the most successful and widely adopted models of non-Newtonian liquids are those called *generalized Newtonian*, where the shear viscosity coefficient μ is no longer constant, but depends on the "amount of shear," namely, on $|\boldsymbol{D}(\boldsymbol{v})|$, with $\boldsymbol{D}$ the stretching tensor defined in (I.0.3).

[32] Navier proposed $(*)$ (or better, an equation equivalent to $(*)$ with $\boldsymbol{b}_* = \boldsymbol{0}$) in alternative to the "adherence" condition (I.1.1), with the objective of explaining the difference between the discharges in glass and copper tubes, as experimentally observed by Girard (1816).

[33] For this type of models and their range of applicability, we refer the interested reader to the monograph of Bird, Armstrong, and Hassager (1987).

For example, in an appropriate range of shear, for liquids such as latex paint, styling gel, molasses, and blood, μ is found to be a decreasing function of $|\boldsymbol{D}(\boldsymbol{v})|$ (*shear-thinning liquids*), whereas for others, such as a mixture of corn starch and water, and clay slurries, μ increases with $|\boldsymbol{D}(\boldsymbol{v})|$ (*shear-thinning liquids*). A prototypical example of the generalized Newtonian model is the one with μ related to $|\boldsymbol{D}(\boldsymbol{v})|$ by the following formula (*power law* model)

$$\mu = \mu_0 + \varepsilon_1 |\boldsymbol{D}(\boldsymbol{v})|^{\varepsilon_2}\,, \tag{$**$}$$

where the (material) constants μ_0, ε_1, and ε_2 satisfy $\mu_0 > 0$, $\varepsilon_1 \geq 0$, and $\varepsilon_2 \in (-1, \infty)$. Thus, for $\varepsilon_1 > 0$, the model is shear-thinning if $\varepsilon_2 \in (-1, 0)$ and shear-thickening if $\varepsilon_2 \in (0, \infty)$, whereas we recover the Newtonian (Navier–Stokes) model if either ε_1 or ε_2 is zero. Now, for a given μ_0, and $\varepsilon_1, \varepsilon_2 > 0$ *arbitrarily small*, one can show (Ladyzhenskaya 1967, §4, Galdi 2008, §2.2.1(b)) that the boundary value problem obtained using the constitutive assumption $(**)$, namely,

$$\left.\begin{array}{l} \nabla \cdot [(\mu_0 + \varepsilon_1 |\boldsymbol{D}(\boldsymbol{v})|^{\varepsilon_2})\boldsymbol{D}(\boldsymbol{v})] = \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla p + \boldsymbol{f} \\ \nabla \cdot \boldsymbol{v} = 0 \end{array}\right\} \ \text{in } \Omega \tag{$***$}$$
$$\boldsymbol{v} = \boldsymbol{v}_* \ \text{ at } \partial\Omega\,,$$

where Ω is a bounded (sufficiently smooth) domain with a possibly *disconnected* boundary, has a solution for any $\boldsymbol{v}_*$, $\boldsymbol{f}$ in a suitable function class, and, more importantly, with $\boldsymbol{v}_*$ satisfying the "natural" compatibility condition (I.1.2). As we have emphasized, unless $\partial\Omega$ is connected, this is an outstanding open question for the Navier–Stokes model, obtained by setting $\varepsilon_1 = 0$ or $\varepsilon_2 = 0$.[34] The above result is conceptually very intriguing, in that it can be rephrased, in common and provocative words, as follows: "what we do not know whether it is true for water, becomes certainly true if we add to water a pinch of corn starch"!

Along the same lines, Galdi & Grisanti (2010) have considered $(***)$, with Ω an exterior region of $\mathbb{R}^2$ and $\boldsymbol{v}_* \equiv \boldsymbol{0}$, along with the asymptotic condition (I.2.4). They have shown that for arbitrary $\varepsilon_1 > 0$ and $\varepsilon_2 \in (-1, 0)$, the resulting boundary value problem possesses a solution for *any choice* of $\boldsymbol{f}$ in an appropriate function class, and for *arbitrary* $\boldsymbol{v}_0 \in \mathbb{R}^2$. As we pointed out in Section I.2, this is another fundamental open question for the Navier–Stokes model.

Section I.2. Among the many other significant "exterior" problems that we can formulate, and that we will not treat in this book, of particular interest is the steady flow of a Navier–Stokes liquid under the action of a given constant shear. This problem, which has received great attention in the applied science community thanks to the fundamental work of Saffman (1965) on the effect

[34] This result can be extended to more general constitutive assumptions than $(**)$; see Galdi, (2008).

of shear on the lift of a sphere, amounts to solving (I.2.4) with $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv \boldsymbol{0}$, along with the following condition at large distances:

$$\lim_{|x|\to\infty} (\boldsymbol{v}(x) - \boldsymbol{v}_\infty) = \boldsymbol{0}\,, \quad \boldsymbol{v}_\infty := \kappa x_2 \boldsymbol{e}_1\,,$$

where κ is a nonzero constant and $\boldsymbol{e}_1$ is the unit vector in the direction x_1. Unfortunately, the classical methods for existence of steady state solutions in exterior domains, with which the reader will become familiar by flipping through the pages of this book, all fail for the above problem. For instance, it seems very unlikely that one could prove a bound of the type (I.2.9) for the field $\boldsymbol{u} := \boldsymbol{v} - \boldsymbol{v}_\infty$, even for small data. I take this opportunity to bring this intriguing open question to the attention of the interested mathematician.

II

Basic Function Spaces and Related Inequalities

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Introduction

In this chapter we shall introduce some function spaces and enucleate certain properties of fundamental importance for further developments. Particular emphasis will be given to what are called *homogeneous Sobolev spaces*, which will play a fundamental role in the study of flow in exterior domains. We shall not attempt, however, to give an exhaustive treatment of the subject, since this is beyond the scope of the book. Therefore, the reader who wants more details is referred to the specialized literature quoted throughout. As a rule, we give proofs where they are elementary or relevant to the development of the subject, or also when the result is new or does not seem to be widely known.

II.1 Preliminaries

In this section we collect a number of preparatory results. After introducing some basic notation, we shall recall the relevant properties of Banach spaces and of certain classical spaces of smooth functions as well. We shall finally define and analyze the properties of special subsets of the Euclidean space.

G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems*, Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-09620-9_2,

II.1.1 Basic Notation[1]

The symbols $\mathbb{N}$ and $\mathbb{N}_+$ denote the set of all non-negative and of all positive natural numbers, respectively.

For X a set, we denote by X^m, $m \in \mathbb{N}_+$, the Cartesian product of m copies of X. Thus, denoting by $\mathbb{R}$ the real line, $\mathbb{R}^n$ is the n-dimensional Euclidean space. Points in $\mathbb{R}^n$ will be denoted by $x = (x_1, \dots, x_n) \equiv (x_i)$ and corresponding vectors by $\boldsymbol{u} = (u_1, \dots, u_n) \equiv (u_i)$. Sometimes, the ith component u_i of the vector $\boldsymbol{u}$ will be denoted by $(\boldsymbol{u})_i$. More generally, for $\boldsymbol{T}$ a tensor of order $m \geq 2$, its generic component $T_{ij\dots kl}$ will be also denoted by $(\boldsymbol{T})_{ij\dots kl}$. The components of the *identity tensor* $\boldsymbol{I}$, are denoted by δ_{ij} (*Kronecker delta*).

The *distance between two points* x and y of $\mathbb{R}^n$ is indicated by $|x-y|$, and we have

$$|x-y| = \left[\sum_{i=1}^{n}(x_i - y_i)^2\right]^{1/2}.$$

More generally, the *distance between two subsets* A *and* B of $\mathbb{R}^n$ is indicated by $\operatorname{dist}(A,B)$, where

$$\operatorname{dist}(A,B) = \inf_{x\in A, y\in B} |x-y|\,.$$

The *modulus of a vector* $\boldsymbol{u}$ is indicated by $|\boldsymbol{u}|$ (or by u) and it is

$$|\boldsymbol{u}| = \left(\sum_{i=1}^{n} u_i^2\right)^{1/2}.$$

Given two vectors $\boldsymbol{u}$, $\boldsymbol{v}$, the second-order tensor having components $u_i v_j$ (*dyadic product of* $\boldsymbol{u}, \boldsymbol{v}$) will be denoted by $\boldsymbol{u} \otimes \boldsymbol{v}$.

The *canonical basis* in $\mathbb{R}^n$ is indicated by

$$\{\boldsymbol{e}_i\} \equiv \big\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\big\}$$

with

$$\boldsymbol{e}_1 = (1,0,\dots,0), \quad \boldsymbol{e}_2 = (0,1,0,\dots,0), \quad \dots, \boldsymbol{e}_n = (0,\dots,0,1).$$

We also set

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$$

$$\mathbb{R}^n_- = \{x \in \mathbb{R}^n : x_n < 0\}\,.$$

[1] For other notation, we refer the reader to footnotes 8, 9, and 10 of Section I.1

For $r > 0$ and $x \in \mathbb{R}^n$ we denote by $B_r(x)$ the (n-dimensional) *open ball of radius* r *centered at* x, i.e.,

$$B_r(x) = \{y \in \mathbb{R}^n : \ |x - y| < r\}.$$

For $r = 1$, we shall put

$$B_1(x) \equiv B(x),$$

and for $x = 0$,

$$B_r(0) \equiv B_r.$$

Unless the contrary is explicitly stated, the Greek letter Ω shall always mean a *domain*, i.e., an open connected set of $\mathbb{R}^n$.

Let $\mathcal{A}$ be an arbitrary set of $\mathbb{R}^n$. We denote by $\overline{\mathcal{A}}$ its *closure*, by $\mathcal{A}^c = \mathbb{R}^n - \mathcal{A}$ its *complementary set* (in $\mathbb{R}^n$), by $\overset{\circ}{\mathcal{A}}$ its *interior*, and by $\partial\mathcal{A}$ its *boundary*. For $n \geq 2$, the boundary of the n-dimensional unit ball centered at the origin (i.e., the n-dimensional unit sphere) is denoted by S^{n-1}:

$$S^{n-1} = \partial B_1.$$

Moreover, $\delta(\mathcal{A})$ is the *diameter of* $\mathcal{A}$, that is,

$$\delta(\mathcal{A}) = \sup_{x,y \in \mathcal{A}} |x - y|.$$

If $\Omega^c \subset B_\rho$ for some $\rho \in (0, \infty)$ and with the origin of coordinates in Ω^c, we set

$$\begin{aligned}
\Omega_r &= \Omega \cap B_r, \quad r > \rho, \\
\Omega^r &= \Omega - \overline{\Omega}_r, \quad r > \rho, \\
\Omega_{r,R} &= \Omega_R - \overline{\Omega}_r, \quad \rho < r < R.
\end{aligned}$$

If $\mathcal{A}$ is Lebesgue measurable and μ_L *is the (Lebesgue) measure in* $\mathbb{R}^n$, we put

$$|\mathcal{A}| = \mu_L(\mathcal{A}).$$

The measure of the n-dimensional unit ball is denoted by ω_n; therefore,

$$\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)},$$

where Γ is the Euler gamma function

By c, c_i, C, C_i, $i = 1, 2, \ldots$, we denote generic positive constants, whose possible dependence on parameters $\xi_1, \ldots, \xi_m$ will be specified whenever it is needed. In such a case, we write $c = c(\xi_1, \ldots, \xi_m)$, $C = C(\xi_1, \ldots, \xi_m)$, or, especially in formulas, $c_{\xi_1,\ldots,\xi_m}$, $C_{\xi_1,\ldots,\xi_m}$, *etc.* Sometimes, we shall use the symbol c to denote a positive constant whose numerical value or dependence on parameters is not essential to our aims. In such a case, c may have several different values in a single computation. For example, we may have, in the same line, $2c \le c$.

For a real function u in Ω, we denote by supp (u) the *support of* u, that is,

$$\mathrm{supp}\,(u) = \overline{\{x \in \Omega : \quad u(x) \neq 0\}}.$$

For a real smooth function u in Ω we set

$$D_j u = \frac{\partial u}{\partial x_j}, \qquad D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j};$$

likewise,

$$\nabla u = (D_1 u, \ldots, D_n u)$$

denotes the *gradient of* u,

$$D^2 u = \{D_{ij} u\}$$

is the *matrix of the second derivatives.* Occasionally, the gradient of u will be indicated by $D^1 u$ or, more simply, by $D\,u$. We also set[2]

$$\Delta u = D_{ii} u$$

is the *Laplacean of* u.

For a vector function $\boldsymbol{u} = (u_1, \ldots, u_n)$, the *divergence of* $\boldsymbol{u}$, $\nabla \cdot \boldsymbol{u}$, is defined by

$$\nabla \cdot \boldsymbol{u} = D_i u_i\,,$$

and, if $n = 3$,

$$\nabla \times \boldsymbol{u} = (D_2 u_3 - D_3 u_2, D_3 u_1 - D_1 u_3, D_1 u_2 - D_2 u_1)$$

denotes the *curl of* $\boldsymbol{u}$. Similarly, if $n = 2$, $\nabla \times \boldsymbol{u}$ has only one component, orthogonal to $\boldsymbol{u}$, given by $(D_1 u_2 - D_2 u_1)$.

If α is an n-tuple of non-negative integers α_i, we set

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

[2] According to Einstein's summation convention, unless otherwise explicitly stated, pairs of identical indices imply summation from 1 to n.

The n-tuple α is called a *multi-index.*

If $\mathcal{D}$ is a domain with $|\mathcal{D}| < \infty$, and $\boldsymbol{u} : \mathcal{D} \to \mathbb{R}^n$, $n \geq 1$, we denote by $\overline{\boldsymbol{u}}_{\mathcal{D}}$ the *mean value of the function* $\boldsymbol{u}$ over the domain $\mathcal{D}$, namely,

$$\overline{\boldsymbol{u}}_{\mathcal{D}} = \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} \boldsymbol{u} \,,$$

whenever the integral is meaningful.

We shall also use the following standard notation, for functions f and g defined in a neighborhood of infinity:

$$f(x) = O(g(x)) \ \text{ means } |f(x)| \leq M_1 |g(x)| \ \text{ for all } |x| \geq M_2,$$

$$f(x) = o(g(x)) \ \text{ means } \lim_{|x| \to \infty} |f(x)|/|g(x)| = 0$$

where M_1, M_2 denote positive constants.

Finally, the symbols $\square$ and $\blacksquare$ will indicate the end of a proof and of a remark, respectively.

II.1.2 Banach Spaces and their Relevant Properties

For the reader's convenience, in this subsection we shall collect all relevant properties of Banach spaces that will be frequently use throughout this book.

Let X be a *vector* (or *linear*) *space* on the field of real numbers, with corresponding operations of sum of two elements, $x + y$, and multiplication of an element x by a real number α, $\alpha\, x$. Then, X is a *normed space* if there exists a map, called *norm*,

$$\| \cdot \|_X : x \in X \to \|x\|_X \in \mathbb{R}$$

satisfying the following conditions, for all $\alpha \in \mathbb{R}$ and all $x, y \in X$:

(1) $\|x\|_X \geq 0$, and $\|x\|_X = 0$ implies $x = 0$;
(2) $\|\alpha x\|_X = |\alpha|\, \|x\|_X$;
(3) $\|x + y\|_X \leq \|x\|_X + \|y\|_X$.

In what follows, X denotes a normed space.

Two norms $\|\cdot\|_X$ and $\|\cdot\|_X^*$ on X are *equivalent* if $c_1 \|\cdot\|_X \leq \|\cdot\|_X^* \leq c_2 \|\cdot\|_X$, for some constants $c_1 \leq c_2$.

A sequence $\{x_k\}$ in X is *convergent* to $x \in X$ if

$$\lim_{k \to \infty} \|x_k - x\|_X = 0 \,, \tag{II.1.1}$$

or, in equivalent notation, $x_k \to x$.

A subset S of X is a *subspace* if $\alpha\, x + \beta\, y$ is in S, for all $x, y \in S$ and all $\alpha, \beta \in \mathbb{R}$.

A subset B of X is *bounded* if there exists a number $M > 0$ such that $\sup_{x \in B} \|x\|_X \le M$.

A subset C of X is *closed* if for every sequence $\{x_k\} \subset C$ such that $x_k \to x$ for some $x \in X$, implies $x \in C$.

The *closure* of a subset S of X consists of those points of $x \in X$ such that $x_k \to x$ for some $\{x_k\} \subset S$.

A subset K of X is *compact* if from every sequence $\{x_k\} \subset K$ we can find a subsequence $\{x_{k'}\}$ and a point $x \in K$ such that $x_{k'} \to x$.

A subset of X is *precompact* if its closure is compact.

A subset S of X is *dense* in X if for any $x \in X$ there is a sequence $\{x_k\} \subset S$ such that $x_k \to x$.

A subset of X is *separable* if it contains a countable dense set. We have the following result (see, e.g. Smirnov 1964, Theorem in §94).

Theorem II.1.1 *Let X be a separable normed space. Then every subset of X is separable.*

A space X is (continuously) *embedded* in a space Y if X is a linear subspace of Y and the identity map $i : X \to Y$ maps bounded sets into bounded sets, that is, $\|x\|_Y \le c\,\|x\|_X$, for some constant c and all $x \in X$. In this case, we shall write

$$X \hookrightarrow Y\,.$$

X is *compactly embedded in* Y if $X \hookrightarrow Y$ and, in addition, i maps bounded sets of X into precompact sets of Y. In such a case we write

$$X \hookrightarrow\hookrightarrow Y.$$

Two linear subspaces X_1, Y_1 of normed spaces X and Y, respectively, are *isomorphic* [respectively, *homeomorphic*] if there is a map L from X_1 onto Y_1, called *isomorphism* [respectively, *homeomorphism*], such that (i) L is linear; (ii) L is a bijection, and, moreover, (iii) $\|L(x)\|_X = \|x\|_Y$ [respectively, $c_1\|x\|_X \le \|L(x)\|_Y \le c_2\|x\|_X$, for some $c_1 \le c_2$], for all $x \in X_1$, where $\|\cdot\|_X$, and $\|\cdot\|_Y$ denote the norms in X and Y.

A sequence $\{x_k\} \subset X$ is called *Cauchy* if

$$\text{given } \varepsilon > 0 \text{ there is } \overline{n} = \overline{n}(\varepsilon) \in \mathbb{N}\text{: } \|x_k - x_{k'}\|_X < \varepsilon \text{ for all } k, k' \ge \overline{n}\,.$$

If every Cauchy sequence in X is convergent to an element of X, then X is called *complete.*

A *Banach space* is a normed space where every Cauchy sequence is there convergent or, equivalently, a Banach space is a *complete normed space.*

If X is not complete, namely, there is at least one Cauchy sequence in X that is not convergent to an element of X, we can nevertheless find a uniquely determined[3] Banach space $\widehat{X}$, with the property that X is isomorphic to a

[3] Up to an isomorphism.

dense subset of $\widehat{X}$. The space $\widehat{X}$ is called (Cantor) *completion* of X, and its elements are classes of equivalence of Cauchy sequences, where two such sequences, $\{x_k\}$, $\{x'_m\}$, are called equivalent if $\lim\limits_{l\to\infty} \|x_l - x'_l\|_X = 0$; see, e.g., Smirnov (1964, §85).

Suppose, now, that on the vector space X we can introduce a real-valued function $(\cdot,\cdot)_X$ defined in $X \times X$, satisfying the following properties for all $x, y, z \in X$ and all $\alpha, \beta \in \mathbb{R}$

(i) $(x, y)_X = (y, x)_X$,
(ii) $(\alpha\, x + \beta\, y, z)_X = \alpha\,(x, z)_X + \beta\,(y, z)_X$,
(iii) $(x, x)_X \geq 0$, and $(x, x)_X = 0$ implies $x = 0$.

Then X becomes a normed space with norm

$$\|x\|_X \equiv \sqrt{(x, x)_X}\,. \tag{II.1.2}$$

The bilinear form $(\cdot,\cdot)_X$ is called *scalar product*, and if X, endowed with the norm (II.1.2), is complete, then X is called *Hilbert space.*

A countable set $\mathfrak{B} \equiv \{x_k\}$ in a Hilbert space X is called a *basis* if (i) $(x_j, x_k) = \delta_{jk}$, for all $x_j, x_k \in \mathfrak{B}$, and $\lim_{N\to\infty} \|\sum_{k=1}^{N}(x, x_k)x_k - x\|_X = 0$, for all $x \in X$.

A linear map $\ell : X \to \mathbb{R}$ on a normed space X, such that

$$s_\ell \equiv \sup_{x\in X;\|x\|_X=1} |\ell(x)| < \infty \tag{II.1.3}$$

is called *bounded linear functional* or, in short, *linear functional* on X. The set, X', of all linear functionals in X can be naturally provided with the structure of vector space, by defining the sum of two functionals ℓ_1 and ℓ_2 as that $\ell \in X'$ such that $\ell(x) = \ell_1(x) + \ell_2(x)$ for all $x \in X$, and the product of a real number α with a functional ℓ as that functional that maps every $x \in X$ into $\alpha\ell(x)$. Moreover, it is readily seen that the map $\ell \in X' \to \|\ell\|_{X'} = s_\ell \in \mathbb{R}$, with s_ℓ defined in (II.1.3), defines a norm in X'. It can be proved that if X is a Banach space, then also X', endowed with the norm $\|\cdot\|_{X'}$, is a Banach space, sometime referred to as *strong dual*; see, e.g. Smirnov (1964, §99).

A Banach space X is naturally embedded into its second dual $(X')' \equiv X''$ via the map $M : x \in X \to J_x \in X''$, where the functional J_x on X' is defined as follows: $J_x(\ell) = \ell(x)$, $\ell \in X'$. One can show that the range, $\mathsf{R}(M)$, of M is closed in X'' and that M is an isomorphism of X onto $\mathsf{R}(M)$; see e.g. Smirnov (1964, Theorem in §99). If $\mathsf{R}(M) = X''$, then X is *reflexive.*

We have the following result (see, e.g. Schechter 1971, Chapter VII, Theorem 1.1, Theorem 3.1 and Corollary 3.2; Chapter VIII, Theorem 1.2).

Theorem II.1.2 *Let X be a Banach space. Then X is reflexive if and only if X' is. Moreover if X' is separable, so is X. Therefore, if X is reflexive and separable, then so is X'. Finally, if X is reflexive, then so is every closed subspace of X.*

A sequence $\{x_k\}$ in a Banach space X is *weakly convergent* to $x \in X$ if

$$\lim_{k\to\infty} \ell(x_k) = x\,, \quad \text{for all } \ell \in X', \tag{II.1.4}$$

or, in equivalent notation, $x_k \xrightarrow{w} x$. In contrast to this latter, convergence in the sense of (II.1.1) will be also referred to as *strong convergence*. It is immediately seen that a strongly convergent sequence is also weakly convergent, while the converse is not generally true, unless X is isomorphic to $\mathbb{R}^n$; see e.g. Schechter (1971, Chapter VIII, Theorem 4.3). The topological definitions given previously (closedness, compactness, etc.) for subsets of X in terms of strong convergence, can be extended to the more general case of weak convergence in an obvious way. We shall then speak of *weakly closed* sets, or *weakly compact* sets, etc. Moreover, we shall say that a sequence $\{x_k\}$ is *weak Cauchy* if the following property holds, for all $\ell \in X'$:

given $\varepsilon > 0$ there is $\overline{n} = \overline{n}(\varepsilon, \ell) \in \mathbb{N}$: $|\ell(x_k - x_{k'})| < \varepsilon$ for all $k, k' \geq \overline{n}$.

A Banach space X is *weakly complete* if every weak Cauchy sequence is weakly convergent to some $x \in X$.

Some significant properties related to weak convergence are collected in the following.

Theorem II.1.3 *Let X be a Banach space. The following properties hold.*

(i) *If $\{x_k\} \subset X$ with $x_k \xrightarrow{w} x$, then there is C independent of k such that $\|x_k\|_X \leq C$. Moreover,*

$$\|x\|_X \leq \liminf_{k\to\infty} \|x_k\|_X\,;$$

see, e.g., Smirnov (1964, §101, Theorem 1 and Theorem 5).

(ii) *The closed unit ball $\{x \in X : \|x\|_X \leq 1\}$, is weakly compact if and only if X is reflexive; see,* e.g., Miranda (1978, §§28, 30).

(iii) *If X is reflexive, then X is also weakly complete; see,* e.g., Smirnov (1964, §101 Theorem 7).

Property (ii) will be sometime referred to as *weak compactness property.*

This property has, in turn, the following interesting consequence.

Theorem II.1.4 *Let X be a reflexive Banach space, and let $\ell \in X'$. Then, there exists $\overline{x} \in X$ such that*

$$\|\ell\|_{X'} = |\ell(\overline{x})|\,, \quad \|\overline{x}\|_X = 1\,. \tag{II.1.5}$$

Proof. If $\ell = 0$, then (II.1.5) is obviously satisfied. So, we assume $\ell \neq 0$. By definition, we have

$$\|\ell\|_{X'} = \sup_{x \in X; \|x\|_X = 1} |\ell(x)| \,.$$

Therefore, there exists a sequence $\{x_k\} \subset X$ such that

$$\|\ell\|_{X'} = \lim_{k \to \infty} |\ell(x_k)| \,, \quad \|x_k\|_X = 1, \text{ for all } k \in \mathbb{N}. \tag{II.1.6}$$

In view of Theorem II.1.3(ii), there exist a subsequence $\{x_{k'}\}$ and $\overline{x} \in X$ such that

$$x_{k'} \xrightarrow{w} \overline{x} \tag{II.1.7}$$

Evaluating (II.1.6) along this subsequence , with the help of Theorem II.1.3(i), we obtain that $\overline{x}$ satisfies the following conditions

$$\|\ell\|_{X'} = |\ell(\overline{x})| \,, \quad \|\overline{x}\|_X \leq 1. \tag{II.1.8}$$

If $\overline{x} = 0$, it follows $\|\ell\|_{X'} = 0$ which was excluded, so that $\overline{x} \neq 0$. Thus, since

$$\|\ell\|_{X'} \geq \frac{|\ell(\overline{x})|}{\|\overline{x}\|_X} \,,$$

from this relation and (II.1.8) we prove the result. □

In the sequel, we shall deal with vector functions, namely, with functions with values in $\mathbb{R}^n$, whose components belong to the same Banach space X. We shall, therefore, recall some basic properties of Cartesian products, X^N, of N copies of X. It is readily checked that X^N can be endowed with the structure of vector space by defining the sum of two generic elements $\boldsymbol{x} = (x_1, \ldots, x_N)$ and $\boldsymbol{y} = (y_1, \ldots, y_N)$, and the product of a real number α with $\boldsymbol{x}$ in the following way

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \ldots, x_N + y_N) \,, \quad \alpha \, \boldsymbol{x} = (\alpha \, x_1, \ldots, \alpha x_N) \,.$$

Furthermore, we may introduce in X^N either one of the following (equivalent) norms (or any other norm equivalent to them)

$$\|\boldsymbol{x}\|_{(q)} \equiv \left(\sum_{i=1}^{N} \|x_i\|_X^q \right)^{1/q} , \; q \in [1, \infty) \,, \quad \|\boldsymbol{x}\|_{(\infty)} \equiv \max_{i \in \{1, \ldots N\}} \|x_i\|_X \,, \;\; \boldsymbol{x} \in X^N \,, \tag{II.1.9}$$

in such a way that (as the reader will prove with no pain) X^N becomes a Banach space.

We have the following.

Theorem II.1.5 *If X is separable, so is X^N. Moreover, X^N is reflexive if so is X,*

Proof. The proof of the first property is obvious, while that of the second one is a consequence of Theorem II.1.3(ii). □

The next result establishes the relation between $(X^N)'$ and $(X')^N$.

Theorem II.1.6 *Every $\mathcal{L} \in (X^N)'$ can be written as follows*

$$\mathcal{L} = \sum_{k=1}^{N} \ell_i \,, \tag{II.1.10}$$

where $\ell_i \in X'$, $i = 1, \ldots, N$ are uniquely determined. Moreover, the map

$$T : \mathcal{L} \in (X^N)' \to (\ell_1, \ldots, \ell_N) \in (X')^N$$

is a homeomorphism of $(X^N)'$ onto $(X')^N$. If, in particular, we endow X^N and $(X^N)'$ with the following norms

$$\|\boldsymbol{x}\|_{X^N} \equiv \|\boldsymbol{x}\|_{(1)} \,, \quad \|\mathcal{L}\|_{(X')^N} = \|\mathcal{L}\|_{(\infty)} \,.$$

then T is an isomorphism.

Proof. The generic element $\mathcal{L} \in X^N$ can be represented as in (II.1.10) where $\ell_1(\boldsymbol{x}) \equiv \mathcal{L}(x_1, 0, \ldots, 0)$, $\ell_2(\boldsymbol{x}) \equiv \mathcal{L}(0, x_2, \ldots, 0)$, etc. Obviously, each functional ℓ_i, $i = 1, \ldots N$, can be viewed as an element of X'. We then consider the map T in the way defined above. It is clear that T is surjective and injective and linear. From (II.1.10), it readily follows that

$$\|\mathcal{L}\|_{(X^N)'} \equiv \sup_{\boldsymbol{x} \in X^N; \|\boldsymbol{x}\|_{X^N}=1} |\mathcal{L}(x)| \leq \|T(\mathcal{L})\|_{(\infty)} \,.$$

Moreover, by definition of supremum, we must have

$$\|\ell_i\|_{X'} \leq \|\mathcal{L}\|_{(X^N)'} \,,$$

so that we conclude $\|\mathcal{L}\|_{(X^N)'} \geq \|T(\mathcal{L})\|_{(\infty)}$, which shows that T is an isomorphism. If, instead, we use any other norm of the type (II.1.9), we can show by a simple calculation that uses (II.3.2) that T is, in general, a homeomorphism. The proof of the lemma is thus completed. □

We next recall the Hahn–Banach theorem and one of its consequences. A proof of these results can be found, e.g., in Schechter (1971, Chapter II Theorem 2.2 and Theorem 3.3).

Theorem II.1.7 *Let M be a subspace of a normed space X. The following properties hold.*

(a) *Let ℓ be a bounded linear functional defined on M, and let*

$$\|\ell\| = \sup_{x \in M; \|x\|_X=1} |\ell(x)| \,.$$

Then, there exists a bounded linear functional, $\overline{\ell}$, defined on the whole of X, such that (i) $\overline{\ell}(x) = \ell(x)$, for all $x \in M$, and (ii) $\|\overline{\ell}\|_{X'} = \|\ell\|$.

(b) *Let* $x_0 \in X$ *be such that*

$$d \equiv \inf_{x \in M} \|x_0 - x\|_X > 0\,.$$

Then, there is $\ell \in X'$ *such that* $\|\ell\|_{X'} = 1/d$, $\ell(x_0) = 1$, *and* $\ell(x) = 0$, *for all* $x \in M$.

We conclude this section by reporting the classical contraction mapping theorem (see, e.g., Kantorovich & Akilov 1964, p. 625), that we shall often use throughout this book in the following form.

Theorem II.1.8 *Let* M *be a closed subset of the Banach space* X, *and let* T *be a map of* M *into itself. Suppose there exists* $\alpha \in (0, 1)$ *such that*

$$\|T(x) - T(y)\|_X \le \alpha \|x - y\|_X\,, \quad \text{for all } x, y \in M.$$

Then, there is a unique $x_0 \in M$ *such that* $T(x_0) = x_0$.

A map satisfying the assumptions of Theorem II.1.8 is called *contraction*.

II.1.3 Spaces of Smooth Functions

We next define some classical spaces of smooth functions and, for some of them, we recall their completeness properties.

Given a non-negative integer k, we let $C^k(\Omega)$ denote the linear space of all real functions u defined in Ω which together with all their derivatives $D^\alpha u$ of order $|\alpha| \le k$ are continuous in Ω. To shorten notations, we set

$$C^0(\Omega) \equiv C(\Omega).$$

We also set

$$C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega).$$

Moreover, by the symbols $C_0^k(\Omega)$ and $C_0^\infty(\Omega)$ we indicate the (linear) subspaces of $C^k(\Omega)$ and $C^\infty(\Omega)$, respectively, of all those functions having compact support in Ω. Furthermore, $C_0^k(\overline{\Omega})$, $0 \le k \le \infty$, denotes the class of restrictions to Ω of functions in $C_0^k(\mathbb{R}^n)$. As before, we put

$$C_0^0(\Omega) \equiv C_0(\Omega), \quad C_0^0(\overline{\Omega}) \equiv C_0(\overline{\Omega}).$$

We next define $C^k(\overline{\Omega})$ ($C(\overline{\Omega})$ for $k = 0$) as the space of all functions u for which $D^\alpha u$ is bounded and uniformly continuous in Ω, for all $0 \le |\alpha| \le k$. We recall (Miranda 1978, §54) that for $k < \infty$, $C^k(\overline{\Omega})$ is a Banach space with respect to the norm

$$\|u\|_{C^k} \equiv \max_{0 \le |\alpha| \le k} \sup_{\Omega} |D^\alpha u|. \tag{II.1.11}$$

Finally, for $\lambda \in (0,1]$ and $k \in \mathbb{N}$, by $C^{k,\lambda}(\overline{\Omega})$ we denote the closed subspace of $C^k(\overline{\Omega})$ consisting of all functions u whose derivatives up to the kth order inclusive are *Hölder continuous* (*Lipschitz continuous* if $\lambda = 1$) in Ω, that is,

$$[u]_{k,\lambda} \equiv \max_{0\le|\alpha|\le k} \sup_{x,y\in\Omega, x\neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\lambda} < \infty.$$

$C^{k,\lambda}(\overline{\Omega})$ is a Banach space with respect to the norm

$$\|u\|_{C^{k,\lambda}} \equiv \|u\|_{C^k} + [u]_{k,\lambda}, \qquad \text{(II.1.12)}$$

(Miranda 1978, §54).

Exercise II.1.1 Assuming Ω bounded, use the Ascoli-Arzelà theorem (see, e.g., Rudin 1987, p. 245) to show that from every sequence of functions uniformly bounded in $C^{k+1,\lambda}(\overline{\Omega})$ it is always possible to select a subsequence converging in the space $C^{k,\lambda}(\overline{\Omega})$.

II.1.4 Classes of Domains and their Properties

We begin with a simple but useful result holding for arbitrary domains of $\mathbb{R}^n$.

Lemma II.1.1 *Let Ω be an arbitrary domain of $\mathbb{R}^n$. Then there exists an open covering, $\mathfrak{O}$, of Ω satisfying the following properties*

(i) *$\mathfrak{O}$ is constituted by an at most countable number of open balls $\{\mathfrak{B}_k\}$, $k \in I \subseteq \mathbb{N}$, such that*

$$\mathfrak{B}_k \subset \Omega\,, \ \text{for all } k \in I\,, \quad \cup_{k\in I}\mathfrak{B}_k = \Omega\,;$$

(ii) *For any family $\mathfrak{F} = \{\mathfrak{B}_l\}$, $l \in I'$ with $I' \subsetneq I$, there is $\mathfrak{B} \in (\mathfrak{O} - \mathfrak{F})$ such that $[\cup_{l\in I'}\mathfrak{B}_l] \cap \mathfrak{B} \neq \emptyset\,;$*

(iii) *For any $\mathfrak{B}, \mathfrak{B}' \in \mathfrak{O}$, there exists a finite number of open balls $\mathfrak{B}_i \in \mathfrak{O}$, $i = 1,\dots,N$, such that*

$$\mathfrak{B} \cap \mathfrak{B}_1 \neq \emptyset\,, \ \ \mathfrak{B}_N \cap \mathfrak{B}' \neq \emptyset\,, \ \ \mathfrak{B}_j \cap \mathfrak{B}_{j+1} \neq \emptyset\,, \ \ j = 1,\dots N-1\,.$$

Proof. Since Ω is open, for each $x \in \Omega$ we may find an open ball $B_{r_x}(x) \subset \Omega$. Clearly, the collection $\mathfrak{C} \equiv \{B_{r_x}(x)\}$, $x \in \Omega$, satisfies $\cup_{x\in\Omega}B_{r_x}(x) = \Omega$. However, since Ω is separable, we may determine an at most countable subcovering, $\mathfrak{O}$, of $\mathfrak{C}$ satisfying condition (i) in the lemma. Next, assume (ii) is not true. Then, there would be at least one family $\mathfrak{F}' = \{\mathfrak{B}_{k'}\}$, $k' \in I'$, with $I' \subsetneq I$ such that

$$\left[\bigcup_{k'\in I'}\mathfrak{B}_{k'}\right]\bigcap\mathfrak{B}=\emptyset\,,\quad\text{for all }\mathfrak{B}\in(\mathfrak{O}-\mathfrak{F}')\,.$$

Consequently, the sets

$$A_1\equiv\bigcup_{k'\in I'}\mathfrak{B}_{k'}\,,\quad A_2\equiv\bigcup_{k\in(I-I')}\mathfrak{B}_k$$

are open, disjoint and satisfy $A_1\cup A_2=\Omega$, contradicting the assumption that Ω is connected. Finally, let $\mathfrak{B},\mathfrak{B}'\in\mathfrak{O}$ and denote their centers by x and x', respectively. Since Ω is open and connected, it is, in particular, arc-connected. Therefore, we may find a curve, γ, joining x and x', that is homeomorphic to the interval $[0,1]$. Let $\mathfrak{O}'\subset\mathfrak{O}$ be a covering of γ. Since γ is compact, we can extract from $\mathfrak{O}'$ a finite covering that satisfies the property stated in the lemma. □

We next present certain classes of domains of $\mathbb{R}^n$, along with their relevant properties. We begin with the following.

Definition II.1.1. Let Ω be a domain with a *bounded* boundary, namely, Ω is either a bounded domain or it is a domain complement in $\mathbb{R}^n$ of a compact (not necessarily connected) set, namely, Ω is an *exterior* domain.[4] Assume that for each $x_0\in\partial\Omega$ there is a ball $B=B_r(x_0)$ and a real function ζ defined on a domain $D\subset\mathbb{R}^{n-1}$ such that in a system of coordinates $\{x_1,\dots,x_n\}$ with the origin at x_0:

(i) The set $\partial\Omega\cap B$ can be represented by an equation of the type $x_n=\zeta(x_1,\dots,x_{n-1})$;
(ii) Each $x\in\Omega\cap B$ satisfies $x_n<\zeta(x_1,\dots,x_{n-1})$.

Then Ω is said to be *of class* C^k (or C^k*-smooth*) [respectively, *of class* $C^{k,\lambda}$ (or $C^{k,\lambda}$*-smooth*), $0<\lambda\le1$] if $\zeta\in C^k(\overline{D})$ [respectively, $\zeta\in C^{k,\lambda}(\overline{D})$]. If, in particular, $\zeta\in C^{0,1}(\overline{D})$, we say that Ω is *locally Lipschitz.* Likewise, we shall say that $\sigma\subset\partial\Omega$ is a *boundary portion of class* C^k [respectively, *of class* $C^{k,\lambda}$] if $\sigma=\partial\Omega\cap B_r(x_0)$, for some $r>0, x_0\in\partial\Omega$ and σ admits a representation of the form described in (i), (ii) with ζ of class C^k [respectively of class $C^{k,\lambda}$]. If, in particular, $\zeta\in C^{0,1}(\overline{D})$, we say that σ is a *locally Lipschitz boundary portion.*

If Ω is sufficiently smooth, of class C^1, for example, then the unit outer normal, $\boldsymbol{n}$, to $\partial\Omega$ is well defined and continuous. However, in several interesting cases, we need less regularity on Ω, but still would like to have $\boldsymbol{n}$ well-defined. In this regard, we have the following result, for whose proof we refer to Nečas (1967, Chapitre II, Lemme 4.2).

Lemma II.1.2 *Let Ω be locally Lipschitz. Then the unit outer normal $\boldsymbol{n}$ exists almost everywhere on $\partial\Omega$.*

[4] Hereafter, the whole space $\mathbb{R}^n$ will be considered a particular exterior domain.

We shall now consider a special class of bounded domains Ω called *star-shaped* (or *star-like*) with respect to a point. For such domains, there exist $\overline{x} \in \Omega$ (which we may, occasionally, assume to be the origin of coordinates) and a continuous, positive function h on the unit sphere such that

$$\Omega = \left\{ x \in \mathbb{R}^n : |x - \overline{x}| < h\left(\frac{x - \overline{x}}{|x - \overline{x}|} \right) \right\}. \tag{II.1.13}$$

Some elementary properties of star-shaped domains are collected in the following exercises.

Exercise II.1.2 Show that Ω is star-shaped with respect to $\overline{x}$ if and only if every ray starting from $\overline{x}$ intersects $\partial\Omega$ at one and only one point.

Exercise II.1.3 Assume Ω star-shaped with respect to the origin and set

$$\Omega^{(\rho)} = \{ x \in \mathbb{R}^n : x = \rho y, \text{ for some } y \in \Omega \}. \tag{II.1.14}$$

Show that $\Omega^{(\rho)} \subset \overline{\Omega}$ if $\rho \in (0,1)$ and $\Omega^{(\rho)} \supset \overline{\Omega}$ if $\rho > 1$.

The following useful result holds.

Lemma II.1.3 *Let Ω be locally Lipschitz. Then, there exist m locally Lipschitz bounded domains $G_1, \dots, G_m$ such that*

(i) $\partial\Omega \subset \cup_{i=1}^m G_i$;
(ii) *The domains $\Omega_i = \Omega \cap G_i$, $i = 1, \dots, m$, are (locally Lipschitz and) star-shaped with respect to every point of a ball B_i with $\overline{B}_i \subset \Omega_i$.*

Proof. Let $x_0 \in \partial\Omega$. By assumption, there is $B_r(x_0)$ and a function $\zeta = \zeta(x')$, $x' = (x_1, \dots, x_{n-1}) \in D \subset \mathbb{R}^{n-1}$ such that

$$|\zeta(\xi') - \zeta(\eta')| < \kappa|\xi' - \eta'|, \quad \xi', \eta' \in D,$$

for some $\kappa > 0$ and, moreover, points $x = (x', x_n) \in \partial\Omega \cap B_r(x_0)$ satisfy

$$x_n = \zeta(x'), \quad x' \in D,$$

while points $x \in \Omega \cap B_r(x_0)$ satisfy

$$x_n < \zeta(x'), \quad x' \in D.$$

We may (and will) take x_0 to be the origin of coordinates. Denote next, by $y_0 \equiv (0, \dots, 0, y_n)$ the point of Ω intersection of the x_n-axis with $B_r(x_0)$ and consider the cone $\Gamma(y_0, \alpha)$ with vertex at y_0, axis x_n, and semiaperture $\alpha < \pi/2$. It is easy to see that, taking α sufficiently small, every ray ρ starting from y_0 and lying in $\Gamma(y_0, \alpha)$ intersects $\partial\Omega \cap B_r(x_0)$ at (one and) only one point. In fact, assume ρ cuts $\partial\Omega \cap B_r(x_0)$ at two points $z^{(1)}$ and $z^{(2)}$ and denote by $\alpha' < \alpha$ the angle formed by ρ with the x_n-axis. Possibly rotating the coordinate system around the x_n-axis we may assume without loss [5]

$$z^{(1)} = (z_1^{(1)}, 0, \ldots, 0, \zeta(z_1^{(1)}, 0, \ldots, 0)), \quad z_1^{(1)} > 0$$

$$z^{(2)} = (z_1^{(2)}, 0, \ldots, 0, \zeta(z_1^{(2)}, 0, \ldots, 0)), \quad z_1^{(2)} > 0$$

and so, at the same time,

$$\tan\alpha' = \frac{z_1^{(1)}}{\zeta(z_1^{(1)}, 0, \ldots, 0) - y_n}$$

$$\tan\alpha' = \frac{z_1^{(2)}}{\zeta(z_1^{(2)}, 0, \ldots, 0) - y_n}$$

implying

$$\frac{|\zeta(z_1^{(1)}, 0, \ldots, 0) - \zeta(z_1^{(2)}, 0, \ldots, 0)|}{|z_1^{(1)} - z_1^{(2)}|} = \frac{1}{\tan\alpha'} \geq \frac{1}{\tan\alpha}.$$

Thus, if (say)

$$\tan\alpha \leq \frac{1}{2\kappa},$$

ρ will cut $\partial\Omega \cap B_r(x_0)$ at only one point. Next, denote by $\sigma = \sigma(z)$ the intersection of $\Gamma(y_0, \alpha/2)$ with a plane orthogonal to x_n-axis at a point $z = (0, \ldots, z_n)$ with $z_n > y_n$, and set

$$R = R(z) \equiv \operatorname{dist}(\partial\sigma, z).$$

Clearly, taking z sufficiently close to y_0 ($z = \overline{z}$, say), $\sigma(\overline{z})$ will be entirely contained in Ω and, further, every ray starting from a point of $\sigma(\overline{z})$ and lying within $\Gamma(y_0, \alpha/2)$ will form with the x_n-axis an angle less than α and so, by what we have shown, it will cut $\partial\Omega \cap B_r(x_0)$ at only one point. Let C be a cylinder with axis coincident with the x_n-axis and such that

$$C \cap \partial\Omega = \Gamma(y_0, \alpha/2) \cap \partial\Omega.$$

Then, setting

$$G = C \cap B_r(x_0),$$

we have that G is locally Lipschitz and that $G \cap \Omega$ is star-shaped with respect to all points of the ball $B_{R(\overline{z})}(\overline{z})$. Since $x_0 \in \partial\Omega$ is arbitrary, we may form an open covering $\mathcal{G}$ of $\partial\Omega$ constituted by domains of the type G. However, $\partial\Omega$ is compact and, therefore, we may select from $\mathcal{G}$ a finite subset $\{G_1, \ldots, G_m\}$ satisfying all conditions in the lemma, which is thus completely proved. $\square$

[5] Clearly, the Lipschitz constant κ is invariant by this transformation.

Other relevant properties related to star-shaped domains are described in the following exercises.

Exercise II.1.4 Assume that the function h in (II.1.13) is Lipschitz continuous, so that, by Lemma II.1.2, the outer unit normal $\boldsymbol{n} = \boldsymbol{n}(x)$ on $\partial\Omega$ exists for a.a. x. Then, setting $F(x) \equiv \boldsymbol{n}(x) \cdot (x - \overline{x})$, show that $\operatorname*{ess\,inf}_{x\in\partial\Omega} F(x) > 0$.

Exercise II.1.5 Assume Ω bounded and locally Lipschitz. Prove that

$$\Omega = \bigcup_{i=1}^{m} \Omega_i\,,$$

where each Ω_i is a locally Lipschitz and star-shaped domain with respect to every point of a ball B_i with $\overline{B}_i \subset \Omega_i$. *Hint:* Use Lemma II.1.3.

We end this section by recalling the following classical result, whose proof can be found, e.g., in Nečas (1967, Chapitre 1, Proposition 2.3).

Lemma II.1.4 *Let K be a compact subset of $\mathbb{R}^n$, and let $\mathcal{O} = \{\mathcal{O}_1, \cdots, \mathcal{O}_N\}$ be an open covering of K. Then, there exist functions ψ_i, $i = 1, \ldots, N$ satisfying the following properties*

(i) $0 \le \psi_i \le 1\,,\ i = 1, \ldots, N\,;$
(ii) $\psi_i \in C_0^\infty(\mathcal{O}_i)\,,\ i = 1, \ldots, N\,;$
(iii) $\sum_{i=1}^{N} \psi_i(x) = 1\,,$ *for all* $x \in K\,.$

The family $\{\psi_i\}$ is referred to as *partition of unity in K subordinate to the covering $\mathcal{O}$.*

II.2 The Lebesgue Spaces L^q

For $q \in [1, \infty)$, let $L^q = L^q(\Omega)$ denote the linear space of all (equivalence classes of) real Lebesgue-measurable functions u defined in Ω such that

$$\|u\|_q \equiv \left(\int_\Omega |u|^q\right)^{1/q} < \infty. \tag{II.2.1}$$

The functional (II.2.1) defines a norm in L^q, with respect to which L^q becomes a Banach space. Likewise, denoting by $L^\infty = L^\infty(\Omega)$ the linear space of all (equivalence classes of) Lebesgue-measurable real-valued functions u defined in Ω with

$$\|u\|_\infty \equiv \operatorname*{ess\,sup}_{\Omega} |u| < \infty \tag{II.2.2}$$

one shows that (II.2.2) is a norm and that L^∞ endowed with this norm is a Banach space. For a proof of the above properties see, e.g., Miranda (1978, §47). For $q = 2$, L^q is a Hilbert space under the scalar product

$$(u,v) \equiv \int_\Omega uv, \qquad u,v \in L^2.$$

Whenever confusion of domains might occur, we shall use the notation

$$\|\cdot\|_{q,\Omega}, \quad \|\cdot\|_{\infty,\Omega}, \quad \text{and} \ \ (\cdot,\cdot)_\Omega.$$

Given a sequence $\{u_m\} \subset L^q(\Omega)$ and $u \in L^q(\Omega)$, $1 \le q \le \infty$, we thus have that $u_m \to u$, namely, $\{u_m\}$ converges (strongly) to u, if and only if

$$\lim_{k\to\infty} \|u_k - u\|_q = 0\,.$$

The following two basic properties, collected in as many lemmas, will be frequently used throughout. The first one is the classical *Lebesgue dominated convergence theorem* (Jones 2001, Chapter 6 §C), while the other one relates convergence in L^q with pointwise convergence; see Jones (2001, Corollary at p. 234)

Lemma II.2.1 *Let $\{u_m\}$ be a sequence of measurable functions on Ω, and assume that*

$$u(x) \equiv \lim_{m\to\infty} u_m(x) \ \ \textit{exists for a.a. } x \in \Omega\,,$$

and that there is $U \in L^1(\Omega)$ such that

$$|u_m(x)| \le |U(x)| \ \ \textit{for a.a } x \in \Omega\,.$$

Then $u \in L^1(\Omega)$ and

$$\lim_{m\to\infty} \int_\Omega u_m = \int_\Omega u\,.$$

Lemma II.2.2 *Let $\{u_m\} \subset L^q(\Omega)$ and $u \in L^q(\Omega)$, $1 \le q \le \infty$, with $u_m \to u$. Then, there exists $\{u_{m'}\} \subseteq \{u_m\}$ such that*

$$\lim_{m'\to\infty} u_{m'}(x) = u(x)\,, \quad \textit{for a.a. } x \in \Omega\,.$$

We want now to collect some inequalities in L^q spaces that will be frequently used throughout. For $1 \le q \le \infty$, we set

$$q' = q/(q-1);$$

one then shows the *Hölder inequality*

$$\int_\Omega |uv| \le \|u\|_q \|v\|_{q'} \tag{II.2.3}$$

for all $u \in L^q(\Omega)$, $v \in L^{q'}(\Omega)$ (Miranda 1978, Teorema 47.I). The number q' is called the *Hölder conjugate of q*. In particular, (II.2.3) shows that the bilinear form (u,v) is meaningful whenever $u \in L^q(\Omega)$ and $v \in L^{q'}(\Omega)$. In case $q=2$, inequality (II.2.3) is referred to as the *Schwarz inequality.* More generally, one has the *generalized Hölder inequality*

$$\int_\Omega |u_1 u_2 \dots u_m| \leq \|u_1\|_{q_1} \|u_2\|_{q_2} \cdot \ldots \cdot \|u_m\|_{q_m}, \tag{II.2.4}$$

where

$$u_i \in L^{q_i}(\Omega),\ 1 \leq q_i \leq \infty, \quad i = 1, \dots, m, \quad \sum_{i=1}^m q_i^{-1} = 1\,.$$

Both inequalities (II.2.3) and (II.2.4) are an easy consequence of the *Young inequality*:

$$ab \leq \frac{\varepsilon a^q}{q} + \varepsilon^{-q'/q} \frac{b^{q'}}{q'} \quad (a, b, \varepsilon > 0) \tag{II.2.5}$$

holding for all $q \in (1,\infty)$. When $q=2$, relation (II.2.5) is known as the *Cauchy inequality.*

Two noteworthy consequences of inequality (II.2.3) are the *Minkowski inequality*:

$$\|u+v\|_q \leq \|u\|_q + \|v\|_q, \quad u, v \in L^q(\Omega), \tag{II.2.6}$$

and the *interpolation* (or *convexity*) *inequality*:

$$\|u\|_q \leq \|u\|_s^\theta \|u\|_r^{1-\theta} \tag{II.2.7}$$

valid for all $u \in L^s(\Omega) \cap L^r(\Omega)$ with $1 \leq s \leq q \leq r \leq \infty$, and

$$q^{-1} = \theta s^{-1} + (1-\theta) r^{-1}, \quad \theta \in [0,1].$$

Another important inequality is the *generalized Minkowski inequality* reported in the following lemma, and for whose proof we refer to Jones (2001, Chapter 11, §E).[6]

Lemma II.2.3 *Let Ω_1, and Ω_2 be domains of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, with $m,n \geq 1$. Suppose that $u : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is a Lebesgue measurable function such that, for some $q \in [1,\infty]$,*

$$\int_{\Omega_2} \left(\int_{\Omega_1} |u(x,y)|^q dx \right)^{1/q} dy < \infty\,.$$

Then,

$$\left(\int_{\Omega_1} \left| \int_{\Omega_2} u(x,y)\, dy \right|^q dx \right)^{1/q} < \infty\,,$$

and the following inequality holds

[6] Actually, it can be proved that (II.2.6) is just a particular case of (II.2.8), hence the adjective "generalized"; see Jones (2001, p. 272).

$$\left(\int_{\Omega_1}\left|\int_{\Omega_2} u(x,y)\,dy\right|^q dx\right)^{1/q} \le \int_{\Omega_2}\left(\int_{\Omega_1}|u(x,y)|^q dx\right)^{1/q} dy\,. \qquad \text{(II.2.8)}$$

Exercise II.2.1 Assume Ω bounded. Show that if $u \in L^\infty(\Omega)$, then

$$\lim_{q\to\infty}\|u\|_q = \|u\|_\infty .$$

Exercise II.2.2 Prove inequality (II.2.5). *Hint*: Minimize the function

$$t^q/q - t + 1/q'.$$

Exercise II.2.3 Prove inequalities (II.2.6) and (II.2.7).

We shall now list some of the basic properties of the spaces L^q. We begin with the following (see, e.g. Miranda 1978, §51).

Theorem II.2.1 *For $1 \le q < \infty$, L^q is separable, $C_0(\Omega)$ being, in particular, a dense subset*

Note that the above property is not true if $q = \infty$, since $C(\overline{\Omega})$ is a closed subspace of $L^\infty(\Omega)$); see Miranda, *loc. cit.*.

Concerning the density of smooth functions in L^q, one can prove something more than what stated in Theorem II.2.1, namely, that every function in L^q, $1 \le q < \infty$, can be approximated by functions from $C_0^\infty(\Omega)$. This fact follows as a particular case of a general *smoothing procedure* that we are going to describe. To this end, given a real (measurable) function u in Ω, we shall write

$$u \in L^q_{loc}(\Omega)$$

to mean

$$u \in L^q(\Omega'), \quad \textit{for any bounded domain } \Omega' \textit{ with } \overline{\Omega'} \subset \Omega.$$

Likewise, we write

$$u \in L^q_{loc}(\overline{\Omega})$$

to mean

$$u \in L^q(\Omega'), \quad \textit{for any bounded domain } \Omega' \subset \Omega.$$

Clearly, for Ω bounded we have $L^q_{loc}(\overline{\Omega}) = L^q(\Omega)$. Now, let $j \in C_0^\infty(\Omega)$ be a non-negative function such that

(i) $j(x) = 0$, for $|x| \ge 1$,

(ii) $\displaystyle\int_{\mathbb{R}^n} j = 1.$

A typical example is

$$j(x) = \begin{cases} c \exp[-1/(1-|x|^2)] & \text{if } |x| < 1 \\ 0 & \text{if } |x| \ge 1, \end{cases}$$

with c chosen in such a way that property (ii) is satisfied. The *regularizer* (or *mollifier*) in the sense of Friedrichs u_ε of $u \in L^1_{loc}(\Omega)$ is then defined by the integral

$$u_\varepsilon(x) = \varepsilon^{-n} \int_{\mathbb{R}^n} j\left(\frac{x-y}{\varepsilon}\right) u(y)dy, \quad \varepsilon < \operatorname{dist}(x, \partial\Omega).$$

This function has several interesting properties, some of which will be recalled now here. First of all, we observe that u_ε is infinitely differentiable at each $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) > \varepsilon$. Moreover, if $u \in L^q_{loc}(\overline{\Omega})$ we may extend it by zero outside Ω, so that u_ε becomes defined for all $\varepsilon > 0$ and all $x \in \mathbb{R}^n$. Thus, in particular, if $u \in L^q(\Omega)$, $1 \le q < \infty$, one can show (Miranda 1978, §51; see also Exercise II.2.10 for a generalization)

$$\begin{aligned} &\|u_\varepsilon\|_q \le \|u\|_q \quad \text{for all } \varepsilon > 0\,, \\ &\lim_{\varepsilon \to 0^+} \|u_\varepsilon - u\|_q = 0. \end{aligned} \tag{II.2.9}$$

Exercise II.2.4 Show that for $u \in C_0(\Omega)$,

$$\lim_{\varepsilon \to 0^+} u_\varepsilon(x) = u(x) \ \text{ holds uniformly in } x \in \Omega.$$

Exercise II.2.5 For $u \in L^q(\Omega)$, $1 \le q < \infty$, show the inequality

$$\sup_{\mathbb{R}^n} |D^\alpha u_\varepsilon(x)| \le \varepsilon^{-n/q - |\alpha|} \|D^\alpha j\|_{q', \mathbb{R}^n} \|u\|_{q,\Omega}, \quad |\alpha| \ge 0.$$

We next observe that, by writing $u_\varepsilon(x)$ as follows:

$$u_\varepsilon(x) = \varepsilon^{-n} \int_{|\xi|<\varepsilon} j\left(\frac{\xi}{\varepsilon}\right) u(x+\xi)d\xi,$$

it becomes apparent that, if u is of compact support in Ω and ε is chosen less than the distance of the support of u from $\partial\Omega$, then $u_\varepsilon \in C_0^\infty(\Omega)$. The latter, together with $(\text{II.2.9})_2$ and the density of C_0 in L^q, yields that $C_0^\infty(\Omega)$ is a dense subspace of $L^q(\Omega)$, $1 \le q < \infty$. The proof of this property, along with some of its consequences, is left to the reader in the following exercises.

Exercise II.2.6 Prove that $C_0^\infty(\Omega)$ is dense in $L^q(\Omega)$, $1 \le q < \infty$. *Hint.* Use the density of $C_0(\Omega)$ in $L^q(\Omega)$ (Miranda 1978, §51) along with the properties of the mollifier.

Exercise II.2.7 Prove the existence of a basis in $L^2(\Omega)$ constituted by functions from $C_0^\infty(\Omega)$. *Hint*: Use the separability of L^2 along with the density of C_0^∞ into L^2.

Exercise II.2.8 Let $u \in L^q(\Omega)$, $1 \leq q < \infty$. Extend u to zero in $\mathbb{R}^n - \Omega$ and continue to denote by u the extension. Show the following *continuity in the mean* property: Given $\varepsilon > 0$ there is $\delta > 0$ such that for every $h \in \mathbb{R}^n$ with $|h| < \delta$ the following inequality holds

$$\int_\Omega |u(x+h) - u(x)|^q \, dx < \varepsilon^q .$$

Hint: Show the property for $u \in C_0^\infty(\Omega)$, then use the density of C_0^∞ in L^q.

Exercise II.2.9 Assume $u \in L^1_{loc}(\Omega)$. Prove that

$$\int_\Omega u\psi = 0, \quad \text{for all } \psi \in C_0^\infty(\Omega), \;\text{ implies } u \equiv 0, \text{ a.e. in } \Omega.$$

Hint: Consider the function

$$\operatorname{sign} u = \begin{cases} 1 \text{ if } u > 0 \\ -1 \text{ if } u \leq 0. \end{cases}$$

For a fixed bounded Ω' with $\overline{\Omega'} \subset \Omega$,

$$\operatorname{sign} u \in L^1(\Omega')$$

and so $\operatorname{sign} u$ can be approximated by functions from $C_0^\infty(\Omega')$.

Exercise II.2.10 Let $u \in L^q(\mathbb{R}^n)$, $1 \leq q < \infty$, and for $z \in \mathbb{R}^n$ and $k \leq n$ set

$$z_{(k)} = (z_1, \ldots, z_k), \quad z^{(k)} = (z_{k+1}, \ldots, z_n) .$$

Moreover, define

$$u_{(k),\varepsilon}(x) = \varepsilon^{-k} \int_{\mathbb{R}^k} j\left(\frac{x_{(k)} - y_{(k)}}{\varepsilon}\right) u(y_{(k)}, y^{(k)}) \, dy_{(k)} .$$

Show the following properties, for each $y^{(k)} \in \mathbb{R}^{n-k}$:

$$\|u_{(k),\varepsilon}\|_{q,\mathbb{R}^k} \leq \|u(\cdot, y^{(k)})\|_{q,\mathbb{R}^k} \quad \text{for all } \varepsilon > 0 ,$$
$$\lim_{\varepsilon \to 0+} \|u_{(k),\varepsilon} - u(\cdot, y^{(k)})\|_{q,\mathbb{R}^k} = 0.$$

Hint: Use the generalized Minkowski inequality, the result in Exercise II.2.8 and Lebesgue dominated convergence theorem (Lemma II.2.1).

Let $v \in L^{q'}(\Omega)$, with q' the Hölder conjugate of q. Then, by (II.2.3), the integral

$$\ell(u) = \int_\Omega vu, \quad u \in L^q(\Omega) \tag{II.2.10}$$

defines a linear functional on L^q. However, for $q \in [1, \infty)$, every linear functional must be of the form (II.2.10). Actually, we have the following *Riesz representation theorem* for whose proof we refer to Miranda (1978, §48).

Theorem II.2.2 *Let ℓ be a linear functional on $L^q(\Omega)$, $q \in [1,\infty)$. Then, there exists a uniquely determined $v \in L^{q'}(\Omega)$ such that representation* (II.2.10) *holds. Furthermore*

$$\|\ell(u)\|_{[L^q(\Omega)]'} \equiv \sup_{\|u\|_q=1} |\ell(u)| = \|v\|_{q'} . \tag{II.2.11}$$

From Theorem II.2.2 we thus obtain the following.

Theorem II.2.3 *The (normed) dual of L^q is isomorphic to $L^{q'}$ for $1 < q < \infty$, so that, for these values of q, L^q is a reflexive space.*

Exercise II.2.11 Show the validity of (II.2.11) when $q \in (1,\infty)$. *Hint*: Use the representation (II.2.10) .

Exercise II.2.12 Let $u \in L^1_{loc}(\Omega)$, and assume that there exists a constant $C > 0$ such that

$$|(u,\psi)| \leq C\|\psi\|_q , \quad \text{for some } q \in [1,\infty) \text{ and all } \psi \in C_0^\infty(\Omega).$$

Show that $u \in L^{q'}(\Omega)$ and that $\|u\|_q \leq C$. *Hint*: $\psi \to (u,\psi)$ defines a bounded linear functional on a dense set of $L^q(\Omega)$. Then use Hahn–Banach Theorem II.1.7 and the Riesz representation Theorem II.2.2.

Riesz theorem also allows us to give a characterization of weak convergence of a sequence $\{u_k\} \subset L^q(\Omega)$ to $u \in L^q(\Omega)$, $1 < q < \infty$. In fact, we have that $u_k \overset{w}{\to} u$ if and only if

$$\lim_{k\to\infty} (v, u_k - u) = 0 , \quad \text{for all } v \in L^{q'}(\Omega),\ q' = q/(q-1).$$

In view of Theorem II.1.3(iii) and Theorem II.2.3, we find that L^q is *weakly complete*, for $q \in (1,\infty)$. In fact, this property continues to hold in the case $q = 1$; see Miranda (1978, Teorema 48.VII).

We wish now to recall the following results related to weak convergence.

Theorem II.2.4 *Let $\{u_m\} \subset L^q(\Omega)$, $1 \leq q \leq \infty$. The following properties hold.*

(i) *If $u_m \overset{w}{\to} u$, for some , $u \in L^q(\Omega)$, then there is C independent of m such that $\|u_m\|_q \leq C$. Moreover,*

$$\|u\|_q \leq \liminf_{m\to\infty} \|u_m\|_q.$$

In addition, if $1 < q < \infty$, and

$$\|u\|_q \geq \limsup_{m\to\infty} \|u_m\|_q,$$

then $u_m \to u$.

(ii) *If* $1 < q < \infty$ *and* $\|u_m\|_q \le C$, *for some* C *independent of* m, *then there exists a subsequence* $\{u_{m'}\}$ *and* $u \in L^q(\Omega)$ *such that* $u_{m'} \xrightarrow{w} u$.

Proof. The statement in (ii) follows from Theorem II.1.3(ii), while the first statement in (i) is a consequence of the general result given in Theorem II.1.3(i). A proof of the second statement in (i) can be found, for example, in Brezis (1983, Proposition III.5(iii) and Proposition III.30). However, for $q = 2$ the proof of (i) becomes very simple and it will be reproduced here. By hypothesis and Riesz theorem we have that for all $v \in L^2$ and $\varepsilon > 0$ there exists $m' \in \mathbb{N}$ such that

$$|(u_m - u, v)| < \varepsilon, \quad \text{for all } m \ge m'.$$

If we choose $v = u_m$, with the help of the Schwarz inequality we find

$$\|u_m\|_2^2 \le \|u\|_2 \|u_m\|_2 + \varepsilon\,.$$

Using Cauchy inequality on the right-hand side of this latter relation we conclude

$$\|u_m\|_2^2 \le \|u\|_2^2 + 2\varepsilon\,,$$

which proves the boundedness of the sequence. We next choose

$$v = u, \quad \varepsilon = \eta\|u\|_2, \quad \eta > 0,$$

to obtain, again with the aid of Schwarz inequality,

$$\|u\|_2 \le \|u_m\|_2 + \eta,$$

which completes the proof of the first part of the statement in (i). The second part is a consequence of the assumption and the identity

$$\|u_m - u\|_2^2 = \|u\|_2^2 + \|u_m\|_2^2 - 2(u_m, u).$$

□

We conclude this section with some observations concerning L^q-spaces of *vector-valued* functions. Let $[L^q(\Omega)]^N$ be the direct product of N copies of $L^q(\Omega)$. Then, as we know from Subsection I.1.2, $[L^q(\Omega)]^N$ is a Banach space with respect to any of the following equivalent norms:

$$\|\boldsymbol{u}\|_{q,(r)} \equiv \left(\sum_{i=1}^{N} \|u_i\|_q^r\right)^{1/r}, \quad r \in [1, \infty) \quad \|\boldsymbol{u}\|_{q,(\infty)} \equiv \max_{i \in \{1,\dots,N\}} \|u_i\|_q\,,$$

where $\boldsymbol{u} = (u_1, \dots, u_N)$. Moreover, in view of Theorem II.2.1, Theorem II.2.3, and Theorem II.1.5, we have.

Theorem II.2.5 $[L^q(\Omega)]^N$ *is separable for* $q \in [1, \infty)$, *and reflexive for* $q \in (1, \infty)$.

Also, the Riesz representation theorem can be suitably extended to this more general case. In fact, let

$$(\boldsymbol{v},\boldsymbol{u}) \equiv \sum_{i=1}^{N}(u_i,v_i)\,, \quad \boldsymbol{u}\in [L^q(\Omega)]^N\,, \quad \boldsymbol{v}\in [L^{q'}(\Omega)]^N\,, 1/q+1/q'=1.$$

In view of Theorem II.1.6 and of Theorem II.2.2 , we then have that for every $\mathcal{L}\in \big([L^q(\Omega)]^N\big)'$, there exist uniquely determined $\boldsymbol{v}\in [L^{q'}(\Omega)]^N$, such that

$$\mathcal{L}(\boldsymbol{u}) = (\boldsymbol{v},\boldsymbol{u})\,,$$

and that the map $M:\mathcal{L}\to \boldsymbol{v}$ is a homeomorphism. Actually, if we endow $[L^q(\Omega)]^N$ with the norm $\|\boldsymbol{u}\|_{q,(q)}\equiv \|\boldsymbol{u}\|_q$, the map M is an isomorphism, as stated in the second part of the following lemma, whose proof can be found in Simader (1972, Lemma 4.2).[7]

Theorem II.2.6 *Let Ω be a domain of $\mathbb{R}^n$, and let $q\in(1,\infty)$. Then, for every $\mathcal{L}\in \big([L^q(\Omega)]^N\big)'$, there exists uniquely determined $\boldsymbol{v}\in [L^{q'}(\Omega)]^N$, such that*

$$\mathcal{L}(\boldsymbol{u}) = (\boldsymbol{v},\boldsymbol{u})\,, \quad \boldsymbol{u}\in [L^q(\Omega)]^N\,.$$

Moreover,

$$\|\mathcal{L}\|_{([L^q(\Omega)]^N)'} \equiv \sup_{\boldsymbol{u}\in[L^q(\Omega)]^N,\ \|\boldsymbol{u}\|_q=1} |\mathcal{L}(\boldsymbol{u})| = \|\boldsymbol{v}\|_q\,.$$

II.3 The Sobolev Spaces $W^{m,q}$ and Embedding Inequalities

Let $u\in L^1_{loc}(\Omega)$. Given a multi-index α, we shall say that a function $u^{(\alpha)}\in L^1_{loc}(\Omega)$ is the αth *generalized* (or *weak*) *derivative of* u if and only if the following relation holds:

$$\int_\Omega uD^\alpha\varphi = (-1)^{|\alpha|}\int_\Omega u^{(\alpha)}\varphi, \quad \text{for all } \varphi\in C_0^\infty(\Omega).$$

It is easy to show that $u^{(\alpha)}$ is uniquely determined (use Exercise II.2.9) and that, if $u\in C^{|\alpha|}(\Omega)$, $u^{(\alpha)}$ is the αth derivative of u in the ordinary sense, and the previous integral relation is an obvious consequence of the well-known Gauss formula. Hereafter, the function $u^{(\alpha)}$, whenever it exists, will be indicated by the symbol $D^\alpha u$.

[7] The assumption made in Simader *loc. cit.*, that Ω is bounded, is completely superfluous, since it is never used in the proof, as it was also independently communicated to me by Professor Simader.

Generalized derivatives have several properties in common with ordinary derivatives. For instance, given two functions u, v possessing generalized derivatives $D_j u$, $D_i v$ we have that $\beta u + \gamma v \quad (\beta, \gamma \in \mathbb{R})$ has a generalized derivative and $D_j(\beta u + \gamma v) = \beta D_j u + \gamma D_j v$. In addition, if

$$uv, \quad uD_j v + vD_j u \in L^1_{loc}(\Omega),$$

then uv has a generalized derivative and the familiar *Leibniz rule* holds:

$$D_j(uv) = uD_j v + vD_j u.$$

The proof of these properties is left to the reader as an exercise.

Exercise II.3.1 Generalized differentiation and differentiation almost everywhere are two distinct concepts. Show that a function φ that is continuous in [0,1] but *not* absolutely continuous admits no generalized derivative. *Hint*: Assume, *per absurdum*, that φ has a generalized derivative Φ. Then, it would follow

$$\varphi(x) = \int_0^x \Phi(t)dt + \varphi(0), \quad x \in (0,1),$$

which gives a contradiction. On the other hand, one can give examples of real, continuous functions f on $[0,1]$ that are differentiable *a.e.* in $[0,1]$ and with $f' \in L^1(0,1)$ which are not absolutely continuous (Rudin 1987, pp. 144-145). In this connection, it is worth noticing the following general result (Smirnov 1964, §110): *a function $u \in L^1_{loc}(\Omega)$ $(\Omega \subset \mathbb{R}^n)$ is weakly differentiable if $u = \widetilde{u}$ a.e. in Ω, with $\widetilde{u}$ absolutely continuous on almost all line segments parallel to the coordinate axes and having partial derivatives locally integrable.*

Exercise II.3.2 Let $u \in L^1_{loc}(\Omega)$ and assume that $D^\alpha u$ exists. Show

$$D^\alpha(u_\varepsilon(x)) = (D^\alpha u)_\varepsilon(x)\,, \quad \text{dist}\,(x, \partial\Omega) > \varepsilon.$$

Exercise II.3.3 Let $\Omega \subset \mathbb{R}^n$, and let $\psi \in C^1(\Omega)$ map Ω onto $\Omega_1 \subset \mathbb{R}^n$, with $\psi^{-1} \in C^1(\Omega_1)$. Assume u possesses generalized derivatives $D_j u$, $j = 1, \ldots, n$, and set $v = u \circ \psi^{-1}$. Show that also v possesses generalized derivatives $D_j v$, $j = 1, \ldots, n$, and that the usual change of variable formula applies:

$$D_i u(x) = \frac{\partial \psi_j}{\partial x_i}\, D_j v(y)\,, \quad y = \psi(x)\,,$$

for a.a. $x \in \Omega$ and $y \in \Omega_1$.

For $q \in [1, \infty]$ and $m \in \mathbb{N}$, we let

$$W^{m,q} = W^{m,q}(\Omega) = \{u \in L^q(\Omega) : D^\alpha u \in L^q(\Omega),\ 0 \le |\alpha| \le m\}\,.$$

In the linear space $W^{m,q}(\Omega)$ we introduce the following norm:

$$\begin{aligned} \|u\|_{m,q} &= \left(\sum_{0\le|\alpha|\le m} \|D^\alpha u\|_q^q \right)^{1/q} \quad \text{if } 1 \le q < \infty \\ \|u\|_{m,\infty} &= \max_{0\le|\alpha|\le m} \|D^\alpha u\|_\infty \qquad \text{if } q = \infty. \end{aligned} \tag{II.3.1}$$

If confusion of domains arises, we shall write $\|u\|_{m,q,\Omega}$ and $\|u\|_{m,\infty,\Omega}$ in place of $\|u\|_{m,q}$ and $\|u\|_{m,\infty}$. Owing to the completeness of the spaces L^q and taking into account the definition of generalized derivative, it is not hard to show that $W^{m,q}$ endowed with the norm (II.3.1) becomes a Banach space, called *Sobolev space* (of order (m,q)). Along with this space, we shall consider its closed subspace $W_0^{m,q} = W_0^{m,q}(\Omega)$, defined as the completion of $C_0^\infty(\Omega)$ in the norm (II.3.1). Clearly, we have (see Exercise II.2.6)

$$W^{0,q} = W_0^{0,q} = L^q.$$

In the special case $q = 2$, $W^{m,q}$ (and thus $W_0^{m,q}$) is a Hilbert space with respect to the scalar product

$$(u,v)_m = \sum_{0\le|\alpha|\le m} (D^\alpha u, D^\alpha v)\,.$$

Exercise II.3.4 Prove that, for any Ω, $W_0^{m,q}(\Omega)$ is a closed subspace of $W^{m,q}(\Omega)$. Prove also $W_0^{m,q}(\mathbb{R}^n) = W^{m,q}(\mathbb{R}^n)$, $1 \le q < \infty$. *Hint*: To show the second assertion, take a function $\varphi \in C^\infty(\mathbb{R}^n)$ with $\varphi(x) = 1$ if $|x| \le 1$, $\varphi(x) = 0$ if $|x| \ge 2$ ("cut-off" function) and set

$$u_m(x) = \varphi(x/m)u(x), \quad u \in W^{m,q}(\mathbb{R}^n), \quad m \in \mathbb{N}.$$

Then, u is approximated in $W^{m,q}(\mathbb{R}^n)$ by $\{(u_m)_\varepsilon\} \subset C_0^\infty(\mathbb{R}^n)$.

Remark II.3.1 Sobolev spaces share several important properties with Lebesgue spaces L^q. Thus, for example, since a closed subspace of a Banach space X is reflexive and separable if X is (see Theorem II.1.1 and Theorem II.1.2), and since $W^{m,q}(\Omega)$ can be naturally embedded in $[L^q(\Omega)]^N$, for a suitable $N = N(m)$, one can readily show, by using Theorem II.2.5 and the fact that $W^{m,q}(\Omega)$ is complete, that $W^{m,q}(\Omega)$ is separable if $1 \le q < \infty$ and reflexive if $1 < q < \infty$; for details, see, e.g., Adams (1975, §3.4). As a consequence, by Theorem II.1.3(ii), we find, in particular, that for $q \in (1,\infty)$, $W^{m,q}$ has the weak compactness property. ■

Exercise II.3.5 Let $u \in L^1_{loc}(\Omega)$ and suppose $\|u_\varepsilon\|_{m,q,B} \le C$, $m \ge 0$, $1 < q < \infty$, where B is an arbitrary open ball with $\overline{B} \subset \Omega$, and C is independent of ε. Show that $u \in W^{m,q}_{loc}(\Omega)$ and that $\|u\|_{m,q,B} \le C$.

Another interesting question is whether elements from $W^{m,q}(\Omega)$ can be approximated by smooth functions. This question is important, for instance, when one wants to establish in $W^{m,q}$ inequalities involving norms (II.3.1). Actually, if such an approximation holds, it then suffices to prove these inequalities for smooth functions only. In the case where Ω is either the whole of $\mathbb{R}^n$ or it is star-shaped with respect to a point, the question is affirmatively answered; cf. Exercise II.3.4 and Exercise II.3.7. In more general cases, we have a fundamental result, given in Theorem II.3.1, which in its second part involves domains having a mild property of regularity, i.e., the *segment property*, which states that, for every $x \in \partial\Omega$ there exists a neighborhood U of x and a vector $\boldsymbol{y}$ such that if $z \in \overline{\Omega} \cap U$, then $z + t\boldsymbol{y} \in \Omega$, for all $t \in (0,1)$.

Exercise II.3.6 Show that a domain having the segment property cannot lie simultaneously on both sides of its boundary.

Theorem II.3.1 *For any domain Ω, every function from $W^{m,q}(\Omega)$, $1 \leq q < \infty$, can be approximated in the norm* (II.3.1)$_1$ *by functions in $C^m(\Omega) \cap W^{m,q}(\Omega)$. Moreover, if Ω has the segment property, it can be approximated in the same norm by elements of $C_0^\infty(\overline{\Omega})$.*

The first part of this theorem is due to Meyers and Serrin (1964), while the second one is given by Adams (1975, Theorem 3.18).

Exercise II.3.7 (Smirnov 1964, §111)**.** Assume Ω star-shaped with respect to the origin. Prove that every function u in $W^{m,q}(\Omega)$, $1 \leq q < \infty$, $m \geq 0$, can be approximated by functions from $C_0^\infty(\overline{\Omega})$. (Compare this result with Theorem II.3.1.) *Hint*: Consider the sequence

$$u_k(x) = \begin{cases} u\left((1 - 1/k)x\right) & \text{if } x \in \Omega^{(k/(k-1))} \\ 0 & \text{if } x \notin \Omega^{(k/(k-1))} \end{cases} \qquad k = 2, 3, \dots ,$$

with $\Omega^{(\rho)}$ defined in (II.1.14). Then, regularize u_k and use (II.2.9) and Exercise II.3.2.

We wish now to prove some basic inequalities involving the norms (II.3.1). Such results are known as *Sobolev embedding theorems* (see Theorem II.3.2 and Theorem II.3.4). To this end, we propose an elementary inequality due to Nirenberg (1959).

Lemma II.3.1 *For all $u \in C_0^\infty(\mathbb{R}^n)$,*

$$\|u\|_{n/(n-1)} \leq \frac{1}{2\sqrt{n}} \|\nabla u\|_1. \tag{II.3.2}$$

Proof. Just to be specific, we shall prove (II.3.2) for $n = 3$, the general case being treated analogously. We have

$$|u(x)| \le \frac{1}{2}\int_{-\infty}^{\infty} |D_1 u| dx_1 \equiv F_1(x_2, x_3)$$

and similar estimates for x_2 and x_3. With the obvious meaning of the symbols we then deduce

$$|2u(x)|^{3/2} \le \left[F_1(x_2,x_3)F_2(x_1,x_3)F_3(x_1,x_2)\right]^{1/2}.$$

Integrating over x_1, and using the Schwarz inequality,

$$\int_{-\infty}^{\infty} |2u(x)|^{3/2} dx_1 \le [F_1(x_2,x_3)]^{1/2}\left(\int_{-\infty}^{\infty} F_2(x_1,x_3)dx_1\right)^{1/2} \times \left(\int_{-\infty}^{\infty} F_3(x_1,x_2)dx_1\right)^{1/2}.$$

Integrating this relation successively over x_2 and x_3 and applying the same procedure, we find

$$2\|u\|_{3/2} \le \left(\int_{\mathbb{R}^3} |D_1 u| \int_{\mathbb{R}^3} |D_2 u| \int_{\mathbb{R}^3} |D_3 u|\right)^{1/3} \le (1/3)\sum_{i=1}^{3}\int_{\mathbb{R}^3} |D_i u|,$$

which, in turn, after employing the inequality [1]

$$(a_1 + a_2 + \ldots + a_m)^q \le m^{q-1}(a_1^q + a_2^q + \ldots + a_m^q), \quad a_i > 0\,, \; q \ge 1 \qquad \text{(II.3.3)}$$

with $m = 3, q = 2$, gives (II.3.2). □

For $q \ge 1$, replacing u with $|u|^q$ in (II.3.2) and using the Hölder inequality, we obtain at once

$$\|u\|_{qn/(n-1)} \le \left(\frac{q}{2\sqrt{n}}\right)^{1/q} \|u\|_q^{1-1/q}\|\nabla u\|_q^{1/q}. \qquad \text{(II.3.4)}$$

Inequalities (II.3.2), (II.3.4), and (II.2.7) allow us to deduce more general relations, which are contained in the following lemma.

Lemma II.3.2 *Let*

$$r \in [q, nq/(n-q)], \quad \textit{if } q \in [1, n),$$

and

$$r \in [q, \infty), \quad \textit{if } q \ge n.$$

Then, for all $u \in C_0^\infty(\mathbb{R}^n)$ *we have*

[1] See Hardy, Littlewood, & Polya 1934, Theorem 16, p. 26.

$$\|u\|_r \le \left(\frac{c_1}{2\sqrt{n}}\right)^{\lambda} \|u\|_q^{1-\lambda}\|\nabla u\|_q^{\lambda}, \tag{II.3.5}$$

where

$$c_1 = \max(q, r(n-1)/n), \quad \lambda = n(r-q)/rq.$$

Proof. We shall distinguish the two cases:

(i) $q \le r \le qn/(n-1)$,
(ii) $r \ge qn/(n-1)$.

In case (i) we have by (II.2.7) and (II.3.4)

$$\|u\|_r \le \|u\|_q^{\theta}\|u\|_{qn/(n-1)}^{1-\theta} \le \left(\frac{q}{2\sqrt{n}}\right)^{(1-\theta)/q} \|u\|_q^{(\theta-1)/q+1}\|\nabla u\|_q^{(1-\theta)/q}$$

with

$$\theta = \frac{r(1-n)+nq}{r}.$$

Substituting the value of θ in the preceding relation furnishes (II.3.5). In case (ii), we replace u in (II.3.2) with $|u|^{r(n-1)/n}$ and apply the Hölder inequality to obtain

$$\|u\|_r^{r(n-1)/n} \le \frac{r(n-1)}{2n\sqrt{n}}\|u\|_{\beta}^{[r(n-1)-n]/n}\|\nabla u\|_q, \qquad \beta = \frac{qr(n-1)-n}{n(q-1)}.$$

Notice that $q \le \beta$. Moreover, it is

$$\beta \le r \ \text{ for } r \le nq/(n-q), \quad \text{if } q < n$$

and

$$\beta \le r \ \text{ for all } r < \infty, \quad \text{if } q \ge n.$$

In either case we may use (II.2.7) to obtain

$$\|u\|_{\beta} \le \|u\|_q^{\theta}\|u\|_r^{1-\theta}, \quad \theta = \frac{r(q-n)+nq}{(r-q)[r(n-1)-n]}.$$

Substituting this inequality in the preceding one gives (II.3.5), and the proof of the lemma is complete. □

Lemma II.3.2 can be extended to include L^q-norms of derivatives of order higher than one. A general multiplicative inequality is given in Nirenberg (1959, p.125). We reproduce here this result, referring the reader to the paper of Nirenberg for a proof. Set

$$|u|_{k,p} \equiv \left(\sum_{|\ell|=k}\int_{\Omega}|D^{\ell}u|^p\right)^{1/p}.$$

We have the following.

Lemma II.3.3 *Let $u \in L^q(\mathbb{R}^n)$, with $D^\alpha u \in L^r(\mathbb{R}^n)$, $|\alpha| = m > 0$, $1 \leq q, r \leq \infty$. Then, $D^\alpha u \in L^s(\mathbb{R}^n)$, $|\alpha| = j$, and the following inequality holds for $0 \leq j < m$ and some $c = c(n, m, j, q, r, a)$:*

$$|u|_{j,s} \leq c\, |u|^a_{m,r} \|u\|_q^{1-a}, \tag{II.3.6}$$

where

$$\frac{1}{s} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q},$$

for all a in the interval

$$\frac{j}{m} \leq a \leq 1,$$

with the following exceptional cases

1. *If $j = 0$, $rm < n$, $q = \infty$ then we make the additional assumption that either $u(x) \to 0$ as $|x| \to \infty$, or $u \in L^{\overline{q}}(\mathbb{R}^n)$ for some $\overline{q} \in (0, \infty)$.*
2. *if $1 < r < \infty$, and $m - j - n/r$ is a nonnegative integer then $(*)$ holds only for a satisfying $j/m \leq a < 1$.*

From Lemma II.3.2 we wish to single out some special inequalities that will be used frequently in the theory of the Navier–Stokes equations. First of all, we have the *Sobolev inequality*

$$\|u\|_r \leq \frac{q(n-1)}{2(n-q)\sqrt{n}}\|\nabla u\|_q, \quad 1 \leq q < n, \; r = nq/(n-q), \tag{II.3.7}$$

derived for the first time by Sobolev (1938) by a complete different method and for $q \in (1, n)$.[2] Inequality (II.3.7), holding a priori only for functions $u \in C_0^\infty(\mathbb{R}^n)$, can be clearly extended, by density, to every $u \in W_0^{1,q}(\Omega)$, $1 \leq q < n$. We then deduce, in particular, that every such function is in $L^r(\Omega)$ with r given in (II.3.7).

Exercise II.3.8 Let $\Omega = B_1$ or $\Omega = \mathbb{R}^n$, $n \geq 2$. Show, by means of a counterexample, that the Sobolev inequality does not hold if $q = n$, that is, a (positive, finite) constant γ independent of u such that

$$\|u\|_\infty \leq \gamma \|\nabla u\|_n, \quad u \in C_0^\infty(\Omega), \; n \geq 2,$$

does not exist. (In this respect, see also Section II.9 and Section II.11.)[3]

Remark II.3.2 In connection with (II.3.7) we would like to make some comments. When Ω is an unbounded domain (in particular, exterior to the closure of a bounded domain) the investigation of the asymptotic properties of a solution u to a system of partial differential equations is strictly related to the

[2] In this regard, see Theorem II.11.3 and Exercise II.11.4.

[3] A sharp version of the Sobolev inequality when $q = n$ and Ω is bounded, is due to Trudinger (1967).

Lebesgue space $L^s(\Omega)$ to which u belongs and, roughly speaking, the behavior of u at large distances will be better known when the exponent s is lower.[4] Now, as we shall see in subsequent chapters, the inherent information we derive from the Navier–Stokes equations in such domains is that u (a generic component of the velocity field) has first derivatives $D_i u$ summable with exponents q_i which, however, *may vary with* x_i, $i = 1, \ldots, n$. Therefore, we may wonder if (II.3.7) can be replaced by another inequality which takes into account this different behavior in different directions and leads to an exponent s of summability for u *strictly* less than the exponent r given in (II.3.7). This question finds its answer within the context of *anisotropic Sobolev spaces* (Nikol'skiĭ 1958). Here, we shall limit ourselves to quote, without proof, an inequality due to Troisi (1969, Teorema 1.2) representing the natural generalization of (II.3.7) to the anisotropic case. Let

$$1 \le q_i < \infty, \quad i = 1, \ldots, n.$$

Then, for all $u \in C_0^\infty(\mathbb{R}^n)$ the following *Troisi inequality* holds:

$$\|u\|_s \le c \prod_{i=1}^n \|D_i u\|_{q_i}^{1/n}, \quad \sum_{i=1}^n q_i^{-1} > 1, \quad s = \frac{n}{\left(\sum_{i=1}^n q_i^{-1} - 1\right)}. \tag{II.3.8}$$

If $q_i = q$, for all $i = 1, \ldots, n$, (II.3.8) reduces to (II.3.7). On the other hand, if for some i (=1, say), $q_1 < q \equiv q_2 = \ldots = q_n$, from (II.3.8) we deduce

$$s = r + \frac{nq(q_1 - q)}{(q - q_1) + q_1(n - q)} < r.$$

■

Other special cases of (II.3.5) are now considered. We choose in Lemma II.3.2 $n = q = 2$ and $r = 4$ to deduce the *Ladyzhenskaya inequality*

$$\|u\|_4 \le 2^{-1/4} \|u\|_2^{1/2} \|\nabla u\|_2^{1/2}, \tag{II.3.9}$$

shown for the first time by Ladyzhenskaya (1958, 1959a, eq. (6)). It should be emphasized that (II.3.9) does *not* hold in three space dimensions with the *same* exponents (see Exercise II.3.9). Rather, for $n = 3$, $q = 2$, and $r = 4$, inequality (II.3.5) delivers

$$\|u\|_4 \le \left(\frac{4}{3\sqrt{3}}\right)^{3/4} \|u\|_2^{1/4} \|\nabla u\|_2^{3/4}. \tag{II.3.10}$$

Furthermore, for $n = 3$, $q = 2$, $r = 6$ the Sobolev inequality (II.3.7) specializes to

[4] It is needless to say that the possibility of lowering the exponent s depends on the particular problem.

$$\|u\|_6 \leq \frac{2}{\sqrt{3}}\|\nabla u\|_2. \tag{II.3.11}$$

In two space dimensions there is no analogue of (II.3.11), and so, in particular, for $n = 2$, a function having all derivatives in $L^2(\mathbb{R}^2)$ need not be in $L^r(\mathbb{R}^2)$, whatever $r \in [1, \infty]$. [5]

Exercise II.3.9 Let φ be the C^∞ "cut-off" function introduced in Exercise II.3.4 and set $u_m(x) = \varphi(x)\exp(-m|x|)$, $m \in \mathbb{N}$. Obviously, $\{u_m\} \subset C_0^\infty(\mathbb{R}^n)$. Show that for $n = 3$ the following inequality holds

$$R(m) \equiv \frac{\|u_m\|_4^4}{\|u_m\|_2^2\|\nabla u_m\|_2^2} \geq c\, m \frac{\int_0^m e^{-y}y^2 dy}{\int_0^m e^{-2y}y^2 dy},$$

with c a positive number independent of m. Since $R(m) \to \infty$ as $m \to \infty$, a constant $\gamma \in (0, \infty)$ such that

$$\|u\|_4 \leq \gamma \|u\|_2^{1/2}\|\nabla u\|_2^{1/2}, \quad u \in C_0^\infty(\mathbb{R}^3),$$

does not exist.

The case $q > n$ of Lemma II.3.2 can be further strengthened, as shown by the following lemma.

Lemma II.3.4 *Let $q > n$. Then, for all $u \in C^1(\overline{B(x)})$ we have*

$$|u(x)| \leq \omega_n^{-1}\|u\|_{1,B(x)} + \omega_n^{-1/q}\left(\frac{q-1}{q-n}\right)^{1-1/q}\|\nabla u\|_{q,B(x)}\,, \tag{II.3.12}$$

and so, in particular, for all $u \in C_0^\infty(\mathbb{R}^n)$,

$$\sup_{x\in\mathbb{R}^n}|u(x)| \leq c_2\omega_n^{-1/q}\|u\|_{1,q,\mathbb{R}^n} \tag{II.3.13}$$

with

$$c_2 = \max\left\{1, \left(\frac{q-1}{q-n}\right)^{(q-1)/q}\right\}.$$

Proof. It is enough to prove (II.3.12), since (II.3.13) follows by using the Hölder inequality in the first term of (II.3.12). From the identity

$$u(x) - u(y) = -\int_0^{|x-y|}\frac{\partial u(x+r\boldsymbol{e})}{\partial r}dr, \quad \boldsymbol{e} = \frac{\boldsymbol{y}-\boldsymbol{x}}{|y-x|}, \tag{II.3.14}$$

[5] For example, for $\alpha \in (0, 1/2)$, take $u(x) = \ln^\alpha|x|$, if $|x| > 1$ and $u(x) = 0$ if $|x| \leq 1$. The problem of the behavior at large spatial distances of functions with gradients in $L^q(\Omega)$, Ω an exterior domain, will be fully analyzed in Section II.7 and Section II.9.

we easily show

$$\omega_n|u(x)| \le \|u\|_{1,B(x)} + \int_{B(x)} |\nabla u(y)|\,|x-y|^{1-n} dy. \tag{II.3.15}$$

Applying the Hölder inequality in the integral in (II.3.15) and dividing the resulting relation by ω_n we prove (II.3.12). □

We want now to draw some consequences from Lemma II.3.2 and Lemma II.3.4. Employing the Young inequality (II.2.5) and the density of $C_0^\infty(\Omega)$ in $W_0^{1,q}(\Omega)$, from (II.3.3), (II.3.5), and (II.3.13) we find, in particular, that a function $u \in W_0^{1,q}(\Omega)$ is also in $L^r(\Omega)$, for all $r \in [q, nq/(n-q)]$, if $1 \le q < n$, and for all $r \ge q$, if $q = n$. Moreover, if $q > n$, u coincides *a.e.* in Ω with a (uniquely determined) function of $C(\overline{\Omega})$. Finally, u obeys the following inequalities:

$$\begin{aligned} &\|u\|_r \le C_1\|u\|_{1,q} \quad 1 \le q < n, \quad q \le r \le \frac{nq}{n-q} \\ &\|u\|_r \le C_2\|u\|_{1,q} \quad q = n, \quad q \le r < \infty \\ &\|u\|_C \le C_3\|u\|_{1,q} \quad q > n \end{aligned} \tag{II.3.16}$$

with $C_i = C_i(n,q,r)$, $i = 1,2,3$. Now, using (II.3.16) and an iterative argument we may generalize (II.3.16) to functions from $W_0^{m,q}(\Omega)$, to obtain the following *embedding theorem* whose proof is left to the reader as an exercise.

Theorem II.3.2 *Let $u \in W_0^{m,q}(\Omega)$, $q \ge 1$, $m \ge 0$. If $mq \le n$ we have*

$$W_0^{m,q}(\Omega) \hookrightarrow L^r(\Omega)$$

for all $r \in [q, \frac{nq}{n-mq}]$ if $mq < n$, and for all $r \in [q,\infty)$ if $mq = n$. In particular, there are constants c_i, $i = 1,2$, depending only on m, q, r and n such that

$$\begin{aligned} &\|u\|_r \le c_1\|u\|_{m,q} \quad \text{for all } r \in [q, \tfrac{nq}{n-mq}], \quad \text{if } mq < n, \\ &\|u\|_r \le c_2\|u\|_{m,q} \quad \text{for all } r \in [q,\infty), \quad \text{if } mq = n, \end{aligned} \tag{II.3.17}$$

Finally, if $mq > n$, each $u \in W_0^{m,q}(\Omega)$ is equal a.e. in Ω to a unique function in $C^k(\overline{\Omega})$, $0 \le k < m - n/q$, and the following inequality holds

$$\|u\|_{C^k} \le c_3\|u\|_{m,q}, \tag{II.3.18}$$

with $c_3 = c_3(m,q,r,n)$.

We wish now to generalize Theorem II.3.2 to the spaces $W^{m,q}(\Omega)$, $\Omega \ne \mathbb{R}^n$. One of the most usual ways of doing this is to construct an (m,q)-*extension map for Ω*. By this we mean that there exists a linear operator $E : W^{m,q}(\Omega) \to W^{m,q}(\mathbb{R}^n)$ such that

(i) $u(x) = [E(u)](x)$, for all $x \in \Omega$
(ii) $\|E(u)\|_{m,q,\mathbb{R}^n} \le C\|u\|_{m,q,\Omega}$,

for some constant C independent of u. It is then not hard to show that inequalities (II.3.17) and (II.3.18) continue to hold in $W^{m,q}(\Omega)$. For instance, to prove (II.3.17) from (i) and (ii), we notice that

$$\|u\|_{r,\Omega} \le \|E(u)\|_{r,\mathbb{R}^n} \le c\,\|E(u)\|_{m,q,\mathbb{R}^n} \le c\,C\|u\|_{m,q,\Omega}.$$

Results on the existence of an extension map can be proved in a more or less complicated way, depending on the *smoothness* of the domain. In this regard, we shall now state a very general result due to Stein (1970, Chapter VI, Theorem 5; see also Triebel 1978, §§4.2.2, 4.2.3) on the existence of suitable extension maps called *universal* or *total* in that they do not depend on the order of differentiability and summability involved. Specifically, we have the following theorem whose rather deep proof will be omitted.

Theorem II.3.3 *Let Ω be locally Lipschitz.*[6] *Then, there exists an (m, q)-extension map for Ω, for all $q \in [1,\infty]$ and $m \ge 0$.*

On the other hand, results similar to those of Theorem II.3.3 can be proved in an elementary way, provided the domain is of class C^m (see, e.g., Lions 1962, Théorème 4.1, and Friedman 1969, Lemma 5.2). This is because, for such a domain, the boundary can be locally straightened by means of the smooth transformation:

$$y_i = x_i \ \text{ if } 1 \le i \le n-1, \quad y_n = x_n - \zeta(x_1,\dots,x_{n-1}).$$

The extension problem is then reduced to the same problem in $\mathbb{R}^n_+$, for which a simple solution is available, as shown by the following exercise.

Exercise II.3.10 For $x \in \mathbb{R}^n$, we put $x' = (x_1,\dots,x_{n-1})$. Let $u \in C_0^\infty(\overline{\mathbb{R}}^n_+)$ and set

$$\mathcal{E}u(x) = \begin{cases} u(x) & \text{if } x_n \ge 0 \\ \displaystyle\sum_{p=1}^{m+1} \lambda_p u(x', -px_n) & \text{if } x_n < 0 \end{cases}$$

where

$$\sum_{p=1}^{m+1} \lambda_p (-p)^\ell = 1, \quad \ell = 0, 1, \dots, m.$$

Show that $\mathcal{E}u \in C_0^m(\mathbb{R}^n)$ and that, moreover, for all $q \in [1,\infty]$ and all $|\beta| \in [0, m]$

$$\|D^\beta \mathcal{E}u\|_{q,\mathbb{R}^n} \le C\|D^\beta u\|_{q,\mathbb{R}^n_+}.$$

Therefore, $\mathcal{E}$ can be extended to an operator $E : W^{m,q}(\mathbb{R}^n_+) \to W^{m,q}(\mathbb{R}^n)$, which is an (m, q)-extension map for $\mathbb{R}^n_+$.

[6] Actually, Stein's theorem applies to much more general domains (with bounded or unbounded boundary) and precisely to those which are "minimally smooth," see Stein (1970, Chapter VI, §3.3).

Exercise II.3.11 Let $u \in W_0^{m,q}(\Omega)$ and set

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega^c. \end{cases}$$

Show that $\widetilde{u} \in W^{m,q}(\mathbb{R}^n)$.

On the strength of Theorem II.3.3 we thus have

Theorem II.3.4 *Suppose Ω locally Lipschitz. Then all conclusions in* Theorem II.3.2 *remain valid if we replace $W_0^{m,q}(\Omega)$ with $W^{m,q}(\Omega)$ for some constants $c_i = c_i(m,q,r,n,\Omega)$, $i = 1,2,3$.*

We wish to remark that, by using alternative methods due to Gagliardo (1958, 1959), one can show the results in Theorem II.3.4 under more general assumptions on Ω (see also Miranda 1978, §58).

Exercise II.3.12 Assume Ω locally Lipschitz. Use Theorem II.3.3 to show that, under the assumptions on r, q, and n stated in Lemma II.3.2 the following inequality holds for $u \in W^{1,q}(\Omega)$:

$$\|u\|_r \le c\,\|u\|_q^{1-\lambda}\,\|u\|_{1,q}^{\lambda}, \tag{II.3.19}$$

where c is independent of u and $\lambda = n(r-q)/rq$.

Exercise II.3.13 Let $u : \Omega \to \mathbb{R}^n$ and let $\boldsymbol{e}$ be a given unit vector. For $h \neq 0$ the quantity

$$\Delta^h u(x) \equiv \frac{u(x+h\boldsymbol{e}) - u(x)}{h}$$

is called the *difference quotient of u along $\boldsymbol{e}$.* (a) Show that, if Ω' is any domain with $\overline{\Omega'} \subset \Omega$, the following properties hold for all $u \in W^{1,q}(\Omega)$:

(i) $\Delta^h u(x) \in L^q(\Omega')$, for all $h < \mathrm{dist}(\Omega', \Omega)$;
(ii) $\|\Delta^h u(x)\|_{q,\Omega'} \le \|\nabla u\|_{q,\Omega}$;
(iii) If $\Omega \equiv \mathbb{R}^n_+$ and $\boldsymbol{e}$ is orthogonal to $\boldsymbol{e}_n$:

$$\|\Delta^h u(x)\|_{q,\mathbb{R}^n_+} \le \|\nabla u\|_{q,\mathbb{R}^n_+}.$$

Hint: For a smooth function u and $\boldsymbol{e}$ parallel to $\boldsymbol{e}_i$ (say) it holds

$$\Delta^h u(x) = \frac{1}{h}\int_0^h D_i u(x_1, \ldots, x_i + \eta, \ldots, x_n)\,d\eta.$$

(b) Conversely, assume $u \in L^q(\Omega)$ and that for all Ω' with $\overline{\Omega'} \subset \Omega$ and for all $h < \mathrm{dist}(\Omega', \Omega)$ it holds $\|\Delta^h u\|_{q,\Omega'} \le C$, with a constant C independent of Ω' and h. Then if $\boldsymbol{e}$ is parallel to $\boldsymbol{e}_i$, show that

(iv) $D_i u$ exists;
(v) $\|D_i u\|_{q,\Omega} \le C$.

We wish to end this section by recalling a useful characterization of the normed dual space $(W_0^{m,q}(\Omega))'$ of the space $W_0^{m,q}(\Omega)$. An analogous result can be given for $W^{m,q}(\Omega)$. A functional ℓ on $W_0^{m,q}(\Omega)$ belongs to $(W_0^{m,q}(\Omega))'$ if and only if

$$\|\ell\|_{(W_0^{m,q}(\Omega))'} \equiv \sup_{\|u\|_{m,q}=1} |\ell(u)| < \infty.$$

Let us consider in $(W_0^{m,q}(\Omega))'$ the subspace constituted by functionals $\mathcal{F}$ of the form

$$\mathcal{F}(u) = (f, u)\,, \quad f \in L^{q'}(\Omega). \tag{II.3.20}$$

Clearly, $\mathcal{F} \in (W_0^{m,q}(\Omega))'$. Setting

$$\|f\|_{-m,q'} = \sup_{u \in W_0^{m,q}(\Omega);\|u\|_{m,q}=1} |\mathcal{F}(u)|, \tag{II.3.21}$$

we easily recognize that (II.3.21) is a norm in $L^{q'}(\Omega)$, and that the following inequalities hold:

$$\begin{aligned} &\|f\|_{-m,q'} \leq \|f\|_{q'} \\ &|\mathcal{F}(u)| \leq \|f\|_{-m,q'}\|u\|_{m,q}. \end{aligned} \tag{II.3.22}$$

Let us denote by $W_0^{-m,q'}(\Omega)$ the *negative Sobolev space* of order $(-m, q')$, obtained by completing $L^{q'}(\Omega)$ in the norm (II.3.21). The following result due to Lax (1955, §2) ensures that for $q \in (1, \infty)$ the two spaces $W_0^{-m,q'}(\Omega)$ and $(W_0^{m,q}(\Omega))'$ can be identified (see also Miranda 1978, §57).

Theorem II.3.5 *The spaces $W_0^{-m,q'}(\Omega)$ and $(W_0^{m,q}(\Omega))'$, $1 < q < \infty$, are isomorphic.*

Throughout this book the value of a functional $\mathcal{F} \in W_0^{-m,q'}(\Omega)$ at $u \in W_0^{m,q}(\Omega)$ will be denoted by

$$\langle \mathcal{F}, u \rangle \quad (\textit{duality pairing}).$$

If, in particular, $\mathcal{F} \in L^{q'}(\Omega)$, we have $\langle \mathcal{F}, u \rangle = (\mathcal{F}, u)$.

Remark II.3.3 A characterization completely similar to that of Theorem II.3.5 can be given also for the space $(W^{m,q}(\Omega))'$. Precisely, denoting by $W^{-m,q'}(\Omega)$ the completion of $L^{q'}(\Omega)$ in the norm

$$\|f\|^*_{-m,q'} = \sup_{u \in W^{m,q}(\Omega);\|u\|_{m,q}=1} |\mathcal{F}(u)|\,,$$

with $\mathcal{F}(u)$ defined in (II.3.20), one shows that $W^{-m,q'}(\Omega)$ and $(W^{m,q}(\Omega))'$, $1 < q < \infty$, are isomorphic; see Miranda *loc. cit.* Notice that, obviously,

$$\|f\|_{-m,q'} \leq \|f\|^*_{-m,q'}\,.$$

■

II.4 Boundary Inequalities and the Trace of Functions of $W^{m,q}$

As a next problem, we wish to investigate if, analogously to what happens for smooth functions, it is possible to ascribe a value at the boundary (the *trace*) to functions in $W^{m,q}(\Omega)$. If Ω is sufficiently regular, the considerations developed in the preceding section assure that this is certainly true if $mq > n$, since, in such a case, every function from $W^{m,q}(\Omega)$ can be redefined on a set of zero measure in such a way that it becomes (at least) continuous up to the boundary. However, if $mq \le n$ we can nevertheless prove some inequalities relating $W^{m,q}$-norms of a smooth function with L^r-norms of the same function at the boundary, which will allow us to define, in a suitable sense, the trace of a function belonging to any Sobolev space of order (m,q), $m \ge 1$. To this end, given a sufficiently smooth domain with a bounded boundary (locally Lipschitz, say) we denote by $L^q(\partial\Omega)$, $1 \le q \le \infty$ the space of (equivalence classes of) real functions u defined on $\partial\Omega$ and such that

$$\|u\|_{q,\partial\Omega} \equiv \left(\int_{\partial\Omega} |u|^q d\sigma\right)^{1/q} < \infty, \quad 1 \le q < \infty,$$

$$\|u\|_{\infty,\partial\Omega} \equiv \operatorname*{ess\,sup}_{\partial\Omega} |u| < \infty, \quad q = \infty,$$

where σ denotes the Lebesgue $(n-1)$-dimensional measure.[1] It can be proved that the space $L^q(\partial\Omega)$ enjoys all the relevant functional properties of the spaces $L^q(\Omega)$. In particular, it is a Banach space with respect to the norm $\|\cdot\|_{q,\partial\Omega}$, $1 \le q \le \infty$, which is separable for $1 \le q < \infty$ and reflexive for $1 < q < \infty$ (see Miranda 1978, §60).

In order to accomplish our objective, we need some preliminary considerations and results that we shall next describe.

We shall often use the classical Gauss divergence theorem for smooth vector functions. It is well known that this theorem certainly holds if the domain is (piecewise) of class C^1. However, we need to consider more general situations and, in this respect, we quote the following result of Nečas (1967, Chapitre 2, Lemme 4.2 and Chapitre 3, Théorème 1.1).

Lemma II.4.1 *Let Ω be a bounded, locally Lipschitz domain in $\mathbb{R}^n$. Then the unit outer normal $\boldsymbol{n}$ exists almost everywhere on $\partial\Omega$ (see* Lemma II.1.2*) and the following identity holds*

$$\int_\Omega \nabla\cdot\boldsymbol{u} = \int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{n},$$

for all vector fields $\boldsymbol{u}$ with components in $C^1(\overline{\Omega})$.

[1] As usual, if no confusion arises, the infinitesimal surface element $d\sigma$ in the integral will be omitted.

A generalization of this result to functions from $W^{1,q}(\Omega)$ will be considered in Exercise II.4.3.

We are now in a position to perform a study on the traces of functions from $W^{m,q}$. Let Ω' be a locally Lipschitz, star-shaped domain (with respect to the origin) and let u be an arbitrary function from $C_0^\infty(\overline{\Omega'})$. From the identity

$$|u|^r D_j x_j = D_j(x_j|u|^r) - x_j D_j|u|^r, \quad r \in [1,\infty)$$

and Lemma II.4.1 we easily deduce

$$\int_{\partial\Omega'} \boldsymbol{x}\cdot\boldsymbol{n}|u|^r \le n\|u\|_{r,\Omega'}^r + r\delta(\Omega')\int_{\partial\Omega'}|u|^{r-1}|\nabla u|. \tag{II.4.1}$$

Using the Hölder inequality in the last integral in (II.4.1) and noting that

$$\operatorname*{ess\,inf}_{x\in\partial\Omega'}(\boldsymbol{x}\cdot\boldsymbol{n}(x)) \equiv c > 0$$

(see Exercise II.1.4), we obtain

$$\|u\|_{r,\partial\Omega'}^r \le (n/c)\|u\|_{r,\Omega'}^r + (r\delta(\Omega')/c)\|u\|_{q'(r-1),\Omega'}^{r-1}\|\nabla u\|_{q,\Omega'}. \tag{II.4.2}$$

We now choose $r \in [q,(n-1)q/(n-q)]$, if $q<n$, and arbitrary $r \ge q$, if $q \ge n$. Observing that $r \le q'(r-1)$, in the light of Exercise II.3.12 (see (II.3.19)), inequality (II.4.2) then furnishes for all $u \in C_0^\infty(\overline{\Omega'})$

$$\begin{aligned}\|u\|_{r,\partial\Omega'} &\le C\left(\|u\|_{q,\Omega'}^{r(1-\lambda)}\|u\|_{1,q,\Omega'}^{r\lambda} + \|u\|_{q,\Omega'}^{(r-1)(1-\lambda)}\|u\|_{1,q,\Omega'}^{1+\lambda(r-1)}\right)^{1/r}\\ &\le 2^{1/r}C\left(\|u\|_{q,\Omega'}^{1-\lambda}\|u\|_{1,q,\Omega'}^{\lambda} + \|u\|_{q,\Omega'}^{(1-\frac{1}{r})(1-\lambda)}\|u\|_{1,q,\Omega'}^{\frac{1}{r}+\lambda(1-\frac{1}{r})}\right)\end{aligned} \tag{II.4.3}$$

where $\lambda = n(r-q)/q(r-1)$, $C = C(n,r,q,\Omega')$, and where we used (II.3.3).

Employing Lemma II.1.3 and Lemma II.1.4, we can now establish (II.4.3) for an arbitrary locally Lipschitz domain Ω. In fact, let $\mathcal{G} = \{G_1,\dots,G_N\}$ be the open covering of $\partial\Omega$ constructed in Lemma II.1.3 and let $\{\psi_i\}$ be a partition of unity in $\partial\Omega$ subordinate to $\mathcal{G}$. Setting $\Omega_i = \Omega\cap G_i$, for $u \in C_0^\infty(\overline{\Omega})$, we have

$$\|u\|_{r,\partial\Omega} = \|\sum_{i=1}^N \psi_i u\|_{r,\partial\Omega} \le \sum_{i=1}^N \|u\|_{r,\partial\Omega\cap G_i} \le \sum_{i=1}^N \|u\|_{r,\partial\Omega_i}\,,$$

and therefore, using in this inequality (II.4.3) with $\Omega' \equiv \Omega_i$, we deduce

$$\|u\|_{r,\partial\Omega} \le 2^{1/r}NC\left(\|u\|_{q,\Omega}^{(1-\lambda)}\|u\|_{1,q,\Omega}^{\lambda} + \|u\|_{q,\Omega}^{(1-\frac{1}{r})(1-\lambda)}\|u\|_{1,q,\Omega}^{\frac{1}{r}+\lambda(1-\frac{1}{r})}\right). \tag{II.4.4}$$

Let now Ω be locally Lipschitz, and denote by γ the linear map which to every function $f \in C_0^\infty(\overline{\Omega})$ associates its value at the boundary $\gamma(f) = f|_{\partial\Omega}$, and let $u \in W^{1,q}(\Omega)$. By Theorem II.3.1, there is a sequence $\{f_k\} \subset C_0^\infty(\overline{\Omega})$ converging to u in $W^{1,q}(\Omega)$. On the other hand, by (II.4.4) this sequence will also converge in $L^r(\partial\Omega)$, for suitable r, to a function $\widetilde{u} \in L^r(\partial\Omega)$. Since, as can be easily shown, $\widetilde{u}$ does not depend on the particular sequence, the map γ can be uniquely extended, by continuity, to a map from $W^{1,q}(\Omega)$ into $L^r(\partial\Omega)$ that ascribes, in a well-defined sense, to every function from $W^{1,q}(\Omega)$ a function on the boundary which, for smooth functions u, reduces to the usual trace $u|_{\partial\Omega}$. This result can be fairly generalized to spaces $W^{m,q}$ with $m > 1$. In fact, from Theorem II.3.4 and an iterative argument based on (II.4.4), we obtain the following result whose proof is left to the reader as an exercise.

Theorem II.4.1 *Let Ω be locally Lipschitz. Assume*

$$r \in [q, q(n-1)/(n-mq)]\,, \text{ if } mq < n,$$

$$r \in [q, \infty)\,, \text{ if } mq \geq n.$$

Then there exists a unique, continuous linear map γ from $W^{m,q}(\Omega)$, $1 \leq q < \infty$, $m \geq 1$, into $L^r(\partial\Omega)$ such that for all $u \in C_0^\infty(\overline{\Omega})$ it is $\gamma(u) = u\,|_{\partial\Omega}$. Furthermore, for $m = 1$ the following inequality holds

$$\|\gamma(u)\|_{r,\partial\Omega} \leq C\left(\|u\|_{q,\Omega}^{(1-\lambda)}\|u\|_{1,q,\Omega}^{\lambda} + \|u\|_{q,\Omega}^{(1-\frac{1}{r})(1-\lambda)}\|u\|_{1,q,\Omega}^{\frac{1}{r}+\lambda(1-\frac{1}{r})}\right), \quad \text{(II.4.5)}$$

where $C = C(n, r, q, \Omega)$ and $\lambda = n(r-q)/q(r-1)$.

Exercise II.4.1 Let Ω be locally Lipschitz. Starting from (II.4.5), show that for any $\varepsilon > 0$, there exists $C = C(n, r, q, \Omega, \varepsilon) > 0$ such that

$$\|\gamma(u)\|_{r,\partial\Omega} \leq C\|u\|_{q,\Omega} + \varepsilon\|\nabla u\|_{q,\Omega}\,,$$

with the exponents q and r subject to the restrictions stated in Theorem II.4.1. *Hint*: Use (II.2.5).

Theorem II.4.1 allows us to define, in a natural way, higher-order traces. Actually, since for $u \in W^{m,q}(\Omega)$ we have $D^\alpha u \in W^{m-\ell,q}(\Omega)$ for $0 \leq |\alpha| \leq \ell < m$, the trace of $D^\alpha u$ is well defined and, moreover, it belongs to $L^r(\partial\Omega)$ for suitable exponents $r \geq 1$. In particular, if Ω is sufficiently regular, we can give a precise meaning to the ℓth *normal derivative on* $\partial\Omega$:

$$\frac{\partial^\ell u}{\partial n^\ell} \equiv \sum_{|\alpha|=\ell} n^\alpha D^\alpha u, \quad n^\alpha = n_1^{\alpha_1} n_2^{\alpha_2} \ldots n_n^{\alpha_n},$$

of every function $u \in W^{m,q}(\Omega)$, $m > \ell \geq 0$. Thus, noticing that $n^\alpha \in L^\infty(\partial\Omega)$, we can construct a linear map

$$\Gamma_{(m)} : W^{m,q}(\Omega) \to [L^r(\partial\Omega)]^m \quad \text{(II.4.6)}$$

with

$$\Gamma_{(m)}(u) = \left(u \equiv \gamma_0(u), \frac{\partial u}{\partial n} \equiv \gamma_1(u), \ldots, \frac{\partial^{m-1} u}{\partial n^{m-1}} \equiv \gamma_{m-1}(u) \right). \tag{II.4.7}$$

Obviously, if $u \in W_0^{m,q}(\Omega)$, $\Gamma_m(u) \equiv 0$ *a.e. on* $\partial\Omega$. The converse result also holds and we have (see Nečas 1967, Chapitre 2, Théorème 4.10, 4.12, 4.13).

Theorem II.4.2 *Let Ω be locally Lipschitz if $m = 1, 2$ and of class $C^{m,1}$ if $m \ge 3$. Assume*

$$u \in W^{m,q}(\Omega), \quad 1 \le q < \infty, \quad m \ge 1,$$

with $\Gamma_m(u) \equiv 0$ a.e on $\partial\Omega$. Then $u \in W_0^{m,q}(\Omega)$.

A more complicated study, which is nonetheless fundamental for solving nonhomogeneous boundary-value problems, is that of determining to which Banach space $\mathcal{B} \subseteq [L^r(\partial\Omega)]^m$ a function $w \equiv (w_0, w_1, \ldots, w_{m-1})$ must belong in order to be considered the trace, via the mapping $\Gamma_{(m)}$, of a function in $W^{m,q}(\Omega)$, i.e., $\gamma_\ell(u) = w_\ell$, for some $u \in W^{m,q}(\Omega)$, for all $\ell = 0, 1, \ldots, m-1$. A counterexample due to J. Hadamard shows that $\mathcal{B}$ is, in general, *strictly* contained in $[L^r(\partial\Omega)]^m$, whatever $r \ge 1$ (Sobolev 1963a, Chapter 2, §5; De Vito 1958). Here we shall only briefly describe the answer to the problem, referring the reader to Gagliardo (1957) and Nečas (1967, Chapitre 2, §§4,5) for a fully detailed description of it. Let us first consider the case $m = 1$. Denote by $W^{1-1/q,q}(\partial\Omega)$ the subspace of $L^q(\partial\Omega)$ constituted by functions u for which the following functional is finite:

$$\|u\|_{1-1/q,q(\partial\Omega)} \equiv \|u\|_{q,\partial\Omega} + \langle\langle u \rangle\rangle_{1-1/q,q}, \tag{II.4.8}$$

where

$$\langle\langle u \rangle\rangle_{1-1/q,q} \equiv \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(y) - u(y')|^q}{|y - y'|^{n-2+q}} d\sigma_y d\sigma_{y'} \right)^{1/q}. \tag{II.4.9}$$

It can be proved (Miranda 1978, §61) that $W^{1-1/q,q}(\partial\Omega)$ is a dense subset of $L^q(\partial\Omega)$ and that it is complete in the norm $\|u\|_{1-1/q,q(\partial\Omega)}$. Furthermore, it is separable for $q \in [1, \infty)$ and reflexive for $q \in (1, \infty)$, and, for Ω smooth enough, the class of smooth functions on $\partial\Omega$ is dense in $W^{1-1/q,q}(\partial\Omega)$. We have the following theorem of Gagliardo (1957), which characterizes the trace operator γ.

Theorem II.4.3 *Let Ω be locally Lipschitz and let $q \in (1, \infty)$. If $u \in W^{1,q}(\Omega)$, then $\gamma(u) \in W^{1-1/q,q}(\partial\Omega)$ and*

$$\|\gamma(u)\|_{1-1/q,q(\partial\Omega)} \le c_1 \|u\|_{1,q,\Omega}. \tag{II.4.10}$$

Conversely, given $w \in W^{1-1/q,q}(\partial\Omega)$, there exists $u \in W^{1,q}(\Omega)$ with $\gamma(u) = w$ such that

$$\|u\|_{1,q,\Omega} \le c_2 \|\gamma(u)\|_{1-1/q,q(\partial\Omega)}. \tag{II.4.11}$$

The constants $c_i, i = 1, 2$, depend only on n, q, and Ω.

Since, by Theorem II.4.2, we have, for Ω locally Lipschitz, $u_1, u_2 \in W^{1,q}(\Omega)$ with $\gamma(u_1) = \gamma(u_2)$ then $u_1 - u_2 \in W_0^{1,q}(\Omega)$, Gagliardo's theorem can be equivalently stated by saying: *The trace operator γ is a linear bounded bijective operator from the quotient space $W^{1,q}(\Omega) \Big/ W_0^{1,q}(\Omega)$ onto the space $W^{1-1/q,q}(\partial\Omega)$.*

Remark II.4.1 Gagliardo proved this result by making a clever use of two elementary inequalities due to G. H. Hardy and C. B. Morrey, respectively. Though the proof of Theorem II.4.3 is well beyond the scope of this monograph, we may wish nevertheless to sketch a demonstration of (II.4.10) in the case when Ω is the square

$$S = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < y < 1\right\}.$$ [2]

We begin to notice that, in view of Theorem II.4.1, it suffices to show that the double surface integral in (II.4.7) is bounded above by the norm of u in $W^{1,q}(S)$, i.e.,

$$\begin{aligned}
&\int_0^1\int_0^1 \left|\frac{u(0,y)-u(0,y')}{y-y}{}'\right|^q dy\,dy' + \int_0^1\int_0^1 \left|\frac{u(1,y)-u(1,y')}{y-y'}\right|^q dy\,dy' \\
&+\int_0^1\int_0^1 \left|\frac{u(x,0)-u(x',0)}{x-x'}\right|^q dxdx' + \int_0^1\int_0^1 \left|\frac{u(x,1)-u(x',1)}{x-x'}\right|^q dx\,dx' \\
&\qquad\qquad\qquad\qquad \le C\,\|u\|_{1,q,S}^q
\end{aligned} \tag{II.4.12}$$

with a constant C independent of u. By Theorem II.3.1, we can assume $u \in C_0^\infty(\overline{S})$. Consider the first integral on the left-hand side of (II.4.11) and denote it by $\mathcal{I}$. Making the change of variables

$$\xi = x + y, \quad \eta = y - x,$$

(a rotation of an angle $\pi/4$) we may write

$$\mathcal{I} = \int_0^1\int_0^1 \left|\frac{U(\eta,\eta)-U(\eta,\eta')}{\eta-\eta'}\right|^q d\eta d\eta',$$

where

$$U(\xi,\eta) \equiv u\left(\frac{\xi-\eta}{2}, \frac{\xi+\eta}{2}\right).$$

Setting

$$\phi(\eta) = U(\eta,\eta)$$

[2] In fact, following Gagliardo, it is not difficult to prove that the case of a general locally Lipschitz domain can be reduced to the present one.

for $0 \le \eta' < \eta \le 1$ we have

$$\frac{|\phi(\eta) - \phi(\eta')|}{\eta - \eta'} \le \frac{1}{\eta - \eta'} \int_{\eta'}^{\eta} \left| \frac{\partial U}{\partial \lambda}(\lambda, \eta') \right| d\lambda + \frac{1}{\eta - \eta'} \int_{\eta'}^{\eta} \left| \frac{\partial U}{\partial \mu}(\eta, \mu) \right| d\mu$$

and thus, by (II.3.3),

$$\begin{aligned} f(\eta, \eta') \equiv \left| \frac{\phi(\eta) - \phi(\eta')}{\eta - \eta'} \right|^q \le 2^{q-1} \Bigg\{ & \left[\frac{1}{|\eta - \eta'|} \int_{\eta'}^{\eta} \left| \frac{\partial U}{\partial \lambda}(\lambda, \eta') \right| d\lambda \right]^q \\ & + \left[\frac{1}{|\eta - \eta'|} \int_{\eta'}^{\eta} \left| \frac{\partial U}{\partial \mu}(\eta, \mu) \right| d\mu \right]^q \Bigg\}. \end{aligned} \tag{II.4.13}$$

We now recall the following inequalities due to G.H. Hardy (Hardy, Littlewood and Polya 1934, p. 240):

$$\begin{aligned} & \int_a^b dx \left| \frac{1}{x - a} \int_a^x f(t) dt \right|^q \le \left(\frac{q}{q-1} \right)^q \int_a^b |f(t)|^q dt, \quad x > a, \ q > 1 \\ & \int_a^b dx \left| \frac{1}{b - x} \int_x^b f(t) dt \right|^q \le \left(\frac{q}{q-1} \right)^q \int_a^b |f(t)|^q dt, \quad x < b, \ q > 1. \end{aligned} \tag{II.4.14}$$

Integrating (II.4.13) first in $\eta \in (\eta', 1]$ and then in $\eta' \in [0, 1]$ and using (II.4.14) we obtain

$$\begin{aligned} \int_0^1 \left(\int_{\eta'}^1 f(\eta, \eta') d\eta \right) d\eta' \le 2^{q-1} \left(\frac{q}{q-1} \right)^q & \left[\int_0^1 d\eta' \int_{\eta'}^1 \left| \frac{\partial U}{\partial \lambda}(\lambda, \eta') \right|^q d\lambda \right. \\ & \left. + \int_0^1 d\eta \int_0^{\eta} \left| \frac{\partial U}{\partial \mu}(\eta, \mu) \right|^q d\mu \right] \\ & \le c \|\nabla u\|_{q,S}^q, \end{aligned} \tag{II.4.15}$$

with c a suitable constant. Interchanging the roles of η and η' in (II.4.15) and noticing that $f(\eta, \eta') = f(\eta', \eta)$ one also has

$$\int_0^1 \left(\int_{\eta}^1 f(\eta, \eta') d\eta' \right) d\eta \le c \|\nabla u\|_{q,S}^q. \tag{II.4.16}$$

Adding (II.4.15) and (II.4.16) we find

$$\mathcal{I} \le 2c \|\nabla u\|_{q,S}^q.$$

Since the other integrals on the left-hand side of (II.4.12) can be analogously increased, the proof of (II.4.12) is accomplished. ■

Exercise II.4.2 According to the method just described, the case $q = 1$ of Theorem II.4.3 is excluded because Hardy's inequalities (II.4.14) hold if $q > 1$. Show, by means of a counterexample, that (II.4.14) does not hold when $q=1$. *Hint* (Gagliardo 1957): Take $f(t)=(t-a)^{-1}(\log(t-a))^{-2}$. (For the characterization of the trace when $m=q=1$, see Gagliardo (1957, Teorema 1.II)).

The extension of Theorem II.4.3 to the space $W^{m,q}(\Omega)$, $m\geq 2$, is formally analogous, provided we introduce a suitable generalization of the space $W^{1-1/q,q}(\partial\Omega)$. To this end, assume Ω of class $C^{m-1,1}$ and let $\{B_k\}$ and $\{\zeta_k\}$, $k=1,2,\ldots,s$, be a family of open balls centered at $x_k\in\partial\Omega$ with $\partial\Omega\subset B_k$, and of functions of class $C^{m-1,1}(\overline{D}_k)$, respectively, defining the $C^{m-1,1}$ – regularity of $\partial\Omega$ in the sense of Definition II.1.1. Assuming that

$$x_n^{(k)}=\zeta_k(x_1^{(k)},\ldots,x_{n-1}^{(k)}),\quad (x_1^{(k)},\ldots,x_{n-1}^{(k)})\in D_k$$

is the representation of $\partial\Omega\cap B_k$, for a function u on $\partial\Omega$ we set

$$u_k=u(x_1^{(k)},\ldots,x_{n-1}^{(k)},\zeta_k(x_1^{(k)},\ldots,x_{n-1}^{(k)}))$$

and define

$$\|u\|_{m-1/q,q(\partial\Omega)}\equiv\sum_{k=1}^{s}\|u_k\|_{m-1/q,q,D_k} \tag{II.4.17}$$

where

$$\|u_k\|_{m-1/q,q,D_k}\equiv\sum_{0\leq|\alpha|\leq m-1}\|D^\alpha u_k\|_{q,D_k}+\langle\langle u_k\rangle\rangle_{m-1/q,q}$$

$$\langle\langle u_k\rangle\rangle_{m-1/q,q}\equiv\sum_{|\alpha|=m-1}\left(\int_{D_k}\int_{D_k}\frac{|D^\alpha u(y)-D^\alpha u(y')|^q}{|y-y'|^{n-2+q}}dydy'\right)^{1/q}. \tag{II.4.18}$$

We next denote by $W^{m-1/q,q}(\partial\Omega)$ the linear space of functions u for which the functional defined by (II.4.17)–(II.4.18) is finite. It can be shown that the definition of $W^{m-1/q,q}(\partial\Omega)$ does not depend on the particular choice of the local representation $\{B_k\}$, $\{\zeta_k\}$ of the boundary. In fact, if $\{B'_{k'}\}$, $\{\zeta'_{k'}\}$ is another such a representation and $\|u\|'_{m-1/q,q(\partial\Omega)}$ is the corresponding functional associated to u, there exist constants c_1, $c_2>0$ such that

$$\|u\|_{m-1/q,q(\partial\Omega)}\leq c_1\|u\|'_{m-1/q,q(\partial\Omega)}\leq c_2\|u\|_{m-1/q,q(\partial\Omega)}$$

(Nečas 1967, Chapitre 3, Lemme 1.1). As in the case of $W^{1-1/q,q}(\partial\Omega)$, one shows that the space $W^{m-1/q,q}(\partial\Omega)$ is a dense subset of $L^q(\partial\Omega)$, which is complete in the norm (II.4.17)–(II.4.17), separable for $q\in[1,\infty)$ and reflexive for $q\in(1,\infty)$ (Nečas 1967, Chapitre 2, Proposition 3.1).

Set

$$\mathcal{W}_{m,q}(\partial\Omega)\equiv W^{m-1/q,q}(\partial\Omega)\times W^{m-1-1/q,q}(\partial\Omega)\times\ldots\times W^{1-1/q,q}(\partial\Omega).$$

We then have the following characterization of the trace operator $\Gamma_{(m)}$ defined in (II.4.6)–(II.4.7) (Nečas 1967, Chapitre 2, Théorème 5.5, 5.8).

Theorem II.4.4 *Let Ω be of class $C^{m-1,1}$, $m \geq 2$. If*

$$u \in W^{m,q}(\Omega), \quad 1 < q < \infty,$$

then

$$\Gamma_{(m)}(u) \in \mathcal{W}_{m,q}(\partial\Omega)$$

and for all $\ell = 0, 1, \ldots, m-1$ it is

$$\|\gamma_\ell(u)\|_{m-\ell-1/q,q(\partial\Omega)} \leq c_1 \|u\|_{m,q,\Omega}. \tag{II.4.19}$$

Conversely, if Ω is of class $C^{m,1}$, given

$$w \in \mathcal{W}_{m,q}(\partial\Omega)$$

there exists $u \in W^{m,q}(\Omega)$ with

$$\Gamma_{(m)}(u) = w$$

and the following inequality holds

$$\|u\|_{m,q,\Omega} \leq c_2 \sum_{\ell=0}^{m-1} \|\gamma_\ell(u)\|_{m-\ell-1/q,q(\partial\Omega)}. \tag{II.4.20}$$

The constants c_i, $i = 1, 2$, depend only on n, m, q, and Ω.

As in the case of the operator γ, the operator $\Gamma_{(m)}$ can also be characterized, in view of Theorem II.4.2 and Theorem II.4.4, as a bounded linear bijection of $W^{m,q}(\Omega)/W_0^{m,q}(\Omega)$ onto $\mathcal{W}_{m,q}(\partial\Omega)$ (topologized in the obvious way).

Remark II.4.2 If Ω is not globally smooth but has a smooth boundary portion σ, we can still define the trace on σ of functions from $W^{m,q}(\Omega)$ and the space $\mathcal{W}_{m,q}(\sigma)$. In particular, inequality (II.4.19) continues to hold with σ in place of $\partial\Omega$ (see Nečas, *loc. cit.*). ∎

Remark II.4.3 Problems of trace on the plane $\{x_n = 0\}$ for functions defined in $\mathbb{R}^{n-1}$ will be considered in Section II.10. ∎

Exercise II.4.3 (Nečas 1967, Chapitre 3, Théorème 1.1)**.** Let Ω be bounded and locally Lipschitz. Show the following Gauss identity:

$$\int_\Omega \Phi \nabla \cdot \boldsymbol{u} = \int_{\partial\Omega} \Phi \boldsymbol{u} \cdot \boldsymbol{n} - \int_\Omega \boldsymbol{u} \cdot \nabla\Phi \tag{II.4.21}$$

for all vectors $\boldsymbol{u}$ with components in $W^{1,q}(\Omega)$ and scalars Φ from $W^{1,r}(\Omega)$ where q and r satisfy

(i) $q^{-1} + r^{-1} \leq (n+1)/n$ if $1 \leq q < n$, $1 \leq r < n$;
(ii) $r > 1$ if $q \geq n$;
(iii) $q > 1$ if $r \geq n$;

Hint: Use Lemma II.4.1 and Theorem II.3.3 and Theorem II.4.1.

Remark II.4.4 An extension of (II.4.21) to functions $\boldsymbol{u}$ with less regularity than that required in Exercise II.4.3 will be given in Section III.2, see (III.2.14). ∎

II.5 Further Inequalities and Compactness Criteria in $W^{m,q}$

We begin to prove some inequalities relating the L^q-norm of a function with that of its first derivatives (Poincaré 1894, §III, and Friedrichs 1933). Throughout this section we shall denote by L_d a layer of width $d > 0$, namely

$$L_d = \{x \in \mathbb{R}^n : -d/2 < x_n < d/2\}.$$

Theorem II.5.1 *Assume $\Omega \subset L_d$, for some $d > 0$. Then, for all $u \in W_0^{1,q}(\Omega)$, $1 \le q \le \infty$,*

$$\|u\|_q \le (d/2)\|\nabla u\|_q. \tag{II.5.1}$$

Proof. It is enough to show the theorem for $u \in C_0^\infty(\Omega)$. For such functions one has

$$|u(x_1, \ldots, x_n)| = \left| \int_{-d/2}^{x_n} \frac{\partial u(x_1, \ldots, \xi)}{\partial \xi}\, d\xi \right| = \left| \int_{x_n}^{d/2} \frac{\partial u(x_1, \ldots, \xi)}{\partial \xi}\, d\xi \right|,$$

which implies

$$|u(x)| \le (1/2) \int_{-d/2}^{d/2} |\nabla u|\, dx_n\,. \tag{II.5.2}$$

From this relation we at once recover (II.5.1) for $q = \infty$. If $q \in [1, \infty)$, employing the Hölder inequality in the right-hand side of (II.5.2) yields

$$|u(x)|^q \le (d^{q-1}/2^q) \int_{-d/2}^{d/2} |\nabla u|^q dx_n$$

which, after integrating over L_d, proves (II.5.1). □

Exercise II.5.1 Inequality (II.5.1) fails, in general, if Ω is not contained in some layer L_d. Suppose, for instance, $\Omega \equiv \mathbb{R}^n$ and consider the sequence

$$u_m = \exp[-|x|/(m+1)],\ m \in \mathbb{N}.$$

Show that

$$\frac{\|u_m\|_q}{\|\nabla u_m\|_q} = \frac{m+1}{q}.$$

Modify this example to prove the invalidity of (II.5.1) for Ω an arbitrary exterior domain or a half-space.

The special case $q = 2$ in (II.5.1) plays an important role in several applications. In particular, it is of great interest in uniqueness and stability questions to determine the smallest constant μ such that

$$\|u\|_2^2 \le \mu \|\nabla u\|_2^2. \tag{II.5.3}$$

The constant μ (sometimes called the *Poincaré constant*) depends on the domain Ω, and when Ω is bounded one easily shows that $\mu = 1/\lambda_1$, where λ_1 is the smallest eigenvalue of the problem

$$-\Delta u = \lambda u \ \text{ in } \Omega\,, \qquad u = 0 \ \text{ at } \partial\Omega\,; \tag{II.5.4}$$

see Sobolev 1963a, Chapter II, §16. An estimate of λ_1 comes from (II.5.1) and one has

$$\lambda_1 \ge 4/[\delta(\Omega)]^2.$$

However, a better estimate can be obtained as a consequence of the following simple argument due to E. Picard (Picone 1946, §160).[1] In fact, assume as before $\Omega \subset L_d$ for some $d > 0$ and consider the function

$$U(x) = \frac{u(x)}{\sin[\pi(x_n + d/2)/d]}, \quad u \in C_0^\infty(\Omega).$$

Since $U(x)$ is bounded in L_d and vanishes at $-d/2$, $d/2$, integrating by parts we find

$$0 \le \int_{-d/2}^{d/2} \left\{ \frac{\partial u}{\partial x_n} - \frac{\pi}{d} u(x) \cot\left[\frac{\pi(x_n + d/2)}{d}\right]\right\}^2 dx_n = \int_{-d/2}^{d/2} \left(\frac{\partial u}{\partial x_n}\right)^2 dx_n$$

$$-\frac{\pi^2}{d^2} \int_{-d/2}^{d/2} u^2 \left\{ \sin^{-2}\left[\frac{\pi(x_n + d/2)}{d}\right] - \cot^2\left[\frac{\pi(x_n + d/2)}{d}\right]\right\} dx_n\,.$$

Hence

$$\int_{-d/2}^{d/2} u^2 dx_n \le (d/\pi)^2 \int_{-d/2}^{d/2} \left(\frac{\partial u}{\partial x_n}\right)^2 dx_n,$$

which implies

$$\|u\|_2 \le (d/\pi)\|\nabla u\|_2.$$

Therefore, one deduces

$$\mu \le d^2/\pi^2$$

and, if Ω is bounded,

$$\mu \le [\delta(\Omega)/\pi]^2.$$

Notice that these estimates are sharp in the sense that when $n = 1$ and $\Omega = L_d$ we have from (II.5.4) $\mu^{-1} = \lambda_1 = [\pi/\delta(\Omega)]^2 = (\pi/d)^2$.

Generalizations of (II.5.1) and (II.5.3) are considered in the following exercises.

[1] This proof was brought to my attention by Professor Luigi Pepe.

Exercise II.5.2 Let $\Omega \subset \{x \in \mathbb{R}^n : -d/2 < x_i < d/2,\ i = 1, \dots, n\}$. Use Picard's argument to show the following estimate for the Poincaré constant μ:

$$\mu \le d^2/n\pi^2.$$

Exercise II.5.3 Let $\Omega \subset L_d$, for some $d > 0$. Show that

$$\|\nabla u\|_2 \le (d/\pi)\|\Delta u\|_2$$

for all $u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$. Thus, in particular,

$$\|u\|_2 \le (d/\pi)^2\|\Delta u\|_2.$$

Hint: Consider the identity: $(u, \Delta u) = -\|\nabla u\|_2$.

Exercise II.5.4 Let Ω be of finite measure and let $u \in W_0^{1,q}(\Omega)$, $1 \le q < \infty$. Show the inequality

$$\|u\|_q \le \beta|\Omega|^{1/n}\|\nabla u\|_q \tag{II.5.5}$$

where

$$\beta = \begin{cases} \dfrac{q(n-1)}{2(n-q)\sqrt{n}} & \text{if } q < n \\[2ex] \dfrac{q}{2\sqrt{n}} & \text{if } q \ge n\,. \end{cases}$$

Hint: Use (II.3.5) and the inequality

$$\|u\|_q \le |\Omega|^{(1/q)-(1/r)}\|u\|_r, \quad r > q.$$

Exercise II.5.5 Let Ω be bounded and let $u \in W_0^{1,q}(\Omega)$, $q > n$. Show that, for all $q_1 \in (n, q)$, the following inequality holds

$$\|u\|_C \le c\,\|u\|_q^{1-q/q_1}\,\|\nabla u\|_q^{q/q_1}\,,$$

with $c = c(n, q, q_1, \Omega)$. *Hint*: From (II.3.18) and (II.5.1) we find $\|u\|_C \le c\,\|\nabla u\|_q$.

Exercise II.5.6 Let Ω be bounded and C^1-smooth, and let $\boldsymbol{u}$ be a vector function with components in $W^{1,q}(\Omega)$, $1 \le q < \infty$, and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ at $\partial\Omega$ ($\boldsymbol{n}$ being the outer normal). Show the inequality

$$\|\boldsymbol{u}\|_q \le C\,\|\nabla \boldsymbol{u}\|_q, \quad C \le \delta(\Omega)(|q-2| + n + 1).$$

Hint (due to L.H. Payne): Integrate the identity:

$$\sum_{i,j=1}^{n} \left(D_i[u_i x_j u_j |u|^{q-2}] - (D_i u_i) x_j u_j |u|^{q-2} - |u|^q - u_i x_j D_i[u_j |u|^{q-2}]\right) = 0.$$

An inequality of the type (II.5.1) continues to hold even though u is not zero at the boundary, provided one replaces u with $u - \overline{u}_\Omega$. We shall begin to prove the following result which traces back to Poincaré (1894).

Lemma II.5.1 *For $a > 0$ let*

$$C = \{x \in \mathbb{R}^n : 0 < x_i < a\}. \tag{II.5.6}$$

Then, for all $u \in W^{1,q}(C)$, $1 \le q < \infty$,

$$\|u - \overline{u}_C\|_q \le na\|\nabla u\|_q. \tag{II.5.7}$$

Proof. For simplicity, we shall give the proof in the case $n = 3$. Clearly, in view of Theorem II.3.1, it is enough to show (II.5.6) for $u \in C^1(\overline{\Omega})$. Consider the identity

$$u(x_1, x_2, x_3) - u(y_1, y_2, y_3) = \int_{y_1}^{x_1} \frac{\partial u}{\partial \xi}(\xi, x_2, x_3)d\xi + \int_{y_2}^{x_2} \frac{\partial u}{\partial \eta}(y_1, \eta, x_3)d\eta$$
$$+ \int_{y_3}^{x_3} \frac{\partial u}{\partial \zeta}(y_1, y_2, \zeta)d\zeta.$$

Integrating over the y-variables and raising to the qth power, we deduce

$$|u(x_1, x_2, x_3) - \overline{u}_C|^q \le |C|^{-q} \left[a^3 \int_0^a |\nabla u(\xi, x_2, x_3)|d\xi\right.$$
$$\left. +a^2 \int_0^a \int_0^a |\nabla u(y_1, \eta, x_3)|dy_1 d\eta + a \int_C |\nabla u| dC\right]^q.$$

Employing in this relation the inequality (II.3.3) along with the Hölder inequality and integrating over the x-variables we obtain

$$\int_C |u - \overline{u}_C|^q \le 3^q a^q \int_C |\nabla u|^q,$$

which completes the proof. □

Remark II.5.1 An extension of (II.5.7) to arbitrary locally Lipschitz domains will be given in Theorem II.5.4. Here, however, we wish to observe that, unlike Theorem II.5.1, some regularity assumptions on Ω are strictly necessary for inequalities of type (II.5.7) to hold, as shown by means of counterexample in Courant & Hilbert (1937, Kapitel VII, §8.2); see also Fraenkel (1979, and §2 in particular). ■

Let us now analyze some consequences of Lemma II.5.1. Suppose Ω is a cube of side a and subdivide it into N equal cubes C_i, each having sides of length $a/N^{1/n}$. Applying (II.5.7) to each cube C_i and using the Minkowski inequality and (II.3.3) one recovers

$$\|u\|_{q,\Omega}^q \leq \sum_{i=1}^N 2^{q-1}\left(\frac{a}{N^{1/n}}\right)^{n(1-q)}\left|\int_{C_i} udC_i\right|^q + \frac{(2na)^q}{2N^{q/n}}\|\nabla u\|_{q,\Omega}^q.$$

Therefore, introducing the N independent functions

$$\psi_i(x) = 2^{(q-1)/q}\left(\frac{a}{N^{1/n}}\right)^{n(1-q)/q}\chi_i(x),$$

with χ_i characteristic function of the cube C_i, from the previous inequality one has the following result due to Friedrichs (1933).

Lemma II.5.2 *Let C be the cube* (II.5.6) *and let*

$$u \in W^{1,q}(C), \quad 1 \leq q < \infty.$$

Then, given an arbitrary positive integer N, there exist N independent functions $\psi_i \in L^\infty(C)$ depending only on C and N such that

$$\|u\|_{q,C}^q \leq \sum_{i=1}^N \left|\int_C \psi_i u\right|^q + \frac{(2na)^q}{2N^{q/n}}\|\nabla u\|_{q,C}^q. \tag{II.5.8}$$

Inequality (II.5.8) is very useful in proving compactness results, as we are about to show. In fact, let Ω be bounded and let $\{u_m\} \subset W_0^{1,q}(\Omega)$, $1 \leq q < \infty$, be uniformly bounded in the norm $\|\cdot\|_{1,q}$. Extending u_m by zero outside Ω and denoting again by u_m such an extension, we thus have that $\{u_m\}$ is uniformly bounded in $W^{1,q}(C)$, for some cube C (see Exercise II.3.11), and therefore, by Lemma II.5.2, Theorem II.2.4(ii) and Theorem II.3.2, it is not difficult to show the existence of a subsequence $\{u_{m'}\}$ that is Cauchy in $L^q(C)$ and, as a consequence, converges strongly in $L^q(\Omega)$. On the other hand, by Lemma II.3.2 and by Exercise II.5.5, it follows that $\{u_{m'}\}$ converges also in $L^r(\Omega)$, for all $r \in [1, nq/(n-q))$, if $q < n$, for all $r \in [1,\infty)$ if $q = n$, while it converges in $C(\overline{\Omega})$ if $q > n$. We have proved the following compact embedding result (see Rellich 1930).

Theorem II.5.2 *Assume Ω bounded, and let $q \in [1,\infty)$. Then*

$$W_0^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)\,,$$

with arbitrary $r \in [1, nq/(n-q))$, if $q < n$, and arbitrary $r \in [1,\infty)$, if $q = n$. Finally, if $q > n$, then $W_0^{1,q}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$

In Theorem II.5.2, when $q < n$, the exponent $q^* = nq/(n-q)$ is excluded. Actually one proves by means of counterexamples that the strong convergence is, in general, ruled out in this case. For, in the ball B_1 consider the sequence of functions

$$u_m(x) = \begin{cases} m^{(n-q)/q}(1-m|x|) & \text{if } |x| < 1/m \\ 0 & \text{if } |x| \geq 1/m \end{cases} \qquad m = 1, 2, \ldots$$

with $q < n$. One has

$$\|\nabla u_m\|_q = C_1, \quad \|u_m\|_{q*} = C_2,$$

with C_1 and C_2 independent of m. Since

$$\lim_{m\to\infty} u_m(x) = 0 \ \ a.e. \text{ in } B_1$$

it follows that no subsequence can converge strongly in $L^{q*}(B_1)$.

Theorem II.5.2 admits the following counterpart in negative Sobolev spaces.

Theorem II.5.3 *Let Ω be bounded. Then $L^q(\Omega) \hookrightarrow\hookrightarrow W_0^{-1,q}(\Omega)$, for any $1 < q < \infty$. Precisely, if $\{u_m\} \subset L^q(\Omega)$ is uniformly bounded, there exists a subsequence $\{u_{m'}\}$ and $u \in L^q(\Omega)$ such that*

$$\lim_{m'\to\infty} \|u - u_{m'}\|_{-1,q} = 0\,.$$

Proof. In view of inequality (II.5.1), we may endow $W_0^{1,q}(\Omega)$ with the equivalent norm $\|\nabla(\cdot)\|_q$. We observe next that, by assumption and by Theorem II.2.4(iii), there are $u \in L^q(\Omega)$ and a subsequence $\{u_{m'}\}$ such that $u_{m'} \xrightarrow{w} u$. Set $U_{m'} = u - u_{m'}$. By Theorem II.3.5 and Theorem II.1.4, for each $m' \in \mathbb{N}$, we can find $w_{m'} \in W_0^{1,q'}(\Omega)$ such that

$$\|U_{m'}\|_{-1,q} = |(U_{m'}, w_{m'})|\,, \quad \|\nabla w_{m'}\|_{q'} = 1\,. \tag{II.5.9}$$

Then, by Theorem II.5.2 and Theorem II.1.3(ii), there exist a subsequence $\{w_{m''}\}$ and $w \in W_0^{1,q'}(\Omega)$ such that $w_{m''} \to w$ in $L^{q'}(\Omega)$, and so (II.5.9) delivers

$$\|U_{m''}\|_{-1,q} \leq |(U_{m''}, w)| + \|U_{m''}\|_q \|w_{m''} - w\|_{q'} \leq |(U_{m''}, w)| + C\,\|w_{m''} - w\|_{q'}\,,$$

which, in turn, gives the desired result since $U_{m''} \xrightarrow{w} 0$ in $L^q(\Omega)$ and $w_{m''} \to w$ in $L^{q'}(\Omega)$. □

Some generalizations of Theorem II.5.2 are proposed to the reader in the following exercises.

Exercise II.5.7 Assume Ω bounded and let $q \in [1, \infty)$, $m \geq 1$. Show that

$$W_0^{m,q}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$$

with arbitrary $r \in [1, nq/(n - mq))$ if $mq < n$ and all $r \in [1, \infty)$ if $mq = n$. Finally, show that if $mq > n$, then $W_0^{m,q}(\Omega) \hookrightarrow\hookrightarrow C^k(\overline{\Omega})$, for all $k \in \mathbb{N}$ such that $0 \leq k < 1 - mq/n$.

Exercise II.5.8 Prove that, when Ω is bounded and locally Lipschitz, Theorem II.5.2 and Exercise II.5.7 continue to hold if $W_0^{m,q}(\Omega)$ is replaced by $W^{m,q}(\Omega)$. *Hint*: Use Theorem II.3.3 and (II.3.19).

We want now to obtain further inequalities as a consequence of the compactness results just derived. The following theorem extends the Poincaré inequality (II.5.7) to more general domains.

Theorem II.5.4 *Let Ω be bounded and locally Lipschitz. Then, for all $u \in W^{1,q}(\Omega)$, $1 \le q < \infty$, we have*

$$\|u - \overline{u}_\Omega\|_q \le c\|\nabla u\|_q, \tag{II.5.10}$$

where $c = c(n, q, \Omega)$.

Proof. To simplify notation, we omit the subscript Ω. If (II.5.10) were not true, a sequence $\{u_m\} \subset W^{1,q}(\Omega)$ would exist such that for all $m \in \mathbb{N}$

$$\overline{u}_m = 0, \quad \|u_m\|_q = 1, \quad \|\nabla u_m\|_q \le 1/m. \tag{II.5.11}$$

Therefore, from $(\text{II.5.11})_{2,3}$ and Exercise II.5.8 there is a subsequence converging in the norm of $W^{1,q}(\Omega)$ to some $u \in W^{1,q}(\Omega)$ which, by (II.5.11), should have $\nabla u = 0$, $\overline{u} = 0$, namely, $u \equiv 0$ *a.e.* in Ω and $\|u\|_q = 1$. This gives a contradiction that proves the theorem. □

Theorem II.5.4 admits several interesting consequences, some of which are left to the reader in the following exercises.

Exercise II.5.9 Let Ω be an arbitrary domain and let $u \in W_{loc}^{1,1}(\Omega)$. Show that, if $Du = 0$, then there is $u_0 \in \mathbb{R}$ such that $u = u_0$ *a.e.* in Ω. Using this result, show that, more generally, if $u \in W_{loc}^{m,1}(\Omega)$ with $D^\alpha u = 0$, $|\alpha| = m$, then $u = P$ *a.e* in Ω, where P is a polynomial of degree $\le m-1$. *Hint*: Use Lemma II.1.1.

Exercise II.5.10 Assume Ω bounded and locally Lipschitz and let $u \in W^{1,q}(\Omega)$. If $q \in [1, n)$, prove the following *Poincaré-Sobolev inequality*:

$$\|u - \overline{u}_\Omega\|_r \le c\|\nabla u\|_q, \tag{II.5.12}$$

where $r = nq/(n-q)$ and $c = c(n, q, \Omega)$. Moreover, show that, if $q > n$, the following inequality holds

$$\|u - \overline{u}_\Omega\|_C \le c_1\|\nabla u\|_q\,. \tag{II.5.13}$$

Hint: Use Theorem II.5.4 and $(\text{II.3.16})_{1,3}$.

Exercise II.5.11 Let $u \in W^{1,q}(B_r(x_0))$, $q > n$. Show that the following inequality holds

$$\max_{x \in B_r(x_0)} |u(x) - u(x_0)| \le c\, r^{1-n/q}\|\nabla u\|_{q,B_r(x_0)}\,,$$

with $c = c(n, q)$. *Hint*: Use (II.5.13) on the unit ball and then rescale the result for a ball of radius r.

Another consequence of Theorem II.5.4 furnishes an interesting generalization of the *Wirtinger inequality* (Hardy, Littlewood, and Polya 1934, p. 185), which we are going to show. Denote by $\nabla^* u$ the projection of ∇u on the unit sphere S^{n-1} in $\mathbb{R}^n$, $n \geq 2$. We have

$$|\nabla^* u|^2 = r^2 \left[|\nabla u|^2 - \left| \frac{\partial u}{\partial r} \right|^2 \right], \quad r = |x|. \tag{II.5.14}$$

For a function f defined on S^{n-1} we may write

$$\|f - \overline{f}\|^q_{q,S^{n-1}} \leq \frac{2^n n}{2^n - 1} \|f - \overline{f}\|^q_{q,\Omega}, \tag{II.5.15}$$

where

$$\overline{f} = |S^{n-1}|^{-1} \int_{S^{n-1}} f dS^{n-1} \tag{II.5.16}$$

and Ω is the spherical shell of radii 1/2 and 1. Noting that

$$\overline{f} = |\Omega|^{-1} \int_\Omega f,$$

we may employ Theorem II.5.4 to obtain

$$\|f - \overline{f}\|^q_{q,\Omega} \leq c^q \|\nabla f\|^q_{q,\Omega} = c_1 \|\nabla^* f\|^q_{q,S^{n-1}}.$$

Thus, combining (II.5.15) with the latter inequality, we deduce the desired *Wirtinger inequality*:

$$\|f - \overline{f}\|_{q,S^{n-1}} \leq c_2 \|\nabla^* f\|_{q,S^{n-1}}, \quad 1 \leq q < \infty, \tag{II.5.17}$$

with $\overline{f}$ defined in (II.5.16), and $c_2 = c_2(n,q)$.

Exercise II.5.12 (Finn and Gilbarg 1957)**.** Show that, for $q = 2$, the smallest constant c_2 for which (II.5.17) holds is $c_2 = (n-1)^{-1/2}$. *Hint*: Consider the associated eigenvalue problem $\Delta^* u + \lambda u = 0$, where Δ^* denotes the Laplace operator on the unit sphere.

In the exercises that follow, we propose to the reader the proof of some useful inequalities, easily obtainable by using the same compactness argument adopted in the proof of Theorem II.5.4.

Exercise II.5.13 Let Ω be bounded and locally Lipschitz and let Σ be an arbitrary portion of $\partial\Omega$ of positive ($(n-1)$-dimensional) measure. Show that for all $u \in W^{1,q}(\Omega)$, $1 \leq q < \infty$, the following inequality holds

$$\|u\|_q \leq c \left(\|\nabla u\|_q + \left| \int_\Sigma u \right| \right) \tag{II.5.18}$$

with $c = c(n, q, \Omega, \Sigma)$.

Exercise II.5.14 Let Ω be bounded and locally Lipschitz, and let $u \in W^{m,q}(\Omega)$. Then, there exists $c = c(n, q, \Omega, \omega)$ such that

$$\|u\|_{m,q} \le c \left(\sum_{|\alpha|=m} \|D^\alpha u\|_q + \int_\omega |u| \right) \tag{II.5.19}$$

where ω is an arbitrary subdomain of Ω of positive (n-dimensional) measure. *Hint*: Use Exercise II.5.9 .

Exercise II.5.15 Let Ω be bounded and locally Lipschitz and let $\boldsymbol{u}$ be a vector function in Ω with components from $W^{1,q}(\Omega)$, $1 \le q < \infty$. Assuming $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ at $\partial\Omega$, show that there exists a constant $c = c(n, q, \Omega)$ such that

$$\|\boldsymbol{u}\|_q \le c\|\nabla \boldsymbol{u}\|_q.$$

Hint: Use Exercise II.5.8.

Exercise II.5.16 (Ehrling inequality) Let Ω be bounded and locally Lipschitz. Show that for any $\varepsilon > 0$ there is $c = c(\varepsilon, n, q, \Omega) > 0$ such that

$$\|\nabla u\|_q \le c\|u\|_q + \varepsilon\|D^2 u\|_q, \tag{II.5.20}$$

for all $u \in W^{2,q}(\Omega)$, $1 \le q < \infty$. The regularity assumption on Ω can be removed if $u \in W_0^{2,q}(\Omega)$. *Hint*: Use Exercise II.5.8 and Theorem II.5.2.

Remark II.5.2 Inequalities of the type given in Exercise II.5.13 and Exercise II.5.14 are relevant in the context of the equivalence of norms in the spaces $W^{m,q}$. A general theorem, that contains these inequalities as a particular case, can be found in Smirnov (1964, §114, Theorem 3). ■

We end this section by giving another significant application of the contradiction-compactness argument used in the proof of Theorem II.5.4, that generalizes the result given in Galdi (2007, Lemma 5.4). To this end, we set

$$\overset{o}{W}{}^{1,q}(\Omega) = \{u \in W^{1,q}(\Omega) : \quad u|_\Sigma = 0\}, \tag{II.5.21}$$

where Σ is an arbitrarily fixed locally Lipschitz boundary portion of $\partial\Omega$. It is easily shown that $\overset{o}{W}{}^{1,q}(\Omega)$ is a closed subspace of $W^{1,q}$ (Exercise II.5.17). Moreover, in view of Exercise II.5.13, we find that a norm equivalent to $\|(\cdot)\|_{1,q}$ is given by $\|\nabla(\cdot)\|_q$, and we shall endow $\overset{o}{W}{}^{1,q}(\Omega)$ with this latter.

We recall that a sequence of of linear functionals, $\{\ell_i\}$, on a Banach space X, is called *complete* if

$$\ell_i(u) = 0\,, \text{ for all } i \in \mathbb{N}, \text{ implies } u = 0 \text{ in } X\,.$$

We have the following result.

Lemma II.5.3 *Let Ω be locally Lipschitz, and let $\{l_i\}$ be a complete sequence of linear functionals on $\overset{o}{W}{}^{1,q}(\Omega)$, $1 < q < \infty$. Then, given $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and a positive constant C such that*

$$\|u\| \leq \varepsilon \|\nabla u\|_q + C \sum_{i=1}^{N} |l_i(u)|,$$

where $\|u\| \equiv \|u\|_r$ with $r \in [1, nq/(n-q))$, if $q < n$, and $r \in [1, \infty)$, if $q = n$, while $\|u\| \equiv \|u\|_C$ if $q > n$. The numbers N and C depend on Ω, ε, q, and also on r if $q \leq n$.

Proof. We give a proof in the case $q < n$, the other two cases being treated in a completely analogous way, with the help of Theorem II.5.2. Thus, assume, by contradiction, that there is $\overline{\varepsilon} > 0$ such that, for all $C > 0$ and all $N \in \mathbb{N}$ we can find at least one $u = u(C, N) \in \overset{o}{W}{}^{1,q}(\Omega)$ such that

$$\|u\|_r \geq \overline{\varepsilon} \|\nabla u\|_q + C \sum_{i=1}^{N} |l_i(u)|.$$

We then fix $N = N_1$ and find a sequence $\{u_m\}$, possibly depending on N_1, such that

$$\|u_m\|_r \geq \overline{\varepsilon} \|\nabla u_m\|_q + m \sum_{i=1}^{N_1} |l_i(u_m)|.$$

Setting $w_m = u_m / \|\nabla u_m\|_q$,[2] from the preceding inequality we find

$$\|w_m\|_r \geq \overline{\varepsilon} + m \sum_{i=1}^{N_1} |l_i(w_m)|, \quad \|\nabla w_m\|_q = 1, \ m \in \mathbb{N}. \tag{II.5.22}$$

From (II.5.22) we then deduce that

$$\|w_m\|_{1,q} \leq C_1 \tag{II.5.23}$$

with $C_1 = C_1(\Omega, \Sigma, q) > 0$. So, by Theorem II.5.2 and by the weak compactness property of the unit closed ball (see Remark II.3.1), there exist a subsequence, again denoted by $\{w_m\}$, and $w^{(1)} \in \overset{o}{W}{}^{1,q}(\Omega)$ such that

$$\begin{aligned} & w_m \to w^{(1)} \ \text{ in } L^r(\Omega) \\ & w_m \overset{w}{\to} w^{(1)} \ \text{ in } \overset{o}{W}{}^{1,q}(\Omega). \end{aligned} \tag{II.5.24}$$

Using these latter properties along with (II.5.22) we infer, on the one hand,

[2] Of course, we may assume, without loss of generality, that $\|\nabla u_m\|_q \neq 0$, for all $m \in \mathbb{N}$.

$$\sum_{i=1}^{N_1} |l_i(w^{(1)})| = 0\,,$$

and, on the other hand,

$$\|w^{(1)}\|_r \geq \overline{\varepsilon}\,.$$

Moreover, from (II.5.22)$_2$, (II.5.23), and (II.5.24) we obtain

$$\|w^{(1)}\|_r + \|w^{(1)}|_{1,q} \leq C_2$$

with $C_2 = C_2(D, S, r, q)$. We next fix $N = N_2 > N_1$ and, by the same procedure, we can find another $w^{(2)} \in \overset{o}{W}{}^{1,q}(\Omega)$ satisfying the same properties as $w^{(1)}$. By iteration, we can thus construct two sequences, $\{N_k\}$ and $\{w^{(k)}\}$, with $\{N_k\}$ increasing and unbounded, such that

$$\begin{aligned}
&\sum_{i=1}^{N_k}|l_i(w^{(k)})| = 0\,, \\
&\|w^{(k)}\|_r + \|w^{(k)}\|_{1,q} \leq C_2 \\
&\|w^{(k)}\|_r \geq \overline{\varepsilon}\,,
\end{aligned} \tag{II.5.25}$$

for all $k \in \mathbb{N}$. By (II.5.25)$_2$ and again by Theorem II.5.2, it follows that there are a subsequence of $\{w^{(k)}\}$, which we continue to denote by $\{w^{(k)}\}$, and a function $w^{(0)} \in \overset{o}{W}{}^{1,q}(\Omega)$ such that

$$\begin{aligned}
&w^{(k)} \to w^{(0)} \quad \text{in } L^q(\Omega) \\
&w^{(k)} \overset{w}{\to} w^{(0)} \quad \text{in } \overset{o}{W}{}^{1,q}(\Omega)\,.
\end{aligned} \tag{II.5.26}$$

In view of (II.5.25)$_3$ and of (II.5.26)$_1$, we must have

$$\|w^{(0)}\|_q \geq \overline{\varepsilon}\,. \tag{II.5.27}$$

We now claim that $w^{(0)} \equiv 0$, contradicting (II.5.27). In fact, if $w^{(0)} \not\equiv 0$, by the completeness of the family of functionals $\{l_i\}$, we must have, for at least one member of the family, $l_{\overline{i}}$, that

$$l_{\overline{i}}(w^{(0)}) \neq 0\,. \tag{II.5.28}$$

By (II.5.26)$_2$, it is

$$\lim_{k\to\infty} l_{\overline{i}}(w^{(k)}) = l_{\overline{i}}(w^{(0)})\,, \tag{II.5.29}$$

while from (II.5.25)$_1$ evaluated at all $N_k > \overline{i}$, we find

$$l_{\overline{i}}(w^{(k)}) = 0\,, \quad \text{for all sufficiently large } k\,.$$

However, in view of (II.5.29), this condition contradicts (II.5.28). Thus, $w^{(0)} = 0$ and the lemma is proved. $\square$

Exercise II.5.17 Show that the space defined in (II.5.21) is a closed subspace of $W^{1,q}(\Omega)$.

Exercise II.5.18 Prove the following abstract formulation of Lemma II.5.3. Let X, Y be Banach spaces with norm $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Suppose that X is reflexive and compactly embedded in Y. Moreover, let $\{\ell_i\}$ be a complete sequence of functionals in X. Show that, given $\varepsilon > 0$ there exist $N = N(\varepsilon) \in \mathbb{N}$ and a constant $C = C(\varepsilon)$ such that

$$\|u\|_Y \le \varepsilon\|u\|_X + C\sum_{i=1}^{N} |\ell_i(u)|\,, \quad \text{for all } u \in X.$$

II.6 The Homogeneous Sobolev Spaces $D^{m,q}$ and Embedding Inequalities

In dealing with boundary-value problems in unbounded domains it can happen that, even for very smooth and rapidly decaying data, the associated solution u does not belong to any space of the type $W^{m,q}$. This is because the behavior at large distances can be different for each derivative of u of a given order and, as a consequence, the corresponding summability properties can be different. As a simple example, consider the Dirichlet problem

$$\Delta u = 0 \ \text{ in } \Omega \equiv \mathbb{R}^3 - \overline{B}_1\,, \qquad u = 1 \ \text{ at } \partial\Omega,$$
$$\lim_{|x|\to\infty} u(x) = 0\,.$$

The solution is $u(x) = 1/|x|$ and we have

$$D^2 u \in L^r(\Omega), \quad 1 < r < \infty,$$
$$\nabla u \in L^s(\Omega), \quad 3/2 < s < \infty,$$
$$u \in L^t(\Omega), \quad 3 < t < \infty.$$

Thus, to formulate boundary-value problems of the above type, one finds it more convenient to introduce spaces more "natural" than the Sobolev spaces $W^{m,q}$, and which, unlike the latter, involve only the derivatives of order m. These classes of functions will be called *homogeneous Sobolev spaces*, and we shall devote this and the next few sections to the study of their relevant properties.

For $m \in \mathbb{N}$ and $1 \le q < \infty$ we define the following linear space (without topology)

$$D^{m,q} = D^{m,q}(\Omega) = \left\{u \in L^1_{loc}(\Omega) : D^\ell u \in L^q(\Omega),\ |\ell| = m\right\}.$$

In order to investigate some preliminary properties of $D^{m,q}$, we introduce the following notation. If u satisfies

$$D^\ell u \in L^q(\Omega'), \quad 0 \le |\ell| \le m, \quad \textit{for all bounded } \Omega' \textit{ with } \overline{\Omega'} \subset \Omega,$$

we shall write

$$u \in W^{m,q}_{loc}(\Omega).$$

Likewise, if

$$D^\ell u \in L^q(\Omega'), \quad 0 \le |\ell| \le m, \quad \textit{for all bounded } \Omega' \subset \Omega$$

we shall write

$$u \in W^{m,q}_{loc}(\overline{\Omega}).$$

We have the following.

Lemma II.6.1 *Let Ω be an arbitrary domain of $\mathbb{R}^n$, $n \ge 2$, and let $u \in D^{m,q}(\Omega)$, $m \ge 0$, $q \in (1,\infty)$. Then $u \in W^{m,q}_{loc}(\Omega)$ and the following inequality holds*

$$\|u\|_{m,q,\omega} \le c \left(\sum_{|\ell|=m} \|D^\ell u\|_{q,\omega} + \|u\|_{1,\omega} \right) \tag{II.6.1}$$

where ω is an arbitrary bounded locally Lipschitz domain with $\overline{\omega} \subset \Omega$. If, in addition, Ω is locally Lipschitz, then $u \in W^{m,q}_{loc}(\overline{\Omega})$, and (II.6.1) *holds for all bounded and locally Lipschitz domains $\omega \subset \Omega$.*

Proof. Clearly, proving that $u \in W^{m,q}(\omega)$, for any ω satisfying the properties stated in the first part of the lemma, implies $u \in W^{m,q}_{loc}(\Omega)$. Let $d = \operatorname{dist}(\partial\omega, \partial\Omega)$ (> 0), and extend u by zero outside Ω. For $d > 1/k > 0$, $k \in \mathbb{N}$, we denote by u_k the regularizer of u corresponding to $\varepsilon = 1/k$. Obviously, $u_k \in W^{m,q}(\omega)$; moreover, by Exercise II.3.2, we have

$$(D^\ell u)_k(x) = (D^\ell u_k)(x), \quad \text{for all } \ell \text{ with } |\ell| = m, \text{ and all } x \in \omega.$$

We may thus use (II.5.19) to find, for any $k, k' \in \mathbb{N}$,

$$\|u_k - u_{k'}\|_{m,q,\omega} \le C \left(\sum_{|\ell|=m} \|(D^\ell u)_k - (D^\ell u)_{k'})\|_{q,\omega} + \|u_k - u_{k'}\|_{1,\omega} \right),$$

for some $C = C(N, q, \omega)$. Observing that, by $(\text{II.2.9})_2$, $(D^\ell u)_k$, $|\ell| = m$, and u_k converge (strongly) in $L^q(\omega)$ and $L^1(\omega)$ to $D^\ell u$ and u, respectively, as $k \to \infty$, from the previous inequality we deduce that $\{u_k\}$ is Cauchy in $W^{m,q}(\omega)$, as well as the validity of (II.6.1). The first part of the lemma is thus proved. In order to show the second part, we begin to observe that, by Exercise II.1.5, we can find a finite number of locally Lipschitz and star-shaped domains Ω_i, $i = 1, \ldots, r$, satisfying the following condition

$$\omega \subseteq \bigcup_{i=1}^{r} \Omega_i \subseteq \Omega .$$

If we thus show that $u \in W^{m,q}(\Omega_i)$ for each $i = 1, \ldots, r$, the stated property follows with the help of Exercise II.5.14. For a fixed i, we extend $u|_{\Omega_i}$ to zero outside Ω_i, and continue to denote by u this extension. By means of a translation in $\mathbb{R}^n$, we may take the point x_i, with respect to which Ω_i is star-shaped, to be the origin of the coordinates. Then, the domains

$$\Omega_i^{(k)} = \{x \in \mathbb{R}^n : (1 - 1/k)\, x \in \Omega_i\}, \quad k \in \mathfrak{N} \equiv \{m \in \mathbb{N} : \ m \geq 2\},$$

satisfy $\Omega_i^{(k)} \supset \overline{\Omega}_i$, for all $k \in \mathfrak{N}$; see Exercise II.1.3. Setting

$$u_k = u_k(x) \equiv u((1 - 1/k)\, x), \quad x \in \Omega_i^{(k)},$$

and $h_0 = \max_{x \in \partial\Omega_i} |x|$, we find that the mollifier, $(u_k)_\varepsilon$, of u_k belongs to $W^{m,q}(\Omega_i)$, if we choose (for example) $\varepsilon = h_0/(2k-2)$. With the aid of (II.2.9)$_1$, we deduce

$$\|u-(u_k)_\varepsilon\|_{1,\Omega_i} \leq \|u-u_\varepsilon\|_{1,\Omega_i} + \|u_\varepsilon - (u_k)_\varepsilon\|_{1,\Omega_i} \leq \|u-u_\varepsilon\|_{1,\Omega_i} + \|u-u_k\|_{1,\Omega_i},$$

which, in turn, by (II.2.9)$_2$ and by Exercise II.2.8, implies

$$\lim_{k\to\infty} \|u - (u_k)_\varepsilon\|_{1,\Omega_i} = 0. \tag{II.6.2}$$

We next set $\chi(x) = D^\ell u(x)$, $|\ell| = m$. Observing that, by Exercise II.3.2 and Exercise II.3.3, it is

$$D^\ell (u_k)_\varepsilon = (1 - 1/k)^m [\chi((1 - 1/k)x)]_\varepsilon \quad x \in \Omega_i,$$

we may repeat an argument similar to that leading to (II.6.2) to show

$$\lim_{k\to\infty} \|D^\ell u - D^\ell (u_k)_\varepsilon\|_{q,\Omega_i} = 0. \tag{II.6.3}$$

Now, with the help of (II.6.2) and (II.6.3), we can use the same procedure used in the proof of the first part of the lemma with $\omega \equiv \Omega_i$, to show the statement contained in the second part. The lemma is thus completely proved. □

Remark II.6.1 From Lemma II.6.1 it follows, in particular, that if Ω is bounded and locally Lipschitz, then $u \in D^{m,q}(\Omega)$ implies $u \in W^{m,q}(\Omega)$, so that $D^{m,q}(\Omega) = W^{m,q}(\Omega)$ algebraically, and, in fact, also topologically, if we endow the space $D^{m,q}(\Omega)$ with the norm $\sum_{|\ell|=m} \|D^\ell u\|_q + \|u\|_1$. On the other hand, if Ω is unbounded in all directions, these latter properties no longer hold, since a priori one loses information on *global* summability of derivatives of order *less* than m, and one can only state *local* properties in the sense specified in Lemma II.6.1. ■

Exercise II.6.1 Let $u \in D^{m,q}(\mathbb{R}^n)$, $n \geq 2$, $m \geq 0$, $q \in (1, \infty)$. Show that $u \in W^{m,q}(B_R)$, for all $R > 0$, and there exists a constant $C = C(R)$ such that

$$\|u\|_{m,q,B_R} \leq C \left(\sum_{\ell=m} \|D^\ell u\|_{q,\mathbb{R}^n} + \|u\|_{1,B_1} \right).$$

Hint: Adapt the arguments used in the proof of the first part of Lemma II.6.1

In $D^{m,q}$ we introduce the seminorm

$$|u|_{m,q} \equiv \left(\sum_{|\ell|=m} \int_\Omega |D^\ell u|^q \right)^{1/q} . \tag{II.6.4}$$

Let $\mathsf{P_m}$ be the class of all polynomials of degree $\leq m-1$ and, for $u \in D^{m,q}$, set

$$[u]_m = \{w \in D^{m,q} : w = u + \mathcal{P},\ \text{for some } \mathcal{P} \in \mathsf{P_m}\}.$$

Denoting by $\dot{D}^{m,q} = \dot{D}^{m,q}(\Omega)$ the space of all (equivalence classes) $[u]_m$, $u \in D^{m,q}$, we see at once that (II.6.4) induces the following norm in $\dot{D}^{m,q}$:

$$|[u]_m|_{m,q} \equiv |u|_{m,q}\,, \quad u \in [u]_m\,. \tag{II.6.5}$$

We shall now show that $\dot{D}^{m,q}$ equipped with the norm (II.6.5) is a Banach space.

Lemma II.6.2 *Let Ω be an arbitrary domain of $\mathbb{R}^n$, $n \geq 2$. Then $\dot{D}^{m,q}(\Omega)$ is a Banach space. In particular, if $q = 2$, it is a Hilbert space with the scalar product*

$$[[u]_m, [v]_m]_m = \sum_{|\ell|=m} \int_\Omega D^\ell u D^\ell v\,, \quad u \in [u]_m\,,\ v \in [v]_m\,.$$

Proof. It is enough to show the first part of the lemma, the second follows easily. We shall consider the case $m = 1$, leaving the more general case as an exercise. We also set $[u]_1 \equiv [u]$. Let $\{[u_s]\}$ be a Cauchy sequence in $\dot{D}^{1,q}(\Omega)$; we have to show the following statements:

(i) For any $\{v_s\}$ with $v_s \in [u_s]$, $s \in \mathbb{N}$, there exists $u \in D^{1,q}(\Omega)$ such that

$$\lim_{s\to\infty} \|D_i v_s - D_i u\|_q = 0\,,\ i = 1, \ldots, n\,;$$

(ii) For any $\{v_s\}$, $\{v'_s\}$, with $v_s, v'_s \in [u_s]$, $s \in \mathbb{N}$, and with u, u' corresponding limits, we have $u' \in [u]$.

It is seen that (ii) easily follows from (i). In fact, since $v_s, v'_s \in [u_s]$, from (i) we have

$$(D_i u, \varphi) = (D_i u', \varphi), \quad \text{for all } \varphi \in C_0^\infty(\Omega),$$

which, in view of Exercise II.5.9, implies (ii). Let us show (i). By the completeness of L^q, we find $V_i \in L^q(\Omega)$, $i = 1, \ldots, n$, with

$$D_i v_s \to V_i \ \text{ in } L^q(\Omega). \tag{II.6.6}$$

Let $\mathfrak{O}$ be the open covering of Ω indicated in Lemma II.1.1 and let $\mathfrak{B}_0 \in \mathfrak{O}$. By the Poincaré inequality and (II.6.6) we deduce the existence of $u^{(0)} \in L^q(\mathfrak{B}_0)$ such that

$$v_s - \overline{v_s}_{\mathfrak{B}_0} \to u^{(0)} \quad \text{in } L^q(\mathfrak{B}_0).$$

Since for all $\varphi \in C_0^\infty(\mathfrak{B}_0)$ it is

$$\int_{\mathfrak{B}_0} V_i \varphi = \lim_{s\to\infty} \int_{\mathfrak{B}_0} D_i v_s \varphi = \lim_{s\to\infty} \int_{\mathfrak{B}_0} (v_s - \overline{v_s}_{\mathfrak{B}_0}) D_i \varphi = - \int_{\mathfrak{B}_0} u^{(0)} D_i \varphi,$$

by definition of the weak derivative, it follows

$$V_i = D_i u^{(0)} \quad \text{a.e. in } \mathfrak{B}_0. \tag{II.6.7}$$

By the property (ii) of $\mathfrak{O}$, we can find $\mathfrak{B}_1 \in (\mathfrak{O} - \mathfrak{B}_0)$ with $\mathfrak{B}_1 \cap \mathfrak{B}_0 \equiv \mathfrak{B}_{1,2} \neq \emptyset$. As before, we show the existence of $u^{(1)} \in L^q(\mathfrak{B}_1)$ such that

$$V_i = D_i u^{(1)} \quad \text{a.e. in } \mathfrak{B}_1. \tag{II.6.8}$$

Thus, $u^{(1)} = u^{(0)} + c$ a.e. in $\mathfrak{B}_{1,2}$, for some $c \in \mathbb{R}$. Therefore, we may modify $u^{(1)}$ by the addition of a constant in such a way that $u^{(1)}$ and $u^{(0)}$ agree a.e. in $\mathfrak{B}_{1,2}$. Continue to denote by $u^{(1)}$ the modified function and define a new function $u^{(0,1)}$ that is equal to $u^{(0)}$ in $\mathfrak{B}_0$ and is equal to $u^{(1)}$ in $\mathfrak{B}_1$. By (II.6.6)–(II.6.8) we deduce that $u^{(0,1)}, D_i u^{(0,1)} \in L^q(\mathfrak{B}_0 \cup \mathfrak{B}_1)$, with $V_i = D_i u^{(0,1)}$ a.e. in $\mathfrak{B}_0 \cup \mathfrak{B}_1$. In view of the property (iii) of the covering $\mathfrak{O}$, we can repeat this procedure to show, by a simple inductive argument, the existence of $u \in L^q_{loc}(\Omega)$ satisfying the statement (i) of the lemma, which is thus completely proved. □

Notation. Sometime, and unless confusion arises, the elements of $\dot{D}^{m,q}(\Omega)$ will be denoted simply by u, instead of $[u]_m$, with u a representative of the class $[u]_m$.

The functional (II.6.4) defines a *norm* in the space $C_0^\infty(\Omega)$. We then introduce the Banach space $D_0^{m,q} = D_0^{m,q}(\Omega)$ as the (Cantor) completion of the normed space $\{C_0^\infty(\Omega), |\cdot|_{m,q}\}$.

Remark II.6.2 Since $C_0^\infty(\Omega)$ can be viewed as a subspace of $\dot{D}^{m,q}(\Omega)$ via the natural map

$$i : u \in C_0^\infty(\Omega) \to i(u) = [u]_m \in \dot{D}^{m,q}(\Omega),$$

it follows that, for any domain Ω, $D_0^{m,q}(\Omega)$ is isomorphic to a closed subspace of $\dot{D}^{m,q}(\Omega)$. More specifically, $[u]_m \in \dot{D}^{m,q}(\Omega)$ belongs to $D_0^{m,q}(\Omega)$ if and only if there is $u \in [u]_m$ and corresponding $\{u_k\} \subset C_0^\infty(\Omega)$ such that $\lim_{k\to\infty} |u_k - u|_{m,q} = 0$. Other characterizations of the spaces $D_0^{m,q}$ will be given in Section II.7. We finally observe that (see Exercise II.2.6)

$$D_0^{0,q}(\Omega) = D^{0,q}(\Omega) = L^q(\Omega), \quad q \geq 1.$$

■

Remark II.6.3 If Ω is contained in a layer, then by means of inequality (II.5.1) and Lemma II.6.1 one can easily show that $\|\cdot\|_{m,q}$ is equivalent to $|\cdot|_{m,q}+\|\cdot\|_q$ and to $|\cdot|_{m,q}$. Therefore, if we endow $W_0^{m,q}(\Omega)$ with this latter norm, we find that $D_0^{m,q}(\Omega)$ and $W_0^{m,q}(\Omega)$ are isomorphic. ■

Exercise II.6.2 Show that $\dot{D}^{m,q}$ and $D_0^{m,q}$ are separable for $1 \le q < \infty$ and reflexive for $1 < q < \infty$. Thus, for $q \in (1,\infty)$ these spaces are weakly complete and the unit closed ball is weakly compact (see Theorem II.1.3(ii)). *Hint* (for $m=1$): Let

$$W = \left\{ w \in [L^q]^n : \; w = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right), \text{ for some } u \in \dot{D}^{1,q} \right\}.$$

W is isomorphic to $\dot{D}^{1,q}$, and, since $\dot{D}^{1,q}$ is complete, W is a closed subspace of $[L^q]^n$. Therefore, W is separable for $1 \le q < \infty$ and reflexive for $1 < q < \infty$ (see Theorem II.2.5, Theorem II.1.1 and Theorem II.1.2), which, in turn, gives the stated properties for $\dot{D}^{1,q}$. Since $D_0^{1,q}$ is isomorphic to a closed subspace of $\dot{D}^{1,q}$, the same properties are true for $D_0^{1,q}$; see also Simader and Sohr (1997, Theorem I.2.2).

Our next goal will be to investigate global properties of functions from $D^{m,q}(\Omega)$, including their behavior at large distances, when Ω is either *an exterior domain* or *a half-space.*

Remark II.6.4 It will be clear from the context that, in fact, most of the results we shall prove continue to hold for a much larger class of domains. This class certainly includes domains Ω for which any function from $D^{1,q}(\Omega)$ can be extended to one from $D^{1,q}(\mathbb{R}^n)$ with preservation of the seminorm $|\cdot|_{1,q}$. For the existence of such extensions, we refer the reader to the classical paper of Besov (1967); see also Burenkov (1976). ■

Our following objective is to prove some embedding inequalities that ensure that derivatives of u of order less than m belong to suitable Lebesgue or weighted-Lebesgue spaces. Such estimates, unlike the bounded-domain case, where they give information on the "regularity" of u, furnish information on the behavior of u at large distances. We begin to derive these inequalities for the case $m=1$ (see Theorem II.6.1, Theorem II.6.3), the general case $m \ge 1$ being treated by a simple iterative argument (see Theorem II.6.4).

We recall that, if $q \in [1,n)$ every $u \in C_0^\infty(\Omega)$, satisfies the Sobolev inequality (II.3.7), that we rewrite below for reader's convenience:

$$\|u\|_s \le \frac{q(n-1)}{2(n-q)\sqrt{n}} |u|_{1,q}\,, \quad \text{for all } q \in [1,n)\,,\; s = nq/(n-q). \tag{II.6.9}$$

We shall next consider certain *weighted inequalities* that (in a less general form) were first considered by Leray (1933, p. 47; 1934, §6) and Hardy (Hardy, Littlewood, and Polya 1934, §7.3). Specifically, if $u \in C_0^\infty(\Omega)$, we have

$$\|u|x-x_0|^{-1}\|_q \le \frac{q}{(n-q)} |u|_{1,q}, \quad \text{for all } q \in [1,n). \tag{II.6.10}$$

In fact, consider the identity

$$\nabla \cdot (\boldsymbol{g}|u|^q) = |u|^q \nabla \cdot \boldsymbol{g} + \boldsymbol{g} \cdot \nabla |u|^q \tag{II.6.11}$$

with

$$\boldsymbol{g} = (\boldsymbol{x} - \boldsymbol{x}_0)\,/|x - x_0|^q. \tag{II.6.12}$$

Since

$$\nabla \cdot \boldsymbol{g} = (n-q)/|x-x_0|^q,$$

integrating (II.6.11) and using the Hölder inequality proves (II.6.10). Notice that if $q > n$ and

$$\Omega^c \supset B_a(x_0), \quad \text{some } a > 0,$$

then by the same token one shows the validity of the following inequality:

$$\|u|x-x_0|^{-1}\|_q \le \frac{q}{(q-n)}|u|_{1,q}, \quad \text{for all } q > n\,; \tag{II.6.13}$$

see also Exercise II.6.7. In case $q = n$ ($\neq 1$) and if

$$\Omega^c \supset B_a(x_0), \quad \text{some } a > 0,$$

we have instead

$$\|u\,[|x-x_0|\ln(|x-x_0|/a)]^{-1}\,\|_n \le \frac{n}{a\,(n-1)}|u|_{1,n}. \tag{II.6.14}$$

To show this latter, we use again identity (II.6.11) with

$$\boldsymbol{g} = -\frac{(\boldsymbol{x}-\boldsymbol{x}_0)}{|x-x_0|^n[\ln(|x-x_0|/a)]^{n-1}}.$$

Since

$$\nabla \cdot \boldsymbol{g} = \frac{a\,(n-1)}{[|x-x_0|\ln(|x-x_0|/a)]^n}\,,$$

substituting into (II.6.11), integrating over Ω, and applying the Hölder inequality to the last term on the right-hand side of (II.6.11) proves (II.6.14).

We shall next analyze if and to what extent inequalities similar to (II.6.9), (II.6.10), (II.6.13), and (II.6.14) continue to hold for functions from $D^{1,q}(\Omega)$, where the domain Ω can be either an exterior domain or a half-space.[1] In order to perform this study, we need to know more about the behavior at large distances of functions of $D^{1,q}(\Omega)$. In this respect we have

Lemma II.6.3 *Let $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, be an exterior domain and let*

$$u \in D^{1,q}(\Omega), \quad 1 \le q < n.$$

Then, there exists a unique $u_0 \in \mathbb{R}$ such that, for all $R > \delta(\Omega^c)$,

$$\int_{S^{n-1}} |u(R,\omega) - u_0|^q d\omega \le \gamma_0 R^{q-n} \int_{\Omega^R} |\nabla u|^q,$$

where $\gamma_0 = [(q-1)/(n-q)]^{q-1}$ if $q > 1$ and $\gamma_0 = 1$ if $q = 1$.

[1] See Remark II.6.4.

Proof. Let $r > R > \delta(\Omega^c)$, and consider first the case $q > 1$. For a smooth u, by the Hölder inequality we have

$$\begin{aligned}\int_R^r \int_{S^{n-1}} \left|\frac{\partial u}{\partial \rho}\right|^q \rho^{n-1} d\rho dS^{n-1} &= \int_{S^{n-1}} \left[\int_R^r \left|\frac{\partial u}{\partial \rho}\right|^q \rho^{n-1} d\rho\right] dS^{n-1} \\ &\geq \int_{S^{n-1}} \left[\frac{\left|\int_R^r \frac{\partial u}{\partial \rho} d\rho\right|^q}{\left(\int_R^r \rho^{(1-n)/(q-1)} d\rho\right)^{q-1}}\right] = \gamma_0^{-1} R^{n-q} \int_{S^{n-1}} |u(r) - u(R)|^q,\end{aligned} \tag{II.6.15}$$

while, by the Wirtinger inequality (II.5.17), it follows that

$$\begin{aligned}&\int_R^r \rho^{n-q-1} \left(\int_{S^{n-1}} |\nabla^* u|^q dS^{n-1}\right) d\rho \\ &\qquad \geq c_1^{-q} \int_R^r \left(\int_{S^{n-1}} |u - \overline{u}|^q dS^{n-1}\right) \rho^{n-q-1} d\rho,\end{aligned}$$

where

$$\overline{f} = (n\omega_n)^{-1} \int_{S^{n-1}} f.$$

Therefore, setting

$$D_r(R) = \int_{\Omega_{R,r}} |\nabla u|^q,$$

and taking into account that, by (II.5.14), $|\partial u/\partial r|^q, (|\nabla^* u|/r)^q \leq |\nabla u|^q$, we find

$$\begin{aligned}D_r(R) &\geq \gamma_0^{-1} R^{n-q} \int_{S^{n-1}} |u(r) - u(R)|^q \\ D_r(R) &\geq c_1^{-q} \int_R^r \left(\int_{S^{n-1}} |u - \overline{u}|^q dS^{n-1}\right) \rho^{n-q-1} d\rho.\end{aligned} \tag{II.6.16}$$

In view of Lemma II.6.1, and with the help of Theorem II.3.1, one shows that (II.6.16) continues to hold for all functions merely satisfying the assumption of the lemma. Letting $R, r \to \infty$, into $(\text{II.6.16})_1$, we deduce that u converges (strongly) in $L^q(S^{n-1})$ to some function u^*. Set

$$u_0 = \overline{u^*}, \quad w = u - u_0.$$

Obviously,

$$\lim_{|x|\to\infty} \int_{S^{n-1}} w(x) = 0. \tag{II.6.17}$$

Rewriting (II.6.16) with w instead of u, we recover the existence of a sequence $\{r_m\} \subset \mathbb{R}_+$, with $\lim_{m\to\infty} r_m = \infty$ such that

$$\lim_{m\to\infty}\int_{S^{n-1}}|w(r_m)-\overline{w}(r_m)|^q=0,$$

which, because of (II.6.17), furnishes

$$\lim_{m\to\infty}\int_{S^{n-1}}|w(r_m)|^q=0.$$

Inserting this information into $(\text{II.6.16})_1$ written with w in place of u and letting $r\to\infty$ completes the proof of the lemma when $q>1$. If $q=1$, we easily show that

$$\int_R^r\int_{S^{n-1}}\left|\frac{\partial u}{\partial\rho}\right|\rho^{n-1}d\rho dS^{n-1}\ge R^{n-1}\int_{S^{n-1}}|u(r)-u(R)|\,.$$

Therefore, replacing (II.6.15) with this latter relation and arguing exactly as before, we show the result also when $q=1$ $\square$

Exercise II.6.3 The previous lemma describes the precise way in which a function u, having first derivatives in $L^q(\Omega)$, $1\le q<n$, Ω an exterior domain, must tend to a (finite) limit at large spatial distances. Show by a counterexample that the condition $q<n$ is indeed necessary for the validity of the result. Moreover, prove that if $q\ge n$ the following estimate holds, for all $r\ge r_0>\max\{1,\delta(\Omega^c)\}$:

$$\int_{S^{n-1}}|u(r,\omega)|^q d\omega\le 2^{q-1}\left(\int_{S^{n-1}}|u(r_0,\omega)|^q d\omega+h(r)|u|^q_{1,q,\Omega_{r_0,r}}\right),\qquad(\text{II.6.18})$$

where

$$h(r)=\begin{cases}(\log r)^{n-1} & \text{if } q=n\\[2ex] [(q-1)/(q-n)]^{q-1}\,r^{q-n} & \text{if } q>n.\end{cases}$$

Finally, using (II.6.18), show

$$\lim_{r\to\infty}(h(r))^{-1}\int_{S^{n-1}}|u(r,\omega)|^q d\omega=0.$$

(For pointwise estimates, see Section II.9.) *Hint*: To show (II.6.18), start with the identity

$$u(r,\omega)=u(r_0,\omega)+\int_{r_0}^r(\partial u/\partial\rho)d\rho,$$

and apply the Hölder inequality.

This preliminary result allows us to prove the following, which answers the question raised previously; see also Finn (1965a), Galdi and Maremonti (1986).

Theorem II.6.1 *Let $\Omega\subseteq\mathbb{R}^n$, $n\ge 2$, be an exterior domain, and let*

$$u\in D^{1,q}(\Omega),\quad 1\le q<\infty\,.$$

The following properties hold.

(i) *If* $q \in [1,n)$, *set*

$$w = u - u_0$$

with u_0 *defined in* Lemma II.6.3. *Then, for any* $x_0 \in \mathbb{R}^n$, *we have*

$$w|x-x_0|^{-1} \in L^q(\Omega^R(x_0)),$$

where

$$\Omega^a(x_0) \equiv \Omega - B_a(x_0), \quad B_a(x_0) \supset \Omega^c,$$

and the following inequality holds:

$$\left(\int_{\Omega^R(x_0)} \left|\frac{w(x)}{x-x_0}\right|^q dx\right)^{1/q} \le \frac{q}{(n-q)} |w|_{1,q,\Omega^R(x_0)}. \tag{II.6.19}$$

If $|x_0| = \alpha R$, *for some* $\alpha \ge \alpha_0 > 1$ *and some* $R > \delta(\Omega^c)$, *we have*

$$\left(\int_{\Omega^R} \left|\frac{w(x)}{x-x_0}\right|^q dx\right)^{1/q} \le c|w|_{1,q,\Omega^R}, \tag{II.6.20}$$

where $c = c(n,q,\alpha_0)$. *Furthermore, if* Ω *is locally Lipschitz, then*

$$w \in L^s(\Omega), \quad s = nq/(n-q), \tag{II.6.21}$$

and for some γ_1 *independent of* u

$$\|w\|_s \le \gamma_1 |w|_{1,q}. \tag{II.6.22}$$

(ii) *If* $q \in [n,\infty)$, *assume* Ω *locally Lipschitz with* $\Omega^c \supset B_a(x_0)$, *for some* $a > 0$, *and set*

$$\mathfrak{w} = \begin{cases} |x-x_0|^{-1} & \text{if } q > n \\ \left(|x-x_0| \ln(|x-x_0|/a)\right)^{-1} & \text{if } q = n\,. \end{cases} \tag{II.6.23}$$

Then, if u *has zero trace at* $\partial\Omega$, *we have* $\mathfrak{w}\, u \in L^q(\Omega)$, *and the following inequality holds, for all* $R > \delta(\Omega^c)$,

$$\|\mathfrak{w}\, u\|_{q,\Omega_R(x_0)} \le C_q\, |u|_{1,q,\Omega_R(x_0)}\,, \tag{II.6.24}$$

where $\Omega_R(x_0) \equiv \Omega \cap B_R(x_0)$, *and* $C_q = q/(q-n)$, *if* $q > n$, *while* $C_q = n/[a\,(n-1)]$, *if* $q = n$.

Proof. As in the proof of Lemma II.6.3, it will be enough to consider smooth functions only. We begin to prove part (i). Let us integrate identity (II.6.11), with w in place of u and $\boldsymbol{g}$ given by (II.6.12), over the spherical shell:

$$\Omega^{R,r}(x_0) \equiv \Omega \cap (B_r(x_0) - B_R(x_0))\,, \quad r > R.$$

We have

$$(n-q)\int_{\Omega^{R,r}(x_0)}\left|\frac{w(x)}{x-x_0}\right|^q dx \le \int_{\partial B_R(x_0)} \boldsymbol{g}\cdot\boldsymbol{n}|w|^q + r^{1-q}\int_{\partial B_r(x_0)}|w|^q$$

$$+q\int_{\Omega^{R,r}(x_0)}|\boldsymbol{g}||w|^{q-1}|\nabla w|,$$

where $\boldsymbol{n}$ is the unit normal to $\partial B_R(x_0)$ pointing toward x_0. This yields that the first term on the right-hand side of this latter equation is non-positive. Thus, estimating the integral over $\partial B_r(x_0)$ with the help of Lemma II.6.3, we deduce

$$(n-q)\int_{\Omega^{R,r}(x_0)}\left|\frac{w(x)}{x-x_0}\right|^q dx \le c_1\int_{\Omega^r(x_0)}|\nabla w|^q + q\int_{\Omega^{R,r}(x_0)}|\boldsymbol{g}||w|^{q-1}|\nabla w|,$$

where $c_1 = c_1(n,q)$. Now, if $q=1$ the result follows by letting $r\to\infty$ into this relation; otherwise, employing Young's inequality (II.2.5) with $\varepsilon = [(q-1)/\lambda(n-q)]^{q-1}$, $0<\lambda<1$, in the last integral at the right-hand side we obtain

$$\int_{\Omega^{R,r}(x_0)}\left|\frac{w(x)}{x-x_0}\right|^q dx \le \frac{c_1}{(n-q)(1-\lambda)}\int_{\Omega^r(x_0)}|\nabla w|^q$$

$$+\frac{(q-1)^{q-1}}{(1-\lambda)\lambda^{q-1}(n-q)^q}\int_{\Omega^{R,r}(x_0)}|\nabla w|^q.$$

We now let $r\to\infty$ into this relation and minimize over λ, thus completing the proof of the first part of the lemma. To show the second part, for $r>(\alpha+2)R$ we set

$$\Omega^{R,r} \equiv \Omega\cap(B_r(x_0)-B_R),$$

and so, operating as before, we derive

$$(n-q)\int_{\Omega^{R,r}}\left|\frac{w(x)}{x-x_0}\right|^q dx \le \int_{\partial B_R} \boldsymbol{g}\cdot\boldsymbol{n}|w|^q + r^{1-q}\int_{\partial B_r(x_0)}|w|^q$$

$$+q\int_{\Omega^{R,r}(x_0)}|\boldsymbol{g}||w|^{q-1}|\nabla w|.$$

If $q>1$, we use Young's inequality in the last integral, then Lemma II.6.3 to estimate the surface integral over $\partial B_r(x_0)$. Letting $r\to\infty$ we may then conclude, as in the proof of the first part of the lemma, the validity of the following inequality:

$$\int_{\Omega^R}\left|\frac{w(x)}{x-x_0}\right|^q dx \le \frac{1}{(n-q)(1-\lambda)}\int_{\partial B_R}\boldsymbol{g}\cdot\boldsymbol{n}|w|^q$$
$$+\frac{(q-1)^{q-1}}{(1-\lambda)\lambda^{q-1}(n-q)^q}\int_{\Omega^R}|\nabla w|^q \tag{II.6.25}$$

for all $\lambda \in (0,1)$. Now, if $x \in \partial B_R$ it is

$$|x - x_0| \geq |x_0| - |x| \geq (\alpha_0 - 1)R,$$

and so

$$|\boldsymbol{g}(x)| \leq |x - x_0|^{1-q} \leq [(\alpha_0 - 1)R]^{1-q}, \quad x \in \partial B_R.$$

From this inequality and Lemma II.6.3 we obtain the following:

$$\int_{\partial B_R} \boldsymbol{g} \cdot \boldsymbol{n} |w|^q \leq \frac{R^{n-q}}{(\alpha_0 - 1)^{q-1}} \int_{S^{n-1}} |w|^q \leq \frac{\gamma_0}{(\alpha_0 - 1)^{q-1}} \int_{\Omega^R} |\nabla w|^q,$$

which, once replaced into (II.6.25), proves (II.6.20) for $q > 1$. The proof for $q = 1$ is similar and therefore is left to the reader. To complete the proof of part (i), it remains to show the last statement. To this end, let $\varphi \in C^1(\mathbb{R})$ be a nondecreasing function such that $\varphi(\xi) = 0$ if $|\xi| \leq 1$ and $\varphi(\xi) = 1$ if $|\xi| \geq 2$. We set for $r > 2R > \delta(\Omega^c)$

$$\begin{aligned} \varphi_R(x) &= \varphi(|x|/R), \\ \chi_r(x) &= 1 - \varphi_r(x), \\ w^{\#}(x) &= \varphi_R(x)\chi_r(x)w(x). \end{aligned}$$

Notice that

$$|\nabla \chi_r(x)| \leq c/r, \quad c = c(\varphi).$$

Evidently, $w^{\#} \in W_0^{1,q}(\Omega)$, and we may apply Sobolev inequality (II.3.7) to deduce

$$\|w^{\#}\|_s \leq \gamma |w^{\#}|_{1,q}, \quad s = nq/(n-q),$$

which, by the properties of φ_R and χ_r, in turn implies

$$\|w^{\#}\|_s \leq c_1 \left(|w|_{1,q} + \|w\|_{q,\Omega_{R,2R}} + \|w|x|^{-1}\|_{q,\Omega_{r,2r}} \right),$$

with $c_1 = c_1(R, \varphi, n, q)$. We now let $r \to \infty$ into this relation. By inequality (II.6.19) the last term on the right-hand side must tend to zero. Using this fact along with the monotone convergence theorem, we recover

$$\|w\|_{s,\Omega^{2R}} \leq c_1 \left(|w|_{1,q} + \|w\|_{q,\Omega_{R,2R}} \right). \tag{II.6.26}$$

We next apply the inequality (II.5.18) to the integral over $\Omega_{R,2R}$ to deduce

$$\|w\|_{s,\Omega^{2R}} \leq c_2 \left(|w|_{1,q} + \left(\int_{\partial B_R \cup \partial B_{2R}} |w|^q \right)^{1/q} \right).$$

Using Lemma II.6.3 in this inequality, we finally obtain

$$\|w\|_{s,\Omega^{2R}} \leq c_3 |w|_{1,q}. \tag{II.6.27}$$

We now want to estimate w "near" $\partial\Omega$. We set

$$\zeta_R(x) = 1 - \varphi(|x|/2R)$$

and notice that

$$\zeta_R w \in W^{1,q}(\Omega).$$

Employing the embedding Theorem II.3.4, we obtain

$$\|w\|_{s,\Omega_{2R}} \le c_4 \left(|w|_{1,q} + \|w\|_{q,\Omega_{2R,4R}} \right).$$

We may now bound the last term on the right-hand side of this relation by $|w|_{1,q}$, in the same way as we did for the analogous term in (II.6.26), thus deducing

$$\|w\|_{s,\Omega_{2R}} \le c_5 |w|_{1,q}.$$

The last claim in part (i) of the lemma then follows from this latter inequality and from (II.6.27). We shall prove the claim in part (ii) when $q > n$, the case $q = n$ being treated in exactly the same way. We integrate (II.6.11) over $\Omega_R(x_0)$, with arbitrary $R > \delta(\Omega^c)$. Recalling that u has zero trace at $\partial\Omega$, we find

$$(q-n)\int_{\Omega_R(x_0)} \frac{|u|^q}{|x-x_0|^q} = -\int_{\partial B_R(x_0)} \boldsymbol{g}\cdot\boldsymbol{n}|u|^q - \int_{\Omega_R(x_0)} \boldsymbol{g}\cdot\nabla|u|^q \,.$$

The surface integral in this relation is non positive, so that, proceeding as in the proof of (II.6.13) we obtain

$$\int_{\Omega_R(x_0)} \frac{|u|^q}{|x-x_0|^q} \le \frac{q}{(q-n)} \int_{\Omega_R(x_0)} |\nabla u|^q \,, \qquad \text{(II.6.28)}$$

which, in turn, by the arbitrarity of R proves the claim. □

Exercise II.6.4 Let $L^q_{\mathfrak{w}}(\Omega)$, $q \ge n \ge 2$, be the class of (measurable) functions v such that $\mathfrak{w}\, v \in L^q(\Omega)$, with $\mathfrak{w}$ defined in (II.6.24). Show that $L^q_{\mathfrak{w}}(\Omega)$ endowed with the norm $\|\mathfrak{w}(\cdot)\|_q$ is a Banach space.

Exercise II.6.5 Let $u \in D^{1,q}(B^R)$, $q \in [1,n)$. Show that u satisfies (II.6.21), with $\Omega \equiv B^R$, with a constant γ_1 independent of R.

Exercise II.6.6 Let $u \in D^{1,q}(B_R(x_0))$, $n \ge 2$, $q > n$, $R > 0$. Show that the following inequality holds

$$\|(u - u(x_0))/|x - x_0|\|_{q,B_R(x_0)} \le q/(q-n)|u|_{1,q,B_R(x_0)} \,.$$

Hint: Integrate (II.6.11) over $B_R(x_0) - B_\varepsilon(x_0)$, $\varepsilon < R$. Then, use the results of Exercise II.5.11 and let $\varepsilon \to 0$. (Notice that $u(x_0)$ is well defined, because, for $q > n$, $D^{1,q}(\Omega) \subset W^{1,q}(B_R(x_0)) \subset C(B_R(x_0))$; see Lemma II.6.1 and Theorem II.3.4.)

Exercise II.6.7 Let Ω be an exterior, locally Lipschitz domain, and assume that $u \in D^{1,q}(\Omega)$, $q > n$, with zero trace at $\partial\Omega$. Show that, for all $R > \delta(\Omega^c)$ and all $x_0 \in \Omega_R$,

$$\|\mathfrak{w}(u - u(x_0))\|_{q,\Omega_R} \leq \frac{q}{q-n}|u|_{1,q,\Omega_R}\,,$$

where $\mathfrak{w}$ is defined in (II.6.23)$_1$. *Hint*: Integrate (II.6.11) over $\Omega_R - B_\varepsilon(x_0)$, for sufficiently small ε. Then use the results of Exercise II.5.11 and let $\varepsilon \to 0$.

Exercise II.6.8 Let Ω be an exterior domain of $\mathbb{R}^n$, $n \geq 2$, and let $u \in D^{1,q}(\Omega)$, $q \in [1, \infty)$, satisfy the following generalized version of "vanishing of the trace" at $\partial\Omega$:

$$\psi\, u \in W_0^{1,q}(\Omega)\,, \text{ for all } \psi \in C_0^\infty(\mathbb{R}^n)\,. \tag{II.6.29}$$

(a) Assume $q \geq n$ and that $\Omega^c \supset B_a(x_0)$, for some $x_0 \in \mathbb{R}^n$ and $a > 0$. Show that u satisfy (II.6.24)

(b) Assume $q \in [1, n)$, and that the constant u_0 associated to u by Lemma II.6.3 is zero. Show that $u \in L^{nq/(n-q)}(\Omega)$ and that there exists $C = C(n, q, \Omega)$ such that

$$\|u\|_{nq/(n-q)} \leq C\,|u|_{1,q}\,.$$

Theorem II.6.1 ensures, in particular, that, for Ω an exterior locally Lipschitz domain and for $q \in [1, n)$, every function from $D^{1,q}(\Omega)$, possibly modified by the addition of a uniquely determined constant, obeys the Sobolev inequality (II.6.22), even though its trace at the boundary need not be zero. Our next goal is to perform a similar analysis, more generally, for Troisi inequality (II.3.8). Specifically, assuming that the seminorms of u appearing on the right-hand side of (II.3.8) are finite, we wish to investigate if $u \in L^r(\Omega)$ and if (II.3.8) holds. To this end, we will use a special "anisotropic cut-off" function whose existence is proved in the next lemma; see Galdi & Silvestre (2007a) and Galdi (2007). The lemma will also include properties of this function which are not immediately needed, but that will be very useful for future purposes; see, e.g., Chapter VIII.

Lemma II.6.4 *For any $\alpha, R > 0$, there exists a function $\psi_{\alpha,R} \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \psi_{\alpha,R}(x) \leq 1$, for all $x \in \mathbb{R}^n$ and satisfying the following properties*

$$\begin{aligned}
&\lim_{R\to\infty} \psi_{\alpha,R}(x) = 1 \quad \textit{uniformly pointwise, for all } \alpha > 0\,,\\
&\left|\frac{\partial\psi_{\alpha,R}}{\partial x_1}(x)\right| \leq \frac{C_1}{R^\alpha}, \quad \left|\frac{\partial\psi_{\alpha,R}}{\partial x_i}(x)\right| \leq \frac{C_1}{R}\,,\ i = 2, \ldots, n\,,\\
&|\Delta\psi_{\alpha,R}(x)| \leq \frac{C_2}{R^2},\\
&(\boldsymbol{e}_1 \times x)\cdot\nabla\psi_{\alpha,R}(x) = 0 \quad \textit{for all } x \in \mathbb{R}^3\,,
\end{aligned} \tag{II.6.30}$$

where C_1, C_2 are independent of x and R. Moreover, the support of $\partial\psi_{\alpha,R}/\partial x_j$, $j = 1, \ldots, n$, is contained in the cylindrical shell $\mathcal{S}_R = \mathcal{S}_R^{(1)} \cap \mathcal{S}_R^{(2)}$ where

$$\mathcal{S}_R^{(1)} = \left\{ x \in \mathbb{R}^n : \ \frac{R}{\sqrt{2}} < r < \sqrt{2}R, \ \right\},$$

$$\mathcal{S}_R^{(2)} = \left\{ x \in \mathbb{R}^n : \frac{R^\alpha}{\sqrt{2}} < |x_1| < \sqrt{2}R^\alpha \right\} \cup \left\{ x \in \mathbb{R}^n : \ -\frac{R^\alpha}{\sqrt{2}} \le x_1 \le \frac{R^\alpha}{\sqrt{2}} \right\}, \tag{II.6.31}$$

and where $r = (x_2^2 + \cdots x_n^2)^{1/2}$. *In addition, the following properties hold for all* $\alpha > 0$

$$\frac{\partial \psi_{\alpha,R}}{\partial x_1} \in L^q(\mathbb{R}^3), \quad \textit{for all } q \ge \tfrac{n-1}{\alpha} + 1, \quad \left\| \frac{\partial \psi_{\alpha,R}}{\partial x_1} \right\|_q \le C_3,$$

$$\|(u - u_0)\, |\nabla \psi_{\alpha,R}|\, \|_s \le C_4\, |u|_{1,s,\Omega_{\frac{R^\beta}{\sqrt{2}}}}, \quad \textit{for all } u \in D^{1,s}(\mathbb{R}^n), 1 \le s < n, \tag{II.6.32}$$

where u_0 *is the constant associated to* u *by* Lemma II.6.3, $\beta = \min\{1, \alpha\}$, *and* C_3, C_4 *are independent of* R.

Proof. Let $\psi = \psi(t)$ be a C^∞, non-increasing real function, such that $\psi(t) = 1$, $t \in [0,1]$ and $\psi(t) = 0$, $t \ge 2$. We set

$$\psi_{\alpha,R}(x) = \psi\left(\sqrt{\frac{x_1^2}{R^{2\alpha}} + \frac{r^2}{R^2}} \right), \ x \in \mathbb{R}^n,$$

so that we find

$$\psi_{\alpha,R}(x) = \begin{cases} 1 \ \text{ if } \dfrac{x_1^2}{R^{2\alpha}} + \dfrac{r^2}{R^2} \le 1 \\[2ex] 0 \ \text{ if } \dfrac{x_1^2}{R^{2\alpha}} + \dfrac{r^2}{R^2} \ge 4. \end{cases} \tag{II.6.33}$$

The first property in (II.6.30) then follows at once. Moreover, since

$$\frac{\partial \psi_{\alpha,R}}{\partial x_1}(x) = \frac{x_1}{R^\alpha \sqrt{x_1^2 + R^{2\alpha-2} r^2}}\, \psi' \left(\sqrt{\frac{x_1^2}{R^{2\alpha}} + \frac{r^2}{R^2}} \right),$$

$$\frac{\partial \psi_{\alpha,R}}{\partial x_i}(x) = \frac{x_i}{R\sqrt{R^{2-2\alpha} x_1^2 + r^2}}\, \psi' \left(\sqrt{\frac{x_1^2}{R^{2\alpha}} + \frac{r^2}{R^2}} \right), \quad i = 2, \ldots n,$$

the uniform bounds for the first derivatives hold with $C := \max_{t \ge 0} |\psi'(t)|$. The estimate for the Laplacean of $\psi_{\alpha,R}$ is easily obtained with C_2 depending on C_1 and $\max_{t\ge 0} |\psi''(t)|$. Moreover, the orthogonality relation $(\text{II.6.30})_4$ is immediate if we take account the above components of $\nabla\psi_{\alpha,R}$ and the fact that $e_1 \times x = -x_3 e_2 + x_2 e_3$. Denote next by Σ the support of $\nabla \psi_{\alpha,R}$. From (II.6.33) we deduce that

$$\Sigma \subset \left\{ x \in \mathbb{R}^n : \ 1 < \frac{x_1^2}{R^{2\alpha}} + \frac{r^2}{R^2} < 4 \right\} \equiv \Sigma_1.$$

Consider the following sets

$$\mathcal{S}_1 = \left\{ x \in \mathbb{R}^n : \frac{x_1^2}{R^{2\alpha}} < \frac{1}{2} \text{ and } \frac{r^2}{R^2} < \frac{1}{2} \right\},$$

$$\mathcal{S}_2 = \left\{ x \in \mathbb{R}^n : \frac{x_1^2}{R^{2\alpha}} > 2 \text{ and } \frac{r^2}{R^2} > 2 \right\}.$$

Clearly, ${\varSigma_1}^c \supset \mathcal{S}_1 \cup \mathcal{S}_2$. Therefore, by de Morgan's law, we get $\varSigma_1 \subset \mathcal{S}_1^c \cap \mathcal{S}_2^c$ and we conclude, from (II.6.31), that $\varSigma_1 \subset \mathcal{S}$, since $\mathcal{S}_1^c \cap \mathcal{S}_2^c = \mathcal{S}$. It remains to prove (II.6.32). The first property follows at once from the estimate for $\partial\psi_{\alpha,R}/\partial x_1$ given in (II.6.30) and the fact that the measure of the support of $\partial\psi_{\alpha,R}/\partial x_1$ is bounded by a constant times $R^{\alpha+n-1}$. Furthermore, we observe that, for all $x \in \mathcal{S}_R$, it is $|x| \leq C\sqrt{(R^{2\alpha} + R^2)}$, with C a positive constant independent of R. Thus, from (II.6.30) we find, with $w \equiv u - u_0$,

$$\|w\,|\nabla\psi_{\alpha,R}|\|_{s,\varOmega} = \|w\,|\nabla\psi_{\alpha,R}|\|_{s,S_R} \leq C_2\,\|w/|x|\|_{s,S_R} \leq C_2\|w/|x|\|_{s,B^{\frac{R^\beta}{\sqrt{2}}}},$$

with C_2 a positive constant independent of R and w. The second property in (II.6.32) then follows from this latter inequality and from (II.6.19). The proof of the lemma is complete. $\square$

We are now in a position to prove the following result.

Theorem II.6.2 *Let $\varOmega \subseteq \mathbb{R}^n$, $n \geq 3$, be an exterior locally Lipschitz domain. Assume $u \in D^{1,2}(\varOmega)$ and*

$$\frac{\partial u}{\partial x_1} \in L^{q_1}(\varOmega), \quad 1 < q_1 < 2.$$

Then, denoting by u_0 the uniquely determined constant associated to u by Lemma II.6.3, *we have*

$$w = u - u_0 \in L^r(\varOmega), \quad r = \frac{2nq_1}{2 + (n-3)q_1},$$

and

$$\|w\|_r^n \leq C\left(\left\|\frac{\partial u}{\partial x_1}\right\|_{q_1} \prod_{i=2}^n \|D_i u\|_2 + |u|_{1,2}^n\right), \tag{II.6.34}$$

with $C = C(q_1, n, \varOmega)$.

Proof. Let $\phi_\rho = \phi_\rho(x)$ be a smooth "cut-off" function that is 1 for $x \in \varOmega_\rho$, it is 0 for $x \in \varOmega^{2\rho}$, and that satisfies $\max_{x\in\varOmega} |\nabla\phi_\rho(x)| \leq M$, with M independent of x. We thus have $w = \phi_\rho w + (1 - \phi_\rho)w \equiv w_1 + w_2$. We begin to show the following property: $D_1 w_2$ and $D_i w_2$, $i = 2, \dots, n$, can be approximated, in $L^{q_1} \cap L^2$ and L^2, respectively, by a sequence of functions from $C_0^\infty(\mathbb{R}^n)$. To this end, we set $\widetilde{w}_{2,k} = \psi_{\alpha,R_k} w_2$, where $\psi_{\alpha,R}$ the function constructed in Lemma

II.6.4 with a choice of α that we specify later in the proof, and where $\{R_k\}$ is an unbounded sequence of positive numbers with R_0 sufficiently large. We thus have that the support of $\widetilde{w}_{2,k}$ is compact in $\mathbb{R}^n$. Therefore, its regularizer, $(\widetilde{w}_{2,k})_\varepsilon$ is in $C_0^\infty(\mathbb{R}^n)$. Observing that $D_j(\widetilde{w}_{2,k})_\varepsilon = (D_j\widetilde{w}_{2,k})_\varepsilon$, $j = 1, \ldots, n$ (see Exercise II.3.2), in view of (II.2.9) we may choose a vanishing sequence $\{\varepsilon_k\}$ such that

$$\lim_{k\to\infty} \|D_j\widetilde{w}_{2,k} - D_j w_{2,k}\|_{s_j} = 0\,, \tag{II.6.35}$$

where $w_{2,k} = (\widetilde{w}_{2,k})_{\varepsilon_k}$, $s_1 \in \{q_1, 2\}$, and $s_j = 2$ for $j = 2, \ldots, n$. By the Minkowski inequality, we also obtain

$$\|D_j w_2 - D_j w_{2,k}\|_{s_j} \le \|D_j w_2 - D_j\widetilde{w}_{2,k}\|_{s_j} + \|D_j\widetilde{w}_{2,k} - D_j w_{2,k}\|_{s_j}\,, \tag{II.6.36}$$

so that, in view of (II.6.35), to show the stated property we have to show that the first term on the right-hand side of (II.6.36) tends to 0 as $k \to \infty$. We now observe that

$$\|D_j w_2 - D_j\widetilde{w}_{2,k}\|_{s_j} \le \|(1 - \psi_{\alpha,R_k})D_j w_2\|_{s_j} + \|D_j\psi_{\alpha,R_k} w_2\|_{s_j}\,, \tag{II.6.37}$$

and so, in view of $(\text{II.6.30})_1$, the property follows if we prove that the second term on the right-hand side of (II.6.37) vanishes as $k \to \infty$. Take $j = 1$ and $s_j = q_1$ first. Since

$$\|D_1\psi_{\alpha,R_k} w_2\|_{q_1} \le \|D_1\psi_{\alpha,R_k}\|_{\frac{2nq_1}{2n-(n-2)q_1}} \|w_2\|_{\frac{2n}{n-2},\Omega^{R_k/\sqrt{2}}}\,,$$

and, by Theorem II.6.1, $w_2 \in L^{2n/(n-2)}(\Omega)$, we take $\alpha \ge (n-1)[2n-(n-2)q_1]/[3nq_1 - 2(n+q_1)]$ to deduce, from the properties of $\psi_{\alpha,R}$,

$$\lim_{k\to\infty} \|D_1 w_2 - D_1\widetilde{w}_{2,k}\|_{q_1} = 0\,. \tag{II.6.38}$$

We next choose $s_j = 2$, $j = 1, \ldots, n$, and obtain, with the help of (II.6.32),

$$\|D_j\psi_{\alpha,R_k} w_2\|_2 \le C\,\|w_2/|x|\|_{2,\Omega^{R_k/\sqrt{2}}}\,,$$

which, by (II.6.19) and (II.6.37) implies

$$\lim_{k\to\infty} \|D_j w_2 - D_j\widetilde{w}_{2,k}\|_2 = 0\,. \tag{II.6.39}$$

From (II.6.35), (II.6.36), (II.6.38), and (II.6.39) it then follows

$$\lim_{k\to\infty} \|D_j w_2 - D_j w_{2,k}\|_{s_j} = 0\,, \quad j = 1, \ldots, n\,, \tag{II.6.40}$$

which proves the desired property. Notice that, by Theorem II.6.1, (II.6.40) yields

$$\lim_{k\to\infty} \|w_2 - w_{2,k}\|_{2n/(n-2)} = 0\,. \tag{II.6.41}$$

We next observe that each function $w_{2,k}$ obeys, in particular, Troisi inequality (II.3.8) with $s = r$, $q_1 = q_1$ and $q_2 = \cdots = q_n = 2$. In fact, this inequality shows also that $\{w_{2,k}\}$ is Cauchy in $L^r(\Omega)$ and thus it converges there to some $\overline{w}$. In view of (II.6.40) and (II.6.41), it is simple to show that $\overline{w} = w_2$, a.e. in Ω, and that the inequality continues to hold also for the function w_2:

$$\|w_2\|_r^n \leq c \left\| \frac{\partial w_2}{\partial x_1} \right\|_{q_1} \prod_{i=2}^{n} \|D_i w_2\|_2 . \tag{II.6.42}$$

Furthermore, by the fact that $w \in L^{2n/(n-2)}(\Omega)$, it follows $w_1 \in L^r(\Omega)$, and since $w = w_1 + w_2$, we deduce $w \in L^r(\Omega)$. It thus remains to prove the validity of (II.6.34) when $\Omega \neq \mathbb{R}^n$. Recalling that $w_2 = \phi_\rho w$, we readily obtain

$$\begin{aligned} \|w_2\|_r^n \leq c_1 \left\| \frac{\partial w}{\partial x_1} \right\|_{q_1} \prod_{i=2}^{n} \|D_i w\|_2 \\ + c_2 \left[(\|w\|_{q_1,\sigma} + |w|_{1,2}) \|w\|_{2,\sigma}^{n-1} + \|w\|_{q_1,\sigma} |w|_{1,2}^{n-1} \right] , \end{aligned} \tag{II.6.43}$$

where σ is the (bounded) support of $\nabla \phi_\rho$. We now suitably apply the Hölder inequality in the σ-terms in square brackets and then use (II.6.22) with $q = 2$. Consequently, (II.6.43) furnishes

$$\|w_2\|_r^n \leq c_1 \left\| \frac{\partial w}{\partial x_1} \right\|_{q_1} \prod_{i=2}^{n} \|D_i w\|_2 + c_3 \, |w|_{1,2}^n . \tag{II.6.44}$$

Finally, from Exercise II.3.12, we readily find that

$$\|w_1\|_r \leq c_4 \left(\|w\|_{2,\sigma'} + |w|_{1,2} \right) ,$$

with σ' the (bounded) support of ϕ_ρ. Then, inequality (II.6.34) follows from this latter inequality, from (II.6.22) with $q = 2$ and (II.6.44). □

Exercise II.6.9 Show that if $\Omega = \mathbb{R}^n$, the last term on the right-hand side of (II.6.34) can be omitted.

We would like now to extend the results of Theorem II.6.1 to the case when Ω is a half-space (see Remark II.6.4).[2] We begin to observe that, given $u \in D^{1,q}(\mathbb{R}^n_+)$, $1 \leq q < \infty$, we may extend it to a function $u' \in D^{1,q}(\mathbb{R}^n)$ satisfying (see Exercise II.3.10)

$$\begin{aligned} u(x) = u'(x), \quad x \in \mathbb{R}^n_+ , \\ |u'|_{1,q,\mathbb{R}^n} \leq c|u|_{1,q,\mathbb{R}^n_+} \leq c|u'|_{1,q,\mathbb{R}^n} . \end{aligned} \tag{II.6.45}$$

If $1 \leq q < n$, by Lemma II.6.3, there is a uniquely determined $u_0 \in \mathbb{R}$ such that $(u' - u_0) \in L^s(\mathbb{R}^n)$, $s = nq/(n-q)$, and, moreover,

[2] As a matter of fact, also Theorem II.6.2 can be extended to $\Omega = \mathbb{R}^n_+$. However, for our purposes, this extension would be irrelevant.

$$\|u' - u_0\|_{s,\mathbb{R}^n} \le \gamma_1 |u'|_{1,q,\mathbb{R}^n}.$$

This relation, together with (II.6.45), then delivers

$$\|u - u_0\|_{s,\mathbb{R}^n_+} \le \gamma_3 |u|_{1,q,\mathbb{R}^n_+},$$

which is what we wanted to show. It is interesting to observe that if u has zero trace at the boundary $x_n = 0$ then $u_0 = 0$.[3] Actually, denoting by $\widehat{u}$ the function obtained by setting $u \equiv 0$ outside $\mathbb{R}^n_+$, one easily shows

$$\widehat{u} \in D^{1,q}(\mathbb{R}^n)$$

$$|\widehat{u}|_{1,q,\mathbb{R}^n} \le |u|_{1,q,\mathbb{R}^n_+}$$

(see Exercise II.6.10). Setting $S^{n-1}_- = S^{n-1} \cap \mathbb{R}^n_-$, by Lemma II.6.3 we deduce

$$|u_0|^q |S^{n-1}_-| \le \int_{S^{n-1}} |\widehat{u}(R,\omega) - u_0|^q d\omega \le \gamma_0 R^{q-n} |\widehat{u}|_{1,q,\Omega^R},$$

for all $R > 0$, which furnishes $u_0 = 0$. By the same token, we can show weighted inequalities of the type (II.6.19) and (II.6.20). Next, if $q \ge n$, we notice that, if u has zero trace at the plane $x_n = 0$, we may apply the results of part (ii) in Theorem II.6.1 to the extension $\widehat{u}$, to show that the same results continue to hold for $\Omega = \mathbb{R}^n_+$, and with an arbitrary $x_0 \in \mathbb{R}^n_-$. Actually, we can prove a somewhat stronger weighted inequality, holding for any $u \in D^{1,q}(\mathbb{R}^n_+)$, $q \in (1,\infty)$, that vanishes at $x_n = 0$. We start with the identity (valid for smooth u)

$$\frac{\partial}{\partial x_n}\left[\frac{|u|^q}{(1+x_n)^{q-1}}\right] = \frac{1}{(1+x_n)^{q-1}}\frac{\partial |u|^q}{\partial x_n} + (1-q)\frac{|u|^q}{(1+x_n)^q}.$$

Integrating this inequality over the parallelepiped $P_{a,b} = \{x \in \mathbb{R}^n_+ : |x'| < b\,,\ x_n \in (0,a)\}$, $x' \equiv (x_1, \cdots, x_{n-1})$, and using the fact that u vanishes at $x_n = 0$ along with the Hölder inequality, we deduce

$$\|u/(1+x_n)\|_{q,P_{a,b}} \le \frac{q}{q-1}|u|_{1,q,P_{a,b}}.$$

Since $D^{1,q}(\mathbb{R}^n_+) \subset W^{1,q}(P_{a,b})$, by a density argument we can extend this latter inequality to functions merely belonging to $D^{1,q}(\mathbb{R}^n_+)$ having zero trace at $x_n = 0$. Thus, in particular, letting $b \to \infty$, we find, for all $a > 0$,

$$\|u/(1+x_n)\|_{q,L_a} \le \frac{q}{q-1}|u|_{1,q,L_a}. \tag{II.6.46}$$

where

$$L_a = \{x \in \mathbb{R}^n_+ :\ x_n \in (0,a)\}. \tag{II.6.47}$$

We may summarize the above considerations in the following.

[3] Notice that since $u \in W^{1,q}(C)$ for every cube C of $\mathbb{R}^n_+$ with a side at $x_n = 0$, the trace of u at $x_n = 0$ is well defined. A more general result for u_0 to be zero is furnished in Exercise II.7.5 and Section II.10.

Theorem II.6.3 *Let $n \geq 2$ and assume*

$$u \in D^{1,q}(\mathbb{R}^n_+), \quad 1 \leq q < \infty.$$

(i) *If $q \in [1, n)$, there exists a uniquely determined $u_0 \in \mathbb{R}$ such that the function*

$$w = u - u_0$$

enjoys the following properties. For any $x_0 \in \mathbb{R}^n$, it is

$$w|x - x_0|^{-1} \in L^q(\Omega^R(x_0)),$$

where

$$\Omega^R(x_0) \equiv \mathbb{R}^n_+ - B_R(x_0)$$

and the following inequality holds:

$$\left(\int_{\Omega^R(x_0)} \left| \frac{w(x)}{x - x_0} \right|^q dx \right)^{1/q} \leq q/(n-q)|w|_{1,q,\Omega^R(x_0)}. \tag{II.6.48}$$

Furthermore, if $x_0 \in \mathbb{R}^n_+$, $|x_0| = \alpha R$, for some $\alpha \geq \alpha_0 > 1$ and some $R > 0$, we have

$$\left(\int_{\Omega^R} \left| \frac{w(x)}{x - x_0} \right|^q dx \right)^{1/q} \leq c|w|_{1,q,\Omega^R}$$

with $\Omega^R = \mathbb{R}^n_+ - B_R$ and $c = c(n, q, \alpha_0)$. In addition,

$$w \in L^s(\mathbb{R}^n_+), \quad s = nq/(n-q) \tag{II.6.49}$$

and for some γ_2 independent of u

$$\|w\|_s \leq \gamma_2 |w|_{1,q}.$$

If the trace of u is zero at $x_n = 0$, then $u_0 = 0$.

(ii) *If $q \geq n$, and u has zero trace at $x_n = 0$ then $\mathfrak{w}$ $u \in L^q(\mathbb{R}^n_+)$ and inequality* (II.6.24) *holds with any $x_0 \in \mathbb{R}^n_-$.*[4]

(iii) *If $q \in (1, \infty)$ and u has zero trace at $x_n = 0$, then $u/(1 + x_n) \in L^q(\mathbb{R}^n_+)$ and inequality* (II.6.46) *holds for all $a > 0$.*

By means of a simple procedure based on the iterative use of (II.6.22) and (II.6.49) one can show the following general embedding theorem for functions in $D^{m,q}(\Omega)$, whose proof is left to the reader as an exercise.

Theorem II.6.4 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be either a locally Lipschitz exterior domain or $\Omega = \mathbb{R}^n_+$, and let $u \in D^{m,q}(\Omega)$, $m \geq 1$, $1 \leq q < \infty$.*

[4] So that (II.6.24) holds with $\Omega_R(x_0) \equiv \mathbb{R}^n_+$.

(a) *If* $q \in [1, n)$, *let* $\ell \in \{1, \dots, m\}$ *be the largest integer such that* $\ell q < n$. *Then there are* ℓ *uniquely determined homogeneous polynomials* $\mathcal{M}_{m-r}$, $r = 1, \dots, \ell$, *of degree* $\leq m - r$ *such that, setting*

$$u_{m-k} = \sum_{r=1}^{k} \mathcal{M}_{m-r}\,, \quad k \in \{1, \dots, \ell\}\,,$$

we have

(i) $(u - u_{m-k}) \in D^{m-k,q_k}(\Omega)\,,$

(ii) $\displaystyle\sum_{k=1}^{\ell} |u - u_{m-k}|_{m-k,q_k} \leq c|u|_{m,q}\,,$

where $q_k = nq/(n - kq)\,.$

(b) *If* $q \geq n$, $\Omega \neq \mathbb{R}^n$, *and the trace of* $D^\alpha u$, $|\alpha| = m - 1$, *is zero at* $\partial\Omega$, *then* $\mathfrak{w}\, D^\alpha u \in L^q(\Omega_R(x_0))$, *with* $\mathfrak{w}$ *and* $\Omega_R(x_0)$ *given in part* (ii) *of* Theorem II.6.1 *and* Theorem II.6.3, *and* (II.6.24) *holds with* $u \equiv D^\alpha u$.

(c) *If* $u \in D^{m,q}(\mathbb{R}^n_+)$, $q \in (1, \infty)$, *and the trace of* $D^\alpha u$, $|\alpha| = m - 1$, *is zero at* $x_n = 0$, *then* $D^\alpha u/(1 + x_n) \in L^q(\mathbb{R}^n_+)$ *and inequality* (II.6.46), *with* $u \equiv D^\alpha u$, *holds for all* $a > 0$.

Our final objective is to establish embedding inequalities for functions from $D^{m,q}(\Omega)$ that vanish at $\partial\Omega$. We wish to prove these results without assuming any regularity on $\partial\Omega$, and so we introduce the following generalized version of "vanishing of traces at the boundary" for $u \in D^{m,q}(\Omega)$ (see Simader and Sohr 1997, Chapter I)

$$\psi u \in W_0^{m,q}(\Omega)\,, \quad \text{for all } \psi \in C_0^\infty(\mathbb{R}^n)\,. \tag{II.6.50}$$

Remark II.6.5 In view of Theorem II.4.2, we find at once that, if Ω has the regularity specified in that theorem, condition (II.6.50) is *equivalent* to the condition $\Gamma_m(u) = 0$ at $\partial\Omega$. ■

Theorem II.6.5 *Let* Ω *be an exterior domain of* $\mathbb{R}^n$, $n \geq 2$, *and let* $u \in D^{m,q}(\Omega)$, $m \geq 1$, $q \in [1, \infty)$, *satisfy* (II.6.50).

(i) *Assume* $\Omega^c \supset B_a$, *for some* $a > 0$. *Then, the following inequality holds for all* $R > \delta(\Omega^c)$

$$\|u\|_{m-1,q,\Omega_R} \leq m\, C\, |u|_{m,q,\Omega_R}\,,$$

where $C = n^{-1/q} R^{1+(n-1)/q} a^{(1-n)/q}$.

(ii) *Assume* $q \in [1, n)$ *and let* $\ell \in \{1, \dots, m\}$ *be the largest integer such that* $\ell q < n$. *Then, if the homogeneous polynomials* $\mathcal{M}_{m-r}$, $r = 1, \dots, \ell$, *defined in* Theorem II.6.4(a) *are all zero, the properties* (i) *and* (ii) *of that theorem hold.*

Proof. For any given $R > \delta(\Omega^c)$, let $\psi \in C_0^\infty(\mathbb{R}^n)$ to be 1 in Ω_{2R} and 0 in Ω^{3R}. By (II.6.50), we know that there is $\{u_s\} \subset W_0^{m,q}(\Omega)$ converging to ψu.

Thus, it is enough to show the statement in (i) for $u \in C_0^\infty(\Omega)$ and for $m = 1$. If we extend u to 0 in Ω^c, we find

$$u(x) = \int_a^{|x|} \frac{\partial u}{\partial r}(r\,x/|x|)dr\,.$$

By using the Hölder inequality in this identity, we derive

$$|u(x)|^q \leq R^{q-1}a^{1-n}\int_a^R |\nabla u|^q r^{n-1}dr\,.$$

Therefore, by multiplying both sides of this inequality by r^{n-1}, and by integrating the resulting relations over $r \in [a,R]$ and again over the unit sphere, we obtain the desired inequality. Under the stated assumptions in part (ii), from Theorem II.6.4(a) we find

$$\sum_{k=1}^{\ell} |u|_{m-k,nq/(n-kq),\Omega^R} \leq C\,|u|_{m,q}\,, \tag{II.6.51}$$

while, by a repeated use of (II.6.9), it follows that

$$\sum_{k=1}^{\ell} |u_s|_{m-k,nq/(n-kq),\Omega_R} \leq C\,|u_s|_{m,q}\,.$$

Passing to the limit $s \to \infty$ in this relation, and recalling the properties of ψ, we deduce

$$\sum_{k=1}^{\ell} |u|_{m-k,\frac{nq}{n-kq},\Omega_R} \leq C\left(\sum_{k=1}^{\ell}|u|_{m-k,q,\Omega_{2R,3R}} + \sum_{k=\ell+1}^{m}|u|_{m-k,q,\Omega_{2R,3R}} + |u|_{m,q}\right).$$

Combining this inequality with (II.6.51), we find

$$\sum_{k=1}^{\ell} |u|_{m-k,nq/(n-kq)} \leq C\,\left(\|u\|_{m-\ell-1,q,\Omega_{2R,3R}} + |u|_{m,q}\right)\,, \tag{II.6.52}$$

and the result follows from (II.6.52) and part (i). □

Exercise II.6.10 Let $u \in D^{1,q}(\Omega)$, $1 \leq q < \infty$. Assume $\Omega \cap B_r(x_0)$ locally Lipschitz for every $x_0 \in \partial\Omega$ and some $r > 0$. Show that if u has zero trace at $\partial\Omega$, then its extension $\widehat{u}$ to $\mathbb{R}^n$, obtained by setting $u \equiv 0$ in Ω^c, is in $D^{1,q}(\mathbb{R}^n)$. *Hint:* Take φ arbitrary from $C_0^\infty(\mathbb{R}^n)$. and let B be an open ball with $B \supset \operatorname{supp}(\varphi)$. Then $\varphi u \in W_0^{1,q}(\Omega \cap B)$, and one can argue as in Exercise II.3.11.

II.7 Approximation of Functions from $D^{m,q}$ by Smooth Functions and Characterization of the Space $D_0^{m,q}$

In the preceding section, we have defined the space $D_0^{m,q}(\Omega)$ as the (Cantor) completion of the normed space $\{C_0^\infty(\Omega), |\cdot|_{m,q}\}$. As such, the generic element of $D_0^{m,q}(\Omega)$ is an equivalence class of Cauchy sequences. Our main objective in this section is to furnish a "concrete" representation of $D_0^{m,q}(\Omega)$, up to an isomorphism, when *Ω is either an exterior domain or a half-space.*

In order to reach this objective, it is of the utmost importance to investigate the conditions under which an element from $D^{m,q}(\Omega)$ can be approximated by functions from $C_0^\infty(\Omega)$ in the seminorm (II.6.4) (see Galdi and Simader 1990, and Remark II.6.4). As a by-product, we shall also find conditions ensuring the validity of this approximation by functions from $C_0^\infty(\overline{\Omega})$. Like we did previously in analogous circumstances, we shall consider the case $m = 1$, leaving the case $m > 1$ to the reader (see Theorem II.7.3 through Theorem II.7.8).

Theorem II.7.1 *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be an exterior domain, and let $u \in D^{1,q}(\Omega)$, $1 \leq q < \infty$. Then, u can be approximated in the seminorm $|\cdot|_{1,q}$ by functions from $C_0^\infty(\Omega)$ under the following assumptions.*

(i) *If $q \in [1, n)$, u satisfies* (II.6.50) *with $m = 1$, and $u_0 = 0$, where u_0 is the constant of* Lemma II.6.3*;*
(ii) *If $q \in [n, \infty)$, u satisfies* (II.6.50) *with $m = 1$,.*

Proof. We shall follow the ideas of Sobolev (1963b), combined with the arguments used in the proof of Theorem II.6.2. Let $\psi \in C_0^\infty(\mathbb{R})$ be nonincreasing with $\psi(\xi) = 1$ if $|\xi| \leq 1/2$ and $\psi(\xi) = 0$ if $|\xi| \geq 1$ and set, for R large enough,

$$\psi_R(x) = \psi\left(\frac{\ln\ln|x|}{\ln\ln R}\right). \tag{II.7.1}$$

Notice that, for a suitable constant $c > 0$ independent of R,

$$|D^\alpha \psi_R(x)| \leq \frac{c}{\ln\ln R}\frac{1}{|x|^m \ln|x|}, \quad |\alpha| = m \geq 1 \tag{II.7.2}$$

and $D^\alpha \psi_R(x) \not\equiv 0$, $|\alpha| \geq 1$, only if $x \in \widetilde{\Omega}_R$, where

$$\widetilde{\Omega}_R = \left\{x \in \Omega : \exp\sqrt{\ln R} < |x| < R\right\}. \tag{II.7.3}$$

Next, let $u \in D^{1,q}(\Omega)$, $q \in [1, \infty)$, satisfying (II.6.50) with $m = 1$, and with $u_0 = 0$ if $q \in [1, n)$. We write $u = (1 - \psi_R)u + \psi_R u$. By (II.6.50) we then have

$$\psi_R u \in W_0^{1,q}(\Omega) \tag{II.7.4}$$

for all $R > \delta(\Omega^c)$. So, given $\varepsilon > 0$ we may find a sufficiently large R and a function $u_{R,\varepsilon} \in C_0^\infty(\Omega)$ such that

$$|u_{R,\varepsilon} - \psi_R u|_{1,q} < \varepsilon,$$

and

$$|u - u_{R,\varepsilon}|_{1,q} \leq \|(1-\psi_R)\nabla u\|_q + \|\nabla\psi_R u\|_q + |u_{R,\varepsilon} - \psi_R u|_{1,q} < 2\varepsilon + \|\nabla\psi_R u\|_q.$$

The lemma will then follow from this inequality, provided we show that the last term on its right-hand side tends to zero as $R \to \infty$. Setting

$$\ell(R) \equiv \|\nabla\psi_R u\|_q, \tag{II.7.5}$$

in view of (II.7.2) and (II.7.3) we can find a constant $c_1 > 0$ such that

$$\ell(R)^q \leq \frac{c_1}{(\ln\ln R)^q}\int_{\exp\sqrt{\ln R}}^R \int_{S^{n-1}} \frac{|u(r,\omega)|^q}{(\ln r)^q} r^{n-q-1} d\omega dr.$$

Now, by Lemma II.6.3 and Exercise II.6.3, recalling that $u_0 = 0$ if $q \in [1,n)$, we have

$$\int_{S^{n-1}} |u(r,\omega)|^q \leq c_2 g(r),$$

where, in particular,

$$g(r) = \begin{cases} (\ln r)^{n-1} & \text{if } q = n \\ r^{q-n} & \text{if } q \neq n\,,\ q \neq 1 \\ r^{1-n}|u|_{1,1,\Omega^r} & \text{if } q = 1. \end{cases}$$

Therefore, if $q = n$ we obtain

$$\ell(R)^n \leq \frac{c_2}{(\ln\ln R)^n}\int_{\exp\sqrt{\ln R}}^R (r\ln r)^{-1} dr \leq c_2(\ln\ln R)^{1-n}; \tag{II.7.6}$$

and if $q \neq n$, $q \neq 1$,

$$\ell(R)^q \leq \frac{c_2}{(\ln\ln R)^q}\int_{\exp\sqrt{\ln R}}^R (\ln r)^{-q} r^{-1} dr \leq \frac{c_2}{(\ln\ln R)^q}\frac{(\ln R)^{(1-q)/2}}{(q-1)}. \tag{II.7.7}$$

Finally, if $q = 1$, we have

$$\ell(R) \leq \frac{c_2}{(\ln\ln R)}\int_{\exp\sqrt{\ln R}}^R (\ln r)^{-1} r^{-1} |u|_{1,1,\Omega^r} dr \leq \frac{c_2}{2}\,|u|_{1,1,\Omega^{\exp\sqrt{\ln R}}}\,. \tag{II.7.8}$$

So, for all $q \in [1,\infty)$, we recover

$$\lim_{R\to\infty} \ell(R) = 0,$$

which completes the proof of the theorem. □

Remark II.7.1 If the trace of u does not vanish at the boundary, that is, if u does not satisfy (II.6.50), Theorem II.7.1 should be suitably modified. In fact, on the one hand, the function $\psi_R u$ does not satisfy the condition (II.7.4) but, rather, it verifies the following one:

$$\psi_R u \in W^{1,q}(\Omega), \quad \text{for all } R > \delta(\Omega^c).$$

So, from Theorem II.3.1 it follows that, if Ω is locally Lipschitz, given $\varepsilon > 0$, we may find a sufficiently large R and a function $u_{R,\varepsilon} \in C_0^\infty(\overline{\Omega})$ such that

$$|u_{R,\varepsilon} - \psi_R u|_{1,q} < \varepsilon$$

and, as in the proof of Theorem II.7.1, we can prove that any $u \in D^{1,q}(\Omega)$ can be approximated in the seminorm $|\cdot|_{1,q}$ by functions from $C_0^\infty(\overline{\Omega})$ for $q \geq n$. However, the same result continues to hold also when $1 \leq q < n$. In fact, it suffices to notice that, for any $u \in D^{1,q}(\Omega)$ with $u_0 \neq 0$, the function $\psi_R(u - u_0)$, with u_0 defined in Lemma II.6.3, is of bounded support in Ω, belongs to $W^{1,q}(\Omega)$ and approaches u in the seminorm $|\cdot|_{1,q}$. We thus have the following.

Theorem II.7.2 *Let Ω be locally Lipschitz, and let $u \in D^{1,q}(\Omega)$. Then, u can be approximated in the norm $|\cdot|_{1,q}$ by functions from $C_0^\infty(\overline{\Omega})$.*

■

Exercise II.7.1 Let Ω be locally Lipschitz. Show that $C_0^\infty(\overline{\Omega})$ is dense in $\dot{D}^{1,q}(\Omega)$.

The technique employed in the proof of Theorem II.7.1 and Theorem II.7.2, along with the results of Theorem II.6.4, allow us to generalize the previous results to the space $D^{m,q}(\Omega)$, $m \geq 1$, in the following theorems, whose proofs we leave to the reader as an exercise.

Theorem II.7.3 *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be an exterior domain and let $u \in D^{m,q}(\Omega)$, $1 \leq q < \infty$, $m \geq 1$. Then $u \in D^{m,q}(\Omega)$ can be approximated in the norm $|\cdot|_{m,q}$ by functions from $C_0^\infty(\Omega)$ under the following assumptions.*

(i) *If $q \in [1, n)$, u satisfies* (II.6.50) *and the following conditions hold:*

$$u_{m-\ell} \equiv 0\,, \tag{II.7.9}$$

where $\ell \in \{1, \ldots, m\}$ is the largest integers such that $\ell q < n$ and the polynomials $u_{m-\ell}$ are defined in Theorem II.6.4.

(ii) *If $q \in [n, \infty)$, u satisfies* (II.6.50)

Theorem II.7.4 *Let Ω be a locally Lipschitz, exterior domain of $\mathbb{R}^n$, $n \geq 2$. Then, every $u \in D^{m,q}(\Omega)$ can be approximated in the seminorm $|\cdot|_{m,q}$ by functions from $C_0^\infty(\overline{\Omega})$.*

We are now in the position to prove a characterization of the space $D_0^{m,q}(\Omega)$. For the sake of argument, we shall first consider the case $m = 1$.

Set

$$\widetilde{D}_0^{1,q}(\Omega) = \begin{cases} \{u \in D^{1,q}(\Omega) : \|u\|_{\frac{nq}{n-q}} < \infty\,,\ u \text{ satisfies (II.6.50) with } m = 1\}\,, \\ \hfill \text{if } q \in [1, n) \\ \\ \{u \in D^{1,q}(\Omega) :\ u \text{ satisfies (II.6.50) with } m = 1\}\,, \text{ if } q \in [n, \infty) \end{cases} \tag{II.7.10}$$

where, if $q \geq n$, we assume $\Omega^c \supset B_a$, for some $a > 0$.

With the help of Exercise II.6.8, it is not difficult to show that $\widetilde{D}_0^{1,q}(\Omega)$, $1 \leq q < \infty$, endowed with the norm $|\cdot|_{1,q}$ is a Banach space, and that this norm is equivalent to the following one

$$\begin{aligned} &|\cdot|_{1,q} + \|\cdot\|_{nq/(n-q)} \ \text{ if } q \in [1, n) \\ &|\cdot|_{1,q} + \|\mathfrak{w}(\cdot)\|_q \ \text{ if } q \in [n, \infty)\,. \end{aligned} \tag{II.7.11}$$

where $\mathfrak{w}$ is defined in (II.6.23).

Theorem II.7.5 *Let Ω be an exterior domain of $\mathbb{R}^n$, $n \geq 2$. Then $D_0^{1,q}(\Omega)$, $q \in [1, \infty)$, is isomorphic to $\widetilde{D}_0^{1,q}(\Omega)$, where $\Omega \neq \mathbb{R}^n$, if $q \geq n$. If $q \geq n$ and $\Omega = \mathbb{R}^n$, then $D_0^{1,q}(\mathbb{R}^n)$ is isomorphic to $\dot{D}^{1,q}(\mathbb{R}^n)$.*

Proof. We first consider the two cases: either (i) $q \in [1, n)$, or (ii) $q \in [n, \infty)$ and $\Omega \neq \mathbb{R}^n$, and begin to construct a suitable map $\mathfrak{T} : D_0^{1,q}(\Omega) \to \widetilde{D}_0^{1,q}(\Omega)$. Let $\widetilde{u}$ be a generic element in $D_0^{1,q}(\Omega)$, that is, an equivalence class of Cauchy sequences, and let $\{u_k\} \in \widetilde{u}$. Then $\{D_j u_k\}$, $j = 1, \ldots, n$, are Cauchy sequences in $L^q(\Omega)$ and, therefore, there exist corresponding $V_j \in L^q(\Omega)$, such that

$$\lim_{k\to\infty} \|D_j u_k - V_j\|_q = 0\,, \quad j = 1, \ldots, n\,. \tag{II.7.12}$$

Moreover, in view of Exercise II.6.4, $\{u_k\}$ is a Cauchy sequence also in $L^{nq/(n-q)}(\Omega)$, if $q \in [1, n)$, and in $L^q_{\mathfrak{w}}(\Omega)$, if $q \geq n$ and $\Omega \neq \mathbb{R}^n$. Thus, there is $u \in L^{nq/(n-q)}(\Omega)$, if $q \in [1, n)$, or $u \in L^q_{\mathfrak{w}}(\Omega)$, if $q \geq n$ and $\Omega \neq \mathbb{R}^n$, such that

$$\begin{aligned} &\lim_{k\to\infty} \|u_k - u\|_{nq/(n-q)} = 0\,, \quad \text{if } q \in [1, n) \\ &\lim_{k\to\infty} \|\mathfrak{w}(u_k - u)\|_q = 0\,, \quad \text{if } q \geq n, \ \ \Omega \neq \mathbb{R}^n\,. \end{aligned} \tag{II.7.13}$$

From the definition of weak derivative and from (II.7.12)–(II.7.13), it immediately follows that $V_j = D_j u$. Next, let $\psi \in C_0^\infty(\mathbb{R}^n)$. We have to show that ψu can be approximated, in $W^{1,q}(\Omega)$–norm, by a sequence $\{v_k\} \subset C_0^\infty(\Omega)$. Take $v_k = \psi u_k$. From (II.7.13) it is clear that $\|\psi u - v_k\|_q \to 0$ as $k \to \infty$. Moreover,

$$|\psi u - v_k|_{1,q} \leq C\,(\,|u - u_k|_{1,q} + \|u - u_k\|_{q,K})$$

with K the support of ψ, so that, from this inequality and (II.7.12), (II.7.13), we find $|\psi u - v_k|_{1,q} \to 0$ as $k \to \infty$, which concludes the proof of the desired property. We may thus infer $u \in \widetilde{D}_0^{1,q}(\Omega)$. Since, as it is readily checked, the function u does not depend on the particular sequence $\{u_k\} \in \widetilde{u}$, we may define a map, $\mathfrak{T}$, that to each $\widetilde{u} \in D_0^{1,q}(\Omega)$ assigns the function $u \in \widetilde{D}_0^{1,q}(\Omega)$ determined in the way described above. Of course, $\mathfrak{T}$ is linear and it is also an isometry, and, in addition,

$$|\widetilde{u}|_{1,q} \equiv \lim_{k\to\infty} |u_k|_{1,q} = |u|_{1,q} \equiv |\mathfrak{T}(\widetilde{u})|_{1,q}.$$

It remains to show that the range of $\mathfrak{T}$ coincides with $\widetilde{D}_0^{1,q}(\Omega)$. This amounts to say that, for each $u \in \widetilde{D}_0^{1,q}(\Omega)$ we can find $\{u_k\} \subset C_0^\infty(\Omega)$ such that $|u_k - u|_{1,q} \to 0$ as $k \to \infty$. However, the validity of this property is assured by Theorem II.7.1. Finally, the case $\Omega = \mathbb{R}^n$ and $q \geq n$. In view of Remark II.6.2, we only have to show that the natural map i is surjective, namely, that for any $[u] \equiv [u]_1 \in \dot{D}_0^{1,q}(\mathbb{R}^n)$, we can find $\{u_k\} \subset C_0^\infty(\Omega)$ such that $|u_k - v|_{1,q} \to 0$, as $k \to \infty$, $v \in [u]$. This property follows from Theorem II.7.1, and the proof of the theorem is complete. □

We may thus summarize the above theorem with the following representation of the spaces $D_0^{1,q}(\Omega)$ (up to an isomorphism).

If $q \in [1, n)$:

$$D_0^{1,q}(\Omega) = \{u \in D^{1,q}(\Omega) : \|u\|_{nq/(n-q)} < \infty\,, u \text{ satisfies (II.6.50) with } m = 1\}, \tag{II.7.14}$$

with equivalent norm given in $(\text{II.7.11})_1$.

If $q \geq n$, and $\Omega^c \supset B_a$, for some $a > 0$:

$$D_0^{1,q}(\Omega) = \{u \in D^{1,q}(\Omega) :\ u \text{ satisfies (II.6.50) with } m = 1\}\,, \tag{II.7.15}$$

with equivalent norm given in $(\text{II.7.11})_2$.

If $q \geq n$ and $\Omega = \mathbb{R}^n$:

$$D_0^{1,q}(\mathbb{R}^n) = \{[u] : u \in D^{1,q}(\mathbb{R}^n)\}\,, \tag{II.7.16}$$

where

$$[u] = \{v \in D^{1,q}(\mathbb{R}^n) \text{ such that } v = u + c\,,\ c \in \mathbb{R}\}\,.$$

By combining Theorem II.7.3 with the arguments used in showing Theorem II.7.5, one is now able to furnish the following representation (up to an isomorphism) of the space $D_0^{m,q}(\Omega)$, for arbitrary $m \geq 1$.

Theorem II.7.6 *Let Ω be an exterior domain of $\mathbb{R}^n$, $n \geq 2$. The following representations hold.*

(i) *If $q < n$, let $\ell \in \{1, \ldots, m\}$ be the largest integer such that $\ell q < n$. If $\ell < m$, we assume $\Omega^c \supset B_a$ for some $a > 0$. Then:*

$$D_0^{m,q}(\Omega) = \left\{ u \in D^{m,q}(\Omega) : \sum_{k=1}^{\ell} |u|_{m-k,\frac{nq}{n-kq}} < \infty \,,\; u \text{ satisfies (II.6.50)} \right\}, \tag{II.7.17}$$

with equivalent norm

$$\|u\|_{m-1,q,\Omega_{R_0}} + \sum_{k=1}^{\ell} |u|_{m-k,nq/(n-kq)} + |u|_{m,q} \,,$$

where R_0 is a fixed number strictly greater than $\delta(\Omega^c)$.

(ii) *If $q \geq n$, assume $\Omega^c \supset B_a$ for some $a > 0$. Then:*

$$D_0^{m,q}(\Omega) = \{ u \in D^{m,q}(\Omega) : \; u \text{ satisfies (II.6.50)} \} \,, \tag{II.7.18}$$

with equivalent norm

$$\|u\|_{m-1,q,\Omega_{R_0}} + |u|_{m,q} \,,$$

where R_0 is a fixed number strictly greater than $\delta(\Omega^c)$.

(iii) *If $q < n$, $mq \geq n$, and $\Omega = \mathbb{R}^n$:*

$$D_0^{m,q}(\mathbb{R}^n) = \left\{ [u]_{m-\ell} \,, u \in D^{m,q}(\Omega) : \; \sum_{k=1}^{\ell} |u|_{m-k,\frac{nq}{n-kq}} < \infty \right\} \tag{II.7.19}$$

where ℓ ($< m$) is the largest integer such that $\ell q < n$, and where, we recall,

$$[u]_{m-\ell} = \{ v \in D^{m,q}(\mathbb{R}^n) : v = u + \mathcal{P}_{m-\ell-1} \} \,,$$

with $\mathcal{P}_{m-\ell-1}$ polynomial of degree $\leq m - \ell - 1$.

(iv) *If $q \geq n$ and $\Omega = \mathbb{R}^n$:*

$$D_0^{m,q}(\mathbb{R}^n) = \{ [u]_m \,, u \in D^{m,q}(\Omega) \} \tag{II.7.20}$$

The proof of the above theorem is quite straightforward. In fact, it is obtained by combining the procedure used in Theorem II.7.5, with the results of Theorem II.7.3 and Theorem II.6.5. We leave the details to the reader.

Exercise II.7.2 Show that the space defined on the right-hand side of (II.7.19) is a Banach space with respect to the norm $|[u]|_{m,q} \equiv |u|_{m,q}$, $u \in [u]_{m-\ell}$. *Hint.* Follow the arguments of the proof of Theorem II.7.1.

Remark II.7.2 From Theorem II.7.6 we deduce that, unless $mq < n$, the space $D_0^{m,q}(\mathbb{R}^n)$ is a Banach space whose elements are equivalence classes of functions that differ by polynomials of suitable degree. In particular, if $q \geq n$, then $D_0^{m,q}(\mathbb{R}^n) = \dot{D}^{m,q}(\mathbb{R}^n)$. In this respect, see also the following exercise. ■

Exercise II.7.3 Let $\{u_k\}$ be a Cauchy sequence in $D_0^{m,q}(\mathbb{R}^n)$, where $mq \geq n$, and let $[u]_m \in \dot{D}^{m,q}(\mathbb{R}^n)$ be such that $|u_k - u|_{m,q} \to 0$ as $k \to \infty$, $u \in [u]_m$. Show, by means of an example, that even though $u \in L^s(B_R)$, for all $s \in [1, q]$ and all $R > 0$, we may have $\|u_k\|_{1,B_R} \to \infty$ as $k \to \infty$, for all sufficiently large R. *Hint* (Deny & Lions 1954, §4): Take $m = 1$, $q = n = 2$ and choose

$$u_k(x) = -\int_{|x|}^{\infty} (t \ln t)^{-1} a_k(t)\, dt\,,$$

where $a_k = a_k(t)$, $k \in \mathbb{N}$, is a smooth, non-negative function of $C_0^\infty(\mathbb{R})$ which is 0 for $t \leq 2$ and for $t \geq k+4$, and it is 1 for $t \in [5/2, 3+k]$. Then $|u_k - u|_{1,2} \to 0$ as $k \to \infty$, where $u(x) = (\sqrt{|x|}\ln x)^{-1} a(x)$, with $a(x) = 0$ for $|x| \leq 2$ and $= 1$ for $|x| \geq 5/2$, while

$$\lim_{k\to\infty} \int_{B_R} |u_k(x)| = \infty\,, \quad \text{for all } R > 5/2\,.$$

Exercise II.7.4 Let Ω be an exterior domain and let $u \in D_0^{2,2}(\Omega)$. Show that

$$|D^2 u|_{2,2} = \|\Delta u\|_2\,.$$

Hint: It is enough to show the identity for $u \in C_0^\infty(\Omega)$.

Results similar to those of Theorem II.7.3 and Theorem II.7.4 can be proved in the case when $\Omega = \mathbb{R}^n_+$. In fact, as we already noticed, every function $u \in D^{m,q}(\mathbb{R}^n_+)$ can be extended to the whole of $\mathbb{R}^n$ to a function u' satisfying (II.6.45). In particular, if the trace $\Gamma_m(u)$ on every (bounded) portion of the plane $x_n = 0$ is identically zero, we may take u' as the function obtained by setting $u \equiv 0$ outside $\mathbb{R}^n_+$. With this and Theorem II.6.4(c) in mind, one can show the following theorems, whose proofs are left to the reader.

Theorem II.7.7 *The following representation holds, for all $m \geq 0$, $q \in [1, \infty)$.*

$$D_0^{m,q}(\mathbb{R}^n_+) = \{u \in D^{m,q}(\mathbb{R}^n_+) :\ \Gamma_m(u) = 0 \text{ on } S\}\,,$$

with S arbitrary bounded domain in the plane $x_n = 0$, with equivalent norm

$$|u|_{m,q} + \|u\|_{m-1,q,L_{a_0}}\,,$$

where L_a is defined in (II.6.47) *and a_0 is a fixed positive number.*

Theorem II.7.8 *Let $u \in D^{m,q}(\mathbb{R}^n_+)$, $m \geq 0$, $q \in [1, \infty)$. Then, u can be approximated in the seminorm $|\cdot|_{m,q}$ by functions from $C_0^\infty(\overline{\mathbb{R}}^n_+)$.*

Remark II.7.3 Unlike the case Ω exterior, Theorem II.7.7 does not explicitly impose any restriction at large distances on the behavior of u when $1 \leq q < n$, such as the vanishing condition (II.7.9) on the polynomials $u_{m-\ell}$. Actually by means of an argument completely analogous to that preceding Theorem II.6.3, one can show that the polynomials $u_{m-\ell}$ are identically zero as a consequence of the vanishing of the trace $\Gamma_m(u)$. ■

Exercise II.7.5 (Coscia and Patria 1992, Lemma 5) Let $u \in D^{1,q}(\mathbb{R}^n_+)$, $1 \le q < n$. By Theorem II.6.3 there is $u_0 \in \mathbb{R}$ such that $u - u_0 \in L^s(\mathbb{R}^n_+)$, $s = nq/(n-q)$. Show that if the trace $\gamma(u)$ at $\Sigma = \{x \in \mathbb{R}^n : x_n = 0\}$ belongs to $L^r(\Sigma)$, for some $r \in [1,\infty)$, then $u_0 = 0$. This fact, together with Theorem II.7.8, implies that every such function can be approximated in the seminorm $|\cdot|_{1,q}$ by functions from $C_0^\infty(\overline{\mathbb{R}}^n_+)$.

II.8 The Normed Dual of $D_0^{m,q}(\Omega)$. The Spaces $D_0^{-m,q}$

We begin to furnish a characterization of the normed dual space $(D_0^{m,q}(\Omega))'$ of $D_0^{m,q}(\Omega)$, when Ω is either an exterior domain or $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$, analogous to the one we described at the end of Section II.3 for the space $W_0^{m,q}(\Omega)$. A (bounded) linear functional $\mathcal{F}$ belongs to $(D_0^{m,q}(\Omega))'$ if and only if

$$\|\mathcal{F}\|_{(D_0^{m,q}(\Omega))'} \equiv \sup_{u \in D_0^{m,q}(\Omega),\ |u|_{m,q}=1} |\mathcal{F}(u)| < \infty\,.$$

Let us first take Ω exterior, $\Omega \neq \mathbb{R}^n$ and satisfying the assumptions of Theorem II.7.6, or $\Omega = \mathbb{R}^n_+$. Consider the functional

$$\mathcal{F}(u) = (f,u), \quad f \in C_0^\infty(\Omega), \quad \text{all } u \in D_0^{m,q}(\Omega). \tag{II.8.1}$$

Applying the Hölder inequality in (II.8.1) we obtain

$$|\mathcal{F}(u)| \le \|f\|_{q'} \|u\|_{q,\Omega_0}, \tag{II.8.2}$$

where $\Omega_0 = \operatorname{supp}(f)$. Then, by Theorem II.7.6 and Theorem II.6.5(i), if Ω is exterior, and by Theorem II.7.7, if $\Omega = \mathbb{R}^n_+$, we find that inequality (II.8.2) implies

$$|\mathcal{F}(u)| \le c\,\|f\|_{q'} |u|_{m,q}$$

with $c = c(\Omega_0)$. We now set

$$|f|_{-m,q'} = \sup_{u \in D_0^{m,q}(\Omega),\ |u|_{m,q}=1} |\mathcal{F}(u)|. \tag{II.8.3}$$

Evidently, (II.8.3) is a norm in $C_0^\infty(\Omega)$. Denote by $D_0^{-m,q'}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in this norm. The following result holds.

Lemma II.8.1 *Let Ω be an exterior domain ($\neq \mathbb{R}^n$) satisfying the assumptions of* Theorem II.7.6, *or $\Omega = \mathbb{R}^n_+$. Then, for any $q \in (1,\infty)$, functionals of the form* (II.8.1) *are dense in $(D_0^{m,q}(\Omega))'$, and $(D_0^{m,q}(\Omega))'$ and $D_0^{-m,q'}(\Omega)$ are isomorphic.*

Proof. Let

$$\mathcal{S} = \{\mathcal{F} \in (D_0^{m,q}(\Omega))' :\ \mathcal{F}(u) = (f,u) \text{ for some } f \in C_0^\infty(\Omega)\}\,.$$

Clearly, $\mathcal{S}$ is a subspace of $(D_0^{m,q}(\Omega))'$. Let us prove that $\mathcal{S}$ is dense in $(D_0^{m,q}(\Omega))'$. In fact, assuming by contradiction that $\overline{\mathcal{S}} \neq (D_0^{m,q}(\Omega))'$, by the Hahn–Banach theorem (see Theorem II.1.7(b)) there exists a nonzero element $Z \in (D_0^{m,q}(\Omega))''$ such that

$$Z(\mathcal{F}) = 0, \text{ for all } \mathcal{F} \in \mathcal{S}.$$

Since $D_0^{m,q}(\Omega)$ is reflexive for $q \in (1, \infty)$ (cf. Exercise II.6.2), the preceding condition implies that there exists a nonzero $z \in D_0^{m,q}(\Omega)$ such that

$$\mathcal{F}(z) = 0, \text{ for all } \mathcal{F} \in \mathcal{S},$$

that is

$$(f, z) = 0, \text{ for all } f \in C_0^\infty(\Omega),$$

that is, $z = 0$, which leads to a contradiction. Following Lax (1955, §2), it is now readily seen that $(D_0^{m,q}(\Omega))'$ and $D_0^{-m,q'}(\Omega)$, $1 < q < \infty$, are isomorphic. To this end, let $\mathcal{L} \in (D_0^{m,q}(\Omega))'$ and let $\{f_k\} \subset C_0^\infty(\Omega)$ be such that the sequence $\mathcal{F}_k \equiv (f_k, u)$, $k \in \mathbb{N}$, $u \in D_0^{m,q}(\Omega)$, converges to $\mathcal{L}$ in the norm $|\cdot|_{(D_0^{m,q}(\Omega))'}$ of $(D_0^{m,q}(\Omega))'$. Since

$$|\mathcal{F}_k|_{(D_0^{m,q}(\Omega))'} = |f_k|_{-m,q'}, \tag{II.8.4}$$

$\{f_k\}$ is a Cauchy sequence in $D_0^{-m,q'}(\Omega)$ converging to some $\mathcal{F} \in D_0^{-m,q'}(\Omega)$. Clearly, $\mathcal{F}$ depends only on $\mathcal{L}$ and not on the particular sequence $\{f_k\}$ and, in addition, it is uniquely determined. Likewise, to each $\mathcal{F} \in D_0^{-m,q'}(\Omega)$ we may uniquely associate an $\mathcal{L} \in (D_0^{m,q}(\Omega))'$, thus establishing the existence of a linear bijection, $\mathscr{L}$, between $(D_0^{m,q}(\Omega))'$ and $D_0^{-m,q'}(\Omega)$. However, from (II.8.4), it follows that $\mathscr{L}$ is an isomorphism, and the proof of the lemma is complete. □

Let us now consider the case $\Omega = \mathbb{R}^n$. For $mq < n$, we employ, in (II.8.1), the Hölder inequality and make use m times of the Sobolev inequality (II.3.7) to deduce

$$|\mathcal{F}(u)| \leq \|f\|_{nq'/(n+q')}\|u\|_{nq/(n-mq)} \leq c\|f\|_{nq'/(n+q')}|u|_{m,q}. \tag{II.8.5}$$

If $mq \geq n$, by Theorem II.7.6 we know that elements from $D_0^{m,q}(\mathbb{R}^n)$ are equivalence classes $[u]_s$ determined by functions that may differ by polynomials $\mathcal{P}_s$ of degree $\leq s - 1$, where

$$\begin{cases} s = m, & \text{if } q \geq n, \\ s = m - \ell, & \text{if } q < n \text{ and } \ell\ (< m) \text{ is the largest integersuch that } \ell q < n. \end{cases} \tag{II.8.6}$$

Thus, if $mq \geq n$, functionals of the type (II.8.1) must satisfy $\mathcal{F}(u_1) = \mathcal{F}(u_2)$ whenever u_1, u_2 belong to the same class $[u]_s$. This is equivalent to the following condition on f:

$$\int_{\mathbb{R}^n} f\,\mathcal{P}_s = 0, \tag{II.8.7}$$

where $\mathcal{P}_s$ is an *arbitrary* polynomial of degree $\le s-1$, with s satisfying (II.8.6). As a consequence, from (II.8.7), for $u \in [u]_s$ we have (with $B_R \supset \mathrm{supp}\,(f)$)

$$|\mathcal{F}(u)| = \left|\int_{B_R} fu\right| = \left|\int_{B_R} f(u+\mathcal{P}_s)\right| \le \|f\|_{q',\mathbb{R}^n}\|u+\mathcal{P}_s\|_{q,B_R}. \tag{II.8.8}$$

We may choose $\mathcal{P}_s$ in such a way that, setting

$$u_s = u - \mathcal{P}_s,$$

it follows

$$\frac{1}{|B_R|}\int_{B_R} D^\alpha u_s = 0, \quad 0 \le |\alpha| \le s.$$

In view of these latter conditions, by a repeated use of the Poincaré inequality (II.5.10) in the last term on the right-hand side of (II.8.8), we obtain

$$|\mathcal{F}(u)| \le c_1\|f\|_{q',\mathbb{R}^n}|u|_{s+1,q,B_R}.$$

Now, if $q \ge n$, from (II.8.6) it is $s = m-1$ and so

$$|u|_{s+1,q,B_R} \le |u|_{m,q,\mathbb{R}^n}.$$

If $q < n$, again from (II.8.6), the Hölder inequality and (II.7.17) of Remark II.7.2, we deduce

$$|u|_{s+1,q,B_R} = |u|_{m-\ell,q,B_R} \le |u|_{m-\ell,nq/(n-\ell q),\mathbb{R}^n} \le c|u|_{m,q,\mathbb{R}^n}.$$

Thus, in all cases, we deduce

$$|\mathcal{F}(u)| \le c_2\|f\|_{q',\mathbb{R}^n}|u|_{m,q,\mathbb{R}^n}. \tag{II.8.9}$$

Once (II.8.9) has been established, we may again use the arguments of Lemma II.8.1 to show that the spaces $(D_0^{m,q}(\mathbb{R}^n))'$ and $D_0^{-m,q'}(\mathbb{R}^n)$, $1 < q < \infty$, are isomorphic.

Thus, for $q \in (1,\infty)$, let us define $\mathfrak{F}_{q,m}(\Omega)$ as the class of functionals (II.8.1), which, if $\Omega = \mathbb{R}^n$ and $n \le mq < \infty$, verify, in addition, (II.8.7) for an arbitrary polynomial $\mathcal{P}_s$ of degree $\le s-1$, with s satisfying (II.8.6). The results just discussed along with those of Lemma II.8.1 can be then summarized in the following.

Theorem II.8.1 *Let $\Omega \subseteq \mathbb{R}^n$ be either an exterior, locally Lipschitz domain, or $\Omega = \mathbb{R}^n_+$ or $\Omega = \mathbb{R}^n$. The completion, $D_0^{-m,q'}(\Omega)$, of $\mathfrak{F}_{q,m}(\Omega)$ in the norm* (II.8.3) *is isomorphic to $(D_0^{m,q}(\Omega))'$.*

Remark II.8.1 If $m = 1$, a restriction of the type (II.8.7) occurs if and only if $q \ge n$. In such a case, $\mathcal{P}_s$ reduces to an arbitrary constant so that condition (II.8.7) becomes

$$\int_{\mathbb{R}^n} f = 0. \tag{II.8.10}$$

■

Hereafter, the value of $\mathcal{F} \in D_0^{-1,q'}(\Omega)$ at $u \in D_0^{1,q}(\Omega)$ (duality pairing) will be denoted by

$$[\mathcal{F}, u].$$

Notice that if, in particular, $\mathcal{F} \in C_0^\infty(\Omega)$, we have

$$[\mathcal{F}, u] = (\mathcal{F}, u).$$

By an obvious continuity argument, the same relation holds, more generally, for all $\mathcal{F} \in L^s(\Omega) \cap D_0^{-1,q'}(\Omega)$, $s \in [1, \infty)$.

Our next goal is to provide a useful representation of functionals on $D_0^{1,q}(\Omega)$, valid for an arbitrary domain Ω, as well as another characterization of the space $(D_0^{1,q}(\Omega))'$. Taking into account that $D_0^{1,q}(\Omega)$ is a closed subspace of $\dot{D}^{1,q}(\Omega)$ (see Remark II.6.2), this representation becomes a particular case of the following important general result.

Theorem II.8.2 *Let Ω be a domain in $\mathbb{R}^n$. Then, for any given $\mathcal{F} \in (\dot{D}^{1,q}(\Omega))'$, $q \in (1, \infty)$, there exists $\boldsymbol{f} \in [L^{q'}(\Omega)]^n$ such that, for all $u \in \dot{D}^{1,q}(\Omega)$,*

$$\mathcal{F}(u) = (\boldsymbol{f}, \nabla u)\,. \tag{II.8.11}$$

Moreover,

$$\|\mathcal{F}\|_{(\dot{D}^{1,q}(\Omega))'} = \|\boldsymbol{f}\|_{q'}\,. \tag{II.8.12}$$

Proof. We recall that, for any $q \in (1, \infty)$, $\dot{D}^{1,q}(\Omega)$ can be viewed as a subspace of $[L^q(\Omega)]^n$, via the map

$$M : u \in \dot{D}^{1,q}(\Omega) \to \boldsymbol{h} \equiv \nabla u \in [L^q(\Omega)]^n\,. \tag{II.8.13}$$

Therefore, given $\mathcal{F} \in (\dot{D}^{1,q}(\Omega))'$, by the Hahn–Banach theorem (see Theorem II.1.7) there exists a (not necessarily unique) functional $\mathcal{L} \in [[L^q(\Omega)]^n]'$, such that

$$\mathcal{L}(\boldsymbol{h}) = \mathcal{F}(u)\,, \quad u \in \dot{D}^{1,q}(\Omega)\,, \tag{II.8.14}$$

and that, moreover, satisfies

$$\|\mathcal{L}\|_{[[L^q(\Omega)]^n]'} = \|\mathcal{F}\|_{(\dot{D}^{1,q}(\Omega))'}\,. \tag{II.8.15}$$

However, by Theorem II.2.6, we have that, corresponding to the functional $\mathcal{L}$, there exists a uniquely determined $\boldsymbol{f} \in [L^{q'}(\Omega)]^n$ such that $\mathcal{L}(\boldsymbol{w}) = (\boldsymbol{f}, \boldsymbol{w})$ for all $\boldsymbol{w} \in [L^q(\Omega)]^n$, with $\|\boldsymbol{f}\|_{q'} = \|\mathcal{L}\|_{[[L^q(\Omega)]^n]'}$. Therefore, the theorem follows from this latter consideration, and from (II.8.14) and (II.8.15). □

We would like to analyze some significant consequences of this result for the space $D_0^{1,q}(\Omega)$. We begin to observe that, since $D_0^{1,q}(\Omega) \subset \dot{D}^{1,q}(\Omega)$, by Theorem II.8.2 the generic linear functional on $D_0^{1,q}(\Omega)$ can be represented as in (II.8.11), for all $u \in D_0^{1,q}(\Omega)$, where the function $\boldsymbol{f} \in [L^{q'}(\Omega)]^n$ is determined up to a function $\boldsymbol{f}_0$ such that

$$(\boldsymbol{f}_0, \nabla u) = 0\,, \quad \text{for all } u \in D_0^{1,q}(\Omega)\,. \tag{II.8.16}$$

Let $\widetilde{L}^{q'}(\Omega)$ be the subspace of $[L^{q'}(\Omega)]^n$ constituted by all those functions satisfying (II.8.16). It is immediately verified that $\widetilde{L}^{q'}(\Omega)$ is closed. Moreover, setting $G_{0,q'}(\Omega) = M(D_0^{1,q'}(\Omega))$, with M defined in (II.8.13), we can readily show that $G_{0,q'}(\Omega)$ is also a closed subspace of $[L^{q'}(\Omega)]^n$; see Exercise II.8.1. Now, let $\boldsymbol{f} \in [L^{q'}(\Omega)]^n$ and consider the problem:

$$\text{Find } w \in D_0^{1,q'}(\Omega) \text{ such that } (\nabla w - \boldsymbol{f}, \nabla u) = 0\,, \text{ for all } u \in D_0^{1,q}(\Omega). \tag{II.8.17}$$

If Ω and $\boldsymbol{f}$ are sufficiently smooth, we can show that this problem is equivalent to the following classical Dirichlet problem

$$\Delta w = \nabla \cdot \boldsymbol{f} \text{ in } \Omega\,, \quad w = 0 \text{ at } \partial\Omega\,, \quad w \in D_0^{1,q'}(\Omega)\,.$$

Lemma II.8.2 *Assume that, for any given* $\boldsymbol{f} \in [L^{q'}(\Omega)]^n$, *problem* (II.8.17) *has one and only one solution* $w \in D_0^{1,q'}(\Omega)$. *Then, the following decomposition holds*

$$[L^{q'}(\Omega)]^n = \widetilde{L}^{q'}(\Omega) \oplus G_{0,q'}(\Omega)\,. \tag{II.8.18}$$

Conversely, if (II.8.18) *holds, then, for any* $\boldsymbol{f} \in [L^{q'}(\Omega)]^n$, *problem* (II.8.17) *is uniquely solvable. Finally, the linear operator* $\Pi_{q'} : \boldsymbol{f} \in [L^{q'}(\Omega)]^n \to \boldsymbol{f}_1 \in G_{0,q'}(\Omega)$ *is a projection (that is,* $\Pi_{q'}^2 = \Pi_{q'}$*) and is continuous.*

Proof. The last statement in the lemma is a consequence of (II.8.18); see Rudin (1973, Theorem 5.16(b)). Since both $L^{q'}(\Omega)$ and $G_{0,q'}(\Omega)$ are closed, in order to prove (II.8.18), under the given assumption, we have to show that (a) $L^{q'}(\Omega) \cap G_{0,q'}(\Omega) = \{0\}$, and that (b) $\boldsymbol{f} = \boldsymbol{f}_0 + \boldsymbol{f}_1$, $\boldsymbol{f}_0 \in \widetilde{L}^{q'}(\Omega)$, $\boldsymbol{f}_1 \in G_{0,q'}(\Omega)$. Suppose there are $\boldsymbol{l} \in \widetilde{L}^{q'}(\Omega)$ and $\boldsymbol{g} = \nabla g \in G_{0,q'}(\Omega)$, for some $g \in D_0^{1,q'}(\Omega)$, such that $\boldsymbol{l} = \boldsymbol{g}$. This means, by definition of $\widetilde{L}^{q'}(\Omega)$ that $(\nabla g, \nabla u) = 0$ for all $u \in D_0^{1,q}(\Omega)$, which, in turn, by the uniqueness assumption on problem (II.8.17), implies $\nabla g = \boldsymbol{l} = 0$. Thus, (a) is proved. Next, for the given $\boldsymbol{f}$, let $w \in D_0^{1,q}(\Omega)$ be the corresponding solution to (II.8.17) and set $\boldsymbol{f}_0 = \boldsymbol{f} - \nabla w$ $(\in \widetilde{L}^{q'}(\Omega))$, and $\boldsymbol{f}_1 = \nabla w$ $(\in G_{0,q'})$. Then, $\boldsymbol{f} = \boldsymbol{f}_0 + \boldsymbol{f}_1$ which proves (b). The converse claim, namely, that (II.8.18) implies the unique solvability of (II.8.17), is almost obvious and, therefore, it is left to the reader as an exercise $\square$

With the help of Theorem II.8.2 and Lemma II.8.2, we can now show the following result.

Theorem II.8.3 *Assume the hypothesis of* Lemma II.8.2 *is satisfied and* $q' \in (1,\infty)$. *Then* $D_0^{1,q'}(\Omega)$ *and* $(D_0^{1,q}(\Omega))'$ *are homeomorphic. Specifically, the linear map*

$$\mathfrak{M} : w \in D_0^{1,q'}(\Omega) \to \mathfrak{M}(w) \in (D_0^{1,q}(\Omega))' , \tag{II.8.19}$$

where

$$[\mathfrak{M}(w), u] = (\nabla w, \nabla u) , \quad \textit{for all } u \in D_0^{1,q}(\Omega) , \tag{II.8.20}$$

is a bijection and, moreover, for some $c = c(q, n, \Omega) > 0$,

$$c\,|w|_{1,q'} \le \|\mathfrak{M}(w)\|_{(D_0^{1,q}(\Omega))'} \le |w|_{1,q'} . \tag{II.8.21}$$

Proof. By assumption, we find that $\mathfrak{M}$ is injective, and, by Theorem II.8.2 ((II.8.11), in particular) and Lemma II.8.2, that $\mathfrak{M}$ is surjective, so that $\mathfrak{M}$ is a bijection. Furthermore, the inequality on the right-hand side of (II.8.21) is an obvious consequence of the Hölder inequality, while the one on the left-hand side follows from the continuity of the projection operator $\Pi_{q'}$ and from (II.8.12). □

In view of the results of Theorem II.8.3, it is of great interest to investigate under what conditions problem (II.8.17) has, for a given $\boldsymbol{f} \in [L^{q'}(\Omega)]^n$, a unique corresponding solution w. As a matter of fact, such unique solvability depends, in general, on the domain Ω and on the exponent q'. In particular, we have the following.

Theorem II.8.4 *Let* Ω *be either* $\mathbb{R}^n$, *or* $\mathbb{R}^n_+$, *or a bounded domain with a boundary of class* C^2. *Then, for all* $q \in (1,\infty)$, *the spaces* $D_0^{1,q'}(\Omega)$ *and* $(D_0^{1,q}(\Omega))'$ *are homeomorphic, in the sense specified in* Theorem II.8.3. *If* Ω *is an exterior domain of class* C^2 *(with* $\partial\Omega \neq \emptyset$*) the same conclusion holds if and only if* $q' \in (n/(n-1), n)$, *if* $n \ge 3$, *and* $q' = 2$, *if* $n = 2$.

We shall not give a proof of this theorem, mainly, because a completely analogous analysis of unique solvability will be carried out in Chapters IV and V, in the more complicated context of the Stokes problem. Here we shall limit ourselves to observe that the restriction on the exponent q', in the case of the exterior domain, comes from the fact that the Dirichlet problem (II.8.17) for $n \ge 3$ looses existence if $1 < q' \le n/(n-1)$ ($q' \in (1,2)$ if $n = 2$), while it lacks of uniqueness if $q' \ge n$, $n \ge 3$ ($q' > 2$, if $n = 2$). For further details, we refer the interested reader to the Notes at the end of this chapter.

Exercise II.8.1 Show that $G_{0,q}(\Omega)$, $q \in [1,\infty)$, is a closed subspace of $L^q(\Omega)$.

Exercise II.8.2 Show that the subspace $\mathcal{S}$ of $(\dot{D}^{1,q}(\Omega))'$, $q \in (1,\infty)$, defined as follows

$$\mathcal{S} = \{u \in C_0^\infty(\Omega) : \; u = \nabla \cdot \boldsymbol{\psi}\,, \text{ for some } \boldsymbol{\psi} \in C_0^\infty(\Omega)\}$$

is dense in $(\dot{D}^{1,q}(\Omega))'$. This result generalizes the one proved by Kozono & Sohr (1991, Corollary 2.3). *Hint*: Use Theorem II.8.2.

II.9 Pointwise behavior at Large Distances of Functions from $D^{1,q}$

We begin to give two classical results of potential theory, in a form suitable to our purposes.

Lemma II.9.1 *Let A be a bounded, locally Lipschitz domain of $\mathbb{R}^n$, $n \geq 2$, and let $w \in C^2(\overline{A})$. The following identity holds for all $x \in A$:*

$$w(x) = \frac{1}{n\omega_n}\int_A \frac{\partial w(y)}{\partial y_i}\frac{(x_i - y_i)}{|x-y|^n}dy - \frac{1}{n\omega_n}\int_{\partial A} w(y)\frac{(x_i-y_i)}{|x-y|^n}N_i(y)d\sigma_y$$

where $\boldsymbol{N} \equiv (N_i)$ is the outer unit normal to ∂A.

Proof. Denote by $\mathcal{E}(x-y)$ the *fundamental solution of Laplace's equation*:

$$\mathcal{E}(x-y) = \begin{cases} (2\pi)^{-1}\log|x-y| & \text{if } n = 2 \\ [n(2-n)\omega_n]^{-1}\,|x-y|^{2-n} & \text{if } n \geq 3. \end{cases} \tag{II.9.1}$$

Employing the (second) *Green's identity*[1]

$$\int_{A_\varepsilon}(v\Delta u - u\Delta v) = \int_{\partial A_\varepsilon}(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N})$$

with $v(y) \equiv w(y)$, $u(y) = \mathcal{E}(x-y)$, $A_\varepsilon = A - B_\varepsilon(x)$ and integrating by parts we deduce

$$\begin{aligned}\int_{A_\varepsilon}\frac{\partial \mathcal{E}(x-y)}{\partial y_i}\frac{\partial w(y)}{\partial y_i} = &\int_{\partial B_\varepsilon} w(y)\frac{\partial \mathcal{E}(x-y)}{\partial y_i}N_i(y)d\sigma_y \\ &+\int_{\partial A} w(y)\frac{\partial \mathcal{E}(x-y)}{\partial y_i}N_i(y)d\sigma_y\end{aligned}$$

which, in turn, by the properties of $\mathcal{E}$ and a standard procedure, proves the result in the limit $\varepsilon \to 0$. □

Lemma II.9.2 *Let*

[1] As is well known, this identity is obtained by means of the Gauss divergence theorem which, by Lemma II.4.1, holds for locally Lipschitz domains and smooth functions u, v.

$$\mathcal{I}_1(x) = \int_{\mathbb{R}^n} \frac{dy}{|x-y|^\lambda |y|^\mu}, \quad \lambda < n, \quad \mu < n.$$

Then, if $\lambda + \mu > n$*, there exists a constant* $c = c(\lambda, \mu, n)$ *such that*

$$|\mathcal{I}_1(x)| \le c|x|^{-(\lambda+\mu-n)}.$$

Moreover, let

$$\mathcal{I}_2(x) = \int_{\mathcal{A}(x)-B_1(x)} \frac{dy}{|x-y|^n \log|y|},$$

with

$$\mathcal{A}(x) = \{y \in \mathbb{R}^n : \kappa_1|x| < |y| < \kappa_2|x|\}, \quad \kappa_1 \in (0,1), \ \kappa_2 \in (1,\infty)$$

and x *satisfying*

$$|x| > 2/\kappa, \quad \kappa = \min\{1-\kappa_1, \kappa_2 - 1, \kappa_1^2\}.$$

Then, there exist positive constants c_1, c_2 *depending only on* κ_1*,* κ_2*, and* n *such that*

$$\mathcal{I}_2(x) \le c_1 + c_2(\log|x|)^{-1}.$$

Proof. Setting

$$x' = \frac{x}{|x|}, \quad y' = \frac{y}{|x|},$$

it follows that

$$|\mathcal{I}_1(x)| \le c|x|^{-(\lambda+\mu-n)} \int_{\mathbb{R}^n} \frac{dy'}{|x'-y'|^\lambda |y'|^\mu} \equiv c|x|^{-(\lambda+\mu-n)}\mathcal{I}.$$

To estimate $\mathcal{I}$, we rotate the coordinates in such a way that x' goes into $x_0 = (1, 0, \ldots 0)$ so that

$$\mathcal{I} = \int_{\mathbb{R}^n} \frac{dy'}{|x_0 - y'|^\lambda |y'|^\mu}.$$

Thus, $\mathcal{I}$ is convergent, since $\lambda < n$, $\mu < n$ and $\lambda + \mu > n$, and it is independent of x. The first estimate is therefore proved. To show the second one, we put $|x| = R$ and perform into $\mathcal{I}_2$ the same change of coordinates operated before to obtain

$$\mathcal{I}_2(x) = \int_{\mathcal{A}'-B_{1/R}(x_0)} \frac{dy'}{|x_0 - y'|^n \log(R|y'|)},$$

where

$$\mathcal{A}' = \{y' \in \mathbb{R}^n : \kappa_1 < |y'| < \kappa_2\}.$$

Being $R^{1/2}|y'| \ge \kappa_1/\kappa^{1/2} > 1$, we have $\log(R|y'|) \ge (\log R)/2$ and so

$$\mathcal{I}_2(x) \leq 2(\log|x|)^{-1}\{I_1 + I_2\},$$

with

$$I_1 = \int_{1/R\leq|x_0-y'|\leq 3\kappa/4} |x_0 - y'|^{-n} dy'$$

$$I_2 = \int_{\mathcal{A}'-B_{3\kappa/4}(x_0)} \frac{dy'}{|x_0 - y'|^n}.$$

Clearly,

$$I_2 = b$$

and, since $\kappa < 1$,

$$I_1 \leq a \log|x|,$$

where a and b are independent of x. The lemma is thus completely proved. $\square$

The result just shown will be used in the proof of the following one; see also Padula (1990, Lemma 2.6).

Theorem II.9.1 *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be an exterior domain and let*

$$u \in D^{1,r}(\Omega) \cap D^{1,q}(\Omega), \quad \textit{for some } r \in [1,\infty) \textit{ and some } q \in (n,\infty). \tag{II.9.2}$$

Then, if $r < n$, there exists $u_0 \in \mathbb{R}$ such that

$$\lim_{|x|\to\infty} |u(x) - u_0| = 0 \quad \textit{uniformly.} \tag{II.9.3}$$

The same conclusion holds if (II.9.2) *is replaced by the following one: there exists $u_0 \in \mathbb{R}$ such that*

$$(u - u_0) \in L^s(\Omega) \cap D^{1,q}(\Omega), \quad \textit{for some } s \in [1,\infty) \textit{ and some } q \in (n,\infty). \tag{II.9.4}$$

Moreover, under the assumption (II.9.2)*, with $r = n$, we find that*

$$\lim_{|x|\to\infty} |u(x)|/(\log|x|)^{(n-1)/n} = 0\,, \quad \textit{uniformly.} \tag{II.9.5}$$

Finally, if

$$u \in D^{1,q}(\Omega), \quad \textit{for some } q \in (n,\infty)\,,$$

we have that

$$\lim_{|x|\to\infty} |u(x)|/|x|^{(q-n)/q} = 0\,, \quad \textit{uniformly.} \tag{II.9.6}$$

Proof. We begin to observe that, by density, (II.3.12) continues to hold for all $u \in W^{1,q}(B(x))$, $q > n$, and consequently, by Lemma II.6.1, for all $u \in D^{1,q}(\Omega)$, $q > n$. Thus, we find

$$|v(x)| \leq c\left(\|v\|_{1,B_1(x)} + |v|_{1,q,B_1(x)}\right), \quad \text{for all } v \in D^{1,q}(\Omega),\ q > n, \qquad \text{(II.9.7)}$$

for some c independent of x. Now, under the assumption (II.9.2), by Theorem II.6.1 there exists $u_0 \in \mathbb{R}$ such that

$$\|u - u_0\|_{nr/(n-r)} < \infty. \qquad \text{(II.9.8)}$$

Relation (II.9.3) then follows with the help of (II.9.8), by setting $v = u - u_0$ in (II.9.7), and then by letting $|x| \to \infty$. Under the assumption (II.9.4), we again use (II.9.7) with $v = u - u_0$, and let $|x| \to \infty$ in the resulting inequality. Let us next prove relation (II.9.6). We take R so large that $\exp\sqrt{\ln R} > 2\delta(\Omega^c)$ and set

$$u^{(1)} = (1 - \psi_R)u,$$

where ψ_R is given in (II.7.1). Putting

$$\Omega^\rho = \Omega - B_\rho, \quad \rho = \exp\sqrt{\ln R},$$

by the properties of the function ψ_R (see (II.7.5), (II.7.7)), it follows for sufficiently large R that

$$|u^{(1)}|_{1,q,\Omega^\rho} \leq |u|_{1,q,\Omega^\rho} + c(\ln\ln R)^{-1}. \qquad \text{(II.9.9)}$$

Moreover, $u^{(1)} \in D^{1,q}(\Omega^\rho)$ and, since $u^{(1)}$ vanishes at $\partial\Omega^\rho$, by Theorem II.7.1 there exists a sequence $\{u_s\}_{s\in\mathbb{N}} \subset C_0^\infty(\Omega^\rho)$ converging to $u^{(1)}$ in the norm $|\cdot|_{1,q}$. For fixed s, $s' \in \mathbb{N}$, we apply Lemma II.9.1 to the function $w(x) \equiv h(x)|x|^{-\gamma}$, where $h(x) = u_s(x) - u_{s'}(x)$ and $A \supset \operatorname{supp}(w)$. We thus have

$$|h(x)||x|^{-\gamma} \leq \int_{\Omega^\rho} |\nabla h(y)||y|^{-\gamma}|x-y|^{1-n}dy$$
$$+\gamma\int_{\Omega^\rho} |h(y)||y|^{-1-\gamma}|x-y|^{1-n}dy.$$

Employing the Hölder inequality and (II.6.13) with $x_0 = 0$, there follows

$$|h(x)||x|^{-\gamma} \leq c|h|_{1,q,\Omega^\rho}\left(\int_{\mathbb{R}^n} |y|^{-\gamma q'}|x-y|^{(1-n)q'}dy\right)^{1/q'},$$

where $q' = q/(q-1)$ and $c = c(n,q)$. Taking $\gamma \in (1 - n/q, n - n/q)$ and since $q > n$, we may estimate the integral over $\mathbb{R}^n$ by means of Lemma II.9.2 to deduce

$$|h(x)||x|^{-\gamma} \leq c|h|_{1,q,\Omega^\rho}|x|^{-\gamma+(q-n)/q}.$$

Recalling the definition of the function h and letting $s, s' \to \infty$, from this latter inequality we obtain

$$|u^{(1)}(x)| \leq c|u^{(1)}|_{1,q,\Omega^\rho}|x|^{(q-n)/q}, \quad \text{for all } x \in \Omega^\rho$$

and so, by the properties of ψ_R and by (II.9.9), it follows that

$$|u(x)| \leq c_1 \left(|u|_{1,q,\Omega^\rho} + (\ln\ln R)^{-1}\right)|x|^{(q-n)/q}, \quad \text{for all } x \in \Omega^\rho,$$

which proves (II.9.6). It remains to show (II.9.5). To this end, let $x \in \Omega$ with $|x| = R$, $R > 2\delta(\Omega^c)$ and sufficiently large. Since

$$u \in W^{1,q}(\Omega_{R/2,2R}) \cap W^{1,n}(\Omega_{R/2,2R}),$$

we may use the density Theorem II.3.1 together with Theorem II.3.4 and Theorem II.4.1 to prove the validity of the identity in the statement of Lemma II.9.1 with $A \equiv \Omega_{R/2,2R}$ and $w(y) \equiv u(y)/(\log|y|)^{(n-1)/n}$. We thus obtain for all $x \in \Omega$ with $|x| = R$

$$|u(x)|/(\log|x|)^{(n-1)/n} \leq c(I_1 + I_2 + I_3 + I_4 + I_5 + I_6), \tag{II.9.10}$$

where $c = c(n)$ and

$$I_1 = \int_{\Omega_{R/2,2R}-B_1(x)} |\nabla u(y)|[(\log|y|)^{1/n}|x-y|]^{1-n}dy,$$

$$I_2 = \int_{B_1(x)} |\nabla u(y)|[(\log|y|)^{1/n}|x-y|]^{1-n}dy,$$

$$I_3 = \int_{\Omega_{R/2,2R}-B_1(x)} |u(y)||y|^{-1}(\log|y|)^{1/n-2}|x-y|^{1-n}dy,$$

$$I_4 = \int_{B_1(x)} |u(y)||y|^{-1}(\log|y|)^{1/n-2}|x-y|^{1-n}dy,$$

$$I_5 = \int_{\partial B_{R/2}} |u(y)|(\log|y|)^{(1-n)/n}|x-y|^{-1}d\sigma_y,$$

$$I_6 = \int_{\partial B_{2R}} |u(y)|(\log|y|)^{(1-n)/n}|x-y|^{-1}d\sigma_y.$$

Set

$$\mathcal{I}(x) \equiv \left(\int_{\Omega_{R/2,2R}-B_1(x)} \frac{dy}{|x-y|^n \log|y|}\right)^{(n-1)/n}.$$

The following estimates are a simple consequence of the Hölder inequality:

$$I_1 \le \mathcal{I}(x)|u|_{1,n,\Omega_{R/2,2R}},$$

$$I_2 \le c_1(\log R)^{(1-n)/n}|u|_{1,q,B_1(x)},$$

$$I_3 \le \mathcal{I}(x)\left(\int_{\Omega_{R/2,2R}} \frac{|u(y)|^n}{(|y|\log|y|)^n}dy\right)^{1/n},$$

$$I_4 \le c_2(\log R)^{(1-2n)/n}\left(\int_{B_1(x)} \frac{|u(y)|^q}{|y|^q}dy\right)^{1/q}.$$

Moreover, since

$$|x-y| \ge \begin{cases} R/2 & \text{for } y \in \partial B_{R/2} \\ R & \text{for } y \in \partial B_{2R} \end{cases},$$

it follows that

$$I_5 + I_6 \le c_3(\log R)^{(1-n)/n}\left\{\left(\int_{S^{n-1}} |u(R/2,\omega)|^n d\omega\right)^{1/n} + \left(\int_{S^{n-1}} |u(2R,\omega)|^n d\omega\right)^{1/n}\right\}.$$

By Lemma II.9.2, we have

$$\mathcal{I}(x) \le c_4 + c_5(\log|x|)^{-1} \tag{II.9.11}$$

while, by Exercise II.6.3, given $\varepsilon > 0$ there is a sufficiently large $\overline{R}$ such that for all $R > \overline{R}$ it holds that

$$\int_{S^{n-1}} |u(R/2,\omega)|^n d\omega + \int_{S^{n-1}} |u(2R,\omega)|^n d\omega \le c_6\,\varepsilon\,(\log R)^{n-1}, \tag{II.9.12}$$

and

$$\int_{\Omega_{R/2,2R}} \frac{|u(y)|^n}{(|y|\log|y|)^n}dy \le c_7\varepsilon \int_{R/2}^{2R} (r\log r)^{-1}dr \le c_8\,\varepsilon. \tag{II.9.13}$$

In addition, from (II.9.6), we find

$$\int_{B_1(x)} \frac{|u(y)|^q}{|y|^q}dy \le c_9 R^{-n}. \tag{II.9.14}$$

Since, clearly, as $R \to \infty$,

$$|u|_{1,n,\Omega_{R/2,2R}}, \quad |u|_{1,q,B_1} = o(1), \tag{II.9.15}$$

in view of (II.9.11)–(II.9.15) we deduce in the limit $R \to \infty$

$$\sum_{i=1}^{6} I_i = o(1) \tag{II.9.16}$$

and (II.9.5) follows from (II.9.10) and (II.9.16). The theorem is therefore completely proved. □

Remark II.9.1 The result just shown applies, with no change to domains Ω that possess an extension property of the type specified in Remark II.6.4, such as a half-space. ■

II.10 Boundary Trace of Functions from $D^{m,q}(\mathbb{R}^n_+)$

Our next objective is to investigate the trace space at the boundary of a function $u \in D^{m,q}(\Omega)$, for $\Omega \equiv \mathbb{R}^n_+$. Actually, if Ω is an exterior domain, there is nothing to add to what was said in Section II.4, since, as shown in Lemma II.6.1, if Ω is locally Lipschitz then $u \in W^{m,q}(\Omega_R)$. On the other hand, if $u \in D^{m,q}(\mathbb{R}^n_+)$ then $u \in W^{m,q}(C)$, for any cube $C \subset \mathbb{R}^n_+$, and therefore, by the results of Section II.4, u possesses a well-defined trace $\Gamma_{(m)}(u)$ at the plane $\Sigma = \{x \in \mathbb{R}^n : x_n = 0\}$ that belongs to the trace space $\mathcal{W}_{m,q}(\Sigma')$, for every *bounded* portion Σ' of Σ. However, from those results we cannot draw any conclusion concerning the finiteness of the norms of $\Gamma_m(u)$ on the *whole* of Σ. Nevertheless, such global information is of primary importance in the resolution of nonhomogeneous boundary-value problems.

A detailed investigation of the properties of the traces on Σ of functions belonging to the spaces $D^{m,q}(\mathbb{R}^n_+)$ has been performed by Kudrjavcev (1966a, 1966b). Here we shall describe some of his results in the case where $m = 1$, since this is the only case we need to consider in the applications. The interested reader is referred to Remark II.10.2 and to the work of Kudrjavcev (1966b, Theorems 2.4′ and 2.7) for generalizations to the case where $m > 1$.

For a function $u \in D^{1,q}(\mathbb{R}^n_+)$, we shall denote throughout by $\overline{u}$ its trace at Σ. From Theorem II.4.1 we derive, in particular, for any bounded (measurable) $\Sigma' \subset \Sigma$,

$$\|\overline{u}\|_{q,\Sigma'} \le c\left(|u|_{1,q,\mathbb{R}^n_+} + \|u\|_{q,B}\right), \tag{II.10.1}$$

where $c = c(\Sigma', n, q, B)$ and B any bounded, locally Lipschitz domain of $\mathbb{R}^n_+$ with $\overline{B} \supset \Sigma'$. Let σ be a non-negative, measurable function in Σ. By the symbol

$$L^q(\Sigma, \sigma), \quad 1 \le q \le \infty,$$

we denote the space of (equivalence classes of) real functions w on Σ that are L^q-summable in with the "weight" σ, namely,

$$\|\sigma w\|_q < \infty.$$

We have

Theorem II.10.1 *Let $\Sigma = \{x \in \mathbb{R}^n : x_n = 0\}$ and $x' = (x_1, \ldots, x_{n-1})$. Then, for any $u \in D^{1,q}(\mathbb{R}^n_+)$ the trace $\overline{u}$ of u at Σ satisfies*

$$\overline{u} \in L^q(\Sigma, \sigma_1), \quad \sigma_1 = (1 + |x'|)^{(1-n)/q - \varepsilon_1},$$

where ε_1 is an arbitrary positive number, and the following inequality holds:

$$\|\sigma_1 \overline{u}\|_{q,\Sigma} \leq c_1 \left(|u|_{1,q,\mathbb{R}^n_+} + \|u\|_{q,B_+} \right),$$

with $c_1 = c_1(n, q, \varepsilon_1)$ and $B_+ = B_1 \cap \mathbb{R}^n_+$. Moreover, if $1 \leq q < n$, we have

$$\overline{u} - u_0 \in L^q(\Sigma, \sigma_2), \quad \sigma_2 = (1 + |x'|)^{(1-q)/q - \varepsilon_2},$$

where u_0 is the constant associated to u by Theorem II.6.3 *and ε_2 is an arbitrary positive number, and the following inequality holds:*

$$\|\sigma_2(\overline{u} - u_0)\|_{q,\Sigma} \leq c_2 |u|_{1,q,\mathbb{R}^n_+},$$

with $c_2 = c_2(n, q, \varepsilon_2)$.

Proof. The proof of the first part of the theorem is found in Kudrjavcev (1966b, Theorem 2.3′) and it will be omitted here. The second part can be obtained by coupling Kudrjavcev's technique with the results of Theorem II.6.3, as we are going to show. For simplicity, we shall consider the case where $n = 2$, leaving to the reader the simple task of establishing the result for $n \geq 3$. Setting

$$\overline{w} = \overline{u} - u_0,$$

we have to prove the following inequality:

$$\int_{-\infty}^{\infty} \sigma_2(x_1)^q |\overline{w}(x_1)|^q dx_1 \leq c_2^q |u|^q_{1,q,\mathbb{R}^2_+}, \quad \sigma_2(x_1) = (1 + |x_1|)^{(1-q)/q - \varepsilon_2}. \tag{II.10.2}$$

Since, by Theorem II.6.3,

$$\begin{gathered} (u - u_0) \in L^{2q/(2-q)}(\mathbb{R}^2_+), \\ \|u - u_0\|_{2q/(2-q)} \leq \gamma_2 |u|_{1,q,\mathbb{R}^2_+}, \end{gathered} \tag{II.10.3}$$

from (II.10.1) we find

$$\int_{-1}^{1} \sigma_2(x_1)^q |\overline{w}(x_1)|^q dx_1 \leq c^q |u|^q_{1,q,\mathbb{R}^2_+},$$

and so to show (II.10.2) it suffices to show

$$\int_{1}^{\infty} \sigma_2(x_1)^q |\overline{w}(x_1)|^q dx_1, \quad \int_{-\infty}^{-1} \sigma_2(x_1)^q |\overline{w}(x_1)|^q dx_1 \leq c_3 |u|^q_{1,q,\mathbb{R}^2_+}. \tag{II.10.4}$$

Let us consider the first integral in (II.10.4). In $\mathbb{R}^2_+$ we introduce a polar coordinate system $\rho \in (0,\infty)$, $\theta \in [0,\pi]$ with θ the angle formed by ρ with the positive x_1-axis. Since

$$x_1 = \rho \cos\theta,$$

$$x_2 = \rho \sin\theta,$$

we have

$$\int_1^\infty \sigma_2(x_1)^q |\overline{w}(x_1)|^q dx_1 = \int_1^\infty \sigma_2(\rho)^q |\overline{w}(\rho,0)|^q d\rho. \tag{II.10.5}$$

Setting

$$w = u - u_0,$$

for $x_1 \geq 1$,

$$\overline{w}(x_1) \equiv \overline{w}(\rho,0) = w(\rho,\theta) - \int_0^\theta \frac{\partial u}{\partial \tau}(\rho,\tau)d\tau.$$

Taking the modulus of both sides of this identity, raising them to the qth power, using (II.3.3) and the Hölder inequality, we find

$$|\overline{w}(\rho,0)|^q \leq c_1 \left(|w(\rho,\theta)|^q + \int_0^\theta \left|\frac{\partial u}{\partial \tau}(\rho,\tau)\right|^q d\tau \right). \tag{II.10.6}$$

Observing that

$$\left|\frac{\partial u}{\partial \theta}(\rho,\theta)\right| \leq \rho |\nabla u|,$$

from (II.10.6) we derive, for all $\alpha \geq 0$,

$$\begin{aligned}\int_1^\infty \frac{|\overline{w}(\rho,0)|^q}{\rho^{\alpha q}} d\rho \leq c_2 \bigg(&\int_1^\infty \int_0^\pi \frac{|w(\rho,\theta)|^q}{\rho^{\alpha q+1}} \rho \, d\rho \, d\theta \\ &+ \int_1^\infty \int_0^\pi \frac{|\nabla u(\rho,\theta)|^q}{\rho^{q(\alpha-1)+1}} \rho \, d\rho \, d\theta \bigg).\end{aligned} \tag{II.10.7}$$

Taking

$$\alpha > 1 - 1/q, \tag{II.10.8}$$

we have for $\rho \geq 1$

$$\rho^{q(\alpha-1)+1} \geq 1. \tag{II.10.9}$$

Further, from (II.10.3) and (II.10.8)

$$\begin{aligned}\int_1^\infty \int_0^\pi \frac{|w(\rho,\theta)|^q}{\rho^{\alpha q+1}} \rho d\rho d\theta &\leq \left(\pi \int_1^\infty \rho^{1-2(\alpha q+1)/q} d\rho \right)^{q/2} \\ &\quad \times \left(\int_{\mathbb{R}^2_+} |w|^{2q/(2-q)} \right)^{(2-q)/q} \\ &\leq c_5 \, |u|^q_{1,q,\mathbb{R}^2_+}.\end{aligned} \tag{II.10.10}$$

Therefore, the first relation in (II.10.4) follows from (II.10.5), (II.10.7), (II.10.9), and (II.10.10). To recover the second one, it is enough to observe that, for $x_1 \leq -1$,

$$\overline{w}(x_1) \equiv \overline{w}(\rho, \pi) = w(\rho, \theta) + \int_\theta^\pi \frac{\partial u(\rho, \tau)}{\partial \tau} d\tau,$$

and to proceed as in the previous case. The theorem is thus completely proved. □

Remark II.10.1 Theorem II.10.1 tells us, in particular, that if $1 \leq q < n$, $\overline{u}$ must tend to the constant u_0 at large distances on Σ, in the sense that for at least a sequence of radii $\{R_m\}$,

$$\lim_{R_m \to \infty} \int_{S^{n-2}} |u(R_m, \omega) - u_0| d\omega = 0,$$

where (R, ω) denotes a system of polar coordinate on Σ. On the other hand, if $q \geq n$, u may even grow at large distance on Σ. ■

Remark II.10.2 We notice, in passing, that Theorem II.10.1 admits of an obvious extension to the case where $m > 1$, in the sense that it selects the weighted L^q-space to which the trace $\overline{u}_\alpha \equiv D^\alpha u$ at Σ, $|\alpha| = m - 1$, of $u \in D^{m,q}(\mathbb{R}^n_+)$ must belong. In particular, if $mq < n$, in the light of Theorem II.6.4, u can be modified by the addition of a suitable polynomial $\mathcal{P}$ in such a way that $\mathsf{u} \equiv u - \mathcal{P}$ and all derivatives of u up to the order $m - 1$ included tend to zero on Σ in the way specified in Remark II.10.1. ■

A weighted space of the type $L^q(\Sigma, \sigma)$, however, does not coincide with the "trace space" of functions from $D^{1,q}(\mathbb{R}^n_+)$. This latter is, in fact, more restricted. To characterize such a space we set, as in the case of a bounded domain,

$$\langle\langle \overline{u} \rangle\rangle_{1-1/q,q} \equiv \left(\int_\Sigma \int_\Sigma \frac{|\overline{u}(x) - \overline{u}(y)|^q}{|x - y|^{n-2+q}} dx dy \right)^{1/q} \tag{II.10.11}$$

and denote by $D^{1-1/q,q}(\Sigma)$ the space of (equivalence classes of) real functions for which the functional (II.10.11) is finite. As in Section II.4, one can show that, provided we identify two functions if they differ by a constant, (II.10.11) defines a norm in $D^{1-1/q,q}(\Sigma)$ and that $D^{1-1/q,q}(\Sigma)$ is complete in this norm.

Exercise II.10.1 (Miranda 1978, Teorema 59.II). Show that

$$u \in W^{1,q}(\Sigma), \quad \text{implies} \quad u \in D^{1-1/q,q}(\Sigma).$$

The following theorem holds, (Kudrjavcev 1966b, Theorems 2.4′ and 2.7 and Corollary 1).

Theorem II.10.2 *Let Σ be as in* Theorem II.10.1 *and let $u \in D^{1,q}(\mathbb{R}^n_+)$, $1 < q < \infty$. Then the trace $\overline{u}$ of u at Σ belongs to $D^{1-1/q,q}(\Sigma)$ and, further,*

$$\langle\langle \overline{u} \rangle\rangle_{1-1/q,q} \leq c_1 |u|_{1,q}$$

with $c_1 = c_1(n,q)$. Conversely, given $\overline{u} \in D^{1-1/q,q}(\Sigma)$, $1 < q < \infty$, there exists $u \in D^{1,q}(\mathbb{R}^n_+)$ such that $\overline{u}$ is the trace of u at Σ and, further,

$$|u|_{1,q} \leq c_2 \, \langle\langle \overline{u} \rangle\rangle_{1-1/q,q},$$

with $c_2 = c_2(n,q)$.

II.11 Some Integral Transforms and Related Inequalities

By *integral transform with kernel K of a function f*, we mean the function Ψ defined by

$$\Psi(x) = \int_\Omega K(x,y) f(y) dy. \tag{II.11.1}$$

Our objective in this section is to present some basic inequalities relating Ψ and f, under different assumptions on the kernel. We shall first consider the situation in which

$$K(x,y) = K(x-y),$$

where $K(\xi)$ is defined in the whole of $\mathbb{R}^n$. In this case, the transform (II.11.1) with $\Omega \equiv \mathbb{R}^n$ is called a *convolution*, and it is also denoted by $K * f$. An example of convolution is the regularizer of f, which we already introduced in Section II.2. For these transforms we have the following classical result due to Young (see, e.g., Miranda 1978, Teorema 10.I).

Theorem II.11.1 *Let*

$$K \in L^s(\mathbb{R}^n), \quad 1 \leq s < \infty.$$

If

$$f \in L^q(\mathbb{R}^n), \quad 1 \leq q \leq \infty, \quad 1/q \geq 1 - 1/s,$$

then

$$K * f \in L^r(\mathbb{R}^n), \quad 1/r = 1/s + 1/q - 1,$$

and the following inequality holds:

$$\|K * f\|_r \leq \|K\|_s \|f\|_q. \tag{II.11.2}$$

Exercise II.11.1 Prove inequality (II.11.2) for the case $q = 1$. *Hint*: Use the generalized Minkowski inequality (II.2.8).

Another class of transforms that will be frequently considered is that defined by kernels K of the form

$$K(x,y) = \frac{k(x,y)}{|y|^\lambda}, \quad \lambda > 0, \quad y \in \Omega, \tag{II.11.3}$$

where $k(x,y)$ is a given regular function. If $0 < \lambda < n$ and $k(x,y) \equiv 1$, the kernel (II.11.3) is referred to as *weakly singular* and the corresponding transform (II.11.1) is called the *Riesz potential.* If $\lambda = n$ and $k(x,y)$ is suitable (see (II.11.15)–(II.11.17)), the kernel and the associated transform are called *singular.* The study in Lebesgue spaces L^s of Riesz potentials finds a fundamental contribution in the celebrated paper of Sobolev (1938) (see Theorem II.11.3), while that related to (multidimensional) singular kernels traces back to the work of Calderón and Zygmund (1956) (see Theorem II.11.4).

When Ω is bounded and K is weakly singular one can easily show elementary estimates for $\Psi = K * f$ in terms of f. For example, if

$$\lambda < n(1 - 1/q)$$

one has the inequality

$$\sup_{x\in\Omega} |\Psi(x)| \le c\|f\|_q \tag{II.11.4}$$

with

$$c = \left(\frac{1}{n - \lambda q'}\right)^{1/q'} \omega_n^{1/q'} \delta(\Omega)^{n/q' - \lambda}. \tag{II.11.5}$$

To show this, it suffices to observe that for all $r > 0$ and $\lambda r < n$,

$$\left(\int_{|x-y|\le R} |x-y|^{-\lambda r} dy\right)^{1/r} \le \left(\frac{1}{n-\lambda r}\right)^{1/r} \omega_n^{1/r} R^{n/r-\lambda}. \tag{II.11.6}$$

Thus, (II.11.4) and (II.11.5) follow from (II.11.1), (II.11.3), (II.11.6), and the Hölder inequality. Actually, one can prove an estimate stronger than (II.11.4) under the same assumption on λ, n, and q. In fact, from (II.11.3) with $k(x,y) = 1$, by the mean value theorem it follows that

$$|K(x-y) - K(z-y)| \le \lambda |x-z| d(y)^{-(\lambda+1)},$$

where $d(y)$ is the distance of y from the segment s with endpoints x and z. Setting $\sigma = |x-z|$ and employing this last inequality, from (II.11.1) we deduce

$$\begin{aligned}|\Psi(x) - \Psi(z)| \le &\int_{|x-y|<2\sigma} |f(y)||x-y|^{-\lambda} dy + \int_{|z-y|<2\sigma} |f(y)||z-y|^{-\lambda} dy \\ &+ \lambda\sigma \int_{\Omega\cap\{|x_0-y|>\sigma\}} |f(y)| d(y)^{-(\lambda+1)} dy\end{aligned} \tag{II.11.7}$$

with x_0 the midpoint of s. Since $d \geq \sigma/2$, by Carnot's theorem it easily follows that $2d \geq |x - x_0|$. Therefore, assuming $\lambda < n(1 - 1/q)$ and employing the Hölder inequality, the last term in (II.11.7) can be increased by

$$C_1 \left(\sigma + \sigma^{n(1-1/q)-\lambda}\right) \|f\|_q \,, \tag{II.11.8}$$

where $C_1 = C_1(\delta(\Omega), n, q, \lambda)$. On the other hand, by an easy calculation that makes use of (II.11.6) and the Hölder inequality, we show that the first two integrals in (II.11.7) can be dominated by

$$C_2 \, \sigma^{n(1-1/q)-\lambda} \|f\|_q,$$

where $C_2 = C_2(n, \lambda)$. Thus, this latter relation along with (II.11.7) and (II.11.8) furnishes

$$|\Psi(x) - \Psi(z)| \leq C \left(\sigma + \sigma^{n(1-1/q)-\lambda}\right) \|f\|_q,$$

where $C = 2\max(C_1, C_2)$. Still retaining the assumption that Ω is bounded, we shall now discuss the case where $\lambda = n(1 - 1/q)$. We set

$$\widetilde{K}(x-y) = \begin{cases} |x-y|^{-\lambda} & \text{if } x, y \in \Omega \\ 0 & \text{if } x, y \notin \Omega \,. \end{cases}$$

Clearly,

$$\Psi(x) = \int_\Omega |x-y|^{-\lambda} f(y) dy = \int_{\mathbb{R}^n} \widetilde{K}(x-y) f(y) dy,$$

and so, by noticing that

$$\widetilde{K} \in L^s(\mathbb{R}^n), \quad \text{for all } s < n/\lambda, \tag{II.11.9}$$

from Young's Theorem II.11.1 it follows that if $f \in L^q(\Omega)$ then

$$\Psi \in L^r(\Omega), \quad 1/r = 1/s + 1/q - 1 \tag{II.11.10}$$

and that the following inequality holds:

$$\|\Psi\|_r \leq c\|f\|_q.$$

Taking into account (II.11.9) and that $\lambda = n(1 - 1/q)$, from (II.11.10) we conclude that

$$\Psi \in L^r(\Omega), \quad \text{for all } r \in [1, \infty).$$

The results established so far are collected in

Theorem II.11.2 *Assume Ω bounded, K weakly singular, and $f \in L^q(\Omega)$, $1 < q < \infty$. Then if $\lambda < n(1 - 1/q)$, the integral transform Ψ defined by*

(II.11.1) *belongs to* $C^{0,\mu}(\overline{\Omega})$ *where* $\mu = \min\{1, n(1-1/q)-\lambda\}$ *and the following estimate holds:*

$$\|\Psi\|_{C^{0,\mu}} \le C_1 \|f\|_q, \tag{II.11.11}$$

with $C_1 = C_1(\delta(\Omega), n, q, \lambda)$. *Moreover, if* $\lambda = n(1-1/q)$, *then* $\Psi \in L^r(\Omega)$ *for all* $r \in [1, \infty)$, *and the following estimate holds:*

$$\|\Psi\|_r \le C_2 \|f\|_q, \tag{II.11.12}$$

with $C_2 = C_2(\delta(\Omega), n, q, \lambda)$.

The complementary situation $\lambda > n(1-1/q)$ is considered in Sobolev's theorem which, in addition, does not require the boundedness of Ω. Precisely, we have (Sobolev 1938; for a simpler proof see Stein 1970, Chapter V)

Theorem II.11.3 *Assume* $f \in L^q(\mathbb{R}^n)$, $1 < q < \infty$, *and* K *weakly singular. Then, if* $\lambda > n(1-1/q)$, *the integral transform* Ψ *defined by* (II.11.1) *with* $\Omega \equiv \mathbb{R}^n$ *belongs to* $L^s(\mathbb{R}^n)$, *where* $1/s = \lambda/n + 1/q - 1$. *Moreover, we have*

$$\|\Psi\|_s \le C \|f\|_q \tag{II.11.13}$$

with $C = C(q, n, \lambda)$.

Remark II.11.1 By means of simple counterexamples one shows that the Sobolev theorem fails either when $q = 1$ or when $s = \infty$ (see Stein 1970, p.119).

Some interesting observations and consequences related to Theorem II.5.1-Theorem II.5.4 are left to the reader in the following exercises. ■

Exercise II.11.2 Show that if (II.11.13) holds, necessarily $1/s = \lambda/n + 1/q - 1$. *Hint*: Use the homogeneity of the Riesz potential.

Exercise II.11.3 For $f \in C_0^\infty(\mathbb{R}^n)$, set $u(x) = (\mathcal{E} * f)(x)$ where $\mathcal{E}$ is the fundamental solution of Laplace's equation (see (II.9.1)). Verify that u is a C^∞ solution of the Poisson equation $\Delta u = f$ in $\mathbb{R}^n$. Moreover, use the Sobolev theorem to show

$$\|\nabla u\|_{nq/(n-q)} \le c\|f\|_q, \quad 1 < q < n.$$

Exercise II.11.4 Assume $u \in W_0^{1,q}(\mathbb{R}^n)$, $1 < q < \infty$. Starting from the representation given in Lemma II.9.1, prove the following assertions:

(i) If $q < n$, then $u \in L^{nq/(n-q)}(\mathbb{R}^n)$ and $\|u\|_{nq/(n-q)} \le \gamma \|\nabla u\|_q$;

Hint: Use Theorem II.11.3. Notice that, without using the Sobolev theorem, (i) is obtained directly from Lemma II.3.2 in a much more elementary way (see (2.6)) and with the following advantages: (a) the case $q = 1$ is included; (b) an explicit estimate of the constant γ can be given.

(ii) If $q = n$, then $u \in L^r(\Omega)$, for all $r \in [n, \infty)$ and for any compact domain Ω. *Hint*: Use Theorem II.11.2.

(iii) If $q > n$, then $u \in C^{0,\mu}(\overline{\Omega})$, $\mu = 1 - n/q$, for any compact domain Ω. *Hint*: Use Theorem II.11.2.

Exercise II.11.5 Let Ω be bounded. Show that every function from $W_0^{1,q}(\Omega)$, $q > n$, satisfies the inequality

$$\|u\|_C \leq c[\delta(\Omega)]^{1-n/q}\|\nabla u\|_q, \tag{II.11.14}$$

with $c = c(n, q)$. *Hint*: Use the representation formula of Lemma II.9.1 together with relations (II.11.4) and (II.11.5).

We shall now consider the case of singular kernels. We say that a kernel of the form (II.11.3) with $x \in \Omega$, $y \in \mathbb{R}^n - \{0\}$ and $\lambda = n$ is *singular* if and only if

(i) For any admissible x, y and every $\alpha > 0$

$$k(x, y) = k(x, \alpha y); \tag{II.11.15}$$

(ii) For every $x \in \Omega$, $k(x, y)$ is integrable on the sphere $|y| = 1$ and

$$\int_{|y|=1} k(x, y)dy = 0; \tag{II.11.16}$$

(iii) There exists $C > 0$, such that[1]

$$\operatorname*{ess\ sup}_{x\in\Omega;|y|=1} |k(x, y)| \leq C\,. \tag{II.11.17}$$

Exercise II.11.6 Show that (II.11.16) is equivalent to the following:

$$\int_{r_1\leq|y|\leq r_2} K(x, y)dy = 0, \tag{II.11.18}$$

for every x and $r_2 > r_1 > 0$.

Condition (II.11.18) allows us to recognize a noteworthy class of singular kernels. Precisely, we have the following simple but useful result, due to L. Bers and M. Schechter, which we state in the form of a lemma (see Bers, John, & Schechter 1964, p. 223).

Lemma II.11.1 *Let $M(x, y)$ be a function on $\Omega \times (\mathbb{R}^n - \{0\})$, differentiable in y and homogeneous of order $1 - n$ with respect to y, that is,*

$$M(x, \alpha y) = \alpha^{1-n}M(x, y), \quad \alpha > 0.$$

[1] This assumption can be weakened; see Calderón & Zygmund (1956, Theorem 2(ii)). However, a weaker assumption would be irrelevant to our purposes.

Assume further that $M_i(x,y) \equiv \partial M(x,y)/\partial y_i$ *satisfies, with some* $C > 0$, *independent of* x,

$$\operatorname*{ess\,sup}_{|y|=1} |M_i(x,y)| \leq C\,.$$

Then $M_i(x,y)$ *is a singular kernel.*

Proof. For all $x \in \Omega$ we have

$$\int_{r_1 \leq |y| \leq r_2} M_i(x,y)dy = \int_{|\eta|=r_2} M(x,\eta)(\eta_i/r_2)d\sigma_\eta$$
$$- \int_{|\eta|=r_1} M(x,\eta)(\eta_i/r_1)d\sigma_\eta,$$

so that (II.11.18) follows by homogeneity. Therefore, setting

$$k(x,y) = M_i(x,y)|y|^n,$$

by assumption and Exercise II.11.6 we conclude that $M_i(x,y) = k(x,y)|y|^{-n}$ is a singular kernel. □

Exercise II.11.7 Let $\mathcal{E}$ be the fundamental solution to Laplace's equation. Show that $D_{ij}\mathcal{E}(x)$ is a singular kernel.

For integral transforms defined by singular kernels we have the following fundamental result due to Calderón & Zygmund (1956, Theorem 2).

Theorem II.11.4 *Assume* $K(x,y)$ *is a singular kernel and let*

$$N(x,y) \equiv K(x,x-y).$$

Then, if

$$f \in L^q(\mathbb{R}^n), \quad 1 < q < \infty,$$

the P.V. *convolution integral*

$$\Psi(x) = \lim_{\epsilon \to 0} \int_{|x-y| \geq \epsilon} N(x,y)f(y)dy \tag{II.11.19}$$

exists for almost all $x \in \Omega$. *Moreover,*

$$\Psi \in L^q(\mathbb{R}^n)$$

and the following inequality holds:

$$\|\Psi\|_q \leq c\|f\|_q. \tag{II.11.20}$$

Exercise II.11.8 Assume K given by (II.11.3), with $k(x,y)$ bounded and $\lambda = n$. Show that, if $f \in C_0^1(\mathbb{R}^n)$, the limit (II.11.19) exists if and only if $k(x,y)$ satisfies condition (II.11.16). *Hint*: Use the identity $(a > \epsilon > 0)$

$$\begin{aligned}\int_{|x-y|\geq\epsilon} N(x,y)f(y)dy = &\int_{|x-y|\geq a} N(x,y)f(y)\\ &+\int_{\epsilon<|x-y|<a}[f(y)-f(x)]N(x,y)dy\\ &+f(x)\int_{\epsilon<|x-y|<a} N(x,y)dy.\end{aligned}$$

Remark II.11.2 Sometimes it is useful to know more about the constant c in (II.11.19) and, particularly, about the way in which it depends on q and k. Here we recall some estimate due to Stein (1970, Chapter II) and to Calderón and Zygmund (1957, §5). Specifically, as far as the dependence on q, one can show:

$$c \leq \begin{cases} c_1/(q-1) & \text{if } 1 < q \leq 2 \\ c_1 q & \text{if } q \geq 2, \end{cases}$$

with $c_1 = c_1(k)$. Likewise, if $A > 0$ is a constant such that

$$\sup_{x\in\Omega,\ |y|=1} |k(x,y)| \leq A,$$

then one has

$$c \leq c_2\, A, \quad c_2 = c_2(q).$$

■

Two important consequences of the Calderón–Zygmund theorem will be considered. The first one is due to Stein (1957) and is contained in the following.

Theorem II.11.5 *Let the assumptions of* Theorem II.11.4 *be satisfied, and suppose, in addition*

$$f(x)|x|^\beta \in L^q(\mathbb{R}^n), \quad \beta \in (-n/q, n(1-1/q)),$$

and that $|k(x,y)| \leq C$, *for some* C *independent of* x *and* y. *Then,*

$$\Psi(x)|x|^\beta \in L^q(\mathbb{R}^n)$$

and the following inequality holds

$$\|\Psi(x)|x|^\beta\|_q \leq c_1 C \|f(x)|x|^\beta\|_q, \tag{II.11.21}$$

where $c_1 = c_1(n,q,\beta)$.

The second consequence is a well-known result of Agmon, Douglis, & Nirenberg (1959, Theorem 3.3), which we are now going to state.

Theorem II.11.6 *Let*

$$K(x', x_n) = \frac{\widetilde{\omega}(x'/|x|, x_n/|x|)}{|x|^{n-1}} \quad x' = (x_1, \ldots, x_{n-1}).$$

Assume that $D_i K$, $i = 1, \ldots, n$, and $D_n^2 K$ are continuous in $\mathbb{R}^n_+$ and bounded in $\mathbb{R}^n_+ \cap S^{n-1}$ by a positive constant κ. Assume further

$$\int_{|x'|=1} \widetilde{\omega}(x', 0) dx' = 0. \tag{II.11.22}$$

Then, setting $\Sigma = \{x \in \mathbb{R}^n : x_n = 0\}$, given

$$\phi \in L^q(\Sigma), \quad \text{with } \langle\langle \phi \rangle\rangle_{1-1/q,q} \text{ finite,}$$

the integral transform

$$u(x', x_n) = \int_\Sigma K(x' - y', x_n)\phi(y')dy' \tag{II.11.23}$$

belongs to $L^q(\mathbb{R}^n_+)$ and the following inequality holds:

$$|u|_{1,q} \le c\kappa \langle\langle\, \phi \,\rangle\rangle_{1-1/q,q}, \tag{II.11.24}$$

with $c = c(n, q)$.

Theorem II.11.4 and Theorem II.11.6 play a fundamental role in the L^q-theory of elliptic partial differential equations, mainly in deriving a priori estimates for solutions (see, e.g., Agmon, Douglis, & Nirenberg 1959). In the following exercises, we shall propose very simple applications of them to the Poisson equation in $\mathbb{R}^n$ and to the Dirichlet problem for the Poisson equation in $\mathbb{R}^n_+$. Other more relevant applications will be derived, along the same lines as those that follow, in Chapter IV, in the context of steady slow motions of a viscous incompressible fluid (Stokes problem).

Exercise II.11.9 For the problem $\Delta u = f$ in $\mathbb{R}^n$ show that there is a solution u such that

(i) If $f \in W^{m,q}(\mathbb{R}^n)$, $m \ge 0$, $1 < q < \infty$, then $u \in \cap_{k=0}^m D^{k+2}(\mathbb{R}^n)$ and the following inequality holds:

$$|u|_{k+2,q} \le c_1 \|f\|_{k,q}, \quad k = 0, 1, ..., m, \ c_1(n, q, k);$$

(ii) If $f \in D_0^{-1,q}(\mathbb{R}^n)$, $m \ge 0$, $1 < q < \infty$, then $u \in D_0^{1,q}(\mathbb{R}^n)$ and the following inequality holds:

$$|u|_{1,q} \le c_2 |f|_{-1,q}, \quad c_2(n, q, k).$$

Hint: Take $f \in C_0^\infty(\mathbb{R}^n)$. Then a solution is given by $u = \mathcal{E} * f$ (see Exercise II.11.3). To show (i), use Theorem II.11.4 and Exercise II.11.7. To show (ii), observe that, for any $\varphi \in L^{q'}(B_r)$ and $i = 1, ..., n$,

$$(D_i u, \varphi) = \int_{\mathbb{R}^n} f(y)\phi(y)dy, \quad \phi = (D_i\mathcal{E}) * \varphi,$$

and that, by Theorem II.11.4,

$$|\phi|_{1,q'} \le c\|\varphi\|_{q',B_r}$$

with c independent of r. Employ, finally, the results of Exercise II.3.4 and Theorem II.8.1.

Exercise II.11.10 It is well known that the function (*Poisson integral*)

$$u(x) = 2\int_\Sigma \phi(y)\frac{\partial \mathcal{E}}{\partial y_n}dy,$$

with $\Sigma = \{x \in \mathbb{R}^n : x_n = 0\}$, $\mathcal{E}$ given in (7.1) and $\phi \in C^m(\overline{\Sigma})$, $m \ge 0$, is a smooth solution to the Dirichlet problem in the half-space:

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \mathbb{R}^n_+, \quad n \ge 2 \\ u &= \phi \quad \text{at } \Sigma \end{aligned} \tag{II.11.25}$$

(see, e.g., Sobolev 1964, Lecture 13). Use Theorem II.11.6 to show that if

$$\phi \in W^{m,q}(\Sigma) \quad \text{and} \quad \sum_{|k|=m} \langle\langle\, D^k\phi\,\rangle\rangle_{1-1/q,q} < \infty, \quad 1 < q < \infty,$$

then

$$|u|_{s+1,q} \le c\sum_{|k|=s} \langle\langle\, D^k\phi\,\rangle\rangle_{1-1/q,q}, \quad \text{for all } s = 0, 1, ..., m,$$

with $c = c(n, q, s)$.

Uniqueness of solutions determined in the preceding exercises can be easily studied by means of the following result, which the reader is invited to prove.

Exercise II.11.11 Let H be harmonic in the whole of $\mathbb{R}^n$. Assume either

(i) $H = \displaystyle\sum_{i=1}^N H_i$, $N \ge 1$, where $\displaystyle\int_{B^\rho} \frac{|H_i(x)|^{q_i}}{(1+|x|)^{\lambda_i}} < \infty$, for some $q_i \in (1,\infty)$, $\rho > 0$, and $\lambda_i \in [0, n]$;

or

(ii) $\displaystyle\lim_{|x|\to\infty} H(x) = 0$.

Show $H \equiv 0$. *Hint*: By the mean value theorem, we have, for each $x \in \mathbb{R}^n$,

$$|H(x)| \le (n\omega_n)^{-1}\int_{S^{n-1}} |H(R,\omega)|d\omega, \quad R = |x-y|.$$

Remark II.11.3 In virtue of this latter result it follows that solutions determined in Exercise II.11.9(i) are unique in $\dot{D}^{2,q}(\mathbb{R}^n)$, while those in (ii) are unique in $D_0^{1,q}(\mathbb{R}^n)$. As far as solutions considered in Exercise II.11.10, their uniqueness is likewise discussed, since if u solves (II.11.25) with $\phi \equiv 0$, then the function

$$\widetilde{u}(x) = \begin{cases} u(x_1, \ldots, x_n) & \text{if } x_n > 0 \\ \\ -u(x_1, \ldots, -x_n) & \text{if } x_n \leq 0 \end{cases}$$

is harmonic (and hence smooth; Weyl 1940, Simader 1992) throughout $\mathbb{R}^n$, including $x_n = 0$; see, for instance, Sobolev (1964, Lecture 13). ■

II.12 Notes for the Chapter

Section II.1. A similar (but different in details) proof of Lemma II.1.3 can be found in Erig (1982, Lemma 5.3).

Section II.3. Inequality (II.3.9) was derived by Ladyzhenskaya (1959a) with a larger value of the constant. In this respect, see also Serrin (1963).

Extensions of Lemma II.3.3 to domains with a (sufficiently smooth) bounded boundary can be found in Friedman (1969, Theorem 10.1) for bounded domains, and in Crispo & Maremonti (2004) for exterior domains.

Sequence of functions like that employed in Exercise II.3.9 can also be used to find the best exponents (for fixed dimension) in certain inequalities relating surface and volume integrals, of the type described in Section II.4 (Galdi, Payne, Proctor, & Straughan 1987).

Section II.4. The way of introducing trace inequalities through star-shaped domains is an intrinsic treatment that does not make a *direct* use of the definition of surface integral by means of local representation of the boundary. For this latter approach see, e.g., Nečas (1967, Chapitre 2 Théorème 4.2) and Adams (1975, Chapter 5 Theorem 5.22).

The constant C in Theorem II.4.1 can be simply estimated if the shape of Ω is particular; in this regard see Galdi, Payne, Proctor, & Straughan (1987).

Section II.5. As already remarked, inequality (II.5.1) fails if Ω is not contained in some layer L_d; see Exercise II.5.1. However, in this latter case, (II.5.1) can be replaced by "weighted" inequalities such as (II.6.10), (II.6.13), and (II.6.14). Furthermore, the choice of the "weight" can be suitably related to the "geometry" of Ω at infinity. For instance, if

$$\Omega \subset \{x \in \mathbb{R}^n : |x'| < g(x_n)\},$$

where g satisfies

$$g(t) > g_0, \quad \text{for some } g_0 > 0,$$

then one has

$$\|u/g(x_n)\|_q \le c|u|_{1,q}, \quad u \in C_0^\infty(\Omega).$$

For this and similar inequalities, we refer, among others, to Elcrat and MacLean (1980), Hurri (1990), and Edmunds & Opic (1993).

The Friedrichs inequality (II.5.8) can be a fundamental tool for treating the convergence of approximating solutions of nonlinear partial differential equations. A nontrivial generalization of (II.5.8) is found in Padula (1986, Lemma 3). Extension of the Friedrichs inequality to *unbounded domains* are considered in Birman & Solomjak (1974).

From Theorem II.5.2 and Theorem II.4.1 it is not hard to prove compactness results involving convergence in boundary norms. For example, we have: if $\{u_k\} \subset W^{1,2}(\Omega)$ (Ω bounded and locally Lipschitz) is uniformly bounded, there is a subsequence $\{u_{m'}\}$ such that $u_{m'} \to u$ in $L^q(\partial\Omega)$ with $q = 2(n-1)/(n-2)$ if $n > 2$ and all $q \in [1,\infty)$ if $n = 2$.

The counterexample to compactness after Theorem II.5.2 is due to Benedek & Panzone (see Serrin 1962).

The Poincaré–Sobolev inequality can be proved for a general class of domains, including those with internal cusps. Such a generalization, which is of interest in the context of capillarity theory of fluids, can be found in Pepe (1978). However, in general, embedding inequalities no longer hold if the domain does not possess a certain degree of regularity. For this type of questions we refer to Adams & Fournier (2003, §4.47).

Section II.6. After the pioneering work of Deny & Lions (1954) on the subject ("Beppo Levi Spaces"), a detailed study of homogeneous Sobolev spaces $\dot{D}^{m,q}(\Omega)$ and $D_0^{m,q}(\Omega)$ along with the study of their relevant properties was performed by the Russian school (Uspenskiĭ 1961, Sobolev 1963b, Sedov 1966, Besov 1967). These authors are essentially concerned with the case where $\Omega = \mathbb{R}^n$. For other detailed analysis of the homogeneous Sobolev spaces we refer the reader also to the work of Kozono & Sohr (1991) and Simader & Sohr (1997), and to Chapter I of the book of Maz'ja (1985).

A central role in the study of properties of functions from $D^{m,q}(\Omega)$ is played by the fundamental Lemma II.6.3 which, for $q = 2$ and $n \ge 3$, was first proved by Payne & Weinberger (1957). A slightly weaker version of it was independently provided by Uspenskiĭ (1961, Lemma 1). The proof given in this book is based on a generalization of the ideas of Payne & Weinberger and is due to me. Another proof has been given by Miyakawa & Sohr (1988, Lemma 3.3), which, however, does not furnish the explicit form of the constant u_0. Concerning this issue, from Lemma II.6.3 it follows that

$$u_0 = \lim_{|x|\to\infty} \int_{S^{n-1}} u(|x|,\omega)d\omega,$$

or also, as kindly pointed out to me by Professor Christian Simader,

$$u_0 = \lim_{R\to\infty} \frac{1}{|\Omega_R|} \int_{\Omega_R} u.$$

Results contained in Exercise II.6.3 generalize part of those established by Uspenskiĭ (1961, Lemma 1), and for $q = n = 2$ they coincide with those of Gilbarg & Weinberger (1978, Lemma 2.1).

Inequality (II.6.20) with $q = 2$ and $n = 3$ is due to Finn (1965a, Corollary 2.2a); see also Birman & Solomjak (1974, Lemma 2.19) and Padula (1984, Lemma 1), while (II.6.22) for $n = 3$ and $q \in (1, 3)$ is proved by Galdi & Maremonti (1986, Lemma 1.3). Theorem II.6.1, in its generality, is due to me.

The inequality in Theorem II.6.5 is due to Simader and Sohr (1997, Lemma 1.2).

Section II.7. The problem of approximation of functions from $D^{m,q}(\Omega)$ when $\Omega = \mathbb{R}^n$ with functions of bounded support was first considered by Sobolev (1963b). In this section we closely follow Sobolev's ideas to generalize his results to more general domains Ω. In this connection, we refer the reader also to the papers of Besov (1967, 1969) and Burenkov (1976).

The elementary proof of the Hardy-type inequality (II.6.10), (II.6.13) and (II.6.14) presented here and based on the use of the "auxiliary" function $\boldsymbol{g}$ was presented for the first time in Galdi (1994a, §2.5). The same approach was successively rediscovered by Mitidieri (2000).

Section II.8. A slightly weaker version of Theorem II.8.2, with a different proof, can be found in Kozono & Sohr (1991, Lemma 2.2).

The proof of the unique solvability of the Dirichlet problem (II.8.17) in the case $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$ is a simple consequence of Exercise II.11.9(ii) and Remark II.11.3. In the case Ω bounded and of class C^∞, a proof was given for the first time by Schechter (1963a, Corollary 5.2). A different proof that requires domains only of class C^2 was later provided by Simader (1972). If Ω is an exterior domain of class C^2, a thorough analysis of the problem can be found in Simader & Sohr (1997, Chapter I). In particular, for $n \geq 3$, the analysis of these authors shows that the problem (II.8.17) has a nonzero one-dimensional null set, if $q' \geq n$. In other words, there exists one and only one non-zero harmonic function $h \in D_0^{1,q'}(\Omega)$, satisfying a normalization condition $\int_{\Omega_R} h^2 = 1$, for some fixed $R > \delta(\Omega)^c$. For instance, if Ω is the exterior of the unit ball in $\mathbb{R}^n$, we have $h(x) = c(|x|^{2-n} - 1)$, for a suitable choice of the constant c depending on R. Consequently, the map $\mathfrak{M}$ defined in (II.8.19)–(II.8.20) is not surjective if $q' \in (1, n/(n-1)]$ and not injective if $q' \in [n, \infty)$.

Section II.9. Results similar to those derived in Theorem II.9.1, in the general context of spaces $D^{m,q}$, $m \geq 1$, have been shown by Mizuta (1989). Estimate (II.9.5) is of a particular interest since, as we shall see in Chapter X, it permits us to derive at once an important asymptotic estimate for solutions to the steady, two-dimensional Navier–Stokes equations in exterior domains having velocity fields with bounded Dirichlet integrals.

Section II.10. The case $1 \leq q < n$ in Theorem II.10.1 is due to me.

Section II.11. If in the Sobolev Theorem II.11.3 one considers the function

$$\psi(x) = \int_{|x-y|\leq R} f(y)|x-y|^{-\lambda} dy,$$

for fixed $R > 0$, the proof of (II.11.13) becomes elementary; however, only for $1/s > \lambda/n + 1/q - 1$ (see Sobolev 1938; 1963a, Chapter 1 §6). For a generalization of the Sobolev theorem in weighted Lebesgue spaces, along the same lines of Theorem II.11.5, we refer to Stein & Weiss (1958).

III

The Function Spaces of Hydrodynamics

O voi, che avete gl'intelletti sani,
mirate la dottrina che s'asconde
sotto il velame degli versi strani!

DANTE, Inferno IX, vv. 61-63

Introduction

Several mathematical problems related to the motion of a viscous, incompressible fluid find their natural formulation in certain spaces of vector functions that can be considered as characteristic of those problems. These functional spaces are of three types, denoted by H_q, H_q^1, and $\mathcal{D}_0^{1,q}$, and are defined as suitable subspaces of *solenoidal functions* of $[L^q]^n$, $[W_0^{1,q}]^n$, and $[D_0^{1,q}]^n$, respectively, $n \geq 2$. Actually, it is *just* the solenoidality restriction that makes these spaces peculiar and, as we shall see, poses problems that otherwise would not arise.

The main objective of this chapter is to study in detail the relevant properties of the above spaces.

If Ω has a compact (and sufficiently smooth) boundary, the function class $H_q = H_q(\Omega)$ can be characterized as the subspace of $[L^q(\Omega)]^n$ of solenoidal vectors in Ω having zero normal components at $\partial\Omega$. The space H_q comes into the picture as a by-product of a more general question related to a certain decomposition of the vector space $[L^q]^n$, the *Helmholtz–Weyl* decomposition. This decomposition plays a fundamental role in the mathematical theory of the Navier–Stokes equations, mainly for the study of *time-dependent* motions. As we shall see, the validity of the decomposition is *equivalent* to the unique solvability of an appropriate *Neumann problem* in weak form. Such a problem is certainly resolvable in domains having a (sufficiently smooth) compact

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boundary and in a half-space. However, there are also domains with boundary of *little regularity* (locally Lipschitz) and domains with smooth *noncompact boundary* where the Neumann problem is *not* uniquely solvable and, therefore, the corresponding Helmholtz–Weyl decomposition does not hold.

The main, basic question that one has to face when dealing with spaces H^1_q and $\mathcal{D}^{1,q}_0$ is related to the very *definition* of the spaces themselves. To see why, let us consider, for the sake of definiteness, the space H^1_q, analogous reasonings being valid for $\mathcal{D}^{1,q}_0$. To study the time-dependent motion of the fluid we need the velocity field $\boldsymbol{v}$ of the particles of the fluid together with its first spatial derivatives to be, at each time, summable in the region of flow Ω to the qth power for some $q \geq 1$; in addition, $\boldsymbol{v}$ has to be solenoidal and vanish at the boundary of Ω. A space of vector functions having such properties (in a generalized sense) can be chosen in either of the following ways:

$$\left\{\textit{completion of } \mathcal{D}(\Omega) \textit{ in the norm of } [W^{1,q}(\Omega)]^n\right\} \equiv H^1_q(\Omega)$$

or

$$\left\{\boldsymbol{v} \in [W^{1,q}_0(\Omega)]^n : \nabla \cdot \boldsymbol{v} = 0 \textit{ in } \Omega\right\} \equiv \widehat{H}^1_q(\Omega),$$

with $\mathcal{D}(\Omega)$ denoting the subclass of $[C^\infty_0(\Omega)]^n$ of solenoidal functions. These spaces may look similar, but in fact a priori they are not, since the condition of solenoidality on their members is imposed before (in $H^1_q(\Omega)$) and after (in $\widehat{H}^1_q(\Omega)$) having taken the completion of $[C^\infty_0(\Omega)]^n$ in the norm of $[W^{1,q}(\Omega)]^n$. Of course, understanding the relationship between $H^1_q(\Omega)$ and $\widehat{H}^1_q(\Omega)$ is a preliminary and fundamental question whose analysis aims to clarify the framework within which the Navier–Stokes problem has to be set. Actually, as pointed out for the first time by Heywood (1976), the coincidence of the two spaces is related to the *uniqueness* of solutions and in particular, in domains for which $H^1_q(\Omega) \neq \widehat{H}^1_q(\Omega)$ the solution may *not* be uniquely determined by the "traditional" initial and boundary data but other extra and appropriate auxiliary conditions are to be prescribed (see Chapters VI and XII).

A primary objective of this chapter will be, therefore, to analyze to some extent for which domains the coincidence of the spaces H^1_q, $\widehat{H}^1_q$ and $\mathcal{D}^{1,q}_0$, $\widehat{\mathcal{D}}^{1,q}_0$ holds and for which it does not. Specifically, we shall see that coincidence is essentially *not* related to the smoothness of the domain but rather to its shape. In particular, the above spaces may not be the same only for domains with a *noncompact boundary*, and we shall provide a large class of such domains for which, in fact, they do *not* coincide.

Another question with which we shall be dealing, and is technically somewhat related to the one just described, is that of the approximation of functions from $H^1_q \cap H^1_r$ [respectively, $\mathcal{D}^{1,q}_0 \cap \mathcal{D}^{1,r}_0$], with $r \neq q$, in the norm of $H^1_q \cap H^1_r$ [respectively, $\mathcal{D}^{1,q}_0 \cap \mathcal{D}^{1,r}_0$], by functions from $\mathcal{D}(\Omega)$. If there were no solenoidality constraints, the question would be rather classical and would find its answer in the standard literature. However, since we are dealing with solenoidal fields, the problem becomes more complicated and we are able to solve it only for a certain class of domains including domains with a smooth enough compact boundary.

Finally, we wish to mention that all problems described previously need a careful study of the properties of the solutions of the equation $\nabla \cdot \boldsymbol{u} = f$, for a suitably ascribed f. Such an auxiliary problem will therefore also be analyzed in great detail.

III.1 The Helmholtz–Weyl Decomposition of the Space L^q

It has been well known, since the work of H. von Helmholtz in electromagnetism (Helmholtz 1870), that any smooth vector field $\boldsymbol{u}$ in $\mathbb{R}^3$ that falls off sufficiently fast at large distances can be uniquely decomposed as the sum

$$\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2 \tag{III.1.1}$$

of a gradient and a curl. In other words, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ can be taken of the form

$$\boldsymbol{u}_1 = \nabla\varphi, \quad \boldsymbol{u}_2 = \nabla \times \boldsymbol{A}, \tag{III.1.2}$$

where φ and $\boldsymbol{A}$ are the *scalar* and *vector potential*, respectively. In fact, setting

$$\boldsymbol{U}(x) = (\mathcal{E} * \boldsymbol{u})\,(x),$$

with $\mathcal{E}$ denoting the fundamental solution of Laplace's equation (II.9.1), it follows that $\Delta\boldsymbol{U}(x) = \boldsymbol{u}(x)$; see Exercise II.11.3. Putting into this equation the identity

$$\Delta\boldsymbol{V} = \nabla(\nabla \cdot \boldsymbol{V}) - \nabla \times (\nabla \times \boldsymbol{V}), \tag{III.1.3}$$

relations (III.1.1) and (III.1.2) follow with

$$\varphi = \nabla \cdot \boldsymbol{U}, \quad \boldsymbol{A} = -\nabla \times \boldsymbol{U}.$$

Much later than 1870, it was recognized that decompositions of the type just described, once formulated in suitable function spaces, become useful tools in the theory of partial differential equations. A systematic study of space decomposition was initiated by Weyl (1940) and continued by Friedrichs (1955), Bykhovski & Smirnov (1960), and others, until the recent work of Simader & Sohr (1992). In this respect, the decomposition of the space of vector functions in Ω having components in $L^q(\Omega)$, which we continue to denote by $L^q(\Omega)$,[1] into the direct sum of certain subspaces is of basic interest

[1] Let X be any space of real functions used in this book. Unless confusion arises, we shall use the same symbol X to denote the corresponding space of vector and tensor-valued functions.

in theoretical hydrodynamics and to this problem we will devote the present section.

We begin to introduce some classes of functions. Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$. Setting

$$\mathcal{D} = \mathcal{D}(\Omega) = \{\boldsymbol{u} \in C_0^\infty(\Omega) : \nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega\},$$

for $q \in [1, \infty)$ we denote by $H_q = H_q(\Omega)$ the completion of $\mathcal{D}$ in the norm of L^q and put

$$G_q = G_q(\Omega) = \left\{\boldsymbol{w} \in L^q(\Omega) : \boldsymbol{w} = \nabla p, \text{ for some } p \in W^{1,q}_{loc}(\Omega)\right\}. \quad \text{(III.1.4)}$$

For $q = 2$ we will simply write H and G in place of H_2 and G_2. Obviously, H_q is a subspace of L^q; moreover, from Exercise III.1.2, it follows that G_q also is a subspace of L^q.

Referring the study of the relevant properties of these spaces to the next section, in the present section we will investigate the validity of the decomposition

$$L^q(\Omega) = G_q(\Omega) \oplus H_q(\Omega), \quad \text{(III.1.5)}$$

where $\oplus$ denotes direct sum operation. In other words, we wish to determine when an arbitrary vector $\boldsymbol{u} \in L^q(\Omega)$ can be *uniquely* expressed as the sum

$$\boldsymbol{u} = \boldsymbol{w}_1 + \boldsymbol{w}_2, \quad \boldsymbol{w}_1 \in G_q(\Omega) \text{ and } \boldsymbol{w}_2 \in H_q(\Omega). \quad \text{(III.1.6)}$$

Remark III.1.1 The validity of the decomposition (III.1.5) implies the existence of a unique *projection operator*

$$P_q : L^q(\Omega) \to H_q(\Omega),$$

that is, of a *linear*, *bounded*, *idempotent* ($P_q^2 = P_q$) operator having $H_q(\Omega)$ as its *range* and G_q as its *null space* (Rudin 1973, §5.15(d)). In the case $q = 2$, we set $P_2 \equiv P$. ■

We shall show that the validity of (III.1.5) is *equivalent* to the unique resolubility of an appropriate (generalized) Neumann problem in Ω ($\mathcal{NP}$, say); see Lemma III.1.2 and also Simader & Sohr (1992). Now, if $q = 2$, *just* employing the Hilbert structure of the space L^2, we prove that (III.1.5) is valid for *any* domain Ω (see Theorem III.1.1), thus obtaining, as a by-product, the solvability of $\mathcal{NP}$ for $q = 2$ in *arbitrary* domains. On the other hand, if $q \neq 2$, in order to obtain (III.1.5) we *directly* address the solvability of $\mathcal{NP}$, which a priori depends on the value of the exponent q, on the "shape" of Ω, and on its regularity. Specifically, if $q \neq 2$, we show that if Ω is either a bounded or an exterior C^2-smooth domain[2] or a half space, then (III.1.5) holds, see Theorem III.1.2. On the other hand, for a certain class of domains with an

[2] The regularity of Ω can be further weakened (Simader & Sohr 1992).

unbounded boundary (no matter how smooth), the corresponding $\mathcal{NP}$ looses either solvability or uniqueness and, therefore, in such a class, the Helmholtz–Weyl decomposition does *not* hold (see Remark III.1.3, Exercise III.1.7 and Bogovskiĭ 1986 and Maslennikova & Bogovskiĭ 1986a, 1986b, 1993). Analogous considerations hold if the domain is bounded and with little regularity; see Fabes, Mendez & Mitrea (1998, Theorem 12.2) and Remark III.1.3.

Exercise III.1.1 Given a reflexive Banach space X and a subset S of X, the *annihilator* $S^\perp$ of S is a subset of the dual space $X^{'}$ defined as

$$S^\perp = \{\ell \in X^{'} : \ell(x) = 0, \text{for all } x \in S\}$$

(Kato 1966, p. 16). If, in particular, X is a Hilbert space, then $S^\perp$ is said to be the *orthogonal complement of* S. In this case, two subsets S_1, S_2 of X are called *orthogonal* if $S_1 \subset S_2^\perp$ (or, equivalently, $S_2 \subset S_1^\perp$). Show that: (a) $S^\perp$ is a closed subspace of $X^{'}$; (b) $H_q^\perp \supset G_{q'}$, $q \in (1, \infty)$, $(1/q + 1/q' = 1)$, so that, for $q = 2$, H and G are orthogonal.

Exercise III.1.2 Show that G_q is a closed subspace of L^q. *Hint:* Use the methods adopted in the proof of Lemma II.6.2.

Fundamental to further development is the characterization of the class of vectors $\boldsymbol{u} \in L^1_{loc}(\Omega)$ that are "orthogonal" to all vectors $\boldsymbol{w} \in \mathcal{D}(\Omega)$, i.e.,

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{w} = 0, \quad \text{for all } \boldsymbol{w} \in \mathcal{D}(\Omega). \tag{III.1.7}$$

Exercise III.1.3 Show that, for $\boldsymbol{u} \in L^1_{loc}(\Omega)$, condition (III.1.7) is equivalent to

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{w} = 0, \quad \text{for all solenoidal } \boldsymbol{w} \in C_0^1(\Omega).$$

If Ω is a simply connected domain in $\mathbb{R}^3$ and $\boldsymbol{u}$ is continuously differentiable one proves at once that $\boldsymbol{u} = \nabla p$ for some smooth single-valued scalar function p. In fact, for arbitrary $\boldsymbol{h} \in C_0^\infty(\Omega)$, let us choose in (III.1.7) $\boldsymbol{w} = \nabla \times \boldsymbol{h}$ and use the identity

$$\nabla \cdot (\boldsymbol{v}_1 \times \boldsymbol{v}_2) = \boldsymbol{v}_2 \cdot \nabla \times \boldsymbol{v}_1 - \boldsymbol{v}_1 \cdot \nabla \times \boldsymbol{v}_2$$

to deduce

$$\int_\Omega \nabla \times \boldsymbol{u} \cdot \boldsymbol{h} = 0, \quad \text{for all } \boldsymbol{h} \in C_0^\infty(\Omega), \tag{III.1.8}$$

which in turn, by Exercise II.2.9, implies $\nabla \times \boldsymbol{u} = 0$. Being Ω simply connected, this last condition furnishes, by the Stokes theorem, $\boldsymbol{u} = \nabla p$, with p a suitable line integral of the differential form

$$\boldsymbol{u} \cdot d\boldsymbol{x} = \sum_{i=1}^3 u_i dx_i.$$

Now, for this procedure to hold, the assumption on the regularity of $\boldsymbol{u}$ can be fairly weakened (see the last part of Lemma III.1.1), while it is crucial the assumption Ω be simply connected; otherwise p need not be single-valued. The aim of the following lemma is to show that the result just proved continues to be valid for *any* domain in $\mathbb{R}^n$. The method we shall employ is based on an idea of Fujiwara & Morimoto (1977, p. 697) and is due to Simader & Sohr (1992).

Lemma III.1.1 *Let Ω be an arbitrary domain in $\mathbb{R}^n$. Suppose that $\boldsymbol{u} \in L^1_{loc}(\Omega)$ verifies* (III.1.7). *Then, there exists a single-valued scalar function $p \in W^{1,1}_{loc}(\Omega)$ such that $\boldsymbol{u} = \nabla p$.*

Proof. Assume first $\boldsymbol{u} \in C(\Omega)$. The proof will be achieved if we show that the line integral of the differential form $\boldsymbol{u} \cdot d\boldsymbol{x}$ is zero along *all* closed nonintersecting polygonals lying in Ω.[3] Let Γ denote any such curve; we may then represent it by a continuous function γ such that

$$\boldsymbol{\gamma} : [0,1] \to \mathbb{R}^n,$$

$$\boldsymbol{\gamma} \in C^\infty([t_i, t_{i+1}]),$$

where $0 = t_0 < t_1 < \ldots < t_k = 1$ and $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(1)$. For $\boldsymbol{w} \in C(\Omega)$ we thus have

$$\int_\Gamma \boldsymbol{w} \cdot d\boldsymbol{x} = \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \boldsymbol{w}(\gamma(t)) \cdot \frac{d\boldsymbol{\gamma}}{dt} dt.$$

Let $\varepsilon_0 = \operatorname{dist}(\Gamma, \partial\Omega)$. For $x \in \Omega$ we set

$$\boldsymbol{\Phi}^\varepsilon(x) = \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} j_\varepsilon(x - \gamma(t)) \frac{d\boldsymbol{\gamma}}{dt} dt,$$

where $j_\varepsilon(\xi) = \varepsilon^{-n} j(\xi/\varepsilon)$ is a mollifying kernel of the type introduced in Section II.2 and $0 < \varepsilon < \varepsilon_0$. Obviously, $\boldsymbol{\Phi}^\varepsilon \in C^\infty(\Omega)$ for all such ε, and since

$$\begin{aligned} \nabla \cdot \boldsymbol{\Phi}^\varepsilon(x) &= \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \nabla_{(x)} j_\varepsilon(x - \gamma(t)) \cdot \frac{d\boldsymbol{\gamma}}{dt} dt \\ &= -\sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \frac{d}{dt} \left[j_\varepsilon(x - \gamma(t)) \right] dt \\ &= -j_\varepsilon(x - \gamma(1)) + j_\varepsilon(x - \gamma(0)) = 0, \end{aligned}$$

[3] Recall that, since Ω is open and connected, it is also polygonally connected. Namely, given $x, x' \in \Omega$ we can find a non-intersecting polygonal joining x with x'.

it follows that $\boldsymbol{\Phi}^\varepsilon \in \mathcal{D}(\Omega)$. Now, by definition of mollifier and by (III.1.7) we have

$$\int_\Gamma \boldsymbol{u}_\varepsilon \cdot d\boldsymbol{x} = \sum_{i=0}^{k} \int_{\mathbb{R}^n} \boldsymbol{u}(x) \cdot \frac{d\gamma}{dt} \left\{ \int_{t_i}^{t_{i+1}} j_\varepsilon(x-\gamma(t))dt \right\} dx = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\Phi}^\varepsilon = 0. \tag{III.1.9}$$

Therefore, letting $\varepsilon \to 0$ in (III.1.9) and using the properties of mollifiers, we find

$$\int_\Gamma \boldsymbol{u} \cdot d\boldsymbol{x} = 0, \quad \text{for all } \Gamma,$$

which implies $\boldsymbol{u} = \nabla p$ with $p \in C^1(\Omega)$. The lemma is therefore proved when $\boldsymbol{u} \in C(\Omega)$. Assume now merely $\boldsymbol{u} \in L^1_{loc}(\Omega)$. Let $\mathfrak{O}$ be the open covering of Ω introduced in Lemma II.1.1, and let $\mathfrak{B}_0 \in \mathfrak{O}$. We choose $\varepsilon > 0$, such that $\varepsilon < \operatorname{dist}(\mathfrak{B}_0, \partial\Omega)$ and let $\boldsymbol{w}$ be an arbitrary vector in $\mathcal{D}(\mathfrak{B}_0)$. From Section II.2 we deduce that the regularization $\boldsymbol{w}_\varepsilon$ of $\boldsymbol{w}$ belongs to $\mathcal{D}(\Omega)$ and thus, by assumption and Fubini's theorem, we obtain

$$\begin{aligned} 0 = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{w}_\varepsilon &= \varepsilon^{-n} \int_{\mathbb{R}^n} \boldsymbol{u}(x) \left[\int_{\mathbb{R}^n} j\left(\frac{x-y}{\varepsilon}\right) \boldsymbol{w}(y)\, dy \right] dx \\ &= \varepsilon^{-n} \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} j\left(\frac{x-y}{\varepsilon}\right) \boldsymbol{u}(x)\, dx \right] \boldsymbol{w}(y) dy = \int_{\mathfrak{B}_0} \boldsymbol{u}_\varepsilon \cdot \boldsymbol{w}. \end{aligned}$$

Since $\boldsymbol{u}_\varepsilon \in C^\infty(\mathfrak{B}_0)$ and $\boldsymbol{w}$ ranges arbitrarily in $\mathcal{D}(\mathfrak{B}_0)$, from the first part of the proof we find $\boldsymbol{u}_\varepsilon = \nabla p_\varepsilon$ in $\mathfrak{B}_0$, for some $p_\varepsilon \in C^\infty(\mathfrak{B}_0)$. Set $\varepsilon = 1/m$, $m \geq m_0 \in \mathbb{N}_+$, and let $m \to \infty$. By an argument completely analogous to that used in the proof of Lemma II.6.2, we show that $p_{1/m}$ converges to some $p^{(0)} \in W^{1,1}(\mathfrak{B}_0)$ such that $\boldsymbol{u} = \nabla p^{(0)}$ a.e. in $\mathfrak{B}_0$. Now, by Lemma II.1.1, we can find $\mathfrak{B}_1 \in \mathfrak{O}$ such that $\mathfrak{B}_0 \cap \mathfrak{B}_1 \equiv \mathfrak{B}_{1,2} \neq \emptyset$, and so, by the same kind of argument, we can find $p^{(1)} \in W^{1,1}(\mathfrak{B}_1)$ such that $\boldsymbol{u} = \nabla p^{(1)}$ a.e. in $\mathfrak{B}_1$. Since $p^{(0)} = p^{(1)} + c$ a.e. in $\mathfrak{B}_{1,2}$, we may modify $p^{(1)}$ by the addition of a constant in such a way that $p^{(0)}$ and $p^{(1)}$ agree on $\mathfrak{B}_{1,2}$. Let us continue to denote by $p^{(1)}$ the modified function and define a new function $p^{(1,2)}$ which equals $p^{(0)}$ on $\mathfrak{B}_0$ and equals $p^{(1)}$ on $\mathfrak{B}_1$. Clearly, $p^{(1,2)} \in W^{1,1}(\mathfrak{B}_0 \cup \mathfrak{B}_1)$ and, furthermore, $\boldsymbol{u} = \nabla p^{(1,2)}$ a.e. in $\mathfrak{B}_0 \cup \mathfrak{B}_1$. In view of the properties of the covering $\mathfrak{O}$, we can repeat this procedure to show, by induction, the existence of a function $p \in W^{1,1}_{loc}(\Omega)$ satisfying the statement in the lemma which is, therefore, completely proved. □

As an immediate consequence of the previous result, we deduce the validity of (III.1.5) for $q = 2$.

Theorem III.1.1 *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \geq 2$. Then $G(\Omega)$ and $H(\Omega)$ are orthogonal subspaces in $L^2(\Omega)$. Moreover*

$$L^2(\Omega) = G(\Omega) \oplus H(\Omega).$$

Proof. From Exercise III.1.1 we know that H and G are orthogonal subspaces in L^2. Moreover, suppose that $\boldsymbol{u}$ belongs to $H^\perp$. By Lemma III.1.1 there exists a scalar function $p \in L^1_{loc}(\Omega)$ such that $\boldsymbol{u} = \nabla p$ and, by Lemma II.6.1, $p \in L^2_{loc}(\Omega)$ thus showing $\boldsymbol{u} \in G$, which completes the proof. □

The study of the validity of the decomposition (III.1.5) when $q \neq 2$ turns out to be more involved, due to the fact that L^q ceases to be a Hilbert space. However, if $q \in (1, \infty)$, it is not hard to show that the decomposition is *equivalent* to the unique solvability of a suitable Neumann problem. Actually, consider the following problem $\mathcal{NP}$: *Given*

$$\boldsymbol{u} \in L^q(\Omega)$$

to find a unique (up to a constant) function $p : \Omega \to \mathbb{R}$ *such that*

(i) $p \in D^{1,q}(\Omega)$;

(ii) $\displaystyle\int_\Omega (\nabla p - \boldsymbol{u}) \cdot \nabla \varphi = 0$, *for all* $\varphi \in D^{1,q'}(\Omega)$.

The reader will check with no pain that if Ω has a sufficiently smooth boundary and $\boldsymbol{u}$ is regular enough, $\mathcal{NP}$ implies the existence of a solution $p \in D^{1,q}(\Omega)$ to the following classical Neumann problem:

$$\begin{aligned} \Delta p &= \nabla \cdot \boldsymbol{u} \ \text{ in } \Omega \\ \frac{\partial p}{\partial n} &= \boldsymbol{u} \cdot \boldsymbol{n} \ \text{ at } \partial\Omega. \end{aligned} \tag{III.1.10}$$

The next lemma gives a characterization of the validity of the Helmholtz–Weyl decomposition.

Lemma III.1.2 *The Helmholtz–Weyl decomposition of* $L^q(\Omega)$, $1 < q < \infty$, *holds if and only if* $\mathcal{NP}$ *is solvable for any* $\boldsymbol{u} \in L^q(\Omega)$.

Proof. Denote by $\mathcal{HW}$ the Helmholtz–Weyl decomposition. Let us first show that $\mathcal{NP}$ implies $\mathcal{HW}$. Given $\boldsymbol{u} \in L^q(\Omega)$, set

$$\boldsymbol{w} = \boldsymbol{u} - \nabla p, \tag{III.1.11}$$

with p (unique) solution to $\mathcal{NP}$. It is easy to see that $\boldsymbol{w} \in H_q(\Omega)$. In fact, by (ii) we deduce

$$\boldsymbol{w} \in G^\perp_{q'}.$$

On the other hand, by Lemma III.1.1 and by the Riesz representation theorem, it is

$$H^\perp_q \subset G_{q'}, \quad 1 < q < \infty.$$

Therefore,

$$\boldsymbol{w} \in (H^\perp_q)^\perp \cap L^q,$$

and so, by well-known properties on annihilators (see, *e.g.*, Kato 1966, p. 136) we conclude $\boldsymbol{w} \in H_q$. To prove $\mathcal{HW}$ completely, it remains to show that the representation obtained for $\boldsymbol{u}$ from (III.1.11) is indeed unique. This amounts to proving that the equality

$$\boldsymbol{w} = \nabla p, \quad \boldsymbol{w} \in H_q(\Omega), \quad p \in D^{1,q}(\Omega) \tag{III.1.12}$$

is possible if and only if $\boldsymbol{w} \equiv \nabla p \equiv 0$. Let us show that this is certainly so in our case. In fact, from (III.1.12) and Exercise III.1.1(b) we have

$$\int_\Omega \nabla p \cdot \nabla \varphi = 0 \quad \text{for all } \varphi \in D^{1,q'}(\Omega),$$

which, in turn, by the uniqueness of solutions to $\mathcal{NP}$ and (III.1.12) implies $\boldsymbol{w} \equiv \nabla p \equiv 0$. Conversely, assume that $\mathcal{HW}$ holds. Then, given $\boldsymbol{u} \in L^q(\Omega)$ we may decompose $\boldsymbol{u}$ as in (III.1.6) where $\boldsymbol{w}_1 = \nabla p$, $p \in D^{1,q}(\Omega)$. Multiplying this relation by $\nabla\varphi$, $\varphi \in D^{1,q'}(\Omega)$, and integrating over Ω, we obtain

$$\int_\Omega (\nabla p - \boldsymbol{u}) \cdot \nabla \varphi = -\int_\Omega \boldsymbol{w}_2 \cdot \nabla \varphi, \quad \boldsymbol{w}_2 \in H_q(\Omega).$$

In view of Exercise III.1.1(b),

$$\int_\Omega \boldsymbol{w}_2 \cdot \nabla \varphi = 0,$$

so that p satisfies (ii). By the uniqueness of the representation (III.1.6), we have that such a p is unique (up to a constant). The lemma is proved. □

Remark III.1.2 Theorem III.1.1 and Lemma III.1.2 imply that, for $q = 2$, the corresponding generalized Neumann problem $\mathcal{NP}$ admits a unique solution in an arbitrary domain Ω. ■

We shall next present a wide class of domains for which $\mathcal{NP}$ is solvable. The simplest situation occurs when $\Omega = \mathbb{R}^n$ for, in this case, for $\boldsymbol{u} \in C_0^\infty(\mathbb{R}^n)$, we can produce an explicit solution, that is (see Exercise II.11.3),

$$p(x) = \int_{\mathbb{R}^n} \mathcal{E}(x-y) \nabla \cdot \boldsymbol{u}(y) dy. \tag{III.1.13}$$

It is easy to show that (III.1.13) satisfies all the requirements. In fact, on the one hand, by Exercise II.11.7 and by the Calderón–Zygmund Theorem II.11.4, it follows that (i) is accomplished and, moreover, that

$$|p|_{1,q} \le c \|\boldsymbol{u}\|_q. \tag{III.1.14}$$

On the other hand, since $\boldsymbol{u}$ vanishes outside a compact set K (say), we also have for sufficiently large R

$$\int_{B_R} (\nabla p - \boldsymbol{u}) \cdot \nabla \varphi = -\int_{\partial B_R} \varphi \frac{\partial p}{\partial n}, \tag{III.1.15}$$

where $\varphi \in D^{1,q'}(\mathbb{R}^n)$. From (III.1.13) it is easily seen that for x outside K

$$\nabla p(x) = O(|x|^{-n}). \tag{III.1.16}$$

Furthermore, φ obeys the estimate (see Exercise II.6.3)

$$\int_{S_n} |\varphi(x)| = o(|x|). \tag{III.1.17}$$

Thus, from (III.1.15)–(III.1.17), in the limit $R \to \infty$ we deduce (ii) of the definition of $\mathcal{NP}$. Finally, uniqueness is obtained with the help of Exercise II.11.11. It is now easy to extend the results just shown to the case when $\boldsymbol{u}$ is an arbitrary function in $L^q(\Omega)$. This will be achieved through a standard approximating procedure based on (III.1.14). Actually, by the density properties recalled in Section II.2, we can approximate $\boldsymbol{u}$ with a sequence $\{\boldsymbol{u}_m\} \subset C_0^\infty(\overline{\Omega})$. For each $\boldsymbol{u}_m$ we solve $\mathcal{NP}$ as before and denote by p_m the corresponding solution. Using (III.1.14) and the uniqueness property we then prove that $\{[p_m]\}$ is a Cauchy sequence in $\dot{D}^{1,q}(\Omega)$, and so, by Lemma II.6.2, there exists $[p] \in \dot{D}^{1,q}(\Omega)$ such that

$$|[p_m] - [p]|_{1,q} \to 0, \quad m \to \infty\,.$$

It is easy to verify that $p' \in [p]$ uniquely satisfies (up to a constant) condition (ii) stated for $\mathcal{NP}$, thus proving the desired decomposition of $L^q(\mathbb{R}^n)$ for all $q \in (1, \infty)$.

Analogous reasoning can be used if Ω is the half space $\mathbb{R}^n_+$. In this case too, in fact, we have an explicit formula for p:

$$p(x) = \int_{\mathbb{R}^n_+} \mathcal{N}(x,y) \nabla \cdot \boldsymbol{u}(y) dy, \quad x \in \mathbb{R}^n_+, \tag{III.1.18}$$

where

$$\mathcal{N}(x,y) \equiv \mathcal{E}(x-y) + \mathcal{E}(x-y^*), \quad y^* = (y_1, \ldots, y_{n-1}, y_n)$$

is the (Neumann) Green's function of the Laplace operator in $\mathbb{R}^n_+$; see Exercise III.1.5. The details of the proof are left to the reader (see also McCracken 1981).

If Ω has a sufficiently smooth bounded boundary, the problem $\mathcal{NP}$ is still solvable but, of course, in a more involved way.

Actually, if Ω is a *bounded* domain of class C^2, a solution to $\mathcal{NP}$ can be determined as a consequence of more general results on elliptic problems established by Lions & Magenes (1962, Teor.4.1), Miranda (1978, §57) and

Schechter (1963a, 1963b); see Fujiwara & Morimoto (1977), Simader (1990, Theorem 4.1), and Simader & Sohr (1992).[4]

Using the above results one can then secure the solvability of $\mathcal{NP}$ for an *exterior* domain. To prove this, we begin to observe that, assuming at first $\boldsymbol{u} \in C_0^\infty(\overline{\Omega})$, the existence of a unique solution p to $\mathcal{NP}$ with $q = 2$ is immediately established; see Remark III.1.2. One can then use the classical estimates of Agmon, Douglis, & Nirenberg (1959, §15) to show

$$p \in C^\infty(\Omega) \cap W^{2,q}(\Omega_r), \quad \text{for all } r > \delta(\Omega^c) \text{ and all } q \geq 1. \tag{III.1.19}$$

Furthermore, p solves (III.1.10). We shall now prove that $p \in D^{1,q}(\Omega)$, $1 < q < \infty$, and the validity of (III.1.14). To reach this goal, we take $\varphi \in C^\infty(\mathbb{R})$ with

$$\varphi(\xi) = \begin{cases} 0 & \text{if } |\xi| \leq 1/2 \\ 1 & \text{if } |\xi| \geq 1 \end{cases}$$

and set

$$\varphi_R(x) = \varphi(|x|/R), \quad w(x) = \varphi_R(x)p(x), \quad R > 2\delta(\Omega^c).$$

From (III.1.10) we have that w solves the problem

$$\begin{aligned} &\Delta w = f \\ &\quad f = \nabla \cdot (p\nabla\varphi_R + \boldsymbol{u}\varphi_R) + \nabla\varphi_R \cdot (\nabla p - \boldsymbol{u}) \equiv f_1 + f_2. \end{aligned} \tag{III.1.20}$$

Clearly, $f_i \in C_0^\infty(\mathbb{R}^n)$, $i = 1, 2$. Also, by using the properties of φ_R and $\boldsymbol{u}$ along with (III.1.10), it is readily seen that

$$\int_{\mathbb{R}^n} f_i = 0, \quad i = 1, 2, \tag{III.1.21}$$

and so, by Theorem II.8.1 (see also Remark II.8.1), $f_i \in D_0^{-1,q}(\mathbb{R}^n)$, $i = 1, 2$, and we may apply the results of Exercise II.11.9(ii) and Exercise II.11.11 to deduce the existence of a unique (up to a constant) solution $w \in D^{1,q}(\mathbb{R}^n)$, $1 < q < \infty$, which further verifies

$$|w|_{1,q} \leq c\,|f|_{-1,q}, \quad 1 < q < \infty. \tag{III.1.22}$$

From $(\text{III.1.20})_2$ it follows for all $\psi \in D_0^{1,q'}(\mathbb{R}^n)$, $1 < q' < \infty$,

$$\left| \int_{\mathbb{R}^n} f_1 \psi \right| \leq c_1 \left(\|p\|_{q,\Omega_R} + \|\boldsymbol{u}\|_q \right) |\psi|_{1,q',\mathbb{R}^n},$$

where $c_1 = c_1(\varphi_R)$. Furthermore, if $1 < q' < n$, by the Sobolev inequality (II.3.7),

[4] These latter two papers require Ω to be only of class C^1.

$$\|\psi\|_{q',\Omega_R} \leq c_2 |\psi|_{1,q',\mathbb{R}^n} ,$$

with $c_2 = c_2(\Omega_R, q')$. Thus, since $\psi \nabla \varphi_R \in W_0^{1,q}(\Omega_R)$, for these values of q' we have, for some $c_3 = c_3(\varphi_R)$,

$$\left| \int_{\mathbb{R}^n} f_2 \psi \right| \leq c_3 \|\nabla p - \boldsymbol{u}\|_{-1,q,\Omega_R} |\psi|_{1,q',\mathbb{R}^n} .$$

If $q' \geq n$, we recall that the generic element of $D_0^{1,q'}(\mathbb{R}^n)$ is an equivalence class, $[\psi]$, constituted by functions that differ, at most, by a constant; see (II.7.16). Thus, pick $\psi \in [\psi]$ and set

$$\psi_0 = \frac{1}{|\Omega_R|} \int_{\Omega_R} \psi .$$

From (III.1.21), with the help of Poincaré inequality (II.5.10), we deduce

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f_2 \psi \right| &= \left| \int_{\mathbb{R}^n} f_2 (\psi - \psi_0) \right| \leq c_4 \|\nabla p - \boldsymbol{u}\|_{-1,q,\Omega_R} |\psi|_{1,q',\mathbb{R}^n} \\ &= c_4 \|\nabla p - \boldsymbol{u}\|_{-1,q,\Omega_R} |[\psi]|_{1,q',\mathbb{R}^n} , \end{aligned}$$

with $c_4 = c_4(\varphi_R, \Omega_R)$. We may then conclude

$$|f|_{-1,q} \leq c_5 \left(\|\boldsymbol{u}\|_q + \|\nabla p\|_{-1,q,\Omega_R} + \|p\|_{q,\Omega_R} \right) .$$

Substituting this inequality into (III.1.22) we find

$$|w|_{1,q} \leq c_6 \left(\|\boldsymbol{u}\|_q + \|\nabla p\|_{-1,q,\Omega_R} + \|p\|_{q,\Omega_R} \right) .$$

Recalling that $w = \varphi_R p$ and property (III.1.19), we conclude that

$$\begin{gathered} p \in D^{1,q}(\Omega), \\ |p|_{1,q,\Omega^R} \leq c_7 \left(\|\boldsymbol{u}\|_q + \|\nabla p\|_{-1,q,\Omega_R} + \|p\|_{q,\Omega_R} \right) , \end{gathered} \tag{III.1.23}$$

where $c_7 = c_7(q, n, \Omega, \varphi_R)$. This proves, in particular, the validity of condition (i) of $\mathcal{NP}$. Moreover,

$$\nabla p = O(|x|^{-n}), \quad |x| \to \infty,$$

see Exercise III.1.4, and so, as in the case $\Omega \equiv \mathbb{R}^n$, we show the validity of identity (ii) of $\mathcal{NP}$. The uniqueness of the solution p is likewise established, see Exercise III.1.4. To complete the proof of the solvability of $\mathcal{NP}$ it remains to extend these results to the case when $\boldsymbol{u}$ merely belongs to $L^q(\Omega)$. To this end, we may proceed exactly as in the case $\Omega \equiv \mathbb{R}^n$ (i.e., by a density argument) provided we show that the solution p just found satisfies the estimate (III.1.14). We shall next prove that this is indeed the case. Set

$$\psi_R(x) = 1 - \varphi_{2R}(x), \quad v(x) = p(x)\psi_R(x).$$

From (III.1.10)$_1$ it follows that the function v satisfies the following problem in Ω_R:

$$\Delta v = \nabla \cdot (p\nabla\psi_R + \boldsymbol{u}\psi_R) + \nabla\psi_R \cdot (\nabla p - \boldsymbol{u}) \equiv \nabla \cdot \boldsymbol{U}_1 + F. \tag{III.1.24}$$

Since, clearly,

$$\int_{\Omega_{2R}} F = 0\,,$$

we may employ Theorem III.3.1 (in the following section) to show the existence of a vector field $\boldsymbol{U}_2 \in W_0^{1,q}(\Omega_{2R})$ such that

$$\nabla \cdot \boldsymbol{U}_2 = F \quad \text{in } \Omega_{2R}\,,$$

$$\|\boldsymbol{U}_2\| \le c\,\|F\|\,,$$

with $c = c(n, q, R)$. Setting

$$\boldsymbol{U} = \boldsymbol{U}_1 + \boldsymbol{U}_2\,,$$

from (III.1.10)$_2$ and (III.1.24) we therefore obtain

$$\Delta v = \nabla \cdot \boldsymbol{U} \quad \text{in } \Omega_{2R},$$

$$\frac{\partial v}{\partial n} = \boldsymbol{U} \cdot \boldsymbol{n} \quad \text{at } \partial\Omega_{2R}.$$

Since Ω_{2R} is bounded (and smooth), we know that the Helmholtz–Weyl decomposition of $L^q(\Omega_{2R})$ holds for all values of $q \in (1, \infty)$ and so, by Lemma III.1.2 and Remark III.1.1, it follows that

$$|v|_{1,q,\Omega_{2R}} \le c_1 \|\boldsymbol{U}\|_{q,\Omega_{2R}}$$

with $c_1 = c_1(n, q, \Omega_R)$. Recalling the definition of v and the estimates for $\|\boldsymbol{U}\|_q$, we deduce

$$|p|_{1,q,\Omega_R} \le c_2 \left(\|\boldsymbol{u}\|_q + \|p\|_{1,q,\Omega_{R,2R}}\right), \tag{III.1.25}$$

where $c_2 = c_2(n, q, R, \Omega_R)$. Combining (III.1.23)$_2$ with (III.1.25) and taking into account that problem (III.1.10) does not change if we modify p by adding a constant to it, it follows that

$$|p|_{1,q,\Omega} \le c_3 \left(\|\boldsymbol{u}\|_q + \|\nabla p\|_{-1,q,\Omega_R} + \|p\|_{q,\Omega_R/\mathbb{R}}\right), \tag{III.1.26}$$

with

$$\|p\|_{q,\Omega_R/\mathbb{R}} \equiv \inf_{k\in\mathbb{R}} \|p + k\|_{q,\Omega_R}.$$

We claim the existence of a positive constant $C = C(n, q, R, \Omega_R)$ such that

$$\|\nabla p\|_{-1,q,\Omega_R} + \|p\|_{q,\Omega_R/\mathbb{R}} \le C\|\boldsymbol{u}\|_q. \tag{III.1.27}$$

Contradicting (III.1.27) implies that there is a sequence $\{\boldsymbol{u}_m\} \subset C_0^\infty(\overline{\Omega})$ and a sequence of corresponding solutions, $\{p_m\}$, to (III.1.10), such that

$$\left.\begin{aligned} \|\boldsymbol{u}_m\|_q &\to 0 \\ \|\nabla p_m\|_{-1,q,\Omega_R} + \|p\|_{q,\Omega_R/\mathbb{R}} &= 1 \end{aligned}\right. \quad \text{as } m \to \infty. \tag{III.1.28}$$

From (III.1.26) and (III.1.28) we obtain that

$$|p_m|_{1,q,\Omega} \le M\,, \tag{III.1.29}$$

for some constant M independent of m. By the weak compactness property of spaces $\dot{D}^{1,q}$, $1 < q < \infty$ (Exercise II.6.2), we find from (III.1.29) the existence of $p \in D^{1,q}(\Omega)$ and of a subsequence $\{p_{m'}\}$ such that

$$(\nabla p_{m'}, \boldsymbol{\varphi}) \to (\nabla p, \boldsymbol{\varphi}) \ \text{ for all } \boldsymbol{\varphi} \in L^{q'}(\Omega).$$

Thus, by this property and (ii) of $\mathcal{NP}$ (that we have previously established for all $\boldsymbol{u} \in C^\infty(\overline{\Omega})$) we find for all $\phi \in D^{1,q'}(\Omega)$

$$0 = \lim_{m'\to\infty} (\boldsymbol{u}_{m'}, \nabla\phi) = \lim_{m'\to\infty} (\nabla p_{m'}, \nabla\phi) = (\nabla p, \nabla\phi),$$

which, by uniqueness, in turn implies

$$\nabla p \equiv 0. \tag{III.1.30}$$

Furthermore, from the compactness results of Exercise II.5.7 and Theorem II.5.3, it follows that $\{p_{m'}\}$ can be chosen to converge to p in $L^q(\Omega_R)$, while $\nabla p_{m'}$ tends to ∇p in $W_0^{-1,q}(\Omega_R)$. As a consequence, from $(\text{III.1.28})_2$, we find that

$$\|\nabla p\|_{-1,q,\Omega_R} + \|p\|_{q,\Omega_R/\mathbb{R}} = 1,$$

which contradicts (III.1.30). Thus, (III.1.27) is established and we may conclude the validity of the Helmholtz–Weyl decomposition of $L^q(\Omega)$, $1 < q < \infty$, for any domain Ω of class C^2.

We have thus proved the following theorem.

Theorem III.1.2 *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$ be either a domain of class C^2 or the whole space or a half-space. Then the Helmholtz–Weyl decomposition holds for $L^q(\Omega)$, for any $q \in (1,\infty)$.* [5]

Remark III.1.3 As already observed, in view of the characterization given in Lemma III.1.2, it is not expected that decomposition (III.1.5) holds for arbitrary domains whenever $q \neq 2$. Actually, one can show that, for certain

[5] Of course, if $q = 2$, it holds for any Ω, see Theorem III.1.1.

smooth domains with an unbounded boundary or for bounded domains with "sharp" corners, the Neumann problem $\mathcal{NP}$ loses either existence or uniqueness for values of q in some range. This problem is analyzed in the work of Maslennikova & Bogovskiĭ (1986a, 1986b, 1993) and Bogovskiĭ (1986), where examples of such domains are given. For instance, if Ω is a domain in the plane that is the complement of a smoothed angle $\vartheta = 2\pi - \theta < \pi$ (see Figure III.1), then $\mathcal{NP}$ loses existence if

$$1 < q < 2/(1 + \pi/\theta)$$

while it loses uniqueness if

$$2/(1 - \pi/\theta) < q \tag{III.1.31}$$

and therefore, for these values of q, the Helmholtz–Weyl decomposition of $L^q(\Omega)$ does not hold; see also Exercise III.1.7.

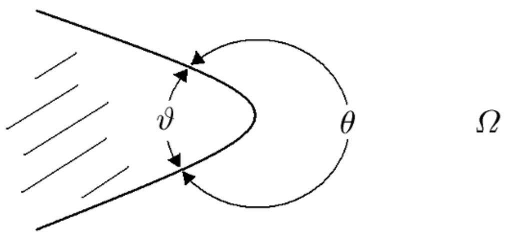

Figure III.1

A counterexample to the validity of the Helmohltz-Weyl decomposition in bounded domains with only locally Lipschitz boundary, is given in Fabes, Mendez & Mitrea (1998, Theorem 12.2); see also the Notes for this Chapter. ■

For other results concerning the resolution of $\mathcal{NP}$ (equivalently, the validity of the Helmholtz–Weyl decomposition) in domains with an unbounded boundary, we refer to the Notes for this Chapter.

Exercise III.1.4 Let $p \in D^{1,q}(\Omega)$ be a (smooth) solution to (III.1.10) with $\boldsymbol{u} \in C_0^\infty(\overline{\Omega})$. Show the following assertions:

(i) $\nabla p = O(|x|^{-n})$ as $|x| \to \infty$;
(ii) If $\boldsymbol{u} \equiv 0$, then $p \equiv \text{const.}$

Hint: (i) Use the methods of Lemma II.9.1 to prove the relation

$$D_i p(x) = \int_{\partial\Omega\cup\partial B_R} p(y)\frac{\partial}{\partial n}(D_i\mathcal{E}(x-y))d\sigma_y - \sum_{j=1}^{n}\int_{\Omega_R} u_j(y)D_{ji}\mathcal{E}(x-y)dy, \tag{III.1.32}$$

where R is so large that Ω_R contains the bounded support K of $\boldsymbol{u}$. Then let $R \to \infty$ into (III.1.32) and employ the results of Exercise II.6.3 and the estimate

$$|D_{ji}\mathcal{E}(\xi)| = O(|\xi|^{-n}) \text{ as } x \to \infty.$$

(ii) Multiply $(\text{III.1.10})_1$ with $\boldsymbol{u} \equiv 0$ by p, integrate by parts over Ω_R, use (i) and let $R \to \infty$.

Exercise III.1.5 We recall that a function $G(x,y)$ is said to be the *Green's function for the Laplace operator* in a domain Ω if $G(x,y) = \mathcal{E}(x-y) + g(x,y)$ with g such that for all $x \in \Omega$

$$\Delta_{(y)} g(x,y) = 0 \quad y \in \Omega$$

and, moreover,

$$g(x,y) = -\mathcal{E}(x-y) \quad y \in \partial\Omega \qquad (\textit{Dirichlet} \text{ or } \textit{first kind}).$$

or

$$\frac{\partial g(x,y)}{\partial n_y} = -\frac{\partial \mathcal{E}(x-y)}{\partial n_y} \quad y \in \partial\Omega \qquad (\textit{Neumann} \text{ or } \textit{second kind}).$$

Assuming Ω bounded and u and Ω sufficiently smooth, use Green's identity (see Lemma II.9.1) to show the following representations

$$u(x) = \int_\Omega G(x,y)\Delta u(y)dy + \int_{\partial\Omega} u(y)\frac{\partial G(x,y)}{\partial n_y}d\sigma_y \qquad \text{(Dirichlet)}, \tag{III.1.33}$$

$$u(x) = \int_\Omega G(x,y)\Delta u(y)dy + \int_{\partial\Omega} \frac{\partial u(y)}{\partial n_y}G(x,y)d\sigma_y \qquad \text{(Neumann)}. \tag{III.1.34}$$

These formulas continue to hold also if Ω is unbounded and u and G are "well behaved" at large distances. In this connection, show that

$$\mathcal{N}(x,y) \equiv \mathcal{E}(x-y) + \mathcal{E}(x-y^*), \quad y^* = (y_1, \ldots, y_{n-1}, -y_n),$$

is Green's function of the second kind for the half-space, while

$$\mathcal{D}(x,y) \equiv \mathcal{E}(x-y) - \mathcal{E}(x-y^*) \tag{III.1.35}$$

is Green's function of the first kind and formulate assumptions on u such that (III.1.33) and (III.1.34) are valid.

Exercise III.1.6 (Fujiwara & Morimoto 1977) Assume that Ω is such that the Helmholtz–Weyl decomposition for $L^q(\Omega)$ holds for all $q \in (1,\infty)$. Show that the adjoint P_q^* of the projection operator P_q (see Remark III.1.1) coincides with $P_{q'}$, $1/q + 1/q' = 1$.

Exercise III.1.7 Let Ω be the "smoothed angle" domain of Figure III.1, with $\theta > \pi$. Show that the homogeneous Neumann problem (III.1.10) with $\boldsymbol{u} \equiv \boldsymbol{0}$ has a *nonzero* solution $p \in D^{1,q}(\Omega)$, for all q satisfying (III.1.31). *Hint*: Let (r,φ) be a polar coordinate system with the origin at the tip of the "smoothed angle". The function

$$\overline{p} = r^{\pi/\theta}\cos\left(\frac{\pi}{\theta}\varphi\right),$$

satisfies $\Delta\overline{p}(r,\varphi) = 0$ and $\partial\overline{p}(r,\varphi)/\partial\boldsymbol{n}|_{\partial\Omega} = 0$, for all (r,θ), $r \ge r_0 > 0$. Moreover, $\overline{p} \in D^{1,q}(\Omega^{2r_0})$ *only* for those q satisfying (III.1.31). The desired solution is then given by $p = \psi\,\overline{p} + p_1$, where $\psi = \psi(r)$ is 0 for $r \le 2r_0$ and is 1 for $r \ge 3r_0$, while p_1 is the unique (up to a constant) solution to the Neumann problem (III.1.10) with $\boldsymbol{u} \equiv -2\nabla\psi\cdot\nabla\overline{p} - \overline{p}\Delta\psi$, and such that $\nabla p_1(r,\varphi) \to 0$ as $r \to \infty$.

III.2 Relevant Properties of the Spaces H_q and G_q

We begin to furnish a simple characterization of elements of $H_q(\Omega)$, $1 < q < \infty$, valid for an *arbitrary* domain Ω. Specifically, we have

Lemma III.2.1 *Let Ω be any domain in $\mathbb{R}^n$, $n \geq 2$. Then, a vector field $\boldsymbol{u}$ in $L^q(\Omega)$, $1 < q < \infty$, belongs to $H_q(\Omega)$ if and only if*

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{h} = 0, \quad \textit{for all } \boldsymbol{h} \in G_{q'}(\Omega). \tag{III.2.1}$$

Proof. Assume (III.2.1) holds. Then employing the same reasoning showed after formula (III.1.11), we deduce $\boldsymbol{u} \in H_q(\Omega)$. Conversely, take $\boldsymbol{u} \in H_q(\Omega)$ and denote by $\{\boldsymbol{u}_m\} \subset \mathcal{D}(\Omega)$ a sequence converging to $\boldsymbol{u}$ in $L^q(\Omega)$. Integrating by parts we show that (III.2.1) is satisfied by each $\boldsymbol{u}_m$ and then, by continuity, by $\boldsymbol{u}$. □

Relation (III.2.1) tells us, in particular, that $\boldsymbol{u}$ is *weakly divergence free*, that is,

$$\int_\Omega \boldsymbol{u} \cdot \nabla\psi = 0, \quad \text{for all } \psi \in C_0^\infty(\Omega) \text{ [1]}$$

and that, in a generalized sense, the "normal component" of $\boldsymbol{u}$ at the boundary is zero. Actually, if Ω is a regular bounded or exterior domain or a half-space and $\boldsymbol{u}$ is a sufficiently smooth function of $L^q(\Omega)$, one can show that $\boldsymbol{u} \in H_q(\Omega)$ if and only if $\nabla \cdot \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ at $\partial\Omega$. To this end, consider first the case where Ω is bounded and locally Lipschitz and let $\boldsymbol{u} \in H_q(\Omega)$. From the Gauss divergence theorem (see Exercise II.4.3) we have for all functions $\varphi \in W^{1,q'}(\Omega)$

$$\int_\Omega \varphi \nabla \cdot \boldsymbol{u} = \int_{\partial\Omega} \gamma(\varphi)\boldsymbol{u} \cdot \boldsymbol{n} - \int_\Omega \boldsymbol{u} \cdot \nabla\varphi, \tag{III.2.2}$$

where $\gamma(\varphi)$ is the trace of φ on $\partial\Omega$. From Lemma III.2.1 and (III.2.2) written, in particular, with $\varphi \in C_0^\infty(\Omega)$ we obtain $\nabla \cdot \boldsymbol{u} = 0$ which, once substituted into (III.2.2), with the aid of Lemma III.2.1 entails

$$\int_{\partial\Omega} \gamma(\varphi)\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \text{for all } \varphi \in W^{1,q'}(\Omega).$$

---

[1] In analogy with the definition of the generalized derivative, one can introduce the notion of generalized (or weak) differential operator, as in fact we already did with the gradient operator (see also Smirnov 1964, §110). Thus, in the case under consideration, we say that a vector $\boldsymbol{u} \in L^1_{loc}(\Omega)$ has a *generalized* (or *weak*) *divergence* $U \in L^q_{loc}(\Omega)$ if and only if

$$\int_\Omega \boldsymbol{u} \cdot \nabla\psi = -\int_\Omega U\psi, \quad \text{for all } \psi \in C_0^\infty(\Omega).$$

As usual, U will be denoted by $\nabla \cdot \boldsymbol{u}$.

Therefore, in view of Gagliardo's Theorem II.4.3, we deduce $\boldsymbol{u}\cdot\boldsymbol{n}=0$ at $\partial\Omega$. Conversely, assume $\nabla\cdot\boldsymbol{u}=0$ in Ω and $\boldsymbol{u}\cdot\boldsymbol{n}=0$ at $\partial\Omega$ and take an arbitrary $\boldsymbol{h}\equiv\nabla\phi\in G_{q'}(\Omega)$. By Lemma II.6.1, it follows that $\phi\in W^{1,q'}(\Omega)$ (this is no longer true if Ω is unbounded[2]) and from (III.2.2) we recover (III.2.1), which implies $\boldsymbol{u}\in H_q(\Omega)$.

If Ω is a locally Lipschitz exterior domain, using (III.2.2) with $\varphi\in C_0^\infty(\overline{\Omega})$ and (III.2.1) we can prove as in the previous case that $\boldsymbol{u}\in H_q(\Omega)$ implies $\nabla\cdot\boldsymbol{u}=0$ in Ω and $\boldsymbol{u}\cdot\boldsymbol{n}=0$ at $\partial\Omega$. To prove the converse relation, however, we should argue in a slightly more complicated way. Let $\boldsymbol{u}$ be a smooth solenoidal function of $L^q(\Omega)$, with $\boldsymbol{u}\cdot\boldsymbol{n}=0$ at $\partial\Omega$ and let ψ_R be the "cut-off" function (II.7.1). Given an arbitrary $\phi\in D^{1,q'}(\Omega)$, we can replace $\varphi=\psi_R\phi$ into (III.2.2) to find

$$\int_\Omega \psi_R(\boldsymbol{u}\cdot\nabla\phi) = -\int_\Omega \phi\nabla\psi_R\cdot\boldsymbol{u}.$$

Since

$$\lim_{R\to\infty}\int_\Omega \psi_R(\boldsymbol{u}\cdot\nabla\phi) = \int_\Omega \boldsymbol{u}\cdot\nabla\phi,$$

in view of Lemma III.2.1 we will show $\boldsymbol{u}\in H_q(\Omega)$ if we prove

$$\lim_{R\to\infty}\int_\Omega \phi\nabla\psi_R\cdot\boldsymbol{u}=0. \tag{III.2.3}$$

Now, by the Hölder inequality we find

$$\left|\int_\Omega \phi\nabla\psi_R\cdot\boldsymbol{u}\right| \le \|\boldsymbol{u}\|_q\|\phi\nabla\psi_R\|_{q',\widetilde{\Omega}_R},$$

where $\widetilde{\Omega}_R$ is defined in (II.7.3). The quantity $\|\phi\nabla\psi_R\|_{q'\widetilde{\Omega}_R}$ formally coincides with (II.7.5) with the replacements $u\to\phi$, $q\to q'$ and so, proceeding as in the part of the proof of Theorem II.7.1 that follows (III.6.5), we establish (III.2.3) and the proof is accomplished.

The above considerations are summarized in the following.

Lemma III.2.2 *Let Ω be a locally Lipschitz domain of $\mathbb{R}^n$, $n\ge 2$, and let $\boldsymbol{u}\in C^1(\overline{\Omega})\cap L^q(\Omega)$, $1<q<\infty$. Then $\boldsymbol{u}\in H_q(\Omega)$ if and only if $\nabla\cdot\boldsymbol{u}=0$ in Ω and $\boldsymbol{u}\cdot\boldsymbol{n}=0$ at $\partial\Omega$.*

Remark III.2.1 By an argument entirely analogous to that just shown, one can prove that the result of Lemma III.2.2 continues to hold for Ω a half-space. ■

If $\boldsymbol{u}$ is no longer assumed regular, we can nevertheless prove that the characterization just described of the space $H_q(\Omega)$ is still valid, provided we give suitable generalizations of the definition of the trace of the normal component

[2] Take, for instance, $\Omega=\{x\in\mathbb{R}^3:|x|>1\}$, $q=2$ and $\phi(x)=|x|^{-1}$. Then $\nabla\phi\in L^q(\Omega)$ while $\phi\notin L^q(\Omega)$.

of $\boldsymbol{u}$ at the boundary and of identity (III.2.2), and provided, of course, that the solenoidality condition is interpreted in the sense of weak derivatives. This will be our next objective, that will be reached by arguments basically due to Temam (1977, Chapter I, §1.3); see also Miyakawa (1982).

For $q \in (1, \infty)$ let

$$\widetilde{H}_q = \widetilde{H}_q(\Omega) = \left\{ \boldsymbol{u} \in L^1_{loc}(\Omega) : \|\boldsymbol{u}\|_{\widetilde{H}_q} < \infty \right\}, \tag{III.2.4}$$

where

$$\|\boldsymbol{u}\|_{\widetilde{H}_q} \equiv \|\boldsymbol{u}\|_q + \|\nabla \cdot \boldsymbol{u}\|_q. \tag{III.2.5}$$

Clearly, the functional (III.2.5) defines a norm in $\widetilde{H}_q$ and, by a simple reasoning, one shows that $\widetilde{H}_q$ is complete under this norm; see Exercise III.2.1.

The following result holds.

Theorem III.2.1 *Let Ω be locally Lipschitz. Then, $C_0^\infty(\overline{\Omega})$ is dense in $\widetilde{H}_q(\Omega)$, for all $q \in [1, \infty)$.*

Proof. Assume first Ω bounded. From Lemma II.1.3 we find a finite open covering of $\overline{\Omega}$, denoted by $\mathcal{G} = \{G_0, G_1, \dots, G_m\}$, with the following properties: (i) $\overline{G_0} \subset \Omega$, (ii) $\partial\Omega \subset \cup_{i=1}^m G_i$, and (iii) $\Omega_i \equiv \Omega \cap G_i$, $i = 1, \dots, m$, is a star-shaped domain with respect to some interior point x_i. We extend $\boldsymbol{u}|_{\Omega_i}$ to zero outside Ω_i, continue to denote by $\boldsymbol{u}$ this extension. Let $\{\psi_i\}_{i=0,1,\dots,m}$ be a partition of unity of $\overline{\Omega}$ subordinate to $\mathcal{G}$ (see Lemma II.1.4), and set $\boldsymbol{u}_i = \psi_i \boldsymbol{u}$. The result then follows if, for any $\varepsilon > 0$, we can find $\boldsymbol{\varphi}_i \in C_0^\infty(\mathbb{R}^n)$ such that

$$\|\boldsymbol{u}_i - \boldsymbol{\varphi}_i\|_{\widetilde{H}_q(\Omega)} < \varepsilon, \quad \text{for all } i = 0, 1, \dots, m. \tag{III.2.6}$$

In fact, setting $\boldsymbol{\Phi} = \sum_{i=0}^m \boldsymbol{\varphi}_i$, and observing that $\sum_{i=0}^m \psi_i(x) = 1$, $x \in \Omega$, from (III.2.6) we obtain

$$\|\boldsymbol{u} - \boldsymbol{\Phi}\|_{\widetilde{H}_q(\Omega)} \le \sum_{i=0}^m \|\boldsymbol{u}_i - \boldsymbol{\varphi}_i\|_{\widetilde{H}_q(\Omega)} < (m+1)\,\varepsilon.$$

Because of the properties of G_0 and of ψ_0, the mollifier $(\boldsymbol{u}_0)_\eta$ of $\boldsymbol{u}_0$ belongs to $C_0^\infty(\Omega)$, for all sufficiently small $\eta > 0$. Therefore, from (II.2.9) and Exercise II.3.2, we at once obtain

$$\|\boldsymbol{u}_0 - (\boldsymbol{u}_0)_\eta\|_{\widetilde{H}_q(\Omega)} \to 0 \text{ as } \eta \to 0. \tag{III.2.7}$$

We next pick $i \in \{1, \dots, m\}$. By means of a translation in $\mathbb{R}^n$, we may take $x_i = 0$. Then, the domain

$$\Omega_i^{(\rho)} = \{x \in \mathbb{R}^n : \rho x \in \Omega_i\} \tag{III.2.8}$$

satisfies $\Omega_i^{(\rho)} \supset \overline{\Omega}_i$, for $\rho \in (0, 1)$; see Exercise II.1.3. Setting

$$\boldsymbol{u}_\rho = \boldsymbol{u}_\rho(x) \equiv \boldsymbol{u}(\rho\, x)\,, \quad x \in \Omega_i^{(\rho)}\,, \tag{III.2.9}$$

and recalling that $\psi_i \in C_0^\infty(G_i)$, by the properties of the mollifier we deduce $\psi_i(\boldsymbol{u}_\rho)_\eta \in C_0^\infty(\mathbb{R}^n)$, for all $\eta > 0$. Since $|x| < \delta(\Omega_i)$, for $x \in \Omega_i$, from the continuity in the mean property (see Exercise II.2.8), given $\varepsilon > 0$, we can find $\rho \in (0,1)$ such that

$$\|\boldsymbol{u} - \boldsymbol{u}_\rho\|_{q,\Omega_i} < \varepsilon\,.$$

Furthermore, from (II.2.9) we also have, for some $\eta = \eta(\rho) > 0$,

$$\|\boldsymbol{u}_\rho - (\boldsymbol{u}_\rho)_\eta\|_{q,\Omega_i} < \varepsilon\,,$$

so that we conclude

$$\|\boldsymbol{u}_i - \psi_i(\boldsymbol{u}_\rho)_\eta\|_{q,\Omega} \le \|\boldsymbol{u} - (\boldsymbol{u}_\rho)_\eta\|_{q,\Omega_i} \le \|\boldsymbol{u} - \boldsymbol{u}_\rho\|_{q,\Omega_i} + \|\boldsymbol{u}_\rho - (\boldsymbol{u}_\rho)_\eta\|_{q,\Omega_i} < 2\,\varepsilon\,. \tag{III.2.10}$$

By the same token and by Exercise II.3.2,

$$\begin{aligned}\|\nabla\cdot(\boldsymbol{u}_i - \psi_i(\boldsymbol{u}_\rho)_\eta)\,\|_{q,\Omega} &\le C\,\|\boldsymbol{u} - (\boldsymbol{u}_\rho)_\eta\|_{q,\Omega_i} + \|\nabla\cdot\boldsymbol{u}_\rho - (\nabla\cdot\boldsymbol{u}_\rho)_\eta\|_{q,\Omega_i} \\ &\quad + \|\nabla\cdot\boldsymbol{u} - \nabla\cdot\boldsymbol{u}_\rho\|_{q,\Omega_i} \\ &\le C\,\varepsilon + \varepsilon + \|\nabla\cdot\boldsymbol{u} - \nabla\cdot\boldsymbol{u}_\rho\|_{q,\Omega_i}\,.\end{aligned} \tag{III.2.11}$$

We now notice that, setting $\chi(x) = \nabla\cdot\boldsymbol{u}(x)$, $x \in \Omega_i$, by Exercise II.3.3 we have $\nabla\cdot\boldsymbol{u}_\rho(x) = \rho\,\chi(\rho\, x)$. As a consequence,

$$\|\nabla\cdot\boldsymbol{u} - \nabla\cdot\boldsymbol{u}_\rho\|_{q,\Omega_i}^q \le (1-\rho)\|\nabla\cdot\boldsymbol{u}\|_{q,\Omega_i}^q + \rho^q \int_{\Omega_i} |\chi(x) - \chi(\rho\, x)|^q\,.$$

Thus, again by the continuity in the mean property (see Exercise II.2.8), for ρ sufficiently close to 1, we deduce $\|\nabla\cdot\boldsymbol{u} - \nabla\cdot\boldsymbol{u}_\rho\|_{q,\Omega_i} < \varepsilon$, which, along with (III.2.7)–(III.2.11) allows us to conclude the validity of (III.2.6). This concludes the proof when Ω is bounded. Next, assume Ω exterior, and, for sufficiently small $\eta > 0$, let ψ_η be a "cut-off" function that is 1 in $\Omega_{1/\eta}$, 0 in $\overline{\Omega^{2/\eta}}$ and satisfies $|\nabla\psi_\eta| \le M\,\eta$, with M independent of η. It is at once recognized that, for any $\boldsymbol{u} \in \widetilde{H}_q(\Omega)$, we have $\boldsymbol{u}_\eta \equiv \psi_\eta \boldsymbol{u} \in \widetilde{H}_q(\Omega)$. Given $\varepsilon > 0$, we choose η such that

$$(1 + M\eta)\|\boldsymbol{u}\|_{q,\Omega^{1/\eta}} + \|\nabla\cdot\boldsymbol{u}\|_{q,\Omega^{1/\eta}} < \varepsilon\,. \tag{III.2.12}$$

Since $\mathrm{supp}\,(\boldsymbol{u}_\eta) \subset \overline{\Omega_{2/\eta}}$, following step by step the proof just given in the case Ω bounded with, this time, $\{\psi_i\}$ partition of unity in $\overline{\Omega_{2/\eta}}$, we show that, corresponding to the given $\varepsilon > 0$, there is $\boldsymbol{\varphi}_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ such that

$$\|\boldsymbol{u}_\eta - \boldsymbol{\varphi}_\varepsilon\|_{\widetilde{H}_q(\Omega)} < \varepsilon\,. \tag{III.2.13}$$

Therefore, by the triangle inequality and the properties of ψ_η, with the help of (III.2.12) and (III.2.13), we conclude

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{\varphi}_\varepsilon\|_{\widetilde{H}_q(\Omega)} &\leq \|\boldsymbol{u}-\boldsymbol{u}_\eta\|_{\widetilde{H}_q(\Omega)} + \|\boldsymbol{u}_\eta-\boldsymbol{\varphi}_\varepsilon\|_{\widetilde{H}_q(\Omega)}\\ &\leq (1+M\eta)\|\boldsymbol{u}\|_{q,\Omega^{1/\eta}} + \|\nabla\cdot\boldsymbol{u}\|_{q,\Omega^{1/\eta}} + \varepsilon < 2\varepsilon\end{aligned}$$

The proof of the theorem is thus completed. □

With the help of Theorem III.2.1 we are now able to define, suitably, the trace of the normal component of $\boldsymbol{u} \in \widetilde{H}_q(\Omega)$ at $\partial\Omega$, provided Ω is locally Lipschitz. Actually, fix $\boldsymbol{u} \in C_0^\infty(\overline{\Omega})$, and consider the following linear functional $F_{\boldsymbol{u}}$ on $W^{1-1/q',q'}(\partial\Omega)$, $1 < q < \infty$:

$$F_{\boldsymbol{u}}(\omega) = \int_{\partial\Omega} \omega\, \boldsymbol{n}\cdot\boldsymbol{u}\,, \quad \omega \in W^{1-1/q',q'}(\partial\Omega).$$

Obviously, this functional is determined once the value of the normal component of $\boldsymbol{u}$ at the boundary is specified. Let $\boldsymbol{n}\cdot$ be the linear map that to each $\boldsymbol{u} \in C_0^\infty(\overline{\Omega})$ prescribes the corresponding functional $F_{\boldsymbol{u}}$ defined above; that is,

$$\boldsymbol{n}\cdot\boldsymbol{u} = F_{\boldsymbol{u}}.$$

Let us denote by $W^{-1/q,q}(\partial\Omega)$ the (strong) dual of $W^{1-1/q',q'}(\partial\Omega)$, and by $\langle\cdot,\cdot\rangle_{\partial\Omega}$ the corresponding duality pair. Using Gagliardo's Theorem II.4.3 one then proves that $\boldsymbol{n}\cdot(\cdot)\,|_{\partial\Omega}$ can be extended to a bounded (linear) operator from $\widetilde{H}_q$ into $W^{-1/q,q}(\partial\Omega)$. In fact, by that theorem we can extend ω to a function $\varphi \in W^{1,q'}(\Omega)$ such that

$$\|\varphi\|_{1,q'} \leq c_1 \|\omega\|_{1-1/q',q'(\partial\Omega)}.$$

Thus, by identity (III.2.2), the Hölder inequality, and (III.2.5) we obtain

$$\begin{aligned}|\langle F_{\boldsymbol{u}},\omega\rangle_{\partial\Omega}| = \left|\int_\Omega (\boldsymbol{u}\cdot\nabla\varphi + \varphi\nabla\cdot\boldsymbol{u})\right| &\leq \|\boldsymbol{u}\|_{\widetilde{H}_q}\|\varphi\|_{1,q'}\\ &\leq c_1\|\boldsymbol{u}\|_{\widetilde{H}_q}\|\omega\|_{1-1/q',q'(\partial\Omega)},\end{aligned}$$

implying

$$\|\boldsymbol{n}\cdot\boldsymbol{u}\|_{W^{-1/q,q}(\partial\Omega)} \leq c_1\|\boldsymbol{u}\|_{\widetilde{H}_q}, \quad \text{for all } \boldsymbol{u} \in C_0^\infty(\overline{\Omega}),$$

which is what we wanted to prove. Now, by the standard procedure used to define generalized traces (see Theorem II.4.1), since, by Theorem III.2.1, $C_0^\infty(\overline{\Omega})$ is dense in $\widetilde{H}_q(\Omega)$, we may extend, by continuity, the map $\boldsymbol{n}\cdot$ to the whole of $\widetilde{H}_q(\Omega)$. Moreover, the following generalization of (II.4.21) and (III.2.2) holds:

$$\langle\boldsymbol{n}\cdot\boldsymbol{u},\omega\rangle_{\partial\Omega} = \int_\Omega \boldsymbol{u}\cdot\nabla\varphi + \int_\Omega \varphi\nabla\cdot\boldsymbol{u}, \quad \varphi \in W^{1,q'}(\Omega), \qquad \text{(III.2.14)}$$

where $\omega = \gamma(\varphi)$ is the trace of φ at $\partial\Omega$.

The above results are summarized in the following

Theorem III.2.2 *Assume Ω locally Lipschitz, and let $\boldsymbol{u} \in \widetilde{H}_q(\Omega)$, $1 < q < \infty$. Then $\boldsymbol{n} \cdot \boldsymbol{u} \in W^{-1/q,q}(\partial\Omega)$ and the generalized Gauss identity* (III.2.14) *holds.*

After having obtained generalizations of the trace of the normal component of a vector field at the boundary and of (III.2.2), it is now straightforward to obtain the desired characterization of *any* element $\boldsymbol{u}$ of the space $H_q(\Omega)$. In fact, by an argument completely analogous to that used in the proof of Lemma III.2.2 we prove the following.

Theorem III.2.3 *Let*

$$H_q'(\Omega) = \{\boldsymbol{u} \in L^q(\Omega) : \nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{n} \cdot \boldsymbol{u} = 0 \text{ at } \partial\Omega\}.$$

Then, for any locally Lipschitz domain Ω of $\mathbb{R}^n$, $n \geq 2$, we have

$$H_q'(\Omega) = H_q(\Omega).$$

Remark III.2.2 The coincidence of the spaces $H_q'(\Omega)$ and $H_q(\Omega)$ can be proved for any (sufficiently smooth) domain for which identity (III.2.14) holds. However, such a coincidence certainly does *not* hold for certain domains with noncompact boundary; see Remark III.4.1. ■

Exercise III.2.1 Show that the space $\widetilde{H}_q(\Omega)$ endowed with the norm (III.2.5) is a Banach space.

Exercise III.2.2 Prove the results of Theorem III.2.3 to the case where $\Omega = \mathbb{R}^n_+$, $n \geq 2$.

Another question that will play an important role later is that of characterizing the kernel of the map $\boldsymbol{n}\cdot$. In this regard, we have the following result, of which Theorem III.2.3 is a special case.

Theorem III.2.4 *Let Ω be a locally Lipschitz domain in $\mathbb{R}^n, n \geq 2$, and let $\widetilde{H}_{0,q} = \widetilde{H}_{0,q}(\Omega)$ designate the completion of $C_0^\infty(\Omega)$ in the norm* (III.2.5). *Then, for $q \in (1, \infty)$ we have that*

$$\widetilde{H}_{0,q}(\Omega) = \left\{\boldsymbol{u} \in \widetilde{H}_q(\Omega) : \boldsymbol{n} \cdot \boldsymbol{u} = 0 \text{ at } \partial\Omega\right\}. \tag{III.2.15}$$

Proof. Denote by $\widetilde{\widetilde{H}}_{0,q}(\Omega)$ the space on the right-hand side of (III.2.15). It is clear that $\widetilde{H}_{0,q}(\Omega)$ is a closed subspace of $\widetilde{\widetilde{H}}_{0,q}(\Omega)$. Therefore, we only have to show that every function from $\widetilde{\widetilde{H}}_{0,q}(\Omega)$ can be approximated by functions from $\mathcal{D}(\Omega)$ in the norm (III.2.5). To this end, we observe that the extension of $\boldsymbol{u} \in \widetilde{\widetilde{H}}_{0,q}(\Omega)$ to the whole of $\mathbb{R}^n$, obtained by setting $\boldsymbol{u} = 0$ outside Ω, is an element of $\widetilde{H}_{0,q}(\mathbb{R}^n)$; see Exercise III.2.3. Let us denote by $\overline{\boldsymbol{u}}$ this extension. Next, let Ω_i, $i = 0, \ldots, m$, and $\{\psi\}_{0,1,\ldots,m}$ be the domains and the partition of unity introduced in the proof of Theorem III.2.1. Also, let $\Omega_i^{(\rho)}$ be the domain defined in (III.2.8) but, this time, with $\rho > 1$, so that $\Omega_i^{(\rho)} \subset \overline{\Omega}_i$. It is then clear that the function $\boldsymbol{u}_i(x) = \psi_i(x)\overline{\boldsymbol{u}}_\rho(x)$, $x \in \Omega_i^{(\rho)}$, with $\boldsymbol{u}_\rho$ defined in (III.2.9), is of compact support in Ω and belongs to $\widetilde{H}_q(\Omega)$, and that its mollifier, $(\psi_i\overline{u})_\eta$, is in $C_0^\infty(\Omega)$, for all sufficiently small $\eta > 0$. The result then follows by using exactly, from now onward, the same procedure used in the proof of Theorem III.2.1. □

Finally, it remains to investigate the properties of the function space $G_q(\Omega)$. However, we notice that members of $G_q(\Omega)$ are gradients of functions belonging to $D^{1,q}(\Omega)$ and, in particular, it is easily shown that, in view of Lemma II.6.2, $G_q(\Omega)$ and $\dot{D}^{1,q}(\Omega)$ are isomorphic via the mapping

$$i : \boldsymbol{u} \in G_q(\Omega) \to i(\boldsymbol{u}) \in \dot{D}^{1,q}(\Omega),$$

where $i(\boldsymbol{u})$ is the class of functions $p \in D^{1,q}(\Omega)$ such that $\boldsymbol{u} = \nabla p$. We may then conclude that all relevant properties of $G_q(\Omega)$ are immediately obtainable from the analogous ones established for the space $\dot{D}^{1,q}(\Omega)$ in Section II.6.

Exercise III.2.3 Show that if $\boldsymbol{u} \in \widetilde{\widetilde{H}}_{0,q}(\Omega)$, with Ω locally Lipschitz, then its extension to $\mathbb{R}^n$, obtained by setting $\boldsymbol{u} = \boldsymbol{0}$ in $\mathbb{R}^n - \Omega$, belongs to $\widetilde{H}_{0,q}(\mathbb{R}^n)$. *Hint*: Use (III.2.14).

III.3 The Problem $\nabla \cdot \boldsymbol{v} = f$

In the proof of several results of this chapter we shall often consider an auxiliary problem whose interest goes well beyond this particular context. Actually, we already encountered it in the proof of Theorem III.1.2, dealing with the Helmholtz–Weyl decomposition of the space $L^q(\Omega)$.

The problem consists, essentially, in representing a scalar function as the divergence of a vector field in suitable function spaces and determining corresponding estimates. The resolution of such a problem is a fundamental tool in several questions of mathematical fluid mechanics and, therefore, we find it convenient to investigate it to some extent.

Let us begin to consider the case when Ω is a *bounded domain* in $\mathbb{R}^n$, $n \geq 2$. The problem is then formulated as follows: *Given*

$$f \in L^q(\Omega)$$

with

$$\int_\Omega f = 0, \tag{III.3.1}$$

to find a vector field $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ *such that*

$$\begin{aligned}&\nabla\cdot\boldsymbol{v}=f\\&\boldsymbol{v}\in W_0^{1,q}(\Omega)\\&|\boldsymbol{v}|_{1,q}\le c\,\|f\|_q\end{aligned}\tag{III.3.2}$$

where $c=c(n,q,\Omega)$.

Notice that (III.3.1) represents a *compatibility condition*, as a consequence of $(\text{III.3.2})_1$ and $(\text{III.3.2})_2$. Also, since Ω is bounded, we may use the inequality (II.5.1) into $(\text{III.3.2})_3$ to deduce the stronger estimate

$$\|\boldsymbol{v}\|_{1,q}\le c_1\|f\|_q.\tag{III.3.3}$$

Problem (III.3.1), (III.3.2) (which, of course, does not admit a unique solution) has been studied by several authors and with different methods (see the Notes for this Chapter). Here, we shall follow the approach of Bogovskiĭ (1979, 1980) based on an explicit representation formula (see (III.3.8)), which requires little regularity for Ω, e.g., Ω locally Lipschitz. In this latter respect it should be emphasized that some regularity on Ω *is* in fact necessary for the solvability of the problem; see Remark III.3.9. We also point out that the difficulty with (III.3.2) relies in the fact that we require that $\boldsymbol{v}$ *vanishes* (in a suitable sense) at $\partial\Omega$. If this condition is removed, resolution of the problem is trivial; see Exercise III.3.1

To begin with, we assume that Ω is of a special shape. Specifically, we have

Lemma III.3.1 *Let $\Omega\subset\mathbb{R}^n$, $n\ge 2$, be star-like with respect to every point of $B_R(x_0)$ with $\overline{B}_R(x_0)\subset\Omega$. Then for any $f\in L^q(\Omega)$, $1<q<\infty$, satisfying* (III.3.1), *problem* (III.3.2) *has at least one solution $\boldsymbol{v}$. Moreover, the constant c in* $(\text{III.3.2})_3$ *admits the following estimate*

$$c\le c_0\,[\delta(\Omega)/R]^n\,(1+\delta(\Omega)/R),\tag{III.3.4}$$

with $c_0=c_0(n,q)$. Finally, if $f\in C_0^\infty(\Omega)$ then $\boldsymbol{v}\in C_0^\infty(\Omega)$.

Proof. Let us assume first $f\in C_0^\infty(\Omega)$. By the change of variables

$$x\to x'=(x-x_0)/R,\tag{III.3.5}$$

we shift the origin of coordinates to the point x_0 and transform $B_R(x_0)$ into $B_1(0)\equiv B$. Moreover, Ω goes into a domain Ω' that is star-like with respect to every point of B with

$$\delta(\Omega')=\delta(\Omega)/R,\tag{III.3.6}$$

while $\boldsymbol{v}$ goes into $\boldsymbol{v}'$, f into f' and equation $(\text{III.3.2})_1$ becomes

$$\nabla\cdot\boldsymbol{v}'=Rf'\equiv F'\quad\text{in }\Omega,\tag{III.3.7}$$

where, of course, ∇ operates on the primed variables. Clearly, F' has mean value zero in Ω' and $F'\in C_0^\infty(\Omega')$. Furthermore, if $\boldsymbol{v}'$, F' satisfy (III.3.7), the transformed functions $\boldsymbol{v}$ and f through the inverse of (III.3.5) satisfy $(\text{III.3.2})_1$. Let now ω be any function from $C_0^\infty(\mathbb{R}^n)$ such that

(i) $\operatorname{supp}(\omega) \subset B$,

(ii) $\displaystyle\int_B \omega = 1$.

We wish to show that the vector field[1]

$$
\begin{aligned}
\boldsymbol{v}(x) &= \int_\Omega F(y) \left[\frac{\boldsymbol{x}-\boldsymbol{y}}{|x-y|^n} \int_{|x-y|}^\infty \omega\left(y + \xi \frac{x-y}{|x-y|} \right) \xi^{n-1} d\xi \right] dy \\
&\equiv \int_\Omega F(y) \boldsymbol{N}(x,y) dy
\end{aligned}
\tag{III.3.8}
$$

solves (III.3.7), where, for simplicity, we have omitted primes. By a straightforward calculation, we easily show that the field $\boldsymbol{v}$ can be written in the following equivalent useful forms

$$
\begin{aligned}
\boldsymbol{v}(x) &= \int_\Omega F(y)(\boldsymbol{x}-\boldsymbol{y}) \left[\int_1^\infty \omega(y + r(x-y)) r^{n-1} dr \right] dy \\
\boldsymbol{v}(x) &= \int_\Omega F(y) \frac{\boldsymbol{x}-\boldsymbol{y}}{|x-y|^n} \left[\int_0^\infty \omega\left(x + r \frac{x-y}{|x-y|} \right) (|x-y| + r)^{n-1} dr \right] dy.
\end{aligned}
\tag{III.3.9}
$$

Making into the integral $(\text{III.3.9})_2$ the change of variable $z = x - y$, we recover at once $\boldsymbol{v} \in C^\infty(\mathbb{R}^n)$. Moreover, from $(\text{III.3.9})_1$, it follows that $\boldsymbol{v}$ is of compact support in Ω. In fact, set

$$
E = \left\{ z \in \Omega : z = \lambda z_1 + (1-\lambda) z_2,\ z_1 \in \operatorname{supp}(F),\ z_2 \in \overline{B},\ \lambda \in [0,1] \right\}.
\tag{III.3.10}
$$

Since Ω is star-like with respect to every point of B, E is a compact subset of Ω. Fix $x \in \Omega - E$. For all $y \in \operatorname{supp}(F)$ and all $r \geq 1$,

$$
y + r(x-y) \notin \overline{B}
$$

and, therefore, $\omega(y + r(x-y)) = 0$, i.e., by $(\text{III.3.9})_1$, $\boldsymbol{v}(x) = 0$. We thus conclude

$$
\boldsymbol{v} \in C_0^\infty(\Omega).
\tag{III.3.11}
$$

Surrounding the point $x \in \Omega$ with a ball $B_\varepsilon(x)$ of radius ε sufficiently small and using integration by parts, from (III.3.8) one has

$$
D_j v_i(x) = \lim_{\varepsilon \to 0} \left(\int_{B_\varepsilon^c(x)} F(y) D_j N_i(x,y) dy + \int_{\partial B_\varepsilon(x)} F(y) \frac{x_j - y_j}{|x-y|} N_i(x,y) d\sigma_y \right).
\tag{III.3.12}
$$

It is simple to show

[1] The equation (III.3.8) is sometimes referred to as "Bogovskiĭ formula." It is a generalization of a similar representation due to Sobolev; see the Notes at the end of this chapter.

$$\lim_{\varepsilon\to 0}\int_{|x-y|=\varepsilon} F(y)\frac{x_j-y_j}{|x-y|}N_i(x,y)d\sigma_y = F(x)\int_\Omega \frac{(x_j-y_j)(x_i-y_i)}{|x-y|^2}\omega(y)dy. \tag{III.3.13}$$

Actually, denoting by I_ε the integral on the left-hand side of (III.3.13),

$$\begin{aligned}\Delta_\varepsilon(x) &\equiv \left|I_\varepsilon(x) - F(x)\int_\Omega \frac{(x_j-y_j)(x_i-y_i)}{|x-y|^2}\omega(y)dy\right| \\ &= \left|\int_{|z|=1}\left\{z_iz_jF(x-\varepsilon z)\int_0^\infty \omega(x+rz)(r+\varepsilon)^{n-1}dr\right\}d\sigma_z\right. \\ &\qquad \left. -F(x)\int_{|z|=1}\left\{z_iz_j\int_0^\infty \omega(x+rz)r^{n-1}dr\right\}d\sigma_z\right|\end{aligned}$$

and so, in the limit $\varepsilon\to 0$ it follows

$$\Delta_\varepsilon(x) \le \int_{|z|=1}|F(x)-F(x-\varepsilon z)|d\sigma_z + o(1),$$

which proves (III.3.13). On the other hand, the first limit on the right-hand side of (III.3.12) exists as a consequence of the Calderón–Zygmund Theorem II.11.4. To see this, we observe that from $(\text{III.3.9})_1$ we have for fixed y

$$\begin{aligned}D_jN_i(x,y) &= D_j\left[(x_i-y_i)\int_1^\infty \omega(y+r(x-y))r^{n-1}dr\right] \\ &= \delta_{ij}\int_1^\infty \omega(y+r(x-y))r^{n-1}dr \\ &\qquad +(x_i-y_i)\int_1^\infty D_j\omega(y+r(x-y))r^n dr \\ &= \frac{\delta_{ij}}{|x-y|^n}\int_0^\infty \omega\left(x+r\frac{x-y}{|x-y|}\right)(|x-y|+r)^{n-1}dr \\ &\qquad +\frac{x_i-y_i}{|x-y|^{n+1}}\int_0^\infty D_j\omega\left(x+r\frac{x-y}{|x-y|}\right)(|x-y|+r)^n dr\end{aligned} \tag{III.3.14}$$

By expanding the powers of n in the last two integrals it easily follows that $D_jN_i(x,y)$ can be decomposed as

$$D_jN_i(x,y) = K_{ij}(x,x-y) + G_{ij}(x,y), \tag{III.3.15}$$

where

$$
\begin{aligned}
K_{ij}(x, x-y) = & \frac{\delta_{ij}}{|x-y|^n} \int_0^\infty \omega \left(x + r \frac{x-y}{|x-y|} \right) r^{n-1} dr \\
& + \frac{x_i - y_i}{|x-y|^{n+1}} \int_0^\infty D_j \omega \left(x + r \frac{x-y}{|x-y|} \right) r^n dr \\
\equiv & \frac{k_{ij}(x, x-y)}{|x-y|^n}
\end{aligned}
\tag{III.3.16}
$$

while G_{ij} admits an estimate of the type

$$
|G_{ij}(x,y)| \leq c \frac{\delta(\Omega)^{n-1}}{|x-y|^{n-1}}, \quad x, y \in \Omega, \tag{III.3.17}
$$

where $c = c(\omega, n)$. It is readily seen that, for each i and j, $K_{ij}(x,z)$ is a singular kernel, i.e., that $k_{ij}(x,z)$ satisfies all conditions (II.11.15)–(II.11.17). Actually, (II.11.15) is at once satisfied. Concerning (II.11.17), we notice that

$$
\begin{aligned}
|k_{ij}(x,z)| \leq & \left| \int_0^\infty \omega\,(x + rz)\, r^{n-1} dr \right| + \left| \int_0^\infty D_j \omega\,(x+rz)\, r^n dr \right| \\
\leq & \|\omega\|_\infty \frac{\delta(\Omega)^n}{n} + \|D_j\omega\|_\infty \frac{\delta(\Omega)^{n+1}}{n+1} \quad \text{for } |z| = 1\,.
\end{aligned}
\tag{III.3.18}
$$

Therefore, also (II.11.17) is satisfied. Furthermore,

$$
\begin{aligned}
\int_{|z|=1} k_{ij}(x,z) = & \,\delta_{ij} \int_{|z|=1} \int_0^\infty \omega(x+rz) r^{n-1} dr \\
& + \int_{|z|=1} z_i \int_0^\infty D_j \omega(x+rz) r^n dr \\
= & \int_{\mathbb{R}^n} [\delta_{ij}\omega(x+y) + y_i D_j \omega(x+y)]\, dy = 0
\end{aligned}
$$

and so condition (II.11.16) is satisfied as well. Consequently, from (III.3.15)–(III.3.17), the first limit on the right-hand side of (III.3.12) exists and (III.3.12) can be rewritten as

$$
\begin{aligned}
D_j v_i(x) = & \int_\Omega K_{ij}(x, x-y) F(y) dy + \int_\Omega G_{ij}(x,y) F(y) dy \\
& + F(x) \int_\Omega \frac{(x_j - y_j)(x_i - y_i)}{|x-y|^2} \omega(y) dy \\
\equiv & \, F_1(x) + F_2(x) + F_3(x),
\end{aligned}
\tag{III.3.19}
$$

where the first integral has to be understood in the Cauchy principal value sense. We next show that (III.3.8) is a solution to (III.3.7). To this end, from (III.3.12)–(III.3.14) and property (ii) of ω we have

$$
\begin{aligned}
\nabla \cdot \boldsymbol{v} = & \int_{\Omega} F(y) \left(n \int_1^{\infty} \omega(y + r(x-y)) r^{n-1} dr \right. \\
& \left. + \sum_{i=1}^{n} \int_1^{\infty} (x_i - y_i) D_i \omega(y + r(x-y)) r^n dr \right) dy \\
& + \sum_{i=1}^{n} F(x) \int_{\Omega} \frac{(x_i - y_i)(x_i - y_i)}{|x-y|^2} \omega(y) dy \\
= & \int_{\Omega} F(y) \left[n \int_1^{\infty} \omega(y + r(x-y)) r^{n-1} dr \right. \\
& \left. + \int_1^{\infty} r^n \left(\frac{d}{dr} \omega(y + r(x-y)) \right) dr \right] dy + F(x) \\
= & -\omega(x) \int_{\Omega} F(x) + F(x)
\end{aligned}
$$

and so, since F has mean value zero over Ω,

$$
\nabla \cdot \boldsymbol{v}(x) = F(x), \quad x \in \Omega, \tag{III.3.20}
$$

which proves (III.3.7). It remains to show that $\boldsymbol{v}$ satisfies $(\text{III.3.2})_3$. For $1 < q < \infty$, from (III.3.16) and (III.3.19), by the Calderón–Zygmund Theorem II.11.4 we obtain

$$
\|F_1\|_q \le c_1 \|F\|_q,
$$

while Young's inequality (II.11.2) and $(\text{III.3.15})_2$ furnish

$$
\|F_2\|_q \le c_2 \delta(\Omega)^n \|F\|_q.
$$

Finally, we obviously have

$$
\|F_3\|_q \le c_3 \delta(\Omega)^n \|F\|_q.
$$

We wish to emphasize that the constants c_2 and c_3 depend on ω, n, q but *not* on Ω. As far as the constant c_1 is concerned, from (III.3.18) and Remark II.11.2 we obtain

$$
c_1 \le c_4 \delta(\Omega)^n (1 + \delta(\Omega)),
$$

where $c_4 = c_4(n, q, \omega)$. Restoring the primed notation, from the previous inequalities we recover

$$
|\boldsymbol{v}'|_{1,q,\Omega'} \le c_5 \delta(\Omega')^n (1 + \delta(\Omega')) \|F'\|_{q,\Omega'},
$$

with $c_5 = c_5(n, q)$. Coming back to the original variables via the inverse of transformation (III.3.4), recalling (III.3.6) and $F' = Rf'$, we obtain that the transformed solution $\boldsymbol{v}$ also satisfies $(\text{III}.3.2)_3$ with a constant c obeying (III.3.4). To complete the proof, we have to show solvability for arbitrary f in $L^q(\Omega)$ (obeying, of course, (III.3.1)). Thus, let $f \in L^q(\Omega)$ satisfy (III.3.1) and let $\{f_m\} \subset C_0^\infty(\Omega)$ be a sequence approximating f in $L^q(\Omega)$. Then, the functions

$$f_m^* = f_m - \varphi \int_\Omega f_m, \quad m \in \mathbb{N}$$

with

$$\varphi \in C_0^\infty(\Omega), \quad \int_\Omega \varphi = 1$$

still approximate f in $L^q(\Omega)$ and, at the same time, they obey (III.3.1) for all $m \in \mathbb{N}$. By what we have just shown, corresponding to each $m \in \mathbb{N}$ we can find a solution $\boldsymbol{v}_m \in C_0^\infty(\Omega)$. By the estimate (III.3.3) and the linearity of problem $(\text{III}.3.2)_{1,2}$, as $m \to \infty$ the sequence $\{\boldsymbol{v}_m\}$ converges (strongly) in $W_0^{1,q}(\Omega)$ to a function $\boldsymbol{v} \in W_0^{1,q}(\Omega)$ that obeys $(\text{III}.3.2)_{1,3}$ in the sense of generalized differentiation. The lemma is therefore proved. □

Remark III.3.1 The result just shown admits of a straightforward generalization to the case when $f \in L^q(\Omega) \cap L^r(\Omega)$, $1 < q, r < \infty$. Specifically, one easily shows that there exists a solution to $(\text{III}.3.2)_1$, which further satisfies

$$\begin{gathered}\boldsymbol{v} \in W_0^{1,q}(\Omega) \cap W_0^{1,r}(\Omega) \\ |\boldsymbol{v}|_{1,q} \leq c\,\|f\|_q \\ |\boldsymbol{v}|_{1,r} \leq c\,\|f\|_r .\end{gathered}$$

■

Remark III.3.2 Formula (III.3.8) allows us to obtain solutions to (III.3.1) and (III.3.2), in a domain Ω star-like with respect to a ball in the sense specified in Lemma III.3.1, that satisfy estimates of the type (III.3.3) in Sobolev spaces $W_0^{m,q}(\Omega)$ of *arbitrary* order. To show this, for two multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, we set $\beta \leq \alpha$ to mean $\beta_i \leq \alpha_i$ for all $i = 1, \dots, n$ and, in such a case, we put

$$D^{\alpha-\beta} \equiv \frac{\partial^{|\alpha|-|\beta|}}{\partial x_1^{\alpha_1-\beta_1} \dots \partial x_n^{\alpha_n-\beta_n}}, \qquad \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \equiv \begin{pmatrix}\alpha_1 \\ \beta_1\end{pmatrix} \cdots \begin{pmatrix}\alpha \\ \beta\end{pmatrix}.$$

Applying the operator D^α to both sides of (III.3.8), integrating by parts, and using the Leibnitz rule we then find for $F \in C_0^\infty(\Omega)$

$$D^\alpha \boldsymbol{v}(x) = \sum_{\beta \leq \alpha} \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \int_\Omega \boldsymbol{N}_\beta(x,y) D^{\alpha-\beta} F(y) dy, \tag{III.3.21}$$

where

$$\boldsymbol{N}_\beta(x,y) = (\boldsymbol{x}-\boldsymbol{y})\int_1^\infty D^\beta\omega(y+r(x-y))r^{n-1}dr. \qquad \text{(III.3.22)}$$

Taking into account that $\boldsymbol{N}_\beta$ has the same properties as $\boldsymbol{N}$, we apply the same reasonings employed before to deduce the following inequality for all $f \in C_0^\infty(\Omega)$

$$\|\nabla\boldsymbol{v}\|_{\ell,q} \le c\,\|f\|_{\ell,q} \qquad \text{(III.3.23)}$$

for all $\ell \ge 0$ and $q \in (1,\infty)$, where c satisfies an estimate of the type (III.3.4). Using (III.3.15) along with a density argument of the type adopted in the last part of the proof of Lemma III.3.1, we thus obtain, in particular, a solution $\boldsymbol{v}$ to (III.3.1) and (III.3.2) for any f in $W_0^{m,q}(\Omega)$. Such a $\boldsymbol{v}$ belongs to $W_0^{m+1,q}(\Omega)$ and satisfies (III.3.23) for all $\ell = 0,\dots,m$. ■

Remark III.3.3 In several applications, the function f depends on a parameter $t \in I$, where I is an interval in $\mathbb{R}$. In such a case, assuming that Ω is star-like with respect to a ball, one immediately obtains from the representation (III.3.8) and the more general (III.3.21), (III.3.22), that if $f(t) \in C_0^\infty(\Omega)$, $t \in I$, is continuous in t in the $W^{m,q}$-norm, then the corresponding $\boldsymbol{v} = \boldsymbol{v}(x,t)$ given by (III.3.8) is continuous in the $W^{m+1,q}$-norm and one has

$$\|\boldsymbol{v}(t_1) - \boldsymbol{v}(t_2)\|_{\ell+1,q} \le c_1\|f(t_1) - f(t_2)\|_{\ell,q}\,, \quad t_1, t_2 \in I\,, \quad \ell = 0,\dots,m\,.$$

Likewise, if f is differentiable in t, with the help of (III.3.21), (III.3.22), we find that the field $\boldsymbol{v}(x,t)$ given by (III.3.8) is also differentiable in t and that

$$\left\|\nabla\left(\frac{\partial\boldsymbol{v}}{\partial t}\right)\right\|_{\ell,q} \le c_2\left\|\frac{\partial f}{\partial t}\right\|_{\ell,q},$$

for all $\ell \ge 0$. Extension of these results to more general domains will be given in Exercise III.3.6 and Exercise III.3.7. ■

Exercise III.3.1 Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \ge 2$, and let $f \in L^q(\Omega)$, $q \in (1,\infty)$. Show that there exists $\boldsymbol{v} \in D^{1,q}(\Omega)$ such that $\nabla\cdot\boldsymbol{v} = f$ in Ω and $|\boldsymbol{v}|_{1,q} \le c\,\|f\|_q$, $c = c(n,q,\Omega)$. *Hint*: Let $\{f_k\} \subset C_0^\infty(\Omega)$ with $f_k \to f$ in $L^q(\Omega)$. Then, $\boldsymbol{v}_k = (\nabla\mathcal{E} * f_k)$ solves $\nabla\cdot\boldsymbol{v}_k = f_k$ in Ω, and, by the Calderón–Zygmund Theorem II.11.4, satisfies $|\boldsymbol{v}_k|_{1,q} \le c\,\|f_k\|_q$.

Our next task is to extend the results of Lemma III.3.1 to the case of more general domains. To this end, we propose

Lemma III.3.2 *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be such that*

$$\Omega = \bigcup_{k=1}^N \Omega_k, \quad N \ge 2,$$

where each Ω_k is a star-shaped domain with respect to some open ball B_k with $\overline{B}_k \subset \Omega_k$, and let $f \in L^q(\Omega)$ satisfy (III.3.1). *Then, there exist N functions f_k such that for all $k = 1,\dots,N$:*

(i) $f_k \in L^q(\Omega)$;
(ii) $\operatorname{supp}(f_k) \subset \overline{\Omega_k}$;
(iii) $\int_{\Omega_k} f_k = 0$;
(iv) $f = \sum_{k=1}^N f_k$;
(v) $\|f_k\|_q \leq C_k \|f\|_q$, *with*

$$C_1 = \left(1 + \frac{|\Omega_1|^{1-1/q}}{|F_1|^{1-1/q}}\right)$$

$$C_k = \left(1 + \frac{|\Omega_k|^{1-1/q}}{|F_k|^{1-1/q}}\right) \prod_{i=1}^{k-1} (1 + |F_i|^{1/q-1} |D_i - \Omega_i|^{1-1/q}), \quad k \geq 2$$

and where $F_i = \Omega_i \cap D_i$ *and* $D_i = \cup_{s=i+1}^N \Omega_s$.[2]

Proof. Define

$$f_1(x) = \begin{cases} f(x) - \dfrac{\chi_1(x)}{|F_1|} \displaystyle\int_{\Omega_1} f & \text{if } x \in \Omega_1 \\ 0 & \text{if } x \in D_1 - \Omega_1 \end{cases}$$

$$g_1(x) = \begin{cases} [1 - \chi_1] f(x) - \dfrac{\chi_1(x)}{|F_1|} \displaystyle\int_{D_1 - \Omega_1} f & \text{if } x \in D_1 \\ 0 & \text{if } x \in \Omega_1 - D_1 \end{cases} \tag{III.3.24}$$

with χ_1 characteristic function of the set F_1. Clearly, it holds that

$$f = f_1 + g_1$$

$$\int_{\Omega_1} f_1 = \int_{D_1} g_1 = 0$$

$$\operatorname{supp}(f_1) \subset \overline{\Omega_1} \;,\quad \operatorname{supp}(g_1) \subset \overline{D_1}$$

$$f_1 \in L^q(\Omega_1) \;,\quad g_1 \in L^q(D_1).$$

By the same token, we split g_1 as

$$g_1 = f_2 + g_2,$$

with f_2 and g_2 belonging to $L^q(\Omega_2)$ and $L^q(D_2)$, respectively, and satisfying

[2] Observe that, since Ω is connected, we can always label the sets F_i in such a way that $|F_i| \neq 0$, for all $i = 1, ..., N-1$.

$$\int_{\Omega_2} f_2 = \int_{D_2} g_2 = 0$$

$$\mathrm{supp}\,(f_2) \subset \overline{\Omega_2}\ ,\ \mathrm{supp}\,(g_2) \subset \overline{D_2}.$$

This procedure gives rise to the following iteration scheme for the determination of the functions f_k. We set $g_0 = f$ and for $k = 1, \dots N-1$

$$g_k(x) = \begin{cases} [1-\chi_k]g_{k-1}(x) - \dfrac{\chi_k(x)}{|F_k|}\displaystyle\int_{D_k-\Omega_k} g_{k-1} & \text{if } x \in D_k \\ 0 & \text{if } x \in \Omega_k - D_k \end{cases} \tag{III.3.25}$$

with $F_k = \Omega_k \cap D_k$ and χ_k characteristic function of F_k; the functions f_k are then given by

$$\begin{aligned} f_k(x) &= \begin{cases} g_{k-1}(x) - \dfrac{\chi_k(x)}{|F_k|}\displaystyle\int_{\Omega_k} g_{k-1} & \text{if } x \in \Omega_k \\ 0 & \text{if } x \in D_k - \Omega_k \end{cases} \qquad k = 1, \dots N-1, \\ f_N(x) &= g_{N-1}(x) \end{aligned} \tag{III.3.26}$$

Relations (III.3.25) and (III.3.26) completely define the functions f_k and prove properties (i)-(iv). To show estimate (v), we observe that from (III.3.26), by the Hölder inequality, for all $k = 1, \dots, N$

$$\|f_k\|_{q,\Omega_k} \le \|g_{k-1}\|_{q,\Omega}\left(1 + |F_k|^{1/q-1}|\Omega_k|^{1-1/q}\right).$$

Therefore, by estimating $\|g_{k-1}\|_{q,\Omega}$ from (III.3.25) in terms of $\|g_{k-2}\|_{q,\Omega}$ and so on for $k-2$ times, we arrive at (v). The lemma is proved. □

Remark III.3.4 A noteworthy class of domains that satisfy the assumption of Lemma III.3.2 is that constituted by domains Ω satisfying the *cone property*. Such a property ensures that there exists a cone Γ [3] such that every point $x \in \partial\Omega$ is the vertex of a finite cone Γ_x congruent to Γ and contained in Ω. To see this, we recall a result of Gagliardo (1958, Teorema 1.I), which states that every bounded domain that satisfies the cone condition can be represented as the union of a finite number of domains, each of which is locally Lipschitz.[4] By virtue of Lemma II.1.3 and Exercise II.1.5, any such domain

[3] Namely, Γ is the intersection of an open ball centered at the origin with a set of the type

$$\{\lambda z : \lambda > 0, z \in \mathbb{R}^n, |z-y| < r\}$$

where $r > 0$ and y is a fixed point in $\mathbb{R}^n$ with $|y| > r$.

[4] Observe that every locally Lipschitz domain satisfies the cone condition; see Exercise III.3.2.

can be, in turn, represented as the union of a finite number of domains each being star-shaped with respect to all points of an open ball that they strictly contain.

■

Lemma III.3.1 and Lemma III.3.2 enable us to show the following result (Bogovskiĭ 1980, Theorem 1 and Lemma 3).

Theorem III.3.1 *Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 2$, such that*

$$\Omega = \cup_{k=1}^{N} \Omega_k, \quad N \geq 1,$$

where each Ω_k is star-shaped with respect to some open ball B_k with $\overline{B}_k \subset \Omega_k$. For instance, Ω satisfies the cone condition.[5] *Then, given $f \in L^q(\Omega)$, $1 < q < \infty$, satisfying* (III.3.1), *there exists at least one solution $\boldsymbol{v}$ to* (III.3.2). *Furthermore, the constant c entering inequality* ((III.3.2)$_3$ *admits the following estimate:*

$$c \leq c_0 C \left(\frac{\delta(\Omega)}{R_0}\right)^n \left(1 + \frac{\delta(\Omega)}{R_0}\right), \tag{III.3.27}$$

where R_0 is the smallest radius of the balls B_k, $c_0 = c_0(n, q)$ and C is an upper bound for the constants C_k given in Lemma III.3.2(v). *Finally, if f is of compact support in Ω so is $\boldsymbol{v}$.*

Proof. We decompose f as in Lemma III.3.2. Then, with the help of Lemma III.3.1, we construct in each domain Ω_k a solution $\boldsymbol{v}_k$ to (III.3.2), corresponding to $f_k, k = 1, \ldots, N$. If we extend $\boldsymbol{v}_k$ to zero outside Ω_k and recall Exercise II.3.11, we deduce that the field

$$\boldsymbol{v} = \sum_{k=1}^{N} \boldsymbol{v}_k$$

belongs to $W_0^{1,q}(\Omega)$ and solves (III.3.2)$_1$ in the whole of Ω. Moreover, again, from Lemma III.3.1 and Lemma III.3.2(v), we have

$$\|\boldsymbol{v}\|_{1,q} \leq \sum_{k=1}^{N} \|\boldsymbol{v}_k\|_{1,q} \leq c \sum_{k=1}^{N} \|f_k\|_q \leq c\, C \|f\|_q, \tag{III.3.28}$$

which completes the proof of the first part of the theorem once we take into account Remark III.3.4. To show the second one, for each Ω_k consider the corresponding domain $\Omega_k^{(\rho)}$, $\rho \in (1/2, 1)$, introduced in Exercise II.1.3. As we know from this exercise,

$$\overline{\Omega_k^{(\rho)}} \subset \Omega_k, \quad \text{for all } k = 1, \ldots, N,$$

[5] See Remark III.3.4 and Remark III.3.5.

and if Ω_k is star-shaped with respect to every point of the ball $B_R(x_{0k})$, then $\Omega_k^{(\rho)}$ enjoys the same property with respect to every point of the ball $B_{\rho R}(x_{0k})$. Let us set

$$\Omega^{(\rho)} = \cup_{k=1}^N \Omega_k^{(\rho)}$$

and denote by $\rho_1 \in (1/2, 1)$ a number such that for all $\rho \in [\rho_1, 1)$ the following properties hold

$$\Omega^{(\rho)} \quad \text{is connected}$$

$$\operatorname{supp}(f) \subset \Omega^{(\rho)}.$$

In virtue of Lemma III.3.2, we can decompose f as the sum of N functions $f_k^{(\rho)}$, where $f_k^{(\rho)}$ satisfy the following properties:

$$f_k^{(\rho)} \in L^q(\Omega_k^{(\rho)}), \quad \operatorname{supp}(f_k^{(\rho)}) \subset \overline{\Omega_k^{(\rho)}}, \quad \int_{\Omega_k^{(\rho)}} f_k^{(\rho)} = 0, \quad k = 1, \ldots, N.$$

Furthermore, taking into account that

$$|\Omega_k^{(\rho)}| = \rho^n |\Omega_k|, \quad |\Omega_k^{(\rho)} \cap \Omega_{k'}^{(\rho)}| = \rho^n |\Omega_k \cap \Omega_{k'}|,$$

from property (v) of Lemma III.3.2 we also have

$$\|f_k^{(\rho)}\|_q \le C \|f\|_q \tag{III.3.29}$$

with a constant C depending on Ω_k but otherwise *independent of* $\rho \in [\rho_1, 1)$. We next solve problem (III.3.1), (III.3.2) in each $\Omega_k^{(\rho)}$ and denote by $\boldsymbol{v}_k^{(\rho)} \in W_0^{1,q}(\Omega_k^{(\rho)})$ the corresponding solution. Extending $\boldsymbol{v}_k^{(\rho)}$ by zero outside $\Omega_k^{(\rho)}$, we obtain that the function

$$\boldsymbol{v}^{(\rho)} = \sum_{k=1}^N \boldsymbol{v}_k^{(\rho)}$$

solves $(\text{III.3.2})_2$, belongs to $W_0^{1,q}(\Omega)$, and is of compact support in Ω. Moreover, proceeding as in (III.3.28) and using (III.3.29) we recover that $\boldsymbol{v}^{(\rho)}$ obeys $(\text{III.3.2})_3$ with a constant c depending on n, q and Ω but independent of ρ, namely, of f. The theorem is completely proved. □

Remark III.3.5 Even though the assumption on the regularity of Ω made in the previous theorem may allow, in principle, for domain even less regular than those satisfying the cone condition, some kind of regularity is indeed necessary for the solvability of problem (III.3.1)–(III.3.2); see Remark III.3.9. For example, Ω can not have an *external* cusp. The question of the "least" requirement on Ω for (III.3.1)–(III.3.2) is studied in Acosta, Durán & Muschietti (2006), where, in particular, for $n = 2$, $q \in (1, 2)$, and Ω simply connected, a complete characterization is furnished in terms of "John domains"; see also the Notes for this Chapter.

■

Remark III.3.6 Remark III.3.1 equally applies to Theorem III.3.1. ■

Remark III.3.7 Theorem III.3.1 leaves out the two limiting cases $q = 1, \infty$. As a matter of fact, in both cases, problem (III.3.2) does *not* have a solution for *all* $f \in L^q(\Omega)$ satisfying (III.3.1). A proof of this assertion when $q = \infty$, is given by Preiss (1997), McMullen (1998, Theorem 2.1) and Bourgain & Brezis (2002, § 2.2); see also Dacorogna, Fusco & Tartar (2004). A proof of the non-solvability of problem (III.3.2) when $q = 1$ for arbitrary $f \in L^1(\Omega)$ satisfying (III.3.1), can be found in Bourgain & Brezis (2003, § 2.1) and in Dacorogna, Fusco & Tartar (2004). The argument of Bourgain & Brezis is elementary and will be reproduced here. Thus, assume that the problem

$$\nabla \cdot \boldsymbol{v} = f\,, \quad \|\boldsymbol{v}\|_{1,1} \le c\,\|f\|_1 \tag{III.3.30}$$

has at least one solution $\boldsymbol{v} \in W_0^{1,1}(\Omega)$, corresponding to an *arbitrarily given* $f \in L^1(\Omega)$ satisfying (III.3.1). Choose $f = g - \overline{g}_\Omega$, where g is any function in $L^1(\Omega)$, and let $u \in C_0^\infty(\Omega)$. From (III.3.30) we thus have

$$(\nabla u, \boldsymbol{v}) = -(u, \nabla \cdot \boldsymbol{v}) = -(u, g - \overline{g}_\Omega)\,,$$

which, by a simple calculation that uses the Hölder inequality and Theorem II.3.2, implies

$$|(u - \overline{u}_\Omega, g)| \le \|\nabla u\|_n \|\boldsymbol{v}\|_{n/(n-1)} \le c\,\|\nabla u\|_n \|\boldsymbol{v}\|_{1,1}\,.$$

From this latter relation, from $(\text{III.3.30})_2$, and from Theorem II.2.2 we readily deduce

$$\|u - \overline{u}_\Omega\|_\infty = \sup_{g \in L^1(\Omega); \|g\|_1 = 1} |(u - \overline{u}_\Omega), g)| \le c\,\|\nabla u\|_n\,,$$

which, by (II.2.6) and (II.5.1), in turn implies the following property

$$\|u\|_\infty \le c\,\|\nabla u\|_n\,, \quad \text{for } \textit{all } u \in C_0^\infty(\Omega)\,,$$

which, as we know from Exercise II.3.8, is not true. ■

Remark III.3.8 If $q > n$, in view of the embedding Theorem II.3.2, any solution $\boldsymbol{v}$ to (III.3.1)–(III.3.2), belongs, in addition, to $L^\infty(\Omega)$.[6] Of course, as we know from Exercise II.3.8, this embedding does not hold if $q = n$ and, therefore, we can not prove, in such a case, $\boldsymbol{v} \in L^\infty(\Omega)$, at least by this kind of argument. However, it is simple to bring examples where, for certain f and Ω, it is indeed possible to produce a solution to (III.3.1)–(III.3.2) which is in $L^\infty(\Omega)$, under the sole assumption that $f \in L^n(\Omega)$. For instance, let $\Omega = B_R$, for some $R > 0$, and assume that $f = f(|x|)$, $f \in L^n(B_R)$. Then, by a straightforward calculation, we prove that a solution to (III.3.1)–(III.3.2) is given by

[6] Actually, to $C(\overline{\Omega})$.

$$\boldsymbol{v}(x) = \begin{cases} \dfrac{\boldsymbol{x}}{|x|^n} \displaystyle\int_0^{|x|} r^{n-1} f(r) dr & \text{if } 0 < |x| \le R\,, \\ 0 \quad \text{if } x = 0\,. \end{cases}$$

It is easy to show that

$$\lim_{|x| \to 0} \boldsymbol{v}(x) = 0\,,$$

which furnishes $\boldsymbol{v} \in C(\overline{\Omega})$. Moreover,

$$\|\boldsymbol{v}\|_\infty \le \left(\frac{1}{n}\right)^{(n-1)/n} \|f\|_n\,.$$

At this point it is natural to ask if such a result can be proved for more general functions f and for (sufficiently smooth, bounded) Ω of arbitrary shape. The answer to this question is positive, and, in fact, by methods completely different than those used here, Bourgain & Brezis (2003, Theorem 3′) have shown the following result, for whose proof we refer to their article.

Theorem III.3.2 *Let Ω be a bounded and locally Lipschitz domain of $\mathbb{R}^n$, $n \ge 2$. Then, for any $f \in L^n(\Omega)$ satisfying* (III.3.1) *there exists a solution $\boldsymbol{v}$ to* (III.3.2) *with $q = n$, which, furthermore, belongs to $C(\overline{\Omega})$ and obeys the following estimate*

$$\|\boldsymbol{v}\|_\infty \le c \, \|f\|_n\,,$$

where $c = c(n, \Omega)$.

■

Another interesting question is the dependence of the constant c entering inequality (III.3.2)$_3$ on the domain Ω. For example, from (III.3.27) we deduce, in particular, that if Ω is a ball, c is independent of the diameter of Ω. This is a particular case of the following lemma whose proof we leave to the reader as an exercise.

Lemma III.3.3 *Let $y_i = \phi_i(x)$, $i = 1, \ldots, n$, be a transformation of $\mathbb{R}^n$ into itself. Then, the constant c in* (III.3.2)$_3$ *does not change if ϕ_i is homothetic,* i.e.,

$$\phi_i(x) = a x_i + b_i, \quad a, b_i \in \mathbb{R}$$

or a rotation, i.e.,

$$\phi_i(x) = \sum_{j=1}^n \mathcal{A}_{ij} x_j, \quad \sum_{j=1}^n \mathcal{A}_{ij} \mathcal{A}_{\ell j} = \delta_{i\ell}.$$

Other questions related to the solvability of (III.3.1) and (III.3.2) are left to the reader in the following exercises.

Exercise III.3.2 Show that if Ω is locally Lipschitz then Ω satisfies the cone condition.

Exercise III.3.3 Show that for $q = 2$ the constant c of inequality $(\text{III.3.2})_3$, in general, cannot be less than one.

Exercise III.3.4 Let Ω be an arbitrary domain of $\mathbb{R}^n$, and let $u \in L^q_{loc}(\Omega)$, $q \in (1, \infty)$. By the Hahn-Banach Theorem II.1.7, there exists a unique $A \in D_0^{-1,q}(\Omega)$ such that

$$(u, \operatorname{div} \boldsymbol{\psi}) = -\langle A, \boldsymbol{\psi} \rangle \,, \quad \text{for all } \boldsymbol{\psi} \in C_0^\infty(\Omega)\,.$$

It is readily checked that A does not depend on q. Moreover, if $D_k u$ exists in the weak sense, $k = 1, \ldots, n$, then $\langle A, \boldsymbol{\psi} \rangle = (\nabla u, \boldsymbol{\psi})$, for all the above specified functions $\boldsymbol{\psi}$. Thus, the above formula can be viewed as a generalization of the definition of weak gradient of u, and the functional A will be still denoted by ∇u. It is obvious that, if $u \in L^q(\Omega)$,

$$|\nabla u|_{-1,q} \le c_1 \|u\|_q \,,$$

for some $c_1 = c_1(q)$. Conversely, suppose Ω bounded and such that problem (III.3.1)–(III.3.2) is solvable in Ω. Show that, if $u \in L^q_{loc}(\Omega)$, with $\nabla u \in W_0^{-1,q}(\Omega)$,[7] then $u \in L^q(\Omega)$, and the following *generalization of the Poincaré's inequality* (II.5.10) holds:[8]

$$\|u - \overline{u}_\Omega\|_q \le c_2 \|\nabla u\|_{-1,q} \,,$$

with $c_2 = c_2(\Omega, q)$. Thus, in particular, if $u \in L^q(\Omega)$, $q \in (1, \infty)$, with

$$\int_\Omega u = 0,$$

for the above types of domain, $\|u\|_q$ and $\|\nabla u\|_{-1,q}$ are equivalent norms. *Hint*: Pick arbitrary $\psi \in C_0^\infty(\Omega)$, and let $\varphi \in C_0^\infty(\Omega)$ with $\int_\Omega \varphi = 1$. Set $f := \psi - \varphi \int_\Omega \psi$, and let $\boldsymbol{v} \in C_0^\infty(\Omega)$ be a solution to (III.3.1)–(III.3.2) corresponding to this f. Then, use the relation

$$(u, f) = (u, \operatorname{div} \boldsymbol{v}) = -\langle \nabla u, \boldsymbol{v} \rangle$$

along with the property (III.3.3) of the function $\boldsymbol{v}$ and the results of Exercise II.2.12.

Remark III.3.9 Poincaré's inequality holds for sufficiently smooth domains, e.g., locally Lipschitz (see Theorem II.5.4), while it fails, in general, for domains with very little regularity, like, for example, those having an external cusp (Courant & Hilbert 1937, Kapitel VII, §8.2; see also Amick 1976 and Fraenkel 1979, §2). Consequently, since by Exercise III.3.4, the validity of Poincaré's inequality is implied by the solvability of problem (III.3.1)–(III.3.2), we conclude that this latter cannot be solved in arbitrary (bounded) domains. ■

[7] Observe that, for bounded Ω, $W_0^{-1,q}(\Omega)$ and $D_0^{-1,q}(\Omega)$ are isomorphic; see Remark II.6.3.

[8] Observe that, by (II.5.1), we immediately find $\|\nabla u\|_{-1,q} \le c\,\|\nabla u\|_q$, $1 \le q \le \infty$, with $c = c(q, \Omega)$.

Remark III.3.10 An elementary example of two-dimensional domain where problem (III.3.1)–(III.3.2), for $q = 2$, can not have a solution for *all* $f \in L^2(\Omega)$ with $\overline{f}_\Omega = 0$ is the following one, due to G. Acosta (see Durán & López García, 2010, p. 423). Let

$$\Omega = \{x \in \mathbb{R}^2 : \ 0 < x_1 < 1, \ |x_2| < x_1^2\},$$

and consider the function

$$u(x) = \frac{1}{x_1^2} - 3, \quad x \in \Omega.$$

Clearly, $u \in L^1(\Omega)$, with $\overline{u}_\Omega = 0$. Moreover,

$$u \notin L^2(\Omega).$$

However,

$$\frac{\partial u}{\partial x_1} = -2\frac{\partial}{\partial x_2}\left(\frac{x_1 x_2}{x_1^4}\right) := \frac{\partial g}{\partial x_2},$$

and since $g \in L^2(\Omega)$, we obtain

$$u \in W_0^{-1,2}(\Omega).$$

Therefore, by Exercise III.3.4, there exists at least one $f \in L^2(\Omega)$ with $\overline{f}_\Omega = 0$ for which problem (III.3.1)–(III.3.2), for $q = 2$, does not have a solution. ■

Exercise III.3.5 Along with problem (III.3.2), one can consider the following non-homogeneous version of it. Given f and $\boldsymbol{a}$ suitably, find a vector field $\boldsymbol{v}$ such that

$$\begin{aligned} &\nabla \cdot \boldsymbol{v} = f \\ &\boldsymbol{v} \in W^{1,q}(\Omega) \\ &\boldsymbol{v} = \boldsymbol{a} \ \text{ at } \partial\Omega. \end{aligned} \tag{III.3.31}$$

Show that, for Ω a bounded and locally Lipschitz domain of $\mathbb{R}^n$, $n \geq 2$, given

$$f \in L^q(\Omega), \quad \boldsymbol{a} \in W^{1-1/q,q}(\partial\Omega), \quad 1 < q < \infty,$$

satisfying

$$\int_\Omega f = \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n},$$

problem (III.3.31) admits at least one solution $\boldsymbol{v}$. Moreover, denoting by $\boldsymbol{A} \in W^{1,q}(\Omega)$ an extension of $\boldsymbol{a}$ (according to Theorem II.4.3), show that this solution satisfies the estimate

$$\|\boldsymbol{v}\|_{1,q} \leq c\,(\|f\|_q + \|\nabla \cdot \boldsymbol{A}\|_q).$$

Therefore, in particular,

$$\|\boldsymbol{v}\|_{1,q} \leq c\left(\|f\|_q + \|\boldsymbol{a}\|_{1-1/q,q(\partial\Omega)}\right). \tag{III.3.32}$$

Exercise III.3.6 Let Ω be as in Theorem III.3.1. Furthermore, let I be an interval in $\mathbb{R}$, and suppose $f = f(t)$ is in $L^{q_i}(\Omega)$, $q_i \in (1,\infty)$, $i = 1,2$, and satisfies (III.3.1), for all $t \in I$. Then, show that problem (III.3.2) has at least one solution $\boldsymbol{v} = \boldsymbol{v}(x,t)$ which, in addition, satisfies

$$|\boldsymbol{v}(t_1) - \boldsymbol{v}(t_2)|_{1,q_i} \le c_1 \|f(t_1) - f(t_2)\|_{q_i}\,, \quad t_1, t_2 \in I\,, \quad i = 1,2\,,$$

for some $c_1 = c_1(q_i, \Omega)$. Moreover, if $\dfrac{\partial f}{\partial t} \in L^{r_i}(\Omega)$, $i = 1,2$, show that $\dfrac{\partial \boldsymbol{v}}{\partial t} \in W_0^{1,r_i}(\Omega)$, $i = 1,2$, and that the following estimate holds

$$\left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|_{1,r_i} \le c_2 \left\|\frac{\partial f}{\partial t}\right\|_{r_i}\,, \quad i = 1,2\,,$$

where $c_2 = c_2(q_i, \Omega)$. *Hint*: Use the argument presented in Remark III.3.3 along with the method used in the proof of Theorem III.3.1.

Unless Ω is star-shaped with respect to a ball, the method of construction of the field $\boldsymbol{v}$ used in the proof of Theorem III.3.1 ensures, in general, no more than the $W^{1,q}$-regularity for $\boldsymbol{v}$, even for $f \in C_0^\infty(\Omega)$. Our next objective is, therefore, to show that if Ω is sufficiently smooth (for example, locally Lipschitz) we may find a solution to (III.3.2) that belongs to $C_0^\infty(\Omega)$ if $f \in C_0^\infty(\Omega)$. The regularity of the domain is required in order to construct a suitable covering of Ω and a corresponding decomposition of f, as the following lemma proves.

Lemma III.3.4 *Let Ω be bounded and locally Lipschitz. Then there exists an open cover $\mathcal{G} = \{G_1, \ldots, G_m, G_{m+1}, \ldots, G_{m+\nu}\}$ of $\overline{\Omega}$ such that*

(i) *$\Omega_i \equiv \Omega \cap G_i$ is star-shaped with respect to an open ball B_i with $\overline{B}_i \subset \Omega_i$ for $i = 1, \ldots, m+\nu$;*
(ii) *$\partial\Omega \subset \cup_{i=1}^m G_i$;*
(iii) *G_i is an open ball with $\overline{B}_i \subset \Omega$ for $i = m+1, \ldots, m+\nu$;*
(iv) *$\Omega = \cup_{i=1}^{m+\nu} \Omega_i$.*

Moreover, if $f \in C_0^\infty(\Omega)$ satisfies (III.3.1), *we have $f = \sum_{i=1}^{m+\nu} f_i$ where*

(v) *$f_i \in C_0^\infty(\Omega_i)$;*
(vi) *$\int_\Omega f_i = 0$;*
(vii) *$f_i = \zeta f + \sum_{k=1}^{m_i} \theta_k \int_\Omega \phi_k f$, where $m_i \in \mathbb{N}$, $\zeta \in C_0^\infty(G_i)$, $\theta_k \in C^\infty(\Omega_i)$ and $\phi_k \in C_0^\infty(\overline{\Omega})$.*
(viii) *$\|f_i\|_{m,q} \le C\|f\|_{m,q}$, for all $m \ge 0$ and $q \ge 1$,*

where C depends only on m, q, and Ω.

Proof. In virtue of Lemma II.1.3, we may find m locally Lipschitz domains $G_1, \ldots, G_m$ such that $\Omega_i \equiv \Omega \cap G_i$ is star-shaped with respect to an open ball B_i with $\overline{B_i} \subset \Omega_i$ and verifying condition (ii). Denote by G_0 a domain with $\overline{G}_0 \subset \Omega$ such that $\{G_0, G_1, \ldots, G_m\}$ forms an open cover of $\overline{\Omega}$. Since Ω is bounded, we may find ν open balls $G_{m+1}, \ldots, G_{m+\nu}$ such that

$$\overline{G}_0 \subset \bigcup_{i=m+1}^{m+\nu} G_i$$

$$\left\{ \bigcup_{i=m+1}^{m+\nu} G_i \right\}^- \subset \Omega.$$

Evidently,

$$\mathcal{G} = \{G_1, \ldots, G_m, G_{m+1}, \ldots, G_{m+\nu}\}$$

is an open cover of Ω with $\overline{\Omega} \subset \cup_{i=1}^{m+\nu} G_i$. Setting

$$\Omega_i \equiv \Omega \cap G_i, \quad i = 1, \ldots, m+\nu$$

it is immediately obtained that properties (i) and (iv) are satisfied. Let us now pick $f \in C_0^\infty(\Omega)$ and show the existence of f_i satisfying (v)-(viii). To this end, let

$$\{\psi_1, \ldots \psi_{m+\nu}\}$$

be a partition of unity subordinate to $\mathcal{G}$; see Lemma II.1.4. Setting

$$D_2 = \bigcup_{i=2}^{m+\nu} \Omega_i, \quad \Psi_2 = \sum_{i=2}^{m+\nu} \psi_i(x),$$

and observing that

$$\psi_1(x) + \Psi_2(x) = 1, \text{ for all } x \in \Omega,$$

we may write $f = f_1 + g_1$, where

$$f_1 = \psi_1 f - \chi_1 \int_\Omega \psi_1 f,$$

$$g_1 = \Psi_2 f - \chi_1 \int_\Omega \Psi_2 f,$$

and

$$\chi_1 \in C_0^\infty(\Omega_1 \cap D_2), \quad \int_\Omega \chi_1 = 1.$$

Since $\psi_1 \in C_0^\infty(G_1)$ and $f \in C_0^\infty(\Omega)$ it immediately follows that

$$f_1 \in C_0^\infty(\Omega_1).$$

Moreover, we have

$$\Psi_2 = \sum_{i=2}^{m} \psi_i(x) + \sum_{i=m+1}^{m+\nu} \psi_i(x)$$

so that, by recalling the definition of $G_2, \ldots, G_m, G_{m+1}, \ldots, G_{m+\nu}$ and that $f \in C_0^\infty(\Omega)$, it follows that

$$g_1 \in C_0^\infty(D_2).$$

In addition, we have, evidently,

$$\int_\Omega f_1 = \int_\Omega g_1 = 0.$$

We may then continue this procedure, using g_1 as the function to be split. Precisely, setting

$$D_3 = \bigcup_{i=3}^{m+\nu} \Omega_i, \quad \Psi_3 = \bigcup_{i=3}^{m+\nu} \psi_i(x),$$

we let

$$f_2 = \psi_2 g_1 - \chi_2 \int_\Omega \psi_2 g_1,$$

$$g_2 = \Psi_3 g_1 - \chi_2 \int_\Omega \Psi_3 f,$$

where

$$\chi_2 \in C_0^\infty(\Omega_2 \cap D_3)$$

$$\int_\Omega \chi_2 = 1.$$

From the expression of g_1 and property (b) of the partition of unity we recover at once that f_2 can be written as

$$f_2 = \psi_2(1-\psi_1)f - \chi_1\psi_2 \int_\Omega (1-\psi_1)f - \chi_2 \int_\Omega \psi_2(1-\psi_1)f$$

$$+\chi_2 \int_\Omega \psi_2\chi_1 \int_\Omega (1-\psi_1)f,$$

which proves $f_2 \in C_0^\infty(\Omega_2)$, along with properties (vii) and (viii). Furthermore,

$$g_1 = f_2 + g_2, \quad g_2 \in C_0^\infty(D_3),$$

$$\int_{\Omega_2} f_2 = \int_{D_3} g_2 = 0.$$

We may then use an iteration scheme of the same type employed in the proof of Lemma III.3.2 to show the validity of (v)-(vii) for all $i = 1, \dots, m+\nu$. $\square$

The result just proved allows us to show the following one.

Theorem III.3.3 *Let Ω be a bounded and locally Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Given*

$$f \in W_0^{m,q}(\Omega), \quad m \geq 0, \quad 1 < q < \infty, \tag{III.3.33}$$

satisfying (III.3.1), *there exists $\boldsymbol{v} \in W_0^{m+1,q}(\Omega)$ satisfying* (III.3.2) *and* (III.3.23), *for all $\ell = 0, \dots, m$. Moreover, if $f \in C_0^\infty(\Omega)$ then $\boldsymbol{v} \in C_0^\infty(\Omega)$.*

Proof. The proof is essentially analogous to that of Theorem III.3.1 once we take into account Lemma III.3.4 (in place of Lemma III.3.2) and Remark III.3.2. Actually, for $f \in C_0^\infty(\Omega)$, we denote by Ω_i and f_i the domains and functions introduced in Lemma III.3.4 and we let $\boldsymbol{v}_i \in C_0^\infty(\Omega_i)$ denote the solution to problem (III.3.1), (III.3.2) in Ω_i whose existence is assured by Lemma III.3.1. In view of Lemma III.3.4(vi) and Remark III.3.2, we have also

$$\|\nabla \boldsymbol{v}_i\|_{\ell,q} \leq C\|f\|_{\ell,q}\,, \quad \ell = 0,\ldots,m\,,$$

for some constant $C = C(n,q,\ell,\Omega)$. Then the field

$$\boldsymbol{v} = \sum_{i=1}^{N} \boldsymbol{v}_i$$

belongs to $C_0^\infty(\Omega)$ and satisfies (III.3.2) in Ω along with the inequality

$$\|\nabla \boldsymbol{v}\|_{\ell,q} \leq C\|f\|_{\ell,q}, \quad \ell = 0,\ldots,m\,, \tag{III.3.34}$$

which proves the second part of the theorem. Let us now assume $\boldsymbol{f}$ merely satisfying (III.3.33) and denote by $\{f_k\} \subset C_0^\infty(\Omega)$ a sequence of functions satisfying (III.3.1) and converging to f in $W_0^{m,q}(\Omega)$. Let $\{\boldsymbol{v}_k\} \subset C_0^\infty(\Omega)$ be the corresponding solutions to (III.3.2). It is readily seen that, by the estimate (III.3.34) and inequality (II.5.1), as $k \to \infty$ the sequence $\{\boldsymbol{v}_k\}$ converges (strongly) in $W_0^{m+1,q}(\Omega)$ to a function $\boldsymbol{v} \in W_0^{m+1,q}(\Omega)$ that solves $(\text{III.3.2})_1$ in the sense of generalized differentiation and satisfies (III.3.34). The theorem is therefore proved. □

Remark III.3.11 The regularity assumption on Ω made in Theorem III.3.3 can be further weakened (Bogovskiĭ 1980, Theorem 2). ■

Remark III.3.12 From the proof of Theorem III.3.3it immediately follows that if

$$f \in W_0^{m,q}(\Omega) \cap W_0^{m,r}(\Omega), \quad m \geq 0, \quad 1 < q, r < \infty,$$

satisfying (III.3.1), we can find one solution to (III.3.2) with

$$\boldsymbol{v} \in W_0^{m+1,q}(\Omega) \cap W_0^{m+1,r}(\Omega) \tag{III.3.35}$$

such that

$$\|\nabla \boldsymbol{v}\|_{m,q} \leq C\|f\|_{m,q}, \quad \|\nabla \boldsymbol{v}\|_{m,r} \leq C\|f\|_{m,r}. \tag{III.3.36}$$

■

From Theorem III.3.3 we obtain the following corollary on the extension of solenoidal fields (Coscia and Galdi, 1997; Bogovskiĭ 1980, Theorem 3).

Corollary III.3.1 *Let Ω be a bounded locally Lipschitz domain of $\mathbb{R}^n$, $n \geq 2$, and let Γ_i, $i = 1, \ldots, k$, $k \geq 1$, denote the connected components of the boundary $\partial\Omega$. Let $\boldsymbol{v}$ be a solenoidal field of $W^{m,q}(\Omega)$, $m \geq 1$, $1 < q < \infty$, satisfying the following conditions*

$$\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad \textit{for all } i = 1, \ldots, k,$$

with $\boldsymbol{n}$ normal component to $\partial\Omega$. Then, given an open ball B with $B \supset \overline{\Omega}$, there exists a solenoidal field $\boldsymbol{V}$ such that

$$\boldsymbol{V} \in W_0^{m,q}(B)$$

$$\boldsymbol{V}(x) = \boldsymbol{v}(x) \ \ \textit{for all } x \in \Omega.$$

Moreover,

$$\|\boldsymbol{V}\|_{m,q,B} \leq c\|\boldsymbol{v}\|_{m,q,\Omega}.$$

for some $c = c(\Omega, m, q, n, B) > 0$.

Proof. From Theorem II.3.3 there exists $\boldsymbol{u} \in W_0^{m,q}(B)$ such that

$$\boldsymbol{u}(x) = \boldsymbol{v}(x) \ \ \text{for all } x \in \Omega$$

$$\|\boldsymbol{u}\|_{m,q,B} \leq c\|\boldsymbol{v}\|_{m,q,\Omega}.$$

However, $\boldsymbol{u}$ need not be solenoidal and to obtain the desired extension $\boldsymbol{V}$ we have to modify $\boldsymbol{u}$ suitably. To this end, denote by ω_i, $i = 1, \ldots, k-1$ the *bounded* connected components of $\mathbb{R}^n - \overline{\Omega}$ and set $\omega_k = B - \overline{\Omega}$. In each ω_i we consider the following problem

$$\nabla \cdot \boldsymbol{w}_i = \nabla \cdot \boldsymbol{u} \text{ in } \omega_i$$

$$\boldsymbol{w}_i \in W_0^{m,q}(\omega_i)$$

$$\|\boldsymbol{w}_i\|_{m,q} \leq c\|\nabla \cdot \boldsymbol{u}\|_{m-1,q}.$$

By assumption,

$$\int_{\omega_i} \nabla \cdot \boldsymbol{u} = \int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad i = 1, \ldots, k.$$

Moreover, being $\nabla \cdot \boldsymbol{v} = 0$ in Ω, it also follows that

$$\nabla \cdot \boldsymbol{u} \in W_0^{m-1,q}(\omega_i), \quad i = 1, \ldots, k.$$

As a consequence, from Theorem III.3.3 we deduce the existence of the fields $\boldsymbol{w}_i$. Set $\boldsymbol{w}_i \equiv 0$ in ω_i^c and denote again by $\boldsymbol{w}_i$ their extensions. We define

$$\boldsymbol{V}(x) = \begin{cases} \boldsymbol{u}(x) - \boldsymbol{w}_i(x) & x \in \omega_i, \ i = 1, \ldots, k, \\ \boldsymbol{v}(x) & x \in \Omega. \end{cases}$$

It is immediately checked that the field $\boldsymbol{V}$ satisfies all the properties stated in the corollary, which is therefore proved. $\square$

Exercise III.3.7 Let Ω be as in Theorem III.3.3. Furthermore, let I be an interval in $\mathbb{R}$, and suppose $f = f(t)$ is in $W_0^{m,q_i}(\Omega)$, $m \geq 0$, $q_i \in (1,\infty)$, $i = 1,2$, and that satisfies (III.3.1), for all $t \in I$. Then, show that, for all $t \in I$, problem (III.3.2) has at least one solution $\boldsymbol{v}(t) \in W_0^{m+1,q_i}(\Omega)$, $i = 1,2$, satisfying (III.3.23) with $q = q_i$, $i = 1,2$, for all $t \in I$, and which, in addition, obeys the following inequality

$$\|\nabla(\boldsymbol{v}(t_1) - \boldsymbol{v}(t_2))\|_{\ell,q_i} \leq c_1\|f(t_1) - f(t_2)\|_{\ell,q_i}\,, \quad t_1, t_2 \in I\,, \quad \ell = 0,\dots,m\,, \quad i = 1,2\,,$$

for some $c_1 = c_1(q_i, \ell, \Omega)$. Moreover, if $\dfrac{\partial f}{\partial t} \in W_0^{k,q_i}(\Omega)$, for $i = 1,2$ and some $k \geq 0$, show that $\dfrac{\partial \boldsymbol{v}}{\partial t} \in W_0^{k+1,q_i}(\Omega)$, $i = 1,2$, and that the following estimate holds

$$\left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|_{l+1,q_i} \leq c_2 \left\|\frac{\partial f}{\partial t}\right\|_{l,q_i}\,, \quad l = 0,\dots,k\,, \quad i = 1,2\,,$$

where $c_2 = c_2(q_i, l, \Omega)$. *Hint*: Use the argument presented in Remark III.3.3 along with the method used in the proof of Theorem III.3.3.

A further question that can be reasonably posed for problem (III.3.1), (III.3.2) is that of finding a solution that further obeys an estimate with f in negative Sobolev spaces, that is,

$$\|\boldsymbol{v}\|_q \leq c\|f\|_{-1,q}. \tag{III.3.37}$$

However, the answer is, in general, negative even when f is in the form of divergence. Actually, if we take

$$f = \nabla \cdot \boldsymbol{g}, \quad \boldsymbol{g} \in \widetilde{H}_q(\Omega), \tag{III.3.38}$$

with $\widetilde{H}_q(\Omega)$ defined in (III.2.4), (III.2.5), the solvability of (III.3.1), (III.3.2), and (III.3.37) would imply the existence of certain solenoidal extensions of boundary data that, as shown by counterexamples, *cannot* exist; see Remark IX.4.4. Nevertheless, the question admits a positive answer if we further restrict the hypothesis on f. Specifically, we shall prove that (III.3.1), (III.3.2), and (III.3.37) have at least one solution for all f of the type

$$f = \nabla \cdot \boldsymbol{g}, \quad \boldsymbol{g} \in \widetilde{H}_{0,q}(\Omega), \tag{III.3.39}$$

where $\widetilde{H}_{0,q}(\Omega)$ is given in (III.2.15). The difference between (III.3.38) and (III.3.39) is that the latter, unlike the former, requires *the vanishing of the normal component of* $\boldsymbol{g}$ *at the boundary.*

One important consequence of this result (see Theorem III.3.5) is that, provided Ω is sufficiently regular, the following inequality holds

$$\|\boldsymbol{v}\|_q \leq c|f|^*_{-1,q}\,, \tag{III.3.40}$$

where

$$|f|^*_{-1,q} = \sup_{u \in D^{1,q'}(\Omega);\, |u|_{1,q'}=1} |(f,u)|\,. \tag{III.3.41}$$

Notice that, since $W_0^{1,q}(\Omega) \subset D^{1,q}(\Omega)$, we have $\|f\|_{-1,q} \leq |f|^*_{-1,q}$, so that (III.3.40) is a *weaker* form of (III.3.37).

We begin to show the following

Lemma III.3.5 *Let Ω, G be bounded, locally Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$, with $\Omega \cap G \equiv \Omega_0$ $(\neq \emptyset)$ star-shaped with respect to a ball B with $\overline{B} \subset \Omega_0$. Let, further, ϕ_k, θ_k, $k = 1, \ldots, m$, ζ, and $\boldsymbol{g}$ be functions such that*

$$\phi_k \in C_0^\infty(\overline{\Omega}), \quad \theta_k \in C_0^\infty(\Omega_0), \quad \zeta \in C_0^\infty(G),$$

$$\boldsymbol{g} \in \widetilde{H}_{0,q}(\Omega), \quad 1 < q < \infty, \quad \nabla \cdot \boldsymbol{g} \in C_0^\infty(\Omega).$$

Set

$$f = \zeta \nabla \cdot \boldsymbol{g} + \sum_{i=1}^{m} \theta_k \int_\Omega \phi_k \nabla \cdot \boldsymbol{g} \tag{III.3.42}$$

and suppose

$$\int_{\Omega_0} f = 0. \tag{III.3.43}$$

Then, there is at least one solution $\boldsymbol{w}$ to the problem

$$\begin{gathered}\nabla \cdot \boldsymbol{w} = f \\ \boldsymbol{w} \in C_0^\infty\ (\Omega) \\ \|\boldsymbol{w}\|_{1,s,\Omega_0} \leq c\|\nabla \cdot \boldsymbol{g}\|_{s,\Omega} \\ \|\boldsymbol{w}\|_{q,\Omega_0} \leq c\|\boldsymbol{g}\|_{q,\Omega},\end{gathered} \tag{III.3.44}$$

where s is arbitrary in $(1, \infty)$ and $c = c(\phi_k, \theta_k, \zeta, s, q, n, B, G, \Omega)$.

Proof. For simplicity, we shall restrict ourselves to discuss the case where $m = 1$ and set $\theta = \theta_1$, $\phi = \phi_1$. Clearly, $f \in C_0^\infty(\Omega_0)$. Then, by (III.3.43) and the assumption made on Ω_0, we can find a solution to the problem by the Bogovskiĭ formula (III.3.8), that is,

$$\begin{aligned}\boldsymbol{w}(x) =& \int_{\Omega_0} f(y) \left[\frac{\boldsymbol{x} - \boldsymbol{y}}{|x-y|^n} \int_{|x-y|}^\infty \omega \left(y + \xi \frac{x-y}{|x-y|} \right) \xi^{n-1} d\xi \right] dy \\ \equiv& \int_{\Omega_0} f(y) \boldsymbol{N}(x,y) dy.\end{aligned} \tag{III.3.45}$$

Thus, repeating the proof of Lemma III.3.1, we obtain that the field $\boldsymbol{w}$ defined in (III.3.45) satisfies $(\text{III.3.44})_{1,2}$ and

$$\|\boldsymbol{w}\|_{1,s} \leq c_1 \|f\|_s,$$

with $c_1 = c_1(n, s, \Omega_0)$. Since

$$\|f\|_s \le c_2 \|\nabla \cdot \boldsymbol{g}\|_s,$$

with $c_2 = c_2(\phi, \theta, \zeta, s, \Omega)$, $(\text{III.3.44})_3$ also follows. It remains to show $(\text{III.3.44})_4$. From (III.3.45) and (III.3.42) we find for $i = 1, \ldots, n$

$$\begin{aligned} w_i(x) &= w_i^{(1)}(x) + w_i^{(2)}(x) \\ w_i^{(1)}(x) &= \int_{\Omega_0} N_i(x,y)\zeta(y) D_j g_j(y) dy \\ w_i^{(2)}(x) &= \left(\int_\Omega \phi \nabla \cdot \boldsymbol{g}\right)\left(\int_{\Omega_0} N_i(x,y)\theta(y)dy\right). \end{aligned} \tag{III.3.46}$$

Using the properties of ω, we obtain

$$\begin{aligned} |\boldsymbol{N}(x,y)| &\le \left| \frac{\boldsymbol{x}-\boldsymbol{y}}{|x-y|^n} \int_{|x-y|}^\infty \omega\left(y + \xi \frac{x-y}{|x-y|}\right) \xi^{n-1} d\xi \right| \\ &\le |x-y|^{1-n} \|\omega\|_\infty \int_0^{\delta(\Omega_0)} \xi^{n-1} d\xi \le c|x-y|^{1-n} \end{aligned} \tag{III.3.47}$$

with $c = c(n, B, \Omega_0)$. Thus, from Young's inequality on convolutions (Theorem II.11.1) it follows that

$$\| \int_{\Omega_0} \boldsymbol{N}(\cdot, y)\, \theta(y) dy \|_{q,\Omega_0} < \infty,$$

and so

$$\|\boldsymbol{w}^{(2)}\|_{q,\Omega_0} \le c_3 \left| \int_\Omega \phi \nabla \cdot \boldsymbol{g} \right|.$$

Since $\boldsymbol{g} \in \widetilde{H}_{0,q}(\Omega)$, the trace $\boldsymbol{n} \cdot \boldsymbol{g}$ of the normal component of $\boldsymbol{g}$ at $\partial\Omega$ is identically vanishing (see Theorem III.2.4), and consequently we have by (III.2.14)

$$\int_\Omega \phi \nabla \cdot \boldsymbol{g} = -\int_\Omega \boldsymbol{g} \cdot \nabla \phi.$$

Therefore,

$$\|\boldsymbol{w}^{(2)}\|_{q,\Omega_0} \le c_4 \|\boldsymbol{g}\|_{q,\Omega}. \tag{III.3.48}$$

Again from (III.2.14) and taking into account $\zeta \in C_0^\infty(G)$, we may integrate by parts into $(\text{III.3.46})_3$ to obtain

$$
\begin{aligned}
w_i^{(1)}(x) = -\lim_{\varepsilon\to 0}\Bigg(\int_{\Omega_0 - B_\varepsilon(x)} & [N_i(x,y)g_j(y)D_j\zeta(y) \\
& +\zeta(y)g_j(y)D_jN_i(x,y)]dy + \int_{\partial B_\varepsilon(x)} N_i(x,y)\zeta(y)g_j(y)\frac{x_j-y_j}{|x-y|}d\sigma_y\Bigg) \\
\equiv -\lim_{\varepsilon\to 0}\sum_{i=1}^{3} I_i(x,\varepsilon). &
\end{aligned}
\tag{III.3.49}
$$

Since by (III.3.47) and Young's inequality on convolutions,

$$
\|\int_{\Omega_0} \boldsymbol{N}(\cdot,y)\boldsymbol{g}\cdot\nabla\zeta(y)dy\|_{q,\Omega_0} \le c_5\|\boldsymbol{g}\|_{q,\Omega},
$$

we have, for a.a. $x \in \Omega$,

$$
\lim_{\varepsilon\to 0} I_1(x,\varepsilon) = \int_{\Omega_0} N_i(x,y)g_j(y)D_j\zeta(y)dy. \tag{III.3.50}
$$

Furthermore, by a reasoning analogous to that leading to (III.3.13) we show for a.a. $x \in \Omega_0$

$$
\lim_{\varepsilon\to 0} I_3(x,\varepsilon) = \zeta(x)g_j(x)\int_{\Omega_0}\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\omega(y)dy. \tag{III.3.51}
$$

Finally, as we know from (III.3.15), (III.3.16), and (III.3.17),

$$
D_jN_i(x,y) = K_{ij}(x,x-y) + G_{ij}(x,y),
$$

where K_{ij} is a Calderón–Zygmund kernel, while G_{ij} is bounded by a weakly singular kernel. Therefore, from Theorem II.11.4 we deduce for a.a. $x \in \Omega_0$

$$
\lim_{\varepsilon\to 0} I_2(x,\varepsilon) = \int_{\Omega_0} K_{ij}(x,x-y)\zeta(y)g_j(y)dy + \int_{\Omega_0} G_{ij}(x,y)\zeta(y)g_j(y)dy, \tag{III.3.52}
$$

where, of course, the first integral has to be understood in the Cauchy principal value sense. From (III.3.49)–(III.3.52), from Theorem II.11.4, $(\text{III.3.15})_2$ and Young's inequality on convolutions we then conclude

$$
\|\boldsymbol{w}^{(1)}\|_{q,\Omega_0} \le c_6\|\boldsymbol{g}\|_{q,\Omega}. \tag{III.3.53}
$$

Thus, estimates $(\text{III.3.44})_4$ becomes a consequence of (III.3.46), (III.3.48), and (III.3.53) and the lemma is proved. □

We are now in a position to prove the following.

Theorem III.3.4 *Let Ω be a bounded, locally Lipschitz domain in $\mathbb{R}^n$, $n \ge 2$. Then, given*

$$\boldsymbol{g} \in \widetilde{H}_{0,q}(\Omega), \quad 1 < q < \infty,$$

there exists at least one solution $\boldsymbol{v}$ *to the problem*

$$\begin{aligned} \nabla \cdot \boldsymbol{v} &= \nabla \cdot \boldsymbol{g} \\ \boldsymbol{v} &\in W_0^{1,q}(\Omega) \\ \|\boldsymbol{v}\|_{1,q} &\leq c\,\|\nabla \cdot \boldsymbol{g}\|_q \\ \|\boldsymbol{v}\|_q &\leq c\,\|\boldsymbol{g}\|_q. \end{aligned} \tag{III.3.54}$$

In particular, if $\nabla \cdot \boldsymbol{g} \in C_0^\infty(\Omega)$*, then* $\boldsymbol{v} \in C_0^\infty(\Omega)$ *and we have*

$$\|\boldsymbol{v}\|_{1,s} \leq c\,\|\nabla \cdot \boldsymbol{g}\|_s, \quad \textit{for all } s \in (1,\infty). \tag{III.3.55}$$

Proof. We first take $\boldsymbol{g}$ such that

$$\nabla \cdot \boldsymbol{g} \in C_0^\infty(\Omega), \quad \boldsymbol{g} \in \widetilde{H}_{0,q}(\Omega), \quad 1 < q < \infty. \tag{III.3.56}$$

In view of Lemma III.3.4, there exist N domains $\Omega_i = \Omega \cap G_i$, with $\{G_i\}$ an open covering of $\overline{\Omega}$, satisfying (i)–(iv) of that lemma. Furthermore, we may write

$$\nabla \cdot \boldsymbol{g} = \sum_{i=1}^{N} f_i$$

with

$$\begin{aligned} & f_i \in C_0^\infty(\Omega_i) \\ & \int_{\Omega_i} f_i = 0, \end{aligned} \tag{III.3.57}$$

and where f_i has the expression

$$f_i = \zeta_i \nabla \cdot \boldsymbol{g} + \sum_{k=1}^{m_i} \theta_k \int_\Omega \phi_k \nabla \cdot \boldsymbol{g}, \quad \text{some } m_i \in \mathbb{N}, \tag{III.3.58}$$

with

$$\phi_k \in C_0^\infty(\overline{\Omega}), \quad \zeta_i \in C_0^\infty(G_i), \quad \theta_k \in C_0^\infty(\Omega_i), \quad k = 1,\dots,m_i,\ i = 1,\dots,N.$$

From (III.3.57) and (III.3.58), with the aid of Lemma III.3.4, for any $i = 1,\dots,N$ we can state the existence of a vector $\boldsymbol{v}_i$ such that for all $s \in (1,\infty)$

$$\begin{aligned} \nabla \cdot \boldsymbol{v}_i &= f_i \\ \boldsymbol{v}_i &\in C_0^\infty(\Omega_i) \\ \|\boldsymbol{v}_i\|_{1,s,\Omega_i} &\leq c\,\|\nabla \cdot \boldsymbol{g}\|_{s,\Omega} \\ \|\boldsymbol{v}_i\|_{q,\Omega_i} &\leq c\,\|\boldsymbol{g}\|_{q,\Omega}. \end{aligned} \tag{III.3.59}$$

Thus, the field

$$\boldsymbol{v} = \sum_{k=1}^{N} \boldsymbol{v}_k$$

satisfies all requirements of the theorem, which is thus proved if $\boldsymbol{g}$ satisfies (III.3.56). Assume, now, $\boldsymbol{g}$ merely belongs to $\widetilde{H}_{0,q}(\Omega)$, $1 < q < \infty$. In view of Theorem III.2.4 we can approximate $\boldsymbol{g}$ by a sequence $\{\boldsymbol{g}_s\} \subset C_0^\infty(\Omega)$. For each s we then establish the existence of $\boldsymbol{v}_s$ solving (III.3.54) with $\boldsymbol{g}_s$ in place of $\boldsymbol{g}$. Because of $(\text{III.3.54})_{3,4}$ we attain the existence of $\boldsymbol{v} \in W_0^{1,q}(\Omega)$ such that

$$\boldsymbol{v}_s \to \boldsymbol{v} \ \text{ in } W_0^{1,q}(\Omega).$$

Evidently, $\boldsymbol{v}$ solves $(\text{III.3.54})_1$ in the generalized sense and, again by $(\text{III.3.54})_{3,4}$ (written for $\boldsymbol{v}_s$) $\boldsymbol{v}$ satisfies $(\text{III.3.54})_{3,4}$. The proof of the theorem is therefore completed. □

An interesting consequence of the result just shown is given in the following.

Theorem III.3.5 *Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 2$, of class C^2. Then, for any $f \in L^q(\Omega)$, $1 < q < \infty$, satisfying* (III.3.1), *there exists a solution $\boldsymbol{v}$ to problem* (III.3.2) *which, in addition, obeys the following estimate*

$$\|\boldsymbol{v}\|_q \leq c\,|f|^*_{-1,q}\,,$$

with $c = c(q, \Omega, n)$, where $|\cdot|^*_{-1,q}$ is defined in (III.3.41).

Proof. For a given f in the statement of the theorem, consider the functional

$$\mathcal{F} : [\varphi] \in \dot{D}^{1,q'}(\Omega) \to (f, \varphi) \in \mathbb{R}\,, \quad \varphi \in [\varphi].$$

Since f satisfies (III.3.1), the value of $\mathcal{F}$ is independent of the choice of $\varphi \in [\varphi]$, and $\mathcal{F}$ is, therefore, well defined. It is easy to show that

$$\mathcal{F} \in (\dot{D}^{1,q'}(\Omega))'\,. \tag{III.3.60}$$

In fact, it is obvious that $\mathcal{F}$ is additive. Furthermore, since f satisfies (II.1.13), by means of (II.5.10) we obtain

$$|(f, \varphi)| = |(f, \varphi - \overline{\varphi}_\Omega)| \leq \|f\|_q \|\varphi - \overline{\varphi}_\Omega\|_{q'} \leq c\,\|f\|_q |\varphi|_{1,q'}\,,$$

which proves (III.3.60), and, moreover,

$$\|\mathcal{F}\|_{(\dot{D}^{1,q'}(\Omega))'} = |f|^*_{-1,q}\,. \tag{III.3.61}$$

Now, by Theorem II.8.2, there is $\boldsymbol{F} \in [L^{q'}(\Omega)]^n$ such that $(f, \varphi) = (\boldsymbol{F}, \nabla\varphi)$, for all $\varphi \in D^{1,q}(\Omega)$ and $\|\boldsymbol{F}\|_q = |f|^*_{-1,q}$. Thus, in view of Lemma III.1.2 and Theorem III.1.2, the Neumann problem

$$-(\nabla w, \nabla \varphi) = (f, \varphi), \text{ for all } \varphi \in D^{1,q'}(\Omega), \tag{III.3.62}$$

has a unique (up to a constant) solution $w \in D^{1,q}(\Omega)$ which, in particular, satisfies the following estimate

$$|w|_{1,q} \le c\,|f|^*_{-1,q}, \tag{III.3.63}$$

with $c = c(n, q, \Omega)$. If we set $\boldsymbol{g} = \nabla w$, from (III.3.62) we get (in the sense of weak derivatives)

$$\nabla \cdot \boldsymbol{g} = f \tag{III.3.64}$$

and, in particular,

$$(\boldsymbol{g}, \nabla\varphi) + (\nabla \cdot \boldsymbol{g}, \varphi) = 0, \quad \text{for all } \varphi \in W^{1,q'}(\Omega).$$

Therefore, by (III.2.14), we find $\boldsymbol{g} \cdot \boldsymbol{n} = 0$, and, by Theorem III.2.4, we conclude that $\boldsymbol{g} \in \widetilde{H}_{0,q}$. The result is then a consequence of (III.3.63), (III.3.64) and of Theorem III.3.4. □

We shall next consider the solvability of problem (III.3.2) for Ω an *exterior domain.* In this case the problem will be suitably reformulated, as the following considerations suggest. Let

$$\Omega = \mathbb{R}^2 - B(0),$$

and $f = f(|x|)$ be smooth and of compact support in Ω. The vector field

$$\boldsymbol{v}(x) = \left(\frac{\boldsymbol{x}}{|x|^2}\right) \int_1^{|x|} \tau\, f(\tau) d\tau$$

solves $(\text{III.3.2})_1$, vanishes on $\partial\Omega$ and, further,

$$|\boldsymbol{v}|_{1,2} = \|f\|_2.$$

However,

$$\boldsymbol{v} \notin L^2(\Omega)$$

and so

$$\boldsymbol{v} \notin W_0^{1,2}(\Omega).$$

This example suggests that, for an exterior domain, the class $W_0^{1,q}(\Omega)$ in problem (III.3.2) should be enlarged to the class $D_0^{1,q}(\Omega)$. Therefore, for such domains, we shall formulate the problem as follows: *Given $f \in L^q(\Omega)$ to find a vector field $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ such that*

$$\begin{gathered} \nabla \cdot \boldsymbol{v} = f \\ \boldsymbol{v} \in D_0^{1,q}(\Omega) \\ |\boldsymbol{v}|_{1,q} \le c\,\|f\|_q \end{gathered} \tag{III.3.65}$$

where $c = c(n, q, \Omega)$.

Remark III.3.13 For Ω an exterior domain, the condition (III.3.1) on f is no longer required. ■

The following existence result holds.

Theorem III.3.6 *Let Ω be a locally Lipschitz, exterior domain of $\mathbb{R}^n$, $n \geq 2$. Then, for any $f \in L^q(\Omega)$, $1 < q < \infty$, there exists a solution to problem* (III.3.65).

Proof. If $\Omega = \mathbb{R}^n$, we at once check, with the help of Exercise II.11.9(i), that the field $\boldsymbol{v} = \nabla\psi$ with $\Delta\psi = f$ gives a solution to the problem. Thus, we assume $\partial\Omega \neq \emptyset$. Let $\{f_m\} \subset C_0^\infty(\Omega)$ be a sequence approximating f in $L^q(\Omega)$, and set

$$\boldsymbol{v}_m = \nabla\psi_m + \boldsymbol{w}_m, \quad m \in \mathbb{N},$$

where

$$\Delta\psi_m = f_m \ \text{ in } \mathbb{R}^n$$

and, for some $R > 2\delta(\Omega_c)$,

$$\begin{aligned} \nabla \cdot \boldsymbol{w}_m &= 0 \ \text{ in } \Omega_R \\ \boldsymbol{w}_m &= -\nabla\psi_m \ \text{ on } \partial\Omega \\ \boldsymbol{w}_m &= 0 \ \text{ on } \partial B_R(0). \end{aligned} \tag{III.3.66}$$

By Exercise II.11.9(i), we find

$$|\psi_m|_{2,q} \leq c\|f_m\|_q. \tag{III.3.67}$$

Moreover, since $\Delta\psi_m = 0$ in Ω^c, we have

$$\int_{\partial\Omega} \nabla\psi_m \cdot \boldsymbol{n} = 0, \quad \text{for all } m \in \mathbb{N}, \tag{III.3.68}$$

and so, from Exercise III.3.5, we deduce the existence of a solenoidal field $\boldsymbol{w}_m \in W_0^{1,q}(\Omega_R)$ solving (III.3.66). We extend $\boldsymbol{w}_m$ to zero outside Ω_R so that

$$\|\boldsymbol{w}_m\|_{1,q} \leq c_1\|\nabla \cdot (\varphi\nabla\psi_m)\|_q, \tag{III.3.69}$$

where $\varphi \in C^1(\mathbb{R}^n)$ and $\varphi = 1$ if $|x| < R/2$, $\varphi = 0$ if $|x| \geq R$. From (III.3.67) and (III.3.69) we thus obtain

$$\|\boldsymbol{w}_m\|_{1,q} \leq c_2\left(\|f_m\|_q + \|\nabla\psi_m\|_{q,\Omega_{R/2,R}}\right). \tag{III.3.70}$$

If $1 < q < n$, since $\nabla\psi_m = O(|x|^{1-n})$ as $|x| \to \infty$, we apply Theorem II.6.1 to deduce

$$\|\nabla\psi_m\|_{nq/(n-q)} \leq c_3|\psi_m|_{2,q}\,, \tag{III.3.71}$$

which, along with the properties of $\boldsymbol{w}_m$ and the characterization (II.7.14), delivers $\boldsymbol{v}_m \in D_0^{1,q}(\Omega)$. Moreover, from (III.3.67), (III.3.70) and (III.3.71) we obtain

$$\|\boldsymbol{w}_m\|_{1,q} \le c_4 \|f_m\|_q. \tag{III.3.72}$$

If $q \ge n$, we add to ψ_m a linear function in $\boldsymbol{x}$ such that

$$\int_{\Omega_{R/2,R}} \nabla \psi_m = 0,$$

and continue to denote the modified fields by ψ_m and $\boldsymbol{v}_m$. Clearly, ψ_m continue to satisfy (III.3.68), while $\boldsymbol{v}_m$, in view of the characterization (II.7.15), belongs to $D_0^{1,q}(\Omega)$. From the last displayed equation and Theorem II.5.4 it follows that

$$\|\nabla \psi_m\|_{q,\Omega_{R/2,R}} \le c_5 |\psi_m|_{2,q}.$$

Employing this inequality back into (III.3.69) and using (III.3.70) gives again (III.3.72). From what we said, and from (III.3.67) and (III.3.72) it then follows that $\boldsymbol{v}_m$ solves the problem

$$\begin{aligned} \nabla \cdot \boldsymbol{v}_m &= f_m \\ \boldsymbol{v}_m &\in D_0^{1,q}(\Omega) \\ |\boldsymbol{v}_m|_{1,q} &\le c\|f_m\|_q \end{aligned} \tag{III.3.73}$$

with $c = c(n, q, \Omega)$. The theorem then easily follows by letting $m \to \infty$ into (III.3.73) (details are left to the reader). □

Remark III.3.14 Theorem III.3.6 continues to hold if Ω satisfies the cone condition (Bogovskiĭ 1980, Theorem 4). ■

Remark III.3.15 From the proof of Theorem III.3.4 it follows that if

$$f \in L^q(\Omega) \cap L^r(\Omega),$$

we can find a solution $\boldsymbol{v}$ to $(\text{III.3.65})_1$ such that $\boldsymbol{v} \in D_0^{1,q}(\Omega) \cap D_0^{1,r}(\Omega)$ and that satisfies

$$|\boldsymbol{v}|_{1,q} \le c\|f\|_q, \quad |\boldsymbol{v}|_{1,r} \le c\|f\|_r.$$

■

Remark III.3.16 A result similar to that of Theorem III.3.3 can also be proved for exterior domains. We shall not show this here, since it will not be needed later, and refer to Bogovskiĭ (1980, Theorem 5) for a proof. However, the reader may wish to give his/her own proof. ■

Exercise III.3.8 Let Ω satisfy the assumptions of Theorem III.3.6. Show that, for any $f \in L^q(\Omega)$ and $\boldsymbol{a} \in W^{1-1/q,q}(\partial\Omega)$, there exists at least one solution to the problem

$$\begin{aligned} \nabla \cdot \boldsymbol{v} &= f \\ \boldsymbol{v} &\in D^{1,q}(\Omega) \\ \boldsymbol{v} &= \boldsymbol{a} \ \text{ at } \partial\Omega \\ |\boldsymbol{v}|_{1,q} &\le c\left(\|f\|_q + \|\boldsymbol{a}\|_{1-1/q,q(\partial\Omega)}\right), \end{aligned}$$

with $c = c(n, q, \Omega)$. Notice that, unlike the case of a bounded domain (see Exercise III.3.5), the condition

$$\int_\Omega f = \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n}$$

is *not* needed. In this respect, show that if

$$f \equiv 0 \quad \text{and} \quad \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} = 0,$$

then $\boldsymbol{v}$ can be taken of bounded support in Ω.

Exercise III.3.9 Let Ω satisfy the assumption of Theorem III.3.6, and assume that $\|(|x|^\alpha + 1)\, f\|_\infty < \infty$, for some $\alpha \in (1, n)$. Show that problem (III.3.65), with $q > n$, has at least one solution which, in addition, satisfies the estimate

$$\|(|x|^{\alpha-1} + 1)\boldsymbol{v}\|_\infty \le c_1 \, \|(|x|^\alpha + 1)\, f\|_\infty \, .$$

Show also that, if $f \in L^q(\Omega)$, $q > n$, with a bounded support, K, then there exists a solution to (III.3.65) which, in addition, satisfies the estimate

$$\|(|x|^{n-1} + 1)\boldsymbol{v}\|_\infty \le c_2 \|f\|_q \, ,$$

with $c_2 = c_2(K, q)$. *Hint*: Use the argument employed in the proof of Theorem III.3.6, along with Lemma II.9.2.

Exercise III.3.10 Let Ω satisfy the assumption of Theorem III.3.6, and let $u \in L^1_{loc}(\Omega)$ with $\nabla u \in D_0^{-1,q}(\Omega)$. Show that $u \in L^q(\Omega)$ and that the following generalization of the Poincaré inequality holds

$$\|u\|_q \le c\, |\nabla u|_{-1,q} \, ,$$

where $c = c(\Omega, q)$. *Hint*: See Exercise III.3.4. (For inequality of this type where only one derivative of u is in $D_0^{-1,q}(\Omega)$ see Galdi 2007, Proposition 1.1.)

Problem (III.3.65) can be considered also for domains with a *noncompact boundary* (Bogovskiĭ 1980, §3; Solonnikov 1981, §2; Padula 1992, Lemma 2.2). In Section 3 of the next chapter (see Theorem IV.3.2) we shall show that it can be solved for $\Omega = \mathbb{R}^n_+$. A detailed study of solvability of (III.3.65) in domains having $m \ge 1$ "exits" at infinity (such as infinite tubes and pipes) has been performed by Solonnikov (1981) for the case $q = 2$. (His results, however, admit of a straightforward generalization to the case $q \in (1, \infty)$.) It should be noted that, in this latter case, the problem need not be solvable, in general, if f merely belongs to $L^q(\Omega)$ and, in fact, some *additional compatibility conditions* are to be assumed on f. For instance, if Ω is the infinite cylinder:

$$\Omega = \{x \in \mathbb{R}^n : |x'| < A\} \, , \quad x' = (x_1, \dots, x_{n-1}), A > 0,$$

one easily checks that the problem

$$\begin{aligned} &\nabla \cdot \boldsymbol{v} = f \\ &\boldsymbol{v} \in D_0^{1,q}(\Omega) \end{aligned} \tag{III.3.74}$$

can admit a solution *only if* f satisfies (III.3.1) and the quantity

$$|f\widetilde{|}_q \equiv \left\{ \int_{-\infty}^{\infty} \left| \int_t^{\infty} \left(\int_{|x'|\le A} f(x', x_n) dx' \right) dx_n \right|^q dt \right\}^{1/q}$$

is finite. Conversely, one can prove that, for any f satisfying (III.3.1) and belonging to the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|f\widetilde{\|}_q \equiv \|f\|_q + |f\widetilde{|}_q,$$

there exists a solution to (III.3.74) that also satisfies the inequality

$$|\boldsymbol{v}|_{1,q} \le c\|f\widetilde{\|}_q$$

(see Solonnikov 1981, Theorem 2). Notice that, in the particular case where f has zero average along the cross-section of the cylinder, namely,

$$\int_{|x'|\le A} f(x', x_n) dx' = 0\,,$$

then $\|f\widetilde{\|}_q = \|f\|_q$.

Remark III.3.17 The solvability of problem (III.3.1), $(\text{III.3.2})_1$ in Hölder spaces $C^{m,\lambda}(\Omega)$ has been investigated by Kapitanskiĭ & Pileckas (1984, §8). In particular, these authors prove the following result (see their Theorem 1 on p. 481). ∎

Theorem III.3.7 *Let Ω be a bounded domain of $\mathbb{R}^n$, $n \ge 2$, of class $C^{m+2,\lambda}$, $m \ge 0$, $\lambda \in (0, 1)$. Then, given $\boldsymbol{f} \in C^{m,\lambda}(\overline{\Omega})$ satisfying* (III.3.1)*, there exists at least one field $\boldsymbol{v} \in C^{m+1,\lambda}(\overline{\Omega})$ vanishing at $\partial\Omega$ such that $\nabla\cdot\boldsymbol{v} = \boldsymbol{f}$. Moreover, $\boldsymbol{v}$ obeys the following estimate*

$$\|\boldsymbol{v}\|_{C^{m+1,\lambda}} \le c\|\boldsymbol{f}\|_{C^{m,\lambda}}$$

with $c = c(m, \lambda, n, \Omega)$.

Remark III.3.18 In connection with Theorem III.3.7, we wish to observe that, in general, the number λ *cannot* be taken to be zero; otherwise, this would imply the existence of certain solenoidal extensions of boundary data that, however, *cannot* exist, as shown by counterexamples, see Section IX.4; see also Dacorogna, Fusco & Tartar (2004). ∎

III.4 The Spaces H_q^1

The study of the dynamical properties of the flow of a viscous, incompressible fluid requires velocity fields of the particles of the fluid which, at each time, are summable to the qth power, $q \geq 1$, along with their first spatial derivatives, are solenoidal and vanish at the boundary of the region where the motion occurs. One is thus led to introduce a space of vector functions having such properties (in a generalized sense) and, in this respect, either of the following two choices seems plausible, namely,

$$\left\{ \textit{completion of } \mathcal{D}(\Omega) \textit{ in the norm of } W^{1,q}(\Omega) \right\} \equiv H_q^1(\Omega)$$

or

$$\left\{ \boldsymbol{v} \in W_0^{1,q}(\Omega) : \nabla \cdot \boldsymbol{v} = 0 \ \ \textit{in } \Omega \right\} \equiv \widehat{H}_q^1(\Omega).$$

(For $q = 2$ we will write $H^1(\Omega)$ and $\widehat{H}^1(\Omega)$, respectively.) Even though these spaces look similar, they are a priori distinct in that the condition of solenoidality on their members is imposed before (in $H_q^1(\Omega)$) and after (in $\widehat{H}_q^1(\Omega)$) taking the completion of $C_0^\infty(\Omega)$ in the norm of $W^{1,q}(\Omega)$. However, one easily shows that for *any* domain Ω the following inclusions hold for all $q \geq 1$:

$$H_q^1(\Omega) \subset \widehat{H}_q^1(\Omega)$$
$$H_q^1(\Omega) \subset H_q(\Omega).$$

The fact that the two spaces can be different for some domains and some q (depending on the space dimension n) can be guessed through the following considerations. Let Ω be the infinite cylinder

$$\left\{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1 \right\}.$$

If $\boldsymbol{v}$ is a solenoidal vector function vanishing on $\partial\Omega$, one readily verifies that the flux ϕ of $\boldsymbol{v}$ through the cross section Σ of Ω[1] at a point x_3 of the axis of the cylinder is a constant independent of x_3, that is,

$$\phi \equiv \int_\Sigma \boldsymbol{v} \cdot \boldsymbol{n} = \text{const.}, \quad \boldsymbol{n} = (0, 0, 1).$$

If $\boldsymbol{v} \in H^1(\Omega)$, one immediately obtains $\phi = 0$. Actually, in such a case, we know that $\boldsymbol{v}$ is approximated by *solenoidal* vector functions *of compact support* in Ω, see Exercise III.4.1. If $\boldsymbol{v} \in \widehat{H}^1(\Omega)$, however, this approximation does not hold a priori but nevertheless we can deduce $\phi = 0$ from the following three observations:

(i) $\left| \int_\Sigma \boldsymbol{v} \cdot \boldsymbol{n} \right| \leq |\Sigma|^{1/2} \left(\int_\Sigma \boldsymbol{v}^2 \right)^{1/2},$

[1] Namely, Σ is the intersection of Ω with a plane orthogonal to the axis of Ω.

(ii) There exists a sequence $\{x_3^{(k)}\} \subset \mathbb{R}$ with $|x_3^{(k)}| \to \infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty}(1+|x_3^{(k)}|)\int_{\Sigma}\boldsymbol{v}^2 = 0,$$

(iii) $|\Sigma|/(1+|x_3|)$ is bounded.

Notice that (ii) is a simple consequence of the hypothesis $\boldsymbol{v} \in W_0^{1,2}(\Omega)$.[2] Assume now that the domain, instead of having "exits" to infinity of bounded cross section (as in the previous case), has a cross section whose area increases sufficiently fast as $x_3 \to \pm\infty$. For example, we may choose

$$\Omega = \left\{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1 + x_3^2\right\}. \tag{III.4.1}$$

We still have $\phi = 0$ for $\boldsymbol{v} \in H^1(\Omega)$, but we can *no longer* deduce the same result if $\boldsymbol{v} \in \widehat{H}^1(\Omega)$. Actually (i) and (ii) also remain true in this case but (iii) *fails*, because now $|\Sigma| = \pi(1+x_3^2)$. Thus one may suspect the existence of a solenoidal vector in $W_0^{1,2}(\Omega)$ which, though satisfying (ii), has a non-zero flux through Σ, thus obtaining $H^1(\Omega) \neq \widehat{H}^1(\Omega)$. We shall see later in this section that such a vector field actually exists.

Exercise III.4.1 Let Ω be a domain of $\mathbb{R}^n$, $n \geq 2$, with $N \geq 1$ "outlets to infinity," i.e.,

$$\Omega = \Omega_0 \cup \Omega_1 \cup \ldots \cup \Omega_N,$$

where Ω_0 is bounded, $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$, and each Ω_i, in possibly different coordinate systems, has the form

$$\Omega_i = \left\{x \in \mathbb{R}^n : x_n > 0, x' \in \Sigma(x_n)\right\},$$

where $\Sigma(x_n)$ is a bounded domain in $\mathbb{R}^{n-1}$ smoothly varying with x_n and $x' = (x_1, \ldots, x_{n-1})$. Show that for all $\boldsymbol{v} \in H_q^1(\Omega)$, $1 \leq q \leq \infty$,

$$\int_{\Sigma_i} \boldsymbol{v} \cdot \boldsymbol{n}_i = 0, \quad i = 1, ..., N.$$

---

[2] In fact, setting

$$g(x_3) = \int_{\Sigma} v^2(x_1, x_2, x_3)dx_1dx_2,$$

by Theorem II.4.1 $g(x_3)$ is well defined. Moreover,

$$\int_{-\infty}^{\infty} g(x_3)dx_3 < \infty. \tag{$*$}$$

Therefore, there exists a sequence $\{x_3^{(k)}\}$ with $|x_3^{(k)}| \to \infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty}(1+|x_3^{(k)}|)g(x_3^{(k)}) = 0.$$

In fact, if this were not true, we should have

$$g(x_3) \geq \ell/(1+|x_3|), \quad \text{for all } x_3 \text{ with } |x_3| > b,$$

and for some $\ell, b > 0$, which contradicts $(*)$.

Hint: Use the fact that $\boldsymbol{v}$ can be approximated by elements of $\mathcal{D}(\Omega)$, together with Theorem II.4.1.

The problem of the relationship between $H^1_q(\Omega)$ and $\widehat{H}^1_q(\Omega)$ is not merely a question of mathematical completeness; rather, as pointed out for the first time by Heywood (1976), the coincidence of the two spaces is tightly linked with the uniqueness of flow of a viscous, incompressible fluid and, in particular, in domains for which $H^1_q(\Omega) \neq \widehat{H}^1_q(\Omega)$ the motion of such a fluid is not uniquely determined by the "traditional" initial and boundary data but other extra and appropriate auxiliary conditions are needed. Referring the reader to Chapters VI and XII for a description of these latter and related results, in the present section we shall only consider the question of investigating for which domains the two spaces coincide and will indicate domains for which they certainly don't. To this end, we subdivide the domains of $\mathbb{R}^n (n \geq 2)$ into three groups:

(a) *bounded domains*;
(b) *exterior domains*;
(c) *domains with a noncompact boundary.*

In cases (a) and (b) one shows $H^1_q(\Omega) = \widehat{H}^1_q(\Omega)$ provided only Ω has a mild degree of smoothness (for example, cone condition, or even less, would suffice); see Theorem III.4.1 and Theorem III.4.2. On the other hand, in case (c) one exhibits examples of domains for which the two spaces are distinct, no matter how smooth their boundary is; see Theorem III.4.4 and Theorem III.4.6. Therefore, the coincidence of $H^1_q(\Omega)$ and $\widehat{H}^1_q(\Omega)$ does not seem related to a high degree of regularity of Ω but rather to its shape. In this connection, we wish to recall a remarkable general result of Maslennikova & Bogovskiĭ (1983, Theorem 5), which states that, if Ω is an *arbitrary* strongly locally Lipschitz domain,[3] then $H^1_q(\Omega) = \widehat{H}^1_q(\Omega)$, for all $q \in [1, n/(n-1)]$.

It is conjectured that, for *all* domains with a *compact* boundary, the two spaces coincide for $q = 2$, but no proof is, to date, available in the general case.[4] Should this coincidence *fail* to hold for *some* domain of the above type, we would have paradoxical situations from the physical point of view. For example, if for some *bounded* domain, $\Omega^\sharp$ (say), the two spaces did not coincide, then the steady-state Stokes boundary-value problem formulated in $\Omega^\sharp$ and corresponding to *zero* body force and *zero* (Dirichlet) data at the boundary would admit a *non-zero and smooth* solution; see Remark IV.1.2. Analogous situation would occur if $\Omega^\sharp$ is an exterior domain; see Remark V.1.2

We finally notice that, as already observed, for the proof of coincidence, it is sufficient to show $\widehat{H}^1_q(\Omega) \subset H^1_q(\Omega)$, the converse inclusion being always satisfied.

[3] This type of regularity extends the local Lipschitz one to the case of domains with a noncompact boundary, see, e.g., Adams (1975, p. 66).

[4] See, however the result of Šverák (1993) mentioned in the Notes for this Chapter.

III.4.1 Bounded Domains

The simplest situation occurs when Ω *is star-shaped* (with respect to the origin, say). For $\rho \in (0,1)$ we know from Exercise II.1.3 that $\overline{\Omega}^{(\rho)} \subset \Omega$. Thus, given a vector $\boldsymbol{v} \in \widehat{H}^1_q(\Omega)$, by extending it by zero outside Ω, we deduce that the family of vectors $\boldsymbol{v}_\rho(x) \equiv \boldsymbol{v}(x/\rho)$, $x \in \mathbb{R}^n$, belongs to $\widehat{H}^1_q(\Omega)$ and is of compact support in Ω. Set

$$h_0 = \min_{x \in \partial\Omega} |x|, \quad \varepsilon(\rho) = h_0(1-\rho)/2$$

and consider the mollification $(\boldsymbol{v}_\rho)_{\varepsilon(\rho)} \equiv \boldsymbol{V}_\rho$. By the results ofSection II.2 on regularizations we obtain that, for each $\rho \in (0,1)$, $\boldsymbol{V}_\rho \in \mathcal{D}(\Omega)$ and that $\boldsymbol{V}_\rho \to \boldsymbol{v}$ in $W^{1,q}(\Omega)$ as $\rho \to 1$, for all $q \in [1,\infty)$. Therefore, for these values of q, $\widehat{H}^1_q(\Omega) \subset H^1_q(\Omega)$ and the two spaces coincide.

Using an idea of Bogovskiĭ (1980, Lemma 4), this result can be generalized to a fairly reach class of bounded domains. Specifically, we have the following.

Theorem III.4.1 *Assume that the bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is the union of a finite number of domains Ω_i each of which is star-shaped with respect to an open ball B_i with $\overline{B}_i \subset \Omega_i$* (e.g., *$\Omega$ satisfies the cone condition*).[5] *Then $\widehat{H}^1_q(\Omega) = H^1_q(\Omega)$.*

Proof. Let us first consider the case $q \in (1,\infty)$. Given $\boldsymbol{v} \in \widehat{H}^1_q(\Omega)$, let $\{\boldsymbol{v}_k\} \subset C_0^\infty(\Omega)$ be a sequence approximating $\boldsymbol{v}$ in $W^{1,q}(\Omega)$ and denote by $\boldsymbol{w}_k$ a solution to (III.3.2) with $f = -\nabla \cdot \boldsymbol{v}_k$. Since f satisfies (III.3.1), by Theorem III.3.1 $\boldsymbol{w}_k$ exists, belongs to $W_0^{1,q}(\Omega)$, and can be taken to be of compact support in Ω. The functions $\boldsymbol{u}_k = \boldsymbol{v}_k + \boldsymbol{w}_k$, $k \in \mathbb{N}$, are thus divergence free and of compact support in Ω. Furthermore, using (III.3.3) for $\boldsymbol{w}_k$, we have

$$\|\boldsymbol{u}_k - \boldsymbol{v}\|_{1,q} \leq \|\boldsymbol{v}_k - \boldsymbol{v}\|_{1,q} + \|\boldsymbol{w}_k\|_{1,q} \leq \|\boldsymbol{v}_k - \boldsymbol{v}\|_q + c\|\nabla \cdot \boldsymbol{v}_k\|_q$$

with c independent of k. Since $\nabla \cdot \boldsymbol{v} = 0$, it follows that

$$\|\nabla \cdot \boldsymbol{v}_k\|_q \to 0 \quad \text{as } k \to \infty,$$

and so the previous inequality furnishes

$$\boldsymbol{u}_k \to \boldsymbol{v} \quad \text{in } W^{1,q}(\Omega), \text{ as } k \to \infty.$$

Let $(\boldsymbol{u}_k)_\varepsilon$ be the regularization of $\boldsymbol{u}_k$. For sufficiently small ε, $(\boldsymbol{u}_k)_\varepsilon$ belongs to $\mathcal{D}(\Omega)$ and approaches $\boldsymbol{u}_k$ in $W^{1,q}(\Omega)$ (see Section II.2 and Exercise II.3.2), which proves $\boldsymbol{v} \in H^1_q(\Omega)$. Let us now prove $H^1_1(\Omega) = \widehat{H}^1_1(\Omega)$. To this end,

[5] See Remark III.3.4 and Remark III.3.5.

it is sufficient to show that every linear functional F defined in $\widehat{H}_1^1(\Omega)$ and vanishing in $H_1^1(\Omega)$ is identically zero. Since $H_q^1(\Omega) \subset H_1^1(\Omega)$ and $\widehat{H}_q^1(\Omega) = H_q^1(\Omega)$ for $q > 1$, we have $F \in \mathcal{S}$ where

$$\mathcal{S} = \left\{ \ell \in \left(\widehat{H}_q^1(\Omega)\right)' : \ell(\boldsymbol{v}) = 0 \text{ for all } \boldsymbol{v} \in H_q^1(\Omega) \right\}.$$

On the other hand, by what we already proved, $\mathcal{S} \equiv 0$ and hence $F \equiv 0$, which completes the proof. □

III.4.2 Exterior Domains

Following the argument of Ladyzhenskaya & Solonnikov (1976), we can prove the following.

Theorem III.4.2 *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be an exterior domain such that, for some $\rho > \delta(\Omega^c)$, Ω_ρ satisfies the assumption of* Theorem III.4.1. *Then,*

$$\widehat{H}_q^1(\Omega) = H_q^1(\Omega), \;\; 1 < q < \infty\,.$$

Proof. We begin with the obvious observation that if Ω_ρ satisfies the assumption of Theorem III.4.1, then also Ω_R does, for all $R > \rho$. Now, let $\psi \in C^1(\mathbb{R})$ with $\psi(\xi) = 1$ if $|\xi| \leq 1$ and $\psi(\xi) = 0$ if $|\xi| \geq 2$ and set $\psi_R(x) = \psi(|x|/R)$, $R > \rho$. For $\boldsymbol{v} \in \widehat{H}_q^1(\Omega)$, denote by $\boldsymbol{w}^{(R)}$ a solution to the problem

$$\begin{aligned} \nabla \cdot \boldsymbol{w}^{(R)} &= -\boldsymbol{v} \cdot \nabla \psi_R \\ \boldsymbol{w}^{(R)} &\in W_0^{1,q}(\Omega_{R,2R}) \\ |\boldsymbol{w}^{(R)}|_{1,q,\Omega_{R,2R}} &\leq c\|\boldsymbol{v} \cdot \nabla \psi_R\|_{q,\Omega}. \end{aligned} \tag{III.4.2}$$

Since the compatibility condition

$$\int_{\Omega_{R,2R}} \boldsymbol{v} \cdot \nabla \psi_R = \int_{\Omega_{R,2R}} \nabla \cdot (\boldsymbol{v}\psi_R) = 0$$

is satisfied, such a field exists, by Theorem III.3.1. Moreover, by Lemma III.3.3 the constant c does *not* depend on R. Also, since $\nabla \psi_R = O(1/R)$ uniformly in x, using inequality (II.5.5) we deduce for some c_1, c_2 *independent* of R

$$\|\boldsymbol{w}^{(R)}\|_{q,\Omega_{R,2R}} \leq c_1 R |\boldsymbol{w}^{(R)}|_{q,\Omega_{R,2R}} \leq c_2 \|\boldsymbol{v}\|_{q,\Omega_{R,2R}}. \tag{III.4.3}$$

Setting $\boldsymbol{w}^{(R)} \equiv 0$ in the complement of $\Omega_{R,2R}$, we define

$$\boldsymbol{v}^{(R)} = \psi_R \boldsymbol{v} + \boldsymbol{w}^{(R)}.$$

Due to (III.4.2), $\boldsymbol{v}^{(R)} \in \widehat{H}_q^1(\Omega_{2R})$. Since Ω_{2R} satisfies the assumption of Theorem III.4.1, we have

$$\widehat{H}_q^1(\Omega_{2R}) = H_q^1(\Omega_{2R}), \quad \text{for all } R > \delta(\Omega^c),$$

and therefore, given $\varepsilon > 0$, we can find $\boldsymbol{v}^{\varepsilon,R} \in \mathcal{D}(\Omega_{2R}) \subset \mathcal{D}(\Omega)$ such that

$$\|\boldsymbol{v}^{(R)} - \boldsymbol{v}^{\varepsilon,R}\|_{1,q,\Omega_{2R}} < \varepsilon.$$

So,

$$\begin{aligned}\|\boldsymbol{v} - \boldsymbol{v}^{\varepsilon,R}\|_{q,\Omega} &\leq \|\boldsymbol{v}^{\varepsilon,R} - \boldsymbol{v}^{(R)}\|_{q,\Omega} + \|\boldsymbol{v} - \boldsymbol{v}^{(R)}\|_{q,\Omega} \\ &< \varepsilon + \|(1-\psi_R)\boldsymbol{v}\|_{q,\Omega} + \|\boldsymbol{w}^{(R)}\|_{q,\Omega_{R,2R}}.\end{aligned}$$

Because of (III.4.3) and the properties of ψ_R, by taking R sufficiently large and ε sufficiently small, we can make the right-hand side of this inequality as small as we please, thus proving

$$\|\boldsymbol{v}^{\varepsilon,R} - \boldsymbol{v}\|_{q,\Omega} \to 0 \ \text{ as } \varepsilon \to 0,\ R \to \infty.$$

By the some token one shows

$$|\boldsymbol{v}^{\varepsilon,R} - \boldsymbol{v}|_{1,q,\Omega} \to 0 \ \text{ as } \varepsilon \to 0,\ R \to \infty,$$

which completes the proof of the coincidence. □

III.4.3 Domains with Noncompact Boundary

It is easy to convince oneself that the method of proof just employed for the exterior case applies with no significant changes to show $\widehat{H}_q^1(\Omega) = H_q^1(\Omega)$, $1 < q < \infty$, for domains Ω with noncompact boundary, provided the following conditions are satisfied for all R greater than a fixed number R_0:

(i) $\Omega_{2R} = \{x \in \Omega : |x| < 2R\}$ and $\Omega_{R,2R} = \{x \in \Omega : R < |x| < 2R\}$ are domains;
(ii) Problem (III.4.2) is solvable with a constant c independent of R;
(iii) Inequality (III.4.3) holds with a constant c_2 independent of R;
(iv) $\widehat{H}_q^1(\Omega_{2R}) = H_q^1(\Omega_{2R})$;

(see Ladyzhenskaya & Solonnikov 1976, Theorem 4.1). In particular, conditions (i)-(iv) are certainly fulfilled if $\Omega = \mathbb{R}^n_+$. Thus, we have.

Theorem III.4.3 $\widehat{H}_q^1(\mathbb{R}^n_+) = H_q^1(\mathbb{R}^n_+)$, $1 < q < \infty$.

Our next task, throughout the rest of this section, will be to investigate the question of coincidence when Ω has "exits" to infinity, namely, when outside a connected, compact subset Ω_0 (say) Ω splits into m disjoint components Ω_i, which in possibly different coordinate systems (depending on i), can be represented as

$$\Omega_i = \{x \in \mathbb{R}^n : x_n > 0, x' \in \Sigma_i(x_n)\},$$

where $\Sigma_i(x_n)$ is a domain in $\mathbb{R}^{n-1}$ smoothly varying with x_n and $x' = (x_1, \ldots, x_{n-1})$. Familiar domains Ω having these properties are, for instance, infinitely long pipes and tubes of possibly varying cross section.

As we have noted at the beginning of this section, the coincidence of $\widehat{H}_q^1(\Omega)$ and $H_q^1(\Omega)$ can be tightly related to the way in which $\Sigma_i(x_n)$ "widens" as $x_n \to \infty$. It would thus be useful to establish a *characterization* of the class of cross sections for which the coincidence holds but, to the best of our knowledge, such a result is not available yet; nonetheless, one can certainly select certain classes of $\Sigma_i(x_n)$ and establish whether or not $\widehat{H}_q^1(\Omega) = H_q^1(\Omega)$. For example, if all Σ_i, $i = 1, \ldots, m$, are independent of x_n, i.e., each Ω_i reduces to a straight semi-infinite cylinder, then, by the same technique used for the case of an exterior domain, it is not hard to show that $\widehat{H}_q^1(\Omega) = H_q^1(\Omega)$, $1 < q < \infty$, provided Ω has a mild degree of regularity; see Exercise III.4.4. On the other hand, if some of the domains $\Sigma_i(x_n)$ become unbounded as $x_n \to \infty$ with a suitable growth rate, then, for the corresponding Ω we may have $\widehat{H}_q^1(\Omega) \neq H_q^1(\Omega)$.

An example of such domains was given for the first time by Heywood (1976), who proved noncoincidence of $\widehat{H}^1(\Omega)$ and $H^1(\Omega)$ when Ω is an "aperture domain," namely, a domain of $\mathbb{R}^n, n > 2$, of the type

$$\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$$

with Ω_0 a bounded subset of the plane $x_n = 0$ containing a unit disk C and, in the same coordinate system,

$$\Omega_1 = \mathbb{R}^n_-, \quad \Omega_2 = \mathbb{R}^n_+,$$

so that the two cross sections Σ_1, Σ_2 reduce to the entire space $\mathbb{R}^{n-1}$. Stated equivalently,

$$\Omega = \{x \in \mathbb{R}^n : x_n \neq 0 \text{ or } x' \in \Omega_0\}, \tag{III.4.4}$$

with $x' = (x_1, \ldots, x_{n-1})$. By using the ideas of Heywood, we now show the following.

Theorem III.4.4 *Let Ω be the "aperture" domain* (III.4.4). *Then*

$$\widehat{H}^1(\Omega) \neq H^1(\Omega).$$

Moreover,

$$d \equiv \dim\left(\widehat{H}^1(\Omega) \,/\, H^1(\Omega)\right) = 1.$$

Proof. Take the origin of coordinates at the center of C, denote by Γ_i, $i = 1, 2$, the cones $\{x \in \Omega_i : x' < |x_n|\}$ and set

$$\sigma_i = \Gamma_i \cap \{|x| = 1\}, \ i = 1, 2.$$

We then let

$$\boldsymbol{b}^*(x) = \frac{\omega(\theta)\boldsymbol{x}}{|x|^n}$$

with

$$\omega(\theta) = \begin{cases} (\cos 2\theta)^2 & \text{for } \theta \in [0, \pi/4] \\ 0 & \text{for } \theta \in [\pi/4, 3\pi/4] \\ -(\cos 2\theta)^2 & \text{for } \theta \in [3\pi/4, \pi], \end{cases}$$

where θ is the angle between the positive x_n axis and the ray joining the point x with the origin. Evidently,

$$\begin{gathered} \boldsymbol{b}^* \in L^1_{loc}(\mathbb{R}^n), \\ \nabla \cdot \boldsymbol{b}^* = 0, \quad \text{for all } x \in \Omega - \{0\} \\ \operatorname{supp}(\boldsymbol{b}^*) \subseteq \overline{\Gamma_1 \cup \Gamma_2}. \end{gathered}$$

Furthermore, by indicating with $S_{i,R}$ the surface $\Omega_i \cap \{|x| = R\}$, it follows that

$$\begin{aligned} \int_{S_{1,R}} \boldsymbol{b}^*(x) \cdot \boldsymbol{n} = \int_{\sigma_1} \omega(\theta) = -\int_{\sigma_2} \omega(\theta) & \\ = -\int_{S_{2,R}} \boldsymbol{b}^*(x) \cdot \boldsymbol{n} \equiv \phi = \text{const.} > 0. & \end{aligned} \tag{III.4.5}$$

Setting

$$\boldsymbol{b} = (\boldsymbol{b}^*)_\varepsilon, \;\; 0 < \varepsilon < 1/2,$$

the regularization of $\boldsymbol{b}^*$, by the properties of regularizations (see Section II.2) we readily deduce $\boldsymbol{b} \in C^\infty(\mathbb{R}^n)$ and that the following conditions hold for all $x \in \Omega$

$$\begin{aligned} \nabla \cdot \boldsymbol{b}(x) &= 0 \\ |\boldsymbol{b}(x)| &\le c\,|x|^{-n+1} \\ |\nabla \boldsymbol{b}(x)| &\le c\,|x|^{-n} \\ |\Delta \boldsymbol{b}(x)| &\le c\,|x|^{-n-1} \end{aligned} \tag{III.4.6}$$

Furthermore,

$$\operatorname{supp}(\boldsymbol{b}) \subset \left\{ |x'| < |x_n| + 1/\sqrt{2} \right\}. \tag{III.4.7}$$

Using $(\text{III.4.6})_{2,3}$, (III.4.7) along with a now standard "cut-off" argument, it follows that

$$\boldsymbol{b} \in W_0^{1,2}(\Omega)$$

and so, in virtue of $(\text{III.4.6})_1$, we may infer

$$\boldsymbol{b} \in \widehat{H}^1(\Omega).$$

However, by the solenoidality of $\boldsymbol{b}$ and (III.4.5), using the properties of regularizations we can select ε so small that

$$\int_{\Omega_0} \boldsymbol{b}\cdot\boldsymbol{n} \equiv \text{const.} > \phi/2 > 0. \tag{III.4.8}$$

Condition (III.4.8) then tells us that $\boldsymbol{b} \notin H^1(\Omega)$. In fact, since every element $\boldsymbol{u}$ from $H^1(\Omega)$ is approximated by functions from $\mathcal{D}(\Omega)$ it is immediately shown that

$$\int_{\Omega_0} \boldsymbol{u}\cdot\boldsymbol{n} = 0. \tag{III.4.9}$$

We may then conclude $\widehat{H}^1(\Omega) \neq H^1(\Omega)$. We shall now prove the second part of the theorem. For simplicity, we shall take the bounded domain Ω_0 just coincident with the unit circle C and begin to show that if $\boldsymbol{u} \in \widehat{H}^1(\Omega)$ satisfies (III.4.9) then $\boldsymbol{u} \in H^1(\Omega)$. Actually, since $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ with $\Omega_1 \equiv \mathbb{R}^n_-$ and $\Omega_2 \equiv \mathbb{R}^n_+$, we set

$$D_1 = \Omega_1 \cap B_1$$
$$D_2 = \Omega_2 \cap B_2$$

and denote by $\boldsymbol{v}_i$ a solenoidal vector field in D_i that equals $\boldsymbol{u}$ at Ω_0, vanishes at $\Omega_i \cap \partial B_1$ and belongs to $W^{1,2}(D_i)$. Since $\boldsymbol{u}$ satisfies (III.4.9), such a field exists in virtue of Exercise III.3.5. Furthermore, it is a simple task to show that, by extending $\boldsymbol{v}_i$ to zero outside D_i, the vector field $\boldsymbol{v}_1 + \boldsymbol{v}_2$ is solenoidal in B_1 and belongs to $W_0^{1,2}(B_1)$. Denoting by $\boldsymbol{u}_i$ the restriction of $\boldsymbol{u}$ to Ω_i, we may thus write

$$\boldsymbol{u} = (\boldsymbol{u}_1 - \boldsymbol{v}_1) + (\boldsymbol{u}_2 - \boldsymbol{v}_2) + (\boldsymbol{v}_1 + \boldsymbol{v}_1) \equiv \boldsymbol{w}_1 + \boldsymbol{w}_2 + \boldsymbol{w}_3.$$

Employing the results on the coincidence of the two spaces for the half-space and for a bounded domain, we deduce

$$\boldsymbol{w}_1 \in H^1(\mathbb{R}^n_-), \quad \boldsymbol{w}_2 \in H^1(\mathbb{R}^n_+), \quad \boldsymbol{w}_3 \in H^1(B_1),$$

and so, since each of these latter spaces is embedded in $H^1(\Omega)$ we conclude $\boldsymbol{u} \in H^1(\Omega)$. Take now the vector $\boldsymbol{b}$ constructed before and multiply it by a constant in such a way that, denoting this new vector by $\boldsymbol{b}$,

$$\int_{\Omega_0} \boldsymbol{b}\cdot\boldsymbol{n} = 1. \tag{III.4.10}$$

(This is allowed, in virtue of (III.4.8).) Take any $\boldsymbol{v} \in \widehat{H}^1(\Omega)$ and let Φ indicate its flux through Ω_0. Then, by (III.4.10),

$$\boldsymbol{v} - \Phi\boldsymbol{b} \equiv \boldsymbol{u}$$

satisfies (III.4.9) and so $\boldsymbol{u} \in H^1(\Omega)$, which proves $d = 1$.

$\square$

Exercise III.4.2 By means of the arguments described in the proof of the preceding theorem, show that for Ω given in (III.4.4):

$$\widehat{H}^1_q(\Omega) \neq H^1_q(\Omega), \quad \text{for all } q > n/(n-1),\, n \geq 2.$$

Moreover, for the above values of q, show $\dim(\widehat{H}^1_q(\Omega)\,/\,H^1_q(\Omega)) = 1\,.$

It is not difficult to convince oneself that Heywood's procedure described in the proof of Theorem III.4.4 should also work for a general class of domains with a finite number of exits to infinity, provided each exit contains a semi-infinite cone. This problem is taken up by Ladyzhenskaya & Solonnikov (1976, Theorem 4.2) who prove, in fact, that $\widehat{H}^1(\Omega) \neq H^1(\Omega)$, whenever $\Omega \subset \mathbb{R}^n$, $n \geq 3$, has $m \geq 2$ (disjoint) exits Ω_i to infinity and provided each Ω_i contains a semi-infinite cone. Furthermore,

$$\dim(\widehat{H}^1(\Omega)\,/\,H^1(\Omega)) = m - 1\,.$$

However, if $n = 2$, then $\widehat{H}^1(\Omega) = H^1(\Omega)$; see also Kapitanskiĭ (1981).

Remark III.4.1 In the light of the proof of Theorem III.4.4, it is easy to show that for the domain (III.4.4), $H'_q(\Omega) \neq H_q(\Omega)$, for all $q > n/(n-1)$, $n \geq 2$, where $H_q(\Omega)$ and $H'_q(\Omega)$ are defined in Section III.2 and Theorem III.2.3. Actually, in view of Lemma III.2.1, it is enough to prove the existence of $\boldsymbol{v} \in H'_q(\Omega)$ such that

$$\int_\Omega \boldsymbol{v} \cdot \nabla\phi = \kappa \neq 0, \quad \text{for some } \phi \in D^{1,q'}(\Omega). \tag{III.4.11}$$

For simplicity, we shall take $\Omega_0 = C$, with C unit disk of $\mathbb{R}^{n-1}$. Putting the origin of coordinates at the center of C and setting $r = |x|$, we choose

$$\phi(x) = \begin{cases} \exp(-r) & \text{if } r \geq 1 \text{ and } x_n > 0 \\ \exp(-1) & \text{if } r \leq 1 \\ 1 - (1 - \exp(-1))\exp(-r+1) & \text{if } r \geq 1 \text{ and } x_n < 0. \end{cases}$$

Clearly, $\phi \in D^{1,q'}(\Omega)$ for any $q' \geq 1$, and

$$\lim_{r\to\infty} \phi(x) = \begin{cases} 0 \text{ if } x_n > 0 \\ 1 \text{ if } x_n < 0. \end{cases} \tag{III.4.12}$$

Taking into account the properties of the field $\boldsymbol{b}$ introduced in the proof of Theorem III.4.4, it immediately follows that

$$\boldsymbol{b} \in H'_q(\Omega) \text{ for all } q > n/(n-1),\, n \geq 2.$$

Furthermore, by $(\text{III.4.6})_1$ and (III.4.7), we find for all $R > 0$

$$\int_{\Omega\cap B_R} \boldsymbol{b}\cdot\nabla\phi = \int_{\Sigma_R^+} \phi\boldsymbol{b}\cdot\boldsymbol{n} + \int_{\Sigma_R^-} \phi\boldsymbol{b}\cdot\boldsymbol{n} = \int_{\Sigma_R^+} \phi\boldsymbol{b}\cdot\boldsymbol{n} + \int_{\Sigma_R^-} (\phi-1)\boldsymbol{b}\cdot\boldsymbol{n} + \int_{\Omega_0} \boldsymbol{b}\cdot\boldsymbol{n}, \tag{III.4.13}$$

where

$$\Sigma_R^+ = \partial B_R \cap \mathbb{R}^n_+, \quad \Sigma_R^- = \partial B_R \cap \mathbb{R}^n_-.$$

Thus, letting $R \to \infty$ into (III.4.13) and recalling (III.4.12), $(\mathrm{III}.4.6)_2$ and (III.4.8) we recover (III.4.11) with $\boldsymbol{v} = \boldsymbol{b}$ and $\kappa = \int_{\Omega_0} \boldsymbol{b}\cdot\boldsymbol{n}$. ■

The aim of the remaining part of this subsection is to analyze all previous problems when the exits Ω_i are bodies of rotation, i.e., in possibly different coordinate systems,

$$\Omega_i = \{x \in \mathbb{R}^n : x_n > 0, |x'| < f_i(x_n)\}, \tag{III.4.14}$$

where f_i are suitable strictly positive functions. This class of domains is interesting in that, provided f_i satisfies a global Lipschitz condition and $\Omega_R \equiv \Omega \cap B_R$ has a mild degree of regularity for all R, we can completely *characterize* the class of f_i for which the coincidence of $\widehat{H}^1_q$ and H^1_q holds, $1 < q < \infty$, and, in the case where it doesn't, we can establish the dimension of the quotient space. The above study is essentially due to Solonnikov & Pileckas (1977), Pileckas (1983), and independently to Bogovskiĭ & Maslennikova (1978) and Maslennikova & Bogovskiĭ (1978, 1981a, 1981b, 1983). We begin to show the following preliminary result.

Lemma III.4.1 *Let*

$$\Omega = \bigcup_{i=1}^{m} \Omega_i$$

be a domain in $\mathbb{R}^n$, $n \geq 2$, where Ω_0 is a compact set and Ω_i, $i = 1, \dots, m$, are disjoint domains that (in possibly different coordinate systems) are of type (III.4.14) *with f_i satisfying the following conditions:*

(i) $f_i(t) \geq f_0 > 0$;
(ii) $|f_i(t_2) - f_i(t_1)| \leq M|t_2 - t_1|$, *for some constants f_0, M and for all t, t_1, $t_2 > 0$.*

Suppose, further, that

$$\Omega_R \equiv \Omega \cap B_R$$

satisfies the cone condition for all $R > \delta(\Omega_0)$ (the origin of coordinates is taken in Ω_0). Then $\widehat{H}^1_q(\Omega) = H^1_q(\Omega)$, $1 < q < \infty$, if and only if all vectors $\boldsymbol{v} \in \widehat{H}^1_q(\Omega)$ have zero flux through the planar cross sections $\Sigma_i = \Sigma_i(x_n)$ of Ω_i,[6] *perpendicular to the axis $x' = 0$ and passing through the point $(0, x_n)$, that is,*

$$\int_{\Sigma_i} \boldsymbol{v}\cdot\boldsymbol{n} = 0, \quad i = 1, \dots, m. \tag{III.4.15}$$

[6] Of course, the flux of $\boldsymbol{v}$ through $\Sigma_i(x_n)$ is independent of x_n.

Proof. For fixed i and all $R > 0$, put

$$\widehat{R}_i = R + f_i(R)/2M$$

and

$$\Omega_{R,\widehat{R}_i} = \Omega_i \cap \left\{ R < x_n < \widehat{R}_i \right\},$$

in the system of coordinates to which the exit Ω_i is referred. Let $\zeta \in C^1(\mathbb{R})$ be such that $\zeta(\xi) = 1$ for $\xi \leq 1$ and $\zeta(\xi) = 0$ for $\xi \geq 2$ and set

$$\zeta_i^R(x) = \zeta\left(2M\left[x_n + (f_i(R)/2M) - R\right]/f_i(R)\right).$$

Obviously, ζ_i^R is equal to one for $x_n \leq R$ and is zero for $x_n \geq \widehat{R}_i$ and, moreover,

$$|\nabla \zeta_i^R| \leq C/f_i(R), \tag{III.4.16}$$

for some C independent of R. Denote by $\boldsymbol{w}_i^R$ a solution to the problem

$$\begin{aligned} \nabla \cdot \boldsymbol{w}_i^R &= -\nabla \zeta_i^R \cdot \boldsymbol{v} \;\; \text{in } \Omega_{R,\widehat{R}_i} \\ \boldsymbol{w}_i^R &\in W_0^{1,q}(\Omega_{R,\widehat{R}_i}) \\ |\boldsymbol{w}_i^R|_{1,q,\Omega_{R,\widehat{R}_i}} &\leq C\|\nabla \zeta_i^R \cdot \boldsymbol{v}\|_{q,\Omega_{R,\widehat{R}_i}}. \end{aligned} \tag{III.4.17}$$

In view of Theorem III.3.1, (III.4.17) is solvable since, as a consequence of (III.4.15), the compatibility condition

$$\int_{\Omega_{R,\widehat{R}_i}} \nabla \zeta_i^R \cdot \boldsymbol{v} = 0$$

is satisfied. Moreover, the constant C in (III.4.17)_3 can be taken independent of R, see Exercise III.4.3. Extend $\boldsymbol{w}_i^R$ to zero outside $\Omega_{R,\widehat{R}_i}$ and set

$$\zeta^R \equiv \sum_{i=1}^m \zeta_i^R, \quad \boldsymbol{w}^R \equiv \sum_{i=1}^m \boldsymbol{w}_i^R, \quad \boldsymbol{v}_1 \equiv \zeta^R \boldsymbol{v} + \boldsymbol{w}^R.$$

We have that $\boldsymbol{v}_1$ is a solenoidal vector field belonging to $W_0^{1,q}(\Omega_{R'})$ for all sufficiently large R'. So, $\boldsymbol{v}_1 \in \widehat{H}_q^1(\Omega_{R'})$ and, by the result of this subsection (a) and the assumption made on $\Omega_{R'}$, $\boldsymbol{v}_1$ can be approximated by elements of $\mathcal{D}(\Omega_{R'}) \subset \mathcal{D}(\Omega)$. Therefore, given $\varepsilon > 0$, we can find $\boldsymbol{v}^{\varepsilon,R'} \in \mathcal{D}(\Omega)$ such that

$$\|\boldsymbol{v}_1 - \boldsymbol{v}^{\varepsilon,R'}\|_{1,q,\Omega} < \varepsilon.$$

We then have

$$\begin{aligned} \|\boldsymbol{v} - \boldsymbol{v}^{\varepsilon,R'}\|_{q,\Omega} &\leq \|\boldsymbol{v}_1 - \boldsymbol{v}^{\varepsilon,R'}\|_{q,\Omega} + \|\boldsymbol{v}_1 - \boldsymbol{v}\|_{q,\Omega} \\ &< \varepsilon + \|(1-\zeta^R)\boldsymbol{v}\|_{q,\Omega} + \sum_{i=1}^m \|\boldsymbol{w}_i^R\|_{q,\Omega_{R,\widehat{R}_i}}. \end{aligned} \tag{III.4.18}$$

From (III.4.16), (III.4.17)$_3$ and (II.5.5) we deduce for some constants c_1, c_2 *independent of* R, and for $i = 1, \ldots, m$,

$$\|\boldsymbol{w}_i^R\|_{q,\Omega_{R,\widehat{R}_i}} \leq c_1 f_i(R) |\boldsymbol{w}_i^R|_{1,q,\Omega_{R,\widehat{R}_i}} \leq c_2 \|\boldsymbol{v}\|_{q,\Omega_{R,\widehat{R}_i}},$$

which, once replaced into (III.4.18), shows that $\boldsymbol{v}^{\varepsilon,R'}$ approaches $\boldsymbol{v}$ in $L^q(\Omega)$ when $\varepsilon \to 0$ and $R' \to \infty$. Likewise, one shows $\nabla \boldsymbol{v}^{\varepsilon,R'} \to \nabla \boldsymbol{v}$ in $L^q(\Omega)$, thus proving the lemma. □

Remark III.4.2 If the domain Ω is itself a body of rotation, i.e.,

$$\Omega = \{x \in \mathbb{R}^n : x_n \in \mathbb{R}, |x'| < f(x_n)\}$$

with f satisfying assumptions (i) and (ii) of Lemma III.4.1, this lemma continues to hold for $q = 1$; see Maslennikova & Bogovskiĭ (1981b, Section 2). ■

Exercise III.4.3 Show that the constant C in (III.4.17)$_3$ can be taken independent of R. *Hint*: Make the change of variables:

$$y' = 2Mx'/f_i(R), \quad y_n = 2M(x_n - R)/f_i(R).$$

By hypothesis (ii) on f_i, the domain $\Omega_{R,\widehat{R}_i}$ goes into

$$\Omega^*_{R,\widehat{R}_i} = \left\{ y \in \mathbb{R}^n : |y'| \leq g_i^R(y_n) \equiv 2M \frac{f_i\left(R + (2M)^{-1} f_i(R) y_n\right)}{f_i(R)}, 0 < y_n < 1 \right\}$$

with

$$M \leq g_i^R(t) \leq 3M, \quad \text{for all } t > 0.$$

$\Omega^*_{R,\widehat{R}_i}$ is thus contained in a ball of *fixed* radius for every R. One then solves (III.4.17) in $\Omega^*_{R,\widehat{R}_i}$ with a constant *independent* of R and retransform the solution to $\Omega_{R,\widehat{R}_i}$, obtaining the desired result.

Let us now consider some consequences of Lemma III.4.1. First of all, if $m = 1$, i.e., Ω has only one exit to infinity, then $\widehat{H}^1_q(\Omega) = H^1_q(\Omega)$ for all $q \in (1, \infty)$. Actually, in this case we may take $\Omega_0 = \Omega \cap \mathbb{R}^n_-$ and so, denoting by $\{\boldsymbol{v}_k\} \subset C_0^\infty(\Omega)$ a sequence of functions approximating $\boldsymbol{v}$ in $W_0^{1,q}(\Omega)$ and by the intersection of Ω with the plane $x_n = 0$, it follows that

$$\int_{\Sigma_0} \boldsymbol{v}_k \cdot \boldsymbol{n} = \int_{\Omega_0} \nabla \cdot \boldsymbol{v}_k.$$

Since $\nabla \cdot \boldsymbol{v} = 0$, the right-hand side of this identity tends to zero as $k \to \infty$ and, by Theorem II.4.1, we deduce (III.4.15), which proves the coincidence.

In the following, we shall assume $m \geq 2$. On every cross section $\Sigma_i = \Sigma_i(x_n)$ we have

$$|\Phi_i|^q \equiv \left| \int_{\Sigma_i} v_n \right|^q \leq C f_i^{(n-1)(q-1)}(x_n) \int_{\Sigma_i} |\boldsymbol{v}|^q,$$

with Φ_i independent of x_n and so

$$|\Phi_i|^q\beta_i \equiv |\Phi_i|^q \int_0^\infty f_i^{(1-n)(q-1)}(x_n)dx_n \le C\|\boldsymbol{v}\|_q^q. \tag{III.4.19}$$

Therefore, if, for some fixed q and n, there are $m-1$ integrals β_i that diverge we conclude $\Phi_i = 0$ for all $i = 1,\dots,m$ and so, $\widehat{H}_q^1(\Omega) = H_q^1(\Omega)$. Actually, if $\beta_i = \infty$ for $i = 1,\dots,m-1$ (say) we have $\Phi_i = 0$ for the corresponding fluxes, but since $\boldsymbol{v}$ is solenoidal in the whole of Ω we also have $\Phi_m = 0$. The circumstance just described happens whenever $1 \le q \le n/(n-1)$. In fact we have the following result that represents, in the particular case of the domain considered by us, a much more general one due to Maslennikova & Bogovskiĭ (1983, Theorem 5) and that we recalled just before subsection (a).

Theorem III.4.5 *Let Ω be as in* Lemma III.4.1 *with $m \ge 2$. Then $\widehat{H}_q^1(\Omega) = H_q^1(\Omega)$ for all $q \in [1, n/(n-1)]$.*

Proof. By assumption (ii) made on $f_i(x_n)$, for x_n sufficiently large ($> x_0$, say) it holds that

$$|f_i(x_n)| \le Cx_n,$$

and so

$$\beta_i \ge C\int_{x_0}^\infty x_n^{(1-n)(q-1)}(x_n)dx_n = \infty, \quad \text{for all } q \in [1, n/(n-1)],$$

and the proof is achieved. □

Suppose now that there are at least two of the quantities β_i (β_1 and β_2, say) that are finite. The fluxes Φ_1 and Φ_2, then, need not be zero and so the spaces $\widehat{H}_q^1(\Omega)$ and $H_q^1(\Omega)$ may not coincide. As a matter of fact, they *are* distinct and $\dim[\widehat{H}_q^1(\Omega)/H_q^1(\Omega)]$ is just the number of β_i, which are finite minus one. This result is due to Pileckas (1983, Theorem 7) for space dimension $n = 2, 3$. Here, coupling the ideas of Maslennikova and Bogovskiĭ (1978) and those of Ladyzhenskaya & Solonnikov (1976) we extend it to arbitrary dimension $n \ge 2$. We commence by introducing an auxiliary function.

Lemma III.4.2 *Let f be a function satisfying properties* (i) *and* (ii) *of* Lemma III.4.1. *Then, given $\sigma < 1/2$, there exists a function $\delta \in C^\infty(\mathbb{R}_+)$ such that, for all $t > 0$ and $m \ge 0$ we have*

(i) $\sigma f(t) \le \delta(t) \le f(t)$,
(ii) $|d^m\delta(t)/dt^m| \le c/f^{m-1}(t)$,

where $c = (m, M)$.

Proof. Let $\{t_k\}$ be a sequence of numbers such that

$$t_{k+1} = t_k + \alpha f(t_k), \quad t_0 = 0,$$

with α a positive parameter to be fixed later. Let $\varphi(\xi)$ denote a C^∞ function on $\mathbb{R}$ with $0 \le \varphi \le 1$ and

$$\varphi(\xi) = \begin{cases} 1 \text{ if } & |\xi| \le 1/2 \\ 0 \text{ if } & |\xi| > \varepsilon + 1/2 \end{cases}$$

for some positive ε. Setting

$$\varphi_k(t) = \varphi((t - c_k)/\ell_k)$$
$$\varepsilon = \eta/\ell_k, \ \eta > 0,$$

with

$$c_k = (t_{k+1} + t_k)/2$$
$$\ell_k = t_{k+1} - t_k$$

and $\eta > 0$ (to be fixed later in the proof) we verify at once that

$$\varphi_k(t) = \begin{cases} 0 \text{ if } & t < t_k - \eta \\ 1 \text{ if } & t_k < t < t_{k+1} \\ 0 \text{ if } & t > t_k + \eta \end{cases} \tag{III.4.20}$$

and

$$|d^m \varphi_k(t)/dt^m| \le c/f^m(t_k), \tag{III.4.21}$$

where $c = c(\alpha, m)$. We now choose

$$\eta < \tfrac{1}{2} \min_k \ell_k,$$

so that

$$t_k + \eta < t_{k+1} - \eta,$$

and set

$$\widetilde{\delta}(t) = \sum_{k=0}^{\infty} f(t_k)\varphi_k(t), \tag{III.4.22}$$

with $\varphi_0(t) \equiv 0$. Given $t \ge 0$, we have three possibilities:

(a) $t \in [t_k - \eta, t_k)$
(b) $t \in [t_k, t_k + \eta)$
(c) $t \in [t_k + \eta, t_{k+1} - \eta)$,

for some $k \ge 0$. Correspondingly, (III.4.20) and (III.4.21) furnish

$$\begin{aligned} \widetilde{\delta}(t) &= f(t_k) + \varphi_{k+1}(t)f(t_{k+1}) && \text{in case (a)} \\ \widetilde{\delta}(t) &= f(t_k)\varphi_k(t) + f(t_{k+1}) && \text{in case (b)} \\ \widetilde{\delta}(t) &= f(t_{k+1}) && \text{in case (c).} \end{aligned} \tag{III.4.23}$$

By the properties of the function f, we obtain in case (a)

$$f(t) \le f(t_k) + M\eta, \quad f(t_k) \le f(t) + M\eta \tag{III.4.24}$$

and

$$f(t_{k+1}) \le f(t) + M|t_{k+1} - t_k| + M\eta = f(t) + \alpha M f(t_k) + M\eta$$

and so, by (III.4.24),

$$f(t_{k+1}) \le (1 + \alpha M) f(t) + (\alpha M^2 + M)\eta. \tag{III.4.25}$$

On the other hand,

$$f(t) \le f(t_{k+1}) + \alpha M f(t_k) + M\eta. \tag{III.4.26}$$

Because

$$f(t_k) \le f(t_{k+1}) + \alpha M f(t_k),$$

by choosing

$$\alpha = \gamma/M, \ \ \gamma < 1,$$

it follows that

$$f(t_k) \le f(t_{k+1})/(1-\gamma)$$

and (III.4.26) yields

$$f(t) \le f(t_{k+1})/(1+\gamma) + M\eta. \tag{III.4.27}$$

Selecting $\eta = \beta f_0/M$, from (III.4.24), (III.4.25), and (III.4.27) we derive

$$\left.\begin{aligned} \frac{f(t)}{1+\beta} \le f(t_k) \le (1+\beta) f(t) \\ \frac{(1-\gamma) f(t)}{1+\beta(1-\gamma)} \le f(t_{k+1}) \le (1+\gamma)(1+\beta) f(t) \end{aligned}\right\} \ (\textit{in case } (\mathrm{a})). \tag{III.4.28}$$

Therefore,

$$\frac{f(t)}{1+\beta} \le \widetilde{\delta}(t) \le (1+\beta)(2+\gamma) f(t) \quad (\textit{in case } (\mathrm{a})). \tag{III.4.29}$$

Likewise, one shows

$$\left.\begin{aligned} \frac{f(t)}{1+\beta} \le f(t_k) \le (1+\beta) f(t) \\ (1-\gamma) f(t) \le f(t_{k+1}) \le [1+\gamma(1+\beta)] f(t) \end{aligned}\right\} \ (\textit{in case } (\mathrm{b})) \tag{III.4.30}$$

and so

$$(1-\gamma)f(t) \le \widetilde{\delta}(t) \le [2+\beta+\gamma(1+\beta)]f(t) \quad (in\ case\ \text{(b)}). \tag{III.4.31}$$

Finally, by (III.4.26) and by the properties of f we easily deduce

$$(1-\gamma)f(t) \le f(t_{k+1}) \le (1-\gamma)f(t) \quad (in\ case\ \text{(c)}), \tag{III.4.32}$$

which gives

$$(1-\gamma)f(t) \le \widetilde{\delta}(t) \le (1-\gamma)f(t) \quad (in\ case\ \text{(c)}). \tag{III.4.33}$$

Collecting (III.4.23), (III.4.29), (III.4.31), and (III.4.33) we find

$$\sigma_1 f(t) \le \widetilde{\delta}(t) \le \sigma_2 f(t) \quad \text{for all } t \ge 0 \tag{III.4.34}$$

with

$$\sigma_1 = \min\{1/(1+\beta), 1-\gamma\}, \quad \sigma_2 = (1+\beta)(2+\gamma).$$

Furthermore, from (III.4.21), (III.4.23), (III.4.28), (III.4.30), and (III.4.32) there follows

$$|d^m\widetilde{\delta}(t)/dt^m| \le \frac{c}{f^{m-1}(t)}$$

with $c = c(m, M)$. Therefore, noting that $\sigma \equiv \sigma_1/\sigma_2$ ($< 1/2$) can be chosen as close to $1/2$ as we please by taking β and γ sufficiently close to zero, we may conclude that the function $\delta(t) = \widetilde{\delta}(t)/\sigma_2$ satisfies all requirements of the lemma, which is therefore completely proved. □

For $\beta \in (0,1)$, set

$$\omega_i(\beta) = \{x \in \Omega_i : \beta f_i(x_n) < |x'| < f_i(x_n)\}.$$

We have

Lemma III.4.3 *Let Ω_i be a body of rotation of type* (III.4.14). *Then there exists a vector $\boldsymbol{b}^i \in C^\infty(\Omega_i)$ such that for all $x \in \Omega_i$ and all $|\alpha| \ge 0$,*

$$\begin{aligned} |D^\alpha \boldsymbol{b}^i(x)| &\le c f_i^{-n+|\alpha|+1}(x_n) \\ \nabla \cdot \boldsymbol{b}^i(x) &= 0 \\ \int_{\Sigma_i} \boldsymbol{b}^i \cdot \boldsymbol{n} &= 1, \end{aligned} \tag{III.4.35}$$

where $\boldsymbol{n}$ is the unit normal to Σ_i in the direction of increase of the coordinate x_n and $c = c(n, \alpha, M)$. Moreover, $\boldsymbol{b}^i$ vanishes in $\omega_i(\beta)$, for any fixed $\beta \in (0,1)$.

Proof. Let $\psi \in C^\infty(\mathbb{R})$, $0 \le \psi \le 1$, with $\psi(t) = 1$ if $t \le 1$ and $\psi(t) = 0$ if $t \ge 2$, and set

$$\Psi(x) = \psi\left(\frac{4}{3}\left[\frac{|x'|^2}{\beta^2\delta_i^2(x_n)} + \frac{1}{2}\right]\right),$$

where δ_i is the function associated to f_i by the preceding lemma. We immediately recognize that

$$\Psi(x) = \begin{cases} 1 \text{ if } & |x'| \leq \frac{1}{2}\beta\delta_i(x_n) \\ 0 \text{ if } & \beta\delta_i(x_n) \leq |x'| \end{cases}$$

and therefore, by property (i) of the function δ_i, Ψ identically vanishes in $\omega_i(\beta)$. The field $\boldsymbol{b}^i$ is then defined as

$$\begin{aligned} b_k^i(x) &= \frac{x_k \delta_i'(x_n)\Psi(x)}{\delta_i^n(x_n)}, \quad k = 1, \dots, n-1 \\ b_n^i(x) &= \frac{\Psi(x)}{\delta_i^{n-1}(x_n)}, \end{aligned} \tag{III.4.36}$$

where $'$ indicates differentiation with respect to x_n. It is easy to check that $\boldsymbol{b}^i$ satisfies (III.4.35)$_2$ and vanishes in $\omega_i(\beta)$ for any fixed $\beta \in (0,1)$. Moreover, differentiating (III.4.36) and taking into account Lemma III.4.2, we also deduce (III.4.35)$_1$. Finally, setting

$$\xi_k = x_k/\delta_i(x_n), \quad k = 1, \dots, n-1,$$

we obtain

$$\int_{\Sigma_i(x_n)} \boldsymbol{b}^i(x) \cdot \boldsymbol{n} = 2\omega_{n-1} \int_{\Sigma_i(x_n)} \Psi(\xi)|\xi|^{n-2} d|\xi| > 0$$

with ω_{n-1} the measure of the $(n-1)$-dimensional unit ball. Thus, (III.4.35)$_3$ also follows, after possible multiplication of $\boldsymbol{b}^i$ by a suitable constant. The lemma is proved. □

We are now in a position to prove the following results complementing those of Theorem III.4.5.

Theorem III.4.6 *Let Ω be as in* Lemma III.4.1, $m \geq 2$. *Assume that the integrals*

$$\int_0^\infty f_i^{(1-n)(q-1)}(t)dt \tag{III.4.37}$$

converge for $i = 1, \dots, \ell$ ($\leq m$), and diverge for $i = \ell+1, \dots, m$. Then, setting

$$K_q^1(\Omega) = \widehat{H}_q^1(\Omega) \,/\, H_q^1(\Omega),$$

for $q \in (n/(n-1), \infty)$ it holds that

$$\dim K_q^1(\Omega) = \ell - 1.$$

In addition, if $\ell \geq 2$, every $\boldsymbol{v} \in \widehat{H}_q^1(\Omega)$ can be uniquely represented as

$$\boldsymbol{v} = \boldsymbol{u} + \sum_{i=1}^{\ell-1} \alpha_i \boldsymbol{d}^i,$$

where $\boldsymbol{u} \in H_q^1(\Omega)$. *Here,*

$$\alpha_i = \sum_{j=1}^{i} \Phi_j,$$

and Φ_j, $j = 1, \ldots, \ell$, *with* $\sum_{j=1}^{\ell} \Phi_j = 0$, *are given by*

$$\Phi_j = \int_{\Sigma_j} \boldsymbol{v} \cdot \boldsymbol{n},$$

with $\boldsymbol{n}$ *normal to* Σ_j *in the direction of increase of the coordinate* x_n*in* Ω_j. *Moreover,* $\{\boldsymbol{d}^i\}$ *is a basis in* $K_q^1(\Omega)$ *constituted by* $C^\infty(\Omega)$ *vector fields that vanish in a neighborhood of* $\partial\Omega$, *in* $\omega_i(\beta)$, $i = 1, \ldots, \ell$, *for all sufficiently small* β *and in* Ω_i, $i = \ell + 1, \ldots, m$, *for sufficiently large* $|x|$. *Finally, for every* $j = 1, \ldots, \ell$ *and* $i = 1, \ldots, \ell - 1$, *the vectors* $\boldsymbol{d}^i$ *satisfy an estimate of type* (III.4.35)$_1$ *along with the relations*

$$\int_{\Sigma_j} \boldsymbol{d}^i \cdot \boldsymbol{n} = \delta_{ij} - \delta_{i+1\,j}.$$

Proof. For $i = 1, \ldots, \ell$, let S_i be the intersection of Ω_i with the plane $x_n = 1$ (in the coordinate system to which Ω_i is referred) and consider the open ball $B_d(x_0^i)$, with x_0^i intersection of S_i with the axis $x' = 0$. We take

$$d < \min\left\{\tfrac{1}{2} \inf_{t>0} f_i(t), 1\right\}$$

so that $\overline{B}_d(x_0^i)$ is contained in Ω_i, while we may assume $B_d(x_0^i)$ strictly containing $S_i \cap \operatorname{supp}(\boldsymbol{b}^i) \equiv \sigma_i$, where $\boldsymbol{b}^i$ are the vectors constructed in Lemma III.4.2. This condition is easily achieved by selecting β in such a way that

$$\beta f_i(2) < (1/2)d,$$

which, in particular, imply

$$\sigma_i \subset B_{d/2}(x_0^i).$$

Let $\psi_i(x)$ be an infinitely differentiable function that is one in $B_{3d/4}(x_0^i)$ and is zero outside $B_d(x_0^i)$. Evidently, $\psi_i(x)$ is one near σ_i. Setting

$$K_i = \{x \in \Omega_i : 0 < x_n < 1\}$$

and

$$\boldsymbol{u}^i(x) = \begin{cases} \boldsymbol{b}^i(x) & \text{if } x \in \Omega_i - K_i \equiv C_i \\ \psi_i(x)\boldsymbol{b}^i(x) & \text{if } x \in \Omega - C_i \end{cases} \tag{III.4.38}$$

we see that $\boldsymbol{u}^i \in C^\infty(\Omega)$ and that $\operatorname{supp}(\boldsymbol{u}^i) \subset \Omega_i$. Moreover, by the assumption made on the integrals (III.4.37) and by $(\text{III.4.35})_1$ we have $\boldsymbol{u}^i \in W^{1,q}(\Omega)$. For $i = 1, \ldots, \ell - 1$ define

$$\boldsymbol{d}^i = \boldsymbol{u}^i - \boldsymbol{u}^{i+1} + \boldsymbol{r}_{i,i+1}, \tag{III.4.39}$$

where $\boldsymbol{r}_{i,i+1}$ satisfies

$$\nabla \cdot \boldsymbol{r}_{i,i+1} = -\nabla\psi_i \cdot \boldsymbol{u}_i + \nabla\psi_{i+1} \cdot \boldsymbol{u}^{i+1} \quad \text{in } \Gamma \equiv \cup_{i=1}^{\ell} \Gamma_i. \tag{III.4.40}$$

Since Γ is locally Lipschitz and, moreover, by $(\text{III.4.35})_3$

$$\int_\Gamma (-\nabla\psi_i \cdot \boldsymbol{u}_i + \nabla\psi_{i+1} \cdot \boldsymbol{u}^{i+1}) = \int_{\sigma_i} \boldsymbol{b}^i \cdot \boldsymbol{n} - \int_{\sigma_{i+1}} \boldsymbol{b}^{i+1} \cdot \boldsymbol{n} = 0$$

we may use Theorem III.3.3 to establish the existence of a solution $\boldsymbol{r}_{i,i+1}$ to (III.4.40). Actually, since the right-hand side of (III.4.40) belongs to $C_0^\infty(\Gamma)$, we may take $\boldsymbol{r}_{i,i+1} \in C_0^\infty(\Gamma)$. Therefore, the fields (III.4.39) are solenoidal, belong to $W^{1,q}(\Omega) \cap C^\infty(\overline{\Omega})$, vanish in a neighborhood of $\partial\Omega$, and coincide in $\Omega_i \cap \{x_n \geq 1\}$, $i = 1, \ldots, \ell$, with the fields $\boldsymbol{b}^i$. Furthermore, for every $j = 1, \ldots, \ell$

$$\int_{\Sigma_j} \boldsymbol{d}^i \cdot \boldsymbol{n} = \int_{\Sigma_j} (\boldsymbol{b}^i - \boldsymbol{b}^{i+1}) \cdot \boldsymbol{n} = \delta_{ij} - \delta_{i+1j},$$

and Lemma III.4.1 implies, in particular,

$$\boldsymbol{d}^i \in \widehat{H}_q^1(\Omega) \quad \text{and} \quad \boldsymbol{d}^i \notin H_q^1(\Omega),$$

so that

$$\widehat{H}_q^1(\Omega) \neq H_q^1(\Omega).$$

Given now $\boldsymbol{v} \in \widehat{H}_q^1(\Omega)$, consider the vector

$$\boldsymbol{u} = \boldsymbol{v} - \sum_{i=1}^{\ell-1} \alpha^i \boldsymbol{d}^i,$$

where

$$\alpha_i = \sum_{j=1}^{i} \Phi_j.$$

Recalling that $\boldsymbol{n}$ denotes the normal to Σ_j in the direction of the increase of the coordinate x_n in Ω_j, we find for all $k = 1, \ldots, \ell$

$$\int_{\Sigma_k} \boldsymbol{u} \cdot \boldsymbol{n} = \Phi_k - \sum_{i=1}^{\ell-1} \alpha^i (\delta_{ik} - \delta_{i+1k}) = \Phi_k - \sum_{j=1}^{k} \Phi_j + \sum_{j=1}^{k-1} \Phi_j = 0.$$

Since the flux of $\boldsymbol{u}$ through the remaining exits Ω_i, $i = \ell+1, \ldots, m$, is zero by assumption and so, from Lemma III.4.1 we deduce $\boldsymbol{u} \in H_q^1(\Omega)$. Moreover, the vectors $\{\boldsymbol{d}^i\}$ are linearly independent, which completes the proof of the theorem. □

Remark III.4.3 We wish to point out a noteworthy generalization of Theorem III.4.6. In fact, it is not necessary to suppose the "exits" Ω_i to be bodies of rotation but, rather, we can assume the following conditions (Pileckas 1983, p. 153; Solonnikov 1981):

(a) $D_i^1 \subset \Omega_i \subset D_i^2$, where, in possibly different coordinate systems:

$$D_i^1 = \{x \in \mathbb{R}^n : x_n > 0, |x'| < f_i(x_n)\}$$

$$D_i^2 = \{x \in \mathbb{R}^n : x_n > 0, |x'| < a_i f_i(x_n), a_i > 1\};$$

(b) In the domains:

$$\Omega_{R,\widehat{R}_i} = \Omega_i \cap \{R < x_n < R + f_i(R)\}$$

problem (III.4.17) is solvable with a constant c independent of R.

■

Remark III.4.4 Two further significant contributions to the problem of the coincidence of the spaces $\widehat{H}_q^1(\Omega)$ and $H_q^1(\Omega)$ are due to Solonnikov (1983) and to Maslennikova & Bogovskiĭ (1983). In the first one (see Theorems 2.5 and 2.6) results are given avoiding assumptions on the shape of the "exits" and imposing only some general restrictions. In the second one, the authors provide necessary conditions and sufficient conditions for the above coincidence to hold, in a class of domains with noncompact boundary which are *only* requested to be strongly locally Lipschitz (see footnote 3 in this section). ■

Exercise III.4.4 Let Ω be a domain of the type considered in Exercise III.4.1. Assume $\Sigma_i(x_n) = \Sigma_{0i}$, $i = 1, \ldots, N$, where each section Σ_{0i} is a locally Lipschitz simply connected domain in $\mathbb{R}^{n-1}$ independent of x_n and $\widehat{H}_q^1(\Omega_R) = H_q^1(\Omega_R)$, $1 < q < \infty$, for all sufficiently large R. Show $\widehat{H}_q^1(\Omega) = H_q^1(\Omega)$, $1 < q < \infty$.

Exercise III.4.5 (Heywood 1976) Show that for any domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, we have $\widehat{H}^1(\Omega) = H^1(\Omega)$ if and only if the only vector $\boldsymbol{v} \in \widehat{H}^1(\Omega)$ such that

$$\int_\Omega (\nabla \boldsymbol{v} : \nabla \boldsymbol{\varphi} + \boldsymbol{v} \cdot \boldsymbol{\varphi}) = 0 \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega)$$

is the null vector.

III.5 The Spaces $\mathcal{D}_0^{1,q}$

In this section we shall investigate the relevant properties of certain function spaces which, among other things, play a fundamental role in the study of steady motions of a viscous incompressible fluid in unbounded domains. These spaces, denoted by $\mathcal{D}_0^{1,q}(\Omega)$, are subspaces of $D_0^{1,q}(\Omega)$ defined as the completion of functions from $\mathcal{D}(\Omega)$ in the norm of $D_0^{1,q}(\Omega)$. If Ω is contained in some finite layer, then, by inequality (II.5.1) we have $\mathcal{D}_0^{1,q}(\Omega) = H_q^1(\Omega)$, otherwise $\mathcal{D}_0^{1,q}(\Omega) \subset H_q^1(\Omega)$.

As in the case of spaces $H_q^1(\Omega)$, it is of great interest to relate $\mathcal{D}_0^{1,q}(\Omega)$ with the space

$$\widehat{\mathcal{D}}_0^{1,q}(\Omega) = \left\{ \boldsymbol{v} \in D_0^{1,q}(\Omega) : \nabla \cdot \boldsymbol{v} = 0 \text{ in } \Omega \right\}$$

and to point out domains Ω for which $\widehat{\mathcal{D}}_0^{1,q}(\Omega) \neq \mathcal{D}_0^{1,q}(\Omega)$, for some $q = q(n)$ (in general, $\mathcal{D}_0^{1,q}(\Omega) \subset \widehat{\mathcal{D}}_0^{1,q}(\Omega)$, for any domain Ω). This is because, as shown for the first time by Heywood (1976), whenever the coincidence does not hold, the "traditional" boundary-value problem for Stokes and Navier–Stokes equations must be supplemented with appropriate extra conditions in order to take into account physically important solutions that would otherwise be excluded. Since for a bounded domain we have

$$\mathcal{D}_0^{1,q}(\Omega) = H_q^1(\Omega), \quad \widehat{\mathcal{D}}_0^{1,q}(\Omega) = \widehat{H}_q^1(\Omega)$$

it follows that, in such a case, that result proved in Theorem III.4.1 also holds for $\mathcal{D}_0^{1,q}$-spaces. Using this fact, we can then repeat verbatim the proofs of Theorem III.4.1 and Theorem III.4.3 and show that they continue to be true also for $\mathcal{D}_0^{1,q}$-spaces. We thus have the following theorem.

Theorem III.5.1 *If Ω is a bounded domain satisfying the assumption of* Theorem III.4.1, *then, for all $q \in [1, \infty)$,*

$$\mathcal{D}_0^{1,q}(\Omega) = \widehat{\mathcal{D}}_0^{1,q}(\Omega)\,. \tag{III.5.1}$$

If Ω is an exterior domain satisfying the assumption of Theorem III.4.2, *then* (III.5.1) *holds for all $q \in (1, \infty)$. Finally, $\mathcal{D}_0^{1,q}(\mathbb{R}_+^n) = \widehat{\mathcal{D}}_0^{1,q}(\mathbb{R}_+^n)$, for all $q \in (1, \infty)$.*

Assume now Ω with a noncompact boundary and having m "exits" Ω_i to infinity. Then, one can show results similar to those established in Subsection 4(c) for spaces $H_q^1(\Omega)$, even though different in some details. Precisely, one shows that if each Ω_i contains a semi-infinite cone, then $\widehat{\mathcal{D}}_0^{1,q}(\Omega) \neq \mathcal{D}_0^{1,q}(\Omega)$ for all $q > 1$. Moreover, if Ω enjoys some further properties, then $\dim \left(\widehat{\mathcal{D}}_0^{1,q}(\Omega) \;/\; \mathcal{D}_0^{1,q}(\Omega) \right) = m - 1$ (Ladyzhenskaya & Solonnikov 1976, Theorem 4.2).[1]

[1] Actually, the proof given by these authors is for $q = 2$. Nevertheless, *mutatis mutandis* it can be easily extended to all $q > 1$.

In the special case when the domains Ω_i are bodies of rotation defined by suitable functions f_i, one can give a complete description of the coincidence of the two spaces in a way similar to that used in Subsection 4(c) for the spaces $H_q^1(\Omega)$. In particular, the proof of Lemma III.4.1 remains unchanged to show the following result.

Lemma III.5.1 *Let Ω be as in* Lemma III.4.1. *Then $\widehat{\mathcal{D}}_0^{1,q}(\Omega) = \mathcal{D}_0^{1,q}(\Omega)$, $1 < q < \infty$, if and only if all vectors $\boldsymbol{v} \in \widehat{\mathcal{D}}_0^{1,q}(\Omega)$ satisfy* (III.4.15).

However, there is a modification in the characterization furnished by Theorem III.4.6, in that the condition imposed on the functions f_i must be appropriately changed. This is due to the fact that now $\boldsymbol{v}$ has only first derivatives in L^q. Thus, using the notations of Subsection 4(c), with the aid of the Hölder inequality and inequality (II.5.5), the flux Φ_i can be increased as follows:

$$
\begin{aligned}
|\Phi_i|^q \equiv \left| \int_{\Sigma_i(x_n)} \boldsymbol{v} \cdot \boldsymbol{n} \right| &\leq C\, |\Sigma_i|^{q-1+q/(n-1)} \int_{\Sigma_i(x_n)} |\nabla \boldsymbol{v}|^q \\
&\leq C_1 f_i^{(n-1)(q-1)+q} \int_{\Sigma_i(x_n)} |\nabla \boldsymbol{v}|^q
\end{aligned}
$$

and, therefore, we find

$$
|\Phi_i|^q \int_0^\infty f_i^{(1-n)(q-1)-q}(x_n)dx_n \leq C_1 |\boldsymbol{v}|_{1,q}^q .
$$

So the vanishing of the fluxes Φ_i is this time related to the finiteness of the integrals

$$
\int_0^\infty f_i^{(1-n)(q-1)-q}(t)dt
$$

instead of integrals (III.4.37). Consequently, Theorem III.4.6 is replaced by the following one whose proof, which follows exactly the same lines as those of Theorem III.4.6, we leave to the reader as an exercise.

Theorem III.5.2 *Let Ω be as in* Lemma III.4.1, *$m \geq 2$. Assume that the integrals*

$$
\int_0^\infty f_i^{(1-n)(q-1)-q}(t)dt \tag{III.5.2}
$$

converge for $i = 1, \ldots, \ell$ ($\leq m$) and diverge for $i = \ell + 1, \ldots, m$. Then, setting

$$
F_0^{1,q}(\Omega) = \widehat{\mathcal{D}}_0^{1,q}(\Omega) \,/\, \mathcal{D}_0^{1,q}(\Omega),
$$

for $q \in (1, \infty)$,

$$
\dim F_0^{1,q}(\Omega) = \ell - 1.
$$

In addition, every $\boldsymbol{v} \in \widehat{\mathcal{D}}_0^{1,q}(\Omega)$ can be uniquely represented as

$$\boldsymbol{v} = \boldsymbol{u} + \sum_{i=1}^{\ell-1} \alpha_i \boldsymbol{d}^i,$$

where $\boldsymbol{u} \in \mathcal{D}_0^{1,q}(\Omega)$*. Here,*

$$\alpha_i = \sum_{j=1}^{i} \Phi_j,$$

and Φ_j, $j = 1, \dots, \ell$, *with* $\sum_{j=1}^{\ell} \Phi_j = 0$, *are given by*

$$\Phi_j = \int_{\Sigma_j} \boldsymbol{v} \cdot \boldsymbol{n},$$

with $\boldsymbol{n}$ *normal to* Σ_j *in the direction of increase of the coordinate* x_n *in* Ω_j*. Moreover,* $\{\boldsymbol{d}^i\}$ *is a basis in* $F_0^{1,q}(\Omega)$ *constituted by* $C^\infty(\Omega)$ *vector fields that vanish in a neighborhood of* $\partial\Omega$*, in* $\omega_i(\beta)$, $i = 1, \dots, \ell$*, for all sufficiently small* β *and in* Ω_i, $i = \ell + 1, \dots, m$*, for sufficiently large* $|x|$*. Finally, for every* $j = 1, \dots, \ell$ *and* $i = 1, \dots, \ell - 1$*, the vectors* $\boldsymbol{d}^i$ *satisfy an estimate of the type* (III.4.35)$_1$ *along with the conditions*

$$\int_{\Sigma_j} \boldsymbol{d}^i \cdot \boldsymbol{n} = \delta_{ij} - \delta_{i+1\,j}.$$

Remark III.5.1 Theorem III.5.3 holds with the restriction from below $q > 1$ instead of $q > n/(n-1)$ as required in Theorem III.4.6. This is because, unlike integrals (III.4.37), integrals (III.5.2) certainly diverge only if $q = 1$; see Theorem III.4.5. ■

Remark III.5.2 The results of Lemma III.5.1 and Theorem III.5.2 continue to hold in the more general case when the domains Ω_i are not necessarily bodies of rotation but only satisfy the requirements listed in Remark III.4.3; see Pileckas (1983, §3). ■

Exercise III.5.1 Let Ω be as in Exercise III.4.4. Assume $\widehat{\mathcal{D}}_0^{1,q}(\Omega_R) = \mathcal{D}_0^{1,q}(\Omega_R)$ for all sufficiently large R. Show $\widehat{\mathcal{D}}_0^{1,q}(\Omega) = \mathcal{D}_0^{1,q}(\Omega)$.

Exercise III.5.2 Let Ω be the "aperture domain" (III.4.4). Show $\widehat{\mathcal{D}}_0^{1,2}(\Omega) \neq \mathcal{D}_0^{1,2}(\Omega)$, for all $n \geq 2$. (Notice that, unlike spaces $\widehat{H}^1(\Omega)$ and $H^1(\Omega)$, the case $n = 2$ *is* included.)

Exercise III.5.3 (Heywood 1976) Show that for any domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, we have $\widehat{\mathcal{D}}_0^{1,2}(\Omega) = \mathcal{D}_0^{1,2}(\Omega)$ if and only if the only vector $\boldsymbol{v} \in \widehat{\mathcal{D}}_0^1(\Omega)$ such that

$$\int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{\varphi} = 0, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega)$$

is the null vector.

Our last objective in this section is to give a representation of functionals on $D_0^{1,q}(\Omega)$ that identically vanish on $\widehat{\mathcal{D}}_0^{1,q}(\Omega)$. As we shall see in the next chapter, this question is tightly linked with the existence of the pressure field associated to the motion of a viscous, incompressible fluid. We have (Pileckas 1983)

Theorem III.5.3 *Assume Ω be such that problem* (III.3.65)[2] *is solvable for any $f \in L^q(\Omega)$ [respectively, $f \in \widehat{L}^q(\Omega) \equiv L^q(\Omega)\,/\,\mathbb{R}$, if Ω is bounded]. Then, given any (bounded) linear functional $\mathcal{F}$ on $D_0^{1,q}(\Omega)$, $1 < q < \infty$, identically vanishing on $\widehat{\mathcal{D}}_0^{1,q}(\Omega)$, there exists a uniquely determined*

$$p \in L^{q'}(\Omega) \quad \Big[\textit{respectively,} \quad p \in \widehat{L}^{q'}(\Omega)\Big]$$

such that $\mathcal{F}$ admits the following representation:

$$\mathcal{F}(\boldsymbol{v}) = \int_\Omega p \nabla \cdot \boldsymbol{v}, \quad \textit{for all } \boldsymbol{v} \in D_0^{1,q}(\Omega). \tag{III.5.3}$$

Proof. Consider the operator

$$A : \boldsymbol{v} \in D_0^{1,q}(\Omega) \to \nabla \cdot \boldsymbol{v} \in L^q(\Omega).$$

Evidently, A is linear and bounded and, by assumption, its range $\mathsf{R}(A)$ coincides with the whole of $L^q(\Omega)$ ($\widehat{L}^q(\Omega)$, for Ω bounded). By a well-known theorem on adjoint equations (Banach closed range theorem), see, e.g., Brezis (1983, Théorème II.18) we then have

$$[\ker(A)]^\perp = \mathsf{R}(A^*), \tag{III.5.4}$$

where $\ker(A)$ is the kernel of A, A^* is its adjoint and $\perp$ means annihilator, cf. Exercise III.1.1. However, it is obvious that

$$\ker(A) = \widehat{\mathcal{D}}_0^{1,q}(\Omega)$$

so that (III.5.4) delivers

$$\Big[\widehat{\mathcal{D}}_0^{1,q}(\Omega)\Big]^\perp = \mathsf{R}(A^*)$$

and, consequently, all functionals $\mathcal{F}$ vanishing on $\widehat{\mathcal{D}}_0^{1,q}(\Omega)$ must be in the range of A^*. On the other hand, by definition, every element in $\mathsf{R}(A^*)$ is of the form $\mathcal{L}(A\boldsymbol{v})$, where $\mathcal{L}$ is a functional on $L^q(\Omega)$ [respectively, $\widehat{L}^q(\Omega)$, if Ω is bounded]. We may then employ the Riesz representation theorem to obtain, for some uniquely determined $p \in L^{q'}(\Omega)$ [respectively, $p \in \widehat{L}^{q'}(\Omega)$],

[2] We require the problem in the form of (III.3.65) instead of (III.3.2) so that the result in the theorem applies also to unbounded domains.

$$\mathcal{F}(\boldsymbol{v}) = \mathcal{L}(A\boldsymbol{v}) = \int_\Omega pA\boldsymbol{v} = \int_\Omega p\nabla\cdot\boldsymbol{v}, \quad \boldsymbol{v}\in D_0^{1,q}(\Omega),$$

which completes the proof of the theorem. □

In view of the results established in Sections III.3 and III.4, from Theorem III.5.3 we obtain the following.

Corollary III.5.1 *Let Ω be a bounded or exterior domain of $\mathbb{R}^n$, $n\ge 2$, that satisfies the cone condition, or $\Omega = \mathbb{R}^n_+$. Then, any bounded linear functional $\mathcal{F}$ on $D_0^{1,q}(\Omega)$, $1<q<\infty$, identically vanishing on $\widehat{\mathcal{D}}_0^{1,q}(\Omega)$ is of the form* (III.5.3) *for some uniquely determined $p\in L^{q'}(\Omega)$ [respectively, $p\in \widehat{L}^{q'}(\Omega)$, if Ω is bounded].*

Remark III.5.3 If Ω is an *arbitrary* domain in $\mathbb{R}^n$, $n\ge 2$, it is not said that a representation of the type (III.5.4) holds in Ω. Actually, the assumptions of Theorem III.5.3 certainly do *not* hold if Ω is not smooth enough. However, since problem (III.3.65) is solvable in every ball B contained in Ω, we may use Theorem III.5.3 to show the following result whose proof we leave to the reader as an exercise.

Corollary III.5.2 *Let Ω be an arbitrary domain of $\mathbb{R}^n$, $n\ge 2$. Suppose $\mathcal{F}$ is a bounded linear functional on $D_0^{1,q}(\Omega')$, $1<q<\infty$, identically vanishing on $\widehat{\mathcal{D}}_0^{1,q}(\Omega')$, where Ω' is any bounded domain of Ω with $\overline{\Omega'}\subset\Omega$. Then, there exists $p\in L^{q'}_{loc}(\Omega)$ such that $\mathcal{F}$ admits the following representation:*

$$\mathcal{F}(\boldsymbol{\psi}) = \int_\Omega p\nabla\cdot\boldsymbol{\psi}, \quad \textit{for all } \boldsymbol{\psi}\in C_0^\infty(\Omega).$$

■

III.6 Approximation Problems in Spaces H_q^1 and $\mathcal{D}_0^{1,q}$

A problem often encountered in the applications is the following. Assume

$$\boldsymbol{v}\in H_q^1(\Omega)\cap\left[\cap_{i=1}^k L^{r_i}(\Omega)\right], \quad 1<q, r_i<\infty,\ i=1,\dots,k.$$

Clearly, $\boldsymbol{v}$ can be approximated by a sequence $\{\boldsymbol{v}'_m\}\subset\mathcal{D}(\Omega)$ (as a member of $H_q^1(\Omega)$) and by a sequence $\{\boldsymbol{v}''_m\}\subset C_0^\infty(\Omega)$ (as a member of $L^{r_i}(\Omega)$). The question now is to establish if $\boldsymbol{v}$ can be approximated by the *same* sequence in *both* spaces or, in other words, taking into account that $\mathcal{D}(\Omega)\subset C_0^\infty(\Omega)$, if there is a sequence $\{\boldsymbol{v}_m\}\subset\mathcal{D}(\Omega)$ such that as $m\to\infty$

$$\boldsymbol{v}_m\to\boldsymbol{v} \ \text{ in } H_q^1(\Omega)\cap\left[\cap_{i=1}^k L^{r_i}(\Omega)\right]. \tag{III.6.1}$$

Of course this problem admits a trivial positive answer when the domain Ω is bounded and q, r_i are suitably related to each other. For example, take $k=1$ and assume that at least one of the following conditions is fulfilled ($r_1\equiv r$):

(i) $r \le q$;
(ii) $q \ge n$;
(iii) $q < n$ and $r \le nq/(n-q)$.

Then (III.6.1) follows at once. Actually, denoting by $\{\boldsymbol{v}_m\} \subset \mathcal{D}(\Omega)$ a sequence converging to $\boldsymbol{v}$ in $H_q^1(\Omega)$, in case (i) we have, by the Hölder inequality,

$$\|\boldsymbol{v} - \boldsymbol{v}_m\|_r \le c\|\boldsymbol{v} - \boldsymbol{v}_m\|_{1,q} \to 0.$$

In case (ii) or (iii) the same conclusion can be drawn by using, instead of the Hölder inequality, the embedding inequalities of Theorem II.3.2. Moreover, the Sobolev inequality (II.3.7) also gives the result for *arbitrary* Ω, provided $1 < q < n$ and $r = nq/(n-q)$.

What can be said in the general case when q, n, and r_i are not necessarily related to each other? The aim of this section is to show that for Ω locally Lipschitz, it is always possible to find a sequence $\{\boldsymbol{v}_m\} \subset \mathcal{D}(\Omega)$ satisfying (III.6.1). An analogous result holds if we replace H_q^1 with $\mathcal{D}_0^{1,q}$.

The proof will be achieved through several intermediate steps. The first step is to introduce a suitable "cut-off" function. This function involves the distance $\delta(x)$ of a point $x \in \Omega$ from the boundary $\partial\Omega$. We need to differentiate $\delta(x)$ but, in fact, $\delta(x)$ is in general not more differentiable than the obvious Lipschitz-like condition $|\delta(x) - \delta(y)| \le |x - y|$. To overcome such a difficulty, we introduce the so-called *regularized distance* in the sense of Stein (1970, p.171). In this respect, we have the following lemma for whose proof we refer the reader to Stein (1970, Chapter VI, Theorem 2).

Lemma III.6.1 *Let Ω be a domain of $\mathbb{R}^n$ and set*

$$\delta(x) = \operatorname{dist}(x, \partial\Omega). \tag{III.6.2}$$

Then there is a function $\rho \in C^\infty(\Omega)$ such that for all $x \in \Omega$

(i) *$\delta(x) \le \rho(x)$;*
(ii) $|D^\alpha \rho(x)| \le \kappa_{|\alpha|+1} [\delta(x)]^{1-|\alpha|}$, $|\alpha| \ge 0$,

where $\kappa_{|\alpha|+1}$ depends only on α and n.

Remark III.6.1 A simple estimate for the constant κ_1 is given by Stein (1970, p.169 and p.171) and one has $\kappa_1 = (20/3)(12)^n$. Moreover, if Ω is sufficiently smooth (depending on $|\alpha|$), and x is sufficiently close to $\partial\Omega$, one can take $\rho = \delta$ and, consequently, $\kappa_1 = \kappa_2 = 1$. ■

Owing to this result, we can prove the following.

Lemma III.6.2 *Let Ω, δ be as in* Lemma III.6.1. *For any $\varepsilon > 0$ set*

$$\gamma(\varepsilon) = \exp(-1/\varepsilon).$$

Then, there exists a function $\psi_\varepsilon \in C^\infty(\overline{\Omega})$ such that

(i) $|\psi_\varepsilon(x)| \le 1$, *for all* $x \in \Omega$;
(ii) $\psi_\varepsilon(x) = 1$, *if* $\delta(x) < \gamma^2(\varepsilon)/2\kappa_1$;
(iii) $\psi_\varepsilon(x) = 0$, *if* $\delta(x) \ge 2\gamma(\varepsilon)$;
(iv) $|\nabla\psi_\varepsilon(x)| \le \kappa_2\varepsilon/\delta(x)$, *for all* $x \in \Omega$,
(v) $|D^\alpha\psi_\varepsilon(x)| \le \kappa\,\varepsilon/\delta^{|\alpha|}(x)\,,\quad |\alpha| \ge 2\,,\quad \varepsilon \in (0,\varepsilon_1],\quad \varepsilon_1 > 0\,,$

where κ_1 *and* κ_2 *are given in* Lemma III.6.1,[1] *while* $\kappa = \kappa(\alpha, n, \varepsilon_1)$.

Proof. Consider the following function of $\mathbb{R}$ into itself:

$$\varphi_\varepsilon(t) = \begin{cases} 1 & \text{if } t < \gamma^2(\varepsilon) \\ \varepsilon \ln(\gamma(\varepsilon)/t) & \text{if } \gamma^2(\varepsilon) < t < \gamma(\varepsilon) \\ 0 & \text{if } t > \gamma(\varepsilon). \end{cases}$$

Clearly, choosing $\eta = \gamma^2/2$, the mollifier $\Phi_\varepsilon \equiv (\varphi_\varepsilon)_\eta$ of φ_ε satisfies $\Phi_\varepsilon(t) = 1$ for $t < \gamma^2/2$, $\Phi_\varepsilon(t) = 0$ for $t > 2\gamma$ and

$$\left.\begin{aligned} &|\Phi'_\varepsilon(t)| \le \varepsilon/t, \\ &\left|\frac{d^k\Phi_\varepsilon}{dt^k}\right| \le c\,\varepsilon/t^k\,,\quad k \ge 1\,,\quad \varepsilon \in (0,\varepsilon_1]\,, \end{aligned}\right\} \text{for all } t \in \mathbb{R}, \tag{III.6.3}$$

where $c = c(k, \varepsilon_1)$. In addition, $|\Phi_\varepsilon(t)| \le 1$. Setting

$$\psi_\varepsilon(x) \equiv \Phi_\varepsilon(\rho(x)),$$

where ρ is the regularized distance of Lemma III.6.1, and recalling statements (i) and (ii) of that lemma, we deduce

$$\psi_\varepsilon(x) = 1 \text{ if } \delta(x) < \gamma^2/2\kappa_1$$

$$\psi_\varepsilon(x) = 0 \text{ if } \delta(x) > 2\gamma.$$

Moreover, from (III.6.3) and Lemma III.6.1 it follows for all $x \in \Omega$

$$|\nabla\psi_\varepsilon(x)| \le \kappa_2\varepsilon/\rho(x) \le \kappa_2\varepsilon/\delta(x)$$

$$|D^\alpha\psi_\varepsilon(x)| \le \kappa\,\varepsilon/\delta^{|\alpha|}(x)\,,\quad |\alpha| \ge 2\,,$$

whenever $\varepsilon \le \varepsilon_1$, for some $\varepsilon_1 > 0$ and with $\kappa = \kappa(\alpha, n, \varepsilon_1)$. The result is therefore completely proved. □

[1] Notice that, by Remark III.6.1,

$$\gamma(\varepsilon) < 2\kappa_1, \quad \text{for all } \varepsilon > 0.$$

Of course, whatever the estimate for κ_1, we can always choose a larger value for κ_1 such that this latter inequality holds.

Lemma III.6.3 *Let Ω be a bounded, locally Lipschitz domain of $\mathbb{R}^n$. Then there exists $c = c(\Omega, q, n)$ such that for all $u \in W_0^{1,q}(\Omega)$, $1 < q < \infty$, we have*

$$\|u\delta^{-1}\|_q \le c|u|_{1,q}$$

where δ is given in (III.6.2).

Proof. It is enough to prove the result for $u \in C_0^\infty(\Omega)$. By (II.5.5)

$$\|u\|_{q,\Omega'} \le c_1|u|_{1,q}$$

for any domain Ω', with $\overline{\Omega}' \subset \Omega$ and where $c_1 = c_1(\Omega)$. To show the estimate "near" the boundary, we recall that $\partial\Omega$ can be covered with a finite number of cylinders of the type

$$C_\sigma = \left\{x' \in D_\zeta \subset \mathbb{R}^{n-1} : \ \zeta(x') - \sigma < x_n < \zeta(x') + \sigma\right\},$$

where ζ is a Lipschitz function locally defining the boundary of Ω. Therefore, setting $V_\sigma = \Omega \cap C_\sigma$, and noting that

$$x_n - \zeta(x') \le c\,\delta(x), \quad x \in V_\sigma,$$

we find

$$\int_{V_\sigma} |u(x)\delta^{-1}(x)|^q \le c\int_{D_\zeta} dx' \left\{\int_{\zeta(x')}^{\zeta(x')+\sigma} |u(x', x_n)|^q |x_n - \zeta(x')|^{-q} dx_n\right\}$$

and the desired estimate follows from the elementary one-dimensional inequality

$$\int_0^\infty |h(t)|^q t^{-q} \le \left(\frac{q}{q-1}\right)\int_0^\infty \left|\frac{dh}{dt}\right|^q, \quad h \in C_0^\infty(\mathbb{R}_+),$$

which can be easily proved by integrating the identity

$$|h(t)|^q t^{-q} = \frac{d}{dt}\left[\frac{t^{1-q}}{1-q}|h(t)|^q\right] - \frac{t^{1-q}}{1-q}\frac{d}{dt}|h(t)|^q.$$

□

We also have

Lemma III.6.4 *Let Ω be as in* Lemma III.6.3. *Suppose*

$$u \in W_0^{1,q}(\Omega) \cap \left[\cap_{i=1}^k L^{r_i}(\Omega)\right]$$

for some $q, r_i \in (1, \infty)$, $i = 1, \ldots, k$. Then, for any $\eta > 0$ there exists $u_\eta \in C_0^\infty(\Omega)$ such that

$$\|u - u_\eta\|_{1,q} + \sum_{i=1}^k \|u - u_\eta\|_{r_i} < \eta.$$

Proof. For simplicity, we show the result for $k = 1$ and set $r_1 = r$. Given $\varepsilon > 0$, we set

$$\vartheta_\varepsilon(x) = 1 - \psi_\varepsilon(x),$$

where the function $\psi_\varepsilon(x)$ has been introduced in Lemma III.6.2. We then have that $|\vartheta_\varepsilon(x)| \le 1$, $\vartheta_\varepsilon(x)$ vanishes in a neighborhood $\mathcal{N}_\varepsilon$ of $\partial\Omega$, $\vartheta_\varepsilon(x) = 1$ for $\delta(x) \ge 2\exp(-1/\varepsilon)$ and

$$|\nabla\vartheta_\varepsilon(x)| \le \varepsilon/\delta(x), \quad \text{for all } x \in \Omega. \tag{III.6.4}$$

Putting

$$u_\varepsilon(x) = \vartheta_\varepsilon(x)u(x),$$

we at once recognize that $u_\varepsilon(x)$ is of compact support in Ω and that

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_s = 0, \quad s = r, q. \tag{III.6.5}$$

Furthermore, from (III.6.4),

$$|u_\varepsilon - u|_{1,q} \le \|(1-\vartheta_\varepsilon)\nabla u\|_q + \varepsilon\|u/\delta\|_q$$

and so, by Lemma III.6.3, we obtain

$$|u_\varepsilon - u|_{1,q} \le \|(1-\vartheta_\varepsilon)\nabla u\|_q + c\varepsilon|u|_{1,q},$$

with c independent of u and ε. Thus

$$\lim_{\varepsilon\to 0} |u_\varepsilon - u|_{1,q} = 0. \tag{III.6.6}$$

However, u_ε can be approximated by its regularizer $(u_\varepsilon)_\rho$ in both spaces $W^{1,q}(\Omega)$ and $L^r(\Omega)$ and since, for ρ small enough, $(u_\varepsilon)_\rho \in C_0^\infty(\Omega)$, the lemma follows from this and from (III.6.5) and (III.6.6). □

We are now in a position to prove the main result.

Theorem III.6.1 *Let Ω be a locally Lipschitz domain of $\mathbb{R}^n$, $n \ge 2$. Assume*

$$\boldsymbol{v} \in H_q^1(\Omega) \cap \left[\cap_{i=1}^k L^{r_i}(\Omega)\right]$$

for some $q, r_i \in (1,\infty)$, $i = 1,\dots,k$. Then, there exists a sequence $\{\boldsymbol{\varphi}_m\} \subset \mathcal{D}(\Omega)$ such that

$$\lim_{m\to\infty} \|\boldsymbol{v}_m - \boldsymbol{v}\|_{1,q} = \lim_{m\to\infty} \sum_{i=1}^k \|\boldsymbol{v}_m - \boldsymbol{v}\|_{r_i} = 0.$$

Proof. Again, for simplicity, we shall treat the case $k = 1$ and set $r_1 = r$. Let us first consider the case Ω bounded. By Lemma III.6.4 there exists a sequence $\{\boldsymbol{\varphi}_m\} \subset C_0^\infty(\Omega)$ satisfying

$$\lim_{m\to\infty} \|\boldsymbol{\varphi}_m - \boldsymbol{v}\|_{1,q} = \lim_{m\to\infty} \|\boldsymbol{\varphi}_m - \boldsymbol{v}\|_r = 0. \tag{III.6.7}$$

Since

$$\boldsymbol{v},\ \nabla\cdot\boldsymbol{v} \in L^r(\Omega),$$

and $\boldsymbol{v}$ has zero trace at the boundary, by Theorem III.2.4 we have

$$\boldsymbol{\varphi}_m - \boldsymbol{v} \in \widetilde{H}_{0,r}(\Omega).$$

In addition,

$$\nabla\cdot(\boldsymbol{\varphi}_m - \boldsymbol{v}) = \nabla\cdot\boldsymbol{\varphi}_m \in C_0^\infty(\Omega),$$

and so, by Theorem III.3.4 there exists $\boldsymbol{w}_m \in C_0^\infty(\Omega)$ such that, for all $m \in \mathbb{N}$,

$$\begin{aligned} \nabla\cdot\boldsymbol{w}_m &= -\nabla\cdot\boldsymbol{\varphi}_m \\ \|\boldsymbol{w}_m\|_{1,q} &\le c\,\|\nabla\cdot\boldsymbol{\varphi}_m\|_q \\ \|\boldsymbol{w}_m\|_r &\le c\,\|\boldsymbol{\varphi}_m - \boldsymbol{v}\|_r, \end{aligned}$$

where c is independent of $\boldsymbol{w}_m$, $\boldsymbol{\varphi}_m$, and $\boldsymbol{v}$. Setting

$$\boldsymbol{v}_m = \boldsymbol{\varphi}_m + \boldsymbol{w}_m,$$

it follows that

$$\begin{aligned} \|\boldsymbol{v} - \boldsymbol{v}_m\|_{1,q} &\le \|\boldsymbol{\varphi}_m - \boldsymbol{v}\|_{1,q} + \|\boldsymbol{w}_m\|_{1,q} \le \|\boldsymbol{\varphi}_m - \boldsymbol{v}\|_{1,q} + c\|\nabla\cdot\boldsymbol{\varphi}_m\|_{1,q} \\ \|\boldsymbol{v} - \boldsymbol{v}_m\|_r &\le \|\boldsymbol{\varphi}_m - \boldsymbol{v}\|_r + \|\boldsymbol{w}_m\|_r \le (1+c)\|\boldsymbol{\varphi}_m - \boldsymbol{v}\|_r, \end{aligned}$$

which by (III.6.7) completes the proof in the case where Ω is bounded. Assume now that Ω is an exterior domain and denote by $\zeta_R \in C_0^\infty(\mathbb{R}^n)$ a "cut-off" function that equals one in Ω_R and zero in Ω^{2R} with

$$|\nabla\zeta_R| \le c_1/R, \tag{III.6.8}$$

with c_1 independent of R. Let $\boldsymbol{w}_R$ be a solution to the problem

$$\begin{aligned} &\nabla\cdot\boldsymbol{w}_R = -\nabla\cdot(\zeta_R\boldsymbol{v}) \\ &\boldsymbol{w}_R \in W_0^{1,q}(\Omega_{R,2R}) \cap W_0^{1,r}(\Omega_{R,2R}) \\ &|\boldsymbol{w}_R|_{1,s,\Omega_{R,2R}} \le c_2\|\nabla\zeta_R\cdot\boldsymbol{v}\|_s, \quad s = q, r. \end{aligned} \tag{III.6.9}$$

By Theorem III.3.1 and Theorem III.3.4 such a solution exists and by Lemma III.3.1 the constant c_2 entering the estimate can be taken independent of R. In view of (III.6.8) and (III.6.9), we also have

$$|\boldsymbol{w}_R|_{1,s,\Omega_{R,2R}} \le c_3 R^{-1} \|\boldsymbol{v}\|_{s,\Omega_{R,2R}}, \quad s = q, r, \tag{III.6.10}$$

where, again, c_3 does not depend on R. Set

$$\boldsymbol{v}_R = \zeta_R \boldsymbol{v} + \boldsymbol{w}_R.$$

Clearly,

$$\boldsymbol{v}_R \in W_0^{1,q}(\Omega_{2R}) \cap L^r(\Omega_{2R}), \quad \nabla \cdot \boldsymbol{v}_R = 0.$$

Since Ω_{2R} is locally Lipschitz, from Subsection 4(a) it follows that

$$\boldsymbol{v}_R \in H_q^1(\Omega_{2R}) \cap L^r(\Omega_{2R}).$$

By the first part of the proof we may then state that for any $\varepsilon > 0$ there is $\boldsymbol{v}_{\varepsilon,R} \in \mathcal{D}(\Omega)$ such that

$$\|\boldsymbol{v}_R - \boldsymbol{v}_{\varepsilon,R}\|_{1,q} + \|\boldsymbol{v}_R - \boldsymbol{v}_{\varepsilon,R}\|_r < \varepsilon$$

and so for $s = q, r$

$$\|\boldsymbol{v} - \boldsymbol{v}_{\varepsilon,R}\|_s \le \|\boldsymbol{v}_R - \boldsymbol{v}_{\varepsilon,R}\|_s + \|\boldsymbol{v} - \boldsymbol{v}_R\|_s < \varepsilon + \|(1-\zeta_R)\boldsymbol{v}\|_s + \|\boldsymbol{w}_R\|_{s,\Omega_{R,2R}}. \tag{III.6.11}$$

Obviously, for all sufficiently large R,

$$\|(1-\zeta_R)\boldsymbol{v}\|_s < \varepsilon. \tag{III.6.12}$$

Moreover, from inequality (II.5.5) and (III.6.10) it follows that

$$\|\boldsymbol{w}_R\|_{s,\Omega_{R,2R}} \le c_4 R |\boldsymbol{w}_R|_{1,s,\Omega_{R,2R}} \le c_5 \|\boldsymbol{v}\|_{s,\Omega_{R,2R}}$$

with c_5 independent of R and so, again for all sufficiently large R,

$$\|\boldsymbol{w}_R\|_{s,\Omega_{R,2R}} < \varepsilon. \tag{III.6.13}$$

From (III.6.11)–(III.6.13) we thus find that $\boldsymbol{v}_{\varepsilon,R} \in \mathcal{D}(\Omega)$ approaches $\boldsymbol{v}$ in $L^q \cap L^r$. It remains to be shown that $\boldsymbol{v}_{\varepsilon,R}$ also approaches $\boldsymbol{v}$ in $D^{1,q}$. However, this is obtained at once since, as before, we prove

$$|\boldsymbol{v} - \boldsymbol{v}_{\varepsilon,R}|_{1,q} < 2\varepsilon + |\boldsymbol{w}_R|_{1,q,\Omega_{R,2R}} < 2\varepsilon + c_3 R^{-1} \|\boldsymbol{v}\|_{q,\Omega_{R,2R}}, \tag{III.6.14}$$

and so for R large enough we deduce

$$|\boldsymbol{v} - \boldsymbol{v}_{\varepsilon,R}|_{1,q} < 3\varepsilon$$

and the proof of the theorem is complete. □

In an analogous manner, we can prove

Theorem III.6.2 *Let Ω be a locally Lipschitz domain of $\mathbb{R}^n$, $n \geq 2$. Assume*

$$\boldsymbol{v} \in \mathcal{D}_0^{1,q}(\Omega) \cap \left[\cap_{i=1}^k L^{r_i}(\Omega)\right],$$

for some $q, r_i \in (1,\infty)$, $i = 1, \dots, k$. Then, there exists a sequence $\{\boldsymbol{v}_m\} \subset \mathcal{D}(\Omega)$ such that

$$\lim_{m\to\infty} |\boldsymbol{v}_m - \boldsymbol{v}|_{1,q} = \lim_{m\to\infty} \sum_{i=1}^k \|\boldsymbol{v}_m - \boldsymbol{v}\|_{r_i} = 0.$$

Proof. The proof goes exactly as in Theorem III.6.1[2] except for one point that demands a little more care. Precisely, again for $k = 1$ and with $r_1 = r$, once we arrive at (III.6.14), we cannot immediately conclude that for R large enough

$$R^{-1}\|\boldsymbol{v}\|_{q,\Omega_{R,2R}} < \varepsilon \tag{III.6.15}$$

because we do not know that $\boldsymbol{v} \in L^q(\Omega)$. To show (III.6.15) we have to argue somewhat differently. If $q \in (1,n)$, by the Hölder inequality we find

$$\|\boldsymbol{v}\|_{q,\Omega_{R,2R}} \leq c_1 R \|\boldsymbol{v}\|_{nq/(n-q),\Omega^R},$$

with $c_1 = c_1(n,q)$ and since, by the Sobolev inequality,

$$\|\boldsymbol{v}\|_{nq/(n-q),\Omega} < \infty,$$

we conclude (III.6.15). If $r \geq q \geq n$ the same conclusion holds since, in such a case, by the Hölder inequality, it follows that

$$R^{-1}\|\boldsymbol{v}\|_{q,\Omega_{R,2R}} \leq c_2 R^{n(1/q-1/r-1/n)}\|\boldsymbol{v}\|_{r,\Omega_{R,2R}},$$

and (III.6.15) is recovered. Finally, if $q \in [n,\infty) \cap (r,\infty)$, from Lemma II.3.3 (see also Exercise III.6.3) we find that if

$$u \in D^{1,q}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n), \tag{III.6.16}$$

then $u \in L^q(\mathbb{R}^n)$ and the following inequality holds

$$\|u\|_q \leq c|u|_{1,q}^{\lambda}\|u\|_r^{1-\lambda}, \tag{III.6.17}$$

with

$$\lambda = \frac{n(q-r)}{r(q-n)+nq} \quad (< 1) \tag{III.6.18}$$

and $c = c(n,q,r)$. Since $\boldsymbol{v} \in D^{1,q}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$, we apply this result to $\boldsymbol{v}$ to deduce $\boldsymbol{v} \in L^q(\mathbb{R}^n)$ and (III.6.15) again follows. The proof is therefore completed. $\square$

[2] Recall that for Ω bounded, $H_q^1(\Omega) = \mathcal{D}_0^{1,q}(\Omega)$.

Approximation problems in the spaces $H_q^1 \cap \left[\cap_{i=1}^k W_0^{1,r_i}(\Omega)\right]$ and in the space $H_q^1 \cap \left[\cap_{i=1}^k D_0^{1,r_i}(\Omega)\right]$ [respectively, $\mathcal{D}_0^{1,q} \cap \left[\cap_{i=1}^k W_0^{1,r_i}(\Omega)\right]$] and $\mathcal{D}_0^{1,q} \cap \left[\cap_{i=1}^k D_0^{1,r_i}(\Omega)\right]$ can be treated by the same technique used before. Their resolution is left to the reader in the following two exercises.

Exercise III.6.1 Under the hypothesis on Ω, q and r_i stated in Theorem III.6.1, show that any function $\boldsymbol{v} \in H_q^1(\Omega) \cap \left[\cap_{i=1}^k W_0^{1,r_i}(\Omega)\right]$ [respectively, $H_q^1(\Omega) \cap \left[\cap_{i=1}^k D_0^{1,r_i}(\Omega)\right]$], can be approximated in the space $H_q^1(\Omega) \cap \left[\cap_{i=1}^k W_0^{1,r_i}(\Omega)\right]$ [respectively, in $H_q^1(\Omega) \cap \left[\cap_{i=1}^k D_0^{1,r_i}(\Omega)\right]$] by functions from $\mathcal{D}(\Omega)$.

Exercise III.6.2 Under the hypothesis on Ω, q, and r_i stated in Theorem III.6.1, show that any function $\boldsymbol{v} \in \mathcal{D}_0^{1,q}(\Omega) \cap \left[\cap_{i=1}^k W_0^{1,r_i}(\Omega)\right]$ [respectively, $\mathcal{D}_0^{1,q}(\Omega) \cap \left[\cap_{i=1}^k D_0^{1,r_i}(\Omega)\right]$], $1 < q, r_i < \infty$, can be approximated in the space $\mathcal{D}_0^{1,q}(\Omega) \cap \left[\cap_{i=1}^k W_0^{1,r_i}(\Omega)\right]$ [respectively, in $\mathcal{D}_0^{1,q}(\Omega) \cap \left[\cap_{i=1}^k D_0^{1,r_i}(\Omega)\right]$] by functions from $\mathcal{D}(\Omega)$.

Exercise III.6.3 Prove the interpolation inequality of Nirenberg given in (III.6.16)–(III.6.18). *Hint*: Use the "cut-off" method of Theorem II.6.3 to approximate any u satisfying (III.6.16) with functions from $C_0^\infty(\mathbb{R}^n)$. Successively, employ inequality (II.3.5) together with the Hölder inequality.

III.7 Notes for the Chapter

Section III.1. As already pointed out, Lemma III.1.1 plays an important role in the theory of the Navier–Stokes equations. It is then not surprising that it has received the attention of many writers who proved it by several methods and under more or less different assumptions on the regularity of $\boldsymbol{u}$. To our knowledge, a first, elementary demonstration of the result was proposed by Hopf (1950/1951, pp.214-215) without, however, giving full details. Hopf's proof, which essentially aims at showing that the line integral of $\boldsymbol{u}$ on every closed loop is zero, was clarified and completed by Prodi (1959) and Ladyzhenskaya (1969). The assumption $n = 3$ implicitly made by these authors is removed, along the same line of method, by Temam (1973, Chapter I, §1.4). A less elementary proof based on de Rahm's theorem on currents, which assumes $\boldsymbol{u}$ to be only a distribution, is given by Lions (1969, pp. 67-69) and Temam (1977, Chapter I, Proposition 1.1). Tartar gives an alternative proof, with $\boldsymbol{u}$ in a negative Sobolev space, based on operator theory (see Temam 1977, Chapter I, Remark 1.9). A similar result is furnished by Giga & Sohr (1989, Corollary 2.2(i)). For other proofs that avoid de Rahm's theorem, see also Fujiwara and Morimoto (1977) and Simon (1991, 1993).

The Helmholtz–Weyl decomposition of the vector space $L^q(\Omega)$ in domains with a compact boundary has been the object of several investigations. In addition to the papers quoted in Section III.1, we refer the reader to von Wahl (1990b) and the bibliography of the work of Simader & Sohr (1992).

We also would like to mention the paper of Fabes, Mendez & Mitrea (1998, §§11,12) where sharp results are furnished for the validity of the decomposition in bounded domains with lower regularity (locally Lipschitz). In particular, in Theorem 12.2 of Fabes, Mendez & Mitrea, *loc. cit.* it is shown the existence of a bounded and Lipschitz domain of $\mathbb{R}^n$, where the Helmholtz–Weyl decomposition fails for all $p \notin [3/2, 3]$.

Concerning the Helmholtz–Weyl decomposition in domains with noncompact boundaries, we would like to mention the following important contributions. For the aperture domain (see (III.4.4)), its validity has been proved by Farwig (1993) and Farwig and Sohr (1996). Miyakawa (1994) showed an analogous result for semi-infinite cylinders and for infinite layers in $\mathbb{R}^n$, $n \geq 2$, using Littlewood-Paley theory. Wiegner (1995) studied the case of a layer in $\mathbb{R}^3$ by means of (partial) Fourier transforms. Thäter (1995) and Sohr and Thäter (1998) proved the validity of the decomposition for infinite cylinders of $\mathbb{R}^n$, $n \geq 2$, based on estimates for imaginary powers of the operator associated to the associated Neumann problem. Finally, an elementary and deep analysis of the decomposition in the case of infinite cylinders and layers of $\mathbb{R}^n$, $n \geq 2$, can be found in the paper by Simader and Ziegler (1998). For further and more recent contributions to this question, we refer to the article of Farwig (2003) and the bibliography therein.

Decompositions of weighted Lebesgue spaces on exterior domains have been studied by Specovius-Neugebauer (1990, 1995) and Fröhlich (2000).

Decompositions in Sobolev and Besov spaces are investigated in Fujiwara and Yamazaki (2007).

Section III.2. The notion of trace on the boundary for functions in $\widetilde{H}_q$, for $q = 2$, was introduced by Temam (1973, Chapter I), starting with identity (III.2.2). The same question was independently addressed by Fujiwara & Morimoto (1977) for $1 < q < \infty$, who generalized the results of Temam by means of a different (and less direct) approach.

Section III.3. The auxiliary problem considered in this section has also been studied in detail by Cattabriga (1961) for $n = 3$ and $q \in (1, \infty)$; Ladyzhenskaya (1969, Chapter I, §2) for $n = 2, 3$ and $q = 2$; Nečas (1967, Chapitre 3, Lemme 7.1) for $n \geq 2$ and $q = 2$; Babuska & Aziz (1972, Lemma 5.4.2) for $n = q = 2$ (see also Oden & Reddy 1976, Lemma 6.3.2); Solonnikov & Ščadilov (1973) and Ladyzhenskaya & Solonnikov (1976) for $n = 2, 3$ and $q = 2$; Amick (1976) for $n \geq 2$ and $q = 2$; Bogovskiĭ (1979, 1980) and Erig (1982), for $n \geq 2$ and $1 < q < \infty$; Pileckas (1980b, 1983) for $n = 2, 3$ and $1 < q < \infty$; Giaquinta & Modica (1982) for $n \geq 2$ and $q = 2$; Solonnikov (1983) and Kapitanskiĭ & Pileckas (1984) for $n \geq 2$ and $q \in (1, \infty)$; Arnold, Scott & Vogelius (1988) for $n = 2$ and $q \in (1, \infty)$; von Wahl (1989, 1990a) for $n = 3$ and $q \in (1, \infty)$; Borchers & Sohr (1990) for $n \geq 2$ and $q \in (1, \infty)$,[1]; Bourgain & Brezis (2003)

[1] Proposition *d*) of Theorem 2.4 of Borchers & Sohr (1990) is not correct as stated. The corrected version is furnished in Corollary 2.2 of Farwig & Sohr (1994a); see also Remark (iii) and the footnote at p. 274 of this latter paper.

when $n \geq 2$ and $q \in (1, \infty)$, and finally, by Durán & Lopéz García (2010) for $n = 2$ and $q \in (1, \infty)$. The main difference among the results proved by these authors relies either upon the method used to construct $\boldsymbol{v}$ or upon the regularity assumptions made on Ω. In particular, the method of Bogovskiĭ is based on the explicit representation formula (III.3.8) which adapts to the divergence operator a well-known formula of Sobolev (1963a, Chapter 7, §4; specifically, see eq. (7.9)). A similar representation for the curl operator has been given by Griesinger (1990a, 1990b). With the exception of Solonnikov & Ščadilov (1973), Bogovskiĭ (1980), Kapitanskiĭ & Pileckas (1984), and Borchers & Sohr (1990), all other mentioned authors consider the case Ω bounded. However, once (III.3.1), (III.3.2) is solved for such domains, the problem for Ω exterior can be directly handled by using the technique of Theorem III.3.6. The case $\Omega = \mathbb{R}^n_+$ requires, apparently, a separate treatment and will be considered in Section IV.3; see Corollary IV.3.1. In this respect, we refer the reader to the papers of Cattabriga (1961), Solonnikov & Ščadilov (1973) and Solonnikov (1973, 1983). These latter two papers deserve particular attention where *explicit* solutions are given (see Solonnikov 1973, formula (2.38), and Solonnikov 1983, Lemma 2.1).

Problem (III.3.1), (III.3.2) can also be solved in Sobolev spaces $W^{s,q}_0(\Omega)$ with s real, Bogovskiĭ (1979, 1980), and in certain weighted Sobolev spaces (Voldrich 1984). In this respect, as we already observed in Remark III.3.5, problem (III.3.1), (III.3.2) can not be solved in (bounded) domains having an external cusp. Nevertheless, Durán & López García (2010) have shown that, for such domains, it can still be solved in suitable weighted Sobolev spaces, with weights depending on the type of cusp.

For Ω exterior, results in weighted Sobolev spaces can be obtained by using, in Theorem III.3.6, Stein's Theorem II.11.5 instead of Theorem II.11.4. In this regard, we refer the reader to the papers of Specovious-Neugebauer (1986) and Lockhart & McOwen (1983). We finally mention that the same type of problem can be analyzed for the equation $\operatorname{curl} \boldsymbol{v} = \boldsymbol{f}$. In addition to the already cited papers of Griesinger, we refer the interested reader to the book of Girault & Raviart (1986) and to the works of Borchers & Sohr (1990), von Wahl (1989, 1990a) and Bolik & von Wahl (1997).

Theorem III.3.4 is due to Galdi (1992a). A different proof of Theorem III.3.5 (with slightly more stringent assumptions on the regularity of Ω, is given in Farwig & Sohr (1994a, Corollary 2.2). Extensions of these results to Sobolev spaces $W^{s,q}_0(\Omega)$ with s real and (suitably) negative are shown in Geissert, Heck, and Hieber (2006, § 2).

The numerical value of the constant c appearing in $(\text{III.3.2})_3$ and $(\text{III.3.65})_3$ is very important for several applications, see, *e.g.*, Chapters VI and XII. In this respect, we refer the reader to the papers of Horgan & Wheeler (1978), Horgan & Payne (1983), Velte (1990), and Stoyan (2001).

A different proof of Theorem III.3.7, originally due to Kapitanskiĭ & Pileckas (1984, Theorem 1), was provided, in a different context, by Dacorogna (2002).

Section III.4. An elementary proof of the coincidence of $H_q^1(\Omega)$ and $\widehat{H}_q^1(\Omega)$ for Ω bounded was first given by Heywood (1976) for $n = 2, 3$ and $q = 2$. His method, which can be easily extended to the cases $n \geq 2$ and $q \in [1, \infty)$, is a generalization of that used for a star-like domain. Actually, he introduces a smooth transformation $T_\rho(x), x \in \Omega$, $\rho \in (0, 1)$ that replaces the simple contraction ρx used for the star-like case. For this procedure to hold it is, of course, necessary that Ω have some regularity, and Heywood shows that C^2 smoothness is sufficient. However, as proved in Theorem 1 of Heywood's paper, the procedure would equally work with much less regularity. A proof of the coincidence for $n \geq 2$ and $q = 2$ using de Rahm's theorem on currents was previously furnished for Ω locally Lipschitz by Lions (1969, pp. 67-68; here the domain must be locally Lipschitz even if it is not explicitly stated) and by Temam (1977, Chapter I, Theorem 1.6). The same result is established by Salvi (1982) using extensions of sequentially continuous functionals. Other, different approaches are employed by Ladyzhenskaya & Solonnikov (1976) for $n = 2, 3$ and $q = 2$ and, along the same lines, by Pileckas (1980b, 1983) for arbitrary $q \in [1, \infty)$ and, for $n \geq 2$ and $q \in [1, \infty)$, by Bogovskiĭ (1980). In particular, Bogovskiĭ requires only that Ω is the union of a finite number of domains each of which is star-shaped with respect to a ball (see Theorem III.4.1); for example, Ω satisfies the cone condition. More recently, Wang & Yang (2008) have shown coincidence for $n = 2, 3$ and $q = 2$, provided Ω has only a kind of segment property. It is, however, still an open question to prove (or disprove) the validity of the coincidence for bounded domains with *no* regularity. In this respect, we wish to mention a result given in Šverák (1993), Remark at p. 12, which shows $\widehat{H}^1(\Omega) = H^1(\Omega)$, where Ω is any bounded open set of $\mathbb{R}^2$ such that, denoting by B an open ball with $\overline{B} \supset \Omega$, we have that the set $\overline{B} - \Omega$ has a finite number of connected components. Probably, it is true that $\widehat{H}^1(\Omega) = H^1(\Omega)$ (or, equivalently, $\widehat{\mathcal{D}}_0^{1,2}(\Omega) = \mathcal{D}_0^{1,2}(\Omega)$), for *any bounded domain in* $\mathbb{R}^n$, $n \geq 2$. It is worth emphasizing that should this not be true for some bounded open connected set $\Omega^\sharp$, the Stokes problem formulated in $\Omega^\sharp$ corresponding to zero body forces and *zero* boundary data would admit a *nonzero* smooth solution; see Remark IV.1.2.

The case of an exterior domain has likewise been analyzed by Heywood (1976), Ladyzhenskaya & Solonnikov (1976) for $n = 2, 3$ and $q = 2$, by Pileckas (1980b, 1983) for $n = 2, 3$ and $q \in (1, \infty)$, and by Bogovskiĭ (1980) for $n \geq 2$ and $q \in (1, \infty)$. All these authors prove the coincidence of $H_q^1(\Omega)$ and $\widehat{H}_q^1(\Omega)$ under the same regularity assumptions made on Ω for the corresponding bounded case. A more elementary proof for $n \geq 2$ and $q \in [1, \infty)$ that requires Ω^c be star-shaped is provided by Bogovskiĭ & Maslennikova (1978) and Maslennikova & Bogovskiĭ (1978).

Lemma III.4.2 is due to me. I have been kindly informed by Professor K. Pileckas that a similar result has been independently proved by Professor V. I. Burenkov and it appears in note 5.3 to Chapter VI of the Russian translation of the book of Stein (1970).

Section III.5. The papers mentioned in the notes to Section III.4 concerning the coincidence of the spaces $H^1_q(\Omega)$ and $\widehat{H}^1_q(\Omega)$ also deal with the same problem for $\mathcal{D}^{1,q}(\Omega)$ and $\widehat{\mathcal{D}}^{1,q}(\Omega)$.

The relation between linear functionals on $D^{1,q}_0(\Omega)$ vanishing on $\mathcal{D}^{1,q}_0(\Omega)$ and the existence of a pressure field for Stokes and Navier–Stokes problems was first recognized by Solonnikov & Ščadilov (1973), who prove Corollary III.5.1 for $q = 2$, $n = 3$ and Ω of class C^2. The same result was rediscovered thirteen years later by Guirguis (1986).

Section III.6. The results given here are due to Galdi (1992a). They will be used in several questions concerning Navier–Stokes equations, such as the validity of the energy identity in exterior domains (see Section X.2). Similar results, with different techniques and much more regularity on the domain, are contained in the works of Giga (1986) and Kozono & Sohr (1992a); see also the Appendix of Masuda (1984) and Lemma 3.8 of Maremonti (1991).

IV

Steady Stokes Flow in Bounded Domains

Ora sia il tuo passo
più cauto: ad un tiro di sasso
di qui ti si prepara
una più rara scena.

E. MONTALE, Ossi di Seppia.

Introduction

We now undertake the study of the mathematical properties of the motion of a viscous incompressible fluid. We shall begin with the simplest situation, namely, that of a steady, infinitely slow motion occurring in a bounded region Ω. The hypothesis of slow motion means that the ratio

$$\frac{|\boldsymbol{v}\cdot\nabla\boldsymbol{v}|}{|\nu\Delta\boldsymbol{v}|}$$

of inertial to viscous forces is vanishingly small, so that we can disregard the nonlinear term into the full (steady) Navier–Stokes equations (I.0.3$_1$). If we introduce reference length L and velocity V, this approximation amounts to assume that the (dimensionless) *Reynolds number*

$$\mathcal{R}=\frac{VL}{\nu}$$

is suitably small.

G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems*, Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-09620-9_4,

The linearization procedure can be performed around a generic solution $\boldsymbol{v}_0, p_0$, say, of equations (I.0.1). In this chapter (and the next two) we shall consider the case where $\boldsymbol{v}_0 \equiv 0$, $p \equiv$ const., so that we recover the following *Stokes equations* (see Stokes 1845)

$$\left.\begin{aligned} \Delta \boldsymbol{v} &= \nabla p + \boldsymbol{f} \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}\right\} \text{ in } \Omega. \tag{IV.0.1}$$

Here we have formally put, without loss, the coefficient of kinematic viscosity ν equal to unity. To system (IV.0.1), we append the usual adherence condition (I.1.1) at the boundary, that is,

$$\boldsymbol{v} = \boldsymbol{v}_* \quad \text{at } \partial\Omega. \tag{IV.0.2}$$

Since Ω is bounded, from (IV.0.1)$_2$and Gauss theorem, it follows that the prescribed velocity field $\boldsymbol{v}_*$ must satisfy the *compatibility condition*:

$$\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} = 0. \tag{IV.0.3}$$

The main objective of this chapter is to show existence, uniqueness, and regularity along with appropriate estimates for solutions to problem (IV.0.1)–(IV.0.3). In doing this, we shall be inspired by the work of Cattabriga (1961) and Galdi & Simader (1990). Specifically, we first give a variational (weak) formulation of the problem and introduce the concept of *q-generalized solution* (for $q = 2$, simply: *generalized solution*). These solutions are essentially characterized by the property of being members of the space $D^{1,q}(\Omega)$ and a priori they do not possess enough regularity to be considered as solutions in the ordinary sense. Following the work of Ladyzhenskaya (1959b), it is simple to show the existence of a generalized solution to (IV.0.1)–(IV.0.3). However, it is a much more difficult job to study its regularity, that is, to show that, under suitable smoothness assumptions on $\boldsymbol{f}$, $\boldsymbol{v}_*$, and Ω, such a solution belongs, in fact, to the Sobolev space $W^{m,q}(\Omega)$ and that it obeys corresponding estimates:

$$\|\boldsymbol{v}\|_{m+2,q} + \|p\|_{m+1,q} \le c \left(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)} \right), \tag{IV.0.4}$$

with $m \ge 0$ and $q \in (1, \infty)$.

Since system (IV.0.1), (IV.0.2) is elliptic in the sense of Douglis-Nirenberg (see Solonnikov 1966, Temam 1977, pp. 33-34), the validity of a weaker form of estimate (IV.0.4), namely,

$$\|\boldsymbol{v}\|_{m+2,q} + \|p\|_{m+1,q} \le c \left(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)} + \|\boldsymbol{v}\|_q + \|p\|_{-1,q} \right)$$

holding for q-generalized solutions, can be obtained directly from the general theory of Agmon, Douglis, & Nirenberg (1964) and Solonnikov (1966) (without, however, providing existence).

Here, to reach our goal, we shall follow a classical approach due to Cattabriga (1961) that relies on the ideas of Agmon, Douglis, & Nirenberg (1959). This method consists in transforming the problem into analogous problems in the whole space and in a half-space by means of the "localization procedure." Now, in $\mathbb{R}^n$ and $\mathbb{R}^n_+$, the task of proving the unique solvability of (IV.0.1), (IV.0.2), and (IV.0.4) is rendered easy by the circumstance that one can furnish an *explicit* solution to the problem. It is worth noticing that such a procedure is completely similar to that employed for the Poisson equation at the end of Section II.11 and that the only tool needed is the Calderón–Zygmund Theorem II.11.4 and its variant as given in Theorem II.11.6. We also wish to emphasize that the study of the Stokes problem in $\mathbb{R}^n$ and in $\mathbb{R}^n_+$ possesses an independent interest and that it will be fundamental for the treatment of other (linear and nonlinear) problems when the region of flow is either an exterior domain or a domain with a suitable unbounded boundary.

By the same arguments, we shall also show existence and uniqueness of q-generalized solutions when $q \neq 2$ and shall derive corresponding estimates, formally obtained by taking in (IV.0.4) $m = -1$ and $q \in (1, \infty)$.

We end with a final remark. As a rule, we shall treat in detail only the physically interesting cases when the relevant region of motion is either a three-dimensional or (for a plane flow) a two-dimensional domain. In particular, all results will be essentially proved for space dimension $n = 2, 3$. However, whenever needed, we shall outline all the main steps to follow in order to generalize the proof to $n \geq 4$.

IV.1 Generalized Solutions. Existence and Uniqueness

In this section we shall prove some existence and uniqueness results for Stokes flow. Following Ladyzhenskaya (1959b), we shall give an integral variational formulation of the problem, which will then be easily solved by the classical Riesz representation theorem.[1] However, the solutions we shall obtain are a priori not smooth enough to be considered as strict solutions of the starting problem; for this reason, they are called *generalized* or *weak*. Nevertheless, in the next sections we will show that provided the force, the velocity at the boundary, and the region of motion are sufficiently regular, weak solutions are, in fact, differentiable solutions of $(\text{IV.0.1})_{1,2}$ in the ordinary sense and assume continuously the boundary data.

To justify the generalized (or weak or variational) formulation, we proceed formally as follows. Let $\boldsymbol{v}$, p be a classical solution to $(\text{IV.0.1})_{1,2}$, for example,

[1] It should be observed that, in spite of its simplicity and elegance, the method of resolution based on the Riesz theorem is not constructive. The more constructive Galerkin method will be considered later, directly in the nonlinear context (see also Chapter VII).

$\boldsymbol{v} \in C^2(\Omega)$, $p \in C^1(\Omega)$. Multiplying (0.1_1) by an arbitrary function $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and integrating by parts we deduce [2]

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) \equiv \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{\varphi} = -\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} \equiv -(\boldsymbol{f}, \boldsymbol{\varphi}). \tag{IV.1.1}$$

Thus, every classical solution to $(\text{IV}.0.1)_1$ satisfies (IV.1.1) for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. Conversely, if $\boldsymbol{v} \in C^2(\Omega)$ and $\boldsymbol{f} \in C(\Omega)$, from (IV.1.1) and Lemma III.1.1 we show the existence of $p \in C^1(\Omega)$ verifying $(\text{IV}.0.1)_1$. On the other hand, we may think of a function $\boldsymbol{v}$ satisfying (IV.1.1) but which is not sufficiently differentiable to be considered a solution to $(\text{IV}.0.1)_1$ (for a suitable choice of p). In this sense (IV.1.1) is a "weak" version of $(\text{IV}.0.1)_1$. For further purposes, we may and shall consider the more general situation in which the right-hand side of (IV.1.1) is defined by a functional $\boldsymbol{f}$ from $D_0^{-1,q}(\Omega)$. We shall then write $\langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle$ instead of $(\boldsymbol{f}, \boldsymbol{\varphi})$ where, we recall, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W_0^{1,q}(\Omega)$ and $W_0^{1,q'}(\Omega)$, $1/q + 1/q' = 1$ (see Section II.3). As far as the regularity of a weak solution is concerned, we merely require a priori $\boldsymbol{v} \in D^{1,q}(\Omega)$ for some $q \in (1, \infty)$, so that the solenoidal condition will be satisfied according to generalized differentiation, while the boundary condition (IV.0.2) is to be understood in the trace sense (see Theorem II.4.4 and Remark II.6.1). If, in particular, the velocity at the boundary is zero, we require $\boldsymbol{v} \in D_0^{1,q}(\Omega)$ which, along with the solenoidality condition, furnishes $\boldsymbol{v} \in \widehat{\mathcal{D}}_0^{1,q}(\Omega)$; see Remark IV.1.2. We may then summarize all the above in the following.

Definition IV.1.1. A field $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ is called a *q-weak* (or *q-generalized*) *solution* to the Stokes problem (IV.0.1), (IV.0.2)[3] if and only if

(i) $\boldsymbol{v} \in D^{1,q}(\Omega)$, for some $q \in (1, \infty)$;
(ii) $\boldsymbol{v}$ is (weakly) divergence-free in Ω;
(iii) $\boldsymbol{v}$ satisfies the boundary condition (IV.0.2) (in the trace sense) or, if the velocity at the boundary is identically zero, $\boldsymbol{v} \in D_0^{1,q}(\Omega)$;
(iv) $\boldsymbol{v}$ verifies the identity

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) = -\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle \tag{IV.1.2}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,q'} \Omega)$, $1/q + 1/q' = 1$.

If $q = 2$, $\boldsymbol{v}$ will be called a *weak* (or *generalized*) *solution.*

Remark IV.1.1 Since Ω is bounded, $D_0^{1,q}(\Omega)$ and $W_0^{1,q}(\Omega)$ are isomorphic; see Remark II.6.3. Furthermore, if Ω is locally Lipschitz, $D^{1,q}(\Omega)$ endowed with a suitable norm, is isomorphic to $W^{1,q}(\Omega)$; see Remark II.6.1. Therefore, if $\boldsymbol{v}_* \equiv 0$ we may equivalently require in (i) that $\boldsymbol{v} \in W_0^{1,q}(\Omega)$, while, if $\boldsymbol{v}_* \not\equiv 0$ and Ω is locally Lipschitz, (i) is equivalent to $\boldsymbol{v} \in W^{1,q}(\Omega)$. ■

[2] As agreed,we shall put, without loss of generality, $\nu = 1$.

[3] Solutions possessing a priori even less regularity than q-weak solutions (the so called *very weak solutions*) will be briefly considered in the Notes for this Chapter.

Remark IV.1.2 If the velocity at the boundary is zero, every q-weak solution belongs to $\widehat{\mathcal{D}}_0^{1,q}(\Omega)$. We recall that, in general, $\widehat{\mathcal{D}}_0^{1,q}(\Omega) \supseteq \mathcal{D}_0^{1,q}(\Omega)$, for each $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and each $q \in (1,\infty)$; see Section III.5. In this regard, it is important to realize that if there exists a domain, $\Omega^\sharp$ such that $\widehat{\mathcal{D}}_0^{1,2}(\Omega^\sharp) \neq \mathcal{D}_0^{1,2}(\Omega^\sharp)$,[4] one *could find a nonzero generalized solution,* $\boldsymbol{v}^\sharp$*, to the Stokes problem* (IV.0.1)–(IV.0.2) *with* $\Omega \equiv \Omega^\sharp$*, corresponding to zero force and zero boundary data.* This fact is an immediate consequence of Exercise III.5.3. Moreover, as it will be shown in Theorem IV.4.3, $\boldsymbol{v}^\sharp \in C^\infty(\Omega)$ and we can find a corresponding pressure field $p^\sharp \in C^\infty(\Omega)$, such that the pair $(\boldsymbol{v}^\sharp, p^\sharp)$ satisfies the problem (IV.0.1)–(IV.0.2) in the ordinary sense! It is clear that such a situation is difficult to explain from a physical point of view, in that the flow $(\boldsymbol{v}^\sharp, p^\sharp)$ should be driven *merely* by the "roughness" of the boundary of the bounded domain where the motion occurs. In fact, if a mild degree of regularity on the boundary is assumed, then $\boldsymbol{v}^\sharp = \nabla p^\sharp = 0$. These considerations add more weight to the conjecture that $\widehat{\mathcal{D}}_0^{1,2}(\Omega) = \mathcal{D}_0^{1,2}(\Omega)$ for *every* bounded domain Ω, but, as we remarked several times in the previous chapter (see, especially, the Notes to Section III.4), no proof of this fact is available to date. ∎

In this section we shall establish existence and uniqueness of weak solutions. The analogous questions for q-weak solutions, arbitrary $q > 1$, will be considered in Section IV.6. Before performing this study, however, we wish to make some preliminary considerations.

Definition IV.1.1 is apparently silent about the pressure field. Actually, this is not true, as we will show. Assume, at first, $\boldsymbol{v}, p$ a classical solution and multiply $(\text{IV.0.1})_1$ by $\boldsymbol{\psi} \in C_0^\infty(\Omega)$ (not necessarily solenoidal). Integrating by parts we obtain, instead of identity (IV.1.2),

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) = -\langle \boldsymbol{f}, \boldsymbol{\psi} \rangle + (p, \nabla \cdot \boldsymbol{\psi}). \tag{IV.1.3}$$

Now, if $\boldsymbol{f}$ has a mild degree of regularity, to every q-weak solution we are able to associate a "pressure field" p in such a way that (IV.1.3) holds and, further, we can give a definition of q-weak solution *equivalent* to Definition IV.1.1, using (IV.1.3) in place of (IV.1.2) as a consequence of the following general result.

Lemma IV.1.1 *Let* Ω *be an arbitrary domain of* $\mathbb{R}^n$, $n \geq 2$, *and let* $\boldsymbol{f} \in W_0^{-1,q}(\Omega')$, $1 < q < \infty$, *for any bounded domain* Ω' *with* $\overline{\Omega'} \subset \Omega$. *A vector field* $\boldsymbol{v} \in W_{loc}^{1,q}(\Omega)$ *satisfies* (IV.1.2) *for all* $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ *if and only if there exists a "pressure field"* $p \in L_{loc}^q(\Omega)$ *such that* (IV.1.3) *holds for every* $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. *If, moreover,* Ω *is bounded and satisfies the cone condition and* $\boldsymbol{f} \in D_0^{-1,q}(\Omega)$, $\boldsymbol{v} \in D^{1,q}(\Omega)$ *then*

[4] By the results of Section III.5, $\Omega^\sharp$ should be less regular than a domain that is the union of a finite number of domains each of which is star-shaped with respect to a ball.

$$p \in L^q(\Omega).$$

Finally, if we normalize p *by the condition*

$$\int_\Omega p = 0, \tag{IV.1.4}$$

the following estimate holds:

$$\|p\|_q \le c\,(\|\boldsymbol{f}\|_{-1,q} + |\boldsymbol{v}|_{1,q})\,. \tag{IV.1.5}$$

Proof. We begin to prove the first part. It is enough to show that (IV.1.2) implies (IV.1.3), the reverse implication being obvious. Let us consider the functional

$$\mathcal{F}(\boldsymbol{\psi}) \equiv (\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) + \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle$$

for $\psi \in D_0^{1,q'}(\Omega')$. By assumption, $\mathcal{F}$ is bounded in $D_0^{1,q'}(\Omega')$ and is identically zero in $\mathcal{D}(\Omega)$ and, therefore, by continuity, in $\mathcal{D}_0^{1,q'}(\Omega')$. If Ω is arbitrary (in particular, has no regularity), from Corollary III.5.2 we deduce the existence of $p \in L^q_{loc}(\Omega)$ verifying (IV.1.3) for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. If Ω is bounded and satisfies the cone condition, by assumption and Corollary III.5.1 there exists a uniquely determined $p' \in L^q(\Omega)$ with

$$\int_\Omega p' = 0$$

such that

$$\mathcal{F}(\boldsymbol{\psi}) = (p', \nabla \cdot \boldsymbol{\psi}), \tag{IV.1.6}$$

for all $\psi \in D_0^{1,q'}(\Omega)$. From (IV.1.3) and (IV.1.6) we find, in particular,

$$(p - p', \nabla \cdot \boldsymbol{\psi}) = 0, \quad \text{for all } \boldsymbol{\psi} \in C_0^\infty(\Omega),$$

implying $p = p'$+const. (see Exercise II.5.9), and so, if we normalize p by (IV.1.4) we may take $p = p'$. Consider the problem

$$\begin{aligned} &\nabla \cdot \boldsymbol{\psi} = |p|^{q-2}p - \frac{1}{|\Omega|}\int_\Omega |p|^{q-2}p \equiv g \\ &\boldsymbol{\psi} \in W_0^{1,q'}(\Omega) \\ &\|\boldsymbol{\psi}\|_{1,q'} \le c_1\|p\|_q^{q-1}, \end{aligned} \tag{IV.1.7}$$

with Ω bounded and satisfying the cone condition. Since

$$\int_\Omega g = 0, \quad g \in L^{q'}(\Omega), \quad \|g\|_{q'} \le c_2\|p\|_q^{q-1},$$

from Theorem III.3.1 we deduce the existence of $\boldsymbol{\psi}$ solving (IV.1.7). If we replace such a $\boldsymbol{\psi}$ into (IV.1.6) and use (IV.1.4) together with the Hölder inequality and inequality $(\text{II.3.22})_2$, we obtain (IV.1.5). The proof is therefore completed. □

Remark IV.1.3 If we relax the normalization condition (IV.1.4) on p, in place of (IV.1.5) one can show, as the reader will easily check, the inequality

$$\inf_{c\in\mathbb{R}} \|p+c\|_q \le c\,(\|\boldsymbol{f}\|_{-1,q} + |\boldsymbol{v}|_{1,q})\,.$$

■

We now pass to the proof of existence and uniqueness of weak solutions.

Theorem IV.1.1 *Let $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, be bounded and locally Lipschitz. For any $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$ and $\boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega)$ verifying*

$$\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} = 0,$$

there exists one and only one weak solution $\boldsymbol{v}$ to the Stokes problem (IV.0.1), (IV.0.2). *Moreover, if we denote by p the corresponding pressure field associated to $\boldsymbol{v}$ by* Lemma IV.1.1, *the following estimate holds:*

$$\|\boldsymbol{v}\|_{1,2} + \|p\|_2 \le c\left(\|\boldsymbol{f}\|_{-1,2} + \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\right), \tag{IV.1.8}$$

where $c = c(n,\Omega)$.

Proof. By the results of Exercise III.3.8 there exists a solenoidal extension $\boldsymbol{V} \in W^{1,2}(\Omega)$ of $\boldsymbol{v}_*$ such that

$$\|\boldsymbol{V}\|_{1,2} \le c_1 \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \tag{IV.1.9}$$

with c_1 independent of $\boldsymbol{V}$ and $\boldsymbol{v}_*$. We look for a generalized solution of the form $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{V}$ where $\boldsymbol{w} \in \mathcal{D}_0^{1,2}(\Omega)$ satisfies the identity

$$(\nabla \boldsymbol{w}, \nabla \boldsymbol{\varphi}) = -\langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle - (\nabla \boldsymbol{V}, \nabla \boldsymbol{\varphi}), \tag{IV.1.10}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$. The right-hand side of (IV.1.10) defines a bounded linear functional in $\mathcal{D}_0^{1,2}(\Omega)$ and so, by the Riesz representation theorem, there exists one and only one $\boldsymbol{w} \in \mathcal{D}_0^{1,2}(\Omega)$ verifying (IV.1.10). This shows existence of a weak solution. To prove uniqueness, denote by $\boldsymbol{v}_1$ another weak solution corresponding to the same data. Evidently, Theorem II.4.2 furnishes that $\boldsymbol{u} \equiv \boldsymbol{v} - \boldsymbol{v}_1$ is an element of $\widehat{\mathcal{D}}_0^{1,2}(\Omega)$ and, therefore, by the results of Section III.5, of $\mathcal{D}_0^{1,2}(\Omega)$. On the other hand by (iv) of Definition IV.1.1 it follows that

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) = 0$$

for all $\varphi \in \mathcal{D}_0^{1,2}(\Omega)$, implying $\boldsymbol{u} = 0$ *a.e.* in Ω. To show estimate (IV.1.8), we take $\boldsymbol{\varphi} = \boldsymbol{w}$ into (IV.1.10), apply the Schwarz inequality and inequality $(\text{II}.3.22)_2$, and use (IV.1.9) and (II.5.1) together with (III.3.14) to obtain for some $c_2 = c_2(n,\Omega)$

$$\|\boldsymbol{w}\|_{1,2} \le c_2\left(\|\boldsymbol{f}\|_{-1,2} + \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\right). \tag{IV.1.11}$$

Estimate (IV.1.8) then follows from (IV.1.4), (IV.1.9), and (IV.1.11). □

Remark IV.1.4 If $\boldsymbol{v}_* \equiv 0$, the existence of a generalized solution is established without regularity assumptions on Ω. ■

Exercise IV.1.1 Theorem IV.1.1 also remains valid if $\nabla \cdot \boldsymbol{v} = g \not\equiv 0$, where g is a suitably ascribed function. Specifically, show that for Ω, $\boldsymbol{f}$, and $\boldsymbol{v}_*$ satisfying the assumption of that theorem and all $g \in L^2(\Omega)$ such that

$$\int_\Omega g = \int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n}$$

there exists one and only one weak solution $\boldsymbol{v}$ to the nonhomogeneous Stokes problem, that is, a field $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ satisfying (i), (iii), and (iv) of Definition IV.1.1, with $q = 2$, and $\nabla \cdot \boldsymbol{v} = g$ (weakly). Show, in addition, that $\boldsymbol{v}$ and the corresponding pressure field p obey the following estimate:

$$\|\boldsymbol{v}\|_{1,2} + \|p\|_2 \le c\left(\|\boldsymbol{f}\|_{-1,2} + \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} + \|g\|_2\right).$$

Hint: Look for a solution of the form $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{V}$ where $\boldsymbol{w}$ verifies (IV.1.7), while $\boldsymbol{V}$ solves $\nabla \cdot \boldsymbol{V} = g$, in Ω, $\boldsymbol{V} = \boldsymbol{v}_*$ at $\partial\Omega$ and use the results of Exercise III.3.8.

IV.2 Existence, Uniqueness, and L^q-Estimates in the Whole Space. The Stokes Fundamental Solution

Our next task is to establish interior and boundary inequalities for solutions to the Stokes problem that will furnish, in particular, that generalized solutions are in fact classical if the domain and data are sufficiently smooth. We first derive these estimates in two special cases, namely, when either $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$. The job here is easier because we are able to furnish solutions of explicit form. To this end, let us introduce the *fundamental solution for the Stokes equation* (IV.0.1), which plays the same role as the fundamental solution of Laplace's equation.[1] Consider the second-order, symmetric tensor field $\boldsymbol{U}$ and the vector field $\boldsymbol{q}$ defined by the relations

$$\begin{aligned} U_{ij}(x-y) &= \left(\delta_{ij}\Delta - \frac{\partial^2}{\partial y_i \partial y_j}\right)\Phi(|x-y|) \\ q_j(x-y) &= -\frac{\partial}{\partial y_j}\Delta\Phi(x-y), \end{aligned} \tag{IV.2.1}$$

where $x, y \in \mathbb{R}^n$, δ_{ij} is the Kronecker symbol and $\Phi(t)$ is an arbitrary function on $\mathbb{R}$, which is smooth for $t \neq 0$. Noticing that $\partial|x-y|/\partial x_i = -\partial|x-y|/\partial y_i$, by a simple calculation from (IV.2.1) one has for $x \neq y$ and all $i, j = 1, \ldots, n$[2]

[1] Actually, all the material presented in this and in the subsequent section will be derived along the same lines of the one developed for the Dirichlet problem for the Poisson equation at the end of Section II.7 (see Exercise II.11.9, Exercise II.11.10, and Exercise II.11.11.

[2] We recall that, according to Einstein's convention, unless otherwise explicitly stated, pairs of identical indices imply summation from 1 to n.

$$\begin{aligned} &\Delta U_{ij}(x-y) + \frac{\partial}{\partial x_i} q_j(x-y) = \delta_{ij}\Delta^2 \Phi(x-y) \\ &\frac{\partial}{\partial x_i} U_{ij}(x-y) = 0. \end{aligned} \tag{IV.2.2}$$

Choose now Φ as the fundamental solution to the biharmonic equation. So, for $n = 3$,

$$\Phi(|x-y|) = -\frac{|x-y|}{8\pi}$$

and the associated fields $\boldsymbol{U}$ and $\boldsymbol{q}$ become (Lorentz 1896)

$$\begin{aligned} U_{ij}(x-y) &= -\frac{1}{8\pi}\left[\frac{\delta_{ij}}{|x-y|} + \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^3}\right] \\ q_j(x-y) &= \frac{1}{4\pi}\frac{x_j-y_j}{|x-y|^3}. \end{aligned} \tag{IV.2.3}$$

Likewise, for $n = 2$,

$$\Phi(|x-y|) = |x-y|^2 \log(|x-y|)/8\pi$$

and we have

$$\begin{aligned} U_{ij}(x-y) &= -\frac{1}{4\pi}\left[\delta_{ij}\log\frac{1}{|x-y|} + \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right] \\ q_j(x-y) &= \frac{1}{2\pi}\frac{x_j-y_j}{|x-y|^2}. \end{aligned} \tag{IV.2.4}$$

Moreover, with the above choice of Φ, from (IV.2.2) it follows that the fields (IV.2.3) and (IV.2.4) satisfy

$$\left.\begin{aligned} &\Delta U_{ij}(x-y) + \frac{\partial}{\partial x_i} q_j(x-y) = 0 \\ &\frac{\partial}{\partial x_i} U_{ij}(x-y) = 0. \end{aligned}\right\} \quad \text{for } x \neq y. \tag{IV.2.5}$$

The pair $\boldsymbol{U}, \boldsymbol{q}$ is called the *fundamental solution of the Stokes equation.*

Remark IV.2.1 In dimension $n > 3$ the fundamental solution is given by (IV.2.1) with

$$\Phi = \begin{cases} -(1/8\pi^2)\log|x-y| & \text{if } n = 4 \\ \left[\Gamma(n/2-2)/16\pi^{n/2}\right]|x-y|^{4-n} & \text{if } n \geq 4. \end{cases}$$

One thus has for all $n \geq 4$

$$U_{ij}(x-y) = -\frac{1}{2n(n-2)\omega_n}\left[\frac{\delta_{ij}}{|x-y|^{n-2}} + (n-2)\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^n}\right]$$

$$q_j(x-y) = \frac{1}{n\omega_n}\frac{x_j-y_j}{|x-y|^n}.$$

■

From (IV.2.3) and (IV.2.4) (and Remark IV.2.1), we may formally compute the asymptotic properties of $\boldsymbol{U}$ and $\boldsymbol{q}$. In particular, the following estimates, as either $|x| \to 0$ or $|x| \to \infty$, are readily established:

$$\begin{aligned} \boldsymbol{U}(x) &= O(\log|x|) \ \text{ if } n = 2, \\ \boldsymbol{U}(x) &= O(|x|^{-n+2}) \ \text{ if } n > 2, \\ D^\alpha \boldsymbol{U}(x) &= O(|x|^{-n-|\alpha|+2}), \quad |\alpha| \geq 1, \quad n \geq 2, \\ D^\alpha \boldsymbol{q}(x) &= O(|x|^{-n-|\alpha|+1}), \quad |\alpha| \geq 0, \quad n \geq 2. \end{aligned} \tag{IV.2.6}$$

Let us now consider the following *nonhomogeneous Stokes problem*

$$\left.\begin{aligned} \Delta \boldsymbol{v} &= \nabla p + \boldsymbol{f} \\ \nabla \cdot \boldsymbol{v} &= g \end{aligned}\right\} \ \text{ in } \mathbb{R}^n, \tag{IV.2.7}$$

where $\boldsymbol{f}$ and g are prescribed functions from $C_0^\infty(\mathbb{R}^n)$. Using (IV.2.3) and (IV.2.4) it is not difficult to prove the existence of solutions to (IV.2.7), verifying suitable L^q-estimates. To reach this goal, we introduce the *Stokes volume potentials*

$$\begin{aligned} \boldsymbol{u}(x) &= \int_{\mathbb{R}^n} \boldsymbol{U}(x-y)\cdot \boldsymbol{F}(y)dy \\ \pi(x) &= -\int_{\mathbb{R}^n} \boldsymbol{q}(x-y)\cdot \boldsymbol{F}(y)dy, \end{aligned} \tag{IV.2.8}$$

where $\boldsymbol{F} \in C_0^\infty(\mathbb{R}^n)$. Since

$$\int_{\mathbb{R}^n} \boldsymbol{U}(x-y)\cdot \boldsymbol{F}(y)dy = \int_{\mathbb{R}^n} \boldsymbol{U}(z)\cdot \boldsymbol{F}(x-z)dz$$

$$\int_{\mathbb{R}^n} \boldsymbol{q}(x-y)\cdot \boldsymbol{F}(y)dy = \int_{\mathbb{R}^n} \boldsymbol{q}(z)\cdot \boldsymbol{F}(x-z)dz$$

one has $\boldsymbol{u}, \pi \in C^\infty(\mathbb{R}^n)$. Moreover, it is easy to show that $\boldsymbol{u}$, π is a solution to (IV.2.7) with $g \equiv 0$ and $\boldsymbol{f} \equiv \boldsymbol{F}$. Actually, it is obvious that $\nabla \cdot \boldsymbol{u} = 0$; also, using (IV.2.1) and Exercise II.11.3, we deduce

$$\begin{aligned} \Delta \boldsymbol{u}(x) - \nabla \pi(x) &= \Delta \int_{\mathbb{R}^n} \Delta \Phi(|x-y|)\boldsymbol{F}(y)dy \\ &= \Delta(\mathcal{E} * \boldsymbol{F})(x) = \boldsymbol{F}(x). \end{aligned} \tag{IV.2.9}$$

We shall now look for a solution $\boldsymbol{v}$, p to (IV.2.7) of the form $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{h}$, $p = \pi$ where $\boldsymbol{u}$ and π are volume potentials corresponding to $\boldsymbol{F} \equiv \boldsymbol{f} - \Delta \boldsymbol{h}$ and

$$\boldsymbol{h} = \nabla(\mathcal{E} * g). \tag{IV.2.10}$$

Since $\Delta \boldsymbol{h} = \nabla g \in C_0^\infty(\mathbb{R}^n)$ and

$$\nabla \cdot \boldsymbol{h} = g, \tag{IV.2.11}$$

from (IV.2.9) and (IV.2.11) we may conclude that $\boldsymbol{v}, p$ is a solution to (IV.2.7). Furthermore, from (IV.2.6) one shows as $|x| \to \infty$ [3]

$$\begin{aligned} \boldsymbol{v}(x) &= O(\log|x|) \quad \text{if } n = 2, \\ \boldsymbol{v}(x) &= O(|x|^{-n+2}) \quad \text{if } n > 2, \\ D^\alpha \boldsymbol{v}(x) &= O(|x|^{-n-|\alpha|+2}), \quad |\alpha| \geq 1, \quad n \geq 2, \\ D^\alpha p(x) &= O(|x|^{-n-|\alpha|+1}), \quad |\alpha| \geq 0, \quad n \geq 2. \end{aligned} \tag{IV.2.12}$$

Let us now derive some L^q-inequalities for $\boldsymbol{v}$, p in terms of g and $\boldsymbol{f}$. From (IV.2.10) and the Calderón–Zygmund Theorem II.7.4 we have

$$|\boldsymbol{h}|_{\ell+1,q} \leq c|g|_{\ell,q}, \quad \text{for all } \ell \geq 0 \tag{IV.2.13}$$

with $c = c(n,q)$. Next, consider the identity with $|\alpha| = \ell$

$$\begin{aligned} D_{ij} D^\alpha u_k(x) &= \int_{\mathbb{R}^n} D_i U_{k\ell}(x-y) D_j D^\alpha F_\ell(y) dy \\ &= \lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} D_{ij} U_{k\ell}(x - y D^\alpha F_\ell(y) dy \\ &\quad + \lim_{\varepsilon \to 0} \int_{|x-y| = \varepsilon} D_i U_{k\ell}(x - y D^\alpha F_\ell(y) n_j(y) d\sigma_y, \end{aligned} \tag{IV.2.14}$$

where n_j is the jth component of the unit outer normal to the sphere $|x-y| = \varepsilon$. From (IV.2.3) and (IV.2.4) one has that $D_i U_{k\ell}$ is homogeneous of degree $1-n$, so that by Lemma II.11.1, $D_{ij} U_{k\ell}$ is a singular kernel. Furthermore, again by that lemma, it follows that

$$\lim_{\varepsilon \to 0} \int_{|x-y|=\varepsilon} D_i U_{k\ell}(x-y) D^\alpha F_\ell(y) n_j(y) d\sigma_y = A_{ijk\ell} D^\alpha F_\ell(x)$$

with $A_{ijk\ell}$ a constant fourth-order tensor. Combining this formula with (IV.2.14) gives

[3] More detailed estimates will be given in Section V.3.

$$D_{ij}D^{\alpha}u_k(x) = \lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon} D_{ij}U_{k\ell}(x-y)D^{\alpha}F_{\ell}(y)dy + A_{ijk\ell}D^{\alpha}F_{\ell}(x),$$

where the integral is to be understood in the Cauchy principal value sense. We may now employ in this identity the Calderón–Zygmund theorem and (IV.2.13) to obtain for all $\ell \geq 0$ and all $q > 1$

$$|\boldsymbol{u}|_{\ell+2,q} \leq c_1 \left(|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q}\right), \tag{IV.2.15}$$

where $c_1 = c_1(n,q)$. Likewise, one proves

$$|\pi|_{\ell+1,q} \leq c_2 \left(|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q}\right). \tag{IV.2.16}$$

From (IV.2.13), (IV.2.15), and (IV.2.16) we thus obtain the following estimate for the solution $\boldsymbol{v}$, p, valid for all $\ell \geq 0$ and all $q > 1$

$$|\boldsymbol{v}|_{\ell+2,q} + |p|_{\ell+1,q} \leq c\left(|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q}\right) \tag{IV.2.17}$$

with $c = c(n,q)$.

Other estimates can be obtained directly from (IV.2.8) and (IV.2.10), by noting that

$$|D_iD^{\alpha}\boldsymbol{u}(x)| + |D^{\alpha}\pi(x)| \leq c_1\int_{\mathbb{R}^n}\frac{|D^{\alpha}\boldsymbol{F}(y)|}{|x-y|^{n-1}}dy$$

$$|D^{\alpha}\boldsymbol{h}(x)| \leq c_2\int_{\mathbb{R}^n}\frac{|D^{\alpha}g(y)|}{|x-y|^{n-1}}dy.$$

If $1 < q < n$, we may thus apply the Sobolev Theorem II.7.3 to obtain

$$|\boldsymbol{v}|_{\ell+1,s_1} + |p|_{\ell,s_1} \leq c_3\left(|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q}\right), \quad s_1 = \frac{nq}{n-q}. \tag{IV.2.18}$$

Likewise, if $1 < q < n/2$, from (IV.2.18) and (II.6.17) we have

$$|\boldsymbol{v}|_{\ell,s_2} \leq c_4\left(|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q}\right), \quad s_2 = \frac{nq}{n-2q}. \tag{IV.2.19}$$

Assume now $\boldsymbol{f}$ and g merely belonging to $W^{m,q}(\mathbb{R}^n)$ and $D^{m+1,q}(\mathbb{R}^n)$, respectively, $m \geq 0$ and $q \in (1,\infty)$. We can approximate them with sequences $\{\boldsymbol{f}_k\}$, $\{g_k\} \subset C_0^{\infty}(\mathbb{R}^n)$. Denoting by $\{\boldsymbol{v}_k, p_k\}$ the corresponding sequence of solutions to (IV.2.7), we see that each solution satisfies (IV.2.17) for all $\ell \in [0,m]$ and, if $1 < q < n$ [respectively, $1 < q < n/2$], it satisfies also (IV.2.18) [respectively, (IV.2.19)]. Employing these estimates together with the weak compactness property of spaces $\dot{D}^{m,q}$ (see Exercise II.6.2), one easily shows the existence of two fields $\boldsymbol{v}$ and p such that

$$\boldsymbol{v} \in \mathcal{B} \equiv \bigcap_{\ell=0}^{m} D^{\ell+2,q}(\mathbb{R}^n), \qquad p \in \mathcal{P} \equiv \bigcap_{\ell=0}^{m} D^{\ell+1,q}(\mathbb{R}^n)$$

and

$$\lim_{k\to\infty}(D^\alpha \boldsymbol{v}_k, \boldsymbol{\psi}) = (D^\alpha \boldsymbol{v}, \boldsymbol{\psi}), \quad 0 \le |\alpha| \le m+2$$

$$\lim_{k\to\infty}\left(D^\beta \nabla p_k, \boldsymbol{\psi}\right) = \left(D^\beta \nabla p, \boldsymbol{\psi}\right) \quad 0 \le |\beta| \le m$$

for all $\boldsymbol{\psi} \in L^{q'}(\mathbb{R}^n)$. This implies, in particular, that the pair $\boldsymbol{v}$, p satisfies (IV.2.7) a.e. in $\mathbb{R}^n$ along with estimates (IV.2.17)–(IV.2.19). Furthermore, by Lemma II.6.1, we have

$$\boldsymbol{v} \in W^{m+2,q}(B_R), \quad p \in W^{m+1,q}(B_R),$$

for all $R > 0$.

Let now $\boldsymbol{v}_1, p_1$ denote another solution to (IV.2.7) corresponding to the same data as $\boldsymbol{v}$, p, with $|\boldsymbol{v}_1|_{\ell+2,q}$ finite, for some $\ell \in [0,m]$. It is then easy to show that $|\boldsymbol{v}_1 - \boldsymbol{v}|_{\ell+2,q} = |p_1 - p|_{\ell+1,q} = 0$.[4] In fact, setting $\boldsymbol{z} = \boldsymbol{v}_1 - \boldsymbol{v}$ and $\tau = p_1 - p$, we obtain

$$\begin{aligned} \Delta \boldsymbol{z} &= \nabla \tau \\ \nabla \cdot \boldsymbol{z} &= 0 \end{aligned} \tag{IV.2.20}$$

a.e. in $\mathbb{R}^n$. It follows at once that

$$\int_{\mathbb{R}^n} \nabla \tau \cdot \nabla \psi = 0$$

for any $\psi \in C_0^\infty(\mathbb{R}^n)$. Since $D^\alpha \nabla \tau \in L^q(\mathbb{R}^n)$, $|\alpha| = \ell$, by a well-known result of Caccioppoli (1937), Cimmino (1938a, 1938b), and Weyl (1940) we then deduce that τ is harmonic, and hence smooth, in the whole space. As a consequence, by Exercise II.11.11 it follows that $D^\alpha \nabla \tau = 0$. Therefore, $(\text{IV.2.20})_1$ furnishes $\Delta D^\alpha \boldsymbol{z} = 0$ and, again by Exercise II.11.11, we have $|\boldsymbol{z}|_{\ell+2,q} = 0$, which is what we wanted to prove.

We collect the results obtained so far in the following.

Theorem IV.2.1 *Given*

$$\boldsymbol{f} \in W^{m,q}(\mathbb{R}^n),\ g \in D^{m+1,q}(\mathbb{R}^n), \quad m \ge 0,\ 1 < q < \infty,\ n \ge 2,$$

there exists a pair of functions $\boldsymbol{v}, p$ such that $\boldsymbol{v} \in W^{m+2,q}(B_R)$, $p \in W^{m+1,q}(B_R)$ for any $R > 0$, satisfying a.e. the nonhomogeneous Stokes system (IV.2.7). *Moreover, for all $\ell \in [0,m]$, $|\boldsymbol{v}|_{\ell+2,q}$ and $|p|_{\ell+1,q}$ are finite and we have*

$$|\boldsymbol{v}|_{\ell+2,q} + |p|_{\ell+1,q} \le c\left(|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q}\right). \tag{IV.2.21}$$

If, in particular, $n/2 \le q < n$, then $|\boldsymbol{v}|_{\ell+1,s_1}$ and $|p|_{\ell,s_1}$, $s_1 = nq/(n-q)$, are finite, and we have

[4] Notice that if $\boldsymbol{v}$ is a solution to (IV.2.7) having $|\boldsymbol{v}|_{\ell+2,q}$ finite, then $|p|_{\ell+1,q}$ is finite too.

$$|\boldsymbol{v}|_{\ell+1,s_1} + |p|_{\ell,s_1} + |\boldsymbol{v}|_{\ell+2,q} + |p|_{\ell+1,q} \le c\,(|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q})\,. \tag{IV.2.22}$$

Furthermore, if $1 < q < n/2$, *then* $|\boldsymbol{v}|_{\ell,s_2}$, $s_2 = nq/(n-2q)$, *is finite and the following inequality holds*

$$|\boldsymbol{v}|_{\ell,s_2} + |\boldsymbol{v}|_{\ell+1,s_1} + |p|_{\ell,s_1} + |\boldsymbol{v}|_{\ell+2,q} + |p|_{\ell+1,q} \le c\,(|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q})\,. \tag{IV.2.23}$$

In the above inequalities, $c = c(n,q,\ell)$. *In addition, if* $\boldsymbol{f}, g \in C_0^\infty(\mathbb{R}^n)$, *then* $\boldsymbol{v}, p \in C^\infty(\mathbb{R}^n)$ *and they have for large* $|x|$ *the asymptotic behavior indicated in* (IV.2.12). *Finally, if* $\boldsymbol{v}_1, p_1$ *is another solution to* (IV.2.6) *corresponding to the data* $\boldsymbol{f}, g$ *with* $|\boldsymbol{v}_1|_{\ell+2,q}$ *finite for some* $\ell \in [0,m]$, *then* $|\boldsymbol{v}_1 - \boldsymbol{v}|_{\ell+2,q} = 0$ *and* $|p_1 - p|_{\ell+1,q} = 0$.

The last part of this section is devoted to show existence and uniqueness of q-weak solutions to (IV.2.7). The results we obtain are similar to those of Theorem IV.1.1 and Exercise IV.1.1, with the difference that now we consider the problem in the general context of spaces $D_0^{1,q}$, $1 < q < \infty$.

To this end, we give the following

Definition IV.2.1 A vector field $\boldsymbol{v}$ is a *q-generalized solution to* (IV.2.7) if and only if

(i) $\boldsymbol{v} \in D_0^{1,q}(\mathbb{R}^n)$;
(ii) $(\nabla\boldsymbol{v}, \nabla\boldsymbol{\varphi}) = -\,[\boldsymbol{f}, \boldsymbol{\varphi}]$, for all $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,q'}(\mathbb{R}^n)$;
(iii) $(\boldsymbol{v}, \nabla\varphi) = -(g, \varphi)$, for all $\varphi \in C_0^\infty(\mathbb{R}^n)$.

Lemma IV.1.1 implies the following result.

Lemma IV.2.1 *Let* $\boldsymbol{f} \in W_0^{-1,q}(B_R)$, *for all* $R > 0$. *Then, to every q-generalized solution in the sense of* Definition IV.1.2, *we may associate a pressure field* $p \in L^q(B_R)$, *all* $R > 0$, *such that*

$$(\nabla\boldsymbol{v}, \nabla\boldsymbol{\psi}) - (p, \nabla\cdot\boldsymbol{\psi}) = -[\boldsymbol{f}, \boldsymbol{\psi}], \quad \textit{for all } \boldsymbol{\psi} \in C_0^\infty(\mathbb{R}^n). \tag{IV.2.24}$$

If, in particular, $\boldsymbol{f} \in D_0^{-1,q}(\mathbb{R}^n)$, *then* $p \in L^q(\mathbb{R}^n)$, *and the following estimate holds*

$$\|p\|_q \le c\,(|\boldsymbol{v}|_{1,q} + \|\boldsymbol{f}\|_{-1,q})\,.$$

We then have

Theorem IV.2.2 *Given*

$$\boldsymbol{f} \in D_0^{-1,q}(\mathbb{R}^n),\ g \in L^q(\mathbb{R}^n), \quad 1 < q < \infty,\ n \ge 2,$$

there exists at least one q-generalized solution, $\boldsymbol{v}$ *to* (IV.2.7). *Moreover, denoting by* p *the pressure field associated to* $\boldsymbol{v}$ *by* Lemma IV.2.1, *we have*

$$|\boldsymbol{v}|_{1,q} + \|p\|_q \le c\,(|\boldsymbol{f}|_{-1,q} + \|g\|_q)\,. \tag{IV.2.25}$$

If $q \in (1,n)$, *then* $\boldsymbol{v} \in L^{nq/(n-q)}(\mathbb{R}^n)$ *and the following inequality holds*

$$\|\boldsymbol{v}\|_{nq/(n-q)} + |\boldsymbol{v}|_{1,q} + \|p\|_q \le c\,(|\boldsymbol{f}|_{-1,q} + \|g\|_q)\,. \tag{IV.2.26}$$

Finally, if $\boldsymbol{v}_1$ *is a* q_1*-generalized solution* ($1 < q_1 < \infty$, q_1 *possibly different from* q) *corresponding to the same* $\boldsymbol{f}$ *and* g, *it follows that* $\boldsymbol{v}_1 = \boldsymbol{v} + \boldsymbol{c}_1$ *a.e. in* $\mathbb{R}^n$, *for some constant vector* $\boldsymbol{c}_1$, *with* $\boldsymbol{c}_1 = 0$ *if* $q < n$ *and* $q_1 < n$, *and, denoting by* p_1 *the pressure field associated to* $\boldsymbol{v}_1$ *by* Lemma IV.2.1, *we have also* $p_1 = p + \text{const.}$ *a.e. in* $\mathbb{R}^n$.

Proof. We begin to observe that if we prove the existence of a q-generalized solution satisfying (IV.2.25), then the validity of (IV.2.26) follows from this equation and Theorem II.7.6. We next notice that it is enough to show the existence result for $\boldsymbol{f}$, $g \in C_0^\infty(\mathbb{R}^n)$ (f_i satisfying (II.8.10) when $q' \ge n$, for all $i = 1,\dots,n$). The general case will then follow by a standard density argument that uses (IV.2.25), the weak compactness property of spaces $D_0^{1,q}$ (see Exercise II.6.2), Theorem II.8.1 and the density of $C_0^\infty(\mathbb{R}^n)$ in $L^q(\mathbb{R}^n)$. Actually, given $\boldsymbol{f} \in D_0^{-1,q}(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, let $\{\boldsymbol{f}_k\}, \{g_k\} \subset C_0^\infty(\mathbb{R}^n)$ be two sequences approximating $\boldsymbol{f}$ and g. If existence of a solution $\{\boldsymbol{v}_k, p_k\}$ is established for each $\boldsymbol{f}_k$ and g_k, by (IV.2.25) and the weak compactness property of $D_0^{1,q}$ and L^q, $1 < q < \infty$ (see Exercise II.6.2 and Theorem II.2.4(ii)), we may find two fields $\boldsymbol{v} \in D_0^{1,q}(\mathbb{R}^n)$ and $p \in L^q(\mathbb{R}^n)$ such that, for all $\phi \in L^{q'}(\Omega)$,

$$\lim_{k\to\infty}(D_i(\boldsymbol{v}_k)_j, \phi) = (D_i v_j, \phi), \quad \lim_{k\to\infty}(p_k, \phi) = (p, \phi), \quad i,j = 1,\dots n,$$

and which, by Theorem II.2.4(i), obey (IV.2.25). Furthermore, since for any $k \in \mathbb{N}$

$$(\nabla \boldsymbol{v}_k, \nabla \boldsymbol{\psi}) - (p_k, \nabla\cdot\boldsymbol{\psi}) = -[\boldsymbol{f}_k, \boldsymbol{\psi}], \quad \text{for all } \boldsymbol{\psi} \in C_0^\infty(\mathbb{R}^n),$$

we take the limit $k \to \infty$ into this identity and use the density properties of C_0^∞ into $D_0^{-1,q}$ and L^q, thus proving existence in the general case. Therefore, we need to show existence for smooth $\boldsymbol{f}$ and g only. In such a case, we know that a solution to the problem is given by $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{h}$, $p = p_1 + p_2$, where $\boldsymbol{h}$ is defined in (IV.2.10) and $\boldsymbol{v}_1 = \boldsymbol{U} * \boldsymbol{f}$, $\boldsymbol{v}_2 = \boldsymbol{U} * \Delta\boldsymbol{h}$, $p_1 = -\boldsymbol{q} * \boldsymbol{f}$, $p_2 = -\boldsymbol{q} * \Delta\boldsymbol{h}$. From (IV.2.13) we obtain

$$|\boldsymbol{v}_2|_{1,q} + |\boldsymbol{h}|_{1,q} \le c_1 \|g\|_q \tag{IV.2.27}$$

with $c_1 = c_1(n,q)$. On the other hand, for fixed $\rho > 0$ and arbitrary $\varphi \in L^{q'}(B_\rho)$ we have (extending φ to zero in B_ρ^c)

$$\begin{aligned} \|D_i(\boldsymbol{v}_1)_j\|_{q,B_\rho} &= \sup_{\|\varphi\|_{q'}\le 1} |(D_i(\boldsymbol{v}_1)_j, \varphi)| \\ &= \sup_{\|\varphi\|_{q'}\le 1} \left| \int_{\mathbb{R}^n} f_r(y) \left[\int_{\mathbb{R}^n} D_i U_{rj}(x-y)\varphi(x)dx \right] dy \right| \end{aligned} \tag{IV.2.28}$$

for all $i, j = 1, \ldots, n$. From Theorem II.7.6 and the Calderón–Zygmund theorem it is easy to show that, for any i and j, the function $\phi_r \equiv D_i U_{rj} * \varphi$ belongs to $D_0^{1,q'}(\mathbb{R}^n)$ and that

$$\|\nabla \phi_r\|_{q',\mathbb{R}^n} \leq c_2 \|\varphi\|_{q',B_\rho},$$

where $c_2 = c_2(n, q)$. From this inequality, (IV.2.28), and the fact that f_r satisfies (II.8.10) if $q' \geq n$, we deduce

$$\|\nabla \boldsymbol{v}_1\|_{q,B_\rho} \leq c_3 |\boldsymbol{f}|_{-1,q}$$

which, since c_3 is independent of ρ, in the limit $\rho \to \infty$ yields

$$|\boldsymbol{v}_1|_{1,q} \leq c_3 |\boldsymbol{f}|_{-1,q}. \tag{IV.2.29}$$

with $c_3 = c_3(n, q)$. As a consequence, (IV.2.25) follows from (IV.2.27), (IV.2.29), and Lemma IV.2.1, and the existence proof is accomplished. It is worth emphasizing that the solution $\boldsymbol{v}, p$ just considered for $\boldsymbol{f}, g \in C_0^\infty(\mathbb{R}^n)$ is a smooth solution to (IV.2.7) and that it satisfies

$$\boldsymbol{v} \in D_0^{1,r}(\mathbb{R}^n), \quad p \in L^r(\mathbb{R}^n) \ \ \textit{for all}\ r \in (1, \infty).$$

With this in mind, we shall now show the uniqueness part. Let $\boldsymbol{v}_1$ be a q_1-generalized solution to (IV.2.7), corresponding to the same $\boldsymbol{f}$ and g. Setting

$$\boldsymbol{w} \equiv \boldsymbol{v}_1 - \boldsymbol{v},$$

from the definition of s-generalized solution it follows that

$$\begin{aligned} (\nabla \boldsymbol{w}, \nabla \boldsymbol{\phi}) &= 0, \quad \text{for all } \boldsymbol{\phi} \in \mathcal{D}_0^{1,q_1'}(\mathbb{R}^n) \cap \mathcal{D}_0^{1,q'}(\mathbb{R}^n) \\ (\boldsymbol{w}, \nabla \varphi) &= 0, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^n). \end{aligned} \tag{IV.2.30}$$

By what we have shown, given $\boldsymbol{F} \in C_0^\infty(\mathbb{R}^n)$, corresponding to $\boldsymbol{f} \equiv \boldsymbol{F}$, $g \equiv \phi \equiv 0$ there exists a smooth solution $\boldsymbol{u}$, τ to (IV.2.7), which further satisfies

$$(\boldsymbol{u}, \tau) \in \mathcal{D}_0^{1,r}(\mathbb{R}^n) \times L^r(\mathbb{R}^n), \quad \textit{for all}\ r \in (1, \infty). \tag{IV.2.31}$$

If $r \leq n/(n-1)$, the function $\boldsymbol{F}$ must verify (II.8.10). We now multiply $(\text{IV.2.7})_1$, written for $\boldsymbol{u}$ and τ, by $\psi_R \boldsymbol{w}$ where ψ_R is the Sobolev "cut-off" function defined in (II.7.1). Integrating by parts over $\mathbb{R}^n$, with the help of Exercise II.4.3, we deduce for all sufficiently large R:

$$\int_{\mathbb{R}^n} \psi_R \nabla \boldsymbol{u} : \nabla \boldsymbol{w} = -\int_{\mathbb{R}^n} (\nabla \psi_R \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{w} - \tau \nabla \psi_R \boldsymbol{w}) - (\boldsymbol{F}, \boldsymbol{w}). \tag{IV.2.32}$$

By the Hölder inequality, we have

$$\left|\int_{\mathbb{R}^n} (\nabla\psi_R \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{w} - \tau \nabla\psi_R \boldsymbol{w})\right| \leq \left(|\boldsymbol{u}|_{1,q_1'} + \|\tau\|_{q_1'}\right) \|\nabla\psi_R \boldsymbol{v}_1\|_{q_1,\widetilde{\Omega}_R}$$
$$+ \left(|\boldsymbol{u}|_{1,q'} + \|\tau\|_{q'}\right) \|\nabla\psi_R \boldsymbol{v}\|_{q,\widetilde{\Omega}_R}, \tag{IV.2.33}$$

where $\widetilde{\Omega}_R$ is defined in (II.7.3) and contains the support of $\nabla\psi_R$. As shown in the proof of Theorem II.7.1,

$$\|\nabla\psi_R \boldsymbol{v}_1\|_{q_1\widetilde{\Omega}_R} + \|\nabla\psi_R \boldsymbol{v}\|_{q,\widetilde{\Omega}_R} \to 0 \quad \text{as } R \to \infty \tag{IV.2.34}$$

and so, letting $R \to \infty$ into (IV.2.32), from (IV.2.31), (IV.2.33), and (IV.2.34) it follows that

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{w}) = (\boldsymbol{F}, \boldsymbol{w}). \tag{IV.2.35}$$

Because of (IV.2.30), we may now take $\boldsymbol{\varphi} = \boldsymbol{u}$ into (IV.2.30) and use (IV.2.35) to find

$$(\boldsymbol{F}, \boldsymbol{w}) = 0, \tag{IV.2.36}$$

which, by the arbitrarity of $\boldsymbol{F}$, in turn implies $\boldsymbol{w} \equiv 0$ *a.e.* in Ω, if both q_1 and q are strictly less than n. Otherwise, since $\boldsymbol{F}$ has to satisfy (II.8.10), we obtain $\boldsymbol{w}$ =const. *a.e.* in Ω. From (IV.2.24) we then recover τ = const. *a.e.* in Ω, which completes the proof of the theorem. □

IV.3 Existence, Uniqueness, and L^q-Estimates in a Half-Space. Evaluation of Green's Tensor

In this section we shall prove results similar to those of Theorem IV.2.1 and Theorem IV.2.2 for the inhomogeneous Stokes problem in the half-space $\mathbb{R}^n_+$, $n \geq 2$. Here the situation is complicated by the fact that the domain has a boundary, even if a simple one. We begin to study the problem

$$\left.\begin{aligned} \Delta \boldsymbol{W} &= \nabla S \\ \nabla \cdot \boldsymbol{W} &= 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^n_+$$
$$\boldsymbol{W} = \boldsymbol{\Phi} \quad \text{at } \Sigma \equiv \{x \in \mathbb{R}^n : x_n = 0\}, \tag{IV.3.1}$$

where $\boldsymbol{\Phi} \in C^m(\Sigma)$ for some $m \geq 1$, $\boldsymbol{\Phi} = O(\log|\xi|)$ as $|\xi| \to \infty$,[1] and $D^\alpha \boldsymbol{\Phi} \in C(\Sigma)$, $1 \leq |\alpha| \leq m$. To this end, we introduce with Odqvist (1930, §2) the *Stokes double-layer potentials (for the half-space)*

$$W_j(x) = 2\int_\Sigma \Phi_i(y) \left[-\delta_{ik} q_j(x-y) + \frac{\partial U_{ij}(x-y)}{\partial y_k} + \frac{\partial U_{kj}(x-y)}{\partial y_i}\right] n_k d\sigma_y$$
$$S(x) = -4\int_\Sigma \Phi_i(y) \frac{\partial q_k(x-y)}{\partial y_i} n_k d\sigma_y, \tag{IV.3.2}$$

[1] We could allow $\boldsymbol{\Phi}$ to "grow" faster. Such a weaker assumption, however, would be unessential for further purposes.

where $\boldsymbol{n}$ $(= -\boldsymbol{e}_n)$ is the outer normal to Σ.[2] Recalling the expressions (IV.2.3) and (IV.2.4) of the Stokes fundamental solution, (IV.3.2) can be rewritten as

$$
\begin{aligned}
W_j(x) &= \int_\Sigma K_{ij}(x'-y',x_n)\Phi_i(y')dy' \\
S(x) &= -D_i \int_\Sigma k(x'-y',x_n)\Phi_i(y')dy'
\end{aligned}
\tag{IV.3.3}
$$

with $z' = (z_1, \ldots, z_{n-1})$ and

$$
\begin{aligned}
K_{ij}(x'-y',x_n) &= \frac{2}{\omega_n}\frac{x_n(x_i-y_i)(x_j-y_j)}{(|x'-y'|^2+x_n^2)^{(n+2)/2}}, \quad y_n = 0, \\
k(x'-y',x_n) &= \frac{4}{n\omega_n}\frac{x_n}{(|x'-y'|^2+x_n^2)^{n/2}}, \quad y_n = 0.
\end{aligned}
\tag{IV.3.4}
$$

We easily show that $\boldsymbol{W}$ and S are C^∞ solutions to $(\text{IV.3.1})_{1,2}$. In fact, it is clear that $\boldsymbol{W}$ and S are smooth; in addition, since $\boldsymbol{q}$ is harmonic (for $x \neq y$) from (IV.2.5) and $(\text{IV.3.2})_1$ we find

$$
\Delta W_j(x) = -2\int_\Sigma \Phi_i(y)\left[\frac{\partial^2 q_j(x-y)}{\partial y_k \partial x_i} + \frac{\partial^2 q_j(x-y)}{\partial x_k \partial y_i}\right] n_k d\sigma_y
$$

and, by $(\text{IV.2.2})_2$,

$$
\begin{aligned}
\frac{\partial W_j(x)}{\partial x_j} &= 2\int_\Sigma \Phi_i(y)\left[-\delta_{ik}\frac{\partial q_j(x-y)}{\partial x_j} + \frac{\partial^2 U_{ij}(x-y)}{\partial x_j \partial y_k} + \frac{\partial^2 U_{kj}(x-y)}{\partial x_j \partial y_i}\right] n_k d\sigma_y \\
&= -2\int_\Sigma \Phi_n(y)\frac{\partial q_j(x-y)}{\partial x_j} d\sigma_y\,.
\end{aligned}
$$

However, it is immediately checked that

$$
\frac{\partial q_i}{\partial x_j} = \frac{\partial q_j}{\partial x_i}, \quad \frac{\partial q_i}{\partial y_j} = \frac{\partial q_j}{\partial y_i}, \quad \frac{\partial q_i}{\partial y_j} = -\frac{\partial q_i}{\partial x_j}, \quad \frac{\partial q_i}{\partial x_i} = \frac{\partial q_i}{\partial y_i} = 0\,, \quad x \neq y\,,
$$

and so,we deduce that $(\text{IV.3.1})_{1,2}$ are satisfied. Also, for all $x' \in \mathbb{R}^{n-1}$ we can prove

$$
\lim_{x_n \to 0} \boldsymbol{W}(x',x_n) = \boldsymbol{\Phi}(x'). \tag{IV.3.5}
$$

Actually, for fixed $\xi \in \mathbb{R}^{n-1}$, we take an $n-1$-dimensional ball C_ε, centered at ξ such that

$$
\sup_{y \in C_\varepsilon} |\boldsymbol{\Phi}(\xi) - \boldsymbol{\Phi}(y)| < \varepsilon. \tag{IV.3.6}
$$

By a direct calculation based on $(\text{IV.3.4})_1$, one shows that the following relations hold:

[2] The functions (IV.3.2) are the analogue of the familiar Poisson integral for the Dirichlet problem for Laplace equation in the half-space, considered at the end of Section II.11.

(i) $\displaystyle\int_{C_\varepsilon} K_{ij}(\xi - y', x_n)dy' = \delta_{ij} + o(1) \quad \text{as } x_n \to 0,$

(ii) $\displaystyle\int_{\Sigma} |K_{ij}(\xi - y', x_n)|dy' \le c$

with c independent of x_n and ξ. Likewise, using the growth properties of $\boldsymbol{\Phi}$, we obtain

(iii) $\displaystyle\int_{\Sigma - C_\varepsilon} K_{ij}(\xi - y', x_n)\Phi_i(y')dy' = o(1) \quad \text{as } x_n \to 0.$

Therefore, using (i) and (iii) we recover as $x_n \to 0$

$$\begin{aligned} W_j(\xi, x_n) - \Phi_j(\xi) =& \int_{C_\varepsilon} K_{ij}(\xi - y', x_n)\Phi_i(y')dy' - \Phi_j(\xi) \\ &+ \int_{\Sigma - C_\varepsilon} K_{ij}(\xi - y', x_n)\Phi_i(y')dy' \\ =& \int_{C_\varepsilon} K_{ij}(\xi - y', x_n)[\Phi_i(y') - \Phi_i(\xi)]dy' + o(1), \end{aligned}$$

which, in view of (IV.3.6) and (ii), in turn implies

$$\limsup_{x_n \to 0} |\boldsymbol{W}(\xi, x_n) - \boldsymbol{\Phi}(\xi)| \le c\,\varepsilon.$$

By the arbitrarity of ε, we deduce (IV.3.5).

We wish to determine some L^q-estimates for $\boldsymbol{W}$ and S in terms of $\boldsymbol{\Phi}$ analogous to those derived for the Dirichlet problem for the Laplace equation at the end of Section II.11, and which for problem (IV.3.1) were proved for the first time by Cattabriga (1961). To be specific, we shall deal with the case $n = 3$, the general case being treated similarly. For $|\alpha| \le m$, from (IV.3.3) we have

$$\begin{aligned} D'^\alpha W_j(x) &= \int_\Sigma K_{ij}(x' - y', x_n)D'^\alpha \Phi_i(y')dy' \\ D'^\alpha S(x) &= -D_i \int_\Sigma k(x' - y', x_n)D'^\alpha \Phi_i(y')dy', \end{aligned} \tag{IV.3.7}$$

where

$$D'^\alpha \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_{n-1}^{\alpha_{n-1}}}, \quad \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1} = |\alpha|. \tag{IV.3.8}$$

We observe that from (IV.3.4) it follows that $K_{ij}(x', x_3)$, $i, j = 1, 2, 3$, and $k(x', x_3)$ are of class C^∞ for $x_3 > 0$, with bounded derivatives of any order in the hemisphere $\{|x|^2 = 1, x_3 > 0\}$. Moreover, since

$$K_{ij}(x', x_3) = \frac{3}{2\pi} \frac{\dfrac{x_i}{|x|} \dfrac{x_j}{|x|} \dfrac{x_3}{|x|}}{|x|^2} \equiv \frac{\Omega_{ij}\left(\dfrac{x'}{|x|}, \dfrac{x_3}{|x|}\right)}{|x|^2}$$

$$k(x', x_3) = \frac{1}{\pi} \frac{\dfrac{x_3}{|x|}}{|x|^2} \equiv \frac{\omega\left(\dfrac{x'}{|x|}, \dfrac{x_3}{|x|}\right)}{|x|^2}$$

and observing that

$$\Omega_{ij}(x', 0) = \omega(x', 0) = 0 \ \text{ for all } x' \neq 0,$$

we conclude that K_{ij} and k satisfy all assumptions of Theorem II.11.6. So, if $D^\alpha \boldsymbol{\Phi}$ is in $L^q(\Sigma)$ and has finite $\langle\langle D^\alpha \boldsymbol{\Phi} \rangle\rangle_{1-1/q,q}$ norm, $1 < q < \infty$, we obtain

$$\nabla D'^\alpha \boldsymbol{W}, \quad D'^\alpha S \in L^q(\mathbb{R}^n_+)$$

together with the following estimate

$$|D'^\alpha \boldsymbol{W}|_{1,q} + \|D'^\alpha S\|_q \leq c \, \langle\langle D^\alpha \boldsymbol{\Phi} \rangle\rangle_{1-1/q,q}, \tag{IV.3.9}$$

where c depends only on q, n (=3), and α. A similar inequality also valid for x_3-derivatives can be easily obtained using (IV.3.9) and the fact that $\boldsymbol{W}, S$ is a solution to (IV.3.1). Let us consider the case $|\alpha| = 1$. Differentiating $(\text{IV.3.1})_2$ with respect to x_3 and employing (IV.3.9), we have

$$\|D_3^2 W_3\|_q \leq \|D_3 D_2 W_2\|_q + \|D_3 D_1 W_1\|_q \leq 2c \langle\langle \nabla \boldsymbol{\Phi} \rangle\rangle_{1-1/q,q}. \tag{IV.3.10}$$

Moreover, setting

$$\boldsymbol{W}' = (W_1, W_2), \quad \Delta' = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \quad \nabla' = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$$

from $(\text{IV.3.1})_1$ we obtain

$$\Delta W_3 = D_3 S$$

$$D_3^2 \boldsymbol{W}' = -\Delta' \boldsymbol{W}' + \nabla' S$$

and thus, again by (IV.3.9),

$$\|D_3^2 \boldsymbol{W}'\|_q + \|D_3 S\|_q \leq 8c \langle\langle \nabla \boldsymbol{\Phi} \rangle\rangle_{1-1/q,q},$$

which, along with (IV.3.10), proves the desired estimate; that is,

$$|\boldsymbol{W}|_{2,q} + |S|_{1,q} \leq c \langle\langle \nabla \boldsymbol{\Phi} \rangle\rangle_{1-1/q,q},$$

where c depends only on n, α, and q. More generally, if $|\alpha| \geq 1$, one can use a similar argument (which we leave to the reader) to show the inequality

$$|\boldsymbol{W}|_{k+1,q} + |\boldsymbol{S}|_{k,q} \le c \sum_{|\alpha|=k} \langle\langle D^\alpha \boldsymbol{\Phi} \rangle\rangle_{1-1/q,q}, \tag{IV.3.11}$$

where $0 \le k \le m$. If $q \in (1, n)$ we may obtain a sharper estimate. In fact, from Theorem II.6.3(i) and (IV.3.11), we deduce, in particular, the existence of constant vectors $\boldsymbol{W}^{(\alpha)}$ such that

$$\sum_{|\alpha|=k} \|D^\alpha \boldsymbol{W} - \boldsymbol{W}^{(\alpha)}\|_{nq/(n-q)} \le c \sum_{|\alpha|=k} \langle\langle D^\alpha \boldsymbol{\Phi} \rangle\rangle_{1-1/q,q} \, .$$

However, by direct inspection, from $(\mathrm{IV}.3.3)_1$ and $(\mathrm{IV}.3.5)_1$, we have $D^\alpha \boldsymbol{W} \to \boldsymbol{0}$ as $x_n \to \infty$, $|\alpha| \ge 0$ which, in turn, implies (as the reader will easily show) $\boldsymbol{W}^{(\alpha)} = \boldsymbol{0}$. Therefore, we conclude

$$|\boldsymbol{W}|_{k,nq/(n-q)} + |\boldsymbol{W}|_{k+1,q} + |\boldsymbol{S}|_{k,q} \le c \sum_{|\alpha|=k} \langle\langle D^\alpha \boldsymbol{\Phi} \rangle\rangle_{1-1/q,q}, \quad \text{if } q \in (1, n) \, . \tag{IV.3.12}$$

The results proved so far continue to hold if we weaken somewhat the regularity assumptions made on $\boldsymbol{\Phi}$. For later purposes (see Section IV.5) it is interesting to consider the case when

$$\boldsymbol{\Phi} \in W^{m,q}(\Sigma) \quad \text{with} \sum_{|\alpha|=m} \langle\langle D^\alpha \boldsymbol{\Phi} \rangle\rangle_{1-1/q,q} < \infty.$$

Under these hypotheses it is simple to show that (IV.3.2) is still a C^∞-solution of $(\mathrm{IV}.3.1)_{1,2}$ though, of course, the boundary value is now attained a priori in a way less regular than (IV.3.5), that is (see Exercise IV.3.1),

$$\lim_{x_n \to 0} \int_{\mathcal{C}} |\boldsymbol{W}(x', x_n) - \boldsymbol{\Phi}(x')|^q dx' = 0, \tag{IV.3.13}$$

where $\mathcal{C}$ is any compact subset of Σ. Furthermore, since by the results of Exercise II.10.1 it follows that $\langle\langle D^\alpha \boldsymbol{\Phi} \rangle\rangle_{1-1/q,q}$ is finite for all $|\alpha| \in [0, m]$, we conclude that

$$D^\alpha \nabla \boldsymbol{W}, \quad D^\alpha S \in L^q(\mathbb{R}^n_+),$$

and that inequality (IV.3.11) holds for these values of α.

We may thus summarize all previous results in the following.

Lemma IV.3.1 *Let $\boldsymbol{\Phi} \in C^m(\Sigma)$, $m \ge 1$, with $\boldsymbol{\Phi}(\xi) = O(\log|\xi|)$ as $|\xi| \to \infty$ and $D^\alpha \boldsymbol{\Phi} \in C(\Sigma)$, $1 \le |\alpha| \le m$. Then the functions $\boldsymbol{W}$, S defined by* (IV.3.3), (IV.3.4) *are of class C^∞ in $\mathbb{R}^n_+$ and satisfy there* (IV.3.1) *and* (IV.3.5). *Moreover, if $\boldsymbol{\Phi} \in D^{k,q}(\Sigma)$ and $\sum_{|\alpha|=k} \langle\langle D^\alpha \boldsymbol{\Phi} \rangle\rangle_{1-1/q,q}$ is finite for some integer $k \in [0, m]$, $1 < q < \infty$, then*

(i) *$|\boldsymbol{W}|_{k+1,q}$ and $|S|_{k,q}$ are finite, and, if $q \in (1, n)$, also $|\boldsymbol{W}|_{k,nq/(n-q)}$ is finite;*

(ii) $\boldsymbol{W}$, S *satisfy inequalities* (IV.3.11), (IV.3.12).

Likewise, let

$$\boldsymbol{\Phi} \in W^{m,q}(\Sigma) \quad \textit{with} \sum_{|\alpha|=m} \langle\langle D^\alpha \boldsymbol{\Phi} \rangle\rangle_{1-1/q,q} < \infty.$$

Then $\boldsymbol{W}$, S *satisfy* (IV.3.1), (IV.3.13) *and statements* (i) *and* (ii) *hold for all integers* $k \in [0, m]$.

Exercise IV.3.1 Show the validity of condition (IV.3.13). *Hint:* Use the same arguments adopted in the proof of (IV.3.5).

We shall next consider the problem

$$\left.\begin{aligned} \Delta \boldsymbol{w} &= \nabla s + \boldsymbol{f} \\ \nabla \cdot \boldsymbol{w} &= g \end{aligned}\right\} \text{ in } \mathbb{R}^n_+ \qquad \text{(IV.3.14)}$$
$$\boldsymbol{v} = 0 \quad \text{at } \Sigma,$$

where $\boldsymbol{f}$, $g \in C_0^\infty(\overline{\mathbb{R}^n_+})$ and shall prove the existence of a solution verifying suitable estimates in terms of the data. This will be done by reducing (IV.3.14) to (IV.3.1) and then using Lemma IV.3.1. First, we make extensions $\boldsymbol{f}_r$ and g_r of $\boldsymbol{f}$ and g to the whole of $\mathbb{R}^n$ in the way suggested in Exercise II.3.10, so that $\boldsymbol{f}_r$, $g_r \in C_0^{r+1}(\mathbb{R}^n)$ for sufficiently large r and

$$\begin{aligned} \|D^\beta \boldsymbol{f}_r\|_{q,\mathbb{R}^n} &\le c \|D^\beta \boldsymbol{f}\|_{q,\mathbb{R}^n_+} \\ \|D^\beta g_r\|_{q,\mathbb{R}^n} &\le c \|D^\beta g\|_{q,\mathbb{R}^n_+} \end{aligned} \qquad 0 \le |\beta| \le r+1, \qquad \text{(IV.3.15)}$$

where c depends only on r, n, and q. Successively, we look for a solution of the form

$$\boldsymbol{w} = \boldsymbol{w}_1 + \widetilde{\boldsymbol{W}}, \quad s = s_1 + \widetilde{S}, \qquad \text{(IV.3.16)}$$

where $\boldsymbol{w}_1$, s_1 is the solution to (IV.2.7) with $\boldsymbol{f} \equiv \boldsymbol{f}_r$ and $g \equiv g_r$, and whose existence is ensured by Theorem IV.2.1, while $\widetilde{\boldsymbol{W}}, \widetilde{S}$ solve

$$\left.\begin{aligned} \Delta \widetilde{\boldsymbol{W}} &= \nabla \widetilde{S} \\ \nabla \cdot \widetilde{\boldsymbol{W}} &= 0 \end{aligned}\right\} \text{ in } \mathbb{R}^n_+ \qquad \text{(IV.3.17)}$$
$$\widetilde{\boldsymbol{W}} = -\boldsymbol{w}_1 \quad \text{at } \Sigma.$$

We shall show that $\boldsymbol{\Phi} \equiv -\boldsymbol{w}_1|_\Sigma$ satisfies the assumptions of the first part of Lemma IV.3.1. Therefore, by that lemma, $\widetilde{\boldsymbol{W}}, \widetilde{S}$ will obey, in particular, estimate (IV.3.11). From Theorem IV.2.1 it follows that $\boldsymbol{\Phi} \in C^\infty(\Sigma)$. Moreover, $\boldsymbol{\Phi}(\xi) = O(\log|\xi|)$ and $D^\alpha \boldsymbol{\Phi} \in C^0(\Sigma)$ for all $|\alpha| \in [1, s]$ if $n = 2$, while

$\boldsymbol{\Phi} \in C^m(\Sigma)$ for all $m \in [0,s]$ if $n=3$. Let us now prove that for any $|\alpha| \in [1,s]$ and any $q>1$

$$
\begin{aligned}
&D^\alpha \boldsymbol{\Phi} \in L^q(\Sigma), \\
&\langle\langle D^\alpha \boldsymbol{\Phi} \rangle\rangle_{1-1/q} < \infty,
\end{aligned}
\tag{IV.3.18}
$$

so that all the hypotheses of the first part of Lemma IV.3.1 are fulfilled. Recalling that $\boldsymbol{w}_1$ is given by $(\text{IV.2.8})_1$ with $\boldsymbol{F} \equiv \boldsymbol{f}_r - \Delta \boldsymbol{h}_r = \boldsymbol{f}_r - \nabla g_r$ where $\boldsymbol{h}_r$ is given by (IV.2.10) with g_r in place of g, from $(\text{IV.2.12})_2$ we find

$$
D^\alpha \boldsymbol{\Phi}(x') = O(|x'|^{-n+1}) \ \text{ as } |x'| \to \infty.
$$

This property, along with the fact that $D^\alpha \boldsymbol{\Phi} \in C^0(\Sigma)$, implies $(\text{IV.3.18})_1$. On the other hand, using Theorem II.10.2 and Theorem IV.2.1, it follows that, for all $|\alpha| \geq 0$,

$$
\begin{aligned}
\langle\langle D^\alpha \nabla \boldsymbol{w}_1 \rangle\rangle_{1-1/q,q} &\leq c\, \|D^\alpha \nabla \boldsymbol{w}_1\|_{q,\mathbb{R}^n_+} \\
&\leq c\left(\|D^\alpha \boldsymbol{f}_r\|_{q,\mathbb{R}^n} + \|D^\alpha g_r\|_{q,\mathbb{R}^n}\right).
\end{aligned}
\tag{IV.3.19}
$$

We may thus apply Lemma IV.3.1 and use (IV.3.16), (IV.3.19), (IV.3.15), and Theorem IV.2.1 to obtain for all $\ell \geq 0$

$$
\begin{aligned}
|\boldsymbol{w}|_{\ell+2,q} + |s|_{\ell+1,q} &\leq |\boldsymbol{w}_1|_{\ell+2,q} + |s_1|_{\ell+1,q} + |\widetilde{\boldsymbol{W}}|_{\ell+2,q} + |\widetilde{S}|_{\ell+1,q} \\
&\leq c\left(|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q}\right),
\end{aligned}
\tag{IV.3.20}
$$

where c depends only on q, ℓ, and n. As in the proof of the previous lemma, we observe that, if $q \in (1,n)$, from Theorem II.6.3, the fact that $D^\alpha \boldsymbol{w} \to \boldsymbol{0}$ as $x_n \to \infty$, $|\alpha| \geq 0$, and (IV.3.20), we deduce that $|\boldsymbol{w}|_{\ell+1,nq/(n-q)}$ is finite and that

$$
|\boldsymbol{w}|_{\ell+1,nq/(n-q)} \leq c\left(|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q}\right). \tag{IV.3.21}
$$

By using exactly the same procedure employed for Theorem IV.2.1, based on the density of $C_0^\infty(\overline{\mathbb{R}^n_+})$ in $W^{l,t}(\mathbb{R}^n_+)$, inequalities (IV.3.20) and (IV.3.21), and the functional properties of spaces $\dot{D}^{m,q}(\mathbb{R}^n_+)$, we can extend the results just proved to the case where

$$
\boldsymbol{f} \in W^{m,q}(\mathbb{R}^n_+), \quad g \in D^{m+1,q}(\mathbb{R}^n_+), \quad m \geq 0, \ 1<q<\infty. \tag{IV.3.22}
$$

Clearly, the corresponding solution, that we continue to denote by $\boldsymbol{w}$, s, is such that

$$
\boldsymbol{w} \in \bigcap_{\ell=0}^{m} D^{\ell+2,q}(\mathbb{R}^n_+), \quad s \in \bigcap_{\ell=0}^{m} D^{\ell+1,q}(\mathbb{R}^n_+), \tag{IV.3.23}
$$

and, if $q \in (1,\infty)$,

$$
\boldsymbol{w} \in \bigcap_{\ell=0}^{m} D^{\ell+1,nq/(n-q)}(\mathbb{R}^n_+). \tag{IV.3.24}
$$

Thus, in particular, by Lemma II.6.1,

$$\boldsymbol{w} \in W^{m+2,q}(C), \quad s \in W^{m+1,q}(C),$$

for all (open) cubes $C \subset \mathbb{R}^n_+$. Finally, $\boldsymbol{w}$ and s satisfy (IV.3.20), and, if $q \in (1,n)$, (IV.3.21) for all $\ell \in [0,m]$, and $\boldsymbol{w}$ has zero trace at the boundary.

We have therefore proved

Lemma IV.3.2 *For any $\boldsymbol{f}, g$ in the class defined by* (IV.3.22), *there exists a solution $\boldsymbol{w}, s$ a.e. to the nonhomogeneous Stokes system* (IV.3.14) *such that*

$$\boldsymbol{w} \in W^{m+2,q}(C), \quad s \in W^{m+1,q}(C)\,,$$

for all open cubes $C \subset \mathbb{R}^n_+$. Moreover, $\boldsymbol{w}, s$ satisfy (IV.3.23) *and, if $q \in (1,n)$, also* (IV.3.24). *In addition, for every $\ell \in [0,m]$, the following inequality holds:*

$$|\boldsymbol{w}|_{\ell+2,q} + |s|_{\ell+1,q} \le c\,(|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q})\;. \tag{IV.3.25}$$

If $q \in (1,n)$, we also have:

$$|\boldsymbol{w}|_{\ell+1,nq/(n-q)} + |\boldsymbol{w}|_{\ell+2,q} + |s|_{\ell+1,q} \le c\,(|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q})\;. \tag{IV.3.26}$$

In the above inequalities, $c = c(n,\ell,q)$.

The next step is to prove uniqueness for such solutions. In this respect, the general result proved in the following theorem is appropriate.

Theorem IV.3.1 *Let $\boldsymbol{u} \in W^{m+2,q}(C)$, $\pi \in W^{m+1,q}(C)$ ($m \ge 0, C$ arbitrary open cube in $\mathbb{R}^n_+$, $n \ge 2$) be a solution a.e. to the Stokes system* (IV.3.14) *with $\boldsymbol{f} \equiv g \equiv 0$. Assume $|\boldsymbol{u}|_{\ell+2,q}$ finite for some $\ell \ge 0$ and some $q \in (1,\infty)$. Then*

$$|\boldsymbol{u}|_{\ell+2,q} = |\pi|_{\ell+1,q} = 0.$$

In particular, if $\ell = 0$, then

$$\boldsymbol{u} = \boldsymbol{a}\,x_n, \quad \pi = \text{const.}$$

with $\boldsymbol{a} = (a_1, \dots, a_{n-1}, 0)$ constant vector.

Proof. To fix the ideas, we shall consider the case $n = 3$ and $\ell = 0$, the general case being handled in a completely analogous way. As in the proof of Theorem IV.2.1, we obtain at once that π is harmonic throughout the half-space. Let $\boldsymbol{u}_{\varepsilon'}$ and $\pi_{\varepsilon'}$be the regularizations of $\boldsymbol{u}$ and π, respectively, with respect to $x' = (x_1, x_2)$. It is readily shown that, for all $\varepsilon' > 0$, $\boldsymbol{u}_{\varepsilon'}$ and $\pi_{\varepsilon'}$ satisfy the same boundary-value problem as $\boldsymbol{u}, \pi$ and that

$$\boldsymbol{w} \equiv D'^2 \boldsymbol{u}_{\varepsilon'}, \quad s \equiv D'^2 \pi_{\varepsilon'}$$

with D' defined in (IV.3.8), are solutions to the following problem

$$\left.\begin{aligned} \Delta \boldsymbol{w} &= \nabla s \\ \nabla \cdot \boldsymbol{w} &= 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3_+ \tag{IV.3.27}$$

$$\boldsymbol{w} = 0 \quad \text{at } \Sigma \equiv \mathbb{R}^2 \times \{0\}.$$

By assumption and the properties of mollifiers one shows

$$\boldsymbol{w}, s, D^2\boldsymbol{w}, \nabla s \in L^q(\mathbb{R}^3_+)$$

and, therefore, by a simple interpolation,

$$\boldsymbol{w} \in W^{2,q}(\mathbb{R}^3_+);$$

(see Exercise IV.3.2). Let now $\boldsymbol{f} \in C_0^\infty(\overline{\mathbb{R}}^3_+)$ and let $\boldsymbol{v}, p$ be the corresponding solution determined in Lemma IV.3.2 with $g \equiv 0$. Evidently,

$$\boldsymbol{v}_1 \equiv D'^2\boldsymbol{v}, \quad p_1 \equiv D'^2 p$$

solve (IV.3.14) with $\boldsymbol{f}$ replaced by $D'^2\boldsymbol{f}$ and $g \equiv 0$. Moreover,

$$\boldsymbol{v}_1 \in W^{2,q'}(\mathbb{R}^3_+), \quad p_1 \in W^{1,q'}(\mathbb{R}^3_+).$$

We next multiply $(\text{IV.3.27})_1$ by $\boldsymbol{v}_1$, integrate by parts, and use Theorem III.1.2 to obtain

$$0 = \int_{\mathbb{R}^3_+} \boldsymbol{w} \cdot D'^2\boldsymbol{f} = \int_{\mathbb{R}^3_+} D'^2\boldsymbol{w} \cdot \boldsymbol{f}.$$

Since $\boldsymbol{f}$ is arbitrary from $C_0^\infty(\overline{\mathbb{R}}^3_+)$ and $\boldsymbol{w} \in L^q(\mathbb{R}^3_+)$, from this relation and (IV.3.27) we derive, in particular, for all $x \in \mathbb{R}^3_+$ and all $\varepsilon' > 0$

$$\begin{aligned} &\boldsymbol{w}(x) \equiv D'^2\boldsymbol{u}_{\varepsilon'}(x) = 0 \\ &D'^2\pi_{\varepsilon'}(x) = c_{\varepsilon'}, \end{aligned} \tag{IV.3.28}$$

where $c_{\varepsilon'}$ depends only on ε'. Since $\pi_{\varepsilon'}$ is harmonic throughout $\mathbb{R}^3_+$ with $\nabla\pi_{\varepsilon'} \in L^q(\mathbb{R}^3_+)$ we deduce with the help of $(\text{IV.3.28})_2$ that $\pi_{\varepsilon'}$ must be constant with respect to x, for all $\varepsilon' > 0$, which finally gives $\pi = \text{const}$. This being established, from

$$\Delta\boldsymbol{u}_{\varepsilon'} = \nabla\pi_{\varepsilon'}$$

it follows that $\boldsymbol{u}_{\varepsilon'}$ is harmonic throughout the half-space. Recalling that $D^2\boldsymbol{u} \in L^q(\mathbb{R}^3_+)$ and $\boldsymbol{u}_{\varepsilon'} = 0$ at Σ, from the property of mollifiers and Remark II.11.3 it is easy to show that $D^2\boldsymbol{u} \equiv 0$, i.e., $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{U}_0 \cdot \boldsymbol{x}$, where $\boldsymbol{u}_0$ and $\boldsymbol{U}_0$ are a constant vector and a constant 3×3 matrix, respectively. Since $\boldsymbol{u}$ is zero at Σ and $\nabla \cdot \boldsymbol{u} = 0$ in $\mathbb{R}^3_+$ one obtains, in conclusion, $\boldsymbol{u} = x_3(a_1, a_2, 0)$, for some $a_1, a_2 \in \mathbb{R}$. The proof of the theorem is thus complete. □

On the strength of the results shown so far, we can prove the following theorem, which represents the analogue of Theorem IV.2.1 for the half-space.

Theorem IV.3.2 *Let $\Sigma = \{x \in \mathbb{R}^n : x_n = 0\}$. For every*

$$\boldsymbol{f} \in W^{m,q}(\mathbb{R}^n_+), \quad g \in D^{m+1,q}(\mathbb{R}^n_+)$$

and

$$\boldsymbol{\Phi} \in W^{m+1,q}(\Sigma) \quad \text{with} \quad \sum_{|\alpha|=m+1} \langle\langle D^\alpha \boldsymbol{\Phi} \rangle\rangle_{1-1/q,q} \quad \text{finite},$$

$m \geq 0$, $1 < q < \infty$, $n \geq 2$, there exists a pair of functions $\boldsymbol{v}, p$ such that

$$\boldsymbol{v} \in W^{m+2,q}(C), \quad p \in W^{m+1,q}(C),$$

for all open cubes $C \subset \mathbb{R}^n_+$, solving a.e. the following nonhomogeneous Stokes system

$$\left.\begin{aligned} \Delta \boldsymbol{v} &= \nabla p + \boldsymbol{f} \\ \nabla \cdot \boldsymbol{v} &= g \end{aligned}\right\} \quad \text{in } \mathbb{R}^n_+ \qquad \text{(IV.3.29)}$$

$$\boldsymbol{v} = \boldsymbol{\Phi} \quad \text{at } \Sigma.$$

Moreover, for all $\ell \in [0, m]$, the seminorms $|\boldsymbol{v}|_{\ell+2,q}$ and $|p|_{\ell+1,q}$ are finite and we have

$$|\boldsymbol{v}|_{\ell+2,q} + |p|_{\ell+1,q} \leq c\Big(|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q} + \sum_{|\alpha|=\ell+1} \langle\langle D^\alpha \phi \rangle\rangle_{1-1/q,q}\Big)\,. \qquad \text{(IV.3.30)}$$

If, in particular, $q \in (1, n)$, then also $|\boldsymbol{v}|_{\ell+1,nq/(n-q)}$ is finite, and, for all $\ell \in [0, m]$, we have

$$|\boldsymbol{v}|_{\ell+1,nq/(n-q)} + |\boldsymbol{v}|_{\ell+2,q} + |s|_{\ell+1,q} \leq c\Big(|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q} + \sum_{|\alpha|=\ell+1} \langle\langle D^\alpha \phi \rangle\rangle_{1-1/q,q}\Big)\,. \qquad \text{(IV.3.31)}$$

In the above inequalities $c = c(n, q, m)$. Furthermore, if $\boldsymbol{v}_1$, p_1 is another solution to (IV.3.29) *corresponding to the same data and, for some $\ell \in [0, m]$, $|\boldsymbol{v}_1|_{\ell+2,q}$ is finite, then $|\boldsymbol{v} - \boldsymbol{v}_1|_{\ell+2,q} = |p - p_1|_{\ell+1,q} = 0$. In particular, if $\ell = 0$, there exists a vector $\boldsymbol{a} = (a_1, \ldots, a_{n-1}, 0)$ such that $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{a}x_n$, $p = p_1 + \text{const}$. Finally, if $\ell = 0$, $q \in (1, n)$ and $|\boldsymbol{v}_1|_{1,nq/(n-q)}$ is finite,*[3] *then $\boldsymbol{a} = \boldsymbol{0}$.*

Proof. A solution to (IV.3.29) is given by $\boldsymbol{v} = \boldsymbol{W} + \boldsymbol{w}$, $p = S + s$, where $\boldsymbol{W}, S$ and $\boldsymbol{w}, s$ are given in Lemma IV.3.1 and Lemma IV.3.2. Then, from those lemmas, we deduce that $\boldsymbol{v}, p$ possesses all the properties claimed in the

[3] Notice that, by Theorem II.6.3(i), we can find a uniquely determined $\boldsymbol{c} \in \mathbb{R}^n$ such that $\|\nabla \boldsymbol{v}_1 + \boldsymbol{c}\|_{nq/(n-q)}$ is finite. So, the request is that $\boldsymbol{c} = \boldsymbol{0}$.

statement. This concludes the proof of existence. Concerning uniqueness, in view of Theorem IV.3.1 we have only to discuss the case when $\boldsymbol{a} = \mathbf{0}$. But this is obvious, since, under the stated assumptions, $\nabla(\boldsymbol{v} - \boldsymbol{v}_1) \in L^{nq/(n-q)}(\mathbb{R}^n_+)$. The proof of the theorem is complete. □

Exercise IV.3.2 Let $u \in L^q(\mathbb{R}^n_+)$ with $D^2 u \in L^q(\mathbb{R}^n_+)$, $1 < q < \infty$. Show that $\nabla u \in L^q(\mathbb{R}^n_+)$. *Hint*: Use Ehrling's inequality (II.5.20) on every unitary cube in $\mathbb{R}^n_+$.

Our next task is to prove existence of *q-generalized solutions to problem* (IV.3.29). In complete analogy with Definition IV.2.1, by this latter we mean a field $\boldsymbol{v}$ such that

(i) $\boldsymbol{v} \in D^{1,q}(\mathbb{R}^n_+)$;
(ii) $(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) = -[\boldsymbol{f}, \boldsymbol{\varphi}]$, for all $\boldsymbol{\varphi} \in \mathcal{D}^{1,q'}_0(\mathbb{R}^n_+)$;
(iii) $(\boldsymbol{v}, \nabla \varphi) = -(g, \varphi)$, for all $\varphi \in C^\infty_0(\mathbb{R}^n_+)$;
(iv) $\boldsymbol{v}$ obeys $(\text{IV.3.29})_3$ in the trace sense (see (IV.3.13)).

In view of Lemma IV.1.1, if $\boldsymbol{f} \in W^{-1,q}_0(C)$, for all open cubes $C \subset \mathbb{R}^n_+$, then, to every q-generalized solution we may associate a pressure field $p \in L^q_{loc}(\mathbb{R}^n_+)$ such that

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) - (p, \nabla \cdot \boldsymbol{\psi}) = -[\boldsymbol{f}, \boldsymbol{\psi}], \ \ \textit{for all } \boldsymbol{\psi} \in C^\infty_0(\mathbb{R}^n).$$

Moreover, if $\boldsymbol{f} \in D^{-1,q}_0(\mathbb{R}^n_+)$, then $p \in L^q(\mathbb{R}^n_+)$.

We have

Theorem IV.3.3 *Given*

$$\boldsymbol{f} \in D^{-1,q}_0(\mathbb{R}^n_+), \quad g \in L^q(\mathbb{R}^n_+)$$

and

$$\boldsymbol{\Phi} \in L^q(\Sigma) \quad \textit{with } \langle\langle \boldsymbol{\Phi} \rangle\rangle_{1-1/q,q} \textit{ finite,}$$

$1 < q < \infty$, $n \geq 2$, *there exists at least one q-generalized solution to* (IV.3.29). *Moreover, this solution satisfies the inequality*

$$|\boldsymbol{v}|_{1,q} + \|p\|_q \leq c\left(|\boldsymbol{f}|_{-1,q} + \|g\|_q + \langle\langle \phi \rangle\rangle_{1-1/q,q}\right), \tag{IV.3.32}$$

where the constant c depends only on q and n. Finally, if $\boldsymbol{v}_1$ is a q_1-generalized solution ($1 < q_1 < \infty$, q_1 possibly different from q) corresponding to the same $\boldsymbol{f}$, g, and $\boldsymbol{\Phi}$, it follows that $\boldsymbol{v}_1 \equiv \boldsymbol{v}$ a.e. in $\mathbb{R}^n_+$ and, consequently, denoting by p_1 the pressure field associated to $\boldsymbol{v}_1$ by Lemma IV.1.1, *we also have* $p_1 \equiv p + \text{const}$ *a.e. in* $\mathbb{R}^n_+$.

Proof. We first show existence. As we know, if $\boldsymbol{f}$, $g \in C^\infty_0(\overline{\mathbb{R}^n_+})$, a smooth solution to (IV.3.29) is given by

$$\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{W}, \quad p = p_1 + S,$$

where

$$
\begin{aligned}
v_{1i}(x) &= h_i(x) + U_{ik} * (f_{sk} - D_k g_s)(x) \equiv h_i(x) + A_i(x) \\
W_i(x) &= \int_\Sigma K_{ij}(x'-y', x_n) A_j(y',0) dy' \\
&\qquad\qquad + \int_\Sigma K_{ij}(x'-y', x_n)[\Phi_j(y') - h_j(y',0)] dy' \\
&\equiv B_i(x) + b_i(x) \\
p_1(x) &= -q_i * (f_{si} - D_i g_s)(x) \\
S(x) &= D_j \int_\Sigma k(x'-y', x_n) A_j(y',0) dy' \\
&\qquad\qquad - \int_\Sigma k(x'-y', x_n)[\Phi_j(y') - h_j(y',0)] dy'.
\end{aligned}
$$

Moreover, $\boldsymbol{h}$ is defined in (IV.2.10) while $\boldsymbol{f}_s$ and g_s are smooth extensions of $\boldsymbol{f}$ and g to $\mathbb{R}^n$ satisfying (IV.3.15). Since $\boldsymbol{h}(y',0) \in L^q(\Sigma)$, by Lemma IV.3.1 we find

$$
|\boldsymbol{b}|_{1,q} \le c_1 \left(\langle\langle \phi \rangle\rangle_{1-1/q,q} + \|g\|_q \right). \tag{IV.3.33}
$$

Let us now estimate the term $\boldsymbol{A} + \boldsymbol{B}$. For fixed $\varphi \in C_0^\infty(B_\rho)$, we have

$$
\begin{aligned}
I \equiv (D_\ell(A_i + B_i), \varphi) \equiv \int_{\mathbb{R}^n_+} \varphi(x) D_\ell \bigg\{ &\int_{\mathbb{R}^n} U_{ik}(x-y)[f_{sk}(y) - D_k g_s(y)] dy \\
&- \int_\Sigma K_{ij}(x'-\eta', x_n) A_j(\eta',0) d\eta' \bigg\} \\
= \int_{\mathbb{R}^n_+} \varphi(x) D_\ell \bigg\{ &\int_{\mathbb{R}^n} U_{ik}(x-y)[f_{sk}(y) - D_k g_s(y)] dy \\
- \int_\Sigma K_{ij}(x'\eta', x_3) &\left[\int_{\mathbb{R}^n} U_{ik}(\eta'-y', y_n)[f_{sk}(y) - D_k g_s(y)] dy \right] d\eta' \bigg\} dx
\end{aligned}
$$

and, therefore, after integration by parts, we arrive at

$$
I = \int_{\mathbb{R}^n} [f_{sk}(y) - D_k g_s(y)] D_\ell z_{ik} dy \tag{IV.3.34}
$$

with

$$
z_{ik}(y) = \int_{\mathbb{R}^n_+} \left\{ U_{ik}(x-y) - \int_\Sigma K_{ij}(x'-\eta', x_3) U_{jk}(\eta'-y', y_n) d\eta' \right\} \varphi(x) dx.
$$

Denote by $Z_{ik}(x,y)$ the function in curly brackets in this integral. It is easy to show that, for every fixed $y \in \overline{\mathbb{R}}^n_-$, it holds that

$$Z_{ik}(x,y) = 0, \quad \text{for all } x \in \mathbb{R}^n_+. \tag{IV.3.35}$$

Actually, for $y \in \overline{\mathbb{R}}^n_-$, by what we already proved in this section and the properties of the tensor $\boldsymbol{U}$, both

$$U_{ik}(x-y)$$

and

$$\int_\Sigma K_{ij}(x'-\eta', x_n) U_{jk}(\eta'-y', y_n) d\eta'$$

as functions of x solve the Stokes system in $\mathbb{R}^n_+$ and assume the same value at Σ. Moreover, they both have second derivatives that are summable in $\mathbb{R}^n_+$ to the qth power, $1 < q < \infty$. Therefore, by Theorem IV.3.1, their difference $\boldsymbol{d}(x)$ (say) can be at most a (suitable) linear function of x_n. However, as is immediately seen, $\boldsymbol{d}(x)$ tends to zero as x_n tends to infinity and (IV.3.35) is therefore established. Setting

$$\zeta_{\ell ik}(y) \equiv D_\ell z_{ik}(y),$$

from (IV.3.35) we obtain, in particular,

$$\zeta_{\ell ik}(y) = 0, \quad \text{for all } y \in \overline{\mathbb{R}}^n_-. \tag{IV.3.36}$$

We shall show next that $\zeta_{\ell ik} \in D^{1,q}(\mathbb{R}^n_+)$ and

$$|\zeta_{\ell ik}|_{1,q} \le c \|\varphi\|_{q,B_\rho} \tag{IV.3.37}$$

for some c independent of ρ. To this end, we observe that

$$\begin{aligned}\zeta_{\ell ik} &= D_\ell \int_{\mathbb{R}^n} U_{ik}(x-y)\varphi(x)dx + D_\ell \int_\Sigma U_{jk}(\eta'-y', y_n)\chi_{ij}(\eta', 0)d\eta' \\ &\equiv \zeta^{(1)} + \zeta^{(2)},\end{aligned} \tag{IV.3.38}$$

where

$$\chi_{ij}(\eta', \eta_n) = \int_{\mathbb{R}^n_+} K_{ij}(x'-\eta', x_n - \eta_n)\varphi(x)dx. \tag{IV.3.39}$$

By the Calderón–Zygmund theorem, we have

$$|\zeta^{(1)}|_{1,q} \le c_1 \|\varphi\|_{q,B_\rho} \tag{IV.3.40}$$

with $c_1 = c_1(q,n)$. Moreover, it is not difficult to show the following two statements:

(i) $U_{jk}(\eta'-y', y_n)$ satisfies the assumptions made on the kernel k in Theorem II.11.6;

(ii) $\chi_{ij}(\eta', 0) \in L^q(\Sigma)$.

We may therefore apply Theorem II.11.6 to deduce

$$|\zeta^{(2)}|_{1,q} \leq c_2 \max_{i,j} \langle\langle \chi_{ij}(\eta', 0) \rangle\rangle_{1-1/q,q} \tag{IV.3.41}$$

with $c_2 = c_2(q,n)$. Applying the trace Theorem II.10.2, from (IV.3.41) we recover

$$|\zeta^{(2)}|_{1,q} \leq c_3 \max_{i,j} |\chi_{ij}|_{1,q}. \tag{IV.3.42}$$

However, employing Lemma II.11.1, it readily follows that for each fixed m, i, and j the kernels $D_m K_{ij}$ are linear combinations of kernels, each satisfying the hypotheses of the Calderón–Zygmund theorem, so that from (IV.3.39) it follows that

$$|\chi_{ij}|_{1,q} \leq c_4 \|\varphi\|_{q,B_\rho},$$

with $c_4 = c_4(q,n)$. This inequality, together with (IV.3.42), furnishes

$$|\zeta^{(2)}|_{1,q} \leq c_5 \|\varphi\|_{q,B_\rho},$$

which, along with (IV.3.40), proves (IV.3.37). In view of Theorem II.7.7, from (IV.3.36) and (IV.3.37) we conclude

$$\begin{aligned} &\zeta_{\ell ik} \in D_0^{1,q}(\mathbb{R}^n_+) \\ &|\zeta_{\ell ik}|_{1,q} \leq c_6 \|\varphi\|_{q,B_\rho} \end{aligned} \tag{IV.3.43}$$

for a constant c_6 independent of ρ. With a view to (IV.3.43) and (IV.3.36), from equation (IV.3.34) we derive

$$\begin{aligned} |I| \equiv |(D_\ell(A_i + B_i), \varphi) &\leq c_7 |\boldsymbol{f}_s - \nabla g_s|_{-1,q,\mathbb{R}^n_+} \|\varphi\|_{q,B_\rho} \\ &\leq c_7 \left(|\boldsymbol{f}|_{-1,q,\mathbb{R}^n_+} + \|g\|_{q,\mathbb{R}^n_+} \right) \|\varphi\|_{q,B_\rho}, \end{aligned}$$

so that, by the arbitrariness of φ and ρ, we obtain

$$|\boldsymbol{A} + \boldsymbol{B}|_{1,q} \leq c_8 \left(|\boldsymbol{f}|_{-1,q} + \|g\|_q \right). \tag{IV.3.44}$$

From (IV.2.13), (IV.3.33), and (IV.3.44) we then conclude

$$|\boldsymbol{v}|_{1,q} \leq c_9 \left(|\boldsymbol{f}|_{-1,q} + \|g\|_q + \langle\langle \phi \rangle\rangle_{1-1/q,q} \right).$$

By similar argument one shows

$$\|p\|_q \leq c_9 \left(|\boldsymbol{f}|_{-1,q} + \|g\|_q + \langle\langle \phi \rangle\rangle_{1-1/q,q} \right)$$

and thus the existence proof is complete, at least for smooth $\boldsymbol{f}$ and g. If $\boldsymbol{f}$ and g merely satisfy the assumptions formulated in the theorem, we can easily establish existence by means of the estimate (IV.3.32) and the usual density argument which, this time, makes use of Theorem II.8.1. The proof of uniqueness is entirely analogous to that given in Theorem IV.2.2 and it is therefore omitted. The theorem is completely proved. □

A simple, interesting consequence of Theorem IV.3.3 is the following.

Corollary IV.3.1 *Given*

$$g \in L^q(\mathbb{R}^n_+) \cap L^{q_1}(\mathbb{R}^n_+), \quad 1 < q, q_1 < \infty,$$

there exists $\boldsymbol{v} \in D_0^{1,q}(\mathbb{R}^n_+) \cap D_0^{1,q_1}(\mathbb{R}^n_+)$ *such that*

$$\begin{aligned} &\nabla \cdot \boldsymbol{v} = g \quad in\ \mathbb{R}^n_+ \\ &|\boldsymbol{v}|_{1,r} \le c\,\|g\|_r, \quad r = q, q_1, \end{aligned} \tag{IV.3.45}$$

where $c = c(n,r)$.

In the last part of this section we shall provide the Green's tensor (of the first kind) for the Stokes system in the half-space. We look for a tensor field $\boldsymbol{G}(x,y) = \{G_{ij}(x,y)\}$ and for a vector field $\boldsymbol{g}(x,y) = \{g_j(x,y)\}$ such that for all $j = 1, \dots, n$:

$$\begin{aligned} \Delta_x G_{ij}(x,y) + \frac{\partial g_j(x,y)}{\partial x_i} &= 0, \quad x, y \in \mathbb{R}^n_+, x \ne y \\ \frac{\partial G_{ij}(x,y)}{\partial x_i} &= 0, \quad x, y \in \mathbb{R}^n_+ \\ G_{ij}(x,y) &= 0, \ x \in \Sigma \equiv \mathbb{R}^{n-1} \times \{0\}, \ y \in \mathbb{R}^n_+ \\ \lim_{|x|\to\infty} G_{ij}(x,y) &= 0, \quad y \in \mathbb{R}^n_+, \end{aligned}$$

and, moreover, as $|x-y| \to 0$

$$G_{ij}(x,y) = U_{ij}(x-y) + o(1), \quad g_j(x,y) = q_i(x-y) + o(1),$$

where $\boldsymbol{U}, \boldsymbol{q}$ is the Stokes fundamental solution (IV.2.3), (IV.2.4). The pair $\boldsymbol{G}, \boldsymbol{g}$ is the *Green's tensor for the Stokes problem in the half-space* and is the vector counterpart of the Green's function for the Laplace operator given in (III.1.35). We can provide an explicit form of $\boldsymbol{G}$ and $\boldsymbol{g}$, see, e.g., Maz'ja, Plamenevskiĭ, & Stupyalis (1974, Appendix 1), and one has for $j = 1, \dots, n$ and $y^* = (y_1, \dots, -y_n)$

$$\begin{aligned} G_{ij}(x,y) &= U_{ij}(x-y) - U_{ij}(x-y^*) + W_{ij}(x,y), \quad i = 1, \dots n-1 \\ G_{nj}(x,y) &= U_{nj}(x-y) + U_{nj}(x-y^*) + W_{nj}(x,y), \\ g_i(x,y) &= q_i(x-y) - q_i(x-y^*) - t_i(x,y), \quad i = 1, \dots n-1 \\ g_n(x,y) &= q_n(x-y) + q_n(x-y^*) - t_n(x,y) \end{aligned} \tag{IV.3.46}$$

where $W_{ij}(x,y)$, $t_i(x,y)$ satisfy for all $j=1,\dots,n$ [4]

$$\begin{aligned}
W_{ij}(x,y) &= -\int_{\Sigma} K_{nj}(x'-\eta',x_n)[U_{in}(\eta'-y',-y_n)\\
&\qquad\qquad -U_{in}(\eta'-y',y_n)]d\eta', \quad i=1,\dots,n-1\\
W_{nj}(x,y) &= -\int_{\Sigma} K_{nj}(x'-\eta',x_n)[U_{nn}(\eta'-y',-y_n)\\
&\qquad\qquad -U_{nn}(\eta'-y',y_n)]d\eta',\\
t_i(x,y) &= -D_n\int_{\Sigma} k(x'-\eta',x_n)[U_{in}(\eta'-y',-y_n)\\
&\qquad\qquad -U_{in}(\eta'-y',y_n)]d\eta', \quad i=1,\dots,n-1\\
t_n(x,y) &= -D_n\int_{\Sigma} k(x'-\eta',x_n)[U_{nn}(\eta'-y',-y_n)\\
&\qquad\qquad -U_{nn}(\eta'-y',y_n)]d\eta',
\end{aligned} \tag{IV.3.47}$$

where the kernels K_{ij} and k are defined in (IV.3.4). In particular, one finds for $n=3$:

$$\begin{aligned}
W_{ij}(x,y) &= \frac{x_3y_3}{4\pi}\frac{\partial^2}{\partial x_j\partial y_i}\left(\frac{1}{|x-y^*|}\right), \quad i,j=1,2,\\
W_{3j}(x,y) &= -\frac{x_3}{4\pi}\frac{\partial}{\partial x_j}\left(\frac{1}{|x-y^*|}\right)+\frac{x_3y_3}{4\pi}\frac{\partial^2}{\partial x_j\partial y_3}\left(\frac{1}{|x-y^*|}\right), \quad j=1,2,\\
W_{i3}(x,y) &= -\frac{y_3}{4\pi}\frac{\partial}{\partial y_i}\left(\frac{1}{|x-y^*|}\right)+\frac{x_3y_3}{4\pi}\frac{\partial^2}{\partial x_3\partial y_i}\left(\frac{1}{|x-y^*|}\right), \quad i=1,2,\\
W_{33}(x,y) &= \frac{1}{4\pi|x-y^*|}-\frac{x_3}{4\pi}\frac{\partial}{\partial x_3}\left(\frac{1}{|x-y^*|}\right)-\frac{y_3}{4\pi}\frac{\partial}{\partial y_3}\left(\frac{1}{|x-y^*|}\right)\\
&\qquad +\frac{x_3y_3}{4\pi}\frac{\partial^2}{\partial x_3\partial y_3}\left(\frac{1}{|x-y^*|}\right),\\
t_i(x,y) &= \frac{y_3}{2\pi}\frac{\partial^2}{\partial x_3\partial y_i}\left(\frac{1}{|x-y^*|}\right), \quad i=1,2,\\
t_3(x,y) &= -\frac{1}{2\pi}\frac{\partial}{\partial x_3}\left(\frac{1}{|x-y^*|}\right)+\frac{y_3}{2\pi}\frac{\partial^2}{\partial x_3\partial y_3}\left(\frac{1}{|x-y^*|}\right),
\end{aligned} \tag{IV.3.48}$$

[4] $U_{ij}(x-y^*)$ is regular for all $x,y\in\mathbb{R}^n_+$. Notice that, unlike the analogous Green's function for the Laplace operator, the function

$$U_{ij}(x-y)-U_{ij}(x-y*)$$

cannot be taken as the Green tensor for the Stokes problem, since the solenoidality condition is not satisfied. Thus, we must modify it by adding functions W_{ij}.

while, for $n = 2$:

$$
\begin{aligned}
W_{11}(x,y) &= \frac{x_2 y_2}{2\pi} \frac{\partial^2}{\partial x_1 \partial y_1} \left(\ln \frac{1}{|x-y^*|} \right), \\
W_{21}(x,y) &= -\frac{x_2}{2\pi} \frac{\partial}{\partial x_1} \left(\ln \frac{1}{|x-y^*|} \right) + \frac{x_2 y_2}{4\pi} \frac{\partial^2}{\partial x_1 \partial y_2} \left(\ln \frac{1}{|x-y^*|} \right), \\
W_{12}(x,y) &= -\frac{y_2}{2\pi} \frac{\partial}{\partial y_1} \left(\ln \frac{1}{|x-y^*|} \right) + \frac{x_2 y_2}{2\pi} \frac{\partial^2}{\partial x_2 \partial y_1} \left(\ln \frac{1}{|x-y^*|} \right), \\
W_{22}(x,y) &= \frac{1}{2\pi} \ln \frac{1}{|x-y^*|} - \frac{x_2}{2\pi} \frac{\partial}{\partial x_2} \left(\ln \frac{1}{|x-y^*|} \right) \\
&\quad - \frac{y_2}{2\pi} \frac{\partial}{\partial y_2} \left(\ln \frac{1}{|x-y^*|} \right) + \frac{x_2 y_2}{2\pi} \frac{\partial^2}{\partial x_2 \partial y_2} \left(\ln \frac{1}{|x-y^*|} \right), \\
t_1(x,y) &= -\frac{y_2}{2\pi} \frac{\partial^2}{\partial x_2 \partial y_1} \left(\ln \frac{1}{|x-y^*|} \right), \\
t_2(x,y) &= -\frac{y_2}{2\pi} \frac{\partial^2}{\partial x_2 \partial y_2} \left(\ln \frac{1}{|x-y^*|} \right).
\end{aligned}
\tag{IV.3.49}
$$

From (IV.3.46)–(IV.3.49) we wish to single out some estimates for $\boldsymbol{G}, \boldsymbol{g}$ that will be useful later. Precisely, by a simple computation, we find for $n = 3$,

$$
\begin{aligned}
|D^\alpha G_{ij}(x,y) &\leq |x-y|^{-1-|\alpha|} \\
|D^\alpha g_i(x,y) &\leq |x-y|^{-2-|\alpha|}
\end{aligned}
\tag{IV.3.50}
$$

and for $n = 2$,

$$
|D^\alpha g_i(x,y)| + |D_k D^\alpha G_{ij}(x,y)| \leq c|x-y|^{-1-|\alpha|}, \tag{IV.3.51}
$$

where $|\alpha| \geq 0$, $c = c(n)$ and D^α, D_k are acting either on x or y.

Exercise IV.3.3 Starting from (IV.3.46), (IV.3.47), prove the following estimates for all $n \geq 3$ and all $|\alpha| \geq 0$

$$
\begin{aligned}
|D^\alpha G_{ij}(x,y)| &\leq c|x-y|^{-n-|\alpha|+2} \\
|D^\alpha g_i(x,y)| + |D_k D^\alpha G_{ij}(x,y)| &\leq c|x-y|^{-n-|\alpha|+1}.
\end{aligned}
\tag{IV.3.52}
$$

IV.4 Interior L^q-Estimates

In this section we will investigate the properties of generalized solutions "far" from the boundary of the region of motion by means of the results established in Section IV.2 for solutions in the whole space. Actually, we consider a pair of functions $\boldsymbol{v}, p$ with $\boldsymbol{v} \in W^{1,1}_{loc}(\Omega)$, $\nabla \cdot \boldsymbol{v} = 0$ in the generalized sense and $p \in L^1_{loc}(\Omega)$ satisfying identity (IV.1.3) for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$ and shall show that, for suitable $\boldsymbol{f}$, the fields $\boldsymbol{v}$, p obey certain L^q-inequalities that imply, among other things, that any weak solution is in fact of class $C^\infty(\Omega)$, provided $\boldsymbol{f}$ enjoys the same property.

We begin to prove some preliminary results. First of all, the regularizations, $\boldsymbol{v}_\varepsilon$, p_ε, of $\boldsymbol{v}$ and p, respectively, obey the Stokes equation in any subdomain Ω_0 with $\overline{\Omega}_0 \subset \Omega$. This result is a straightforward consequence of the following very general one.

Lemma IV.4.1 *Let Ω be an arbitrary domain of $\mathbb{R}^n$, $n \geq 2$. Suppose that $\boldsymbol{v} \in L^1_{loc}(\Omega)$ obeys the conditions*

$$\begin{aligned} &(\boldsymbol{v}, \Delta\boldsymbol{\varphi}) = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle\,, \quad \textit{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega)\,, \\ &(\boldsymbol{v}, \nabla\psi) = 0\,, \quad \textit{for all } \psi \in C_0^\infty(\Omega), \end{aligned} \tag{IV.4.1}$$

where $\boldsymbol{f}$ satisfies either (i) $\boldsymbol{f} \in L^1_{loc}(\Omega)$, or (ii) $\boldsymbol{f} \in \mathcal{D}_0^{-1,q}(\omega)$, for all ω with $\overline{\omega} \subset \Omega$, and some $q > 1$. Then, for any domain Ω_0 with $\overline{\Omega}_0 \subset \Omega$, we have

$$\left.\begin{aligned} \Delta\boldsymbol{v}_\varepsilon &= \nabla p^{(\varepsilon)} + \boldsymbol{g}_\varepsilon \\ \nabla \cdot \boldsymbol{v}_\varepsilon &= 0 \end{aligned}\right\} \quad \textit{in } \Omega_0\,, \tag{IV.4.2}$$

for some $p^{(\varepsilon)} \in C^\infty(\Omega)$, and where $\boldsymbol{g} = \boldsymbol{f}$, in case (i), while, in case (ii), $\boldsymbol{g}_\varepsilon = \nabla \cdot \boldsymbol{F}_\varepsilon$, where $\boldsymbol{F} \in L^q(\Omega_0)$ satisfies $\langle \boldsymbol{f}, \boldsymbol{\chi} \rangle = (\boldsymbol{F}, \nabla\boldsymbol{\chi})$, for all $\boldsymbol{\chi} \in \mathcal{D}(\Omega_0)$ (see Theorem II.1.6 *and* Theorem II.8.2*).*

Proof. Let $\boldsymbol{\varphi} \in \mathcal{D}(\Omega_0)$. Then $\boldsymbol{\varphi}_\varepsilon \in \mathcal{D}(\Omega)$, whenever $\varepsilon < \operatorname{dist}(\Omega_0, \partial\Omega)$, and can be thus replaced in $(\text{IV.4.1})_1$. Using this latter relation, by a straightforward calculation that uses Exercise II.3.2, we then show

$$(\Delta\boldsymbol{v}_\varepsilon - \boldsymbol{g}_\varepsilon, \boldsymbol{\varphi}) = 0\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega_0)\,.$$

Therefore, $(\text{IV.4.2})_1$ follows from this equation and from Lemma III.1.1. The second equation in (IV.4.2) can be proved from (IV.4.1) in exactly the same way. □

The previous result trivially implies the following one.

Lemma IV.4.2 *Let Ω be an arbitrary domain of $\mathbb{R}^n$, $n \geq 2$, and let $\boldsymbol{v}, p$ with $\boldsymbol{v} \in W^{1,1}_{loc}(\Omega)$, $\nabla \cdot \boldsymbol{v} = 0$, and $p \in L^1_{loc}(\Omega)$ satisfy* (IV.1.3) *for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$ and with $\boldsymbol{f} \in L^1_{loc}(\Omega)$. Then the regularizations, $\boldsymbol{v}_\varepsilon$, p_ε,*

$$\left.\begin{aligned} \Delta\boldsymbol{v}_\varepsilon &= \nabla p_\varepsilon + \boldsymbol{f}_\varepsilon \\ \nabla \cdot \boldsymbol{v}_\varepsilon &= 0 \end{aligned}\right\} \quad \textit{in } \Omega_0\,, \tag{IV.4.3}$$

for all domains Ω_0 with $\overline{\Omega}_0 \subset \Omega$.

Employing Theorem IV.2.1 and Lemma IV.4.2 we shall show the following interior regularity result.

Theorem IV.4.1 *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \geq 2$. Let $\boldsymbol{v}$ be weakly divergence-free with $\nabla \boldsymbol{v} \in L^q_{loc}(\Omega)$, $1 < q < \infty$,*[1] *and satisfying* (IV.1.2) *for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. Then, if*

$$\boldsymbol{f} \in W^{m,q}_{loc}(\Omega), \quad m \geq 0,$$

it follows that

$$\boldsymbol{v} \in W^{m+2,q}_{loc}(\Omega), \quad p \in W^{m+1,q}_{loc}(\Omega)$$

where p is the pressure field associated to $\boldsymbol{v}$ by Lemma IV.1.1. *Further, the following inequality holds:*

$$|\boldsymbol{v}|_{m+2,q,\Omega'} + |p|_{m+1,q,\Omega'} \leq c\left(\|\boldsymbol{f}\|_{m,q,\Omega''} + \|\boldsymbol{v}\|_{1,q,\Omega''-\Omega'} + \|p\|_{q,\Omega''-\Omega'}\right) \tag{IV.4.4}$$

where Ω', Ω'' are arbitrary bounded subdomains of Ω with $\overline{\Omega'} \subset \Omega''$, $\overline{\Omega''} \subset \Omega$, and $c = c(n, q, m, \Omega', \Omega'')$.

Proof. Consider a "cut-off" function $\varphi \in C^\infty(\mathbb{R}^n)$ that is one in $\overline{\Omega'}$ and zero outside Ω''.[2] Choosing in (IV.4.4) $\Omega_0 \supset \overline{\Omega''}$ and multiplying (IV.4.3) by φ, after a simple manipulation we obtain that the functions

$$\boldsymbol{u} = \varphi \boldsymbol{v}_\varepsilon, \quad \pi = \varphi p_\varepsilon$$

satisfy

$$\begin{aligned} \Delta \boldsymbol{u} &= \nabla \pi + \boldsymbol{f}_1 + \boldsymbol{f}_c \\ \nabla \cdot \boldsymbol{u} &= g, \end{aligned} \tag{IV.4.5}$$

where

$$\begin{aligned} \boldsymbol{f}_1 = \varphi \boldsymbol{f}_\varepsilon, \quad \boldsymbol{f}_c &= -p_\varepsilon \nabla \varphi + 2\nabla \varphi \cdot \nabla \boldsymbol{v}_\varepsilon + \boldsymbol{v}_\varepsilon \Delta \varphi \\ g &= \nabla \varphi \cdot \boldsymbol{v}_\varepsilon. \end{aligned} \tag{IV.4.6}$$

Problem (IV.4.5) can be considered in the whole of $\mathbb{R}^n$, by extending $\boldsymbol{v}_\varepsilon$, p_ε and $\boldsymbol{f}_\varepsilon$ to zero outside Ω''. Since $D^2\boldsymbol{u} \in L^q(\mathbb{R}^n)$ we may apply Theorem IV.2.1 with $m = 0$ to deduce the following estimate

$$\begin{aligned} \|D^2\boldsymbol{u}\|_q + \|\nabla \pi\|_q \leq c_1 \big(&\|\varphi \boldsymbol{f}_\varepsilon\|_q + \|\nabla(\nabla \varphi \cdot \boldsymbol{v}_\varepsilon)\|_q \\ &\|\nabla \varphi \cdot \nabla \boldsymbol{v}_\varepsilon\|_q + \|\boldsymbol{v}_\varepsilon \Delta \varphi\|_q + \|p_\varepsilon \nabla \varphi\|_q\big), \end{aligned} \tag{IV.4.7}$$

where $c_1 = c_1(n, q)$. From (IV.4.6), (IV.4.7) and from the properties of φ we obtain

---

[1] Notice that, by Lemma II.6.1, $\boldsymbol{v} \in L^q_{loc}(\Omega)$. The assumption on $\boldsymbol{v}$ can be weakened. We shall show this directly for the nonlinear case in Chapter IX.

[2] For instance, we may choose φ as the regularization $(\chi_{\Omega'})_\varepsilon$ of the characteristic function of the domain Ω' and take ε sufficiently small.

$$\|D^2\boldsymbol{v}_\varepsilon\|_{q,\Omega'} + \|\nabla p_\varepsilon\|_{q,\Omega'} \le c\left(\|\boldsymbol{f}_\varepsilon\|_q + \|\boldsymbol{v}_\varepsilon\|_{1,q,\Omega''-\Omega'} + \|p_\varepsilon\|_{q,\Omega''-\Omega'}\right).$$

Letting $\varepsilon \to 0$ into this inequality and recalling the properties of the mollifier (II.2.9) along with the definition of weak derivative, one thus proves the theorem for $m = 0$. The general case is now treated by induction. Assuming that the theorem holds for $m = \ell - 1$, $\ell \ge 1$, we shall show it for $m = \ell$. By hypothesis we then have

$$\boldsymbol{v} \in W^{\ell+1,q}_{loc}(\Omega), \quad p \in W^{\ell,q}_{loc}(\Omega)$$

and, moreover,

$$|\boldsymbol{v}|_{\ell+1,q\Omega^\sharp} + |p|_{\ell,q,\Omega^\sharp} \le c\left(\|\boldsymbol{f}\|_{\ell,q,\Omega''} + \|\boldsymbol{v}\|_{1,q,\Omega''-\Omega^\sharp} + \|p\|_{q,\Omega''-\Omega^\sharp}\right) \quad \text{(IV.4.8)}$$

where $\overline{\Omega'} \subset \Omega^\sharp$, $\overline{\Omega^\sharp} \subset \Omega''$. We now choose φ in (IV.4.5) as a C^∞ function that is one in $\overline{\Omega'}$ and zero outside $\Omega^\sharp$. Applying Theorem IV.2.1 to solutions to (IV.4.5) and recalling (IV.4.8) we thus deduce

$$\begin{aligned}|\boldsymbol{v}_\varepsilon|_{\ell+2,q,\Omega'} + |p_\varepsilon|_{\ell+1,q,\Omega'} &\le c_1\left(\|\boldsymbol{f}\|_{\ell,q,\Omega^\sharp} + \|\boldsymbol{v}_\varepsilon\|_{\ell+1,q,\Omega^\sharp-\Omega'} + \|p_\varepsilon\|_{q,\Omega^\sharp-\Omega'}\right)\\ &\le c_1\left(\|\boldsymbol{f}\|_{\ell,q,\Omega''} + \|\boldsymbol{v}\|_{1,q,\Omega''-\Omega'} + \|p\|_{q,\Omega''-\Omega'}\right).\end{aligned}$$

This inequality, in the limit $\varepsilon \to 0$, then proves the validity of the theorem for arbitrary $m \ge 0$. □

Remark IV.4.1 It is of some interest to observe that if $\Omega'' - \Omega'$ satisfies the cone condition, the term containing the pressure on the right-hand side of (IV.4.4) can be removed, provided we modify p by the addition of a suitable constant. Furthermore, if $\Omega''' \supset \overline{\Omega''}$ with $\Omega''' - \Omega''$ satisfying the cone condition, then the term $\|v\|_{1,q,\Omega''-\Omega'}$ can be replaced by $\|v\|_{q,\Omega'''-\Omega''}$; see Remark IV.4.2. ■

The next result provides a sharpened version of that just proved. In this respect, we observe that if $\boldsymbol{v} \in W^{1,r}_{loc}(\Omega)$ satisfies (IV.1.2) for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$, with $\boldsymbol{f} \in W^{-1,q}_0(\omega)$, for all bounded subdomains ω with $\overline{\omega} \subset \Omega$, and where a priori $r \neq q$, by Lemma IV.1.1 we can associate to $\boldsymbol{v}$ a pressure field p satisfying (IV.1.2) with $p \in L^\mu_{loc}(\Omega)$, $\mu = \min(r, q)$. We have

Theorem IV.4.2 *Let Ω satisfy the assumption of* Theorem IV.4.1. *Assume $\boldsymbol{v}$ is weakly divergence-free with $\nabla\boldsymbol{v} \in L^r_{loc}(\Omega)$, $1 < r < \infty$, and satisfies identity* (IV.1.3). *Then, if*

$$\boldsymbol{f} \in W^{m,q}_{loc}(\Omega), \quad m \ge 0, \quad 1 < q < \infty,$$

it follows that

$$\boldsymbol{v} \in W^{m+2,q}_{loc}(\Omega), \quad p \in W^{m+1,q}_{loc}(\Omega),$$

where p is the pressure field associated to $\boldsymbol{v}$ by Lemma IV.1.1.

Proof. By Theorem IV.4.1, it is enough to show

$$\nabla \boldsymbol{v} \in L^q_{loc}(\Omega).^3 \tag{IV.4.9}$$

If $r \geq q$ the assertion is obvious. Therefore, take $q > r$. Then $\boldsymbol{f} \in L^r_{loc}(\Omega)$ and, by Theorem IV.4.1 and the embedding Theorem II.3.4, we deduce

$$\boldsymbol{v} \in W^{1,r_1}_{loc}(\Omega)$$

with $r_1 = nr/(n-r)$ $(> r)$ if $r < n$ and for arbitrary $r_1 > 1$ if $r \geq n$. In the latter case (IV.4.9) follows. If $q \leq r_1 < n$ we again draw the same conclusion. So, assume $1 < r_1 < q$. Then $\boldsymbol{f} \in L^{r_1}_{loc}(\Omega)$, and Theorem IV.4.1 and Theorem II.3.4 imply

$$\boldsymbol{v} \in W^{1,r_2}_{loc}(\Omega),$$

with $r_2 = nr_1/(n-r_1) = nr/(n-2r)$ $(> r_1)$ if $1 < r_1 < n$ and arbitrary $r_2 > 1$, whenever $r_1 \geq n$. If either $r_2 \geq q$ or $r_1 \geq n$, (IV.4.9) follows; otherwise we iterate the above procedure a *finite* number of times until (IV.4.9) is established. □

Combining the result just proved with Theorem II.3.4 (specifically, inequality (II.3.18)) we at once obtain the following theorem concerning interior regularity of q-weak solutions.

Theorem IV.4.3 *Let $\boldsymbol{v}$ be a q-weak solution to the Stokes problem* (IV.0.1), (IV.0.2) *corresponding to $\boldsymbol{f} \in C^\infty(\Omega)$. Then, $\boldsymbol{v}, p \in C^\infty(\Omega)$ where p is the pressure field associated to $\boldsymbol{v}$ by* Lemma IV.1.1.

Intermediate regularity results are directly obtainable from Theorem IV.4.1 and the embedding Theorem II.3.4 and are left to the reader as an exercise. Other regularity results in Hölder norms can be obtained from the results of Section IV.7.

Exercise IV.4.1 (Ladyzhenskaya 1969). In the case where $q = 2$, Theorem IV.4.1 is obtained in an elementary way. Denote by φ the "cut-off" function of that theorem, multiply $(\text{IV.4.3})_1)$ by $\varphi^2 \Delta \boldsymbol{v}_\varepsilon$ and integrate by parts to show (IV.4.4) with $m = 0$, $q = 2$. (Observe that if $\zeta \in C^2_0(\Omega)$, $\|D^2\zeta\|_2 = \|\Delta\zeta\|_2$.) Use then the induction procedure to prove the general case $m \geq 0$.

Exercise IV.4.2 Show that Theorem IV.4.1 also holds when $\nabla \cdot \boldsymbol{v} = g \not\equiv 0$, provided $g \in W^{m+1,q}_{loc}(\Omega)$. In such a case, the term

$$\|g\|_{m+1,q,\Omega''}$$

must be added to the right-hand side of (IV.4.4).

We shall next consider interior estimates for q-generalized solutions. Specifically, we have the following theorem.

[3] See footnote 1 in this section.

Theorem IV.4.4 *Let Ω, Ω', Ω'', and $\boldsymbol{v}$ be as in* Theorem IV.4.1. *Suppose $\boldsymbol{f} \in W_0^{-1,q}(\omega)$, for all bounded domains ω with $\overline{\omega} \subset \Omega$. Then the following inequality holds:*

$$\|\boldsymbol{v}\|_{1,q,\Omega'} + \|p\|_{q,\Omega'} \leq c\left(\|\boldsymbol{f}\|_{-1,q,\Omega''} + \|\boldsymbol{v}\|_{q,\Omega''-\Omega'} + \|p\|_{-1,q,\Omega''-\Omega'}\right), \tag{IV.4.10}$$

where p is the pressure field associated to $\boldsymbol{v}$ by Lemma IV.1.1.

Proof. Let φ be as in Theorem IV.4.1 and set

$$\boldsymbol{u} = \varphi\boldsymbol{v}, \quad \pi = \varphi p\,.$$

From the assumption and Lemma IV.1.1, we know that $(\boldsymbol{v}, p)$ satisfies (IV.1.3) for all $\boldsymbol{\psi} \in D_0^{1,q'}(\Omega'')$. Therefore, choosing into this relation $\boldsymbol{\psi} = \varphi\boldsymbol{\phi}$, $\boldsymbol{\phi} \in D_0^{1,q'}(\mathbb{R}^n)$, we then readily deduce that $\boldsymbol{u}$, π satisfy the identities

$$\begin{aligned} (\nabla\boldsymbol{u}, \nabla\boldsymbol{\phi}) - (\pi, \nabla\cdot\boldsymbol{\phi}) &= -[\boldsymbol{f}_1, \boldsymbol{\phi}], \quad \text{for all } \boldsymbol{\phi} \in D_0^{1,q'}(\mathbb{R}^n), \\ (\boldsymbol{u}, \nabla\chi) &= -(g, \chi), \quad \text{for all } \chi \in D_0^{1,q'}(\mathbb{R}^n) \end{aligned} \tag{IV.4.11}$$

with

$$\begin{aligned} &\boldsymbol{f}_1 = \boldsymbol{f}_\varphi - p\nabla\varphi + 2\nabla\varphi\cdot\nabla\boldsymbol{v} + \boldsymbol{v}\Delta\varphi \\ &[\boldsymbol{f}_\varphi, \boldsymbol{\phi}] := \langle\boldsymbol{f}, \varphi\,\boldsymbol{\phi}\rangle \\ &g = \nabla\varphi\cdot\boldsymbol{v}. \end{aligned} \tag{IV.4.12}$$

Applying Theorem IV.2.2 to the above problem we then obtain that $\boldsymbol{u}$ and π obey the inequality

$$|\boldsymbol{u}|_{1,q} + \|\pi\|_q \leq c(|\boldsymbol{f}_1|_{-1,q} + \|g\|_q). \tag{IV.4.13}$$

On account of (IV.4.12), it follows that

$$|\boldsymbol{f}_1|_{-1,q} \leq c_1\left(|\boldsymbol{f}_\varphi|_{-1,q} + |p\nabla\varphi + 2\nabla\varphi\cdot\nabla\boldsymbol{v} + \boldsymbol{v}\Delta\varphi|_{-1,q}\right) \tag{IV.4.14}$$

and

$$\|g\|_q \leq \|\nabla\varphi\cdot\boldsymbol{v}\|_q \leq c_2\|\boldsymbol{v}\|_{q,\Omega''-\Omega'}. \tag{IV.4.15}$$

Let $\boldsymbol{\phi}$ be arbitrary element from $D_0^{1,q'}(\mathbb{R}^n)$. We distinguish the two cases:

(i) $n/(n-1) < q < \infty$,
(ii) $1 < q \leq n/(n-1)$.

In case (i), $q' < n$ and so, from the Sobolev inequality (II.3.7), we derive

$$\|\boldsymbol{\phi}\|_{q',\Omega''-\Omega'} \leq c_3|\boldsymbol{\phi}|_{1,q'},$$

which, after a simple calculation, allows us to show that

$$
\begin{aligned}
&|\boldsymbol{f}_{\varphi}|_{-1,q} + |p\nabla\varphi + 2\nabla\varphi \cdot \nabla \boldsymbol{v} + \boldsymbol{v}\Delta\varphi|_{-1,q} \\
&\qquad\qquad \le c_4 \left(\|\boldsymbol{f}\|_{-1,q,\Omega''} + \|\boldsymbol{v}\|_{q,\Omega''-\Omega'} + \|p\|_{-1,q,\Omega''-\Omega'}\right).
\end{aligned}
\tag{IV.4.16}
$$

In case (ii) choosing $\boldsymbol{\psi} = \varphi\, \boldsymbol{e}_i$ into (IV.1.3) delivers

$$
(\nabla v_i, \nabla\varphi) - (p, D_i\varphi) = -\,[f_i, \varphi]\,, \quad \text{for all } i = 1, \ldots, n.
$$

As a consequence, observing that

$$
\int_{\mathbb{R}^n} (2\nabla\varphi \cdot \nabla \boldsymbol{v} + \boldsymbol{v}\Delta\varphi) = \int_{\mathbb{R}^n} \nabla\varphi \cdot \nabla \boldsymbol{v}
$$

we find

$$
[\boldsymbol{f}_{\varphi} - p\nabla\varphi + 2\nabla\varphi \cdot \nabla \boldsymbol{v} + \boldsymbol{v}\Delta\varphi, \boldsymbol{\phi}] = [\boldsymbol{f}_{\varphi} - p\nabla\varphi + 2\nabla\varphi \cdot \nabla \boldsymbol{v} + \boldsymbol{v}\Delta\varphi, \boldsymbol{\phi} + \boldsymbol{C}],
\tag{IV.4.17}
$$

for any constant vector $\boldsymbol{C}$. Choosing

$$
\boldsymbol{C} = -\frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} \boldsymbol{\phi}, \quad \mathcal{M} \equiv \Omega'' - \Omega',
$$

from Poincaré's inequality (II.5.10) we deduce

$$
\|\boldsymbol{\phi} + \boldsymbol{C}\|_{q',\mathcal{M}} \le c_5 |\boldsymbol{\phi}|_{1,q'}.
\tag{IV.4.18}
$$

Since

$$
\begin{aligned}
&\big|\,[\boldsymbol{f}_{\varphi} - p\nabla\varphi + 2\ \nabla\varphi \cdot \nabla \boldsymbol{v} + \boldsymbol{v}\Delta\varphi, \boldsymbol{\phi} + \boldsymbol{C}]\big| \\
&\qquad\qquad \le c_5 \left(\|\boldsymbol{f}\|_{-1,q,\Omega''} + \|\boldsymbol{v}\|_{q,\mathcal{M}} + \|p\|_{-1,q,\mathcal{M}}\right) \|\boldsymbol{\phi} + \boldsymbol{C}\|_{q',\mathcal{M}}
\end{aligned}
\tag{IV.4.19}
$$

from (IV.4.17)–(IV.4.19) we conclude, also in case (ii), the validity of (IV.4.16). The theorem becomes then a consequence of (IV.4.13)–(IV.4.16) and of the properties of the function φ. □

Remark IV.4.2 If, in the previous theorem, we modify p by the addition of a constant, then we can remove the term involving the pressure on the right-hand side of (IV.4.10), provided $\Omega''-\Omega'$ satisfies the cone condition. Therefore, under these conditions, we obtain, in particular, the following estimate

$$
\|\boldsymbol{v}\|_{1,q,\Omega'} + \inf_{k\in\mathbb{R}} \|p + k\|_{q,\Omega'} \le c\left(\|\boldsymbol{f}\|_{-1,q,\Omega''} + \|\boldsymbol{v}\|_{q,\Omega''-\Omega'}\right).
\tag{IV.4.20}
$$

To see this, set, for simplicity, $D = \Omega'' - \Omega'$, and let $g \in W_0^{1,q}(D)$ be arbitrary. Moreover, let $\phi \in C_0^\infty(D)$ be a fixed function with $\int_D \phi = |D|$. We then consider the problem

$$
\nabla \cdot \boldsymbol{\psi} = g - \phi\, \overline{g}_D\,, \quad \boldsymbol{\psi} \in W_0^{2,q}(D)\,, \quad \|\boldsymbol{\psi}\|_{2,q} \le c\, \|g\|_{1,q}\,.
\tag{IV.4.21}
$$

In view of Theorem III.3.3, (IV.4.21) has at least one solution. We thus replace this solution in relation (IV.1.3) and integrate by parts to obtain

$$(p+k,g) = \langle \boldsymbol{f}, \boldsymbol{\psi}\rangle - (\boldsymbol{v}, \Delta\boldsymbol{\psi}) \,, \tag{IV.4.22}$$

where $k = -\overline{(p\phi)}_D$. Thus, from (IV.4.22), with the help of the estimate in (IV.4.21), and by the arbitrariness of g, we at once deduce

$$\|p+k\|_{-1,q,D} \le c\,(\|\boldsymbol{f}\|_{-1,q,D} + \|\boldsymbol{v}\|_{q,D}) \ ,$$

which is what we wanted to prove. ∎

The result to follow is a sharpened version of the preceding one. In this respect, we recall that if $\boldsymbol{v} \in W^{1,r}_{loc}(\Omega)$ satisfies (IV.1.2) for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ with $\boldsymbol{f} \in W_0^{-1,q}(\omega)$, ω as in Theorem IV.4.4 and a priori $r \ne q$, we can always associate with $\boldsymbol{v}$ a pressure field p satisfying (IV.1.3). In particular, $p \in L^{\mu}_{loc}(\Omega)$ where $\mu = \min(r,p)$. We have

Theorem IV.4.5 *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \ge 2$. Suppose $\boldsymbol{v} \in W^{1,r}_{loc}(\Omega)$, $1 < r < \infty$, is weakly divergence-free and satisfies* (IV.1.2) *for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. Then, if $\boldsymbol{f} \in W_0^{-1,q}(\omega)$, $1 < q < \infty$, for all bounded domains ω with $\overline{\omega} \subset \Omega$ it follows that*

$$\boldsymbol{v} \in W^{1,q}_{loc}(\Omega), \quad p \in L^q_{loc}(\Omega),$$

where p is the pressure field associated to $\boldsymbol{v}$ by Lemma IV.1.1.

Proof. If $r \ge q$ there is nothing to prove. We then take $q > r$. Consider problem (IV.4.11)–(IV.4.12). If $r \ge n$, by the embedding Theorem II.3.4 we easily deduce

$$\boldsymbol{v} \in L^q_{loc}(\Omega), \quad p \in W_0^{-1,q}(\omega)$$

for all bounded ω with $\overline{\omega} \subset \Omega$. and consequently, repeating the reasonings used in the proof of Theorem IV.4.4 we obtain, in particular,

$$\begin{aligned} |\boldsymbol{f}_1|_{-1,t} &\le c\,(\|\boldsymbol{f}\|_{-1,t,\Omega''} + \|\boldsymbol{v}\|_{t,\Omega''} + \|p\|_{-1,t,\Omega''}) \\ \|g\|_t &\le c\|\boldsymbol{v}\|_{t,\Omega''} \end{aligned} \tag{IV.4.23}$$

with $t = q$. Therefore, by Theorem IV.2.2, we deduce the assertion of the theorem. If $r < n$, again by Theorem II.3.4, we obtain

$$\boldsymbol{v} \in L^{r_1}_{loc}(\Omega), \quad p \in W_0^{-1,r_1}(\omega), \quad r_1 = rn/(r-n)$$

and so the assertion again follows from (IV.4.23) with $t = r_1$ and Theorem IV.2.2 if $r_1 \ge q$. If $r_1 < q$, from (IV.4.23) it still follows that

$$\boldsymbol{f}_1 \in D_0^{-1,r_1}(\mathbb{R}^n)$$

and so, using once more Theorem IV.2.2,

$$\boldsymbol{v} \in W^{1,r_1}_{loc}(\Omega), \quad p \in L^{r_1}_{loc}(\Omega) \quad (r_1 > r).$$

This "bootstrap" argument becomes of the same type as that of Theorem IV.4.2 and then, proceeding as in the proof of that theorem, we arrive at the desired conclusion. □

Exercise IV.4.3 Show that Theorem IV.4.4 continues to hold if $\nabla \cdot \boldsymbol{v} = g \not\equiv 0$, where $g \in L^q_{loc}(\Omega)$. The inequality in the theorem is then modified by adding the term

$$\|g\|_{q,\Omega''}$$

to its right-hand side.

Exercise IV.4.4 It is just worth noting that interior estimates of the type proved in Theorem IV.4.1 and Theorem IV.4.4 are also valid for the "scalar" case, namely, the Poisson equation $\Delta u = f$. In fact, let $u \in W^{1,q}_{loc}(\Omega)$, $q \in (1, \infty)$, satisfy $(\nabla u, \nabla \psi) = \langle f, \psi \rangle$ for all $\psi \in C_0^\infty(\Omega)$. Show that, if $f \in W^{m,q}_{loc}(\Omega)$, $m \geq -1$, then necessarily $u \in W^{m+2,q}_{loc}(\Omega)$, and the following estimate holds

$$\|u\|_{m+2,q,\Omega'} \leq c\left(\|f\|_{m,q,\Omega''} + \|u\|_{q,\Omega''}\right),$$

for all Ω'' and Ω' as in the above theorems, and with c independent of v and f.

IV.5 L^q-Estimates Near the Boundary

We wish now to determine L^q-estimates analogous to those of Theorem IV.4.1, but in a subdomain of Ω abutting a suitably smooth portion σ of the boundary. This will then allow us, in particular, to obtain regularity results for generalized solutions up to the boundary. Following the method outlined by Cattabriga (1961) and based on the work of Agmon, Douglis, & Nirenberg (1959), the strategy we shall adopt is to introduce a suitable change of variables so that, locally "near" the boundary, the Stokes problem goes over into a similar problem in the half-space. The desired estimate will then follow directly from Theorem IV.3.2 and Theorem IV.3.3.

Assume Ω has a boundary portion σ of class C^2. Without loss, we may rotate the coordinate system with the origin at a point $x_0 \in \sigma$ in such a way that, if we denote by $\zeta = \zeta(x_1, \ldots, x_{n-1})$ the function representing σ,

$$\nabla \zeta(0) = 0. \tag{IV.5.1}$$

(This means that the axes $x_1, \ldots, x_{n-1}$ are in the tangent plane at σ, at the point x_0.) Next, we denote by Ω_0 any bounded subdomain of Ω with $\sigma = \partial\Omega_0 \cap \partial\Omega$ and consider a pair of functions

$$\boldsymbol{v} \in W^{2,q}(\Omega_0), \quad p \in W^{1,q}(\Omega_0), 1 < q < \infty,$$

solving *a.e.* the Stokes problem in Ω_0 corresponding to $\boldsymbol{f} \in L^q(\Omega_0)$ and $\boldsymbol{v}_* \in W^{2-1/q,q}(\sigma)$. If we direct the positive x_n-axis into the interior of Ω, for sufficiently small $d > 0$ the cylinder

$$\omega = \{x \in \Omega : |x'| < d, \zeta < x_n < \zeta + 2d\}$$

is contained in Ω_0. Let $\varphi \in C^\infty(\mathbb{R}^n)$ with $\varphi = 0$ in $\Omega - \omega$ and $\varphi = 1$ in $\overline{\omega'}$ with

$$\omega' = \{x \in \Omega : |x'| < \delta, \zeta < x_n < \zeta + 2\delta, \delta < d\}\,.$$

If we make the change of variables

$$y_i' = x_i', \quad y_n = x_n - \zeta, \tag{IV.5.2}$$

the functions $\boldsymbol{v}$, p, $\boldsymbol{f}$, $\boldsymbol{v}_*$, and φ go over into $\widehat{\boldsymbol{v}}$, $\widehat{p}$, $\widehat{\boldsymbol{f}}$, $\widehat{\boldsymbol{v}}_*$, and $\widehat{\varphi}$, respectively, while ω and ω' are transformed into the cubes

$$\widehat{\omega} = \{y \in \mathbb{R}^n : |y'| < d, 0 < y_n < 2d\}$$

and

$$\widehat{\omega}' = \{y \in \mathbb{R}^n : |y'| < \delta, 0 < y_n < 2\delta\}\,,$$

respectively. Moreover, setting

$$\boldsymbol{u} = \widehat{\varphi}\widehat{\boldsymbol{v}}, \quad \pi = \widehat{\varphi}\widehat{p}$$

and extending all the fields to zero outside $\widehat{\omega}$, after a simple calculation we find[1]

$$\left.\begin{aligned} \Delta \boldsymbol{u} &= \nabla \pi + \boldsymbol{F} \\ \nabla \cdot \boldsymbol{u} &= g \end{aligned}\right\} \text{ in } \mathbb{R}^n_+ \tag{IV.5.3}$$
$$\boldsymbol{u} = \boldsymbol{\Phi} \;\text{ at } \Sigma \equiv \mathbb{R}^{n-1} \times \{0\},$$

where

$$\begin{aligned} F_i &\equiv \widehat{\varphi}\widehat{f}_i + D_j(b_{ji}\pi) + D_j(a_{jk}D_k u_i) + \alpha_i \widehat{p} + \beta \widehat{v}_i + \gamma_j D_j \widehat{v}_i \\ g &\equiv D_j(c_{ji}u_i) + \eta_i \widehat{v}_i \\ \Phi_i &\equiv \widehat{\varphi}\widehat{v}_{*i} \end{aligned} \tag{IV.5.4}$$

and a_{jk}, b_{jk}, c_{jk}, α_j, β, γ_j, and η_j are continuously differentiable functions in the closure of $\widehat{\omega}$. Moreover, the functions a_{jk}, b_{jk}, and c_{jk} are bounded by $A|\nabla\zeta|$ with a constant A independent of d, while α_j, β, γ_j, and η_j are zero outside $\widehat{\omega}$. Notice that by (IV.5.1)

[1] Notice that if f and $\widehat{f}$ are related by transformation (IV.5.2),

$$\frac{\partial f}{\partial x_i} = \frac{\partial \widehat{f}}{\partial y_i} - \frac{\partial \widehat{f}}{\partial y_n}\frac{\partial \zeta}{\partial x_i}, \quad i = 1, \dots n-1; \qquad \frac{\partial f}{\partial x_n} = \frac{\partial \widehat{f}}{\partial y_n};$$

and

$$\frac{\partial \widehat{f}}{\partial y_i} = \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_n}\frac{\partial \zeta}{\partial y_i}, \quad i = 1, \dots n-1.$$

$$a_{jk}(0) = b_{jk}(0) = c_{jk}(0) = 0. \tag{IV.5.5}$$

From (IV.5.4) we readily obtain

$$\begin{aligned}\|\boldsymbol{F}\|_{q,\mathbb{R}^n_+} &\leq c_1 \Big(\|\widehat{\boldsymbol{f}}\|_{q,\widehat{\omega}} + \|\widehat{p}\|_{q,\widehat{\omega}} \\ &\qquad + \|\widehat{\boldsymbol{v}}\|_{1,q,\widehat{\omega}} + a\|D^2\boldsymbol{u}\|_{q,\mathbb{R}^n_+} + b\|\nabla\pi\|_{q,\mathbb{R}^n_+}\Big) \qquad \text{(IV.5.6)} \\ \|g\|_{1,q,\mathbb{R}^n_+} &\leq c_2 \left(\|\widehat{\boldsymbol{v}}\|_{1,q,\widehat{\omega}} + c\|D^2\boldsymbol{u}\|_{q,\mathbb{R}^n_+}\right),\end{aligned}$$

where c_1 and c_2 can be taken independent of d, while

$$a = \max_{j,k}\max_{\widehat{\omega}}(a_{jk})$$

$$b = \max_{j,k}\max_{\widehat{\omega}}(b_{jk})$$

$$c = \max_{j,k}\max_{\widehat{\omega}}(c_{jk}).$$

Furthermore,

$$\boldsymbol{\Phi} \in W^{1,q}(\Sigma), \quad \langle\langle \nabla\boldsymbol{\Phi}\rangle\rangle_{1-1/q,q} < \infty,$$

see Exercise IV.5.1, and

$$D^2\boldsymbol{u} \in L^q(\mathbb{R}^n_+).$$

We may thus applyTheorem IV.3.2 with $m = 0$ to system (IV.5.3) to deduce, in particular, that $\boldsymbol{u}$ and π satisfy inequality (IV.3.30). We then have

$$\|\widehat{\boldsymbol{f}}\|_{q,\widehat{\omega}} + \|\widehat{\boldsymbol{v}}\|_{1,q,\widehat{\omega}} + \|\widehat{p}\|_{q,\widehat{\omega}} \leq c_3 \left(\|\boldsymbol{f}\|_{q,\omega} + \|\boldsymbol{v}\|_{1,q,\omega} + \|p\|_{q,\omega}\right) \tag{IV.5.7}$$

and, by Exercise IV.5.1, (II.4.19), and the properties of the function φ, we have also

$$\langle\langle \nabla\boldsymbol{\Phi}\rangle\rangle_{1-1/q,q} \leq c_4\|\widehat{\varphi}\widehat{\boldsymbol{v}}_*\|_{2-1/q,q(\Sigma)} \leq c_5\|\boldsymbol{v}_*\|_{2-1/q,q(\sigma)}, \tag{IV.5.8}$$

where c_3, c_4, and c_5 do not depend on d, $\boldsymbol{f}$, $\boldsymbol{v}$, p, and $\boldsymbol{\Phi}$. Collecting (IV.5.6)–(IV.5.8) and using (IV.3.30) we deduce

$$\begin{aligned}[1 - c_6(a+c)]\|D^2\boldsymbol{u}\|_{q,\mathbb{R}^n_+} &+ (1 - c_7 b)\|\nabla\pi\|_{q,\mathbb{R}^n_+} \\ &\leq c_8 \left(\|\boldsymbol{f}\|_{q,\omega} + \|\boldsymbol{v}_*\|_{2-1/q,q(\sigma)} + \|\boldsymbol{v}\|_{1,q,\omega} + \|p\|_{q,\omega}\right),\end{aligned} \tag{IV.5.9}$$

where, again, c_6, c_7, and c_8 do not depend on d, $\boldsymbol{f}$, $\boldsymbol{v}$, p, and $\boldsymbol{\Phi}$. Recalling (IV.5.5), we can take d as small to satisfy the conditions

$$(a + c) < 1/c_6, \quad b < 1/c_7.$$

Thus, from (IV.5.9) and the obvious inequalities:

$$\|w\|_{\ell,q,\omega'} \leq \|\varphi w\|_{\ell,q,\omega} \leq c_9\|\widehat{\varphi}\widehat{w}\|_{\ell,q,\mathbb{R}^n_+} \leq c_{10}\|\varphi w\|_{\ell,q,\omega} \tag{IV.5.10}$$

holding for all $\ell \geq 0$ and for a suitable choice of c_9, c_{10}, we finally obtain

$$\begin{aligned}\|\boldsymbol{v}\|_{2,q,\omega'}&+ \|p\|_{1,q,\omega'} \\ &\leq c_{11}\left(\|\boldsymbol{f}\|_{q,\omega} + \|\boldsymbol{v}_*\|_{2-1/q,q(\sigma)} + \|\boldsymbol{v}\|_{1,q,\omega} + \|p\|_{q,\omega}\right),\end{aligned}$$

with c_{11} independent of $\boldsymbol{v}$, p, $\boldsymbol{f}$ and $\boldsymbol{v}_*$. It is now easy to generalize this estimate to the case when

$$\boldsymbol{v} \in W^{m+2,q}(\Omega_0), \quad p \in W^{m+2,q}(\Omega_0), \ m > 0.$$

The corresponding assumptions on the data are then

$$\boldsymbol{f} \in W^{m,q}(\Omega_0), \quad \boldsymbol{v}_* \in W^{m+2-1/q,q}(\sigma) \tag{IV.5.11}$$

and, further, σ of class C^{m+2}. To this end, it is sufficient to use Theorem IV.3.2 in its generality along with an inductive argument entirely analogous to that employed in Theorem IV.4.1. If we do this we obtain

$$\begin{aligned}\|\boldsymbol{v}\|_{m+2,q,\omega'}&+ \|p\|_{m+1,q,\omega'} \\ &\leq c_{12}\left(\|\boldsymbol{f}\|_{m,q,\omega} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\sigma)} + \|\boldsymbol{v}\|_{1,q,\omega} + \|p\|_{q,\omega}\right)\end{aligned} \tag{IV.5.12}$$

which is the general estimate we wanted to show.

Following Cattabriga (1961), we shall now prove that for (IV.5.12) to hold it is enough to assume only

$$\boldsymbol{v} \in W^{1,q}(\Omega_0), \quad p \in L^q(\Omega_0). \tag{IV.5.13}$$

More precisely, suppose $\boldsymbol{v}$, p is such a pair of fields satisfying identity (IV.1.3) for all $\boldsymbol{\psi} \in C_0^\infty(\Omega_0)$ and some $\boldsymbol{f}$, along with the condition $\boldsymbol{v} = \boldsymbol{v}_*$ at σ (in the trace sense). Then if $\boldsymbol{f}$ and $\boldsymbol{v}_*$ verify (IV.5.12) for $m \geq 0$ and σ is of class C^{m+2} it follows that the norms of $\boldsymbol{v}$ and p on the left-hand side of (IV.5.13) are finite and (IV.5.13) holds. We shall prove this assertion for $m = 0$, the general case being treated by a simple iteration and therefore left to the reader. By performing on $\boldsymbol{v}$ and p the same kind of transformation made previously to arrive at (IV.5.3), we can deduce that this time $\boldsymbol{u}$ and π are q-generalized solutions to the inhomogeneous Stokes problem in the half-space (see the definition before Theorem IV.3.3) corresponding to the data $\boldsymbol{F}$, g and $\boldsymbol{\Phi}$ defined in $(\text{IV.5.4})_3$. In particular we have

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{\psi}) - (\pi, \nabla \cdot \boldsymbol{\psi}) = -(\boldsymbol{F}, \boldsymbol{\psi}), \quad \text{for all } \boldsymbol{\psi} \in D_0^{1,q'}(\mathbb{R}^n_+). \tag{IV.5.14}$$

Replace $\boldsymbol{\psi}$ by the difference quotient $\Delta^h \boldsymbol{\psi}$, where the variation is taken with respect to any of the first $n-1$ coordinates, see Exercise II.3.13. Since

$$\int_{\mathbb{R}^n_+} \Delta^h \psi_i w = -\int_{\mathbb{R}^n_+} \psi_i \Delta^{-h} w,$$

we immediately deduce that $\Delta^h \boldsymbol{u}$ and $\Delta^h \pi$ are again a q-generalized solution corresponding to the free terms $\Delta^h \boldsymbol{F}$, $\Delta^h g$ and to the boundary data $\Delta^h \boldsymbol{\Phi}$. Recalling that

$$\|\Delta^h w\|_q \leq \|\nabla w\|_q,$$

see Exercise II.3.13(iii), from (IV.5.4) and (IV.5.7) it follows that

$$\begin{aligned}|\Delta^h \boldsymbol{F}|_{-1,q,\mathbb{R}^n_+} \leq c_1 \big(&\|\boldsymbol{f}\|_{q,\omega} + \|p\|_{q,\omega} + \|\boldsymbol{v}\|_{1,q,\omega} \\ &+a\|\Delta^h \nabla \boldsymbol{u}\|_{q,\mathbb{R}^n_+} + b\|\Delta^h \pi\|_{q,\mathbb{R}^n_+}\Big)\end{aligned}$$

$$\|\Delta^h g\|_{q,\mathbb{R}^n_+} \leq c_2 \left(\|\boldsymbol{v}\|_{1,q,\omega} + c\|\Delta^h \nabla \boldsymbol{u}\|_{q,\mathbb{R}^n_+}\right)$$

with c_1, c_2 independent of d and h. Since

$$\Delta^h \boldsymbol{u} \in D^{1,q}(\mathbb{R}^n_+), \quad \Delta^h \pi \in L^q(\mathbb{R}^n_+),$$

we may use this latter inequality, the results of Exercise IV.5.1 and estimate (IV.3.32) together with the property

$$\nabla \Delta^h \boldsymbol{u} = \Delta^h \nabla \boldsymbol{u}$$

to deduce

$$\begin{aligned}[1 - c_1(a+c)]\|\Delta^h \nabla \boldsymbol{u}\|_{q,\mathbb{R}^n_+} &+(1-c_1 b)\|\Delta^h \pi\|_{q,\mathbb{R}^n_+} \\ &\leq c_3 \left(\|\boldsymbol{f}\|_{q,\omega} + \|\boldsymbol{v}_*\|_{2-1/q,q(\sigma)} + \|\boldsymbol{v}\|_{1,q,\omega} + \|p\|_{q,\omega}\right)\end{aligned} \tag{IV.5.15}$$

with c_3 independent of h and d. Taking d so small that

$$(a+c) < 1/c_1, \quad b < 1/c_2,$$

from (IV.5.15) it follows that

$$\|\Delta^h \nabla \boldsymbol{u}\|_{q,\mathbb{R}^n_+} + \|\Delta^h \pi\|_{q,\mathbb{R}^n_+} \leq c_4 \left(\|\boldsymbol{f}\|_{q,\omega} + \|\boldsymbol{v}_*\|_{2-1/q,q(\sigma)} + \|\boldsymbol{v}\|_{1,q,\omega} + \|p\|_{q,\omega}\right),$$

with c_4 independent of h. From the properties of the difference quotient (see Exercise II.3.13), we then conclude that $\nabla' \nabla \boldsymbol{u}$ and $\nabla' \pi$ exist[2] and belong to $L^q(\mathbb{R}^n_+)$. Moreover,

$$\|\nabla' \nabla \boldsymbol{u}\|_{q,\mathbb{R}^n_+} + \|\nabla' \pi\|_{q,\mathbb{R}^n_+} \leq c_4 \left(\|\boldsymbol{f}\|_{q,\omega} + \|\boldsymbol{v}_*\|_{2-1/q,q(\sigma)} + \|\boldsymbol{v}\|_{1,q,\omega} + \|p\|_{q,\omega}\right). \tag{IV.5.16}$$

From $(\text{IV.5.3})_2$ and (IV.5.16) it is not hard to show

[2] We recall that ∇' is defined below (IV.3.10).

$$\boldsymbol{u} \in D^{2,q}(\mathbb{R}^n_+), \quad \pi \in D^{1,q}(\mathbb{R}^n_+). \tag{IV.5.17}$$

To fix the ideas, consider the case $n = 3$. From (IV.5.16) and $(\text{IV.5.3})_2 II.1.8$ it follows that

$$(D_3^2 u_3 + c_{3i} D_3^2 u_i) \in L^q(\mathbb{R}^n_+). \tag{IV.5.18}$$

Moreover, integrating (IV.5.14) by parts and again employing (IV.5.16) one can prove for all $i = 1, 2, 3$,

$$[(1 + a_{33}) D_3^2 u_i - (\delta_{3i} + b_{3i}) D_3 \pi] \in L^q(\mathbb{R}^n_+). \tag{IV.5.19}$$

Conditions (IV.5.18) and (IV.5.19) then yield

$$\begin{aligned}(1 + a_{33}) \{ [c_{31} \ D_3^2 u_1 + \ c_{32} D_3^2 u_2 + (1 + c_{33}) D_3^2 u_3] \\ - \ D_3 \pi \, [c_{31} b_{31} + c_{32} + (1 + b_{33})(1 + c_{33})] \} \in L^q(\mathbb{R}^n_+).\end{aligned} \tag{IV.5.20}$$

Because of (IV.5.5) we can choose d so small that the coefficient of $D_3\pi$ is strictly positive. From (IV.5.18) and (IV.5.20) we thus derive

$$D_3 \pi \in L^q(\mathbb{R}^n_+), \tag{IV.5.21}$$

which, in view of (IV.5.19), gives

$$D_3^2 u_i \in L^q(\mathbb{R}^n_+). \tag{IV.5.22}$$

From (IV.5.16), (IV.5.21), and (IV.5.22) we then conclude the validity of (IV.5.17). On the other hand, from (IV.5.10), (IV.5.13) and (IV.5.17) we can finally assert

$$\boldsymbol{v} \in W^{2,q}(\omega'), \quad p \in W^{1,q}(\omega'),$$

which is what we wanted to show.

The results obtained so far can then be summarized in the following

Theorem IV.5.1 *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \geq 2$, with a boundary portion σ of class C^{m+2}, $m \geq 0$. Let Ω_0 be any bounded subdomain of Ω with $\partial\Omega_0 \cap \partial\Omega = \sigma$. Further, let*

$$\boldsymbol{v} \in W^{1,q}(\Omega_0), \quad p \in L^q(\Omega_0), \quad 1 < q < \infty,$$

be such that

$$\begin{aligned}(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) &= -\langle \boldsymbol{f}, \boldsymbol{\psi} \rangle + (p, \nabla \cdot \boldsymbol{\psi}), \quad \textit{for all } \boldsymbol{\psi} \in C_0^\infty(\Omega_0), \\ (\boldsymbol{v}, \nabla \varphi) &= 0 \quad \textit{for all } \varphi \in C_0^\infty(\Omega_0), \\ \boldsymbol{v} &= \boldsymbol{v}_* \quad \text{at } \sigma.\end{aligned}$$

Then, if

$$\boldsymbol{f} \in W^{m,q}(\Omega_0), \quad \boldsymbol{v}_* \in W^{m+2-1/q,q}(\sigma),$$

we have

$$\boldsymbol{v} \in W^{m+2,q}(\Omega'), \quad p \in W^{m+1,q}(\Omega'),$$

for any Ω' satisfying:

(i) $\Omega' \subset \Omega_0$;
(ii) $\partial\Omega' \cap \partial\Omega$ *is a strictly interior subregion of* σ.

Finally, the following estimate holds

$$\|\boldsymbol{v}\|_{m+2,q,\Omega'} + \|p\|_{m+1,q,\Omega'} \le c\left(\|\boldsymbol{f}\|_{m,q,\Omega_0} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\sigma)} + \|\boldsymbol{v}\|_{1,q,\Omega_0} + \|p\|_{q,\Omega_0}\right), \tag{IV.5.23}$$

where $c = c(m, n, q, \Omega', \Omega_0)$.

Remark IV.5.1 A consideration similar to that made in Remark IV.4.1 applies also to the estimate (IV.5.23). ■

From this theorem and Theorem IV.4.3 we thus obtain, in particular, the following result concerning global regularity of q-generalized solutions.

Theorem IV.5.2 *Let* $\boldsymbol{v}$ *be a* q*-generalized solution to the Stokes problem* (IV.0.1), (IV.0.2). *Then if* Ω *is of class* C^∞, $\boldsymbol{f} \in C^\infty(\overline{\Omega})$ *and* $\boldsymbol{v}_* \in C^\infty(\partial\Omega)$,[3] *it follows that* $\boldsymbol{v}$, $p \in C^\infty(\overline{\Omega})$, *where* p *is the pressure field associated to* $\boldsymbol{v}$ *by* Lemma IV.1.1.

As in the case of interior regularity, intermediate smoothness results can be obtained from Theorem IV.4.1, Theorem IV.5.1, and the embedding Theorem II.3.4. We leave them to the reader as an exercise. Regularity in Hölder spaces can be obtained from the results of Section IV.7.

Exercise IV.5.1 (Agmon, Douglis, & Nirenberg 1959). Let $\Omega \subset \mathbb{R}^n$ have a smooth boundary portion σ and let $\phi \in W^{m-1/q,q}(\sigma)$, $m \ge 1$, $1 < q < \infty$, be of compact support in σ. Perform the change of variables (IV.5.2) for a sufficiently regular ζ, and denote by $\widehat{\phi}$, $\widehat{\phi}$, $\widehat{\sigma}$ the transforms of ϕ and σ under this change, so that $\widehat{\phi}$ can be considered defined in the whole of $\mathbb{R}^{n-1}$ and of compact support in $\widehat{\sigma}$. Show the existence of a constant c independent of ϕ such that

$$c^{-1}\|\phi\|_{m-1/q,q(\sigma)} \le \|\widehat{\phi}\|_{m-1/q,q(\mathbb{R}^{n-1})} \le c\|\phi\|_{m-1/q,q(\sigma)}.$$

Hint: Let $\Omega_1 = \Omega \cap B$, with $\partial\Omega_1 \cap \partial\Omega = \sigma$ and B a ball centered at $x_0 \in \sigma$. Denote by $\widehat{\Omega}_1$ the transform of Ω_1. By Theorem II.10.2, we may find $v \in W^{m,q}(\mathbb{R}^n_+)$, of compact support in the closure of $\widehat{\Omega}_1$ and such that $v = \widehat{\phi}$ at $\widehat{\sigma}$, $v = 0$ at $\partial\widehat{\Omega}_1 - \widehat{\sigma}$, and, moreover,

$$\|v\|_{m,q,\mathbb{R}^n_+} \le c_1\|\widehat{\phi}\|_{m-1/q,q(\mathbb{R}^{n-1})}.$$

If u is the transform of v under the inverse of (IV.5.2), one has $u = \phi$ at σ and

$$\|v\|_{m,q,\mathbb{R}^n_+} \le c_1\|u\|_{m,q,\Omega_1} \le c_2\|v\|_{m,q,\mathbb{R}^n_+} \le c_3c_1\|\widehat{\phi}\|_{m-1/q,q(\mathbb{R}^{n-1})},$$

which proves the first inequality. The second one is proved analogously.

[3] Namely, $\boldsymbol{v}_*$ is infinitely differentiable along the boundary.

Exercise IV.5.2 Show that Theorem IV.5.1 also holds when $\nabla\cdot\boldsymbol{v} = g \not\equiv 0, g \in W^{m+1,q}(\Omega_0)$, provided we add the term

$$\|g\|_{m+1,q,\Omega_0}$$

on the right-hand side of (IV.5.20).

Exercise IV.5.3 Let Ω, σ and Ω_0, and Ω' be as in Theorem IV.5.1. Let $u \in W^{1,q}(\Omega_0)$, $q \in (1,\infty)$, satisfy the following condition

$$\begin{aligned}&(\nabla u, \nabla\psi) = -\langle f, \psi\rangle\,, \quad \text{for all } \psi \in C_0^\infty(\Omega_0)\\ &u = u_* \quad \text{at } \sigma\,.\end{aligned}$$

Show that, if $f \in W^{m,q}(\Omega_0)$, $u_* \in W^{m+2-1/q,q}(\sigma)$, $m \geq 0$, then necessarily $u \in W^{m+2,q}(\Omega')$, and the following estimate holds

$$\|u\|_{m+2,q,\Omega'} \leq c\left(\|f\|_{m,q,\Omega_0} + \|u_*\|_{m+2-1/q,q,(\sigma)}\right),$$

with c independent of u, f and u_*.

We conclude this section by giving an estimate near the boundary involving the L^q-norm of $\nabla\boldsymbol{v}$ and p. Specifically we have

Theorem IV.5.3 *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \geq 2$, with a boundary portion σ of class C^1.*[4] *Let, further, Ω', Ω_0, $\boldsymbol{v}$, and p be as in* Theorem IV.5.1. *Then if*

$$\boldsymbol{f} \in W_0^{-1,q}(\Omega_0), \quad \boldsymbol{v}_* \in W^{1-1/q,q}(\sigma),$$

the following inequality holds:

$$\begin{aligned}&\|\boldsymbol{v}\|_{1,q,\Omega'} + \|p\|_{q,\Omega'}\\ &\qquad \leq c\left(\|\boldsymbol{f}\|_{-1,q,\Omega_0} + \|\boldsymbol{v}_*\|_{1-1/q,q(\sigma)} + \|\boldsymbol{v}\|_{q,\Omega_0} + \|p\|_{-1,q,\Omega_0}\right).\end{aligned} \tag{IV.5.24}$$

Proof. Transforming $\boldsymbol{v}$ and p into $\boldsymbol{u}$ and π, respectively, as before, we readily obtain that $\boldsymbol{u}$ and π obey (IV.5.3), (IV.5.4) in the weak form. Successively, we apply the results of Theorem IV.3.3 to $\boldsymbol{u}$, π and derive, in particular, that they obey the inequality

$$|\boldsymbol{u}|_{1,q} + \|\pi\|_q \leq c_1\left(|\boldsymbol{F}|_{-1,q} + \|g\|_q\right). \tag{IV.5.25}$$

Recalling (IV.5.4) and (IV.5.10) it is not difficult to show that

$$\begin{aligned}&|\boldsymbol{F}|_{-1,q,\mathbb{R}^n_+} \leq c_2\left(\|\boldsymbol{f}\|_{-1,q,\omega} + \|p\|_{-1,q,\omega} + \|\boldsymbol{v}\|_{q,\omega} + a\,\|\nabla\boldsymbol{u}\|_{q,\mathbb{R}^n_+} + b\|\pi\|_{q,\mathbb{R}^n_+}\right)\\ &\|g\|_{q,\mathbb{R}^n_+} \leq c_3\left(\|\boldsymbol{v}\|_{q,\omega} + c\|\nabla\boldsymbol{u}\|_{q,\mathbb{R}^n_+}\right).\end{aligned} \tag{IV.5.26}$$

Therefore, (IV.5.24) becomes a consequence of (IV.5.25), (IV.5.26), and (IV.5.10). □

[4] Notice that we only need σ of class C^1. In fact the result can be extended to σ Lipschitz but with "not too sharp" corners; see Galdi, Simader, & Sohr (1994).

Remark IV.5.2 A consideration similar to that made in Remark IV.4.2 applies also to the estimate (IV.5.24). ■

Exercise IV.5.4 Show that Theorem IV.5.3 continues to hold if $\nabla \cdot \boldsymbol{v} = g \not\equiv 0$, where $g \in L^q(\Omega_0)$. The inequality (IV.5.21) is then modified by adding the term

$$\|g\|_{q,\Omega_0}$$

on its right-hand side.

IV.6 Existence, Uniqueness, and L^q-Estimates in a Bounded Domain

The interior and boundary inequalities demonstrated in Section IV.4 and Section IV.5 allow us to derive L^q-estimates for q-generalized solutions holding for the whole of Ω, in the case where Ω is bounded and suitably regular. Specifically, setting

$$\|w\|_{k,q/\mathbb{R}} = \inf_{c\in\mathbb{R}} \|w + c\|_{k,q} \tag{IV.6.1}$$

we have

Lemma IV.6.1 *Let $\boldsymbol{v}$ be a q-generalized solution to the Stokes problem* (IV.0.1), (IV.0.2) *in a bounded domain Ω of $\mathbb{R}^n$, $n \ge 2$, of class C^{m+2}, $m \ge 0$, corresponding to*

$$\boldsymbol{f} \in W^{m,q}(\Omega), \quad \boldsymbol{v}_* \in W^{m+2-1/q,q}(\partial\Omega).$$

Then,

$$\boldsymbol{v} \in W^{m+2,q}(\Omega), \quad p \in W^{m+1,q}(\Omega),$$

where p is the pressure field associated to $\boldsymbol{v}$ by Lemma IV.1.1. *Moreover, the following inequality holds:*

$$\|\boldsymbol{v}\|_{m+2,q} + \|p\|_{m+1,q/\mathbb{R}} \le c\left(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)}\right) \tag{IV.6.2}$$

with $c = c(m, n, q, \Omega)$.

Proof. By covering $\overline{\Omega}$ with a finite number of open balls, from Theorem IV.4.1 and Theorem IV.5.1 we deduce

$$\boldsymbol{v} \in W^{m+2,q}(\Omega), \quad p \in W^{m+1,q}(\Omega)$$

and the validity of the inequality

$$|\boldsymbol{v}|_{m+2,q} + |p|_{m+1,q} \le c_1\left(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)} + \|p\|_q + \|\boldsymbol{v}\|_{1,q}\right).$$

We add to both sides of this inequality the L^q-norms of all derivatives of $\boldsymbol{v}$ [respectively, of p] up to the order $m+1$ [respectively, m] and employ Ehrling's inequality (II.5.20) to derive

$$\|\boldsymbol{v}\|_{m+2,q} + \|p\|_{m+1,q} \le c_2 \left(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)} + \|p\|_q + \|\boldsymbol{v}\|_q \right). \tag{IV.6.3}$$

Clearly, (IV.6.3) remains unaffected if we replace p with $p+c$, for any $c \in \mathbb{R}$. Thus, taking the $\inf_{c\in\mathbb{R}}$ of both sides of this new inequality, we obtain

$$\begin{aligned} \|\boldsymbol{v}\|_{m+2,q} + \|p\|_{m+1,q/\mathbb{R}} \le c_2 \big(\|\boldsymbol{f}\|_{m,q} &+ \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)} \\ &+ \|p\|_{q/\mathbb{R}} + \|\boldsymbol{v}\|_q \big). \end{aligned} \tag{IV.6.4}$$

It is easy to show that, provided the solution is unique, we can drop the last two terms on the right-hand side of (IV.6.4). In fact, it is enough to show the existence of a constant c_3 independent of the data and the particular solution such that

$$\|\boldsymbol{v}\|_q + \|p\|_{q/\mathbb{R}} \le c_3 \left(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)} \right). \tag{IV.6.5}$$

If (IV.6.5) were not true, a sequence would exist such as

$$\{\boldsymbol{v}_k\} \subset W^{m+2,q}(\Omega), \quad \{p_k\} \subset W^{m+1,q}(\Omega),$$

with

$$\|\boldsymbol{v}_k\|_q + \|p_k\|_{q/\mathbb{R}} = 1, \quad \text{for all } k \in \mathbb{N},$$

while the right-hand side of (IV.6.5) tends to zero. By (IV.6.3) we then have

$$\|\boldsymbol{v}_k\|_{m+2,q} + \|p_k\|_{m+1,q/\mathbb{R}}$$

uniformly bounded in k and therefore we may select a subsequence which, by the compactness result of Exercise II.5.8, converges strongly to limits

$$\boldsymbol{u} \in W^{1,q}(\Omega), \quad \pi \in L^q(\Omega),$$

respectively, with

$$\|\boldsymbol{u}\|_q + \|\pi\|_{q/\mathbb{R}} = 1.$$

However, this last relation is easily contradicted. Actually, it is immediately shown that $\boldsymbol{u}$ is a q-generalized solution to the Stokes problem in Ω corresponding to $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv 0$ and so, by the uniqueness hypothesis, $\boldsymbol{u} \equiv 0$, $\pi = \text{const}$. The proof of the lemma is therefore completed, once we have shown the following result. □

Lemma IV.6.2 *Let Ω be a bounded, C^2-smooth domain of $\mathbb{R}^n$. If $\boldsymbol{v}$ is a q-weak solution to the Stokes problem* (IV.0.1), (IV.0.2) *corresponding to zero data, then $\boldsymbol{v} \equiv 0$, $p \equiv$ const. a.e. in Ω, where p is the pressure field associated to $\boldsymbol{v}$ by* Lemma IV.1.1.

Proof. If $q \ge 2$, by the uniqueness part of Theorem IV.1.1 we already know that the previous statement is true (even assuming less smoothness on Ω). If $q < 2$, from the first part of Lemma IV.6.1 we have

$$\boldsymbol{v} \in W^{2,q}(\Omega), \quad p \in W^{1,q}(\Omega),$$

and so, by the embedding Theorem II.3.2, it follows that

$$\boldsymbol{v} \in W^{1,r_1}(\Omega), \quad p \in L^{r_1}(\Omega), \quad r_1 = nq/(n-q).$$

Now, if $r_1 \geq 2$ we are finished; otherwise, by the first part of Lemma IV.6.1 we have

$$\boldsymbol{v} \in W^{2,r_1}(\Omega), \quad p \in W^{1,r_1}(\Omega),$$

and so, again by Theorem II.3.2, it follows that

$$\boldsymbol{v} \in W^{1,r_2}(\Omega), \quad p \in L^{r_2}(\Omega), \quad r_2 = nq/(n-2q) \quad (> r_1).$$

If $r_2 \geq 2$ the proof is achieved; otherwise,

$$\boldsymbol{v} \in W^{2,r_2}(\Omega), \quad p \in W^{1,r_2}(\Omega)$$

and we continue this procedure as many times as needed until we arrive to show, after a *finite* number of steps,

$$\boldsymbol{v} \in W^{1,2}(\Omega).$$

The lemma is therefore completely proved. □

We now turn our attention to the question of existence of q-generalized solutions. When $q = 2$ the answer is already furnished in (IV.1.1). In the general case we argue as follows. Given

$$\boldsymbol{f} \in W^{m,q}(\Omega), \quad \boldsymbol{v}_* \in W^{m+2-1/q,q}(\partial\Omega), \quad 1 < q < \infty,$$

with

$$\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} = 0, \tag{IV.6.6}$$

let us approximate them with sequences $\{\boldsymbol{f}_k\}, \{\boldsymbol{v}_{*k}\}$ of sufficiently smooth functions. We can always assume

$$\int_{\partial\Omega} \boldsymbol{v}_{*k} \cdot \boldsymbol{n} = 0.$$ [1]

[1] Actually, let $\{\boldsymbol{v}_k\}$ be a sequence of smooth functions tending to $\boldsymbol{v}_*$ in $W^{2-1/q,q}(\partial\Omega)$, and let ϕ be a smooth function with $\int_{\partial\Omega} \phi = 1$. The sequence

$$\boldsymbol{v}_{*k} = \boldsymbol{v}_k - \phi \int_{\partial\Omega} \boldsymbol{v}_k \cdot \boldsymbol{n}, \quad k \in \mathbb{N}$$

is smooth, tends to $\boldsymbol{v}$ in $W^{2-1/q,q}(\partial\Omega)$, and satisfies

$$\int_{\partial\Omega} \boldsymbol{v}_{*k} \cdot \boldsymbol{n} = 0.$$

Denote by

$$\{\boldsymbol{v}_k\}, \quad \{p_k\}$$

the corresponding solutions whose existence is ensured by Theorem IV.1.1. From Lemma IV.6.1 we have for all $k \in \mathbb{N}$:

$$\boldsymbol{v}_k \in W^{2,2}(\Omega), \quad p_k \in W^{1,2}(\Omega).$$

If $n = 2$, the embedding Theorem II.3.4 tells us

$$\boldsymbol{v}_k \in W^{1,r}(\Omega), \quad p_k \in L^r(\Omega), \quad \text{for any } r \in (1, \infty)$$

and so Lemma IV.6.1 ensures

$$\boldsymbol{v}_k \in W^{2,q}(\Omega), \quad p_k \in W^{1,q}(\Omega)$$

and estimate (IV.6.1) holds. We then let $k \to \infty$ and use (IV.6.1) to obtain for some $\boldsymbol{v} \in W^{2,q}(\Omega)$, $p \in W^{1,q}(\Omega)$

$$\boldsymbol{v}_k \to \boldsymbol{v} \ \text{ in } W^{2,q}(\Omega),$$

$$p_k \to p \ \text{ in } W^{1,q}(\Omega).$$

Clearly, $\boldsymbol{v}$, p solve *a.e.* the Stokes system (IV.0.1) corresponding to $\boldsymbol{f}$, while $\boldsymbol{v}$ equals $\boldsymbol{v}_*$ at the boundary in the trace sense. For $n > 2$, we have

$$\boldsymbol{v}_k \in W^{1,r}(\Omega), \quad p_k \in L^r(\Omega), \quad \text{for any } r \in (1, 2n/(n-2)).$$

Thus, if $2 < n \le 4$, we again use Lemma IV.6.1 and Theorem II.3.4 to deduce

$$\boldsymbol{v} \in W^{2,q}(\Omega), \quad p \in W^{1,q}(\Omega).$$

We then proceed as in the case where $n = 2$. For $n > 4$, by a double application of Lemma IV.6.1 and Theorem II.3.4 we have

$$\boldsymbol{v}_k \in W^{1,r}(\Omega), \quad p_k \in L^r(\Omega), \quad \text{for any } r \in (1, 2n/(n-4))$$

and, by the same token, we recover existence if $4 < n \le 6$, and so forth. Existence of solutions for all $1 < q < \infty$ and *any* space dimension can therefore be fully established.

By means of a similar procedure, we may also show existence of q-weak solutions corresponding to arbitrary

$$\boldsymbol{f} \in W_0^{-1,q}(\Omega), \quad \boldsymbol{v}_* \in W^{1-1/q,q}(\partial\Omega), \quad 1 < q < \infty,$$

with $\boldsymbol{v}_*$ satisfying (IV.6.3) and Ω of class C^2. In fact, if $\boldsymbol{v}$ is a q-weak solution, from Theorem IV.4.4 and Theorem IV.5.3 we derive

$$\|\boldsymbol{v}\|_{1,q} + \|p\|_q \le c\left(\|\boldsymbol{f}\|_{-1,q} + \|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)} + \|\boldsymbol{v}\|_q + \|p\|_{-1,q}\right) \qquad \text{(IV.6.7)}$$

where $c = c(n, q, \Omega)$ and p is the pressure field associated to $\boldsymbol{v}$ by Lemma IV.1.1. From (IV.1.3) it is apparent that the inequality just obtained remains unaffected if we replace p with $p + c$, $c \in \mathbb{R}$. We then recover

$$\|\boldsymbol{v}\|_{1,q} + \|p\|_{q/\mathbb{R}} \leq c \left(\|\boldsymbol{f}\|_{-1,q} + \|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)} + \|\boldsymbol{v}\|_q + \|p\|_{-1,q/\mathbb{R}}\right).$$

The last two terms on the right-hand side of this relation can be increased by the data:

$$\|\boldsymbol{v}\|_q + \|p\|_{-1,q/\mathbb{R}} \leq C \left(\|\boldsymbol{f}\|_{-1,q} + \|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)}\right), \tag{IV.6.8}$$

with $C = C(q, n, \Omega)$. This can be proved by the same contradiction argument used to show (IV.6.5). In fact, if (IV.6.8) were not true, there would exist a sequence of solutions

$$\{\boldsymbol{v}_k, p_k\} \subset W^{1,q}(\Omega) \times (L^q(\Omega) \,/\, \mathbb{R})$$

with

$$\|\boldsymbol{v}_k\|_q + \|p_k\|_{-1,q/\mathbb{R}} = 1, \quad \text{for all } k \in \mathbb{N},$$

corresponding to data $\{\boldsymbol{f}_k, \boldsymbol{v}_{*k}\}$ converging to zero in the space $W^{-1,q}(\Omega) \times W^{1-1/q,q}(\partial\Omega)$. However, by the compactness results of Theorem II.5.3 and Exercise II.5.8, we find

$$\{\boldsymbol{v}, p\} \in W^{1,q}(\Omega) \times L^q(\Omega) \tag{IV.6.9}$$

such that

$$\boldsymbol{v}_k \text{ converges to } \boldsymbol{v} \quad \text{weakly in } W^{1,q}(\Omega), \text{ strongly in } L^q(\Omega)$$

$$p_k \text{ converges to } p \quad \text{weakly in } L^q(\Omega) \,/\, \mathbb{R}, \text{ strongly in } W^{-1,q}(\Omega).$$

Since $\boldsymbol{v}, p$ is a solution to the Stokes problem with $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv 0$, by Lemma IV.6.2 it follows that, as $k \to \infty$,

$$\boldsymbol{v} \equiv 0, \quad p = \text{const}, \quad \text{in } \Omega,$$

and therefore (IV.6.9) cannot hold. We then conclude the validity of the inequality

$$\|\boldsymbol{v}\|_{1,q} + \|p\|_q \leq c \left(\|\boldsymbol{f}\|_{-1,q} + \|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)}\right).$$

By means of this relation, we may argue as before to prove existence of q-generalized solutions.

The results shown so far in this section are collected in the following main theorem.

Theorem IV.6.1 *Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 2$. The following properties hold.*

(a) *Suppose Ω of class C^{m+2}, $m \geq 0$. Then, for any*

$$\boldsymbol{f} \in W^{m,q}(\Omega), \quad \boldsymbol{v}_* \in W^{m+2-1/q,q}(\partial\Omega), \quad 1 < q < \infty,$$

with

$$\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} = 0,$$

there exists one and only one pair $\boldsymbol{v}$, p[2] *such that*

(i) $\boldsymbol{v} \in W^{m+2,q}(\Omega)$, $p \in W^{m+1,q}(\Omega)$;

(ii) *$\boldsymbol{v}, p$ verify the Stokes system* (IV.0.1) *a.e. in Ω and $\boldsymbol{v}$ satisfies* (IV.0.2) *in the trace sense.*

In addition, this solution obeys the inequality

$$\|\boldsymbol{v}\|_{m+2,q} + \|p\|_{m+1,q/\mathbb{R}} \leq c_1 \left(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)} \right), \quad \text{(IV.6.10)}$$

where $c_1 = c_1(n, m, q, \Omega)$.

(b) *Suppose Ω of class C^2. Then, for every*

$$\boldsymbol{f} \in W_0^{-1,q}(\Omega), \quad \boldsymbol{v}_* \in W^{1-1/q,q}(\partial\Omega), \quad 1 < q < \infty,$$

there exists one and only one q-generalized solution $\boldsymbol{v}$ to the Stokes problem (IV.0.1), (IV.0.2). *This solution satisfies the inequality*

$$\|\boldsymbol{v}\|_{1,q} + \|p\|_{q/\mathbb{R}} \leq c_2 \left(\|\boldsymbol{f}\|_{-1,q} + \|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)} \right), \quad \text{(IV.6.11)}$$

where p is the pressure field associated to $\boldsymbol{v}$ by Lemma IV.1.1.

Exercise IV.6.1 Let $\boldsymbol{u} \in H_q^1(\Omega)$, $1 < q < \infty$, with Ω a C^2-smooth bounded domain. Show that there exists $c = c(n, q, \Omega)$ such that

$$\|\boldsymbol{u}\|_{1,q} \leq c \sup_{\boldsymbol{\varphi} \in H_{q'}^1(\Omega), \boldsymbol{\varphi} \neq \boldsymbol{0}} \left\{ \frac{|(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi})|}{\|\boldsymbol{\varphi}\|_{1,q'}} \right\}. \quad \text{(IV.6.12)}$$

Hint: The map $\boldsymbol{\varphi} \in H_{q'}^1(\Omega) \subset W_0^{1,q'}(\Omega) \to (\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi})$ defines a linear functional. Therefore, by the Hahn-Banach Theorem II.1.7, there is $\overline{\boldsymbol{f}} \in W_0^{-1,q}(\Omega)$ such that $\langle \overline{\boldsymbol{f}}, \boldsymbol{\varphi} \rangle = (\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi})$, $\boldsymbol{\varphi} \in H_{q'}^1(\Omega)$ and with $\|\overline{\boldsymbol{f}}\|_{-1,q}$ equal to the right-hand side of (IV.6.12). Consider then the Stokes problem with $\boldsymbol{v}_* \equiv \boldsymbol{0}$ and $\boldsymbol{f} \equiv \overline{\boldsymbol{f}}$ and apply the results of Theorem IV.6.1(b).

Exercise IV.6.2 Suppose $\boldsymbol{v}, p$ solves the Stokes problem (IV.0.1) with $\Omega \equiv B_R$, and suppose also $\boldsymbol{v} \in W^{m+2,q}(\Omega)$, $\boldsymbol{f} \in W^{m,q}(\Omega)$, for some $m \geq 0$, $q \in (1, \infty)$, and $\boldsymbol{v}_* \equiv \boldsymbol{0}$. Show that there exists a constant c *independent of* R such that

$$|\boldsymbol{v}|_{m+2,q} \leq c \, \|\boldsymbol{f}\|_{m,q}.$$

[2] p is determined up to a constant that may be fixed by requiring $\overline{p}_\Omega = 0$. In such a case, the term $\|p\|_{m+1,q/\mathbb{R}}$ can be replaced by $\|p\|_{m+1,q}$.

Exercise IV.6.3 Show that the first [respectively, second] part of Theorem IV.6.1 continues to hold if $\nabla \cdot \boldsymbol{v} = g \not\equiv 0$, with $g \in W^{m+1,q}(\Omega)$ [respectively, $g \in L^q(\Omega)$] and

$$\int_\Omega g = \int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n}.$$

Inequality (IV.6.10) [respectively, (IV.6.11)] is then accordingly modified by adding to its right-hand side the term

$$\|g\|_{m+1,q}, \quad \text{[respectively, } \|g\|_q\text{]}.$$

Hint: Use Exercise IV.4.2, Exercise IV.5.2, Exercise IV.4.3, and Exercise IV.5.4, together with the reasonings employed to arrive at Theorem IV.6.1.

We end this section by proving a further useful estimate satisfied by the pressure field p, in addition to those already provided by (IV.6.10) and (IV.6.11). Specifically, we have

Theorem IV.6.2 *Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 2$, of class C^2, and let $(\boldsymbol{v}, p) \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$ be a solution to* (IV.0.1), *corresponding to $\boldsymbol{f} \in H_q(\Omega)$, for some $q \in (1, \infty)$.*[3] *Furthermore, we normalize p by the condition $\overline{p}_\Omega = 0$. Then, given $\varepsilon > 0$, there exists $c = c(\Omega, n, q, r, \varepsilon)$ such that*

$$\|p\|_q \leq c\,|\boldsymbol{v}|_{1,q} + \varepsilon\,\|\boldsymbol{v}\|_{2,r}\,,$$

for any $r \in [q(n-1)/n, q]$, if $q > n/(n-1)$, and any $r \in (1, q]$, if $q \leq n/(n-1)$.

Proof. We dot-multiply both sides of (IV.0.1) by $\nabla\varphi$, $\varphi \in W^{1,q'}(\Omega)$, to obtain

$$\int_\Omega \Delta\boldsymbol{v} \cdot \nabla\varphi = \int_\Omega \nabla p \cdot \nabla\varphi\,, \quad \text{for all } \varphi \in W^{1,q'}(\Omega)\,. \tag{IV.6.13}$$

We then choose φ as the solution to the following Neumann problem

$$\Delta\varphi = g - \overline{g}_\Omega \text{ in } \Omega\,, \ \left.\frac{\partial\varphi}{\partial\boldsymbol{n}}\right|_{\partial\Omega} = 0\,, \ \int_\Omega \varphi = 0\,, \tag{IV.6.14}$$

where $g \in L^{q'}(\Omega)$. From a classical result of Agmon, Douglis & Nirenberg (1959, §15), it follows that this problem has one and only one solution $\varphi \in W^{2,q'}(\Omega)$, which, in addition, satisfies the estimate

$$\|\varphi\|_{2,q'} \leq c\,\|g\|_{q'}\,, \tag{IV.6.15}$$

for some $c = c(\Omega, q, n)$. Thus, integrating by parts on the right-hand side of (IV.6.13), using (IV.6.14), and the condition $\overline{p}_\Omega = 0$, we obtain

$$\int_\Omega \Delta\boldsymbol{v} \cdot \nabla\varphi = -\int_\Omega p\,g\,. \tag{IV.6.16}$$

[3] In view of the Helmholtz–Weyl decomposition Theorem III.1.2, we may assume, without loss, $\boldsymbol{f} \in H_q(\Omega)$, instead of $\boldsymbol{f} \in L^q(\Omega)$.

On the other hand, integrating by parts the left-hand side of this latter relation we find

$$\int_{\Omega} \Delta \boldsymbol{v} \cdot \nabla \varphi = \int_{\Omega} [\nabla \cdot (\nabla \boldsymbol{v} \cdot \nabla \varphi) - \nabla \boldsymbol{v} : \nabla\nabla\varphi] = \int_{\partial\Omega} \boldsymbol{n} \cdot \nabla \boldsymbol{v} \cdot \nabla \varphi - \int_{\Omega} \nabla \boldsymbol{v} : \nabla\nabla\varphi \,.$$

From this equation and (IV.6.16), we may conclude

$$\int_{\Omega} p\, g = \int_{\Omega} \nabla \boldsymbol{v} : \nabla\nabla\varphi - \int_{\partial\Omega} \boldsymbol{n} \cdot \nabla \boldsymbol{v} \cdot \nabla \varphi \,. \tag{IV.6.17}$$

From the Hölder inequality and (IV.6.15), we have

$$\left| \int_{\Omega} \nabla \boldsymbol{v} : \nabla\nabla\varphi \right| \leq c\, |\boldsymbol{v}|_{1,q} \|g\|_{q'} \,. \tag{IV.6.18}$$

Moreover, again by the Hölder inequality, the trace Theorem II.4.1, and (IV.6.15), we find

$$\begin{aligned} \left| \int_{\partial\Omega} \boldsymbol{n} \cdot \nabla \boldsymbol{v} \cdot \nabla \varphi \right| &\leq \|\nabla \boldsymbol{v}\|_{r,\partial\Omega} \|\nabla \varphi\|_{r',\partial\Omega} \leq c\, \|\nabla \boldsymbol{v}\|_{r,\partial\Omega} \|\varphi\|_{2,q'} \\ &\leq c\, \|\nabla \boldsymbol{v}\|_{r,\partial\Omega} \|g\|_{q'} \,, \end{aligned}$$

where $r' \in [q', q'(n-1)/(n-q')]$, if $q' < n$ and $r' \in [q', \infty)$, if $q' \geq n$. (These conditions are equivalent to those given for the exponent r in the statement of the theorem.) Finally, on the right-hand side of this last displayed relation, we employ the inequality given in Exercise II.4.1, to obtain

$$\left| \int_{\partial\Omega} \boldsymbol{n} \cdot \nabla \boldsymbol{v} \cdot \nabla \varphi \right| \leq \left(c_\varepsilon\, |\boldsymbol{v}|_{1,q} + \varepsilon\, \|\boldsymbol{v}\|_{2,r}\right) \|g\|_{q'} \,. \tag{IV.6.19}$$

Since g is an arbitrary element of $L^{q'}(\Omega)$, the theorem follows from (IV.6.17)–(IV.6.19) and Theorem II.2.2. □

By combining the previous result with inequality (IV.6.10), we immediately obtain the following

Corollary IV.6.1 *Let Ω, $\boldsymbol{v}$, p, $\boldsymbol{f}$ and r be as in* Theorem IV.6.2. *Then, for any $\varepsilon > 0$ there exists $c = c(\Omega, n, q, r, \varepsilon)$ such that*

$$\|p\|_q \leq c\, |\boldsymbol{v}|_{1,q} + \varepsilon \left(\|\boldsymbol{f}\|_r + \|\boldsymbol{v}_*\|_{2-1/r,r(\partial\Omega)} \right) \,,$$

where $\boldsymbol{v}_*$ is the trace of $\boldsymbol{v}$ at $\partial\Omega$.[4]

[4] In view of the trace Theorem II.4.4, $\boldsymbol{v}_* \in W^{2-1/q,q}(\partial\Omega)$, as a consequence of the assumption $\boldsymbol{v} \in W^{2,q}(\Omega)$.

IV.7 Existence and Uniqueness in Hölder Spaces. Schauder Estimates

Existence and uniqueness results similar to those proved in Lemma IV.6.1 and Theorem IV.6.1 can also be obtained in Hölder spaces $C^{k,\lambda}(\overline{\Omega})$, together with corresponding estimates (*Schauder estimates*). The procedure is the same as the one used for Sobolev spaces $W^{m,q}(\Omega)$; that is, one first shows existence, uniqueness, and the validity of corresponding estimates for solutions in $\mathbb{R}^n$ and $\mathbb{R}^n_+$ and, subsequently, one specializes the results to a (sufficiently smooth) bounded domain by means of the "localization procedure" used in the proof of Lemma IV.6.1.

However, to obtain existence in $\mathbb{R}^n$ and $\mathbb{R}^n_+$, instead of the Calderón–Zygmund Theorem II.11.4 and Theorem II.11.6, we have to employ their counterparts in Hölder spaces, namely, the Hölder-Lichtenstein-Giraud theorem; see, e.g., Bers, John, & Schechter (1964, pp. 223-224), and Theorem 3.1 of Agmon, Douglis, & Nirenberg (1959), respectively.

Since estimates in Hölder norms will not play any relevant role in this book, we shall not give details of their derivation, limiting ourselves to quote the main results without proofs. In this regard, it should be observed that they can be obtained, as a particular case, from the work of Agmon, Douglis, & Nirenberg (1964) and Solonnikov (1966) since, as already observed, the Stokes system is elliptic in the sense of Douglis-Nirenberg. Thus, from the uniqueness Lemma IV.6.2 and the results of Agmon, Douglis, & Nirenberg (1964, Theorem 9.3 and Remarks 1 and 2 that follow the theorem) we have

Theorem IV.7.1 *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, of class $C^{m+2,\lambda}$, $m \geq 0$, $\lambda \in (0,1)$, and let $\boldsymbol{v}$, p be a solution to the Stokes problem* (IV.0.1), (IV.0.2) *with*

$$\boldsymbol{v} \in C^{2,\lambda}(\overline{\Omega}), \quad p \in C^{1,\lambda}(\overline{\Omega}).$$

Then, if

$$\boldsymbol{f} \in C^{m,\lambda}(\overline{\Omega}), \quad \boldsymbol{v}_* \in C^{m+2,\lambda}(\partial\Omega),$$

we have

$$\boldsymbol{v} \in C^{m+2,\lambda}(\overline{\Omega}), \quad p \in C^{m+1,\lambda}(\overline{\Omega}),$$

and the following estimate holds:

$$\|\boldsymbol{v}\|_{C^{m+2,\lambda}} + \inf_{c\in\mathbb{R}} \|p+c\|_{C^{m+1,\lambda}} \leq c\left(\|\boldsymbol{f}\|_{C^{m,\lambda}} + \|\boldsymbol{v}_*\|_{C^{m+2,\lambda}(\partial\Omega)}\right), \quad \text{(IV.7.1)}$$

where $c = c(m, \lambda, \Omega, n)$.

Concerning existence, we have (Solonnikov 1966, Theorem 3.1),

Theorem IV.7.2 *Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 2$,*[1]*of class $C^{2,\lambda}$, $\lambda \in (0,1)$. Then, given*

[1] Actually, in Solonnikov's paper the result is proved for $n = 3$. However, the technique employed there can be extended to the case where $n \geq 2$.

$$\boldsymbol{f} \in C^{2,\lambda}(\overline{\Omega}), \quad \boldsymbol{v}_* \in C^{2,\lambda}(\partial\Omega),$$

there exists one and only one solution $\boldsymbol{v}$, p[2] *to* (IV.0.1), (IV.0.2) *such that*

$$\boldsymbol{v} \in C^{2,\lambda}(\overline{\Omega}), \quad p \in C^{1,\lambda}(\overline{\Omega}).$$

Remark IV.7.1 Theorem IV.7.1 and Theorem IV.7.2 continue to hold when $g \equiv \nabla \cdot \boldsymbol{v} \not\equiv 0$. In such a case, one has to assume

$$g \in C^{m+1,\lambda}(\overline{\Omega})$$

and to add the term

$$\|g\|_{C^{m+1,\lambda}}$$

on the right-hand side of (IV.7.1), while for Theorem IV.7.2 we have to take $g \in C^{1,\lambda}(\overline{\Omega})$. ■

IV.8 Green's Tensor, Green's Identity and Representation Formulas

Theorem IV.6.1 allows us, in particular, to construct the *Green's tensor solution for the Stokes problem* in a bounded domain Ω. Actually, for fixed $y \in \Omega$, let us consider the functions $A_{ij}(x,y), a_i(x,y)$ such that for all $i,j = 1,\dots,n$

$$\begin{aligned} \Delta_x A_{ij}(x,y) + \frac{\partial a_j(x,y)}{\partial x_i} &= 0, \quad x \in \Omega, \\ \frac{\partial A_{ij}(x,y)}{\partial x_i} &= 0, \quad x \in \partial\Omega \\ A_{ij}(x,y) = U_{ij}(x-y)&, \quad x \in \partial\Omega. \end{aligned} \tag{IV.8.1}$$

From Theorem IV.6.1, we know that $A_{ij}(x,y)$, $a_i(x,y)$ exist and, for Ω of class C^{m+2}, they satisfy

$$\|A_{ij}\|_{m+2,q} + \|a_i\|_{m+1,q/\mathbb{R}} \le c_y(\omega, n, m, q),$$

where c_y does not depend on y for all y exceeding a fixed distance d from $\partial\Omega$. In analogy with the Laplace operator, we define

$$\begin{aligned} G_{ij}(x,y) &= U_{ij}(x-y) - A_{ij}(x,y) \\ g_j(x,y) &= q_i(x-y) - a_i(x,y), \end{aligned} \tag{IV.8.2}$$

[2] p is determined up to a constant that may be fixed by requiring $\int_\Omega p = 0$. In such a case, the norm involving p on the left-hand side of (IV.7.1) can be replaced by $\|p\|_{C^{m+1,\lambda}}$.

which is the Green's tensor solution for the Stokes problem in the bounded domain Ω (Odqvist 1930, §5). It is not difficult to show (Odqvist 1930, p. 358) that the tensor field $\boldsymbol{G}$ satisfies the following *symmetry condition*

$$G_{ij}(x,y) = G_{ji}(y,x). \tag{IV.8.3}$$

Moreover, from (IV.8.1), (IV.8.2), and the properties of the tensor $\boldsymbol{U}$ it follows, in particular, for any $n > 2$, that

$$|G_{ij}(x,y)| \le c_d |x-y|^{2-n}, \quad |D_k G_{ij}(x,y)| \le c_d |x-y|^{1-n} \tag{IV.8.4}$$

for all $x \in \overline{\Omega}$ and for all $y \in \Omega$ with dist $(y, \partial\Omega) \ge d > 0$, $c_d \to \infty$ as $d \to 0$. By using (IV.8.3) analogous estimates can be obtained interchanging the roles of x and y. If $n = 3$ and Ω is of class $C^{1,\lambda}$, $\lambda \in (0,1)$, relations (IV.8.4) can be extended to all $x, y \in \overline{\Omega}$ and one has

$$|G_{ij}(x,y)| \le c\,|x-y|^{-1}, \quad |D_k G_{ij}(x,y)| \le c\,|x-y|^{-2}, x, y \in \overline{\Omega}, \tag{IV.8.5}$$

with a constant $c = c(\Omega)$; see Odqvist (1930, Satz XVIII) and Cattabriga (1961, pp. 335-336). Observe that estimates (IV.8.5) formally coincide with the same estimates in the case of a half-space; see (IV.3.50). Extension of (IV.8.5) to higher dimension can be obtained by the results of Solonnikov (1970). We also refer the reader to this paper for further evaluations related to $\boldsymbol{G}$, $\boldsymbol{g}$.

We shall next derive several useful representation formulas for solutions to the Stokes problem. To this end, we recall that the *Cauchy stress tensor* $\boldsymbol{T} \equiv \{T_{ij} = T_{ij}(\boldsymbol{v}, p)\}$ associated with a flow $\boldsymbol{v}$, p is given by

$$T_{ij} = -p\delta_{ij} + 2D_{ij}, \tag{IV.8.6}$$

where

$$D_{ij} = D_{ij}(\boldsymbol{v}) \equiv \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{IV.8.7}$$

is the *stretching tensor.* If $\boldsymbol{u}$, π are sufficiently regular vector and scalar fields, respectively, and assuming that Ω is a bounded domain of class C^1, we may integrate by parts to obtain the identities

$$\begin{aligned}
\int_\Omega \nabla \cdot \boldsymbol{T}(\boldsymbol{v},p)\cdot \boldsymbol{u} &= -\int_\Omega \boldsymbol{T}(\boldsymbol{v},p) : \nabla\boldsymbol{u} + \int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{T}(\boldsymbol{v},p)\cdot\boldsymbol{n}\\
\int_\Omega \nabla \cdot \boldsymbol{T}(\boldsymbol{u},\pi)\cdot \boldsymbol{v} &= -\int_\Omega \boldsymbol{T}(\boldsymbol{u},\pi) : \nabla\boldsymbol{v} + \int_{\partial\Omega} \boldsymbol{v}\cdot\boldsymbol{T}(\boldsymbol{u},\pi)\cdot\boldsymbol{n},
\end{aligned} \tag{IV.8.8}$$

where $\boldsymbol{n}$ is the unit outer normal at $\partial\Omega$. By the symmetry of $\boldsymbol{T}$ and taking $\boldsymbol{v}$ and $\boldsymbol{u}$ solenoidal,

$$\int_\Omega \boldsymbol{T}(\boldsymbol{v},p) : \nabla\boldsymbol{u} = \int_\Omega \boldsymbol{T}(\boldsymbol{u},\pi) : \nabla\boldsymbol{v}.$$

Therefore, from this relation, (IV.2.2), and the identity

$$\nabla \cdot \boldsymbol{T} = -\nabla p + \Delta \boldsymbol{v} \tag{IV.8.9}$$

and the analogous one for $\boldsymbol{u}$ and π, we obtain

$$\int_\Omega [(\Delta \boldsymbol{v} - \nabla p) \cdot \boldsymbol{u} - (\Delta \boldsymbol{u} - \nabla \pi) \cdot \boldsymbol{v}] = \int_{\partial\Omega} [\boldsymbol{u} \cdot \boldsymbol{T}(\boldsymbol{v}, p) - \boldsymbol{v} \cdot \boldsymbol{T}(\boldsymbol{u}, \pi)] \cdot \boldsymbol{n}. \tag{IV.8.10}$$

Relation (IV.8.10) is the *Green's identity for the Stokes system.* By using standard procedures, it is easy to derive from (IV.8.10) a representation formula for $\boldsymbol{v}$ and p (Odqvist 1930, §2). In fact, we choose for fixed j and $x \in \Omega$

$$\begin{aligned} \boldsymbol{u}(y) &= \boldsymbol{u}_j(x-y) \equiv (U_{1j}, U_{2j}, \ldots, U_{nj}) \\ \pi(y) &= q_j(x-y), \end{aligned} \tag{IV.8.11}$$

where $\boldsymbol{U}$, $\boldsymbol{q}$ is the fundamental solution (IV.2.3), (IV.2.4), and substitute them into (IV.8.10) with Ω replaced by $\Omega_\varepsilon \equiv \Omega - B_\varepsilon(x)$. Setting $\boldsymbol{f} = \Delta \boldsymbol{v} - \nabla p$, we obtain

$$\begin{aligned} \int_{\Omega_\varepsilon} \boldsymbol{f}(y) \cdot \boldsymbol{u}_j(x-y) dy = \int_{\partial\Omega} &[\boldsymbol{u}_j(x-y) \cdot \boldsymbol{T}(\boldsymbol{v}, p)(y) \\ &\quad - \boldsymbol{v}(y) \cdot \boldsymbol{T}(\boldsymbol{u}_j, q_j)(x-y)] \cdot \boldsymbol{n} d\sigma_y \\ + \int_{\partial B_\varepsilon(x)} &[\boldsymbol{u}_j(x-y) \cdot \boldsymbol{T}(\boldsymbol{v}, p)(y) \\ &\quad - \boldsymbol{v}(y) \cdot \boldsymbol{T}(\boldsymbol{u}_j, q_j)(x-y)] \cdot \boldsymbol{n} d\sigma_y. \end{aligned} \tag{IV.8.12}$$

Clearly,

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \boldsymbol{f}(y) \cdot \boldsymbol{u}_j(x-y) dy &= \int_\Omega \boldsymbol{f}(y) \cdot \boldsymbol{u}_j(x-y) dy \\ \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon} \boldsymbol{u}_j(x-y) \cdot \boldsymbol{T}(\boldsymbol{v}, p)(y) \cdot \boldsymbol{n} d\sigma_y &= 0. \end{aligned} \tag{IV.8.13}$$

Moreover, since

$$T_{k\ell}(\boldsymbol{u}_j, q_j) = \frac{1}{\omega_n} \frac{(x_k - y_k)(x_\ell - y_\ell)(x_j - y_j)}{|x-y|^{n+2}}, \tag{IV.8.14}$$

by a simple calculation one shows

$$\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon} \boldsymbol{v}(y) \cdot \boldsymbol{T}(\boldsymbol{u}_j, q_j)(x-y) \cdot \boldsymbol{n}(y) d\sigma_y = -v_j(x) \tag{IV.8.15}$$

and so from (IV.8.11)–(IV.8.15) we finally deduce the following representation formula for v_j, $j = 1, \ldots, n$, valid for all $x \in \Omega$:

$$v_j(x) = \int_\Omega U_{ij}(x-y)f_i(y)dy - \int_{\partial\Omega} [U_{ij}(x-y)T_{i\ell}(\boldsymbol{v},p)(y) \\ -v_i(y)T_{i\ell}(\boldsymbol{u}_j,q_j)(x-y)]n_\ell(y)d\sigma_y. \qquad \text{(IV.8.16)}$$

To give a similar representation for the pressure p, we begin to observe that for $\boldsymbol{f}$ smooth enough (e.g., Hölder continuous) the volume potentials

$$W_j(x) = \int_\Omega U_{ij}(x-y)f_i(y)dy,$$

$$S(x) = -\int_\Omega q_j(x-y)f_i(y)dy$$

are (at least) of class $C^2(\Omega)$ and $C^1(\Omega)$, respectively (Odqvist 1930, Satz 1; see also Section IV.7) and that, moreover, it is (see Exercise IV.8.1)

$$\Delta \boldsymbol{W}(x) - \nabla S(x) = \boldsymbol{f}(x), \quad x \in \Omega. \qquad \text{(IV.8.17)}$$

We next observe that from (IV.8.16)

$$\frac{\partial p}{\partial x_j} + f_j = \Delta v_j = \Delta W_j + \int_{\partial\Omega} [v_i \Delta T_{i\ell}(\boldsymbol{u}_j,q_j) - (\Delta U_{ij})T_{i\ell}(\boldsymbol{v},p)]n_\ell, \qquad \text{(IV.8.18)}$$

which, by (IV.8.17) and $(\text{IV.2.5})_1$ in turn implies

$$\frac{\partial p}{\partial x_j} = \frac{\partial S}{\partial x_j} + \int_{\partial\Omega} \left[\frac{\partial q_i}{\partial x_j} T_{i\ell}(\boldsymbol{v},p) + v_i \Delta T_{i\ell}(\boldsymbol{u}_j,q_j)\right] n_\ell.$$

Observing that q_j is harmonic (for $x \neq y$) we also have

$$\Delta T_{i\ell}(\boldsymbol{u}_j,q_j) = -\delta_{i\ell}\Delta q_j + \frac{\partial}{\partial x_\ell}\Delta U_{ij} + \frac{\partial}{\partial x_i}\Delta U_{\ell j} = -2\frac{\partial^2 q_j}{\partial x_i \partial x_\ell},$$

for $x \neq y$, which, once substituted into (IV.8.18) and upon using the relation $\partial q_j/\partial x_\ell = \partial q_\ell/\partial x_j$, yields for all $x \in \Omega$

$$p(x) = -\int_\Omega q_i(x-y)f_i(y)dy + \int_{\partial\Omega} [q_j(x-y)T_{i\ell}(\boldsymbol{v},p)(y) \\ -2v_i(y)\frac{\partial q_\ell(x-y)}{\partial x_i}]n_\ell(y)d\sigma_y. \qquad \text{(IV.8.19)}$$

Identity (IV.8.19) gives the representation formula for the pressure.

Exercise IV.8.1 Prove the validity of equation (IV.8.17). *Hint*: Set $w = \mathcal{E} * f$ ($f \equiv 0$ in Ω^c). From potential theory it is well known that, for f Hölder continuous, it is (at least) $w \in C^2(\Omega)$ and, moreover, $\Delta w = f$ in Ω, see Kellog (1929, Chapter VI, §3).

Formulas (IV.8.16) and (IV.8.19) can be easily extended to derivatives of arbitrary order. Actually, observing that for any multi-index α

$$\Delta(D^\alpha \boldsymbol{v}) - \nabla(D^\alpha p) = D^\alpha \boldsymbol{f},$$

one readily shows, for all $x \in \Omega$,

$$\begin{aligned} D^\alpha v_j(x) = \int_\Omega U_{ij}(x-y) D^\alpha f_i(y) dy - \int_{\partial\Omega} [U_{ij}(x-y) T_{i\ell}(D^\alpha \boldsymbol{v}, D^\alpha p)(y) \\ -D^\alpha v_i(y) T_{i\ell}(\boldsymbol{u}_j, q_j)(x-y)] n_\ell(y) d\sigma_y \end{aligned} \tag{IV.8.20}$$

and

$$\begin{aligned} D^\alpha p(x) = -\int_\Omega q_i(x-y) D^\alpha f_i(y) dy + \int_{\partial\Omega} [q_j(x-y) T_{i\ell}(D^\alpha \boldsymbol{v}, D^\alpha p)(y) \\ -2 D^\alpha v_i(y) \frac{\partial q_\ell(x-y)}{\partial x_i} n_\ell(y)] d\sigma_y. \end{aligned} \tag{IV.8.21}$$

Relations (IV.8.20) and (IV.8.21) were obtained under the assumption of suitable regularity on $\boldsymbol{v}$ and p. Nevertheless, it is not difficult to extend them to the case when velocity and pressure fields belong to suitable Sobolev spaces. Precisely, we have

Theorem IV.8.1 *Let Ω be a bounded domain of $\mathbb{R}^n$, $n \ge 2$, of class C^{m+2}, $m \ge 0$, and let $\boldsymbol{v} \in W^{m+2,q}(\Omega)$, $p \in W^{m+1,q}(\Omega)$ be a solution to* (IV.0.1) *corresponding to $\boldsymbol{f} \in W^{m,q}(\Omega)$, $1 < q < \infty$. Then, $\boldsymbol{v}$ and p obey* (IV.8.20) *and* (IV.8.21), *respectively, for all $|\alpha| \in [0, m]$ and almost all $x \in \Omega$.*

Proof. We prove (IV.8.20), the proof of (IV.8.21) being entirely analogous. Let $\boldsymbol{v}_*$ be the trace of $\boldsymbol{v}$ on $\partial\Omega$. From Theorem II.4.4 we then have $\boldsymbol{v}_* \in W^{m+2-1/q,q}(\partial\Omega)$. Denote by $\{\boldsymbol{f}_k\}$, $\{\boldsymbol{v}_{*k}\}$ two sequences of smooth functions approximating $\boldsymbol{f}$ and $\boldsymbol{v}_*$, respectively, in the spaces to which they belong, and by $\{\boldsymbol{v}_k, p_k\}$ the corresponding solutions to (IV.0.1) and (IV.0.2). By what we have seen, for all $|\alpha| \in [0, m]$, these solutions obey (IV.8.20) and (IV.8.21) with $\boldsymbol{v}_k$ in place of $\boldsymbol{v}$ and with $\boldsymbol{f}_k$ in place of $\boldsymbol{f}$. Denote by $(\text{IV.8.20})_k$ these relations and let $k \to \infty$. In this limit, from Lemma IV.6.1 we obtain

$$\begin{aligned} \boldsymbol{v}_k &\to \boldsymbol{v} \quad \text{in } W^{m+2,q}(\Omega) \\ p_k &\to p \quad \text{in } W^{m+1,q}(\Omega). \end{aligned} \tag{IV.8.22}$$

Set

$$V_j(\boldsymbol{b}) = \int_\Omega U_{ij}(x-y) D^\alpha b_i(y) dy$$

$$\begin{aligned} B_j(\boldsymbol{b}, s) = \int_{\partial\Omega} [D^\alpha b_i(y)\, T_{i\ell}(\boldsymbol{u}_j, q_j)(x-y) \\ -U_{ij}(x-y) T_{i\ell}(D^\alpha \boldsymbol{b}, D^\alpha s)(y)] n_\ell(y) d\sigma_y \end{aligned}$$

and

$$P(\boldsymbol{b}) = \int_{\Omega} q_i(x-y) D^{\alpha} b_i(y) dy$$

$$\beta(\boldsymbol{b}, s) = \int_{\partial\Omega} [q_i(x-y)\; T_{i\ell}(D^{\alpha}\boldsymbol{b}, D^{\alpha}s)(y)$$

$$-2\int_{\partial\Omega} D^{\alpha} b_i(y) \frac{\partial q_\ell(x-y)}{\partial x_i} n_\ell(y) d\sigma_y$$

and assume first $q > n/2$. From the estimates (IV.2.6) for $\boldsymbol{U}$ we have, if $n > 2$,

$$I_1 \equiv |V_j(\boldsymbol{f}_k) - V_j(\boldsymbol{f})| \leq c\| |x-y|^{2-n} \|_{q/(q-1)} \|\boldsymbol{f}_k - \boldsymbol{f}\|_{m,q} \leq c_1 \|\boldsymbol{f}_k - \boldsymbol{f}\|_{m,q} \tag{IV.8.23}$$

while, if $n = 2$,

$$I_1 \leq c\| \log|x-y| \|_{q/(q-1)} \|\boldsymbol{f}_k - \boldsymbol{f}\|_{m,q} \leq c_2 \|\boldsymbol{f}_k - \boldsymbol{f}\|_{m,q}. \tag{IV.8.24}$$

Moreover, for any fixed $x \in \Omega$, from Theorem II.4.1 it follows for all $q > 1$ that

$$\begin{aligned} |B_j(\boldsymbol{v}_k, p_k) - B_j(\boldsymbol{v}, p)| &\leq c_3 \left(\|\boldsymbol{v}_k - \boldsymbol{v}\|_{m+1,q(\partial\Omega)} + \|p_k - p\|_{m,q(\partial\Omega)} \right) \\ &\leq c_4 \left(\|\boldsymbol{v}_k - \boldsymbol{v}\|_{m+2,q,\Omega} + \|p_k - p\|_{m+1,q,\Omega} \right) \end{aligned} \tag{IV.8.25}$$

where c_4 depends on $\operatorname{dist}(x, \partial\Omega) \equiv d$ ($c_3 \to \infty$ as $d \to 0$). Also, from the embedding Theorem II.3.4, we have $D^{\alpha}\boldsymbol{v} \in C(\overline{\Omega})$ and, as $k \to \infty$,

$$D^{\alpha}\boldsymbol{v}_k \to D^{\alpha}\boldsymbol{v} \;\text{ in } C(\overline{\Omega}). \tag{IV.8.26}$$

Relations (IV.8.22)–(IV.8.26) show (IV.8.20) if $q > n/2$, and then for all $q > 1$ if $n = 2$. Assume now $1 < q \leq n/2$, $n > 2$. Let Ω' be any subdomain of Ω with $\overline{\Omega}' \subset \Omega$. Using the Minkowski inequality several times we obtain

$$\begin{aligned} \|D^{\alpha} v_j - V_j(\boldsymbol{f}) - B_j(\boldsymbol{v}, p)\|_{q,\Omega'} &\leq \|\boldsymbol{v}_k - \boldsymbol{v}\|_{m,q,\Omega'} + \|V_j(\boldsymbol{f}_k) - V_j(\boldsymbol{f})\|_{q,\Omega'} \\ + \|B_j(\boldsymbol{v}_k, p_k) - B_j(\boldsymbol{v}, p)\|_{q,\Omega'} &+ \|D^{\alpha}(v_{kj}) - V_j(\boldsymbol{f}_k) - B_j(\boldsymbol{v}_k, p_k)\|_{q,\Omega'}. \end{aligned} \tag{IV.8.27}$$

If $q = n/2$, from (IV.2.6) and Theorem II.11.2 we derive that V_j is a bounded transformation of $L^{n/2}(\Omega)$ into $L^r(\Omega)$, for all $r \in (1, \infty)$, while, if $q \in (1, n/2)$, again from (IV.2.6) and Sobolev Theorem II.11.3, V_j is a bounded transformation of $L^q(\Omega)$ into $L^{nq/(n-2q)}(\Omega)$. Therefore, observing that $q < nq/(n-2q)$, in either case we derive

$$\|V_j(\boldsymbol{f}_k) - V_j(\boldsymbol{f})\|_{q,\Omega'} \leq c_5 \|\boldsymbol{f}_k - \boldsymbol{f}\|_{m,q,\Omega}. \tag{IV.8.28}$$

Thus, recalling that $\boldsymbol{v}_k$ satisfies $(\text{IV.8.20})_k$ identically, from (IV.8.22), (IV.8.25), (IV.8.27), and (IV.8.28) we conclude the validity of (IV.8.20) also for $q \in (1, n/2]$. The proof of (IV.8.21) is entirely analogous, provided we distinguish the two cases $n \geq q$ and $n < q$. We leave details to the reader. The proof of the theorem is complete. $\square$

Representation formulas that involve only the body force and the velocity at the boundary can be obtained if we make use of the Green's tensor solution (IV.8.2). Actually, applying (IV.8.10) with

$$\begin{aligned}\boldsymbol{u}(y) &= \boldsymbol{A}_j(x,y) \equiv (A_{1j}, A_{2j}, \ldots, A_{nj})\\ \pi(y) &= a_j(x,y)\end{aligned}$$

and taking into account that $\boldsymbol{A}$, $\boldsymbol{a}$ are smooth fields solving (IV.8.1), we find

$$\int_\Omega \boldsymbol{f}(y)\cdot\boldsymbol{A}_j(x,y)dy = \int_{\partial\Omega}[\boldsymbol{u}_j(x-y)\cdot\boldsymbol{T}(\boldsymbol{v},p)(y) - \boldsymbol{v}(y)\cdot\boldsymbol{T}(\boldsymbol{A}_j,a_j)(x,y)]\cdot\boldsymbol{n}d\sigma_y, \tag{IV.8.29}$$

where, we recall, $\boldsymbol{u}_j \equiv (U_{1j}, U_{2j}, \ldots, U_{nj})$. Subtracting (IV.8.29) from (IV.8.16) and bearing in mind the definition of $\boldsymbol{G}$ given in (IV.8.2) we then conclude

$$v_j(x) = \int_\Omega G_{ij}(x,y)f_i(y)dy - \int_{\partial\Omega}[v_i(y)T_{i\ell}(\boldsymbol{G}_j, g_j)(x,y)]n_\ell(y)d\sigma_y, \tag{IV.8.30}$$

where $\boldsymbol{G}_j \equiv (G_{1j}, G_{2j}, \ldots, G_{nj})$. Finally, along the same lines leading to (IV.8.19), one proves the following formula:

$$p(x) = -\int_\Omega g_i(x-y)f_i(y)dy - 2\int_{\partial\Omega} v_i(y)n_\ell(y)d\sigma_y. \tag{IV.8.31}$$

Exercise IV.8.2 Let $\boldsymbol{v}$ be a q-generalized solution to the Stokes problem (IV.0.1), (IV.0.2) in a half-space, corresponding to smooth data of bounded support. Show that $\boldsymbol{v}$ and the corresponding pressure p satisfy the representation (IV.8.30), (IV.8.31) with $\boldsymbol{G}$ and $\boldsymbol{g}$ given in (IV.3.46).

IV.9 Notes for the Chapter

Section IV.1. The first existence and uniqueness result for the Stokes problem in a bounded domain is due to Korn (1908), under the restriction $\nabla\cdot\boldsymbol{f} = 0$. The problem of existence with no restriction on the body force was solved for Ω a ball by Boggio (1910), Crudeli (1925a, 1925b), and Oseen (1927, §§9.1,9.2). In particular, Oseen determines explicitly the Green's tensor for the Stokes problem in a ball. Existence and uniqueness in full generality, with no restriction on $\boldsymbol{f}$ or the shape of Ω, was provided by Lichtenstein (1928), in the wake of the work of Umberto Crudeli.

The existence of a pressure field associated to a q-generalized solution along with the validity of the corresponding estimate was first established by Cattabriga (1961) for space dimension $n = 3$. The same result was obtained, with a much simpler proof, in the case $q = 2$ (generalized solutions) by Solonnikov & Ščadilov (1973) and it was successively rediscovered, essentially along the same methods, by Amick (1976); see also Temam (1977, Chapter I, Lemma 2.1).

Section IV.2. The material contained in this section is taken, basically, from Galdi & Simader (1990). However, the uniqueness part of Theorem IV.2.2 is due to me. Similar results can be found in Cattabriga (1961), Borchers & Miyakawa (1990, Proposition 3.7 (*iii*)), and Kozono & Sohr (1991, §2.2).

Existence and uniqueness of solutions in weighted Lebesgue and homogeneous Sobolev spaces can be immediately obtained by using, in the proofs of Theorem IV.2.1 and Theorem IV.2.2, Stein's Theorem II.11.5 in place of Calderón–Zygmund Theorem II.11.4; see Pulidori (1993). For similar results in the two-dimensional case, we also refer to Durán & López García (2010).

Section IV.3. The guiding ideas are taken from the work of Cattabriga (1961, §§2,3). However, all theorems in this section are due to me. In this respect, I am grateful to the late Professor Lamberto Cattabriga for the inspiring and enjoyable conversations I had with him, in the winter of 1987, on the existence part of Theorem IV.3.3.

A weaker version of the estimates contained in Theorem IV.3.2 and Theorem IV.3.3 is given by Borchers & Miyakawa (1988, Theorem 3.6) and by Maslennikova & Timoshin (1990, Theorem 1). Results in weighted L^q spaces can be found in Borchers & Pileckas (1992).

The special case of Theorem IV.3.2 corresponding to $\boldsymbol{f} \equiv \boldsymbol{0}$, $g \equiv 0$ and $q = 2$ is proved by Tanaka (1995), by the Fourier transform method. More general (slip) boundary conditions are also considered.

The Green's tensor for a three-dimensional half-space was determined for the first time by Lorentz (1896); see also Oseen (1927, §9.7).

Section IV.4. Theorem IV.4.1 (for $n = 3$) is essentially due to Cattabriga (1961, §5), while Theorem IV.4.2, Theorem IV.4.4, and Theorem IV.4.5 are due to me.

A result similar to that proved in Remark IV.4.2 was first shown by Šverák & Tsai (2000, Theorem 2.2). In fact, Remark IV.4.2 is motivated by their work.

Section IV.5. The results contained in this section are a generalization to $n \geq 2$ of those proved by Cattabriga (1961, §5) for $n = 3$.

An improved version of the results stated in Remark IV.5.1 and Remark IV.5.2 can be found in Kang (2004).

Section IV.6. Theorem IV.6.1 plays a central role in the mathematical theory of the Navier–Stokes equations. In the case where $n = 3$ it was shown for the first time by Cattabriga (1961, Teorema at p.311). The same result of Cattabriga for $m \geq 0$ was announced by Solonnikov (1960) and a full proof, based on the theory of hydrodynamical potentials, appeared later in 1963 in the first edition of the book by Ladyzhenskaya (1969) (in this regard, see also Deuring, von Wahl, & Weidemaier (1988) and the book of Varnhorn (1994)). Sobolevski (1960) proved a weaker result in the special case $m = 0$ and $q = 2$. In their study on the unique solvability of steady-state Navier–Stokes equations, Vorovich and Youdovich (1961, Theorem 2) showed Cattabriga's result for $m \geq 0$ and $q \geq 6/5$. Finally, we wish to mention the ingenious work of Krzywcky (1961), where estimates for the Stokes problem are obtained from the Weyl decomposition of the space L^2.

Since the appearance of these papers, several works have been published which, among other things, investigate the possibility of generalizing Cattabriga's theorem in the following two directions: (i) extension to arbitrary dimension $n \geq 2$; (ii) extension to less regular domains. To the best of our knowledge, the first attempt toward direction (i) is due to Temam (1973, Chapter I). However Temam's arguments work only when $q \geq 2$ and $m \geq 0$, if $n \geq 3$, and for arbitrary $m \geq -1$, $q \in (1, \infty)$ if $n = 2$. In particular, the proof of this latter result is achieved by showing that the Stokes problem is equivalent to a suitable biharmonic problem. In this respect, we refer the reader to the paper of Simader (1992), where an interesting analysis between these two problems is carried out for any $n \geq 2$. Another contribution along direction (i), in the case where $m = 2$, is due to Giga (1981, Proposition 2.1), who uses a theorem of Geymonat (1965, Theorem 3.5) on the invariance of the index of the operator associated to an elliptic system in the sense of Douglis-Nirenberg. This method requires, however, Ω of class C^∞. Ghidaglia (1984) has extended Cattabriga's theorem to arbitrary $n \geq 2$ when $q = 2$. In this respect, it is worth mentioning the contribution of Beirão da Veiga (1998) where results similar to those of Ghidaglia are proved, but under much less regularity on Ω. However, the most important feature of this paper is that the author avoids potential and/or general elliptic equation theories, while he uses, instead, only the elementary estimate $\|u\|_{2,2} \leq c\,\|f\|_2$ for the unique solution $u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ of the scalar Poisson equation $\Delta u = f$.

The full generalization of the results of Cattabriga to arbitrary space dimension $n \geq 2$, i.e. Theorem IV.6.1, was established for the first time, independently, by Kozono & Sohr (1991) and Galdi & Simader (1990, Theorem 2.1). (Actually, the proof given by the former authors requires slightly more regularity on Ω than that stated in Theorem IV.6.1.) Concerning (ii), Amrouche & Girault (1990, 1991), suitably coupling the work of Grisvard (1985) and Giga (1981), have proved Theorem IV.6.1 with $m \geq 0$, for Ω of class $C^{m+1,1}$, and with $m = -1$, for Ω of class $C^{1,1}$. Galdi, Simader, & Sohr (1994) extend Theorem IV.6.1 with $m = -1$ to locally Lipschitz domains with "not too sharp" corners, or to arbitrary domains of class C^1. If $n = 3$ and $q \in [3/2, 3]$ their result continues to hold for Ω locally Lipschitz only, provided $\partial\Omega$ is connected; see Shen (1995). The Stokes problem in non-smooth domains has also been addressed by Kellogg & Osborn (1976) for Ω a convex polygon, and by Voldřich (1984) for arbitrary $\Omega \subset \mathbb{R}^n$, $n \geq 2$, but in weighted Sobolev spaces. Dauge (1989) and Kozlov, Maz'ja and Schwab (1994) have considered extensions of the work of Kellogg and Osborn to three-dimensional domains.

Existence and uniqueness for the Stokes problem in corners and cones has been studied by Solonnikov (1982) and Deuring (1994), respectively. A comprehensive analysis of these questions can be found in the monograph of Nazarov and Plamenevskiĭ (1994).

Along with the definition of weak solution, one can introduce the notion of *very weak solutions* (Giga, 1981, §2; see also Conca, 1989). Specifically, set

$$\mathcal{C}_0^2(\Omega) = \{\boldsymbol{\psi} \in C^2(\overline{\Omega}) : \boldsymbol{\psi}|_{\partial\Omega} = \mathbf{0}\}\,, \tag{$*$}$$

and formally dot-multiply both sides of (IV.0.1)$_1$ by $\boldsymbol{\psi} \in \mathcal{C}_0^2(\Omega)$. After integrating by parts over Ω and taking into account (IV.0.1)$_2$ and (IV.0.2), we find

$$\int_\Omega \boldsymbol{v}\cdot \Delta\boldsymbol{\psi} = -\int_\Omega p\,\mathrm{div}\,\boldsymbol{\psi} + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\psi} - \int_{\partial\Omega} \boldsymbol{n}\cdot\nabla\boldsymbol{\psi}\cdot\boldsymbol{v}_*\,.$$

These considerations lead to the following *definition.* A field $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ is called a very weak solution to the Stokes problem (IV.0.1)–(IV.0.2) if (i) $\boldsymbol{v} \in H_q(\Omega)$, some $q \in (1,\infty)$, and (ii) $\boldsymbol{v}$ satisfies the following relation

$$(\boldsymbol{v}, \Delta\boldsymbol{\psi}) = \langle \boldsymbol{f}, \boldsymbol{\psi}\rangle - \langle \boldsymbol{n}\cdot\nabla\boldsymbol{\psi}, \boldsymbol{v}_*\rangle_{\partial\Omega}\,, \quad \text{for all } \boldsymbol{\psi} \in \mathcal{C}_0^2(\Omega) \text{ with } \mathrm{div}\,\boldsymbol{\psi} = 0 \text{ in } \Omega\,,$$

where, we recall, $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denotes the duality pairing between the spaces $W^{-1/q,q}(\partial\Omega)$ and $W^{1-1/q',q'}(\partial\Omega)$; see Section III.2 . Notice that $\boldsymbol{v}$ is not required to possess any derivative, hence the attribute of "very weak". It is also clear that a q-weak solution is also very weak, and that the converse need not be true. Existence and uniqueness of very weak solutions has been shown, for Ω of class C^∞, by Giga (1981, Proposition 2.2). In particular, the author proves that, for any $\boldsymbol{f} \in D_0^{-1,q}(\Omega)$ and any $\boldsymbol{v}_* \in W^{-1/q,q}(\partial\Omega)$, there exists a unique corresponding very weak solution. Furthermore, we can find a "pressure field" $p \in W_0^{-1,q}(\Omega)$ such that

$$(\boldsymbol{v}, \Delta\boldsymbol{\psi}) = -\langle p, \mathrm{div}\,\boldsymbol{\psi}\rangle + \langle \boldsymbol{f}, \boldsymbol{\psi}\rangle + \langle \boldsymbol{n}\cdot\nabla\boldsymbol{\psi}, \boldsymbol{v}_*\rangle_{\partial\Omega}\,, \quad \text{for all } \boldsymbol{\psi} \in \mathcal{C}_0^2(\Omega)\,.$$

More recently, the question of existence, uniqueness, continuous dependence and regularity of very weak solutions (also for the full nonlinear problem, see Notes for Chapter IX) has been analyzed in detail by Galdi, Simader & Sohr (2005, Theorem 3 and Lemma 4). Among others, one main result of this paper explains and characterizes the way in which a very weak solution attains the boundary value $\boldsymbol{v}_*$, a thing that a priori is not completely obvious; see also Marušic-Paloka (2000, Section 3). It turns out that the normal component of $\boldsymbol{v}$ at $\partial\Omega$ is a well defined member of $W^{-1/q,q}(\partial\Omega)$, as a consequence of the fact that $\boldsymbol{v}$ belongs to $H_q(\Omega)$; see Theorem III.2.3. Moreover, let $\widehat{W}^{1,q}(\Omega)$ denote the completion of $W^{1,q}(\Omega)$ in the norm

$$\|\boldsymbol{u}\|_{\widehat{W}^{1,q}(\Omega)} \equiv \|\boldsymbol{u}\|_q + \|P_q\Delta\boldsymbol{u}\|_{-1,q}\,,$$

with P_q projection operator of $L^q(\Omega)$ onto $H_q(\Omega)$. Then, if Ω is of class $C^{2,1}$, and $\boldsymbol{u} \in \widehat{W}^{1,q}(\Omega)$ we have that $\boldsymbol{n}\times\boldsymbol{u}|_{\partial\Omega}$ is well defined as an element of $\in W^{-1/q,q}(\partial\Omega)$, and the corresponding trace operator is continuous; see Galdi, Simader & Sohr, *loc. cit.*, Theorem 1. In addition, it can be shown that every very weak solution corresponding to $\boldsymbol{f} \in W_0^{-1,q}(\Omega)$ and $\boldsymbol{v}_* \in W^{-1/q,q}(\partial\Omega)$, belongs, in fact, to the space $\widehat{W}^{1,q}(\Omega)$; see Galdi, Simader & Sohr, *loc. cit.*, Lemma 1. Higher regularity results can be derived from Lemma IV.6.1.

Galdi & Varnhorn (1996) have proved the maximum modulus theorem for the Stokes system. Specifically, they show that for solutions to (IV.0.1)–(IV.0.3) with $\boldsymbol{f} \equiv 0$ and Ω of class C^2, the estimate

$$\max_{\Omega} |\boldsymbol{v}| \leq C \max_{\partial\Omega} |\boldsymbol{v}_*| \tag{$**$}$$

holds, with $C = C(n, \Omega)$. In general, the constant C is ≥ 1, and, in fact, it is easy to bring examples where $C > 1$. The same result, under more general assumptions on the data, has been independently obtained by Maremonti (2002). Previous contributions to this problem are due to Naumann (1988), and to Maremonti & Russo (1994) who first proved $(**)$ for the two-dimensional case.[1] An interesting question is to furnish a bound for the constant C appearing in $(**)$. This problem has been addressed, by an elegant method, by Kratz (1997a, 1997b).

Section IV.8. Integral representations of various types for the general nonhomogeneous Stokes problem along with their comparative analysis are given in the paper by Valli (1985). In this paper some errors in analogous formulas given by other authors are also pointed out.

[1] The results of Galdi & Varnhorn, and of Maremonti & Russo cover the case of more general domains such as the exterior ones. In this respect, it seems interesting to notice that, in the case where Ω is a half-space, the estimate $(**)$ with suitable $C = C(n) > 0$ immediately follows, in arbitrary dimension $n \geq 2$, from the representation $(\text{IV.3.3})_1$ and the estimate (ii) given after (IV.3.6).

V

Steady Stokes Flow in Exterior Domains

... Tu stesso ti fai grosso
col falso immaginar, sì che non vedi
ciò che vedresti, se l'avessi scosso.

DANTE, Paradiso I, vv. 88-90.

Introduction

In this chapter we shall analyze the Stokes problem in an exterior domain. Specifically, assuming that the region of flow Ω is a domain coinciding with the complement of a compact set (not necessarily connected) we wish to establish existence, uniqueness, and the validity of corresponding estimates for the velocity field $\boldsymbol{v}$ and the pressure field p of a steady flow in Ω governed by the Stokes approximation, i.e.,

$$\left.\begin{aligned} \Delta \boldsymbol{v} &= \nabla p + \boldsymbol{f} \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}\right\} \text{ in } \Omega \qquad \text{(V.0.1)}$$
$$\boldsymbol{v} = \boldsymbol{v}_* \ \text{ at } \partial\Omega,$$

where $\boldsymbol{f}$, $\boldsymbol{v}_*$ are prescribed fields and where, as usual, we have formally set the coefficient of kinematic viscosity to be one. Of course, since Ω is unbounded, we have to assign also the velocity at infinity, which we do as follows:

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = \boldsymbol{0}\,. \qquad \text{(V.0.2)}$$

There is a physically significant special case of problem (V.0.1)–(V.0.2) that, in fact, constitutes the main motivation for its study. Precisely, consider the slow motion of a rigid body $\mathcal{B}$, with impermeable walls, that moves with

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prescribed translational velocity, $\boldsymbol{v}_0$, and angular velocity, $\boldsymbol{\omega}$, in an otherwise quiescent viscous liquid. By "slow" we mean that all nonlinear terms related to the inertial forces of the liquid can be disregarded compared to the linear one related to viscous forces. This happens if the appropriate Reynolds number is vanishingly small, that is,

$$\max\{|\boldsymbol{v}_0|, |\boldsymbol{\omega}|d\} \ll \frac{\nu}{d}\,, \tag{V.0.3}$$

where $d = \delta(\mathcal{B})$. We further assume that the liquid fills the whole space, Ω, outside $\mathcal{B}$, and that body forces are negligible. Then, problem (V.0.1), (V.0.2) with $\boldsymbol{f} \equiv \mathbf{0}$, and $\boldsymbol{v}_* = \boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{x}$, describes the motion of the liquid referred to a frame attached to the body $\mathcal{B}$.

a priori, we are not expecting that (V.0.1), (V.0.2) may fully describe, even qualitatively, the physics of the problem at low Reynolds number. This is because, if the Stokes approximation of a flow can be fairly reasonable near the bounding wall of the body, where the viscous forces are predominant, it need not be equally reasonable at large distances where the effects related to those forces become less important. Let us consider, for instance, a unit ball, S, (with impermeable walls) moving with a small (in the sense of (V.0.3)) translational velocity $\boldsymbol{v}_0$ and zero angular velocity in a viscous liquid that fills the whole space and is at rest at infinity. Then, by what we said, the motion of the liquid can be described by (V.0.1), (V.0.2) with $\Omega \equiv \mathbb{R}^3 - \overline{S}$, $\boldsymbol{f} \equiv \mathbf{0}$, and $\boldsymbol{v}_* = \boldsymbol{v}_0$. In such a case, Stokes derived in 1851 a remarkable and explicit solution $\boldsymbol{v}_S$, p_S given by (see Stokes 1851, §39)

$$\begin{aligned}\boldsymbol{v}_S(x) &= \frac{3}{4}\nabla \times \left[|x|^2\,\nabla \times \left(\frac{\boldsymbol{v}_0}{|x|}\right)\right] + \frac{1}{4}\nabla \times \nabla \times \left(\frac{\boldsymbol{v}_0}{|x|}\right)\\ p_S(x) &= \frac{3}{2}\boldsymbol{v}_0 \cdot \nabla\left(\frac{1}{|x|}\right).\end{aligned} \tag{V.0.4}$$

Employing this solution one can easily compute the force exerted by the liquid on the sphere and find results that are significantly in agreement with the experiment. [1] However, for the same solution it is apparent that $\boldsymbol{v}(x) = \boldsymbol{v}(-x)$ and, therefore, according to Stokes approximation, there is *no wake region* behind S in contrast with what should be expected in the actual flow.

Similar incongruities are observed if S rotates with a constant and small (in the sense of (V.0.3)) angular velocity $\boldsymbol{\omega}$, without translating (i.e. $\boldsymbol{v}_0 = \mathbf{0}$). In this situation, the solution to (V.0.1), (V.0.2) with $\boldsymbol{f} \equiv \mathbf{0}$, and $\boldsymbol{v}_* = \boldsymbol{\omega} \times \boldsymbol{x}$ is given by (see Lamb 1932, §334)

$$\boldsymbol{v} = \boldsymbol{\omega} \times \frac{\boldsymbol{x}}{|x|^3}\,, \quad p = p_0\,, \tag{V.0.5}$$

[1] A most remarkable example is the Erenhaft-Millikan experiment for determining the elementary electronic charge, where one uses the Stokes law of resistance derived from (V.0.4) (Perucca 1963, Vol II, p.670).

where p_0 is an arbitrary constant. From (V.0.5) it follows that the component of the velocity along the axis parallel to $\boldsymbol{\omega}$ is identically zero. However, this is at odds with the well-known experimental observation that the sphere acts like a "centrifugal fan", receiving the liquid near the poles and throwing it away at the equator; see Stokes (1845) and Lamb (1932, p. 589). This fact was first theoretically explained by Bickley (1938).

Another –and maybe more famous– difficulty arises when one replaces the sphere S with an infinite straight cylinder $\mathcal{C}$ moving with translational velocity $\boldsymbol{v}_0$ in a direction perpendicular to its axis, and zero angular velocity (Stokes 1851, §45). In this situation, the motion of the liquid is planar and, therefore, one may write

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = -\frac{\partial \psi}{\partial r},$$

where $\psi = \psi(r,\theta)$ is the stream function and (r,θ) is a polar coordinate system in the relevant plane of flow orthogonal to the axis of $\mathcal{C}$. Assuming the radius of $\mathcal{C}$ to be one, problem (V.0.1), (V.0.2) with

$$\Omega = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 > 1\},$$

$\boldsymbol{f} \equiv \boldsymbol{0}$ and $\boldsymbol{v}_* = \boldsymbol{v}_0$, can be written, in terms of ψ, as

$$\begin{aligned}
&\Delta^2\psi = 0 \quad \text{in } \Omega \\
&\frac{\partial \psi}{\partial \theta} = U\cos\theta \quad \text{at } r = 1 \\
&\frac{\partial \psi}{\partial r} = U\sin\theta \quad \text{at } r = 1 \\
&\lim_{r\to\infty} \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \lim_{r\to\infty} \frac{\partial \psi}{\partial r} = 0,
\end{aligned} \tag{V.0.6}$$

where, without loss of generality, we have taken $\boldsymbol{v}_0 = (U,0,0)$. A solution to (V.0.6) can be sought in the form

$$\psi(r,\theta) = F(r)G(\theta).$$

Owing to $(\text{V.0.6})_{3,4}$, we find

$$G(\theta) = \sin\theta,$$

and so, by an easy calculation, we show that $F(r)$ satisfies a fourth-order Euler equation whose general integral is

$$F(r) = Ar^{-1} + Br + Cr\log r + Dr^3,$$

$A,\ldots,D$ being arbitrary constants to be fixed so as to match the boundary conditions and conditions at infinity in (V.0.6). To satisfy the latter it is necessary to take $B = C = D = 0$. Moreover, $(\text{V.0.6})_2$ implies $A = U$, while $(\mathrm{V}.0.6)_3$ requires $A = -U$. This is possible if and only if $A = U = 0$. Thus, $\boldsymbol{v} \equiv 0$, $p =$ const. is the only possible solution (of the particular form chosen) of the problem, which tells us that the cylinder can not move. Stokes then concluded with the following (wrong, in retrospect) statement; see Stokes (1851, p. 63):

> "It appears that the supposition of steady motion is inadmissible."

This is the original formulation of the famous *Stokes paradox*, which plays a fundamental role in the study of plane steady flow, also in the nonlinear context (see Section XII.4). The Stokes paradox, in other different and more general forms, will be considered and discussed in several sections of this chapter. [2]

The situation just described is similar to that of the well-known Laplace equation with homogeneous Dirichlet boundary data in the exterior of a unit circle, where the function $\mu(x) = \log|x|$ is a solution to the problem and there are no non-zero solutions that behave at infinity as $o(\mu(x))$. In fact, also for the exterior plane Stokes problem, from the reasonings previously developed we can construct solutions analogous to μ and find the following two independent solutions

$$\begin{aligned}
v_1^{(1)} &= 2\log|x| + 2x_2^2/|x|^2 + (x_1^2 - x_2^2)/|x|^4 - 1,\\
v_2^{(1)} &= -2\frac{x_1 x_2}{|x|^2}\left(1 - |x|^{-2}\right),\\
\pi^{(1)} &= 4x_1/|x|^2,\\
v_1^{(2)} &= -2\frac{x_1 x_2}{|x|^2}\left(1 - |x|^{-2}\right),\\
v_2^{(2)} &= 2\log|x| + 2x_1^2/|x|^2 + (x_2^2 - x_1^2)/|x|^4 - 1,\\
\pi^{(2)} &= 4x_2/|x|^2.
\end{aligned} \tag{V.0.7}$$

As in the case of the sphere, also for the planar flow past a cylinder one can exhibit a solution corresponding to the case when $\mathcal{C}$ rotates (without translating) with constant angular velocity $\boldsymbol{\omega}$ around its axis in a quiescent liquid. Precisely, the appropriate solution is given by (see Lamb 1932, §§ 333, 336)

$$\boldsymbol{v} = \boldsymbol{\omega} \times \frac{\boldsymbol{x}}{|x|^2}, \quad p = p_0, \tag{V.0.8}$$

where p_0 is an arbitrary constant. However, it is interesting to observe that, unlike the case of the sphere, the velocity field in (V.0.8) is also a solution to the *full nonlinear* Navier–Stokes problem, corresponding to the pressure field

$$p = p_0 - \frac{1}{2}\frac{|\boldsymbol{\omega}|^2}{|x|^2}. \tag{V.0.9}$$

[2] It is interesting to notice that a general proof of the Stokes paradox, independent of the particular form of the solution and the shape of the body appeared only in 1938; see Kotschin, Kibel, & Rose (1954, pp. 361-366).

Other contradictions and paradoxes related to problem (V.0.1), (V.0.2) will be mentioned in the introduction to Chapter VII. There, and in the subsequent Chapter VIII, to avoid these contradictions, we will consider an approximation different from that of Stokes, obtained by linearizing the Navier–Stokes equation around a *non-zero* rigid motion solution (*Oseen, and generalized Oseen approximations*).

The main objective of this chapter is to investigate unique solvability for problem (V.0.1), (V.0.2) and to see to what extent it is possible to prove, for the obtained solutions, estimates analogous to those derived in the preceding chapter in the case of a bounded domain. Now, while existence and uniqueness of generalized solutions together with corresponding estimates are proved (for $n > 2$) by a direct extension of the method employed for the bounded domain (in fact, even with an *arbitrary* flux of $\boldsymbol{v}_*$ at $\partial\Omega$), the problem of determining analogous results for q-generalized solutions ($q \neq 2$) is more complicated and demands a preliminary, detailed study of asymptotic properties at large distances. Of course, as in the case of the Poisson equation in exterior domains, we are not expecting that the theory holds in Sobolev spaces $W^{m,q}$ but, rather, in *homogeneous* Sobolev spaces $D^{m,q}$. However, as we shall see, even enlarging the class of functions to which solutions belong, such results can be proved if there is a certain restriction on q depending on the number of space dimension n. This fact can be qualitatively explained as follows. We begin to consider smooth body forces of compact support in Ω and for these we show the unique solvability of problem (V.0.1), (V.0.2) in a function class $\mathcal{F}_q$ (say) along with suitable estimates that represent the natural generalization to the exterior domain of those determined in Theorem IV.6.1 for a bounded domain. Successively, given $\boldsymbol{f}$ in an arbitrary Sobolev space $W^{m-2,q}$, $m \geq 2$, we approximate it by a sequence from $C_0^\infty(\Omega)$ and analyze the convergence of the corresponding solutions to a solution to (V.0.1), (V.0.2) in the class $\mathcal{F}_q$ by means of the preceding estimates. Now, if q is sufficiently small ($1 < q < n/2$) every function in $\mathcal{F}_q$ satisfies (V.0.2), in a suitable sense, and the above procedure is convergent to a uniquely determined solution to (V.0.1), (V.0.2); on the other hand, if q is large enough ($q \geq n/2$) the elements of $\mathcal{F}_q$ need not verify (V.0.2) and, moreover, (V.0.1) with $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv 0$ admits nonzero solutions in the class $\mathcal{F}_q$, which form a finite dimensional space Σ_q. Therefore, for $q \geq n/2$, our procedure gives rise to a solution satisfying a priori *only the Stokes system* (V.0.1) and the corresponding estimates are available only in the quotient space $\mathcal{F}_q/\Sigma_q$.

An analogous situation occurs when $\boldsymbol{f}$ is a functional on $D_0^{1,q'}$ and solutions are sought in the space $D^{1,q}$ (weak solutions). In fact, when $q \geq n$ ($q > 2$, if $n = 2$), also in this class there is a nonempty null space $\mathcal{S}_q$ to the Stokes system (V.0.1). It then follows that such solutions are not unique if $q \geq n$ ($q > 2$, if $n = 2$), while they can exist for $1 < q \leq n/(n-1)$ ($1 < q < 2$, if $n = 2$) if and only if the data satisfy a compatibility condition of the Fredholm type. This latter property has some interesting consequences and, in particular, it leads to a necessary and sufficient condition for the existence of planar solutions to (V.0.1), (V.0.2). Specifically, these solutions can exist if and only if $\boldsymbol{f}$ and $\boldsymbol{v}_*$ verify a suitable relation. Of course, if $\boldsymbol{f} \equiv \boldsymbol{0}$, the choice $\boldsymbol{v}_* = \boldsymbol{v}_0$, with $\boldsymbol{v}_0$ constant vector, does not satisfy such a relation, in accordance with the Stokes paradox. However, this relation may be satisfied for other, physically significant, choices of $\boldsymbol{v}_*$; see Exercise V.7.1.

In the light of what I have described so far, a question that naturally arises is if, by suitably restricting the function class of body forces, it is possible to determine "stronger" estimates that would ensure that the limit solution, obtained by the density procedure previously mentioned, "remembers" the condition at infinity (V.0.2). Such a problem is, in fact, resolvable provided $\boldsymbol{f} = \nabla \cdot \boldsymbol{F}$ with $\boldsymbol{F}$ decaying sufficiently fast at large distances (see Section V.8) and, as we shall see in Chapter X, these results will be decisive in the solvability of the nonlinear problem with zero velocity at infinity.

V.1 Generalized Solutions. Preliminary Considerations and Regularity Properties

In analogy with the case where Ω is bounded, we begin to give a variational formulation of the Stokes problem (V.0.1), (V.0.2). To this end, multiplying (V.0.1) by $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and integrating by parts over Ω, we formally obtain[1]

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) = -[\boldsymbol{f}, \boldsymbol{\varphi}]. \tag{V.1.1}$$

Definition V.1.1. A vector field $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ is called a *q-weak* (or *q-generalized*) *solution* to (V.0.1), (V.0.2) if for some $q \in (1, \infty)$ the following properties are met:

(i) $\boldsymbol{v} \in D^{1,q}(\Omega)$;
(ii) $\boldsymbol{v}$ is (weakly) divergence-free in Ω;
(iii) $\boldsymbol{v}$ assumes the value $\boldsymbol{v}_*$ at $\partial\Omega$ (in the trace sense) or, if the velocity at the boundary is zero, $\boldsymbol{v} \in D_0^{1,q}(\Omega)$;
(iv) $\displaystyle\lim_{|x|\to\infty} \int_{S^{n-1}} |\boldsymbol{v}(x)| = 0$;
(v) $\boldsymbol{v}$ verifies (V.1.1) for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$.

If $q = 2$, $\boldsymbol{v}$ will be called a *weak* (or *generalized*) *solution* to (V.0.1), (V.0.2).

Remark V.1.1 If $\boldsymbol{v}_* = 0$, condition (iii) tells us that $\boldsymbol{v}$ assumes the homogeneous boundary data in the sense of the Sobolev space $W_0^{1,2}$ and no regularity is needed on Ω; see Remark II.6.5. On the other hand, if $\boldsymbol{v}_* \neq \boldsymbol{0}$, according to the trace theory of Section II.5, Ω has to be (at least) locally Lipschitz. ■

[1] As agreed, we are taking $\nu = 1$. Furthermore, as in the case where Ω is bounded, we are considering the more general case that $\boldsymbol{f}$ is an element of $D_0^{-1,q}(\Omega)$, so that, since Ω is an exterior domain, we replace $(\boldsymbol{f}, \boldsymbol{\varphi})$ with $[\boldsymbol{f}, \boldsymbol{\varphi}]$, where, we recall, $[\cdot, \cdot]$ denotes the duality pairing between $D_0^{1,q}(\Omega)$ and $D_0^{1,q'}(\Omega)$.

Remark V.1.2 If $\boldsymbol{v}_* = \mathbf{0}$, a q-weak solution belongs to $\widehat{\mathcal{D}}_0^{1,q}(\Omega)$. This follows directly from conditions (ii) and (iii) of Definition V.1.1. Consequently, for domains such that $\widehat{\mathcal{D}}_0^{1,q}(\Omega) \supset \mathcal{D}_0^{1,q}(\Omega)$ the definition of q-generalized solution using this latter space could then be more restrictive than Definition V.1.1. In this regard, we recall that, by virtue of the results of Theorem III.5.1, the two spaces coincide if Ω satisfies some mild regularity requirement like, for instance, cone condition. However, to date, it is still to ascertain if there exists an exterior domain, $\Omega^\sharp$, for which $\widehat{\mathcal{D}}_0^{1,2}(\Omega^\sharp) \neq \mathcal{D}_0^{1,2}(\Omega^\sharp)$. If such $\Omega^\sharp$ exists, then, as in the case of a bounded domain (see Remark IV.1.2) we could prove, by means of Exercise III.5.3 and Theorem V.2.1, the existence of at least one *smooth, nonzero* solution, $\boldsymbol{v}^\sharp, p^\sharp$, to (V.0.1),(V.0.2) in $\Omega^\sharp$, corresponding to $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv \mathbf{0}$. In addition, $\boldsymbol{v}^\sharp$ would tend to zero as $|x| \to \infty$ uniformly pointwise, and $\boldsymbol{v}^\sharp, p^\sharp$ would have the asymptotic behavior specified in Theorem V.3.2. ■

From Lemma IV.1.1, it follows that to every q-weak solution we can associate a suitable pressure field p. Namely, if $\boldsymbol{f} \in W_0^{-1,q}(\Omega')$, $1 < q < \infty$, for any bounded subdomain Ω' with $\overline{\Omega'} \subset \Omega$, there exists $p \in L^q_{loc}(\Omega)$ such that

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) = -[\boldsymbol{f}, \boldsymbol{\psi}] + (p, \nabla \cdot \boldsymbol{\psi}) \tag{V.1.2}$$

for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. However, if Ω is locally Lipschitz and $\boldsymbol{f} \in D_0^{-1,q}(\Omega)$ we have the following global result.

Lemma V.1.1 *Let Ω be a locally Lipschitz exterior domain in $\mathbb{R}^n$ and let $\boldsymbol{v}$ be a q-generalized solution to* (V.0.1), (V.0.2). *Then, if*

$$\boldsymbol{f} \in D_0^{-1,q}(\Omega),$$

there exists a unique $p \in L^q(\Omega)$ satisfying (V.1.2) *for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. Furthermore, the following inequality holds*

$$\|p\|_q \leq c\left(|\boldsymbol{f}|_{-1,q} + |\boldsymbol{v}|_{1,q}\right). \tag{V.1.3}$$

Proof. The functional

$$\mathcal{F}(\boldsymbol{\psi}) = (\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) + [\boldsymbol{f}, \boldsymbol{\psi}]$$

is bounded for $\boldsymbol{\psi} \in D_0^{1,q'}(\Omega)$ and vanishes for $\boldsymbol{\psi} \in \mathcal{D}_0^{1,q'}(\Omega)$. The existence and uniqueness of p is then a direct consequence of Corollary III.5.1. Consider, next, the problem

$$\begin{aligned} &\nabla \cdot \boldsymbol{\psi} = |p|^{q-2}p \\ &\boldsymbol{\psi} \in D_0^{1,q'}(\Omega) \\ &|\boldsymbol{\psi}|_{1,q'} \leq c_1 \|p\|_q. \end{aligned} \tag{V.1.4}$$

Since $p \in L^q(\Omega)$, by Theorem II.4.2, there exists at least one $\boldsymbol{\psi}$ satisfying (V.1.4). Replacing such a $\boldsymbol{\psi}$ into (V.1.2) and using the Hölder inequality then shows (V.1.3). □

The next step is to investigate the regularity of q-generalized solutions. Since regularity is a local property, such a study is most easily performed by means of the results shown in Section IV.4 and Section IV.5. Specifically, from Theorem IV.4.1 and Theorem IV.5.1 we have the following result, whose proof is left to the reader as an exercise.

Theorem V.1.1 *Let $\boldsymbol{f} \in W^{m,q}_{loc}(\Omega)$, $m \geq 0$, $1 < q < \infty$, and let*

$$\boldsymbol{v} \in W^{1,q}_{loc}(\Omega), \quad p \in L^q_{loc}(\Omega)$$ [2]

with $\boldsymbol{v}$ weakly divergence-free, satisfy (V.1.2) *for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. Then*

$$\boldsymbol{v} \in W^{m+2,q}_{loc}(\Omega), \; p \in W^{m+1,q}_{loc}(\Omega).$$

In particular, if $\boldsymbol{f} \in C^\infty(\Omega)$, then

$$\boldsymbol{v}, \quad p \in C^\infty(\Omega).$$

Also, if Ω is of class C^{m+2} and $\boldsymbol{f} \in W^{m,q}_{loc}(\overline{\Omega})$, $\boldsymbol{v}_* \in W^{m+2-1/q,q}(\partial\Omega)$, then

$$\boldsymbol{v} \in W^{m+2,q}_{loc}(\overline{\Omega}), \quad p \in W^{m+1,q}_{loc}(\overline{\Omega}).$$

In particular, if Ω is of class C^∞ and $\boldsymbol{f} \in C^\infty(\Omega)$, $\boldsymbol{v}_* \in C^\infty(\partial\Omega)$ then $\boldsymbol{v}$, $p \in C^\infty(\overline{\Omega'})$, for all bounded $\Omega' \subset \Omega$.

V.2 Existence and Uniqueness of Generalized Solutions for Three-Dimensional Flow

In this section we shall be concerned with the well-posedness of the Stokes problem when the region of flow is a three-dimensional exterior domain.[1] The two-dimensional case, being related to the Stokes paradox, is in general not solvable. Actually, as will be proved in Section V.7, it admits a solution *if and only if* the data obey suitable restrictions; see also Theorem V.2.2. Furthermore, the problem of existence of q-generalized solutions for any $q > 1$ will be treated in Section V.5.

[2] As a matter of fact, both assumptions on $\boldsymbol{v}$ and p can be replaced by the following one:

$$\nabla \boldsymbol{v} \in L^q_{loc}(\Omega),$$

with $\boldsymbol{v}$ satisfying (V.1.1) for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. This is because, by Lemma II.6.1, $\boldsymbol{v} \in W^{1,q}_{loc}(\Omega)$ and, by Lemma IV.1.1, we infer the existence of $p \in L^q_{loc}(\Omega)$ satisfying (V.1.2).

[1] Actually, n-dimensional with $n \geq 3$; see Remark V.2.1.

Theorem V.2.1 *Let Ω be a locally Lipschitz exterior domain of $\mathbb{R}^3$. Given*

$$\boldsymbol{f} \in D_0^{-1,2}(\Omega), \quad \boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega),$$

there exists one and only one generalized solution to the Stokes problem (V.0.1), (V.0.2). *This solution satisfies for all $R > \delta(\Omega^c)$ the following estimate*

$$\|\boldsymbol{v}\|_{2,\Omega_R} + |\boldsymbol{v}|_{1,2} + \|p\|_2 \leq c\left\{|\boldsymbol{f}|_{-1,2} + \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\right\} \tag{V.2.1}$$

where p is the pressure field associated to $\boldsymbol{v}$ by Lemma V.1.1 *and $c = c(\Omega, R)$, $c \to \infty$ as $R \to \infty$. Furthermore,*

$$\int_{S^2} |\boldsymbol{v}(x)| = o(1/\sqrt{|x|}) \ \text{ as } |x| \to \infty. \tag{V.2.2}$$

Proof. The proof of existence and uniqueness goes exactly as in Theorem IV.1.1, provided we make a suitable extension of $\boldsymbol{v}_*$. In this respect, it is worth noticing that it is not required that the flux of $\boldsymbol{v}_*$ on $\partial\Omega$ be zero. Set

$$\Phi = \int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n}, \quad \boldsymbol{\sigma}(x) = -\Phi\, \nabla\mathcal{E}(x),$$

where $\mathcal{E}$ is the fundamental solution to the Laplace equation and with the origin of coordinates taken in $\dot{\Omega}^c$. Clearly,[2]

$$\Delta\boldsymbol{\sigma} = 0 \ \text{ in } \Omega,$$

$$\int_{\partial\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{n} = \Phi.$$

Putting $\boldsymbol{w}_* = \boldsymbol{v}_* - \boldsymbol{\sigma}$, it follows that

$$\int_{\partial\Omega} \boldsymbol{w}_* \cdot \boldsymbol{n} = 0$$

and we can apply the results in Exercise III.3.8 to construct a solenoidal field $\boldsymbol{V}_1 \in W^{1,2}(\Omega)$, vanishing outside Ω_ρ, for some $\rho > \delta(\Omega^c)$, that equals $\boldsymbol{w}_*$ on $\partial\Omega$ and, moreover,

$$\|\boldsymbol{V}_1\|_{1,2,\Omega_\rho} \leq c_1 \|\boldsymbol{w}_*\|_{1/2,2(\partial\Omega)} \tag{V.2.3}$$

with $c = c(\Omega_\rho)$. On the other hand we have, clearly,

$$\|\boldsymbol{w}_*\|_{1/2,2(\partial\Omega)} \leq c_2 \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)},$$

so that (V.2.3) implies

[2] Recall that $\boldsymbol{n}$ is the unit *outer* normal to $\partial\Omega$.

$$\|\boldsymbol{V}_1\|_{1,2,\Omega_\rho} \le c_3 \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \tag{V.2.4}$$

with $c_3 = c_3(\Omega, \rho)$. A generalized solution to the exterior problem is then sought in the form

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{V}_1 + \boldsymbol{\sigma},$$

where $\boldsymbol{w} \in \mathcal{D}_0^{1,2}(\Omega)$ solves

$$(\nabla \boldsymbol{w}, \nabla \boldsymbol{\varphi}) = -[\boldsymbol{f}, \boldsymbol{\varphi}] - (\nabla \boldsymbol{V}, \nabla \boldsymbol{\varphi}),$$

with

$$\boldsymbol{V} = \boldsymbol{V}_1 + \boldsymbol{\sigma}.$$

Existence, uniqueness, and estimate (V.2.1) are proved along the same lines of Lemma IV.1.1, provided we use Lemma V.1.1 instead of Lemma IV.1.1 and note that, since $\Delta\boldsymbol{\sigma} = 0$ in Ω, we have

$$\int_\Omega \nabla \boldsymbol{\sigma} : \nabla \boldsymbol{\varphi} = 0\,,$$

for any $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$. To show estimate (V.2.2) we notice that for $|x|$ sufficiently large

$$\begin{aligned}\int_{S^2} |\boldsymbol{v}(x)| &\le c_4 \int_{S^2} (|\boldsymbol{w}(x)| + |\Phi|\,|\nabla \mathcal{E}(x)|)\\ &= c_4 \int_{S^2} |\boldsymbol{w}(x)| + O(1/|x|^2),\end{aligned}$$

and, since $\boldsymbol{w} \in \mathcal{D}_0^{1,2}(\Omega)$, by Theorem II.7.6 and Lemma II.6.3 it follows

$$\int_{S^2} |\boldsymbol{w}(x)| = o(1/\sqrt{|x|}),$$

which furnishes (V.2.2). The proof of the theorem is then completed. □

Exercise V.2.1 Show that Theorem V.2.1 holds also when $\nabla \cdot \boldsymbol{v} = g \not\equiv 0$, where g is a prescribed function of $L^2(\Omega)$. In this case (V.2.1) is modified accordingly, by adding the term $\|g\|_2$ on its right-hand side. Notice that, unlike the case where Ω is bounded (see Exercise IV.1.1), no relation between g and $\boldsymbol{v}_*$ is needed.

Remark V.2.1 If in Theorem V.2.1, $\boldsymbol{v}_* = \boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{x} \equiv \boldsymbol{V}$, the solenoidal extension of the boundary velocity can be performed in an elementary way. Specifically, we need a field $\boldsymbol{a}$ that equals $\boldsymbol{V}$ near $\partial\Omega$, equals $\boldsymbol{0}$ at large distances, and has first derivatives in $L^2(\Omega)$. Thus, assuming without loss of generality $\boldsymbol{v}_0 = (v_0, 0, 0)$, we may take (Borchers 1992)

$$\boldsymbol{a} = -\tfrac{1}{2}\nabla \times \left[\nabla \times (\zeta \boldsymbol{v}_0 x_2^2) + \zeta x^2 \boldsymbol{\omega})\right], \tag{V.2.5}$$

where ζ is an arbitrary function from $C_0^\infty(\overline{\Omega})$ that is zero near $\partial\Omega$ and one far from $\partial\Omega$. Consequently, in particular, if $\boldsymbol{v}_* = \boldsymbol{V}$, existence of a generalized solution is proved without regularity assumptions on Ω. ■

Remark V.2.2 In spatial dimension $n > 3$ the results of Theorem V.2.1 continue to hold with estimate (V.2.2) replaced by

$$\int_{S^{n-1}} |\boldsymbol{v}(x)| = o(1/|x|^{n/2-1}) \ \text{ as } |x| \to \infty.$$

In the case of plane motions, however, we have a different situation that resembles the Stokes paradox mentioned at the beginning of the chapter. Actually, using the same method of proof, we can still construct a field $\boldsymbol{v}$ satisfying conditions (i)-(iii) and (v) of Definition V.1.1. However, we are not able, for such a $\boldsymbol{v}$, to check the validity of (iv), that is, to control the behavior of the solution at large distances. This is because functions having a finite Dirichlet integral in two space dimension need not tend to a finite limit at infinity, even in a generalized sense; see Section II.6 and Section II.9. Nevertheless, as will be shown in Theorem V.3.2 (see also Remark V.3.5), every such solution *does* tend to a well-defined vector, $\boldsymbol{v}_\infty$, at infinity, whenever the body force is of compact support. However, we *cannot* conclude

$$\boldsymbol{v}_\infty = \boldsymbol{0}. \tag{V.2.6}$$

Actually, (V.2.6) is in general not true, and in Section V.7 we shall prove that (V.2.6) holds *if and only if* the data satisfy certain restrictions. The meaning of the vector $\boldsymbol{v}_\infty$ will be clarified in Section VII.8, within the context of a singular perturbation theory based on the Oseen approximation. Here, we end by pointing out the following *Stokes paradox for generalized solutions* (Heywood 1974).[3] ■

Theorem V.2.2 *Let $\boldsymbol{v}$ be a weak solution to the Stokes problem in an exterior, locally Lipschitz two-dimensional domain corresponding to $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv 0$. Then $\boldsymbol{v} = 0$ a.e. in Ω.*

Proof. By assumption,

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) = 0, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega), \tag{V.2.7}$$

where $\boldsymbol{v} \in D^{1,2}(\Omega)$ and $\boldsymbol{v} = 0$ at $\partial\Omega$ (in the trace sense). By Theorem II.7.1 with $q = n = 2$, it follows that $\boldsymbol{v} \in D_0^{1,2}(\Omega)$ and, since $\boldsymbol{v}$ is solenoidal, this implies $\boldsymbol{v} \in \widehat{\mathcal{D}}_0^{1,2}(\Omega)$. On the other hand, since Ω is locally Lipschitz, by Theorem III.5.1, we find

$$\widehat{\mathcal{D}}_0^{1,2}(\Omega) = \mathcal{D}_0^{1,2}(\Omega)$$

and so $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega)$, which together with (V.2.7) completes the proof of the theorem. □

[3] In a private conversation in the summer of 2003, Olga Ladyzhenskaya pointed out to me that a result entirely analogous to Theorem V.2.2 is stated, without proof, at p. 43 of Ladyzhenskaya (1969).

V.3 Representation of Solutions. Behavior at Large Distances and Related Results

In order to perform an L^q-theory in exterior domains of the type performed for bounded domains in Section IV.6, we need to know more about the asymptotic behavior of solutions at great distances. We shall see, in particular, that under suitable conditions of "growth" at infinity, they behave exactly as the Stokes fundamental solution, provided the force $\boldsymbol{f}$ is of compact support. All this will be proved as a consequence of some representation formulas we are about to derive. In principle, this can be done quite straightforwardly from the results of Section IV.8. Actually we may write (IV.8.20) and (IV.8.21) on $\Omega \cap B_R(x)$, for sufficiently large R, then let $R \to \infty$ and require that the surface integrals calculated at $\partial B_R(x)$ converge to zero. However, this method would impose too severe restrictions a priori on the behavior of $\boldsymbol{v}$ and p at large distances and the results obtained under such assumptions would be of no use for further purposes. We therefore employ another technique introduced by Fujita (1961) in the nonlinear context, which is based on a suitable "truncation" of the fundamental solution.

Let $\psi = \psi(t)$ be a C^∞-function in $\mathbb{R}$ that equals one for $|t| \le 1/2$ and zero for $|t| \ge 1$. Setting

$$\psi_R(x-y) = \psi\left(\frac{x-y}{R}\right), \quad R > 0,$$

there follows

$$\begin{aligned} &\psi_R(x-y) = \begin{cases} 1 \text{ if } |x-y| \le R/2 \\ 0 \text{ if } |x-y| \ge R \end{cases} \\ &|D^\alpha \psi_R(x-y)| \le M R^{-|\alpha|}, \quad |\alpha| \ge 0 \end{aligned} \tag{V.3.1}$$

where M is independent of x, y. The *Stokes-Fujita truncated fundamental solution* $U_{ij}^{(R)}$, $q_j^{(R)}$ is then defined by (IV.2.1) with Φ replaced by $\psi_R \Phi$. Evidently, from $(\text{V.3.1})_1$ we have

$$U_{ij}^{(R)}(x-y) = U_{ij}(x-y), \quad q_i^{(R)}(x-y) = q_i(x-y), \quad \text{if } |x-y| \le R/2, \tag{V.3.2}$$

while

$$U_{ij}^{(R)}(x-y) \equiv q_i^{(R)}(x-y) \equiv 0, \quad \text{if } |x-y| \ge R. \tag{V.3.3}$$

Moreover, from (IV.2.2) it follows for $x \ne y$ that

$$\begin{aligned} &\Delta U_{ij}^{(R)}(x-y) + \frac{\partial}{\partial x_i} q_j^{(R)}(x-y) = H_{ij}^{(R)}(x-y) \\ &\frac{\partial}{\partial x_\ell} U_{\ell j}^{(R)}(x-y) = 0 \end{aligned} \tag{V.3.4}$$

where $H_{ij}^{(R)}$ is defined by

$$H_{ij}^{(R)}(x-y) = \begin{cases} 0 & \text{if } x = y \\ \delta_{ij}\Delta^2(\psi_R \Phi)(x-y) & \text{if } x \neq y. \end{cases}$$

Since $\Phi(x-y)$ is biharmonic for $x \neq y$ and all derivatives of $\psi_R(x-y)$ vanish unless $R/2 \leq |x-y| \leq R$, we recover that $H_{ij}^{(R)}$ is indefinitely differentiable and vanishes unless $R/2 \leq |x-y| \leq R$. Also, for $u \in L^1_{loc}(\Omega)$ it is $H_{ij}^{(R)} * u \in C^\infty(\mathbb{R}^n)$. Finally, from (V.3.1) and the properties of Φ we at once obtain the following uniform bounds as $R \to \infty$

$$|D^\alpha H_{ij}^{(R)}(x-y)| = \begin{cases} O(\log R/R^{2+|\alpha|}), & \text{if } n = 2 \\ O(R^{-n-|\alpha|}) & \text{if } n > 2. \end{cases} \qquad |\alpha| \geq 0 \tag{V.3.5}$$

Consider now the Green's formula (IV.8.10) in a domain Ω, not necessarily bounded, with

$$\boldsymbol{u}(y) = \boldsymbol{u}_j^{(R)}(x-y) \equiv \left(U_{1j}^{(R)}, U_{2j}^{(R)}, \ldots, U_{nj}^{(R)}\right)$$

$$\pi(y) = q_j^{(R)}(x-y).$$

Such a procedure is meaningful, since, whatever the domain Ω may be, in view of (V.3.3) the integration is always made on a *bounded* region (the support of $\psi_R(x-y)$). By repeating all the steps leading to (IV.8.20) and (IV.8.21) and recalling (V.3.2) we thus readily obtain

$$\begin{aligned} D^\alpha v_j(x) = & \int_\Omega U_{ij}^{(R)}(x-y) D^\alpha f_i(y) dy \\ & - \int_{\partial\Omega} [U_{ij}^{(R)}(x-y) T_{i\ell}(D^\alpha \boldsymbol{v}, D^\alpha p)(y) \\ & \qquad - D^\alpha v_i(y) T_{i\ell}(\boldsymbol{u}_j^{(R)}, q_j^{(R)})(x-y)] n_\ell(y) d\sigma_y \\ & - \int_\Omega H_{ij}^{(R)}(x-y) D^\alpha v_i(y) dy. \end{aligned} \tag{V.3.6}$$

Likewise, setting

$$S^{(R)}(x) = \int_\Omega q_i^{(R)}(x-y) D^\alpha f_i(y) dy,$$

we have

$$\begin{aligned} \frac{\partial(D^\alpha p)}{\partial x_j} = \frac{\partial S^{(R)}}{\partial x_j}(x) + & \int_{\partial\Omega} \left[\frac{\partial q_i^{(R)}(x-y)}{\partial x_j} T_{i\ell}(D^\alpha \boldsymbol{v}, D^\alpha p)(y) \right. \\ & \left. - 2D^\alpha v_i(y) \frac{\partial^2 q_\ell(x-y)}{\partial x_j \partial x_i} \right] n_\ell(y) d\sigma_y \\ & - \int_\Omega \Delta H_{ij}^{(R)}(x-y) D^\alpha v_i(y) dy; \end{aligned} \tag{V.3.7}$$

see Exercise V.3.1. Notice that if $R < \operatorname{dist}(x, \partial\Omega)$, formulas (V.3.6) and (V.3.7) do not require any regularity assumption on Ω. The following result holds.

Lemma V.3.1 *Let Ω be an arbitrary domain of $\mathbb{R}^n$. Let $\boldsymbol{v} \in W^{1,r}_{loc}(\Omega)$, $1 < r < \infty$, be weakly divergence-free and satisfy* (V.1.1) *for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and some $r \in (1, \infty)$. Then, if $\boldsymbol{f} \in W^{m,q}_{loc}(\Omega)$, $1 < q < \infty$, it follows that $\boldsymbol{v} \in W^{m+2,q}_{loc}(\Omega)$ and, moreover, for all fixed $d > 0$, for all $|\alpha| \in [0, m]$ and for almost all $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) > d$, $\boldsymbol{v}$ obeys the identity*

$$D^\alpha v_j(x) = \int_{B_d(x)} U^{(d)}_{ij}(x-y) D^\alpha f_i(y) dy - \int_{\beta(x)} H^{(d)}_{ij}(x-y) D^\alpha v_i(y) dy \quad \text{(V.3.8)}$$

where $\beta(x) = B_d(x) - B_{d/2}(x)$.

Proof. The first part of the lemma has already been proved in Theorem IV.4.2. Concerning the validity of (V.3.8), we notice that if $\boldsymbol{v}$ and p are smooth, it follows from (V.3.6), by taking $R = d$ and recalling the properties of $H^{(d)}_{ij}$. The validity of (V.3.8) under the hypothesis of the lemma is recovered by adopting exactly the same procedure used in the proof of Theorem IV.8.1, and we leave it to the reader. □

Remark V.3.1 The assumptions of Lemma V.3.1 do *not* require Ω to be an exterior domain. Rather, Ω can be *any* domain of $\mathbb{R}^n$. Actually, Lemma V.3.1 will be applied in Chapter VI to the study of the asymptotic behavior of Stokes flow in domains with noncompact boundaries. ■

Remark V.3.2 For future purposes, we observe that a representation analogous to that furnished in Lemma V.3.1 also holds for solutions to the Poisson equation $\Delta u = f$ in an arbitrary domain Ω of $\mathbb{R}^n$. More precisely, set

$$\mathcal{E}^{(R)}(x-y) = \psi_R(x-y)\mathcal{E}(x-y), \tag{V.3.9}$$

where $\mathcal{E}$ is the fundamental solution of Laplace's equation (II.9.1) and ψ_R is given in (V.3.1). In a strict analogy with the Stokes-Fujita truncated fundamental solution, one has

$$\mathcal{E}^{(R)}(x-y) = \begin{cases} \mathcal{E}(x-y) & \text{if } |x-y| \le R/2 \\ 0 & \text{if } |x-y| \ge R. \end{cases} \tag{V.3.10}$$

Furthermore, for all $x \neq y$,

$$\Delta\mathcal{E}^{(R)}(x-y) = H^{(R)}(x-y) \tag{V.3.11}$$

with

$$H^{(R)}(x-y) = \begin{cases} 0 & \text{if } x = y \\ \Delta\left(\psi_R(x-y)\mathcal{E}(x-y)\right) & \text{if } x \neq y. \end{cases} \tag{V.3.12}$$

Clearly, the function $H^{(R)}$ is infinitely differentiable and vanishes unless $R/2 \leq |x-y| \leq R$, and by the properties of ψ_R,

$$|D^\alpha H^{(R)}(x-y)| \leq C \begin{cases} \log R/R^{2+|\alpha|}, & \text{if } n=2 \\ R^{-n-|\alpha|} & \text{if } n>2. \end{cases} \tag{V.3.13}$$

Repeating step by step the proof of Lemma II.9.1 with $\mathcal{E}^{(R)}$ in place of $\mathcal{E}$ and using (V.3.11) we deduce for almost all $x \in \Omega$, with $\operatorname{dist}(x, \partial\Omega) > R$, the following general representation formula

$$D^\alpha u(x) = \int_{B_R(x)} \mathcal{E}^{(R)}(x-y) D^\alpha f(y) dy - \int_{B_R(x)-B_{R/2}(x)} H^{(R)}(x-y) D^\alpha u(y) dy, \tag{V.3.14}$$

with $|\alpha| \geq 0$. ■

Exercise V.3.1 Let Ω be of class C^{m+2}, $m \geq 0$, and let $\boldsymbol{v} \in W^{m+2,q}(\Omega_\rho)$, $p \in W^{m+1,q}(\Omega_\rho)$ solve *a.e.* the Stokes system $(\text{V.0.1})_{1,2}$, corresponding to $\boldsymbol{f} \in W^{m,q}(\Omega_\rho)$, $1 < q < \infty$, all $\rho > \delta(\Omega^c)$. Show the validity of (V.3.6) for almost all $x \in \Omega$ and all $R > \delta(\Omega^c)$. Assuming, further, that the support S of $\boldsymbol{f}$ is bounded, show the validity of (V.3.7) for almost all $x \in \Omega$ and all R for which $B_R(x) \supset \mathsf{S}$. *Hint*: (V.3.6) is shown by the same technique of Theorem IV.8.1. For (V.3.7), the only term that demands little care is that involving $S^{(R)}$. However, for $B_R(x) \supset \mathsf{S}$ we have

$$S^{(R)}(x) = \int_\Omega q_i(x-y) D^\alpha f_i(y) dy \equiv S(x)$$

and, since $D_j q_i$ is a singular kernel, under the stated assumptions on $\boldsymbol{f}$ the function $D_j S$ belongs to $L^q(\Omega_r)$ for all $r > \delta(\Omega^c)$ and one has

$$\|D_j S\|_{q,\Omega_r} \leq c \|D^\alpha \boldsymbol{f}\|_{q,\Omega};$$

see, e.g., Mikhlin (1965, §29).

Lemma V.3.1 allows us to derive some information concerning the pointwise asymptotic behavior of q-weak solutions. For instance, we have[1]

Theorem V.3.1 *Let Ω be an arbitrary exterior domain in $\mathbb{R}^n$, $n \geq 2$, and let $\boldsymbol{v} \in D^{1,q}(\Omega^R) \cap L^s(\Omega^R)$, for some $R > \delta(\Omega^c)$ and some $q, s \in (1, \infty)$. Assume further that $\boldsymbol{v}$ is weakly divergence-free and satisfies* (V.1.1), *for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ with $\boldsymbol{f} \in W^{m,r}(\Omega^R)$, $m \geq 0$, $n/2 < r < \infty$. Then*

$$\lim_{|x| \to \infty} D^\alpha \boldsymbol{v}(x) = \boldsymbol{0}, \quad \textit{for all } |\alpha| \in [0, m].$$

[1] The hypotheses of the following theorem are not optimal on $\boldsymbol{v}$. However, they will suffice for further purposes. Furthermore, the theorem holds under alternative assumptions on $\boldsymbol{f}$. We shall formulate these latter directly in the nonlinear context; see Section X.5.

Proof. We show the result for $n \geq 3$, the case $n = 2$ being treated analogously. For fixed $d > 0$ and all $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) > d$, we have

$$\left| \int_{B_d(x)} U_{ij}^{(d)}(x-y) D^\alpha f_i(y) dy \right| \leq c \| |x-y|^{2-n} \|_{q/(q-1), B_d(x)} \| \boldsymbol{f} \|_{m,q,B_d(x)}$$
$$\leq c_1 \| \boldsymbol{f} \|_{m,q,B_d(x)} \tag{V.3.15}$$

and

$$\left| \int_{\beta(x)} H_{ij}^{(d)}(x-y) \;\; D^\alpha v_i(y) dy \right| = \left| \int_{\beta(x)} D^\alpha H_{ij}^{(d)}(x-y) v_i(y) dy \right|$$
$$\leq c \max_{i,j} \| H_{ij}^{(d)}(x-y) \|_{m,s',\beta(x)} \| \boldsymbol{v} \|_{s,\beta(x)} \tag{V.3.16}$$
$$\leq c_1 \| \boldsymbol{v} \|_{s,B_d(x)},$$

where we have exploited the properties of the function $H_{ij}^{(d)}$. The theorem is then a consequence of this fact, (V.3.15), (V.3.16), and Lemma V.3.1. □

Remark V.3.3 From Theorem V.3.1 and the equation of motion (V.0.1) we can immediately derive a pointwise behavior of the pressure field p at large distances. For example, if $\boldsymbol{f}$ satisfies the assumption of that theorem with $m \geq 2$ and, further, $D^\alpha \boldsymbol{f}(x)$ tends to zero as $|x| \to \infty$, $|\alpha| \in [0, m-2]$ we have

$$\lim_{|x| \to \infty} D^\alpha \nabla p(x) = 0, \quad 0 \leq |\alpha| \leq m-2.$$

■

Exercise V.3.2 Let $\boldsymbol{v}$ satisfy the hypotheses of Theorem V.3.1, with the possible exception of condition (iv) of Definition V.1.1. Assuming $\boldsymbol{f} \in W^{m,r}(\Omega^R)$, $r > n$, show

$$\lim_{|x| \to \infty} D^\alpha \nabla \boldsymbol{v}(x) = 0, \quad \text{for all } |\alpha| \in [0, m].$$

Theorem V.3.1 is silent about the rate of decay of solutions. However, if $\boldsymbol{f}$ is of compact support we can obtain sharp estimates for $\boldsymbol{v}$, p and for their derivatives of arbitrary order. In fact, we have

Theorem V.3.2 *Let Ω be a C^2-smooth, exterior domain and let $\boldsymbol{v} \in W_{loc}^{2,q}(\overline{\Omega})$, $q \in (1, \infty)$, be weakly divergence-free and satisfy* (V.1.1) *for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ with $\boldsymbol{f} \in L^q(\Omega)$. Assume, further, that the support of $\boldsymbol{f}$ is bounded. Then, if at least one of the following conditions is satisfied as $|x| \to \infty$:*

(i) $|\boldsymbol{v}(x)| = o(|x|)$

(ii) $\displaystyle\int_{|x| \leq r} \frac{|\boldsymbol{v}(x)|^t}{(1+|x|)^{n+t}} dx = o(\log r)$, *some* $t \in (1, \infty)$, *there exist vector and scalar constants* $\boldsymbol{v}_\infty$, p_∞ *such that for almost all* $x \in \Omega$

$$\begin{aligned} v_j(x) = v_{\infty j} &+ \int_\Omega U_{ij}(x-y) f_i(y)dy - \int_{\partial\Omega} [U_{ij}(x-y)T_{i\ell}(\boldsymbol{v},p)(y) \\ &-v_i(y)T_{i\ell}(\boldsymbol{u}_j, q_j)(x-y)]n_\ell(y)d\sigma_y \\ &\equiv v_{\infty j} + v_j^{(1)}(x) \end{aligned} \tag{V.3.17}$$

and

$$\begin{aligned} p(x) = p_\infty &- \int_\Omega q_i(x-y) f_i(y)dy + \int_{\partial\Omega} [q_j(x-y)T_{i\ell}(\boldsymbol{v},p)(y) \\ &-2v_i(y)\frac{\partial q_\ell(x-y)}{\partial x_i} n_\ell(y)]d\sigma_y \\ &\equiv p_\infty + p^{(1)}(x). \end{aligned} \tag{V.3.18}$$

Moreover, as $|x| \to \infty$, $\boldsymbol{v}^{(1)}(x)$ *and* $p^{(1)}(x)$ *are infinitely differentiable and there the following asymptotic representations hold:*

$$\begin{aligned} v_j^{(1)}(x) &= \mathcal{T}_i U_{ij}(x) + \sigma_j(x) \\ p^{(1)}(x) &= -\mathcal{T}_i q_i(x) + \eta(x), \end{aligned} \tag{V.3.19}$$

where

$$\mathcal{T}_i = -\int_{\partial\Omega} T_{i\ell}(\boldsymbol{v},p)n_\ell + \int_\Omega f_i \tag{V.3.20}$$

and, for all $|\alpha| \geq 0$,

$$\begin{aligned} D^\alpha \boldsymbol{\sigma}(x) &= O(|x|^{1-n-|\alpha|}) \\ D^\alpha \eta(x) &= O(|x|^{-n-|\alpha|}). \end{aligned} \tag{V.3.21}$$

Proof. Let us observe that, since the support S of $\boldsymbol{f}$ is compact, $\boldsymbol{v}$ and p are infinitely differentiable at each point of $\Omega - \mathsf{S}$, as follows from Theorem IV.4.1. We begin to show that (V.3.19)–(V.3.21) are a consequence of (V.3.17), (V.3.18). Actually we have

$$\begin{aligned} v_j^{(1)}(x) = \mathcal{T}_i U_{ij}(x) &+ \int_{\partial\Omega} v_i(y) T_{il}(\boldsymbol{u}_j, q_j)(x-y)n_\ell d\sigma_y \\ &+ \int_\Omega [U_{ij}(x-y) - U_{ij}(x)] f_i(y)dy \\ &- \int_{\partial\Omega} [U_{ij}(x-y) - U_{ij}(x)] T_{i\ell}(\boldsymbol{v},p)(y)n_\ell(y)d\sigma_y. \end{aligned} \tag{V.3.22}$$

On the other hand, from (IV.8.14) we deduce

$$|D^\alpha T_{ik}(\boldsymbol{u}_j, q_j)(x-y)| = O(|x|^{1-n-|\alpha|}), \quad |\alpha| \geq 0, \tag{V.3.23}$$

uniformly with respect to y in a bounded set. Likewise, since

$$|D^\alpha \left(U_{ij}(x-y) - U_{ij}(x)\right)| = \left| y_\ell \frac{\partial}{\partial x_\ell} D^\alpha \left(U_{ij}(x-\beta y)\right) \right|,$$

where $\beta \in [0,1]$, by (IV.2.6) it follows

$$|D^\alpha \left(U_{ij}(x-y) - U_{ij}(x)\right)| = O(|x|^{1-n-|\alpha|}), \quad |\alpha| \geq 0, \tag{V.3.24}$$

again uniformly in y in a bounded set. Thus, by observing that $\boldsymbol{v}^{(1)}(x)$ is infinitely differentiable for all $x \in \Omega - \mathsf{S}$, relations $(\text{V.3.19})_1$, (V.3.20), and (V.3.21) follow from (V.3.22)–(V.3.24). The analogous estimate for p can be shown in an entirely similar way. To prove (V.3.17), we take R so large that $\mathsf{S} \subset B_R(x)$. Therefore, for such an R,

$$\int_\Omega U_{ij}^{(R)}(x-y) f_i(y) dy = \int_\Omega U_{ij}(x-y) f_i(y) dy. \tag{V.3.25}$$

From Exercise V.3.1 we know that, under the stated assumptions, $\boldsymbol{v}(x)$ obeys (V.3.6) with $\alpha = 0$ for almost all $x \in \Omega$. Therefore, from (V.3.22) and (V.3.25) we find, for almost all $x \in \Omega$,

$$\boldsymbol{v}(x) = \boldsymbol{v}^{(1)}(x) + \boldsymbol{v}^{(2)}(x), \tag{V.3.26}$$

where

$$v_i^{(2)}(x) \equiv -\int_\Omega H_{ij}^{(R)}(x-y) v_j(y) dy. \tag{V.3.27}$$

Since $\boldsymbol{v} - \boldsymbol{v}^{(1)}$ is independent of R, so is $\boldsymbol{v}^{(2)}$. Let us show that

$$D^2 \boldsymbol{v}^{(2)} \equiv 0. \tag{V.3.28}$$

Actually, from (V.3.21) and (V.3.27) we deduce for a suitable constant c independent of R

$$|D^2 \boldsymbol{v}^{(2)}(x)| \leq c \frac{\log^\alpha R}{R^{n+2}} \int_{\Omega_{R/2,R}(x)} |\boldsymbol{v}|, \tag{V.3.29}$$

where $\alpha = 1$ if $n = 2$, $\alpha = 0$ if $n > 2$ and $\Omega_{R,2R}(x) = \{y \in \Omega : R/2 < |x-y| < R\}$. It is easy to prove that under the assumptions (i) and (ii) the right-hand side of (V.3.29) tends to zero as $R \to \infty$. In fact,

$$\frac{\log^\alpha R}{R^{n+2}} \int_{\Omega_{R/2,R}(x)} |\boldsymbol{v}| \leq \begin{cases} c_1 \dfrac{\log^\alpha R}{R} & (\textit{assumption (i)}) \\[2ex] c_2 \dfrac{(\log R)^{\alpha+1/t}}{R} & (\textit{assumption (ii)}) \end{cases}$$

with c_1, c_2 independent of R. So, there exists an $n \times n$ matrix $\boldsymbol{A}$ with $\mathrm{trace}(\boldsymbol{A}) = 0$ and a vector $\boldsymbol{v}_\infty$ such that

$$\boldsymbol{v}^{(2)} = \boldsymbol{A} \cdot \boldsymbol{x} + \boldsymbol{v}_\infty.$$

On the other hand, by using either (i) or (ii) and observing that

$$\boldsymbol{v}^{(1)}(x) = o(|x|) \quad \text{as } |x| \to \infty,$$

we readily show

$$\boldsymbol{A} = \boldsymbol{0},$$

thus establishing (V.3.17). Finally, since

$$\int_\Omega \Delta H_{ij}^{(R)}(x-y) v_i(y) dy = \Delta v_i^{(2)}(x),$$

identity (V.3.18) follows from (V.3.7), with $\alpha = 0$, and (V.3.28). □

From the proof just given it comes out that one may weaken assumptions (i) or (ii) on condition that polynomials in $\boldsymbol{v}$ of suitable degree are added to the right-hand sides of (V.3.17) and (V.3.18). In particular, we wish to single out the following result, which will be of interest for later purposes.

Theorem V.3.3 *Replace assumptions* (i) *and* (ii) *of* Theorem V.3.2 *with*

$$D^2\boldsymbol{v} \in L^q(\Omega), \quad \textit{for some } q \in [1, \infty), \tag{V.3.30}$$

the other assumptions remaining unaltered. Then, there exist a scalar p_∞, a vector $\boldsymbol{v}_\infty$, and an $n \times n$ matrix $\boldsymbol{V}_\infty$ with $\mathrm{trace}(\boldsymbol{V}_\infty) = 0$, *such that*

$$\begin{aligned} \boldsymbol{v}(x) &= \boldsymbol{v}_\infty + \boldsymbol{V}_\infty \cdot \boldsymbol{x} + \boldsymbol{v}^{(1)}(x) \\ p(x) &= p_\infty + p^{(1)}(x), \end{aligned} \tag{V.3.31}$$

where $\boldsymbol{v}^{(1)}$ and $p^{(1)}$ are defined in (V.3.19).

Proof. To show (V.3.31) it is enough to show that (V.3.30) implies (V.3.28). In this respect we have

$$\begin{aligned} |D^2\boldsymbol{v}^{(2)}(x)| &\le \int_{\Omega_{R/2,R}(x)} |H_{ij}^{(R)}(x-y)||D^2\boldsymbol{v}(y)|dy \\ &\le c\frac{\log^\alpha R}{R^{n/q}} \|D^2\boldsymbol{v}\|_{q,\Omega_{R/2,R}(x)}, \end{aligned}$$

where c does not depend on R and $\alpha = 1$ if $n = 2$ and $\alpha = 0$ if $n > 2$. Letting $R \to \infty$ into this relation proves (V.3.28). □

Remark V.3.4 In Theorem V.3.2 and Theorem V.3.3 *no hypothesis is made about the behavior of the pressure at infinity*; rather, it is derived as a consequence of the behavior assumed on the velocity field. ■

Remark V.3.5 As pointed out in Remark V.2.2, for Ω a two-dimensional exterior domain, by the method of Theorem V.2.1 we can construct a field $\boldsymbol{v}$ satisfying (i)-(iii) and (v) of Definition V.1.1, with $q = 2$. However, we are not able to check condition (iv), for a prescribed $\boldsymbol{v}_\infty \in \mathbb{R}^2$. Nevertheless, if $\boldsymbol{f}$ is of compact support, Theorem V.3.2 implies that $\boldsymbol{v}$ *does* tend to a certain vector $\boldsymbol{v}_\infty \in \mathbb{R}^2$. In fact, since $\boldsymbol{v} \in D^{1,2}(\Omega)$, it is then simple to show that

$$\int_{\partial\Omega^R} \boldsymbol{T}(\boldsymbol{v}, p) \cdot \boldsymbol{n} = 0, \tag{V.3.32}$$

where R is taken so large that Ω_R includes the support of $\boldsymbol{f}$ and p is the pressure field associated to p by Lemma V.1.1. To prove (V.3.32) we notice that from (V.0.1) and from the definition of $\boldsymbol{T}$ we have, for all $S > R$,[2]

$$\int_{\partial\Omega^R} \boldsymbol{T}(\boldsymbol{v}, p) \cdot \boldsymbol{n} = \int_{\partial\Omega^S} \boldsymbol{T}(\boldsymbol{v}, p) \cdot \boldsymbol{n} = \int_{\partial\Omega^S} (2\boldsymbol{D}(\boldsymbol{v}) \cdot \boldsymbol{n} + p\boldsymbol{n}).$$

However, since $\boldsymbol{v} \in D^{1,2}(\Omega^R)$ and, by Lemma V.1.1, the corresponding pressure field $p \in L^2(\Omega^R)$, we can find a sequence $\{S_k\}$, $S_k \to \infty$ as $k \to \infty$, along which the last integral on the right-hand side of the preceding identities tends to zero, thus proving (V.3.32). Theorem V.3.2 then ensures the existence of a well-defined vector $\boldsymbol{v}_\infty$ to which $\boldsymbol{v}$ tends at large distances. In general, $\boldsymbol{v}_\infty$ *cannot* be prescribed a priori (in particular, cannot be zero) unless the data verify a suitable restriction, see Section V.7. Notice, also, that, in the particular case when $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv \boldsymbol{0}$, describing the slow, plane motion of the liquid past a cylinder, we can take $\Omega^R \equiv \Omega$ in (V.3.32), thus obtaining that *the total force exerted by the liquid on the cylinder is zero.* This is another form of the Stokes paradox; see also Section V.7. ■

Let us now derive some significant implications of Theorem V.3.2. We begin with a uniqueness result for q-generalized solutions.

Theorem V.3.4 *Let $\boldsymbol{v}$ be a q-generalized solution to the Stokes problem* (V.0.1), (V.0.2) *in an exterior, three-dimensional*[3] *domain of class C^2, corresponding to $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv \boldsymbol{0}$. Then $\boldsymbol{v} \equiv \boldsymbol{0}$.*

Proof. From Lemma V.1.1 and the regularity results of Theorem IV.4.1 and Theorem IV.5.1, we derive

$$\boldsymbol{v} \in W^{2,q}_{loc}(\overline{\Omega}) \cap C^\infty(\Omega), \quad p \in W^{1,q}_{loc}(\overline{\Omega}) \cap C^\infty(\Omega), \quad \text{for all } q \in (1, \infty).$$

We may then apply Theorem V.3.2 to deduce

$$\boldsymbol{v} = O(|x|^{-1}), \quad p, \nabla\boldsymbol{v} = O(|x|^{-2}). \tag{V.3.33}$$

[2] Observe that $\boldsymbol{v}, p \in C^\infty(\Omega^R)$.

[3] The result continues to hold in any space dimension $n \geq 3$.

For fixed $R > \delta(\Omega^c)$, we dot-multiply $(\text{V.0.1})_1$ by $\boldsymbol{v}$, integrate by parts over Ω_R (this is allowed by Exercise II.4.3), and take into account $(\text{V.0.1})_2$. We thus deduce

$$\int_{\Omega_R} \nabla \boldsymbol{v} : \nabla \boldsymbol{v} = \int_{\partial B_R} \boldsymbol{n} \cdot (\nabla \boldsymbol{v} \cdot \boldsymbol{v} - p\boldsymbol{v}).$$

Estimating the surface integral through (V.3.33), and letting $R \to \infty$ then proves the result. □

Similar uniqueness results can be obtained for regular solutions possessing a suitable behavior at large distances. For example, we have the following result which for space dimension $n = 2$ furnishes another form of the *Stokes paradox*, already considered for generalized solutions in Theorem V.2.1. The proof is much like that of Theorem V.3.4 and, therefore, it will be omitted.

Theorem V.3.5 *Let $\boldsymbol{v}, p$ be a regular solution to the Stokes system* (V.0.1), *in a C^1-smooth exterior domain of $\mathbb{R}^n$, corresponding to $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv \boldsymbol{0}$. Then, if as $|x| \to \infty$*

$$\boldsymbol{v}(x) = \begin{cases} o(\log|x|) & \text{if } \; n = 2 \\ o(1) & \text{if } \; n > 2, \end{cases}$$

it follows that $\boldsymbol{v} \equiv \boldsymbol{0}$.

Other consequences of Theorem V.3.2 are left to the reader in the following exercises.

Exercise V.3.3 Let $\boldsymbol{v}$, $\boldsymbol{f}$, and Ω satisfy the assumptions of Theorem V.3.2. Show that $\|\boldsymbol{v} - \boldsymbol{v}_0\|_q = |\boldsymbol{v}|_{1,r} = \infty$ for all $q \in (1, n]$ and all $r \in (1, n/(n-1)]$, unless $\mathcal{T} = 0$.

Exercise V.3.4 Prove the following result of Liouville type. Let $\boldsymbol{v}, p$ be a regular Stokes flow in $\mathbb{R}^n$, corresponding to zero or, more generally, potential-like body force. Then if $\boldsymbol{v}$ is bounded, it follows that $\boldsymbol{v} = \text{const}$.

Exercise V.3.5 Let $\Omega \equiv \mathbb{R}^n$. Prove that if $\boldsymbol{v}$ and $\boldsymbol{f}$ satisfy the assumptions of Theorem V.3.2 the following asymptotic formulas hold:

$$\boldsymbol{v}(x) = \boldsymbol{v}_\infty + \boldsymbol{U}(x) \cdot \int_{\mathbb{R}^n} \boldsymbol{f} + \boldsymbol{\sigma}(x),$$

$$p(x) = p_\infty - \boldsymbol{q}(x) \cdot \int_{\mathbb{R}^n} \boldsymbol{f} + \eta(x),$$

where $\boldsymbol{v}_\infty$, p_∞ are vector and scalar constants, while $\boldsymbol{\sigma}$ and η satisfy (V.3.21).

Exercise V.3.6 Show the following "scalar" version of Theorem V.3.2. Let Ω be a domain of class C^2 and let $u \in W^{2,q}_{loc}(\overline{\Omega})$, some $q \in (1, \infty)$, be a solution to $\Delta u = f$ in Ω, where $f \in L^q(\Omega)$ is of bounded support. Then, if at least one of the following conditions is satisfied as $|x| \to \infty$:

(i) $|u(x)| = o(|x|)$

(ii) $\displaystyle\int_{|x|\le r} \frac{|u(x)|^t}{(1+|x|)^{n+t}}\, dx = o(\log r)$, *some* $t \in (1,\infty)$,

there exist $u_\infty \in \mathbb{R}$ such that for almost all $x \in \Omega$

$$
\begin{aligned}
u(x) &= u_\infty + \int_\Omega \mathcal{E}(x-y) f(y)dy - \int_{\partial\Omega} [\mathcal{E}(x-y)\frac{\partial u}{\partial y_\ell}(y) - u(y)\frac{\partial \mathcal{E}}{\partial y_\ell}(x-y)] n_\ell(y) d\sigma_y \\
&\equiv u_\infty + u^{(1)}(x)
\end{aligned}
.
$$

Moreover, as $|x| \to \infty$, $u^{(1)}(x)$ is infinitely differentiable and there the following asymptotic representations hold:

$$u^{(1)}(x) = a\,\mathcal{E}(x) + \sigma(x)\,,$$

where

$$a = -\int_{\partial\Omega} \frac{\partial u}{\partial x_\ell} n_\ell + \int_\Omega f$$

and, for all $|\alpha| \ge 0$,

$$D^\alpha \sigma(x) = O(|x|^{1-n-|\alpha|})\,.$$

Hint: Reproduce the same type of argument adopted in the proof of Theorem V.3.2, by replacing the (tensor) Stokes-Fujita truncated fundamental solution with the (scalar) Laplace truncated fundamental solution defined in (V.3.9)–(V.3.13).

V.4 Existence, Uniqueness, and L^q-Estimates: Strong Solutions

Our next objective is to investigate to what extent the results proved in Section IV.6 can be generalized to the case when the region of motion is an exterior one. Specifically, in the present section we shall be concerned with existence, uniqueness, and L^q-estimates of *strong* solutions to the Stokes problem (V.0.1), (V.0.2), *i.e.*, solutions with velocity fields possessing at least second derivatives, while in Section 5 we analyze the same question for *q-generalized* solutions.

To begin, we shall study some properties of solutions $\{\boldsymbol{v}, p\}$ to the Stokes *system*

$$
\left.
\begin{aligned}
\Delta \boldsymbol{v} &= \nabla p + \boldsymbol{f} \\
\nabla\cdot\boldsymbol{v} &= 0
\end{aligned}
\right\}\ \text{in } \Omega
\qquad \text{(V.4.1)}
$$

$$\boldsymbol{v} = \boldsymbol{v}_* \ \text{ at } \partial\Omega,$$

with Ω an exterior domain in $\mathbb{R}^n$ $(n \ge 2)$. Notice that the velocity field $\boldsymbol{v}$ need not satisfy a priori any prescribed condition at infinity. For this reason we prefer to call (V.4.1) a Stokes "system" instead of a Stokes "problem".

We have

Lemma V.4.2 *Let $\boldsymbol{v}, p$ be a solution to* $(V.4.1)_{1,2}$. *Assume $\boldsymbol{v} \in W^{2,q}_{loc}(\overline{\Omega})$, $p \in W^{1,q}_{loc}(\overline{\Omega})$ for some $q \in (1,\infty)$, and for some $r \in (1,\infty)$ and some $R > 2\delta(\Omega^c)$*

$$\boldsymbol{v} \in D^{2,r}(\Omega^R). \tag{V.4.2}$$

Then, if $\boldsymbol{f} \in L^q(\Omega)$ it follows that

$$\boldsymbol{v} \in D^{2,q}(\Omega), \quad p \in D^{1,q}(\Omega).$$

Proof. Denote by $\varphi = \varphi(|x|)$ a C^∞-function in Ω that is zero for $|x| \le \rho$ and equals one for $|x| \ge R/2$, $\delta(\Omega^c) < \rho < R/2$. Setting $\boldsymbol{u} = \varphi \boldsymbol{v}$ and $\pi = \varphi p$ we then have that $\boldsymbol{u}$ and π solve in $\mathbb{R}^n$ the system

$$\begin{aligned} \Delta \boldsymbol{u} &= \nabla \pi + \boldsymbol{f}_1 \\ \nabla \cdot \boldsymbol{u} &= g, \end{aligned} \tag{V.4.3}$$

where

$$f_{1i} = \varphi f_i + T_{ik}(\boldsymbol{v}, p) D_k \varphi + D_k(v_k D_i \varphi + v_i D_k \varphi), \quad g = \boldsymbol{v} \cdot \nabla \varphi, \tag{V.4.4}$$

with $\boldsymbol{T}$ defined in (IV.8.6). Clearly, $\boldsymbol{f}_1 \in L^q(\mathbb{R}^n)$ and $g \in W^{1,q}(\mathbb{R}^n)$ and so we may apply Theorem V.2.1 to prove the existence of a solution $\boldsymbol{u}^* \in D^{2,q}(\mathbb{R}^n)$, $\pi^* \in D^{1,q}(\mathbb{R}^n)$. Letting $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u}^*$, $\tau = \pi - \pi^*$, we show

$$D^2 \boldsymbol{w}(x) = \nabla \tau(x) = 0, \quad \text{for all } x \in \mathbb{R}^n. \tag{V.4.5}$$

Actually, in $\mathbb{R}^n$,

$$\begin{aligned} \Delta \boldsymbol{w} &= \nabla \tau \\ \nabla \cdot \boldsymbol{w} &= 0 \end{aligned} \tag{V.4.6}$$

and, therefore, $\Delta(\nabla \tau) = 0$ in $\mathbb{R}^n$, which implies $\Delta(\Delta \boldsymbol{w}) = \boldsymbol{0}$ in $\mathbb{R}^n$.[1] Denoting by ψ_i the ith component of $\Delta \boldsymbol{w}$, we apply the mean value theorem for harmonic functions (e.g., Gilbarg & Trudinger 1983, Theorem 2.1) to deduce for all $x \in \mathbb{R}^n$ and with $s = |x - y|$

$$\psi_i(x) = \frac{1}{\omega_n s^n} \int_{B_s(x)} \{\Delta u_i - \Delta u_i^*\}\, dy\,.$$

So, by the Hölder inequality, (V.4.2), and the fact that $D^2 \boldsymbol{u}^* \in L^q(\mathbb{R}^3)$, we get

$$|\psi_i(x)| \le c[s^{-n(1-1/r')} + s^{-n(1-1/q')}]$$

for $i = 1, \ldots, n$. Letting $s \to \infty$ in this relation gives $\Delta \boldsymbol{w} \equiv \boldsymbol{0}$ in $\mathbb{R}^n$, which, in turn, by $(V.4.6)_1$, delivers $(V.4.5)_2$. As a consequence, from $(V.4.6)_1$ it follows that

[1] Notice that, by Theorem V.1.1, $\boldsymbol{w}, \tau \in C^\infty(\mathbb{R}^n)$.

$$\Delta(D^2\boldsymbol{w}) = 0 \ \text{ in } \mathbb{R}^n.$$

Thus, arguing as before, one shows $(\text{V.4.5})_1$, completing the proof of the lemma. □

Remark V.4.6 If Ω is of class C^2, the conclusion of the preceding lemma can be reached under the assumptions that $\boldsymbol{v}$ satisfies (V.4.1) and, moreover,

$$\boldsymbol{f} \in L^q(\Omega), \quad \boldsymbol{v} \in D^{2,r}(\Omega), \quad \boldsymbol{v}_* \in W^{2-1/q,q}(\partial\Omega). \tag{V.4.7}$$

To this end, it is enough to show that (V.4.7) implies

$$\boldsymbol{v} \in W^{2,q}_{loc}(\overline{\Omega}), \quad p \in W^{1,q}_{loc}(\overline{\Omega}). \tag{V.4.8}$$

If $r \geq q$ the assertion is obvious. Therefore, take $q > r$. From the embedding Theorem II.3.4 and hypothesis (V.4.7) on $\boldsymbol{v}$ we readily conclude $\boldsymbol{v} \in W^{1,r_1}_{loc}(\overline{\Omega})$ with $1 < r_1 \leq nr/(n-r)$ $(> r)$ if $r < n$ and for arbitrary $r_1 > 1$ if $r \geq n$. In the latter case it follows that $\boldsymbol{v} \in W^{1,q}_{loc}(\overline{\Omega})$ and by Theorem IV.4.1

$$\boldsymbol{v} \in W^{2,q}_{loc}(\Omega), \quad p \in W^{1,q}_{loc}(\Omega). \tag{V.4.9}$$

If $q \leq r_1 < n$ we again draw the same conclusion. So, assume $1 < r_1 < q$. Then $\boldsymbol{f} \in L^{r_1}(\Omega)$ and Theorem IV.4.1 along with Theorem II.3.4 implies $\boldsymbol{v} \in W^{1,r_2}_{loc}(\overline{\Omega})$ with $1 < r_2 \leq nr_1/(n-r_1)$ $(> r_1)$ if $1 < r_1 < n$ and for arbitrary $r_2 > 1$, whenever $r_1 \geq n$. If either $r_2 \geq q$ or $r_1 \geq n$ we recover (V.4.9); otherwise we iterate the above procedure as many times as needed, until (V.4.9) is established. Properties (V.4.9) and the trace Theorem II.4.4 furnish $\boldsymbol{v} \in W^{2-1/q,q}(\partial\Omega_R)$ for all $R > \delta(\Omega^c)$. By Theorem IV.6.1 there exists a solution $\boldsymbol{v}_1$, p_1 to the Stokes problem in Ω_R corresponding to the body force $\boldsymbol{f}$, which equals $\boldsymbol{v}$ at the boundary $\partial\Omega_R$ such that $\boldsymbol{v}_1 \in W^{2,q}(\Omega_R)$, $p_1 \in W^{1,q}(\Omega_R)$. Thus, $\boldsymbol{u} \equiv \boldsymbol{v} - \boldsymbol{v}_1$ is a solution to the homogeneous Stokes problem in Ω_R with $\boldsymbol{u} \in W^{1,r}(\Omega_R)$, since $q > r$. By Lemma IV.6.2 we then have $\boldsymbol{u} \equiv 0$ and (V.4.8) is accomplished. ■

We shall next establish for solutions to (V.4.1) an estimate that is the counterpart for exterior domains of estimate (IV.6.3) already proved for bounded domains. Precisely we have

Lemma V.4.3 *Let $\boldsymbol{v}$, p be a solution to* (V.4.1) *in an exterior domain $\Omega \subseteq \mathbb{R}^n$ of class C^{m+2}, $n \geq 2$, $m \geq 0$, corresponding to $\boldsymbol{f} \in W^{m,q}(\Omega)$, $\boldsymbol{v}_* \in W^{m+2-1/q,q}(\partial\Omega)$, $1 < q < \infty$. Assume*

$$\boldsymbol{v} \in D^{2,q}(\Omega).$$

Then $\boldsymbol{v} \in D^{k+2,q}(\Omega)$, $p \in D^{k+1}(\Omega)$ for all $k = 0, 1, \ldots, m$, and for any $R > \delta(\Omega^c)$ it holds that

$$\begin{aligned} &\|\boldsymbol{v}\|_{1,q,\Omega_R} + \sum_{k=0}^{m} \{|\boldsymbol{v}|_{k+2,q} + |p|_{k+1,q}\} \\ &\leq c\left(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)} + \|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{q,\Omega_R}\right), \end{aligned} \tag{V.4.10}$$

where $c = c(n, m, q, R)$.

Proof. As in the proof of the previous lemma, we transform (V.4.1) into (V.4.3). Since $\boldsymbol{u} \in D^{2,q}(\mathbb{R}^n)$, from Theorem IV.2.1 it follows that $\boldsymbol{u} \in D^{k+2,q}(\mathbb{R}^n)$, $\pi \in D^{k+1,q}(\mathbb{R}^n)$ for every $k = 0, \ldots, m$. Furthermore,

$$\sum_{k=0}^{m} \{|\boldsymbol{u}|_{k+2,q} + |\pi|_{k+1,q}\} \leq c_1 \left(\|\boldsymbol{f}\|_{m,q} + \|\nabla\varphi \cdot \boldsymbol{v}\|_{m+1,q} + \|\boldsymbol{v}\Delta\varphi\|_{m,q} + \|p\nabla\varphi\|_{m,q}\right), \tag{V.4.11}$$

where $c_1 = c_1(n, m, q)$. Inequality (V.4.11) then implies

$$\sum_{k=0}^{m} \{|\boldsymbol{v}|_{k+2,q,\Omega^{R/2}} + |p|_{k+1,q,\Omega^{R/2}}\} \leq c_2 \left(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}\|_{m+1,q,\Omega_{R/2}} + \|p\|_{m,q,\Omega_{R/2}}\right). \tag{V.4.12}$$

Consider now problem (V.4.1) in Ω_R and use estimate (IV.6.3) to deduce

$$\|\boldsymbol{v}\|_{m+2,q,\Omega_R} + \|p\|_{m+1,q} \leq c_3(\|\boldsymbol{f}\|_{m,q,\Omega_R} + \|\boldsymbol{v}\|_{m+2-1/q,q(\partial\Omega_R)} + \|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{q,\Omega_R}), \tag{V.4.13}$$

Setting $\sigma \equiv \partial\Omega_R \cap \Omega$, by the trace Theorem II.4.4 we have

$$\|\boldsymbol{v}\|_{m+2-1/q,q(\sigma)} \leq c_4 \left(|\boldsymbol{v}|_{m+2,q,\Omega^{R/2}} + \|\boldsymbol{v}\|_{m+1,q,\Omega_R}\right). \tag{V.4.14}$$

Combining (V.4.12)–(V.4.14) we derive

$$\|\boldsymbol{v}\|_{1,q,\Omega_R} + \sum_{k=0}^{m} \{|\boldsymbol{v}|_{k+2,q} + |p|_{k+1,q}\}$$
$$\leq c_5 \left(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)} + \|\boldsymbol{v}\|_{m+1,q,\Omega_R} + \|p\|_{m,q,\Omega_R}\right),$$

and therefore applying Ehrling's inequality (see Exercise II.5.16) to the last two terms on the right-hand side of this last inequality, we finally deduce (V.4.10) and the lemma is proved. □

In a complete analogy to the case where Ω is bounded, we wish now to investigate whether the local norms involving $\boldsymbol{v}$ and p on the right-hand side of (V.4.10) can be dropped out. Proceeding as in Section IV.6 (see the proof of Lemma IV.6.1) we may try to use a contradiction argument to show the inequality

$$\|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{q,\Omega_R} \leq c \left(\|f\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)}\right),$$

which in turn would imply

$$\|\boldsymbol{v}\|_{1,q,\Omega_R} + \sum_{k=0}^{m} \{|\boldsymbol{v}|_{k+2,q} + |p|_{k+1,q}\} \leq c\left(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)}\right). \tag{V.4.15}$$

However, this argument needs the uniqueness of solutions to the homogeneous Stokes problem $(\mathrm{V.4.1})_0$ (i.e., (V.4.1) with $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv 0$) in the class of those functions for which the norms appearing on the left-hand side of (V.4.15) are finite. On the other hand, it is hopeless to determine uniqueness in such a class, unless we can control *in some sense* the behavior of $\boldsymbol{v}$ at infinity. Now, if $1 < q < n/2$, this can be done as a consequence of the double application of Theorem II.6.1. However, if $q \geq n/2$ we do not have this control any more, and there could be nonzero solutions to $(\mathrm{V.4.1})_0$ in $D^{2,q}(\Omega)$. We shall call these solutions *exceptional.* A typical example of an exceptional solution is given by $\boldsymbol{h}(x), \pi(x)$, with $\boldsymbol{h} \equiv \boldsymbol{v}_0 - \boldsymbol{v}_S$, $\pi \equiv p_S$, and $\boldsymbol{v}_S, p_S$ Stokes solution past a sphere; see (V.0.4). We emphasize that the existence of exceptional solutions is related to the fact that a function in $D^{2,q}(\Omega)$, even though approximable by functions of bounded support (see Theorem II.7.4), need not "recall" the zero value at infinity of the approximating functions, since the approximating procedure has been performed in a norm which, in general, does not control the behavior at infinity.

Notwithstanding this difficulty, we are able to characterize the space of exceptional solutions and to determine its dimension $d = d(n, q)$. Specifically, it comes out that *d is always finite* and that $d = 0$ if $1 < q < n/2$, $d = n$ if $n/2 \leq q < n$, and $d = n^2 + n - 1$ if $q \geq n$, see Lemma V.4.4. On the strength of this result we then show the existence of solutions $\boldsymbol{v}, p \in D^{2,q}(\Omega) \times D^{1,q}(\Omega)$ that satisfy estimate (V.4.15) *modulo exceptional solutions*; see Lemma V.4.5. However, because $d = 0$ if $1 < q < n/2$, for these values of q the validity of (V.4.15) is established.

We begin to characterize the space of exceptional solutions. To this end, we set

$$\widetilde{D}^{2,q}(\Omega) = D^{2,q}(\Omega), \quad \text{if } q \geq n$$

$$\widetilde{D}^{2,q}(\Omega) = \left\{u \in D^{2,q}(\Omega) : |u|_{1,r} < \infty, r = \frac{nq}{n-q}\right\} \quad \text{if } \frac{n}{2} \leq q < n$$

$$\widetilde{D}^{2,q}(\Omega) = \left\{u \in D^{2,q}(\Omega) : \|u\|_s + |u|_{1,r} < \infty, s = \frac{nq}{n-2q}, r = \frac{nq}{n-q}\right\} \quad \text{if } 1 < q < \frac{n}{2}. \tag{V.4.16}$$

From Theorem II.7.4 we know that, for Ω exterior and locally Lipschitz, every function from $\widetilde{D}^{2,q}(\Omega)$ can be approximated by functions from $C_0^\infty(\overline{\Omega})$ in the seminorm $|\cdot|_{2,q}$.

If a solution $\boldsymbol{v}$ to $(\text{V.4.1})_0$ is in $D^{2,q}(\Omega)$ for some $q \geq 1$, the corresponding pressure field p is evidently in $D^{1,q}(\Omega)$. Denote by Σ_q the subspace of $\widetilde{D}^{2,q}(\Omega) \times D^{1,q}(\Omega)$ formed by solutions $\boldsymbol{v}, p$ to $(\text{V.4.1})_0$. We have

Lemma V.4.4 *Let Ω be an exterior domain of class C^2 and set $d = \dim(\Sigma_q)$. Then,*

$$d = \begin{cases} n + n^2 - 1 & \text{if } q \geq n \\ n & \text{if } q \in [n/2, n) \\ 0 & \text{if } q \in (1, n/2). \end{cases}$$

Proof. Let us first consider the case where $n > 2$. We begin to show the following two assertions:

(i) For any $\boldsymbol{v}_\infty \in \mathbb{R}^n - \{\boldsymbol{0}\}$ there is a unique (nonzero) solution $\boldsymbol{v}, p \in C^\infty(\Omega)$ to $(\text{V.4.1})_0$ such that

$$\lim_{|x| \to \infty} |\boldsymbol{v}(x) - \boldsymbol{v}_\infty| = 0.$$

This solution verifies the condition

$$\boldsymbol{v} \in \widetilde{D}^{2,q}(\Omega), \quad \text{for } q \geq n/2.$$

(ii) For any second order tensor $\boldsymbol{A} \equiv \{A_{ij}\}$, $A_{ij} \not\equiv 0$, with $\text{trace}(\boldsymbol{A}) = 0$, there is a unique (nonzero) solution $\boldsymbol{v}, p \in C^\infty(\Omega)$ to $(\text{V.4.1})_0$ such that

$$\lim_{|x| \to \infty} |\boldsymbol{v}(x) - \boldsymbol{A} \cdot \boldsymbol{x}| = 0.$$

This solution verifies the condition

$$\boldsymbol{v} \in \widetilde{D}^{2,q}(\Omega), \quad \text{for } q \geq n.$$

To prove (i), we observe that, if we denote by $\boldsymbol{v}_1, p_1$ the solution constructed in Theorem V.2.1 corresponding to $\boldsymbol{f} \equiv \boldsymbol{0}$, and $\boldsymbol{v}_* = -\boldsymbol{v}_\infty$, the pair $\boldsymbol{v} \equiv \boldsymbol{v}_1 + \boldsymbol{v}_\infty$, $p \equiv p_1$, satisfies all requirements. In fact, it belongs to C^∞, by Theorem IV.4.3, and by Theorem IV.5.1,

$$\boldsymbol{v} \in W^{2,t}_{loc}(\overline{\Omega}), \quad p \in W^{1,t}_{loc}(\overline{\Omega}), \quad \text{for any } t \geq 1. \tag{V.4.17}$$

Also, $\boldsymbol{v}$ and p satisfy the asymptotic expansion (V.3.17)–(V.3.21) which, in particular, furnishes that $\boldsymbol{v} \to \boldsymbol{v}_\infty$ uniformly. Finally, again by (V.3.17)–(V.3.21) and (V.4.17), we deduce

$$\boldsymbol{v} \in D^{1,r}(\Omega) \cap D^{2,q}(\Omega), \quad \text{for all } r > n/(n-1) \tag{V.4.18}$$

and since

$$\boldsymbol{v} \notin L^s(\Omega), \quad \text{for any } s \in (1, \infty),$$

it follows that

$$\boldsymbol{v} \in \widetilde{D}^{2,q}(\Omega), \quad \text{for } q \geq n/2.$$

The uniqueness of the solution is a consequence of Theorem V.3.4.

To prove (ii), we begin to make a suitable solenoidal extension of the field $\boldsymbol{V}_0 \equiv \boldsymbol{A} \cdot \boldsymbol{x}$. Let $\boldsymbol{w}$ denote a solution to the problem

$$\begin{aligned} \nabla \cdot \boldsymbol{w} = \nabla\varphi \cdot \boldsymbol{V}_0 \equiv g \ \text{ in } \Omega' \\ \boldsymbol{w} = 0 \ \text{ at } \partial\Omega', \end{aligned} \tag{V.4.19}$$

where φ is the "cut-off" function used in the proof of Lemma V.4.2 and Ω' is a locally Lipschitz subdomain of Ω that contains the support of φ. Since

$$\int_{\Omega'} g = 0, \quad g \in C_0^\infty(\Omega'),$$

by Theorem III.3.3 we can take $\boldsymbol{w} \in C_0^\infty(\Omega)$. Setting

$$\boldsymbol{a}(x) = (1-\varphi)\boldsymbol{V}_0 - \boldsymbol{w}, \tag{V.4.20}$$

by $(\text{V.4.19})_1$ $\boldsymbol{a}$ is solenoidal, belongs to $C^\infty(\Omega)$, vanishes near $\partial\Omega$, and equals $\boldsymbol{V}_0$ at large distances. Since

$$D^2\boldsymbol{a} \in C_0^\infty(\Omega), \tag{V.4.21}$$

we may use the same procedure adopted in the proof of Theorem V.2.1 to show the existence of a generalized solution $\boldsymbol{v}$ to $(\text{V.4.1})_0$ such that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}$, with $\boldsymbol{u} \in D_0^{1,2}(\Omega)$. Employing Theorem IV.4.3 and Theorem IV.5.1 we deduce, as before, that $\boldsymbol{v}$ and the corresponding pressure p are of class $C^\infty(\Omega)$ and satisfy (V.4.17). Using the asymptotic expansion (V.3.17), (V.3.19), and (V.3.21) for $\boldsymbol{u}$ and recalling (V.4.21), we deduce

$$D^2\boldsymbol{v} \in L^q(\Omega), \quad \text{for all } q > 1.$$

Since $\boldsymbol{v}$ does not belong to any space $D^{1,r}(\Omega)$ nor to any $L^s(\Omega)$, from (V.4.16) we conclude

$$\boldsymbol{v} \in \widetilde{D}^{2,q}(\Omega), \quad \text{for } q \geq n,$$

which completes the proof of (ii).

Now, let $\boldsymbol{h}_i, \pi_i$, $i = 1, \ldots, n$ be the solutions to $(\text{V.4.1})_0$ of type (i) corresponding to the n orthonormal vectors $\boldsymbol{v}_{\infty i} = \boldsymbol{e}_i$. Likewise, let $\boldsymbol{u}_{ij}, \tau_{ij}$ be the n^2-1 solutions to $(\text{V.4.1})_0$ of the type (ii) corresponding to the n^2-1 matrices of zero trace $\boldsymbol{E}_{ij}$, where

$$\boldsymbol{E}_{ij} = \begin{cases} \boldsymbol{e}_i \otimes \boldsymbol{e}_j & \text{if } \ i \neq j \\ \boldsymbol{e}_i \otimes \boldsymbol{e}_j - \boldsymbol{e}_n \otimes \boldsymbol{e}_n & \text{if } \ i = j \neq n. \end{cases}$$

It is readily seen that the system constituted by the n^2+n-1 vectors $\{\boldsymbol{h}_i, \boldsymbol{u}_{ij}\}$ is linearly independent. Actually, assume *per absurdum* that there are nonidentically zero constants α_i, $\alpha_{ij} \in \mathbb{R}$ such that

$$\alpha_i \boldsymbol{h}_i(x) + {\textstyle\sum'_{ij}} \alpha_{ij} \boldsymbol{u}_{ij}(x) = 0 \;\text{ for all } x \in \Omega,$$

where the prime means that the term $i = j = n$ is excluded from the summation. From $(\text{V.3.17})_1$, $(\text{V.3.19})_1$, and $(\text{V.3.21})_1$ we would then obtain for all sufficiently large $|x|$

$$\alpha_i \boldsymbol{e}_i + {\textstyle\sum'_{ij}} \alpha_{ij} \boldsymbol{E}_{ij} \cdot \boldsymbol{x} = O(1/|x|^{n-2}),$$

which implies

$$\alpha_i \boldsymbol{e}_i = {\textstyle\sum'_{ij}} \alpha_{ij} \boldsymbol{E}_{ij} = 0;$$

that is,

$$\alpha_i = \alpha_{ij} = 0, \quad \text{for all } i, j,$$

leading to a contradiction. Now, if $\boldsymbol{v}, p$ is a solution to $(\text{V.4.1})_0$ with $\boldsymbol{v} \in \widetilde{D}^{2,q}(\Omega)$, for some $q > 1$, from Theorem V.3.3 we deduce the existence of $\boldsymbol{v}_\infty \in \mathbb{R}^n$ and of a traceless matrix $\boldsymbol{B}$ such that as $|x| \to \infty$

$$\boldsymbol{v}(x) = \boldsymbol{v}_\infty + \boldsymbol{B} \cdot \boldsymbol{x} + O(1/|x|^{n-2}). \tag{V.4.22}$$

Clearly, by (V.4.16), we must have

(a) $\boldsymbol{v}_\infty = \boldsymbol{B} = 0$, if $1 < q < n/2$;

(b) $\boldsymbol{B} = 0$, if $n/2 \le q < n$.

In case (a), by Theorem V.3.5, we have $\boldsymbol{v} \equiv 0$ and so $d = 0$. In case (b) we may write $\boldsymbol{v}_\infty = v_i \boldsymbol{e}_i$, for some $v_i \in \mathbb{R}, i = 1, \ldots, n$. Therefore

$$\boldsymbol{w} \equiv \boldsymbol{v} - v_i \boldsymbol{h}_i, \quad z \equiv p - v_i \pi_i$$

is a solution to $(\text{V.4.1})_0$ with $\boldsymbol{w} = o(1)$ as $|x| \to \infty$ and so, again by Theorem V.3.5, we deduce $\boldsymbol{w} \equiv 0$, which shows $d = n$ if $n/2 \le q < n$. Finally, if $q \ge n$, we may write $\boldsymbol{B} = B_{ij} \boldsymbol{E}_{ij}$ and thus, setting

$$\boldsymbol{w} \equiv \boldsymbol{v} - v_i \boldsymbol{h}_i - B_{ij} \boldsymbol{E}_{ij}, \quad z \equiv p - v_i \pi_i - B_{ij} \tau_{ij},$$

we again derive that $\boldsymbol{w}$ and z solve $(\text{V.4.1})_0$ with $\boldsymbol{w} = o(1)$ as $|x| \to \infty$, which yields $\boldsymbol{w} \equiv 0$, namely, $d = n^2 + n - 1$. The proof of the theorem is then accomplished if $n > 2$. Let us consider the case where $n = 2$. We begin to show the existence of two independent solutions $\boldsymbol{h}_i, \pi_i$, $i = 1, 2$, to $(\text{V.4.1})_0$ with

$$\boldsymbol{h}_i \in \widetilde{D}^{2,q}(\Omega), \quad 1 \equiv n/2 < q < n \equiv 2.$$

To this end, set

$$\boldsymbol{u}_i = \boldsymbol{U} \cdot \boldsymbol{e}_i, \quad s_i = -\boldsymbol{q} \cdot \boldsymbol{e}_i,$$

where $\boldsymbol{U}$, $\boldsymbol{q}$ is the two-dimensional Stokes fundamental solution. We look for solutions of the form

$$\boldsymbol{h}_i = \boldsymbol{u}_i + \boldsymbol{v}_i, \quad \pi_i = s_i + p_i \quad i = 1, 2,$$

where

$$\left.\begin{aligned} \Delta \boldsymbol{v}_i &= \nabla p_i \\ \nabla \cdot \boldsymbol{v}_i &= 0 \end{aligned}\right\} \text{ in } \Omega \tag{V.4.23}$$
$$\boldsymbol{v}_i = -\boldsymbol{u}_i \quad \text{at } \partial\Omega.$$

By Exercise III.3.5 we may extend $-\boldsymbol{u}_i$ at the boundary to a solenoidal function $\overline{\boldsymbol{v}}_i \in W^{1,2}(\Omega)$ of compact support in Ω. We then use the technique of Theorem V.2.1 to show the existence of a weak solution $\boldsymbol{v}_i$ to (V.4.23) of the form

$$\boldsymbol{v}_i = \boldsymbol{w}_i + \overline{\boldsymbol{v}}_i, \quad \boldsymbol{w}_i \in D_0^{1,2}(\Omega)$$

with

$$p_i \in L^2(\Omega).$$

Actually, such a solution is of class $C^\infty(\Omega)$, by virtue of Theorem IV.4.3. It is easy to prove that

$$\int_{\partial B_r} \boldsymbol{T}(\boldsymbol{v}_i, p_i) \cdot \boldsymbol{n} = 0 \tag{V.4.24}$$

for all $r > \delta(\Omega^c)$. Actually, writing $(\text{V.4.23})_1$ in the form

$$\nabla \cdot \boldsymbol{T}(\boldsymbol{v}_i, p_i) = 0,$$

we have

$$\int_{\partial B_r} \boldsymbol{T}(\boldsymbol{v}_i, p_i) \cdot \boldsymbol{n} = \int_{\partial B_R} \boldsymbol{T}(\boldsymbol{v}_i, p_i) \cdot \boldsymbol{n} = \int_{\partial B_R} (2\boldsymbol{D}(\boldsymbol{w}_i) \cdot \boldsymbol{n} - p_i \boldsymbol{n}) \tag{V.4.25}$$

for all $R > r$. By the summability properties of $\boldsymbol{w}_i, p_i$ we easily establish the existence of a sequence $\{R_k\} \subset \mathbb{R}_+$ with

$$\lim_{k\to\infty} R_k = \infty,$$

along which the right-hand side of (V.4.25) vanishes, which in turn implies (V.4.24). From (V.3.17)–(V.3.21) we then obtain for large enough $|x|$ that $\boldsymbol{v}_i(x), p_i(x)$ admit the following representation:

$$\begin{aligned} \boldsymbol{v}_i(x) &= \boldsymbol{v}_{0i} + O(1/|x|) \\ p_i(x) &= O(1/|x|^2) \end{aligned} \tag{V.4.26}$$

for some constants $\boldsymbol{v}_{0i}$ and so

$$\begin{aligned}\boldsymbol{h}_i(x) &= \boldsymbol{v}_{0i} + \boldsymbol{U}(x)\cdot\boldsymbol{e}_i + O(1/|x|)\\ \pi_i(x) &= -\boldsymbol{q}(x)\cdot\boldsymbol{e}_i + O(1/|x|^2).\end{aligned}\tag{V.4.27}$$

The solutions $\boldsymbol{h}_i, \pi_i$, $i=1,2$, are linearly independent and, further,

$$\boldsymbol{h}_i \in \widetilde{D}^{2,q}(\Omega), \quad q > n/2 \ (\equiv 1).\tag{V.4.28}$$

In fact, if

$$\alpha_1\boldsymbol{h}_1(x) + \alpha_2\boldsymbol{h}_2(x) = 0, \quad \text{for all } x\in\Omega,$$

from $(\text{V.4.27})_1$ we would obtain for some $\boldsymbol{v}_0 \in \mathbb{R}^2$

$$(\alpha_1\boldsymbol{e}_1 + \alpha_2\boldsymbol{e}_2)\cdot\boldsymbol{U}(x) = \boldsymbol{v}_0 + O(1/|x|),$$

which can be attained if and only if $\boldsymbol{v}_0 = \alpha_1 = \alpha_2 = 0$. Moreover, by (V.4.27) and the regularity properties of $\boldsymbol{h}_i$ near the boundary (see (V.4.17)),

$$\boldsymbol{h}_i \in D^{1,r}(\Omega)\cap D^{2,q}(\Omega), \quad r > n/(n-1) \equiv 2, \quad q>1$$

and since $\boldsymbol{h}_i \notin L^s(\Omega)$ for any $s\in(1,\infty)$ we obtain (V.4.28). As in the case where $n>2$, we shall next construct n^2-1 $(\equiv 3)$ independent solutions $\boldsymbol{u}_{ij}$, τ_{ij} with

$$\boldsymbol{u}_{ij} \in \widetilde{D}^{2,q}(\Omega), \quad q \geq n \ (\equiv 2).$$

Specifically, we look for solutions to $(\text{V.4.1})_0$ of the form

$$\boldsymbol{u}_{ij} = \boldsymbol{a}_{ij} + \boldsymbol{w}_{ij},$$

where $\boldsymbol{a}_{ij}$ are solenoidal extensions of type (V.4.20) of the fields $\boldsymbol{V}_{0ij} = \boldsymbol{E}_{ij}\cdot\boldsymbol{x}$, while $\boldsymbol{w}_{ij}$ solve the problem

$$\left.\begin{aligned}\Delta\boldsymbol{w}_{ij} &= \nabla\tau_{ij} - \Delta\boldsymbol{a}_{ij}\\ \nabla\cdot\boldsymbol{w}_{ij} &= 0\end{aligned}\right\}\ \text{in } \Omega$$

$$\boldsymbol{w}_{ij} = 0 \ \text{ at } \partial\Omega.$$

Since $\boldsymbol{a}_{ij}$ satisfies (V.4.21), we apply the technique of Theorem V.2.1 to deduce the existence of

$$\boldsymbol{w}_{ij} \in D_0^{1,2}(\Omega), \quad \tau_{ij} \in L^2(\Omega).$$

As before, by Theorem V.3.2, for all sufficiently large $|x|$ we have

$$\begin{aligned}\boldsymbol{w}_{ij}(x) &= \boldsymbol{w}_{\infty ij} + O(1/|x|)\\ \tau_{ij}(x) &= O(1/|x|^2)\end{aligned}$$

for some constants $\boldsymbol{w}_{\infty ij}$, and so

$$\begin{aligned}\boldsymbol{u}_{ij}(x) &= \boldsymbol{w}_{\infty ij} + \boldsymbol{E}_{ij}\cdot\boldsymbol{x} + O(1/|x|^2)\\ \tau_{ij}(x) &= O(1/|x|^2).\end{aligned}\tag{V.4.29}$$

As in the case where $n>2$ one shows that $\boldsymbol{u}_{ij} \in \widetilde{D}^{2,q}(\Omega)$, for all $q \geq n \equiv 2$ and that the five vectors $\{\boldsymbol{h}_i, \boldsymbol{u}_{ij}\}$ form a linear independent system. Now, if $\boldsymbol{v}, p$ is a solution to $(\text{V.4.1})_0$ with $\boldsymbol{v} \in \widetilde{D}^{2,q}(\Omega)$, for some $q>1$, from Theorem V.3.3 we obtain that for large $|x|$, $\boldsymbol{v}(x)$ satisfies (V.4.22), with $\boldsymbol{B}=0$ if $1<q<2$. Reasoning exactly as in the case where $n>2$ one shows $d=n\equiv 2$ if $q<n\equiv 2$ and $d=n+n^2-1\equiv 5$ if $q\geq n\equiv 2$, thus completing the proof of the lemma. □

With the aid of Lemma V.4.3 and Lemma V.4.4 we can now obtain in the case of exterior domains a result analogous to that proved, for bounded domains, in the first part of Theorem IV.6.1. To this end, for fixed $R>\delta(\Omega^c)$ and $\ell \geq 0$, $\nu \geq 1$ we set

$$\|u\|_{\nu,R;\ell,q} \equiv \|u\|_{\nu-1,q,\Omega_R} + \sum_{i=1}^{\ell+\nu} |u|_{i,q,\Omega}.$$

The following lemma holds.

Lemma V.4.5 *Let Ω, $\boldsymbol{f}$, $\boldsymbol{v}_*$ satisfy the same assumptions of* Lemma V.4.3 *and let $\boldsymbol{v} \in \widetilde{D}^{2,q}(\Omega)$ be a solution to* (V.4.1) *corresponding to $\boldsymbol{f}$ and $\boldsymbol{v}_*$. Then $\boldsymbol{v} \in D^{k+2,q}(\Omega)$, $p \in D^{k+1,q}(\Omega)$ for all $k=0,1,\dots,m$ and if $q \geq n$ we have*

$$\inf_{(\boldsymbol{h},\pi)\in\Sigma_q} \{\|\boldsymbol{v}-\boldsymbol{h}\|_{2,R;m,q} + \|p-\pi\|_{1,R;m,q}\} \leq c\left(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)}\right); \tag{V.4.30}$$

if $n/2 \leq q < n$:

$$\inf_{(\boldsymbol{h},\pi)\in\Sigma_q} \{|\boldsymbol{v}-\boldsymbol{h}|_{1,r} + \|p-\pi\|_r + \|\boldsymbol{v}-\boldsymbol{h}\|_{2,R;m,q} + \|p-\pi\|_{2,R;m,q}\} \leq c\left(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)}\right), \tag{V.4.31}$$

where $r=nq/(n-q)$; and if $1<q<n/2$:

$$\|\boldsymbol{v}\|_s + |\boldsymbol{v}|_{1,r} + \|p\|_r + \|\boldsymbol{v}\|_{2,R;m,q} + \|p\|_{1,R;m,q}\} \leq c\left(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)}\right), \tag{V.4.32}$$

where $s=nq/(n-2q)$.

Proof. In view of Lemma V.4.3, we have to show only the validity of (V.4.30)–(V.4.32). Consider first the case where $n<q$. Taking into account that $(\boldsymbol{h},\pi)$ solves the homogeneous system $(\text{V.4.1})_0$, from (V.4.10) we derive

$$\inf_{(\boldsymbol{h},\pi)\in\Sigma_q} \{\|\boldsymbol{v}-\boldsymbol{h}\|_{2,R;m,q} + \|p-\pi\|_{1,R;m,q}\} \leq c\left(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)} + \inf_{(\boldsymbol{h},\pi)\in\Sigma_q}\{|\boldsymbol{v}-\boldsymbol{h}|_{q,\Omega_R} + \|p-\pi\|_{q,\Omega_R}\}\right). \tag{V.4.33}$$

We claim the existence of a constant c_1 independent of $\boldsymbol{v}, p, \boldsymbol{f}$, and $\boldsymbol{v}_*$ such that

$$\inf_{(\boldsymbol{h},\pi)\in\Sigma_q}\{|\boldsymbol{v}-\boldsymbol{h}|_{q,\Omega_R}+\|p-\pi\|_{q,\Omega_R}\}\le c_1(\|\boldsymbol{f}\|_{m,q}+\|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)}). \tag{V.4.34}$$

Actually, if (V.4.34) were not true, we could select two sequences $\{\boldsymbol{f}_s\}\subset W^{m,q}(\Omega)$, $\{\boldsymbol{v}_{*s}\}\subset W^{m+2-1/q,q}(\partial\Omega)$ with

$$\begin{gathered}\boldsymbol{f}_s\to 0\ \text{ in } W^{m,q}(\Omega)\\ \boldsymbol{v}_{*s}\to 0\ \text{ in } W^{m+2-1/q,q}(\partial\Omega)\end{gathered} \tag{V.4.35}$$

as $s\to\infty$, while the corresponding solutions $\{\boldsymbol{v}_s,p_s\}$ satisfy

$$\inf_{(\boldsymbol{h},\pi)\in\Sigma_q}\{|\boldsymbol{v}_s-\boldsymbol{h}|_{q,\Omega_R}+\|p_s-\pi\|_{q,\Omega_R}\}=1\ \text{ for all } s\in\mathbb{N}. \tag{V.4.36}$$

On the other hand, (V.4.35), (V.4.36), and (V.4.33) imply

$$\inf_{(\boldsymbol{h},\pi)\in\Sigma_q}\{|\boldsymbol{v}_s-\boldsymbol{h}|_{2,R;m,q}+\|p_s-\pi\|_{2,R;m,q}\}\le M \tag{V.4.37}$$

with M a constant independent of s. By the property of the infimum, inequality (V.4.37) furnishes the existence of a sequence of solutions $\{\overline{\boldsymbol{v}}_s\equiv\boldsymbol{v}_s-\boldsymbol{h}_s$, $\overline{p}_s\equiv p_s-\pi_s\}$ for some $(\boldsymbol{h}_s,\pi_s)\in\Sigma_q$ such that

$$\|\overline{\boldsymbol{v}}_s\|_{1,q,\Omega_R}+\|D^2\overline{\boldsymbol{v}}_s\|_{q,\Omega}+\|\overline{p}_s\|_{q,\Omega_R}+\|\nabla\overline{p}_s\|_{q,\Omega}\le 2M. \tag{V.4.38}$$

By the weak compactness of the space $W^{m,q}(\Omega)$, $1<q<\infty$, and by the compactness results of Exercise II.5.8, we deduce the existence of a subsequence $\{\boldsymbol{v}_{s'},p_{s'}\}$ and two pairs $(\boldsymbol{v}^{(1)},p^{(1)})\in W^{1,q}(\Omega_R)\times L^q(\Omega_R)$ and $(\boldsymbol{V},P)\in L^q(\Omega)\times L^q(\Omega)$ such that

$$\begin{gathered}\boldsymbol{v}_{s'}\to\boldsymbol{v}^{(1)},\ \ p_{s'}\to p^{(1)}\ \text{ weakly in } W^{1,q}(\Omega_R),\text{ strongly in } L^q(\Omega_R),\\ D^2\boldsymbol{v}_{s'}\overset{w}{\to}\boldsymbol{V},\ \ \nabla p_{s'}\overset{w}{\to}P\ \text{ in } L^q(\Omega).\end{gathered} \tag{V.4.39}$$

By the definition of weak derivative, it readily follows that $D^2\boldsymbol{v}^{(1)}$ and $\nabla p^{(1)}$ exist in Ω_R and that $\boldsymbol{V}=D^2\boldsymbol{v}^{(1)}$, $\boldsymbol{P}=\nabla p^{(1)}$ in Ω_R. Fix now R_1 >R. In Exercise V.4.7 the following inequality can be proved

$$\|u\|_{q,\Omega_{R_1}}\le c_1\left(\|\nabla u\|_{q,\Omega_{R_1}}+\|u\|_{q,\Omega_R}\right)\ \text{ for all } R_1>R,$$

where $c_1=c_1(\Omega_R,\Omega_{R_1},q)$, and therefore from (V.4.38) we deduce

$$\|\boldsymbol{v}_{s'}\|_{1,q,\Omega_{R_1}}+\|p_{s'}\|_{q,\Omega_{R_1}}\le M_1.$$

Thus, from $\{\boldsymbol{v}_{s'},p_{s'}\}$ we can select a subsequence $\{\boldsymbol{v}_{s''},p_{s''}\}$ such that

$$\boldsymbol{v}_{s''}\to\boldsymbol{v}^{(2)},\ \ p_{s''}\to p^{(2)}\ \text{ weakly in } W^{1,q}(\Omega_{R_1}),\ \text{strongly in } L^q(\Omega_{R_1}),$$

where

$$(\boldsymbol{v}^{(2)}, p^{(2)}) \in W^{1,q}(\Omega_{R_1}) \times L^q(\Omega_{R_1}).$$

Clearly,

$$\begin{gathered}\boldsymbol{v}^{(2)} = \boldsymbol{v}^{(1)} \text{ and } p^{(2)} = p^{(1)} \text{ in } \Omega_R,\\ \boldsymbol{V} = D^2\boldsymbol{v}^{(2)}, \quad \boldsymbol{P} = \nabla p^{(1)} \text{ in } \Omega_R.\end{gathered}$$

Iterating this procedure along a denumerable number of strictly increasing domains of the type $\Omega_{R_m}, m \in \mathbb{N}$, invading Ω, and using the classical diagonalization method, we can eventually define a pair $\overline{\boldsymbol{v}}$, $\overline{p}$ in Ω with $\overline{\boldsymbol{v}}, \overline{p} \in W^{1,q}(\Omega_\rho)$, for all $\rho > \delta(\Omega^c)$ and, moreover, $D^2\overline{\boldsymbol{v}}$, $\nabla\overline{p} \in L^q(\Omega)$. It is simple to check that $\overline{v}, \overline{p}$ solve the *homogeneous* Stokes system and since, by (V.4.16), $D^{2,q}(\Omega) = \widetilde{D}^{2,q}(\Omega)$ for $q \geq n$, by Lemma V.4.4 we must have

$$\overline{\boldsymbol{v}} = \overline{\boldsymbol{h}}, \quad \overline{p} = \overline{\pi}, \quad \text{for some } (\overline{\boldsymbol{h}}, \overline{\pi}) \in \Sigma_q. \tag{V.4.40}$$

As a consequence, by $(\text{V.4.39})_1$ and (V.4.40), it follows that

$$\begin{aligned}&\limsup_{s'\to\infty}\Big(\inf_{(\boldsymbol{h},\pi)\in\Sigma_q}\{\ \|\boldsymbol{v}'_s - \boldsymbol{h}\|_{q,\Omega_R} + \|p_{s'} - \pi\|_{q,\Omega_R}\}\Big)\\ &\qquad\leq \lim_{s'\to\infty}\big(\|\boldsymbol{v}_{s'} - \boldsymbol{h}_{s'} - \overline{\boldsymbol{h}}\|_{q,\Omega_R} + \|p_{s'} - \pi_{s'} - \overline{\pi}\|_{q,\Omega_R}\big) = 0,\end{aligned}$$

which contradicts (V.4.36). Thus (V.4.34) holds and the lemma follows when $q \geq n$. If $n/2 \leq q < n$, we know from Theorem II.7.4 and Theorem II.6.1 that $\boldsymbol{v}$ obeys the inequality

$$|\boldsymbol{v}|_{1,r} \leq c_1\|D^2\boldsymbol{v}\|_q, \tag{V.4.41}$$

where $r = nq/(n-q)$. Likewise, by possibly adding a suitable constant to p, we have

$$\|p\|_r \leq c_2|p|_{1,q} \leq c_3\|D^2\boldsymbol{v}\|_q.$$

Therefore, in such a case, (V.4.33) can be strengthened by including in the curly brackets on its left-hand side the quantity

$$|\boldsymbol{v} - \boldsymbol{h}|_{1,r} + \|p - \pi\|_r.$$

Repeating the procedure adopted for the case where $q \geq n$, we obtain this time that the limit function $\overline{\boldsymbol{v}}$ also belongs to $D^{1,r}(\Omega)$ implying, in view of the characterization given in (V.4.16), $\overline{\boldsymbol{v}} \in \widetilde{D}^{2,q}(\Omega)$. Also, $\overline{\boldsymbol{v}} = \boldsymbol{h}$, $\overline{p} = \pi$ for some $(\boldsymbol{h}, p) \in$ and so, reasoning as before, we then prove (V.4.34) and, consequently, (V.4.31). Finally, if $1 < q < n/2$, in conjunction with (V.4.41), from Theorem II.6.1 we establish the validity of the inequality

$$\|\boldsymbol{v}\|_s \leq c_2|\boldsymbol{v}|_{1,r} \leq c_3\|D^2\boldsymbol{v}\|_q$$

for $s = nq/(n-2q)$. Then the limit function $\overline{\boldsymbol{v}}$ belongs to $L^s(\Omega) \cap D^{1,r}(\Omega) \cap D^{2,q}(\Omega)$ and so, by characterization (V.4.16), $\overline{\boldsymbol{v}} \in \widetilde{D}^{2,q}(\Omega)$. Again reasoning as before, we show (V.4.34) and arrive at (V.4.32). The proof of the lemma is complete. $\square$

Exercise V.4.7 Let Ω be an exterior, locally Lipschitz domain of $\mathbb{R}^n$, $n \geq 2$ and let $u \in L^q(\Omega_R), \nabla u \in L^q(\Omega_{R_1})$, $R_1 > R > \delta(\Omega^c)$. Use a contradiction argument based on compactness to show the inequality

$$\|u\|_{q,\Omega_{R_1}} \leq c(\|\nabla u\|_{q,\Omega_{R_1}} + \|u\|_{q,\Omega_R}),$$

where $c = c(\Omega_R, \Omega_{R_1}, q)$.

Concerning the behavior at large distances of a solution $\boldsymbol{v} \in \widetilde{D}^{2,q}(\Omega)$, we have the following result.

Lemma V.4.6 *Let Ω be an exterior domain in $\mathbb{R}^n$, and let $\boldsymbol{v}$ be a solution to* (V.4.1)$_{1,2}$ *corresponding to $\boldsymbol{f} \in L^t(\Omega)$, with $\boldsymbol{v} \in \widetilde{D}^{2,q}(\Omega)$. Then, if $1 < q < n$ and $t > n$ we have*

$$\lim_{|x|\to\infty} \nabla \boldsymbol{v}(x) = 0 \tag{V.4.42}$$

uniformly, while, if $1 < q < n/2$ and $t > n/2$

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = 0 \tag{V.4.43}$$

uniformly.

Proof. From Lemma V.3.1 we have the following representation for $\boldsymbol{v}$:

$$v_j(x) = \int_{B_d(x)} U_{ij}^{(d)}(x-y) f_i(y) dy - \int_{\beta(x)} H_{ij}^{(d)}(x-y) v_i(y) dy \tag{V.4.44}$$

with $\beta(x) = B_d(x) - B_{d/2}(x)$. By (V.4.16)$_2$ we derive that, if $1 < q < n$, then $D_k v_j \in L^r(\Omega)$, $r = nq/(n-q)$, and so differentiating (V.4.44) and recalling the properties of $U_{ij}^{(d)}$ and $H_{ij}^{(d)}$ we deduce

$$|D_k \boldsymbol{v}_j(x)| \leq c \left(\| |x-y|^{-n+1} \|_{t',B_d(x)} \|\boldsymbol{f}\|_{t,B_d(x)} + \|\nabla \boldsymbol{v}\|_{r,B_d(x)} \right\} .$$

Since $t' < n/(n-1)$, it follows that

$$\| |x-y|^{-n+1} \|_{t',B_d} \leq c_1$$

and so the preceding inequality implies (V.4.42). If $1 < q < n/2$, from (V.4.16) we derive $\boldsymbol{v} \in L^s(\Omega)$, $s = nq/(n-2q)$ and from (V.4.44) we deduce

$$|v_j(x)| \leq c \left(\| |x-y|^{-n+2} \|_{t',B_d(x)} \|\boldsymbol{f}\|_{t,B_d} + \|\boldsymbol{v}\|_{s,B_d(x)} \right) ,$$

which shows (V.4.43). □

We shall next prove some existence results in the class of velocity fields belonging to $\widetilde{D}^{2,q}(\Omega)$.

Theorem V.4.6 *Let Ω be an exterior domain of class C^{m+2}, $m \geq 0$. Given $\boldsymbol{f} \in W^{m,q}(\Omega)$, $\boldsymbol{v}_* \in W^{m+2-1/q,q}(\partial\Omega)$, $1 < q < \infty$, there exists a unique solution to* (V.4.1) *such that*

$$\boldsymbol{v}, p \in \widetilde{D}^{2,q}(\Omega) \times D^{1,q}(\Omega) \,/\, \Sigma_q.$$

Moreover,

$$\boldsymbol{v} \in \bigcap_{k=0}^{m} D^{k+2,q}(\Omega), \quad p \in \bigcap_{k=0}^{m} D^{k+1,q}(\Omega)$$

and estimates (V.4.30)–(V.4.32) *are satisfied.*

Proof. We approximate $\boldsymbol{f}$ and $\boldsymbol{v}_*$ by functions $\{\boldsymbol{f}_s\} \subset C_0^\infty(\overline{\Omega})$, $\{\boldsymbol{v}_{*k}\} \subset W^{m+2-1/r,r}(\partial\Omega)$ any $r \in (1,\infty)$, respectively. From Theorem V.2.1, for all $s \in \mathbb{N}$ there exists a generalized solution $\boldsymbol{v}_s, p_s \in D^{1,2}(\Omega) \times L^2(\Omega)$ corresponding to $\boldsymbol{f}_s, \boldsymbol{v}_{*s}$, and tending to $\boldsymbol{0}$ as $|x| \to \infty$ in the case where $n > 2$. (If $n = 2$ this limit is undetermined.) Using Theorem IV.4.1 and Theorem IV.6.1 one readily establishes (as in the proof shown in Remark V.4.6) that (at least) $\boldsymbol{v}_s \in W^{2,q}(\Omega_R)$, $p_s \in W^{1,q}(\Omega_R)$ for all $R > \delta(\Omega^c)$. From this information and Theorem V.3.2 it follows that if $n > 2$

$$\begin{gathered}
\boldsymbol{v}_s \in L^t(\Omega) \;\; t > n/(n-2), \\
\nabla \boldsymbol{v}_s \in L^r(\Omega) \;\; r > n/(n-1), \\
D^2 \boldsymbol{v}_s \in L^q(\Omega) \;\; q > 1,
\end{gathered}$$

while, if $n = 2$,

$$\nabla \boldsymbol{v}_s \in L^r(\Omega) \;\; r \geq n/(n-1), \quad D^2 \boldsymbol{v}_s \in L^q(\Omega) \;\; q > 1,$$

so that from (V.4.16) we obtain $\boldsymbol{v} \in \widetilde{D}^{2,q}(\Omega)$ for all $q > 1$. (Notice that the case $1 < q < n/2$ is excluded if $n = 2$). The solutions $(\boldsymbol{v}_s, p_s)$ will then satisfy (V.4.30)–(V.4.32), depending on the values of q and n. Assume $q \geq n$. Given $\varepsilon > 0$ from (V.4.30) and from the linearity of problem (V.4.1), for s', s'' sufficiently large , we deduce

$$\inf_{(\boldsymbol{h},\pi) \in \Sigma_q} \{|\boldsymbol{v}_{s'} - \boldsymbol{v}_{s''} - \boldsymbol{h}|_{2,q} + |p_{s'} - p_{s''} - \pi|_{1,q}\} < \varepsilon. \tag{V.4.45}$$

This relation implies that $(\boldsymbol{v}_s, p_s)$ is a Cauchy sequence in the quotient space $\widetilde{D}^{2,q}(\Omega) \times D^{1,q}(\Omega) \,/\, \Sigma_q$ and so, by a classical result of functional analysis,[2] there is an element $(\boldsymbol{v}, p) \in \widetilde{D}^{2,q}(\Omega) \times D^{1,q}(\Omega)$ to which $\boldsymbol{v}_s, p_s$ tend in the quotient norm defined by the left hand side of (V.4.45). Consequently, in view of Lemma V.4.5, the theorem follows if $q \geq n$. Likewise, if $n/2 \leq q < n$, from (V.4.31) we deduce

[2] See, e.g., Schechter (1971, Chapter III, Theorem 5.3).

$$\inf_{(\boldsymbol{h},\pi)\in\Sigma_q} \{|\boldsymbol{v}_{s'} - \boldsymbol{v}_{s''} - \boldsymbol{h}|_{1,r} + |\boldsymbol{v}_{s'} - \boldsymbol{v}_{s''} - \boldsymbol{h}|_{2,q} + |p_{s'} - p_{s''} - \pi|_{1,q}\} < \varepsilon,$$

namely, $(\boldsymbol{v}_s, p_s)$ is a Cauchy sequence in $\widetilde{D}^{2,q}(\Omega) \times D^{1,q}(\Omega) \,/\, \Sigma_q$ and we prove the theorem as before. Finally, if $1 < q < n/2$, from Lemma V.4.4 it is $\Sigma_q = \emptyset$ and $\boldsymbol{v}_{s'} - \boldsymbol{v}_{s''}$, $p_{s'} - p_{s''}$ satisfy (V.4.25)$_3$, thus being a Cauchy sequence in $\widetilde{D}^{2,q}(\Omega) \times D^{1,q}(\Omega)$ and the result again follows. The proof of the theorem is therefore completed. □

Let us now consider some consequences of Theorem V.4.6. Taking into account the property of the infimum, we immediately obtain

Theorem V.4.7 *Assume Ω, $\boldsymbol{f}$, and $\boldsymbol{v}_*$ satisfy the assumptions of* Theorem V.4.6. *Then there exists a solution $\boldsymbol{v}$, p to* (V.4.1) *obeying the estimate*

$$\begin{aligned}\|\boldsymbol{v}\|_{1,q,\Omega_R} + \|p\|_{q,\Omega_R} + \sum_{k=0}^{m} \left(|\boldsymbol{v}|_{k+2,q} + |p|_{k+1,q}\right) \\ \le c\left(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)}\right),\end{aligned}$$

where $c = c(n, q, m, \Omega, R)$.

We also have

Theorem V.4.8 *Suppose Ω, $\boldsymbol{f}$, and $\boldsymbol{v}_*$ satisfy the assumptions of* Theorem V.4.6. *Assume, in addition, $\boldsymbol{f} \in L^t(\Omega)$, $\boldsymbol{v}_* \in W^{2-1/t,t}(\partial\Omega)$, for some $1 < t < n/2$. Then, there exists one and only one solution $\boldsymbol{v}$, p to* (V.4.1) *such that*

$$\begin{aligned}&\boldsymbol{v} \in \widetilde{D}^{2,t}(\Omega) \cap \left[\cap_{k=0}^{m} D^{k+2,q}(\Omega)\right] \\ &p \in D^{1,t}(\Omega) \cap \left[\cap_{k=0}^{m} D^{k+1,q}(\Omega)\right] \cap L^{nt/(n-t)}(\Omega).\end{aligned}$$

Furthermore, $\boldsymbol{v}$ and p obey the following estimate

$$\begin{aligned}\|\boldsymbol{v}\|_s + |\boldsymbol{v}|_{1,r} + |\boldsymbol{v}|_{2,t} + \|p\|_r + |p|_{1,t} + \sum_{k=0}^{m} \left(|\boldsymbol{v}|_{k+2,q} + |p|_{k+1,q}\right) \\ \le c\left(\|\boldsymbol{f}\|_t + \|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{2-1/t,t(\partial\Omega)} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)}\right)\end{aligned} \tag{V.4.46}$$

with $r = nt/(n-t)$, $s = nt/(n-2t)$, and $c = c(n, q, t, m, \Omega)$. Finally, we have, as $|x| \to \infty$,

$$\begin{aligned}&\int_{S_n} |\boldsymbol{v}(x)| = o\left(1/|x|^{n/r-1}\right) \\ &\int_{S_n} |\nabla \boldsymbol{v}(x)| = o\left(1/|x|^{n/t-1}\right) \\ &\int_{S_n} |p(x)| = o\left(1/|x|^{n/t-1}\right)\end{aligned} \tag{V.4.47}$$

and, if $q > n$,

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = \lim_{|x|\to\infty} \nabla \boldsymbol{v}(x) = 0 \tag{V.4.48}$$

uniformly.

Proof. From Theorem V.4.6 we deduce the existence of a solution to (V.4.1) such that

$$\boldsymbol{v} \in \widetilde{D}^{2,t}(\Omega), \quad p \in D^{1,t}(\Omega),$$

and verifying

$$\|\boldsymbol{v}\|_s + |\boldsymbol{v}|_{1,r} + |\boldsymbol{v}|_{2,t} + \|p\|_r + |p|_{1,t} \le c(\|\boldsymbol{f}\|_t + \|\boldsymbol{v}_*\|_{2-1/t,t(\partial\Omega)}). \tag{V.4.49}$$

However, since $\boldsymbol{f} \in W^{m,q}(\Omega)$, $\boldsymbol{v}_* \in W^{m-2+1/q,q}(\partial\Omega)$, from Lemma V.4.2 and Lemma V.4.3 it follows that $\boldsymbol{v} \in D^{k+2,q}(\Omega)$, $p \in D^{k+1,q}(\partial\Omega)$, $k = 0, \dots, m$, and that, moreover,

$$\begin{aligned}\sum_{k=0}^{m} (|\boldsymbol{v}|_{k+2,q} + |p|_{k+1,q}) \le c \big(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)} \\ + \|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{q,\Omega_R}\big)\,.\end{aligned} \tag{V.4.50}$$

Given $\varepsilon > 0$, one can prove the following inequality (see Exercise V.4.8)

$$\|u\|_{\kappa,\Omega_R} \le c\|u\|_{\sigma,\Omega_R} + \varepsilon|u|_{1,\kappa,\Omega_R}, \tag{V.4.51}$$

for all $\kappa, \sigma > 1$, with $c = c(\varepsilon, \kappa, \sigma, \Omega_R)$. Using (V.4.51) we obtain

$$\|p\|_{q,\Omega_R} \le c\|p\|_r + \varepsilon|p|_{1,q,\Omega_R}, \tag{V.4.52}$$

while using it twice furnishes

$$\|\boldsymbol{v}\|_{q,\Omega_R} \le c_1\|\boldsymbol{v}\|_s + c_2|\boldsymbol{v}|_{1,r} + \varepsilon\|D^2\boldsymbol{v}\|_{q,\Omega_R}. \tag{V.4.53}$$

Using (V.4.52) and (V.4.53) on the right-hand side of (V.4.50) and employing (V.4.49) allows us to recover the estimate (V.4.46). Furthermore, relations (V.4.47) follow from Lemma II.6.3, while (V.4.48) is a consequence of Lemma V.4.6. Finally, uniqueness is easily implied by Theorem V.3.4. The theorem is, therefore, completely proved. □

Exercise V.4.8 Use a contradiction argument based on the compactness results of Exercise II.5.8 to show the validity of inequality (V.4.51).

Exercise V.4.9 The results proved in this section continue to hold if, more generally, $\nabla \cdot \boldsymbol{v} = g \not\equiv 0$. In particular, show the validity of Theorem V.4.6 in such a case, if $g \in D^{m+1,q}(\Omega)$ and provided the term $|g|_{m+1,q} + \|g\|_{q,\Omega_R}$ is added on the right-hand side of the estimates (V.4.30)–(V.4.32). Notice that, unlike the case where Ω is bounded, no compatibility condition is required between g and $\boldsymbol{v}_*$.

V.5 Existence, Uniqueness, and L^q-Estimates: q-generalized Solutions

In the present section we shall investigate the existence and uniqueness of q-generalized solutions to system (V.4.1) and the validity of corresponding estimates. As in Section V.3, we shall see that these results heavily depend on how q and n are related. However, this time, if $1 < q \le n/(n-1)$ ($1 < q < n/(n-1)$ for $n = 2$) the above mentioned solutions *exist if and only if the data satisfy a suitable compatibility condition*; see (V.5.4). As a by-product, our theory will furnish a general representation formula for functionals on $\mathcal{D}_0^{1,q'}(\Omega)$.

In order to simplify matters, we assume that the velocity field $\boldsymbol{v}_*$ at the boundary is identically zero. Generalizations to the more general non-homogeneous case are left to the reader in Exercise V.5.1. We therefore consider the following system

$$\left.\begin{aligned} \Delta \boldsymbol{v} &= \nabla p + \boldsymbol{f} \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}\right\} \text{ in } \Omega \qquad \boldsymbol{v} = 0 \text{ at } \partial\Omega, \tag{V.5.1}$$

We have

Definition V.5.1. A vector field $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ is called a *q-generalized solution to the Stokes system* (V.5.1) if and only if $\boldsymbol{v} \in \mathcal{D}_0^{1,q}(\Omega)$ and, furthermore,

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) = -[\boldsymbol{f}, \boldsymbol{\varphi}], \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}_0^{1,q'}(\Omega). \tag{V.5.2}$$

Remark V.5.1 Unlike the definition of q-generalized solutions given in Section V.1 for the Stokes *problem* (V.0.1), (V.0.2), in the case under consideration q-generalized solutions need not tend to zero at infinity; actually, as we shall see, this happens if and only if $n/(n-1) < q < n$, see also Remark V.1.1. ■

From Lemma V.1.1 it follows that, provided Ω is locally Lipschitz and $\boldsymbol{f} \in D_0^{-1,q}(\Omega)$, to any q-generalized $\boldsymbol{v}$ to (V.5.1) we can uniquely associate a pressure field $p \in L^q(\Omega)$ such that

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) - (p, \nabla \cdot \boldsymbol{\psi}) = -[\boldsymbol{f}, \boldsymbol{\psi}], \quad \text{for all } \boldsymbol{\psi} \in \mathcal{D}(\Omega). \tag{V.5.3}$$

As in the case of strong solutions, a fundamental role in our treatment is played by *exceptional* q-generalized solutions, i.e., vector fields $\boldsymbol{v} \in \mathcal{D}_0^{1,q}(\Omega)$ solving (V.5.1) with $\boldsymbol{f} \equiv 0$ (denoted from now on by $(\text{V.5.1})_0$). Their geometric structure is characterized in the following lemma.

Lemma V.5.1 *Let $\Omega \subset \mathbb{R}^n$ be an exterior domain of class C^2. Denote by $\mathcal{S}_q$ the subspace of $\mathcal{D}_0^{1,q}(\Omega) \times L^q(\Omega)$ constituted by q-generalized solutions $(\boldsymbol{v}, p)$ to* $(\mathrm{V}.5.1)_0$*. Then, if $1 < q < n$ ($1 < q \le n$ for $n = 2$) $\mathcal{S}_q = \{0\}$, while if $q \ge n$ ($q > n$ for $n = 2$) $\dim(\mathcal{S}_q) = n$.*

Proof. Assume $1 < q < n$. From Lemma II.6.2 in the limit $|x| \to \infty$, it follows that

$$\int_{S_n} |\boldsymbol{v}(x)| = o(1).$$

Therefore, $\boldsymbol{v}$ is a q-generalized solution to the Stokes problem (V.0.1), (V.0.2), according to Definition V.1.1 corresponding to identically vanishing data and so, in view of Theorem V.3.4, we have $\boldsymbol{v} \equiv 0$ if $1 < q < n$. Also, if $q = n = 2$, from (V.5.2)

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) = 0 \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$$

and so we may take $\boldsymbol{\varphi} = \boldsymbol{v}$ to obtain again $\boldsymbol{v} \equiv 0$, which completes the proof of the first part of the lemma. Assuming next $q \ge n$ ($q > n$ if $n = 2$), consider the pairs $(\boldsymbol{h}_i, \pi_i)$ of solutions to $(\mathrm{V}.5.1)_0$ constructed in the proof of Lemma V.4.4. By what we have seen there, these solutions are linearly independent and, moreover,

$$\boldsymbol{h}_i \in D^{1,q}(\Omega) \quad \text{for all } q > n/(n-1).$$

Therefore, from Theorem II.7.6 and Theorem III.5.1,

$$\boldsymbol{h}_i \in \mathcal{D}_0^{1,q}(\Omega) \quad \text{for all } q \ge n \quad (q > n \text{ if } n = 2)$$

and the proof of the lemma is achieved. □

Remark V.5.2 A basis $\{\boldsymbol{h}_i, \pi_i\}$ in $\mathcal{S}_q$ can be sometime explicitly exhibited. For example, if Ω is exterior to a sphere, it is immediately seen that $\boldsymbol{h}_i, \pi_i$ can be taken just as follows:

$$\boldsymbol{h}_i = \boldsymbol{e}_i - \boldsymbol{v}_S^{(i)}, \quad \pi_i = p_S^{(i)},$$

where $\boldsymbol{v}_S^{(i)}, p_S^{(i)}$ is the Stokes solutions (V.0.4), corresponding to $\boldsymbol{v}_0 = \boldsymbol{e}_i$, $i = 1, 2, 3$, respectively. Likewise if Ω is exterior to a circle, a basis is constituted by the two independent solutions (V.0.7). ■

Lemma V.5.1 has an important consequence, that is, a *q-generalized solution to* (V.5.1) *with* $1 < q \le n/(n-1)$ ($1 < q < n$ if $n = 2$) *exist only if the body force $-\boldsymbol{f}$ satisfies the compatibility condition*

$$[\boldsymbol{f}, \boldsymbol{h}] = 0, \quad \text{for all } (\boldsymbol{h}, \pi) \in \mathcal{S}_{q'}. \tag{V.5.4}$$

In fact, condition (V.5.4) is also sufficient to prove existence of q-generalized solutions for the values of q specified above. In order to show this, we premise the following general result that will be useful also for other purposes.

Lemma V.5.2 *Let $\boldsymbol{u}_i$, $i = 1, \ldots, N$, be N independent functions in $D_0^{1,q'}(\Omega)$, $1 < q' < \infty$. Then, the following properties hold.*

(i) *There exist N elements, $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_N$, of $D_0^{-1,q}(\Omega)$, $q = q'/(q'-1)$, satisfying the conditions*

$$[\boldsymbol{\ell}_i, \boldsymbol{u}_j] = \delta_{ij}\,, \quad i, j = 1, \ldots, N\,,$$

and such that every $\boldsymbol{f} \in D_0^{-1,q}(\Omega)$ can be represented as follows

$$\boldsymbol{f} = \boldsymbol{w} + \sum_{i=1}^{N} [\boldsymbol{f}, \boldsymbol{u}_i]\, \boldsymbol{\ell}_i\,,$$

where

$$[\boldsymbol{w}, \boldsymbol{u}_i] = 0\,, \quad i = 1, \ldots, N\,.$$

(ii) *For any given $\boldsymbol{f} \in D_0^{-1,q}(\Omega)$ such that*

$$[\boldsymbol{f}, \boldsymbol{u}_i] = 0\,, \quad i = 1, \ldots, N \tag{V.5.5}$$

there exists a sequence, $\{\boldsymbol{f}_m\} \subset C_0^\infty(\Omega)$, whose elements satisfy

$$(\boldsymbol{f}_m, \boldsymbol{u}_i) = 0\,, \quad \textit{for all } m \in \mathbb{N} \textit{ and all } i = 1, \ldots, N\,, \tag{V.5.6}$$

and, in addition,

$$\lim_{m\to\infty} |\boldsymbol{f} - \boldsymbol{f}_m|_{-1,q} = 0\,.$$

Proof. We begin with a suitable decomposition of the space $L^{q'}(\Omega)$. Let $\boldsymbol{g}_i$, $i = 1, \ldots, N$, be independent elements of $L^q(\Omega)$. We want to show that there exist $\boldsymbol{L}_i \in C_0^\infty(\Omega)$, $i = 1, \ldots, N$, such that

$$(\boldsymbol{L}_i, \boldsymbol{g}_j) = \delta_{ij}\,, \quad i, j = 1, \ldots, N\,. \tag{V.5.7}$$

The proof of (V.5.7) will be given by induction.[1] Suppose $N = 1$. Then, there exists $\boldsymbol{\psi} \in C_0^\infty(\Omega)$ such that $(\boldsymbol{\psi}, \boldsymbol{g}) \neq 0$, because, otherwise, $\boldsymbol{g} \equiv \boldsymbol{0}$, which contradicts the assumption. We then choose $\boldsymbol{L} = \boldsymbol{\psi}/(\boldsymbol{\psi}, \boldsymbol{g})$, thus proving (V.5.7) for $N = 1$. Next, assume that there exist $\widetilde{\boldsymbol{L}}_i \in C_0^\infty(\Omega)$, $i = 1, \ldots, N-1$, $N \geq 2$, such that

$$(\widetilde{\boldsymbol{L}}_i, \boldsymbol{g}_j) = \delta_{ij}\,, \quad i, j = 1, \ldots, N-1\,, \tag{V.5.8}$$

then we show that there are $\boldsymbol{L}_i \in C_0^\infty(\Omega)$, $i = 1, \ldots, N$ satisfying (V.5.7). In fact, set

$$\boldsymbol{\gamma} = -\sum_{j=1}^{N-1} (\widetilde{\boldsymbol{L}}_j, \boldsymbol{g}_N)\boldsymbol{g}_j + \boldsymbol{g}_N\,. \tag{V.5.9}$$

[1] I owe the proof of this property to Professor Christian Simader and Dr. Thorsten Riedl.

Then, by the same token, there is $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$ such that

$$(\boldsymbol{\varphi}, \boldsymbol{\gamma}) \neq 0\,, \tag{V.5.10}$$

and so, setting

$$\boldsymbol{l} = \boldsymbol{\varphi} - \sum_{j=1}^{N-1} (\boldsymbol{\varphi}, \boldsymbol{g}_j)\widetilde{\boldsymbol{L}}_j\,, \tag{V.5.11}$$

and using the induction hypothesis (V.5.8), we find

$$(\boldsymbol{l}, \boldsymbol{g}_k) = (\boldsymbol{\varphi}, \boldsymbol{g}_k) - (\boldsymbol{\varphi}, \boldsymbol{g}_k) = 0\,, \quad k = 1, \ldots, N-1\,, \tag{V.5.12}$$

whereas, using (V.5.9) and (V.5.10), we obtain

$$\begin{aligned}(\boldsymbol{l}, \boldsymbol{g}_N) &= (\boldsymbol{\varphi}, \boldsymbol{g}_N) - \sum_{j=1}^{N-1} (\boldsymbol{\varphi}, \boldsymbol{g}_j)(\boldsymbol{L}_j, \boldsymbol{g}_N) \\ &= (\boldsymbol{\varphi}, \boldsymbol{\gamma}) + \sum_{j=1}^{N-1} (\boldsymbol{L}_j, \boldsymbol{g}_N)(\boldsymbol{\varphi}, \boldsymbol{g}_j) - \sum_{j=1}^{N-1} (\boldsymbol{\varphi}, \boldsymbol{g}_j)(\boldsymbol{L}_j, \boldsymbol{g}_N) \\ &= (\boldsymbol{\varphi}, \boldsymbol{\gamma}) \neq 0\,.\end{aligned} \tag{V.5.13}$$

Therefore, from (V.5.11)–(V.5.13), we deduce that, defining

$$\boldsymbol{L}_N = \frac{\boldsymbol{l}}{(\boldsymbol{l}, \boldsymbol{g}_N)}\,, \quad \boldsymbol{L}_i = \widetilde{\boldsymbol{L}}_i - (\widetilde{\boldsymbol{L}}_i, \boldsymbol{g}_N)\boldsymbol{L}_N\,, \ i = 1, \ldots, N-1\,,$$

the functions $\boldsymbol{L}_1, \ldots, \boldsymbol{L}_N$ obey property (V.5.7) that, consequently, is completely proved. Next, let $\boldsymbol{F} \in L^q(\Omega)$ be arbitrary, and set

$$\boldsymbol{W} = \boldsymbol{F} - \sum_{i=1}^{N} (\boldsymbol{F}, \boldsymbol{g}_i)\boldsymbol{L}_i\,,$$

where $\boldsymbol{L}_i \in C_0^\infty(\Omega)$ satisfy (V.5.7). Then, clearly, in view of (V.5.7), we have, on the one hand,

$$(\boldsymbol{W}, \boldsymbol{g}_i) = 0\,, \quad i = 1, \ldots, N\,, \tag{V.5.14}$$

and, on the other hand,

$$\boldsymbol{F} = \boldsymbol{W} + \sum_{i=1}^{N} (\boldsymbol{F}, \boldsymbol{g}_i)\boldsymbol{L}_i\,, \tag{V.5.15}$$

which, since $\boldsymbol{W}$ is uniquely determined, furnishes the desired decomposition of the space $L^q(\Omega)$. Now, pick $\boldsymbol{f} \in D_0^{-1,q}(\Omega)$. From Theorem II.1.6 and Theorem II.8.2 we know that there exists $\boldsymbol{F} \in L^q(\Omega)$ such that

$$[\boldsymbol{f}, \boldsymbol{\psi}] = (\boldsymbol{F}, \nabla\boldsymbol{\psi})\,. \tag{V.5.16}$$

Thus, if we choose, in particular, $\boldsymbol{g}_k = \nabla \boldsymbol{u}_k$, the statements in part (i) of the lemma follow from this representation and from (V.5.7), (V.5.14), and (V.5.15). We shall now show part (ii). Let $\boldsymbol{f} \in D_0^{-1,q}(\Omega)$, and let $\boldsymbol{F}$ be a corresponding function in $L^q(\Omega)$ satisfying (V.5.16). Since $\boldsymbol{f}$ must satisfy (V.5.5), we find that $\boldsymbol{F}$ obeys the following conditions

$$(\boldsymbol{F}, \nabla \boldsymbol{u}_i) = 0\,, \quad \imath = 1, \ldots, N\,. \tag{V.5.17}$$

Denote by $\{\boldsymbol{F}_m\} \subset C_0^\infty(\Omega)$ a sequence such that $\boldsymbol{F}_m \to \boldsymbol{F}$ in L^q, and set

$$\boldsymbol{W}_m = \boldsymbol{F}_m - \sum_{i=1}^{N} (\boldsymbol{F}_m, \nabla \boldsymbol{u}_i) \boldsymbol{L}_i\,.$$

With the help of (V.5.7) and of (V.5.17), we easily establish the following properties

$$\begin{aligned} &\{\boldsymbol{W}_m\} \subset C_0^\infty(\Omega) \\ &(\boldsymbol{W}_m, \nabla \boldsymbol{u}_k) = 0\,, \text{for all } m \in \mathbb{N}_+ \text{ and all } k \in \{1, \ldots, N\}\,, \\ &\boldsymbol{W}_m \to \boldsymbol{F} \ \text{ in } L^q(\Omega)\,. \end{aligned}$$

The last statement of the lemma then follows by setting $\boldsymbol{f}_m = \nabla \cdot \boldsymbol{W}_m$, $m \in \mathbb{N}$. □

We are now in a position to prove the following.

Theorem V.5.1 *Let Ω be an exterior domain in $\mathbb{R}^n$ of class C^2. Then, for every $\boldsymbol{f} \in D_0^{-1,q}(\Omega)$ satisfying* (V.5.4) *if $1 < q \le n/(n-1)$ $(1 < q < n/(n-1)$ if $n = 2)$ there exists one and only one q-generalized solution to* (V.5.1) *such that*

$$(\boldsymbol{v}, p) \in \mathcal{D}_0^{1,q}(\Omega) \times L^q(\Omega) \,/\, \mathcal{S}_q.$$

Moreover, this solution verifies

$$\inf_{(\boldsymbol{h},\pi)\in\mathcal{S}_q} \{|\boldsymbol{v} - \boldsymbol{h}|_{1,q} + \|p - \pi\|_q\} \le c|\boldsymbol{f}|_{-1,q}. \tag{V.5.18}$$

Proof. As in the proof of Theorem V.4.6, it is enough to show the result for functions $\boldsymbol{f} \in C_0^\infty(\Omega)$, that, when $1 < q \le n/(n-1)$ $(1 < q < n/(n-1)$ if $n = 2)$, satisfy, in addition, the condition

$$(\boldsymbol{f}, \boldsymbol{h}_i) = 0\,, \quad \text{for all } i = 1, \ldots, n\,. \tag{V.5.19}$$

In fact, the general case will be a consequence of inequality (V.5.18), of the density of $C_0^\infty(\Omega)$ into $D_0^{-1,q}(\Omega)$ and of Lemma V.5.2. Thus, for a smooth $\boldsymbol{f}$ we construct a solution $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega)$, $p \in L^2(\Omega)$ by the methods employed in Theorem V.2.1 (see also Remark V.2.1). This solution satisfies

$$|\boldsymbol{v}|_{1,2} + \|p\|_2 \le c\,|\boldsymbol{f}|_{-1,2}$$

which, in particular, proves the theorem in the special case where $q = n/(n-1)$, $n = 2$. From Theorem IV.4.2 and Theorem IV.6.1 it follows that

$$\boldsymbol{v} \in C^\infty(\Omega) \cap W^{2,q}(\Omega_R), \quad p \in C^\infty(\Omega) \cap W^{1,q}(\Omega_R) \tag{V.5.20}$$

for all $R > \delta(\Omega^c)$ and $q > 1$. Moreover, from Theorem V.3.2, we obtain

$$\nabla \boldsymbol{v} \in L^q(\Omega^{R/2}), \quad p \in L^q(\Omega^{R/2})$$

for all $q > n/(n-1)$. This property, together with (V.5.20), with the aid of Theorem II.7.1 and Theorem III.5.1 in turn implies

$$\boldsymbol{v} \in \mathcal{D}_0^{1,q}(\Omega), \quad p \in L^q(\Omega), \quad q > n/(n-1). \tag{V.5.21}$$

Applying Theorem IV.2.2 to system (V.4.3)–(V.4.4) we readily deduce

$$\|\nabla \boldsymbol{v}\|_{q,\Omega^{R/2}} + \|p\|_{q,\Omega^{R/2}} \le c\,(|\varphi \boldsymbol{f} + \boldsymbol{T}(\boldsymbol{v},p)\cdot\nabla\varphi|_{-1,q} + \|\boldsymbol{v}|\nabla\varphi|\|_q)\,. \tag{V.5.22}$$

Since $q' < n$, from Sobolev inequality (II.3.7) and for all $\boldsymbol{\Phi} \in D_0^{1,q'}(\mathbb{R}^n)$

$$|(\varphi \boldsymbol{f}, \boldsymbol{\Phi})| \le |\boldsymbol{f}|_{-1,q} |\varphi \boldsymbol{\Phi}|_{1,q'} \le c|\boldsymbol{f}|_{-1,q}|\boldsymbol{\Phi}|_{1,q'}. \tag{V.5.23}$$

Likewise, taking into account that $\Phi_i D_j \varphi \in W_0^{1,q}(\Omega_R)$, $i,j = 1,\dots,n$, from (V.3.1) we have

$$\begin{aligned} |(\boldsymbol{T}(\boldsymbol{v},p)\cdot\nabla\varphi, \boldsymbol{\Phi})| &\le \sum_{i,j=1}^n (|(D_i v_j, \Phi_j D_i\varphi)| + |(D_j v_i, \Phi_j D_i \varphi)| \\ &\qquad\qquad + |(p\delta_{ij}, \Phi_j D_i \varphi)|) \\ &\le c\left(\|\boldsymbol{v}\|_{q,\Omega_{R/2}} + \|p\|_{-1,q,\Omega_{R/2}}\right)|\boldsymbol{\Phi}|_{1,q'}. \end{aligned} \tag{V.5.24}$$

Therefore, (V.5.22)–(V.5.24) imply

$$\|\nabla \boldsymbol{v}\|_{q,\Omega^{R/2}} + \|p\|_{q,\Omega^{R/2}} \le c\left(|\boldsymbol{f}|_{-1,q} + \|\boldsymbol{v}\|_{q,\Omega_{R/2}} + \|p\|_{-1,q,\Omega_{R/2}}\right). \tag{V.5.25}$$

Moreover, from estimate (IV.6.8) in Ω_R we also obtain

$$\|\boldsymbol{v}\|_{1,q,\Omega_R} + \|p\|_{q,\Omega_R} \le c\left(|\boldsymbol{f}|_{-1,q} + \|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R} + \|\boldsymbol{v}\|_{1-1/q,q(\partial B_R)}\right), \tag{V.5.26}$$

where we used the obvious inequality

$$\|\boldsymbol{f}\|_{-1,q,\Omega_R} \le |\boldsymbol{f}|_{-1,q}.$$

By employing the traceTheorem II.4.4 at the boundary term in (V.5.26) we deduce

$$\|\boldsymbol{v}\|_{1,q,\Omega_R} + \|p\|_{q,\Omega_R} \le c\left(|\boldsymbol{f}|_{-1,q} + \|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R} + \|\nabla \boldsymbol{v}\|_{q,\Omega^{R/2}}\right). \tag{V.5.27}$$

The solution $\boldsymbol{v}, p$ will then satisfy (V.5.21) and, by (V.5.25), (V.5.27), the inequality

$$|\boldsymbol{v}|_{1,q} + \|p\|_q \le c\,(\,|\boldsymbol{f}|_{-1,q} + \|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R}\,) \tag{V.5.28}$$

for all $q > n/(n-1)$. Let us show that if $\boldsymbol{f}$ satisfies (V.5.19) the properties just shown continue to hold when $1 < q \le n/(n-1)$ ($1 < q < n/(n-1)$ if $n = 2$). We already know that $\boldsymbol{v}$ and p satisfy (V.5.20) for all $q > 1$. Also, $\boldsymbol{v}$ and p obey the asymptotic expansion (V.3.17), (V.3.18), and (V.3.19). If $n > 2$, since $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega)$ we find $\boldsymbol{v}_\infty = \boldsymbol{0}$, and so to show $\boldsymbol{v} \in \mathcal{D}_0^{1,q}(\Omega)$, $1 < q \le n/(n-1)$, by Theorem II.7.1 it is necessary and sufficient to prove that the vector $\boldsymbol{\mathcal{T}}$ defined in (V.3.20) is zero. Likewise, for $n = 2$, since it is readily shown that $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega)$ implies $\boldsymbol{\mathcal{T}} = 0$ (see Remark V.3.5), to prove $\boldsymbol{v} \in \mathcal{D}_0^{1,q}(\Omega)$, $1 < q < n/(n-1)$, again by Theorem II.7.1 it is necessary and sufficient to prove $\boldsymbol{v}_\infty = \boldsymbol{0}$. From Green's formula applied in Ω_R we have for all $R > \delta(\Omega^c)$

$$-\int_{\Omega_R} \boldsymbol{f}\cdot\boldsymbol{h}_i = \int_{\partial B_R} \{\boldsymbol{h}_i \cdot \boldsymbol{T}(\boldsymbol{v},p)\cdot\boldsymbol{n} - \boldsymbol{v}\cdot\boldsymbol{T}(\boldsymbol{h}_i,\pi_i)\cdot\boldsymbol{n}\}\,. \tag{V.5.29}$$

By this relation and the asymptotic properties of $(\boldsymbol{h}_i, \pi_i)$ (see Lemma V.4.4), and of $(\boldsymbol{v}, p)$ (see Theorem V.3.2) it follows for $n > 2$

$$\begin{aligned}\int_{\Omega_R} \boldsymbol{f}\cdot\boldsymbol{h}_i = -\boldsymbol{e}_i\cdot&\int_{\partial B_R}\boldsymbol{T}(\boldsymbol{v},p)\cdot\boldsymbol{n}\\ &+\int_{\partial B_R}\{(\boldsymbol{e}_i-\boldsymbol{h}_i)\cdot\boldsymbol{T}(\boldsymbol{v},p)\cdot\boldsymbol{n}+\boldsymbol{v}\cdot\boldsymbol{T}(\boldsymbol{h}_i,\pi_i)\cdot\boldsymbol{n}\}\\ =-\boldsymbol{e}_i\cdot&\int_{\partial B_R}\boldsymbol{T}(\boldsymbol{v},p)\cdot\boldsymbol{n}+O(1/R^{n-2})\end{aligned}$$

and so, by (V.5.19),

$$\boldsymbol{e}_i\cdot\int_{\partial B_R}\boldsymbol{T}(\boldsymbol{v},p)\cdot\boldsymbol{n} = O(1/R^{n-2}),\quad i=1,2,3. \tag{V.5.30}$$

On the other hand, by taking Ω_R so that $\Omega_R \cap \operatorname{supp}(\boldsymbol{f}) = \emptyset$, from (V.5.1) we have

$$\boldsymbol{\mathcal{T}}\cdot\boldsymbol{e}_i = \boldsymbol{e}_i\cdot\int_{\partial B_R}\boldsymbol{T}(\boldsymbol{v},p)\cdot\boldsymbol{n} = O(1/R^{n-2}),$$

which entails $\boldsymbol{\mathcal{T}} = 0$, thus proving $\boldsymbol{v} \in \mathcal{D}_0^{1,q}(\Omega)$, $p \in L^q(\Omega)$, $1 < q \le n/(n-1)$, for $n \ge 3$. Suppose now $n = 2$. As already noticed $\boldsymbol{\mathcal{T}} = 0$ for solutions $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega)$, $p \in L^2(\Omega)$ and so from (V.3.17)–(V.3.19) it comes out that

$$\begin{aligned}\boldsymbol{T}(\boldsymbol{v},p) &= O(1/|x|^2)\\ \boldsymbol{v} &= \boldsymbol{v}_\infty + O(1/|x|).\end{aligned}$$

Also, by (V.4.27) we have

$$\boldsymbol{h}_i = O(\log|x|)$$
$$\boldsymbol{T}(\boldsymbol{h}_i, \pi_i) = O(1/|x|).$$

Consequently, (V.5.29) furnishes for all $\varepsilon \in (0,1)$

$$\boldsymbol{v}_\infty \cdot \int_{\partial B_R} \boldsymbol{T}(\boldsymbol{h}_i, \pi_i) \cdot \boldsymbol{n} = O(1/R^{1-\varepsilon}). \tag{V.5.31}$$

However, a comparison between the general expansion formulas (V.3.17)–(V.3.21) and (V.4.27) reveals

$$\int_{\partial B_R} \boldsymbol{T}(\boldsymbol{h}_i, \pi_i) \cdot \boldsymbol{n} = \boldsymbol{e}_i,$$

which once replaced into (V.5.31) yields $\boldsymbol{v}_\infty = \boldsymbol{0}$. Therefore, we conclude $\boldsymbol{v} \in \mathcal{D}_0^{1,q}(\Omega), p \in L^q(\Omega)$, $1 < q \le n/(n-1)$ if $n = 2$. We shall next establish the validity of (V.5.28). As in the case where $q > n/(n-1)$ we arrive at inequality (V.5.22). Now, taking into account that $\varphi = 0$ near $\partial\Omega$ and $\varphi = 1$ in Ω^R we have

$$\begin{aligned}\int_\Omega \boldsymbol{f}_1 &= \int_\Omega (\varphi \boldsymbol{f} + \boldsymbol{T}(\boldsymbol{v}, p) \cdot \nabla\varphi) \\ &= \int_{\Omega_{2R}} \varphi(\boldsymbol{f} - \nabla \cdot \boldsymbol{T}) + \int_{\partial\Omega} \varphi \boldsymbol{T} \cdot \boldsymbol{n} + \int_{\partial B_R} \varphi \boldsymbol{T} \cdot \boldsymbol{n} = \int_{\partial B_R} \boldsymbol{T} \cdot \boldsymbol{n}\end{aligned}$$

and so, by what we have shown,

$$\int_\Omega \boldsymbol{f}_1 = 0. \tag{V.5.32}$$

By (V.5.32), in view of Theorem II.8.1 (see also Remark II.8.1), the functional

$$[\boldsymbol{\Phi}] \in D_0^{1,q'}(\mathbb{R}^n) \to (\boldsymbol{f}_1, \boldsymbol{\Phi}), \quad \boldsymbol{\Phi} \in [\boldsymbol{\Phi}], \quad q' \ge n$$

is well defined and independent of the particular choice of the function $\boldsymbol{\Phi}$ in the equivalence class $[\boldsymbol{\Phi}]$. Thus, for any such $\boldsymbol{\Phi}$, we define

$$\widehat{\boldsymbol{\Phi}} = \boldsymbol{\Phi} - \frac{1}{|\Omega_R|} \int_{\Omega_R} \boldsymbol{\Phi},$$

so that by Poincaré's inequality (II.5.10)

$$\|\widehat{\boldsymbol{\Phi}}\|_{q',\Omega_R} \le c|\boldsymbol{\Phi}|_{1,q'}. \tag{V.5.33}$$

From (V.5.32) it follows that

$$(\boldsymbol{f}_1, \boldsymbol{\Phi}) = (\boldsymbol{f}_1, \widehat{\boldsymbol{\Phi}}) = (\varphi \boldsymbol{f}, \widehat{\boldsymbol{\Phi}}) + (\boldsymbol{T}(\boldsymbol{v}, p) \cdot \nabla\varphi, \widehat{\boldsymbol{\Phi}}),$$

and so we may proceed as in (V.5.23), (V.5.24) by using this time (V.5.33) instead of the Sobolev inequality to reach estimate (V.5.25). Since (V.5.26) holds for all $q > 1$, we may finally establish, in the same way as in the case where $q > n/(n-1)$, the validity of (V.5.28). Once (V.5.28) is recovered, we obtain from it

$$\begin{aligned}\inf_{(\boldsymbol{h},\pi)\in\mathcal{S}_q} &\{|\boldsymbol{v}-\boldsymbol{h}|_{1,q} + \|p-\pi\|_q\} \\ &\le c_1 \left\{ |\boldsymbol{f}|_{-1,q} + \inf_{(\boldsymbol{h},\pi)\in\mathcal{S}_q} [|\boldsymbol{v}-\boldsymbol{h}|_{q,\Omega_R} + \|p-\pi\|_{-1,q,\Omega_R}] \right\}.\end{aligned} \tag{V.5.34}$$

Using a contradiction argument entirely analogous to that of Theorem V.4.6 and based on compactness results of Exercise II.5.8 and Theorem II.5.3, we can show

$$\inf_{(\boldsymbol{h},\pi)\in\mathcal{S}_q} \{|\boldsymbol{v}-\boldsymbol{h}|_{q,\Omega_R} + \|p-\pi\|_{-1,q,\Omega_R}\} \le c_2|\boldsymbol{f}|_{-1,q}$$

which, once replaced into (V.5.34), yields (V.5.18). Existence is then fully carried out. The uniqueness of solutions just determined is also immediately established and therefore the proof of the theorem is complete. □

A significant consequence of Theorem V.5.1 is a general representation of functionals on the space $\mathcal{D}_0^{1,q}(\Omega)$, $1 < q < \infty$. Specifically, we have the following result; see Galdi & Simader (1990, Section 7).

Theorem V.5.2 *Let Ω be as in* Theorem V.5.1, *and let $\boldsymbol{f} \in \mathcal{D}_0^{-1,q}(\Omega)$. The following properties hold.*

(i) *If $q > n/(n-1)$ ($q \ge n/(n-1)$ if $n = 2$) then $\boldsymbol{f}$ can be represented as*

$$[\boldsymbol{f}, \boldsymbol{\varphi}] = (\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}), \quad \boldsymbol{\varphi} \in \mathcal{D}_0^{1,q'}(\Omega), \tag{V.5.35}$$

with $\boldsymbol{v}$ uniquely determined if $q < n$ ($q \le n$ if $n = 2$), while $\boldsymbol{v}$ is determined up to a function $\boldsymbol{h}$, if $q \ge n$ ($q > n$ if $n = 2$), where $(\boldsymbol{h}, \pi) \in \mathcal{S}_q$.

(ii) *If $1 < q \le n/(n-1)$ ($1 < q < n/(n-1)$, if $n = 2$), there exist a uniquely determined vector function $\boldsymbol{v} \in \mathcal{D}_0^{1,q}(\Omega)$, and n functions $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n$ with $\boldsymbol{r}_i \in D^{1,q}(\Omega)$, $i = 1, \ldots, n$, uniquely determined up to a constant, such that for all $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,q'}(\Omega)$ and for any fixed basis $\{\boldsymbol{h}_i, \pi_i\}$ in $\mathcal{S}_{q'}$ we have*

$$[\boldsymbol{f}, \boldsymbol{\varphi}] = (\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) + \sum_{i=1}^{n} [\boldsymbol{f}, \boldsymbol{h}_i](\nabla \boldsymbol{r}_i, \nabla \boldsymbol{\varphi}).$$

Proof. Since $\mathcal{D}_0^{1,q}(\Omega)$ is a subspace of $D_0^{1,q}(\Omega)$, by the Hahn-Banach Theorem II.1.7(a), there exists $\boldsymbol{F} \in D_0^{-1,q}(\Omega)$ such that $[\boldsymbol{F}, \boldsymbol{\varphi}] = [\boldsymbol{f}, \boldsymbol{\varphi}]$, for all $\boldsymbol{\varphi} \in$

$\mathcal{D}_0^{1,q'}(\Omega)$. The result stated in part (i) is then an obvious consequence of Theorem V.5.1. We now pass to the proof of part (ii). Again by the Hahn-Banach Theorem II.1.7(a), we extend $\boldsymbol{f}$ to some $\boldsymbol{F} \in D_0^{-1,q}(\Omega)$. Then, from Lemma V.5.2, we find

$$\boldsymbol{F} = \boldsymbol{w} + \sum_{i=1}^{n} [\boldsymbol{f}, \boldsymbol{h}_i]\, \boldsymbol{\ell}_i\,, \tag{V.5.36}$$

where $[\boldsymbol{w}, \boldsymbol{h}_i] = 0$, $i = 1, \ldots, n$, and where we employ the fact that, since $\boldsymbol{h}_i \in \mathcal{D}_0^{1,q'}(\Omega)$, $q' \geq n$ ($q' > n$, if $n = 2$), we have $[\boldsymbol{f}, \boldsymbol{h}_i] = [\boldsymbol{F}, \boldsymbol{h}_i]$. From Theorem V.5.1 it then follows that there exists a unique $\boldsymbol{v} \in \mathcal{D}_0^{1,q}(\Omega)$ such that

$$[\boldsymbol{w}, \boldsymbol{\varphi}] = (\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi})\,, \quad \boldsymbol{\varphi} \in \mathcal{D}_0^{1,q'}(\Omega)\,. \tag{V.5.37}$$

Furthermore, from Theorem II.1.6 and Theorem II.8.2 we may find $\boldsymbol{L}_i \equiv \{(L_{kl})_i\} \in L^q(\Omega)$, $i = 1, \ldots, n$, such that

$$[\boldsymbol{\ell}_i, \boldsymbol{\varphi}] = (\boldsymbol{L}_i, \nabla \boldsymbol{\varphi}) \equiv ((L_{kl})_i, D_k \varphi_l)\,, \quad \boldsymbol{\varphi} \in \mathcal{D}_0^{1,q'}(\Omega)\,.$$

We further apply to each $\boldsymbol{L}_i$ the Helmholtz decomposition Theorem III.1.2 to obtain $\boldsymbol{L}_i = \boldsymbol{R}_i + \nabla \boldsymbol{r}_i$, where $\boldsymbol{R}_i \equiv \{(R_{kl})_i\}$, $i = 1, \ldots, n$, satisfy $((R_{kl})_i, D_k \phi) = 0$, for all $\phi \in D_0^{1,q'}(\Omega)$, and all $l = 1, \ldots, n$. The last displayed equation then becomes

$$[\boldsymbol{\ell}_i, \boldsymbol{\varphi}] = (\nabla \boldsymbol{r}_i, \nabla \boldsymbol{\varphi})\,, \quad \boldsymbol{\varphi} \in \mathcal{D}_0^{1,q'}(\Omega)\,. \tag{V.5.38}$$

The proof then follows from (V.5.36)–(V.5.38). □

From the previous results we obtain the following one.

Corollary V.5.1 *Let Ω be as in* Theorem V.5.1 *and let $\boldsymbol{v} \in \mathcal{D}_0^{1,q}(\Omega)$. Then, if $1 < q < n$ ($1 < q \leq n$ if $n = 2$)*

$$|\boldsymbol{v}|_{1,q} \leq c \sup_{\boldsymbol{\varphi} \in \mathcal{D}_0^{1,q'}, \boldsymbol{\varphi} \neq \boldsymbol{0}} \frac{|(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi})|}{|\boldsymbol{\varphi}|_{1,q'}}.$$

Proof. It follows at once from Theorem V.5.1 and Theorem V.5.2. □

Exercise V.5.1 Let Ω be as in Theorem V.5.1. Show that given $\boldsymbol{f} \in D_0^{-1,q}(\Omega)$, $\boldsymbol{v}_* \in W^{1-1/q,q}(\partial\Omega)$, $g \in L^q(\Omega)$, $q > n/(n-1)$ ($q \geq n/(n-1)$ if $n = 2$) there exists $\boldsymbol{v} \in D^{1,q}(\Omega)$ solving (V.5.2), which equals $\boldsymbol{v}_*$ on $\partial\Omega$ in the trace sense and with $\nabla \cdot \boldsymbol{v} = g$ in the generalized sense. Show, further, that existence of the above type continues to hold if $1 < q \leq n/(n-1)$ ($1 < q < n/(n-1)$, if $n = 2$) provided the following compatibility condition is satisfied:

$$[\boldsymbol{f}, \boldsymbol{h}] + (g, \pi) + \int_{\partial\Omega} (\boldsymbol{n} \cdot \nabla \boldsymbol{h} \cdot \boldsymbol{v}_* - \pi \boldsymbol{v}_* \cdot \boldsymbol{n}) = 0$$

for all $(\boldsymbol{h}, \pi) \in \mathcal{S}_q$. Prove also that $\boldsymbol{v}$ and the corresponding pressure p $(\in L^q(\Omega))$ verify the estimate

$$\inf_{(\boldsymbol{h},\pi)\in\mathcal{S}_q} \{|\boldsymbol{v} - \boldsymbol{h}|_{1,q} + \|p - \pi\|_q\} \le c(|\boldsymbol{f}|_{-1,q} + \|g\|_q + \|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)}).$$

Finally, show that, if $1 < q < n$, $\boldsymbol{v}$ tends to zero as $|x| \to \infty$ in the following sense

$$\int_{S^{n-1}} |\boldsymbol{v}(x)| = o(1/|x|^{n/q-1}).$$

The last part of this section is devoted to the proof of a "regularization at infinity" for q-generalized solutions. In this respect, we recall that if $\boldsymbol{v} \in D^{1,q}(\Omega)$ satisfies (V.1.1) for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$, with $\boldsymbol{f} \in D_0^{-1,r}(\omega)$, for all bounded subdomain ω with $\overline{\omega} \subset \Omega$, and where a priori $r \neq q$, by Lemma IV.1.1 we can associate to $\boldsymbol{v}$ a pressure field p satisfying (V.1.2) with $p \in L^\mu_{loc}(\Omega)$, $\mu = \min(r, q)$.

Theorem V.5.3 *Let Ω be an exterior domain of $\mathbb{R}^n$, let $\boldsymbol{v} \in D^{1,q}(\Omega)$, $1 < q < \infty$, be weakly divergence-free satisfying* (V.1.1) *for all $\varphi \in \mathcal{D}(\Omega)$, and let $\rho > \delta(\Omega^c)$. Then, the following properties hold*

(i) *If*

$$\boldsymbol{f} \in D_0^{-1,r}(\Omega^\rho), \quad r > n/(n-1),$$

we have

$$\boldsymbol{v} \in D^{1,r}(\Omega^R), \quad p \in L^r(\Omega^R) \tag{V.5.39}$$

for all $R > \rho$.

(ii) *If*

$$\boldsymbol{f} \in L^s(\Omega^\rho), \quad 1 < s < \infty,$$

we have

$$\boldsymbol{v} \in D^{2,s}(\Omega^R), \quad p \in D^{1,s}(\Omega^R). \tag{V.5.40}$$

for all $R > \rho$.

In both cases, p is the pressure field associated to $\boldsymbol{v}$ by Lemma IV.1.1.

Proof. The fields

$$\boldsymbol{u} = \varphi \boldsymbol{v}, \quad \pi = \varphi p$$

solve the weak formulation of problem (V.4.3)–(V.4.4) in $\mathbb{R}^n$, namely,

$$\begin{aligned} (\nabla \boldsymbol{u}, \nabla \boldsymbol{\psi}) - (\pi, \nabla \cdot \boldsymbol{\psi}) &= -[\boldsymbol{f}_1, \boldsymbol{\psi}], \quad \text{for all } \boldsymbol{\psi} \in C_0^\infty(\mathbb{R}^n), \\ (\boldsymbol{u}, \nabla \chi) &= -(g, \chi), \quad \text{for all } \chi \in C_0^\infty(\mathbb{R}^n), \end{aligned} \tag{V.5.41}$$

where $(i = 1, \ldots, n)$

$$
\begin{aligned}
&f_{1i} = (f_\varphi)_i + T_{ik}(\boldsymbol{v}, p) D_k\varphi + D_k(v_k D_i\varphi + v_i D_k\varphi),\\
&[\boldsymbol{f}_\varphi, \boldsymbol{\psi}] := [\boldsymbol{f}, \varphi\boldsymbol{\psi}]\\
&g = \boldsymbol{v}\cdot\nabla\varphi,
\end{aligned}
$$

and $\boldsymbol{T}$ is defined in (IV.8.6). By Theorem IV.4.5,

$$\boldsymbol{v} \in W^{1,r}_{loc}(\Omega), \quad p \in L^r_{loc}(\Omega)$$

and so

$$g \in L^r(\mathbb{R}^n). \tag{V.5.42}$$

Furthermore, reasoning exactly as in the proof of Theorem V.5.1, one easily shows that for $r > n/(n-1)$

$$\boldsymbol{f}_1 \in D_0^{-1,r}(\mathbb{R}^n). \tag{V.5.43}$$

In view of Theorem IV.2.2, (V.5.42), and (V.5.43) we establish the existence of a solution $\boldsymbol{u}_1, \pi_1$ to (V.5.41) such that

$$\boldsymbol{u}_1 \in D^{1,r}(\mathbb{R}^n), \quad \pi_1 \in L^r(\mathbb{R}^n), \tag{V.5.44}$$

and, by the uniqueness part of the same theorem we deduce

$$\nabla(\boldsymbol{u}_1 - \boldsymbol{u}) \equiv 0, \quad \pi_1 - \pi \equiv \text{const.} \tag{V.5.45}$$

Since $\varphi = 1$ in Ω^R, conditions (V.5.44) and (V.5.45), after a possible modification of p by adding a constant (which causes no loss), prove (V.5.39). Assume now $\boldsymbol{f} \in L^s(\Omega^\rho)$. By Theorem IV.4.2 we deduce

$$\boldsymbol{v} \in W^{2,s}_{loc}(\Omega), \quad p \in W^{1,s}_{loc}(\Omega) \tag{V.5.46}$$

and so $\boldsymbol{u}, \pi$ solve (V.4.3)–(V.4.4) a.e. in $\mathbb{R}^n$. (V.5.46) implies

$$\boldsymbol{f}_1 \in L^s(\mathbb{R}^n), \quad g \in W^{1,s}(\mathbb{R}^n).$$

We may then apply Theorem II.3.1 to (V.4.3)–(V.4.4) to establish the existence of a solution $\boldsymbol{u}_1, \pi_1$ such that

$$\boldsymbol{u}_1 \in D^{2,s}(\mathbb{R}^n), \quad \pi_1 \in D^{1,s}(\mathbb{R}^n). \tag{V.5.47}$$

Setting $\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}$, by Lemma V.3.1 we obtain

$$D_k D_\ell w_j(x) = \int_{\beta(x)} \Big(H^{(d)}_{ij}(x-y) D_k D_\ell u_{1i}(y) dy - D_k H^{(d)}_{ij}(x-y) D_\ell u_i(y) \Big)\, dy \tag{V.5.48}$$

for all $x \in \mathbb{R}^n$ and all $d > 0$. By properties (V.3.5) of $H^{(d)}_{ij}$, relation (V.5.48), with the help of the Hölder inequality, implies for all sufficiently large d,

$$|D_k D_\ell w_j(x)| \le c \log d \left(d^{-n/r} \|D^2 \boldsymbol{u}_1\|_{s,\beta(x)} + d^{-n/q} \|\nabla \boldsymbol{u}\|_{q,\beta(x)} \right).$$

Letting $d \to \infty$ into this inequality and recalling that $\varphi = 1$ in Ω_R proves $(\text{V.5.45})_2$. Consequently, $(\text{V.5.41})_1$ yields

$$(\pi_1 - \pi, \nabla \cdot \psi) = 0, \quad \text{for all } \psi \in C_0^\infty(\mathbb{R}^n),$$

which, by (V.5.46), in turn delivers $(\text{V.5.45})_2$. The theorem is completely proved. □

Remark V.5.3 For future reference, we wish to observe that results analogous to those of Theorem V.5.3 are valid for the following Dirichlet problem for the Poisson equation:

$$\Delta v = f \ \text{ in } \Omega, \quad v = v_* \text{ at } \partial\Omega.$$

In particular, if $v \in D^{1,q}(\Omega)$, for some $q \in (1,\infty)$, and $f \in L^s(\Omega^\rho)$, $1 < s < \infty$, then $v \in D^{2,s}(\Omega^R)$ for all $r > \rho$. The proof of this assertion, completely similar to (and simpler than) that of Theorem V.5.3, is left to the reader as an exercise. ■

V.6 Green's Tensor and Some Related Properties

The results established in the previous two sections allow us to prove the existence of the Green's tensor for the Stokes problem in a (sufficiently smooth) exterior domain. Actually, for fixed $y \in \Omega$, let us consider the functions $A_{ij}(x,y)$, $a_i(x,y)$ such that for all $i,j = 1,\dots,n$ and all $y \in \Omega$

$$\begin{aligned}
&\Delta_x A_{ij}(x,y) + \frac{\partial a_j(x,y)}{\partial x_i} = 0, \quad x \in \Omega, \\
&\frac{\partial A_{ij}(x,y)}{\partial x_i} = 0, \quad x \in \partial\Omega \\
&A_{ij}(x,y) = U_{ij}(x-y), \quad x \in \partial\Omega \\
&\lim_{|x|\to\infty} A_{ij}(x,y) = 0.
\end{aligned} \tag{V.6.1}$$

From Theorem V.4.8, we know that $A_{ij}(x,y)$ and $a_i(x,y)$ exist and, from Theorem V.1.1, that they are smooth in Ω. Thus, in analogy with the case of a bounded domain, we have that the fields

$$\begin{aligned}
G_{ij}(x,y) &= U_{ij}(x-y) - A_{ij}(x,y) \\
g_i(x,y) &= q_i(x-y) - a_i(x,y)
\end{aligned}$$

define the Green's tensor for the Stokes problem in the exterior domain Ω (Finn 1965a, §2.6). It is not difficult to show along the same lines of Odqvist

(1930, p. 358, see Finn 1965, *loc. cit.*) that the tensor field $\boldsymbol{G}$ satisfies the following *symmetry condition*

$$G_{ij}(x,y) = G_{ji}(y,x).$$

In the rest of this book, we shall not make use of the Green's tensor solution, and, therefore, here we shall not provide a detailed study of its properties. Nevertheless, we would like to point out some features of $\boldsymbol{G}$, that do not appear to be widely known; see, e.g., Babenko (1980, Proposition I).

More specifically, from the result obtained in Theorem V.5.1 we will show that, if $\Omega^c \supset B_a$, some $a > 0$, then the tensor $\boldsymbol{G}$ does *not* satisfy certain estimates which, on the other hand, are known to hold for the same quantity in a bounded domain (see (IV.8.4)) and in a half-space (see (IV.3.3)).[1] For instance, the Green's tensor for the Stokes problem in an exterior three-dimensional domain does *not* satisfy the following estimate:

$$|\nabla G_{ij}(x,y)| \le c|x-y|^{-2}, \tag{V.6.2}$$

for all $x, y \in \Omega$, $x \neq y$, and with ∇ operating on either x or y. Actually, let $\boldsymbol{F}$ be a second-order tensor field with $F_{ij} \in C_0^\infty(\Omega)$ and such that

$$(\nabla \cdot \boldsymbol{F}, \boldsymbol{h}) \neq 0 \ \text{ for all } \boldsymbol{h} \in \mathcal{S}_q,\ q > 3\,. \tag{V.6.3}$$

For example, we may take $\boldsymbol{F} = \psi \nabla \boldsymbol{h}$, where $\psi = \psi(|x|)$ is a smooth, non negative function such that $\psi(|x|) = 0$ if either $|x| \le R$ or $|x| \ge 2R$, $R > \delta(\Omega^c)$. We then have

$$(\nabla \cdot \boldsymbol{F}, \boldsymbol{h}) = -\int_{\Omega_{R,2R}} \psi \nabla \boldsymbol{h} : \nabla \boldsymbol{h}\,,$$

which is, of course, non-zero. Now, in view of (V.6.1) and the properties of $\boldsymbol{G}, \boldsymbol{g}$, it is immediately recognized that the fields

$$\boldsymbol{v}(x) = \int_\Omega \boldsymbol{G}(x,y) \cdot (\nabla \cdot \boldsymbol{F}(y))\,dy, \qquad p(x) = -\int_\Omega \boldsymbol{g}(x,y) \cdot \nabla \cdot \boldsymbol{F}(y) dy$$

define a solution to the Stokes system (V.5.1) with $\boldsymbol{f} = \nabla \cdot \boldsymbol{F}$. Furthermore, since

$$\boldsymbol{v}(x) = -\int_\Omega \nabla \boldsymbol{G}(x,y) \cdot \boldsymbol{F}(y) dy,$$

and $\boldsymbol{F}$ is of bounded support, the validity of (V.6.2) would imply

$$\boldsymbol{v}(x) = O(|x|^{-2}). \tag{V.6.4}$$

This property, with the aid of Theorem V.3.2, then furnishes that the vector $\boldsymbol{\mathcal{T}}$ defined in (V.3.20) must be zero. Thus, from (V.3.19) and (V.3.21) we obtain

[1] Actually, if $\Omega = \mathbb{R}^n$, then $\boldsymbol{G} \equiv \boldsymbol{U}$, and $\boldsymbol{G}$ obeys the same type of estimates holding for a bounded domain and a half-space; see (IV.2.6).

$$\nabla \boldsymbol{v}(x) = O(|x|^{-3}).$$

Such a condition, along with (V.6.4) and the fact that $\boldsymbol{v}$ vanishes at the boundary, leads, by Theorem II.7.6, to the conclusion that $\boldsymbol{v} \in D_0^{1,q}(\Omega)$, for *all* $q \in (1, \infty)$. By Theorem V.5.1, however, this is possible if and only if

$$(\nabla \cdot \boldsymbol{F}, \boldsymbol{h}) = 0 \ \text{ for all } \boldsymbol{h} \in \mathcal{S}_q,\ q > 3,$$

contradicting (V.6.3). As a consequence, the invalidity of (V.6.2) is proved.[2]

However, one can prove (in three dimensions) that the following estimate, *weaker* than (V.6.2), *does* hold:[3]

$$\int_\Omega |\nabla G_{ij}(x,y)||y|^{-2} dy \le c\,|x|^{-1}, \quad x \in \Omega \tag{V.6.5}$$

and that

$$|G_{ij}(x,y)| \le c\,|x-y|^{-1}\,, \quad x, y \in \Omega\,, x \neq y;$$

see Finn (1965a) Theorem 3.1 and §2.6.

V.7 A Characterization of Certain Flows with Nonzero Boundary Data. Another Form of the Stokes Paradox

We wish to investigate the meaning of condition (V.5.4) in the context of slow motions of a viscous flow past a body, subject to zero body force and zero velocity at infinity. This last condition imposes, in fact, no serious restriction, since the Stokes system is invariant if we change $\boldsymbol{v}$ into $\boldsymbol{v}+\boldsymbol{a}$, for any constant vector $\boldsymbol{a}$. In order to make the presentation clearer, we shall limit ourselves to considering smooth regions of motion and smooth velocity fields at the boundary as well, leaving to the reader the (routine) task of extending the results to less regular situations.

Consider the problem

$$\begin{aligned} &\left.\begin{aligned} \Delta \boldsymbol{v} &= \nabla p \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}\right\} \text{ in } \Omega \\ &\boldsymbol{v} = \boldsymbol{v}_* \ \text{ at } \partial\Omega, \\ &\lim_{|x|\to\infty} \boldsymbol{v}(x) = 0. \end{aligned} \tag{V.7.1}$$

Let us begin to show that (V.5.4) is equivalent to the following requirement on $\boldsymbol{v}_*$:

[2] With ∇ operating on y. The symmetry property of $\boldsymbol{G}$ allows us to draw the same conclusion if ∇ operates on x.

[3] Notice that, if a tensor function $\boldsymbol{G}$ satisfies (V.6.2), then (V.6.5) follows from Lemma II.9.2.

$$\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{T}(\boldsymbol{h}_i, \pi_i) \cdot \boldsymbol{n} = 0, \quad \text{for all } i = 1, 2, \ldots, n, \tag{V.7.2}$$

where $\{\boldsymbol{h}_i, \pi_i\}$ is a basis in $\mathcal{S}_q$ constructed in the preceding section. In fact, we write

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{V}_1 + \boldsymbol{\sigma},$$

where

$$\boldsymbol{\sigma}(x) = \Phi \nabla \mathcal{E}(x)$$

$$\Phi = \int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n},$$

and $\boldsymbol{V}_1$ is a smooth solenoidal extension in Ω of compact support of the field

$$\boldsymbol{v}_*(x) - \sigma(x), \quad x \in \partial\Omega.$$

Thus, (V.7.1) can be equivalently rewritten as

$$\left.\begin{aligned} \Delta \boldsymbol{w} &= \nabla p + \boldsymbol{f} \\ \nabla \cdot \boldsymbol{w} &= 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\boldsymbol{w} = 0 \text{ at } \partial\Omega, \tag{V.7.3}$$
$$\lim_{|x|\to\infty} \boldsymbol{w}(x) = 0,$$

where

$$\boldsymbol{f} = -\Delta \boldsymbol{V}_1.$$

Clearly

$$\boldsymbol{f} \in D_0^{-1,q}(\Omega), \quad \text{for all } q \in (1, \infty)$$

and condition (V.5.4) furnishes

$$[\boldsymbol{f}, \boldsymbol{h}_i] = -\int_{\Omega} \Delta \boldsymbol{V}_1 \cdot \boldsymbol{h}_i = -\int_{\Omega} \Delta(\boldsymbol{V}_1 + \sigma) \cdot \boldsymbol{h}_i = \int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{T}(\boldsymbol{h}_i, \pi_i) \cdot \boldsymbol{n} = 0$$

which proves (V.7.2). Suppose now $\Omega \subset \mathbb{R}^3$.[1] It is easy to show that a solution to (V.7.1) *verifies* (V.7.2) *if and only if the following condition holds*:

$$\int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{v}, p) \cdot \boldsymbol{n} = 0. \tag{V.7.4}$$

From a physical point of view, this means that, within the approximation we are considering, the net external force applied to the body is zero. This happens, for example, if the body is *self-propelled*; Pukhnacev (1990a, 1990b), Galdi (1999a). In fact, if (V.7.2) is satisfied, then by Theorem V.5.1 there is a solution $\boldsymbol{w}$ to (V.7.3) in the class $D_0^{1,q}(\Omega)$, $1 < q \le 3/2$. This in turn implies

[1] We may take $\Omega \subset \mathbb{R}^n, n \ge 3$.

that a solution $\boldsymbol{v}$ to (V.7.1) exists in the class $D^{1,q}(\Omega)$, $1 < q \le 3/2$. In view of Theorem V.3.2, however, such a circumstance is possible only if (V.7.4) is satisfied (see Exercise V.3.3). Conversely, assume we have a solution to (V.7.1) satisfying (V.7.4). Then, again by Theorem V.3.2, we have $\boldsymbol{v} \in D^{1,q}(\Omega)$, $1 < q \le 3/2$, and so $\boldsymbol{w} \in D_0^{1,q}(\Omega)$ and, by Theorem V.5.1, (V.7.2) is satisfied. It is interesting to observe that if $\mathbb{R}^3 - \overline{\Omega} = B_1$, from (V.0.4) we have (see Remark V.5.2)

$$\frac{\partial(\boldsymbol{h}_i)_j}{\partial x_l} = -\frac{3}{2}x_i x_j x_l + \frac{3}{2}x_l\delta_{ij}$$

$$\pi_i = -\frac{3}{2}x_i$$

so that condition (V.7.2) becomes

$$\int_{\partial\Omega} \boldsymbol{v}_* = 0. \tag{V.7.5}$$

Let us next consider the case $\Omega \subset \mathbb{R}^2$. We then show that *a solution to* (V.7.1) *exists if and only if condition* (V.7.2) *is satisfied.* Since, as we shall see, the vector field $\boldsymbol{v}_* =$ const. does not verify (V.7.2) this latter statement can be interpreted as another form of the *Stokes paradox.* Assume (V.7.2) holds. Then, by Theorem V.5.1, there is a solution $\boldsymbol{w}$ to (V.7.3) and, consequently, a solution $\boldsymbol{v}$ to (V.7.1). Conversely, assume that there is a solution $\boldsymbol{v}$ to (V.7.1); then $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{V}_1 - \boldsymbol{\sigma}$ is a solution to (V.7.3) which, by Theorem V.3.2, belongs to the class $\mathcal{D}_0^{1,q}(\Omega)$, $1 < q < 2$. As a consequence, by Theorem V.5.1, condition (V.7.2) must be satisfied. We now show that $\boldsymbol{v}_* = \boldsymbol{v}_0$ does not verify (V.7.2) for any nonzero choice of the constant vector $\boldsymbol{v}_0$. This is because from (V.4.27) and Theorem V.3.2 it follows that

$$\int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{h}_i, \pi_i)\cdot\boldsymbol{n} = -\boldsymbol{e}_i, \quad i = 1, 2,$$

and therefore (V.7.2) would imply

$$\boldsymbol{v}_0 \cdot \boldsymbol{e}_i = 0, \quad i = 1, 2;$$

that is,

$$\boldsymbol{v}_0 = \boldsymbol{0}.$$

If Ω is the exterior of a unit circle, from (0.5) we deduce again that (V.7.2) is equivalent to (V.7.5).

Exercise V.7.1 Prove that for $\Omega \subset \mathbb{R}^3$, the field $\boldsymbol{v}_* =$ const. does not verify (V.7.2). Give a physical interpretation of this fact. Moreover, for Ω the exterior of the closed unit ball centered at the origin, show that (V.7.2) is satisfied by

$$\boldsymbol{v}_* = \boldsymbol{\omega} \times \boldsymbol{x} \tag{V.7.6}$$

and that this finding is in agreement with (V.0.5). Finally, suppose that Ω is the exterior of the closed unit circle centered at the origin and lying in the plane $x_3 = 0$.

Show that condition (V.7.2) is again equivalent to (V.7.5), and that it is satisfied by the field (V.7.6), with $\boldsymbol{\omega}$ directed along the x_3-axis, and by the field $\boldsymbol{v}_* = \kappa x_2 \boldsymbol{e}_1$, $\kappa \neq 0$ (simple shear flow).

V.8 Further Existence and Uniqueness Results for q-generalized Solutions

One undesired feature concerning the q-generalized solutions constructed in Section V.5 is the fact that their existence and uniqueness are recovered only if we restrict suitably the range of values of q, *i.e.*, $q \in (n/(n-1), n)$. Now, while the restriction from below ($q > n/(n-1)$) is necessary unless $\boldsymbol{f}$ satisfies the compatibility condition (V.5.4), the restriction from above ($q < n$) is due to the circumstance that the estimates we are able to derive for solutions under the *sole* assumption $\boldsymbol{f} \in D_0^{-1,q}(\Omega)$ are not sufficient to guarantee their uniqueness. However, we may wonder if, taking $\boldsymbol{f}$ from a suitable subclass of the space $D_0^{-1,q}(\Omega)$, we could remove the restriction $q < n$. Following the work of Galdi & Simader (1994), in the present section we shall show that, for $n \geq 3$, this is indeed the case provided $\boldsymbol{f}$ is of the type $\nabla \cdot \boldsymbol{F}$, with $\boldsymbol{F}$ a second-order tensor field (i.e., $f_i = D_k F_{ki}$) such that either $\|(|x|^{n-1}+1)\boldsymbol{F}\|_\infty$ or $\|(|x|^2+1)\boldsymbol{F}\|_\infty$ is finite. Since the estimates we shall derive guarantee that the solution tends to zero at large distances, we are not expecting that a similar result holds also in the case of plane motions for, as we have learned from the preceding section, a two-dimensional solution to the Stokes problem tends to zero if and only if the data satisfy compatibility condition (V.5.4).

We begin to show a simple approximation lemma.

Lemma V.8.1 *Suppose*

$$(1+|x|^\alpha)\mathcal{F} \in L^\infty(\mathbb{R}^n), \quad \alpha > 0, n \geq 2.$$

Then, there exists a sequence $\{\mathcal{F}_h\} \subset C_0^\infty(\mathbb{R}^n)$ *such that*

$$\begin{aligned} &\lim_{h\to\infty} \|\mathcal{F}_h - \mathcal{F}\|_s = 0 \quad \textit{for all } s > n/\alpha, \\ &\|(|x|^\alpha+1)\mathcal{F}_h\|_\infty \leq 2(2^{\alpha-1}+1)\|(|x|^\alpha+1)\mathcal{F}\|_\infty. \end{aligned} \tag{V.8.1}$$

Proof. Let ψ_h, $h \in \mathbb{N}$, be smooth functions in $\mathbb{R}^n$ such that

$$\begin{aligned} &|\psi_h(x)| \leq 1 \\ &\psi_h(x) = \begin{cases} 1 \text{ if } |x| \leq h \\ 0 \text{ if } |x| \geq 2h. \end{cases} \end{aligned} \tag{V.8.2}$$

Set

$$\mathcal{F}_h(x) = \psi_h(x)(\mathcal{F}(x))_\varepsilon, \quad \varepsilon = 1/h,$$

where, as usual, $(\cdot)_\varepsilon$ denotes mollification. Clearly, $\{\mathcal{F}_h\} \subset C_0^\infty(\mathbb{R}^n)$. Observing that $\mathcal{F} \in L^s(\mathbb{R}^n)$ for each $s > n/\alpha$, we find in the limit $k \to \infty$

$$\begin{aligned}\|\mathcal{F}_h - \mathcal{F}\|_s &\le \|(\mathcal{F})_{1/h} - \mathcal{F}\|_s + \|(1-\psi_h)(\mathcal{F})_{1/h}\|_s \\ &\le 2\|(\mathcal{F})_{1/h} - \mathcal{F}\|_s + \|(1-\psi_h)\mathcal{F}\|_s \to 0\end{aligned}$$

as a consequence of (V.8.2), of property $(\text{II.2.9})_2$ of mollifiers and of the dominated convergence theorem of Lebesgue given in Lemma II.2.1. Relation $(\text{V.8.1})_1$ is then acquired. From the definition of mollifier, we obtain for all $\varepsilon \in (0,1]$

$$\begin{aligned}(|x|^\alpha + 1)|(\mathcal{F}(x))_\varepsilon| \le \varepsilon^{-n}\int_{\mathbb{R}^n} j\left(\frac{x-y}{\varepsilon}\right)(|y|^\alpha + 1)|\mathcal{F}(y)|dy \\ + \varepsilon^{-n}\int_{\mathbb{R}^n} j\left(\frac{x-y}{\varepsilon}\right)||x|^\alpha - |y|^\alpha||\mathcal{F}(y)|dy \\ \equiv I_1 + I_2.\end{aligned}$$

Now

$$I_1 \le \|(|x|^\alpha + 1)\mathcal{F}\|_\infty\, \varepsilon^{-n}\int_{\mathbb{R}^n} j\left(\frac{x-y}{\varepsilon}\right)dy = \|(|x|^\alpha + 1)\mathcal{F}\|_\infty. \tag{V.8.3}$$

Furthermore, for $|x-y| \le \varepsilon \le 1$

$$|x| \le |x-y| + |y| \le 1 + |y|$$

and so, in view of inequality (II.3.3) (with $n \equiv 2$ and $q \equiv \alpha$) we derive

$$||x|^\alpha - |y|^\alpha| \le (2^\alpha + 1)(1 + |y|^\alpha).$$

Therefore, recalling that $j\left(\frac{x-y}{\varepsilon}\right) = 0$ for $|x-y| \ge \varepsilon$, it follows that

$$I_2 \le (2^\alpha + 1)\|(|x|^\alpha + 1)\mathcal{F}\|_\infty \varepsilon^{-n}\int_{\mathbb{R}^n} j\left(\frac{x-y}{\varepsilon}\right)dy = (2^\alpha + 1)\|(|x|^\alpha + 1)\mathcal{F}\|_\infty,$$

and the lemma is completely proved. □

We are now in a position to show the following intermediate result.

Lemma V.8.2 *Assume $\boldsymbol{G}$, $\boldsymbol{f}_1$, and g are a given second-order tensor, vector, and scalar field, respectively, in $\mathbb{R}^n$, $n \ge 3$, satisfying*

$$\begin{aligned}&(1 + |x|^\alpha)\boldsymbol{G} \in L^\infty(\mathbb{R}^n), \\ &\boldsymbol{f}_1, g \in L^q(\mathbb{R}^n), \quad \textit{for each } q > n/\alpha, \\ &\operatorname{supp}(\boldsymbol{f}_1), \quad \operatorname{supp}(g) \subset B_{R/2}, \quad \textit{for some } R > 0,\end{aligned}$$

where α is either 2 or $n-1$. Then, the problem

$$\begin{aligned}(\nabla \boldsymbol{u}, \nabla \boldsymbol{\psi}) - (\pi, \nabla \cdot \boldsymbol{\psi}) = (\boldsymbol{G}, \nabla \boldsymbol{\psi}) - (\boldsymbol{f}_1, \boldsymbol{\psi}), \quad \textit{for all } \boldsymbol{\psi} \in C_0^\infty(\mathbb{R}^n),\\ (\boldsymbol{u}, \nabla \chi) = -(g, \chi), \quad \textit{for all } \chi \in C_0^\infty(\mathbb{R}^n),\end{aligned} \tag{V.8.4}$$

admits at least one solution $\boldsymbol{u}, \pi$ such that

$$\begin{gathered}\boldsymbol{u} \in D^{1,q}(\mathbb{R}^n), \quad \pi \in L^q(\mathbb{R}^n), \quad \textit{for all } q > n/\alpha,\\ (1 + |x|^{\alpha-1})\boldsymbol{u} \in L^\infty(\mathbb{R}^n).\end{gathered}$$

Moreover, this solution satisfies the estimate

$$\begin{aligned}\| |x|^{\alpha-1} \boldsymbol{u}\|_\infty + |\boldsymbol{u}|_{1,q,\mathbb{R}^n} + \|\pi\|_{q,\mathbb{R}^n}\\ \le c\left(\|(|x|^\alpha + 1)\boldsymbol{G}\|_\infty + \|\boldsymbol{f}_1\|_{-1,q,B_R} + \|g\|_{q,B_R}\right)\end{aligned} \tag{V.8.5}$$

with $c = c(n,q)$. Finally, if $\boldsymbol{u}', \pi'$ is another pair satisfying (V.8.4) *with the same data as $\boldsymbol{u}, \pi$ and with*

$$\boldsymbol{u}' \in W^{1,r}_{loc}(\mathbb{R}^n) \cap L^s(B^c_\rho), \quad \pi' \in L^r_{loc}(\mathbb{R}^n)$$

for some $r, s \in (1, \infty)$ and $\rho > 0$, then $\boldsymbol{u} \equiv \boldsymbol{u}', \pi \equiv \pi' + \mathrm{const.}$ a.e. in $\mathbb{R}^n$.

Proof. We approximate $\boldsymbol{G}$ with functions $\{\boldsymbol{G}_h\} \subset C_0^\infty(\mathbb{R}^n)$ of the type constructed in Lemma V.8.1. In addition, by the elementary properties of mollifiers, we see that the functions

$$\boldsymbol{f}_{1h} = (\boldsymbol{f}_1)_{1/h}, \quad g_h = (g)_{1/h}, \quad h \in \mathbb{N}, h \ge h_0 > 4/R$$

belong to $C_0^\infty(B_{3R/4})$ and satisfy, as $h \to \infty$,

$$\|\boldsymbol{f}_{1h} - \boldsymbol{f}_1\|_{-1,q,B_R} + \|g_h - g\|_{q,B_R} \to 0, \quad \text{for all } q > n/\alpha.$$

Let us consider the following problem for all $h \ge h_0$:

$$\left.\begin{aligned}\Delta \boldsymbol{u}_h &= \nabla \pi_h + \nabla \cdot \boldsymbol{G}_h + \boldsymbol{f}_{1h}\\ \nabla \cdot \boldsymbol{u}_h &= g_h\end{aligned}\right\} \text{ in } \mathbb{R}^n \tag{V.8.6}$$

$$\lim_{|x| \to \infty} \boldsymbol{u}_h(x) = 0.$$

Proceeding as in Section IV.2, we look for a solution to (V.8.6) of the form

$$\boldsymbol{u}_h = \boldsymbol{w}_h + \boldsymbol{h}_h, \quad \pi_h = \tau_h,$$

where $\boldsymbol{w}_h$ and τ_h are the volume potentials (IV.2.8) corresponding to the body force $\nabla \cdot \boldsymbol{G}_h + \boldsymbol{f}_{1h}$ and $\boldsymbol{h}_h$ is given in (IV.2.10) with g_h in place of g. We begin to furnish estimates for $\boldsymbol{h}_h$. From Calderón–Zygmund Theorem II.11.4 we immediately deduce

$$|\boldsymbol{h}_h|_{1,q} \le c_1 \|g_h\|_{q,B_R}, \tag{V.8.7}$$

with $c_1 = c_1(n,q)$. Moreover, we have for $|x| \ge 2R$,

$$|\boldsymbol{h}_h(x)| \le c_2 \int_{B_{R/2}} |g_h(y)||x-y|^{2-n} dy \le c_3 |x|^{2-n} \|g_h\|_{q,B_R},$$

and so

$$\|(|x|^{\alpha-1}+1)\boldsymbol{h}_h\|_{\infty,B_{2R}^c} \le c_4 \|g_h\|_{q,B_R}, \tag{V.8.8}$$

with $c_4 = c_4(n,q,R)$. We shall next estimate $\boldsymbol{w}_h$. From $(\text{IV.2.8})_1$ it follows (omitting the index h)

$$w_i(x) = -\int_{\mathbb{R}^n} D_k U_{ij}(x-y) G_{kj}(y) dy + \int_{\mathbb{R}^n} U_{ij}(x-y) f_{1j}(y) dy = \mathcal{G}_1 + \mathcal{G}_2. \tag{V.8.9}$$

Clearly, again from the Calderón–Zygmund theorem, we deduce for all $q \in (1,\infty)$

$$|\mathcal{G}_1|_{1,q} \le c_5 \|\boldsymbol{G}\|_q. \tag{V.8.10}$$

Moreover, denoting by ψ_R a C^∞-function which is one in $B_{3R/4}$ and zero outside B_R, we obtain, for all $\boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^n)$

$$|(\boldsymbol{f}_1, \boldsymbol{\varphi})| = |(\boldsymbol{f}_1, \psi_R \boldsymbol{\varphi})| \le \|\boldsymbol{f}\|_{-1,q,B_R} \|\psi_R \boldsymbol{\varphi}\|_{1,q',B_R}. \tag{V.8.11}$$

Now, if $q > n/(n-1)$, by the Sobolev inequality (II.3.7) we easily show

$$\|\psi_R \boldsymbol{\varphi}\|_{1,q',B_R} \le c_7 |\boldsymbol{\varphi}|_{1,q',\mathbb{R}^n},$$

with $c_7 = c_7(r,q,n)$ and so (V.8.11) yields

$$|\boldsymbol{f}_1|_{-1,q,\mathbb{R}^n} \le c_7 \|\boldsymbol{f}_1\|_{-1,q,B_R}.$$

Therefore, repeating the same argument employed in Section IV.2 (see (IV.2.27)–(IV.2.28)) we recover for all $q > n/(n-1)$

$$|\mathcal{G}_2|_{1,q} \le c_8 \|\boldsymbol{f}_1\|_{-1,q,B_R}. \tag{V.8.12}$$

Collecting (V.8.7), (V.8.9), (V.8.10), and (V.8.12) furnishes

$$|\boldsymbol{u}_h|_{1,q} \le c_9 (\|\boldsymbol{G}_h\|_q + \|\boldsymbol{f}_{1h}\|_{-1,q,B_R} + \|g_h\|_{q,B_R}). \tag{V.8.13}$$

Moreover, recalling the expression $(\text{IV.2.8})_2$ for τ_h $(= \pi_h)$ and reasoning as in (V.8.10) we readily prove

$$\|\pi_h\|_q \le c_{10} (\|\boldsymbol{G}_h\|_q + \|\boldsymbol{f}_1\|_{-1,q,B_R}).$$

This latter inequality and (V.8.13) then yield

$$|\boldsymbol{u}_h|_{1,q} + \|\pi_h\|_q \le c_{11} (\|\boldsymbol{G}_h\|_q + \|\boldsymbol{f}_{1h}\|_{-1,q,B_R} + \|g_h\|_{q,B_R}), \quad \text{for all } q > n/\alpha. \tag{V.8.14}$$

We next show the pointwise estimate for $\boldsymbol{w}_h$. From (V.8.9) and from the expression of the tensor $\boldsymbol{U}$ (omitting the index h),

$$|\mathcal{G}_1(x)| \le c_{12}\|(|x|^\alpha + 1)\boldsymbol{G}\|_\infty \int_{\mathbb{R}^n} |x-y|^{1-n}|y|^{-\alpha}dy,$$

and so, Lemma II.9.2 implies

$$\|(|x|^{\alpha-1} + 1)\mathcal{G}_1\|_{\infty,B^c_{2R}} \le c_{13}\|(|x|^\alpha + 1)\boldsymbol{G}\|_\infty. \tag{V.8.15}$$

We have also

$$|\mathcal{G}_2(x)|x|^{\alpha-1}| = \left|\int_{B_{R/2}} \boldsymbol{f}_1(y)\cdot \boldsymbol{B}(x,y)dy\right|, \tag{V.8.16}$$

where, for $i = 1,\dots,n$,

$$B_j(x,y) = \psi_R(y)U_{ij}(x-y)|x|^{\alpha-1}. \tag{V.8.17}$$

Since for $y \in B_R$ and $|x| \ge 2R$ it is

$$|\boldsymbol{U}(x-y)| + |\nabla \boldsymbol{U}(x-y)| \le c|x|^{1-\alpha}$$

with $c = c(R,n)$, from (V.8.16) and (V.8.17) we find

$$\begin{aligned}|\mathcal{G}_2|x|^{\alpha-1}| &\le \|\boldsymbol{f}_1\|_{-1,q,B_R}\left[\int_{B_R}(|\boldsymbol{B}(x,y)|^{q'} + |\nabla_y \boldsymbol{B}(x,y)|)dy\right]^{1/q'}\\ &\le c_{14}\|\boldsymbol{f}_1\|_{-1,q,B_R}.\end{aligned} \tag{V.8.18}$$

Thus, from (V.8.8), (V.8.9), (V.8.15), (V.8.18), and property $(\text{V.8.1})_2$ of $\boldsymbol{G}_h$ we recover

$$\|(|x|^{\alpha-1} + 1)\boldsymbol{u}_h\|_{\infty,B^c_{2R}} \le c_{15}(\|(|x|^\alpha + 1)\boldsymbol{G}\|_\infty + \|\boldsymbol{f}_{1h}\|_{-1,q,B_R} + \|g_h\|_{q,B_R}). \tag{V.8.19}$$

We next pass to the limit $h \to \infty$. From the linearity and from the uniqueness of problem (V.8.6), by virtue of (V.8.14) and by the properties of the approximating functions $\boldsymbol{G}_h$, $\boldsymbol{f}_{1h}$, and g_h we obtain, in particular, that the sequence $\{\boldsymbol{u}_h, \pi_h\}$ is a Cauchy sequence in $D^{1,q}_0(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ for all $q \in (n/\alpha, \infty)$ and, by the Sobolev inequality (II.3.7), it is also a Cauchy sequence in $L^s(\mathbb{R}^n)$, $s = nq/(n-q)$, for all $q \in (1,n)$. We may then assert the existence of two fields $\boldsymbol{u}, \pi$ such that

$$\boldsymbol{u} \in D^{1,q}_0(\mathbb{R}^n), \quad \pi \in L^q(\mathbb{R}^n), \quad \text{for all } q > n/\alpha$$

with

$$\begin{aligned}&\lim_{h\to\infty}|\boldsymbol{u}_h - \boldsymbol{u}|_{1,q} = \lim_{h\to\infty}\|\pi_h - \pi\|_q = 0 \ \text{ for all } q > n/\alpha\\ &\lim_{h\to\infty}|\boldsymbol{u}_h - \boldsymbol{u}|_{nq/(n-q)} = 0 \ \text{ for all } q \in (1,n).\end{aligned} \tag{V.8.20}$$

After multiplying $(\mathrm{V}.8.6)_1$ by $\boldsymbol{\psi} \in C_0^\infty(\mathbb{R}^n)$ and $(\mathrm{V}.8.6)_2$ by $\chi \in C_0^\infty(\mathbb{R}^n)$, integrating by parts and using (V.8.20) we deduce at once that $\boldsymbol{u}, \pi$ solves (V.8.4). In addition, again by (V.8.20) and (V.8.14), it follows that $\boldsymbol{u}$ and π satisfy the estimate

$$|\boldsymbol{u}|_{1,q}+\|\pi\|_q \leq c_{16}(\|\boldsymbol{G}\|_q+\|\boldsymbol{f}_1\|_{-1,q,B_R}+\|g\|_{q,B_R}), \text{ for all } q > n/\alpha. \quad (\mathrm{V}.8.21)$$

Furthermore, by $(\mathrm{V}.8.20)_2$ and Lemma II.2.2, we see that we can select a subsequence $\{\boldsymbol{u}_{h'}\}$, say, which converges pointwise to $\boldsymbol{u}$, *a.e.* in $\mathbb{R}^n$. Consequently, by passing to the limit $h' \to \infty$ into (V.8.19) we conclude

$$\|(|x|^{\alpha-1}+1)\boldsymbol{u}\|_{\infty,B_{2R}^c} \leq c_{15}(\|(|x|^\alpha+1)\boldsymbol{G}\|_\infty + \|\boldsymbol{f}_1\|_{-1,q,B_R} + \|g\|_{q,B_R}). \quad (\mathrm{V}.8.22)$$

Finally, since by the embedding Theorem II.3.4,

$$\|\boldsymbol{u}\|_{\infty,B_{2R}} \leq c\|\boldsymbol{u}\|_{1,r,B_{2R}}, \quad r > n,$$

estimate (V.8.5) becomes a consequence of this last inequality, of (V.8.21) and of (V.8.22). Concerning uniqueness, let $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}'$, $p = \pi - \pi'$. From (V.8.4), the assumptions made on $\boldsymbol{u}', \pi'$ and the regularity results of Theorem IV.4.2 we deduce that $\boldsymbol{v}, p$ is a C^∞-smooth solution to the homogeneous Stokes problem

$$\left.\begin{aligned} \Delta \boldsymbol{v} &= \nabla p \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}\right\} \text{ in } \mathbb{R}^n.$$

From Lemma V.3.1 we then have

$$v_i(x) = \int_{\beta(x)} H_{ij}^{(d)}(x-y)(u_i(y)+u_i'(y))dy, \quad (\mathrm{V}.8.23)$$

where, we recall, $\beta(x) = B_d(x) - B_{d/2}(x)$. Since $\boldsymbol{u} \in L^q(\mathbb{R}^n)$, for all $q > n/\alpha$ and $\boldsymbol{u}' \in L^s(B_\rho^c)$, for some $\rho > 0$, using the Hölder inequality into (V.8.23) and taking into account that $\|H_{ij}^{(d)}\|_t \leq M$, independently of x and for all $t \geq 1$, we easily show that $\boldsymbol{v}(x)$ tends to zero pointwise as $|x|$ tends to infinity. Theorem V.3.5 allows us to conclude $\boldsymbol{v} \equiv 0$, $p \equiv$ const. and the lemma is completely proved. □

The following result furnishes an extension of the one just proved to the case of an exterior domain and it represents the main contribution of this section. For simplicity, we shall state it for homogeneous boundary data, i.e., $\boldsymbol{v}_* \equiv 0$, referring the reader to Exercise V.8.1 for the more general case $\boldsymbol{v}_* \neq 0$.

Theorem V.8.1 *Let $\Omega \subset \mathbb{R}^n, n \geq 3$, be an exterior domain of class C^2. Suppose that the second-order tensor field $\boldsymbol{F}$ in Ω satisfies*

$$(1+|x|^\alpha)\boldsymbol{F} \in L^\infty(\Omega),$$

with α either 2 or $n-1$. Then, the problem

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) - (p, \nabla \cdot \boldsymbol{\psi}) = (\boldsymbol{F}, \nabla \boldsymbol{\psi}) \quad \text{for all } \boldsymbol{\psi} \in C_0^\infty(\mathbb{R}^n), \tag{V.8.24}$$

admits one and only one solution $\boldsymbol{v}, p$ such that

$$\boldsymbol{v} \in \mathcal{D}_0^{1,q}(\Omega), \quad p \in L^q(\Omega), \quad \text{for each } q > n/\alpha,$$
$$(1 + |x|^{\alpha-1})\boldsymbol{v} \in L^\infty(\Omega).$$

Moreover, this solution satisfies the following estimate

$$\|(|x|^{\alpha-1} + 1)\boldsymbol{v}\|_\infty + |\boldsymbol{v}|_{1,q} + \|p\|_q \le c\|(|x|^\alpha + 1)\boldsymbol{F}\|_\infty, \tag{V.8.25}$$

for each $q > n/\alpha$ and with $c = c(n, q, \Omega)$.

Proof. Since $\boldsymbol{F} \in L^q(\Omega)$ with arbitrary $q > n/\alpha \ge n/(n-1)$, from Theorem V.5.1 we know that there exists a unique (q-generalized) solution $\boldsymbol{v}, p$ to (V.8.24) such that

$$\boldsymbol{v} \in \mathcal{D}_0^{1,q}(\Omega), \quad p \in L^q(\Omega) \ \text{ for all } q \in (n/\alpha, n). \tag{V.8.26}$$

Moreover, by Sobolev inequality (II.3.7), we have

$$\boldsymbol{v} \in L^{nq/(n-q)}(\Omega), \quad \text{for all } q \in (n/\alpha, n). \tag{V.8.27}$$

Finally, from the regularity results of Theorem IV.4.2 and Theorem IV.6.1, we readily find

$$\boldsymbol{v} \in W^{1,q}(\Omega_R), \quad p \in L^q(\Omega_R), \quad \text{for all } q > n/\alpha \text{ and all } R > \delta(\Omega^c). \tag{V.8.28}$$

Let φ be the "cut-off" function of Lemma V.4.2. Problem (V.8.24) then goes into problem (V.8.4) with $\boldsymbol{u} = \varphi \boldsymbol{v}, \pi = \varphi p$ and

$$\begin{aligned} G_{ij} &= \varphi F_{ij} \\ f_{1i} &= T_{ik}(\boldsymbol{v}, p) D_k \varphi + D_k(v_k D_i \varphi + v_i D_k \varphi) - F_{ki} D_k \varphi \\ g &= \boldsymbol{v} \cdot \nabla \varphi; \end{aligned}$$

see also (V.4.4). Thus, from (V.8.26)–(V.8.28), from Lemma V.8.2 and (V.5.24) we obtain for all $q > n/\alpha$

$$\begin{aligned} &\|(|x|^{\alpha-1} + 1)\boldsymbol{u}\|_{\infty,\mathbb{R}^n} + |\boldsymbol{u}|_{1,q,\mathbb{R}^n} + \|\pi\|_{q,\mathbb{R}^n} \\ &\qquad \le c\left(\|(|x|^\alpha + 1)\boldsymbol{F}\|_{\infty,\Omega} + \|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R}\right). \end{aligned}$$

Recalling that φ is equal to one in $\Omega^{R/2}$, from this inequality it follows that

$$\begin{aligned} &\|(|x|^{\alpha-1} + 1)\boldsymbol{v}\|_{\infty,\Omega^{R/2}} + |\boldsymbol{v}|_{1,q,\Omega^{R/2}} + \|p\|_{q,\Omega^{R/2}} \\ &\qquad \le c\left(\|(|x|^\alpha + 1)\boldsymbol{F}\|_{\infty,\Omega} + \|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R}\right). \end{aligned} \tag{V.8.29}$$

If we add (V.8.29) to (V.5.26), by reasoning exactly as we did to obtain (V.5.28), we find

$$\begin{aligned}\|\boldsymbol{v}\|_{1,q,\Omega_R} + \|(|x|^{\alpha-1}+1)\boldsymbol{v}\|_{\infty,\Omega^{R/2}} + |\boldsymbol{v}|_{1,q,\Omega} + \|p\|_{q,\Omega} \\ \le c\left(\|(|x|^{\alpha}+1)\boldsymbol{F}\|_{\infty,\Omega} + \|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R}\right).\end{aligned} \tag{V.8.30}$$

We now claim the existence of a constant $\kappa = \kappa(n, q, \Omega, R)$ such that

$$\|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R} \le \kappa\|(|x|^{\alpha}+1)\boldsymbol{F}\|_{\infty,\Omega}. \tag{V.8.31}$$

To show the validity of (V.8.31), we use the usual contradiction argument. Actually, the invalidity of (V.8.31) would imply the existence of a sequence $\{\boldsymbol{F}_m\}$ verifying the assumptions of the theorem for each $m \in \mathbb{N}$ and of a corresponding sequence of solutions $\{\boldsymbol{v}_m, p_m\}$ such that

$$\begin{gathered}\|\boldsymbol{v}_m\|_{q,\Omega_R} + \|p_m\|_{-1,q,\Omega_R} = 1 \\ \|(|x|^{\alpha}+1)\boldsymbol{F}_m\|_{\infty,\Omega} \le 1/m.\end{gathered} \tag{V.8.32}$$

Since, clearly, for all $s > n/(\alpha - 1)$

$$\|\boldsymbol{v}_m\|_{s,\Omega^{R/2}} \le c\||x|^{\alpha-1}\boldsymbol{v}_m\|_{\infty,\Omega^{R/2}},$$

from (V.8.32) and (V.8.30) we deduce, in particular,

$$\|\boldsymbol{v}_m\|_{1,q,\Omega_R} + \|\boldsymbol{v}_m\|_{s,\Omega^{R/2}} + |\boldsymbol{v}_m|_{1,q,\Omega} + \|p_m\|_{q,\Omega} \le M,$$

for a constant M independent of m. From the weak compactness of reflexive Lebesgue spaces and the strong compactness results of Exercise II.5.8 and Theorem II.5.3, it is easy to show the existence of a subsequence, denoted again by $\{\boldsymbol{v}_m, p_m\}$, and of two fields $\boldsymbol{v}, p$ such that

$$\begin{aligned}&\boldsymbol{v} \in L^s(\Omega^{2R}) \cap D^{1,q}(\Omega) \cap W^{1,q}(\Omega_R) \\ &p \in L^q(\Omega) \\ &\nabla\boldsymbol{v}_m \overset{w}{\to} \nabla\boldsymbol{v} \ \text{ in } L^q(\Omega) \\ &p_m \overset{w}{\to} p \ \text{ in } L^q(\Omega) \\ &\boldsymbol{v}_m \to \boldsymbol{v} \ \text{ in } L^q(\Omega_R) \\ &p_m \to p \ \text{ in } W_0^{-1,q}(\Omega_R).\end{aligned} \tag{V.8.33}$$

It is immediately seen that $\boldsymbol{v}$ is a q-generalized solution to the Stokes system (V.5.1) (see Definition V.5.1) corresponding to $\boldsymbol{v}_* \equiv 0$ and by virtue of $(\text{V.8.32})_2$ to $\boldsymbol{F} \equiv 0$. Moreover, by $(\text{V.8.33})_1$, we deduce that, in the exterior of a ball of sufficiently large radius, $\boldsymbol{v}$ is in L^s, for $s > n/\alpha$ and so $\boldsymbol{v}$ is a q-generalized solution to the Stokes problem (V.0.1), (V.0.2) with

$\boldsymbol{F} \equiv \boldsymbol{v}_* \equiv \boldsymbol{v}_\infty \equiv 0$. Thus, recalling that $p \in L^q(\Omega)$, from Theorem V.3.4 we conclude $\boldsymbol{v} \equiv p \equiv 0$ in Ω. However, by virtue of $(\mathrm{V}.8.33)_{5,6}$, this conclusion contradicts $(\mathrm{V}.8.32)_1$ and, therefore, (V.8.31) is proved. From (V.8.31) and (V.8.30) we then obtain, in particular,

$$\|(|x|^{\alpha-1}+1)\boldsymbol{v}\|_{\infty,\Omega^{R/2}} + |\boldsymbol{v}|_{1,q} + \|p\|_q \le c\|(|x|^\alpha+1)\boldsymbol{F}\|_\infty, \quad \text{for all } q > n/\alpha. \tag{V.8.34}$$

Finally, since by the embedding Theorem II.3.4

$$\|\boldsymbol{v}\|_{\infty,\Omega_{R/2}} \le c\|\boldsymbol{v}\|_{1,r,\Omega_{R/2}}, \quad r > n,$$

estimate (V.8.5) becomes a consequence of this last inequality and (V.8.34). The proof of the theorem is complete. □

Exercise V.8.1 Let Ω and $\boldsymbol{F}$ be as in Theorem V.8.1. Show that, given

$$\boldsymbol{v}_* \in W^{1-1/q,q}(\partial\Omega), \quad g \in L^q(\Omega), \quad q > n/\alpha,$$

the problem

$$\begin{aligned}(\nabla\boldsymbol{v}, \nabla\boldsymbol{\psi}) - (\pi, \nabla\cdot\boldsymbol{\psi}) &= (\boldsymbol{F}, \nabla\boldsymbol{\psi}) \quad \text{for all } \boldsymbol{\psi} \in C_0^\infty(\mathbb{R}^n),\\ (\boldsymbol{v}, \nabla\chi) &= -(g, \chi), \quad \text{for all } \chi \in C_0^\infty(\mathbb{R}^n),\end{aligned}$$

admits one and only one solution such that

$$\boldsymbol{v} \in D^{1,q}(\Omega), \quad p \in L^q(\Omega), \quad (1+|x|^\alpha)\boldsymbol{v} \in L^\infty(\Omega).$$

Moreover, show that for all $R > \delta(\Omega^c)$ this solution satisfies the estimates

$$\|(|x|^{\alpha-1}+1)\boldsymbol{v}\|_{\infty,\Omega^R} + |\boldsymbol{v}|_{1,q} + \|p\|_q \le c(\|(|x|^\alpha+1)\boldsymbol{F}\|_\infty + \|g\|_q + \|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)}),$$

where we can take $\Omega^R \equiv \Omega$ if $q > n$.

V.9 Notes for the Chapter

Section V.1. The first existence and uniqueness theorems for the Stokes problem in an exterior domain Ω is due to Boggio (1910), for Ω^c a closed ball. In the same hypothesis on Ω, Oseen (1927, §§9.3,9.4) furnishes the explicit form of the Green's tensor. For an arbitrary exterior domain, Lamb (1932) has given a formal series development of a generic solution in terms of spherical harmonics. The first existence and uniqueness result in the general case can be found in the work of Odqvist (1930, §4).

The variational formulation (V.1.1) has been introduced by Ladyzhenskaya (1959b, §2). Lemma V.1.1 with $q = 2$ and Ω of class C^2 is due to Solonnikov & Ščadilov (1973, §3).

Section V.2. A weaker version of Theorem V.2.1 is proved by Finn (1965a, Theorem 2.5) and Ladyzhenskaya (1969, Chapter 2, §2). Seemingly, Finn has been the first to recognize that, for existence, the condition of zero flux of $\boldsymbol{v}_*$ through the boundary is not necessary (see Finn, *loc. cit.*, Remark on p. 371).

The solenoidal extension of the boundary data, given in (V.2.5), in the case $\boldsymbol{\omega} = \mathbf{0}$ is due to Ladyzhenskaya (1969, p. 41).

Section V.3. Lemma V.3.1 generalizes Lemma 4.2 of Fujita (1961). Theorem V.3.2, Theorem V.3.4, and Theorem V.3.5 are an extension of classical results due to Chang & Finn (1961). A weaker version of the latter can be found in Finn & Noll (1957). Theorem V.3.3 is due to me; see also Galdi & Simader (1990).

Section V.4. All results and methods are originally due to me. Notwithstanding, mainly in the literature of the early nineties, one can find a number of contributions by several authors, that cover, in part, some of these results. However, their approach is different than the one I introduced here.

Weaker versions of Lemma V.4.3 with $m = 0$, $n = 3$ and $q = 2$ were originally given by Masuda (1975, Proposition 1 (iii)) and Heywood (1980, Lemma 1).

Theorem V.4.6, for $m = 0$, $n = 3$ and $1 < q < 3/2$, was shown for the first time by Solonnikov (1973, Theorem 2.3). Generalizations of this result to higher values of q were first investigated by Maremonti & Solonnikov (1986); see also Maremonti & Solonnikov (1985). The extension of Solonnikov's result to arbitrary dimension $n \geq 3$ can be deduced from the work of Borchers & Sohr (1987). Lemma V.4.3 and Lemma V.4.4 and Theorem V.4.6 in the particular case where $m = 0$ and $n = 3$ can be deduced from the work of Maslennikova & Timoshin (1989, 1990). A way of avoiding quotient spaces in Theorem V.4.6 is to modify suitably the conditions at infinity. This view has been considered by Maremonti & Solonnikov (1990).

The validity of (V.4.15) with $m = 0$ in a more restricted class of functions has been disproved by Borchers & Miyakawa (1992). The results contained in Theorem V.4.8 have been the object of several researches. In this regard, we refer the reader to the work of Sohr & Varnhorn (1990), Kozono & Sohr (1991), Deuring (1990a, 1990b, 1990c, 1991), and Deuring & von Wahl (1989).

Existence, uniqueness, and estimates for strong solutions in weighted Sobolev spaces have been studied by Choquet-Bruhat & Christodoulou (1981), Specovius-Neugebauer (1986), Farwig (1990), Girault & Sequeira (1991) and Pulidori (1993).

Section V.5. Here we follow the ideas of Galdi & Simader (1990). Theorem V.5.1 in the case $n \geq 3$, $q \in (n/(n-1), n)$ and Ω of class $C^{2,\lambda}$, $\lambda > 0$, was first obtained by H. Kozono and H. Sohr in a preprint of 1989 and published later in 1991. In particular, in this paper we find a first systematic study of the Stokes problem in exterior domain in homogeneous Sobolev spaces. The estimates contained in Theorem V.5.1 when $q \in (1, n/(n-1)]$ were first derived by W. Borchers and T. Miyakawa in 1989 and published later in 1990. Generalizations of Theorem V.5.1 along the lines of Exercise V.5.1 are considered by Kozono & Sohr (1992b) and Farwig, Simader and Sohr (1993).

Most of the above results are reobtained, basically by the same methods, in the paper by Maslennikova & Timoshin (1994)

Theorem V.5.3 is due to me.

Weak solutions in weighted Sobolev spaces have been analyzed by Girault & Sequeira (1991), Pulidori (1993), Pulidori & Specovius-Neugebauer (1995) and Specovius-Neugebauer (1996).

Weak solutions in Lorentz spaces have been studied by Kozono & Yamazaki (1998).

Section V.7. Results of this section are essentially due to Galdi & Simader (1990), or else can be obtained as corollary to their work. However, the Stokes paradox, as presented here, was first formulated in the particular case of a domain exterior to a circle by Avudainayagam, Jothiram & Ramakrishna (1986). For further results related to the plane, exterior Stokes problem, in addition to the classical papers of Finn & Noll (1957) and Chang & Finn (1961), we refer the reader to the work of Sequeira (1981, 1983, 1986) and of Hsiao & McCamy (1981). Problem (V.7.1), (V.7.4) is related to the steady motion of a viscous fluid past a self-propelled body that is moving at constant small velocity. For this type of questions, see Pukhnacev (1990a, 1990b) and Galdi (1999a, 2002).

Section V.8. For results related to Theorem V.8.1, we refer to the paper of Novotný & Padula (1995).

VI

Steady Stokes Flow in Domains with Unbounded Boundaries

Nel dritto mezzo del campo maligno
vaneggia un pozzo assai largo e profondo
di cui suo loco dicerò l'ordigno.

DANTE, Inferno XVIII, vv. 4-6

Introduction

So far, with the exception of the half-space, we have considered flows occurring in domains with a *compact* boundary. Nevertheless, from the point of view of the applications it is very important to consider flows in domains Ω having an *unbounded* boundary, such as channels or pipes of possibly varying cross section. In studying these problems, however, due to the particular geometry of the region of flow, completely new features, which we are going to explain, appear. To this end, assume Ω to be an unbounded domain of $\mathbb{R}^n$ with $m > 1$ "exits" to infinity, of the type (see Section III.4.3)

$$\Omega = \bigcup_{i=0}^{m} \Omega_i,$$

where Ω_0 is a smooth compact subset of Ω while Ω_i, $i = 1, \ldots, m$, are disjoint domains which, in possibly different coordinate systems (depending on Ω_i) have the form

$$\Omega_i = \big\{x \in \mathbb{R}^n : x_n > 0, x' \equiv (x_1, \ldots, x_{n-1}) \in \Sigma_i(x_n)\big\}.$$

Here $\Sigma_i = \Sigma_i(x_n)$ are smoothly varying, simply connected domains in $\mathbb{R}^{n-1}$, bounded for each $x_n > 0$ with

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$$|\Sigma_i(x_n)| \geq \Sigma_0 = \text{const.} > 0.$$

To fix the ideas, we suppose that Ω has only two exits. Denote by Σ any bounded intersection of Ω with an $(n-1)$-dimensional plane, which in Ω_i reduces to Σ and by $\boldsymbol{n}$ a unit vector orthogonal to Σ, oriented from Ω_1 toward Ω_2, say. Owing to the incompressibility of the liquid and assuming adherence conditions at the boundary, we at once deduce that the flux Φ through Σ of the velocity field $\boldsymbol{v}(x', x_n)$ associated with a given motion is a constant, that is,

$$\Phi \equiv \int_\Sigma \boldsymbol{v} \cdot \boldsymbol{n} = \text{const.} \tag{VI.0.1}$$

Therefore, a natural question that arises is that of establishing existence of a flow subject to a given flux. Clearly, this condition alone may not be enough to determine the flow uniquely and, similarly to what we did for motions in exterior domains, we must prescribe a velocity field $\boldsymbol{v}_{\infty i}$ as $|x| \to \infty$ in the exits Ω_i. However, unlike the case of flows past a body, $\boldsymbol{v}_{\infty i}$ need *not* be constant and, in fact, if $\Phi \neq 0$, the corresponding $\boldsymbol{v}_{\infty i}$ *can be a constant vector only if*

$$\lim_{|x| \to \infty} |\Sigma_i(x_n)| = \infty. \tag{VI.0.2}$$

To see this, we observe that if $\boldsymbol{v}_{\infty i} = \text{const.}$ and $\boldsymbol{v}(x) \to \boldsymbol{v}_{\infty i}$ as $|x| \to \infty$ in Ω_i, uniformly (say), by the adherence conditions at the boundary it follows that $\boldsymbol{v}_{\infty i} = 0$ and so (VI.0.1) implies (VI.0.2) whenever $\Phi \neq 0$. Thus, if $|\Sigma_i|$ is uniformly bounded, $\boldsymbol{v}_{\infty i}$ can not be a constant and one has to figure out how to prescribe it. There are remarkable cases where $\boldsymbol{v}_{\infty i}$ is easily prescribed; this happens when the exits Ω_i, $i = 1, 2$, are cylindrical, namely,

$$\Sigma_i(x_n) = \Sigma_{0i} = \text{const.},$$

such as in tubes or pipes. In these situations it is reasonable to expect that the flow corresponding to a given flux Φ should tend, as $|x| \to \infty$, to the *Poiseuille solution of the Stokes equation in Ω_i corresponding to the flux Φ*, that is, to a pair $(\boldsymbol{v}_0^{(i)}, p_0^{(i)})$ where

$$\boldsymbol{v}_0^{(i)} = v_0^{(i)}(x')\boldsymbol{e}_n, \quad \nabla p_0^{(i)} = -C_i \boldsymbol{e}_n \tag{VI.0.3}$$

with $C_i = C_i(\Phi)$ (see Exercise VI.0.1), such that

$$\begin{gathered} \sum_{j=1}^{n-1} \frac{\partial^2 v_0^{(i)}(x')}{\partial x_j^2} = -C_i \ \text{ in } \Sigma_i, \\ v_0^i = 0 \ \text{ at } \partial\Sigma_i. \end{gathered} \tag{VI.0.4}$$

Thus, if $n = 3$ and the sections are circles of radius R_i, the solution to (VI.0.3), (VI.0.4) is the *Hagen–Poiseuille flow*

$$v_0^{(i)}(x') = C_i R_i^2 (1 - |x'|^2/R_i^2).$$

Likewise, for $n = 2$ and Ω_i a layer of depth d_i, $v_0^{(i)}$ reduces to the *Poiseuille flow*

$$v_0^{(i)}(x') = C_i d_i^2 (1 - x_1^2/d_i^2).$$

The problem of determining a motion in a region Ω with cylindrical exits, subject to a given flux Φ and tending in each exit to the Poiseuille solution corresponding to Φ, is known as *Leray's problem*; see Ladyzhenskaya (1959b, p. 175).

However, if it happens that one of the sections Σ_i is *only* uniformly bounded but *not* constant, then, in general, one does not know the explicit form of $\boldsymbol{v}_{\infty i}$ and, alternatively, one can prescribe at large distance in the exits a "growth" condition (Ladyzhenskaya & Solonnikov 1980, Problem 1.1). Of course, this condition must be such that, in the class of solutions verifying it, uniqueness is preserved. Moreover, once existence is established, one should successively try to analyze the structure of solutions as $|x| \to \infty$.

A further problem that arises when Σ is bounded is that the approach of generalized solutions used for flows in exterior domains (Theorem VI.2.1) is *not* directly applicable and one has to modify it appropriately. This fact is easily seen to be a consequence of (VI.0.1). In fact, using the Schwarz inequality and inequality (II.5.5) we obtain

$$|\Phi|^2 \le C|\Sigma|^{(n+1)/(n-1)} \int_\Sigma \nabla \boldsymbol{v} : \nabla \boldsymbol{v} \tag{VI.0.5}$$

which, for $|\Sigma|$ *uniformly bounded, implies an unbounded Dirichlet integral for* $\boldsymbol{v}$:

$$|\boldsymbol{v}|_{1,2} = \infty, \quad \text{unless } \Phi = 0.$$

Let us next suppose that Σ satisfies (VI.0.2). We may then prescribe a uniform (zero) velocity field $\boldsymbol{v}_{\infty i}$ at large distances in Ω_i, $i = 1, 2$. We shall distinguish the following two possibilities:

(i) $\displaystyle\int_0^\infty |\Sigma_i|^{-(n+1)/(n-1)} dx_n < \infty, \quad i = 1, 2,$

(ii) $\displaystyle\int_0^\infty |\Sigma_i|^{-(n+1)/(n-1)} dx_n = \infty, \quad i = 1, 2.$

In case (i) the condition of prescribed flux is compatible with the approach of generalized solutions, as a consequence of (VI.0.5). Nevertheless one must be careful in choosing the function space where such solutions are to be sought. Actually, if one required $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega)$, by the results of Section III.5 one would automatically impose zero flux through and would therefore exclude a priori all those solutions having $\Phi \neq 0$. Instead, one should look for solutions in the *larger* space $\widehat{\mathcal{D}}_0^{1,2}(\Omega)$, where the condition $\Phi \neq 0$ is allowed.

In case (ii) the non-zero flux condition again becomes noncompatible with the existence of generalized solutions. However, if

$$G_i \equiv \int_0^\infty |\Sigma|^{[(1-n)(q-1)-q]/(n-1)} dx_n < \infty, \quad i = 1, 2, \quad \text{some } q > 2,^1 \quad \text{(VI.0.6)}$$

since from (VI.0.1), the Hölder inequality and inequality (II.5.5)

$$|\Phi|^q G_i \le |\boldsymbol{v}|_{1,q},$$

we deduce that now *q-generalized* solutions may still exist and that the "natural" space where they should be sought is $\widehat{\mathcal{D}}_0^{1,q}(\Omega)$.

A last possibility arises when (VI.0.2) holds but the integrals G_i are infinite for *any* value of $q > 1$. In this case it is not clear in which space the problem has to be formulated.

Finally, we mention that, with the obvious modifications, all the above reasonings apply to the circumstance when one section Σ_1 (say) is bounded and the other is unbounded, as well as to the case where Ω has more than two exits to infinity.

The question of the unique solvability of the Stokes (and, more generally, nonlinear Navier–Stokes) problem in domains of the above types has been investigated by several authors. In particular, Amick (1977, 1978) first proved solvability when the sections are constant (see Chapter XII), giving an affirmative answer to Leray's problem.[2] The case of an unbounded cross section was first posed and uniquely solved by Heywood (1976, Theorem 11) in the special situation of the so-called *aperture domain*:

$$\Omega = \{x \in \mathbb{R}^n : x_n \neq 0 \text{ or } x' \in S\} \qquad \text{(VI.0.7)}$$

with S a bounded domain of $\mathbb{R}^{n-1}$ (see Section III.4.3, (III.4.4)). Successively, under general assumptions on the "growth" of Σ, the problem was thoroughly investigated by Amick & Fraenkel (1980) (see also Amick (1979) and Remark 3.1) when Ω is a domain in the plane having two exits to infinity. In particular, the authors show existence of solutions and pointwise asymptotic decay of the corresponding velocity fields.[3]. However, uniqueness is left out. It is interesting to observe that, unlike the case of an exterior domain, for the general class of regions of flow considered by Amick and Fraenkel, there is no Stokes paradox; see also Section VI.2 and Section VI.4.

The entire question was independently reconsidered within a different approach by Ladyzhenskaya & Solonnikov (1980), Solonnikov (1981,1983) and their associates; see Notes to this chapter. When Σ is uniformly bounded, these authors show, among other things, unique solvability in a class of solutions having a Dirichlet integral that is finite on every bounded subset Ω' of Ω and that may "grow" with a certain rate depending on Σ, as $\Omega' \to \Omega$; see also

[1] Notice that since $|\Sigma| \ge \Sigma_0 > 0$, in case (ii) the integrals G_i are infinite for any $q \le 2$.

[2] Under "small" flux condition in the nonlinear case; see Chapter XI.

[3] Under "small" flux condition in the nonlinear case, if Σ has a certain rate of "growth."

Remark 1.1. If, in particular, the exits Ω_i are cylindrical, the solution to the Stokes problem corresponding to a given flux tends in a well-defined sense to the corresponding Poiseuille solution in Ω_i. Likewise, if Σ is unbounded and satisfies condition (i), they prove existence of generalized solutions in $\widehat{\mathcal{D}}_0^{1,2}(\Omega)$ corresponding to $\boldsymbol{v}_{\infty i} \equiv 0$ and to a prescribed flux.

In case (ii), there is the remarkable contribution of Pileckas (1996a, 1996b, 1996c, 1997) who shows that in the particular case when each Ω_i is a body of revolution of the type

$$\left\{x \in \mathbb{R}^2 : \ x_n > 0, \ |x_1| < f_i(x_n)\right\}, \tag{VI.0.8}$$

the problem is uniquely solvable for any prescribed flux, provided f_i. satisfies (VI.0.6)[4] and a "global" Lipschitz condition (see (ii) at the beginning of Section 3). Furthermore, Pileckas shows that the decay rate of solutions is related to the inverse power of the functions f_i.

In the present chapter we prove existence and uniqueness of solutions to the Stokes problem in a domain with exits, when these exits have either constant sections Σ_i or unbounded Σ_i satisfying (i). Moreover, we shall perform an analysis of the pointwise asymptotic behavior either when Σ_i is constant. We also give some decay results when $\Sigma_i(x_n)$ becomes suitably unbounded as $|x| \to \infty$, and the exits are body of revolution as in (VI.0.8). However, these results are not sharp and we refer the reader to the cited papers of Pileckas for more complete results, obtained by completely different methods.

For simplicity, we shall describe the results in details only when the number m of exits is two, leaving to the reader the (simple) task of generalizing them to the case $m > 2$, and to the case when some of the exits are cylindrical, while the others have an unbounded section verifying (i).

Finally, in the last section of the chapter, we shall furnish a full treatment of the Stokes problem in the aperture domain (VI.0.7), which includes existence, uniqueness and L^q-estimates of solutions together with their asymptotic behavior. Unlike the previously mentioned cases, for domain (VI.0.7) the situation is rendered easier by the fact that the problem can be reduced to a similar problem in a half space where explicit representations of solutions are known; see Section IV.3 and Exercise IV.8.1.

Exercise VI.0.1 Show that for solutions to (VI.0.3), (VI.0.4) there is a one-to-one correspondence between the pressure drop $-C_i$ and the flux

$$\Phi_i = \int_{\Sigma_i} v_0^{(i)}(x')dx'.$$

In particular, show the existence of a positive constant $c_P = c_P(\Sigma_i, n)$ such that $C_i = c_P\Phi_i$. The constant c_P will be called the *Poiseuille constant. Hint*: Consider the following problem

$$\Delta\psi_i = -1\,, \ \text{in } \Sigma_i\,, \quad \psi_i|_{\partial\Sigma_i} = 0\,,$$

[4] In this case, we have $|\Sigma_i| = c(n)f_i^{n-1}$.

and use the linearity of problem (VI.0.3).

VI.1 Leray's Problem: Existence, Uniqueness, and Regularity

Let us consider a liquid performing a steady slow motion in a domain Ω ($\subset \mathbb{R}^n$) of class C^∞ [1] with two cylindrical ends, namely,

$$\Omega = \bigcup_{i=0}^{2} \Omega_i ,$$

with Ω_0 a compact subset of Ω and Ω_i, $i = 1,2$, disjoint domains, which in possibly different coordinate systems, are given by

$$\Omega_1 = \{x \in \mathbb{R}^n : x_n < 0, x' \in \Sigma_1\}$$
$$\Omega_2 = \{x \in \mathbb{R}^n : x_n > 0, x' \in \Sigma_2\}.$$

Here, Σ_i, $i = 1,2$, are C^∞-smooth, simply connected, bounded domains of the plane, if $n = 3$, while $\Sigma_i = (-d_i, d_i)$, $d_i > 0$, if $n = 2$. We denote by Σ a *cross section* of Ω, that is, any bounded intersection of Ω with an $(n-1)$-dimensional plane which in Ω_i reduces to Σ_i. Moreover, $\boldsymbol{n}$ indicates a unit vector orthogonal to Σ and oriented from Ω_1 toward Ω_2 (so that $\boldsymbol{n} = -\boldsymbol{e}_n$ in Ω_1 and $\boldsymbol{n} = \boldsymbol{e}_n$ in Ω_2).

The aim of this section is to solve the following *Leray's problem*: *Given $\Phi \in \mathbb{R}$, to determine a solution $\boldsymbol{v}, p$ to the Stokes system*

$$\left.\begin{aligned} \Delta \boldsymbol{v} &= \nabla p \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}\right\} \quad \text{in } \Omega \tag{VI.1.1}$$

such that

$$\begin{gathered} \boldsymbol{v} = 0 \quad \text{at } \partial\Omega \\ \int_\Sigma \boldsymbol{v} \cdot \boldsymbol{n} = \Phi \end{gathered} \tag{VI.1.2}$$

and

$$\boldsymbol{v} \to \boldsymbol{v}_0^{(i)} \quad \text{in } \Omega_i \text{ as } |x| \to \infty, \tag{VI.1.3}$$

[1] Namely, for every $x_0 \in \Omega$ there exists $r = r(x_0)$ such that $\partial\Omega \cap B_r(x_0)$ is a boundary portion of class C^∞. This assumption will imply, in particular, that the solutions we will determine are of class C^∞. Of course, we may relax the smoothness of Ω at the cost, however, of obtaining less regular solutions. Extension of results under weaker regularity assumptions on the boundary are left to the reader as an exercise.

where $\boldsymbol{v}_0^{(i)}$ *are the velocity fields* (VI.0.3) and (VI.0.4) *of the Poiseuille flow in* Ω_i *corresponding to the flux* Φ*.*

We shall now give a generalized formulation of this problem, similar to that furnished in Chapters IV and V for flows in domains with a compact boundary. Multiplying $(\text{VI.1.1})_1$ by $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and integrating by parts we deduce formally

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) = 0, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega). \tag{VI.1.4}$$

We have

Definition VI.1.1. A vector field $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ is called a *weak* (or *generalized*) *solution to Leray's problem* (VI.1.1)–(VI.1.3) if and only if

(i) $\boldsymbol{v} \in W^{1,2}_{loc}(\overline{\Omega})$;
(ii) $\boldsymbol{v}$ verifies (VI.1.4);
(iii) $\boldsymbol{v}$ is (weakly) divergence free in Ω;
(iv) $\boldsymbol{v}$ vanishes on $\partial\Omega$ (in the trace sense);
(v) $\displaystyle\int_\Sigma \boldsymbol{v} \cdot \boldsymbol{n} = \Phi$ (in the trace sense);
(vi) $(\boldsymbol{v} - \boldsymbol{v}_0^{(i)}) \in W^{1,2}(\Omega_i)$, $i = 1, 2$.

Evidently, conditions (ii)-(v) translate in a generalized form the corresponding properties (VI.1.1) and (VI.1.2), while (i) ensures a certain degree of regularity. Also, it is easy to see that (vi) implies the validity of (VI.1.3) in a well-defined sense. Actually, from the trace inequality of Theorem II.4.1 we deduce, with $\boldsymbol{w} \equiv \boldsymbol{v} - \boldsymbol{v}_0^{(2)}$,

$$\int_{\Sigma_2} |\boldsymbol{w}(x', x_n)|^2 \leq c \int_{t>x_n} \int_{\Sigma_2} \left[(\boldsymbol{w}^2 + \nabla \boldsymbol{w} : \nabla \boldsymbol{w}) \right] dt$$

where the constant c is independent of x_n. So, by (vi),

$$\int_{\Sigma_2} |\boldsymbol{w}(x', x_n)|^2 \to 0, \quad \text{as } |x| \to \infty \text{ in } \Omega_2,$$

and similarly in Ω_1.

Furthermore, it can be shown that to every weak solution we can associate a corresponding pressure field p. Actually, directly from Lemma IV.1.1 we find

Lemma VI.1.1 *Let* $\boldsymbol{v}$ *be a generalized solution to Leray's problem. Then there exists* $p \in L^2_{loc}(\Omega)$ *such that*

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) = (p, \nabla \cdot \boldsymbol{\psi}), \quad \textit{for all } \boldsymbol{\psi} \in C_0^\infty(\Omega). \tag{VI.1.5}$$

It is also simple to establish the smoothness of a weak solution $\boldsymbol{v}$ and the corresponding pressure field p. In fact, taking into account the regularity of Ω, from Theorem IV.4.1 and Theorem IV.5.1 we at once deduce the following result.

Theorem VI.1.1 *Let* $\boldsymbol{v}$ *be a weak solution to Leray's problem* (VI.1.1), (VI.1.3) *and let* p *be the pressure associated to* $\boldsymbol{v}$ *by* Lemma VI.1.1. *Then* $\boldsymbol{v}, p \in C^\infty(\overline{\Omega'})$, *for any bounded domain* $\Omega' \subset \Omega$.

The objective of the remaining part of this section will be to prove existence and uniqueness of a weak solution to Leray's problem. To this end, we need a suitable extension $\boldsymbol{a}$ (say) of the Poiseuille velocity fields $\boldsymbol{v}_0^{(i)}$ which will play the same role played by the field (V.2.5) which, in the case of an exterior domain, is used to extend the rigid body velocity field $\boldsymbol{V}$. Let us denote by $\boldsymbol{a}(x)$ a vector field enjoying the following properties:

(i) $\boldsymbol{a} \in W^{2,2}_{loc}(\overline{\Omega})$;
(ii) $\nabla \cdot \boldsymbol{a} = 0$ in Ω;
(iii) $\boldsymbol{a} = 0$ at $\partial\Omega$;
(iv) $\boldsymbol{a} = \boldsymbol{v}_0^{(1)}$ in Ω_1^R, $\boldsymbol{a} = \boldsymbol{v}_0^{(2)}$ in Ω_2^R, for some $R > 0$, where, for $a > 0$,

$$\Omega_1^a = \{x \in \Omega_1 : x_n < -a\}$$
$$\Omega_2^a = \{x \in \Omega_2 : x_n > a\}.$$

A way of constructing such a field will be described. Let $\zeta_i(x)$, $i = 1, 2$, be functions from $C^\infty(\mathbb{R}^n)$ such that

$$\zeta_i(x) = \begin{cases} 1 \text{ if } x \in \overline{\Omega_1^R} \\[2ex] 0 \text{ if } x \in \Omega - \overline{\Omega_1^{R/2}} \end{cases}$$

and set

$$\boldsymbol{V}(x) = \sum_{i=1}^{2} \zeta_i(x) \boldsymbol{v}_0^{(i)}.$$

Clearly, $\boldsymbol{V} \in C^\infty(\mathcal{A}_R)$ where

$$\mathcal{A}_R \equiv \Omega - \left[\overline{\Omega_1^R} \cup \overline{\Omega_2^R}\right].$$

Consider the problem

$$\nabla \cdot \boldsymbol{w} = -\nabla \cdot \boldsymbol{V} \quad \text{in } \mathcal{A}_R$$
$$\boldsymbol{w} \in W_0^{2,2}(\mathcal{A}_R)$$
$$\|\boldsymbol{w}\|_{2,2,\mathcal{A}_R} \le c \|\nabla \cdot \boldsymbol{V}\|_{1,2,\mathcal{A}_R}.$$

Since $\nabla \cdot \boldsymbol{V} \in W_0^{1,2}(\mathcal{A}_R)$ and

$$\int_{\mathcal{A}_R} \nabla \cdot \boldsymbol{V} = 0,$$

$\boldsymbol{w}$ exists, in view of Theorem III.3.3. Extend $\boldsymbol{w}$ to zero outside $\mathcal{A}_R$ and denote again by $\boldsymbol{w}$ such an extension. Evidently $\boldsymbol{w} \in W^{2,2}(\Omega)$ and so the field

$$\boldsymbol{a}(x) = \boldsymbol{V}(x) + \boldsymbol{w}(x)$$

satisfies all requirements (i)-(iv) listed previously.

We look for a generalized solution to (VI.1.1)–(VI.1.3) of the form

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a},$$

where

$$\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega).$$

Since, by inequality (II.5.5)

$$\mathcal{D}_0^{1,2}(\Omega) \subset W_0^{1,2}(\Omega),$$

$\boldsymbol{v}$ satisfies (i) and (iii)-(vi) of Definition 1.1, while, from (VI.1.4), $\boldsymbol{u}$ must solve the equation

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) = (\Delta \boldsymbol{a}, \boldsymbol{\varphi}), \quad \text{for all } \varphi \in \mathcal{D}(\Omega). \tag{VI.1.6}$$

The existence of $\boldsymbol{u}$ is readily established by means of the Riesz representation theorem. To this end, it suffices to show that the right-hand side of (VI.1.6) defines a linear functional in $\mathcal{D}_0^{1,2}(\Omega)$, i.e.,

$$|(\Delta \boldsymbol{a}, \boldsymbol{\varphi})| \leq c|\boldsymbol{\varphi}|_{1,2}, \tag{VI.1.7}$$

for some constant c (depending on $\boldsymbol{a}$) and for all $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$. We split Ω as follows:

$$\Omega = \Omega_1^R \cup \left[\Omega - \left(\Omega_1^R \cup \Omega_2^R\right)\right] \cup \Omega_2^R \equiv \Omega_1^R \cup \Omega_{0R} \cup \Omega_2^R, \tag{VI.1.8}$$

and observe that in each Ω_i^R the field $\boldsymbol{a}$ coincides with the Poiseuille solution $\boldsymbol{v}_0^{(i)}$ satisfying (VI.0.4). Therefore, we have

$$\int_{\Omega_i^R} \Delta \boldsymbol{a} \cdot \boldsymbol{\varphi} = \int_{\Omega_i^R} \Delta \boldsymbol{v}_0^{(1)} \cdot \boldsymbol{\varphi} = -C_1 \int_{-\infty}^{R} \left[\int_{\Sigma_1} \boldsymbol{\varphi} \cdot \boldsymbol{n} d\Sigma_1\right] dx_n = 0 \tag{VI.1.9}$$

since $\boldsymbol{\varphi}$ carries no flux. Likewise

$$\int_{\Omega_2^R} \Delta \boldsymbol{a} \cdot \boldsymbol{\varphi} = 0. \tag{VI.1.10}$$

In Ω_{0R}, by the Schwarz inequality and inequality (II.5.5), we have

$$\left| \int_{\Omega_{0R}} \Delta \boldsymbol{a} \cdot \varphi \right| \leq c\|\Delta \boldsymbol{a}\|_{2,\Omega_{0R}} |\boldsymbol{\varphi}|_{1,2,\Omega} \tag{VI.1.11}$$

and so (VI.1.7) follows from (VI.1.9)–(VI.1.11) and property (i) of $\boldsymbol{a}$. Existence is then acquired. To show uniqueness, let $\boldsymbol{v}_1$ be another weak solution corresponding to the same flux Φ and Poiseuille velocity fields $\boldsymbol{v}_0^{(i)}$. Then, it is readily shown that $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v}_1$ belongs to $\mathcal{D}_0^{1,2}(\Omega)$. In fact, we have in Ω_1^R

$$\boldsymbol{w} = \boldsymbol{u} - (\boldsymbol{v}_1 - \boldsymbol{v}_0^{(1)}) + (\boldsymbol{a} - \boldsymbol{v}_0^{(1)}) = \boldsymbol{u} - (\boldsymbol{v}_1 - \boldsymbol{v}_0^{(1)})$$

and, likewise, in Ω_2^R

$$\boldsymbol{w} = \boldsymbol{u} - (\boldsymbol{v}_1 - \boldsymbol{v}_0^{(2)}).$$

Therefore, taking into account condition (vi) of Definition VI.1.1 and that $\boldsymbol{v}$ is in $C^\infty(\Omega')$, for any bounded $\Omega' \subset \Omega$, we obtain

$$\boldsymbol{w} \in D^{1,2}(\Omega).$$

Since $\boldsymbol{w}$ is zero at the boundary, from Exercise VI.1.1 we have

$$\boldsymbol{w} \in D_0^{1,2}(\Omega)$$

and, $\boldsymbol{w}$ being solenoidal, we conclude

$$\boldsymbol{w} \in \widehat{\mathcal{D}}_0^{1,2}(\Omega).$$

However, by Exercise III.5.1,

$$\widehat{\mathcal{D}}_0^{1,2}(\Omega) = \mathcal{D}_0^{1,2}(\Omega)$$

so that

$$\boldsymbol{w} \in \mathcal{D}_0^{1,2}(\Omega).$$

This having been established, from (VI.1.4) it follows that

$$(\nabla \boldsymbol{w}, \nabla \boldsymbol{\varphi}) = 0, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega),$$

implying $\boldsymbol{w} = 0$ *a.e.* in Ω, which is what we wanted to prove.

For a more general uniqueness result we refer the reader to Exercise VI.2.2.

Exercise VI.1.1 Let Ω be an infinite "distorted pipe" of the type specified above and let $w \in D^{1,2}(\Omega)$. Assume that the trace of w at $\partial\Omega$ is zero. Show $w \in D_0^{1,2}(\Omega)$. *Hint*: Let ψ_R be a C^∞ function in Ω that is one in Ω_{0R} (see (VI.1.8)) and vanishes in Ω_i^{2R}, $i = 1, 2$. Then $\psi_R w \to w$ in $D_0^{1,2}(\Omega)$. Moreover, since $w = 0$ at $\partial\Omega$, $\psi_R w$ belongs to $W_0^{1,2}(\Omega_{02R})$ and therefore it can be approximated there by functions from $C_0^\infty(\Omega_{02R}) \subset C_0^\infty(\Omega)$.

In order to solve Leray's problem completely, it remains to study the asymptotic behavior of $\boldsymbol{v}$. This will follow as a corollary to a general result concerning estimates of solutions to the Stokes problem in a semi-infinite channel which we are going to derive.

Lemma VI.1.2 *Let*

$$\Omega = \{x \in \mathbb{R}^n : x_n > 0, x' \in \Sigma\}$$

with Σ a C^∞-smooth, bounded, and simply connected domain in $\mathbb{R}^{n-1}$. Given $\boldsymbol{f} \in C^\infty(\Omega')$, Ω' any bounded subset of Ω, denote by $\boldsymbol{u}, \tau \in C^\infty(\Omega')$ a solution to the problem

$$\left.\begin{aligned} \Delta \boldsymbol{u} &= \nabla \tau + \boldsymbol{f} \\ \nabla \cdot \boldsymbol{u} &= 0 \end{aligned}\right\} \text{ in } \Omega \qquad \text{(VI.1.12)}$$
$$\boldsymbol{u} = 0 \quad \text{at } \partial\Omega - \Sigma_0$$

with

$$\Sigma_0 = \left\{x \in \overline{\Omega} : x_n = 0\right\}.$$

For $s \geq 1$ and $\delta \in (0, s]$, set

$$\omega_s = \{x \in \Omega : s < x_n < s+1\},$$
$$\omega_{s,\delta} = \{x \in \Omega : s - \delta < x_n < s + \delta + 1\}.$$

Then, for all $m \geq 0$ and $q \geq 1$ the following estimate holds

$$\|\boldsymbol{u}\|_{m+2,q,\omega_s} + \|\nabla \tau\|_{m,q,\omega_s} \leq c\left(\|\boldsymbol{f}\|_{m,q,\omega_{s,\delta}} + \|\boldsymbol{u}\|_{1,q,\omega_{s,\delta}}\right) \qquad \text{(VI.1.13)}$$

where $c = c(m, q, n, \delta, \Sigma)$.[2]

Proof. Evidently, it is enough to prove (VI.1.13) for $s = 1$, since the estimate for arbitrary $s \geq 1$ follows by making the change of variable $x_n \to x_n - \xi$, $\xi \geq 0$. This will automatically imply that the constant c is independent of s. Choose $\psi \in C^\infty(\mathbb{R})$ such that $\psi(t) = 0$ for $t \leq 1$, $\psi(t) = 1$ for $t \geq 2$ and put

$$\psi_k(x) = \psi\left(k(k+1)x_n - k^2 + 2\right)\left[1 - \psi\left(k(k+1)x_n - k(2(k+1)+1) + 1\right]\right.$$

with k a positive integer. Of course,

$$\psi_k(x) = \begin{cases} 0 \text{ if } & x_n \leq 1 - 1/k \\ 1 \text{ if } & 1 - 1/(1+k) \leq x_n \leq 2 + 1/(1+k) \\ 0 \text{ if } & x_n \geq 2 + 1/k. \end{cases}$$

We also put

$$U_k = \{x \in \Omega : 1 - 1/(k+1) \leq x_n \leq 2 + 1/(1+k)\}.$$

Let $k_1 \in \mathbb{N}$ be such that

$$U_{k_1} \subseteq \omega_{1,\delta}.$$

For $k > k_1$, by setting

$$\boldsymbol{w}_k = \psi_k \boldsymbol{u}, \quad q_k = \psi_k \tau,$$

[2] The assumptions of regularity on $\boldsymbol{f}, \boldsymbol{u}$, and τ are made for the sake of simplicity. Actually, for fixed $m \geq 0$, $s \geq 1$, and $\delta \in (0, s]$, it would suffice to suppose $\boldsymbol{u} \in W^{1,q}(\omega_{s,\delta}), \tau \in L^q(\omega_{s,\delta})$, and $\boldsymbol{f} \in W^{m,q}(\omega_{s,\delta})$.

from (VI.1.11) it follows that

$$\left.\begin{aligned} \Delta \boldsymbol{w}_k &= \nabla q_k + \boldsymbol{f}_k + \boldsymbol{F}_k \\ \nabla \cdot \boldsymbol{w}_k &= g_k \end{aligned}\right\} \text{ in } D \qquad \text{(VI.1.14)}$$
$$\boldsymbol{w}_k = 0 \quad at\ \partial D,$$

where D is any C^∞-smooth, bounded domain containing U_{k_1} and

$$\begin{aligned} \boldsymbol{f}_k &= \psi_k \boldsymbol{f}, \\ \boldsymbol{F}_k &= 2\nabla\psi_k \cdot \nabla \boldsymbol{u} + \boldsymbol{u}\Delta\psi_k - \tau\nabla\psi_k, \\ g_k &= \nabla\psi_k \cdot \boldsymbol{u}. \end{aligned}$$

Fix $k_2 > k_1$, $k_2 \in \mathbb{N}$, and apply to (VI.1.14) the results of Theorem IV.6.1 with $m = 0$ along with those contained in Exercise IV.6.3 to obtain

$$\|\boldsymbol{u}\|_{2,q,U_{k_2}} \le c_1 \left(\|\boldsymbol{f}\|_{q,U_{k_1-1}} + \|\boldsymbol{F}_{k_2}\|_{q,U_{k_1-1}} + \|g_{k_2}\|_{1,q,U_{k_1-1}} \right). \qquad \text{(VI.1.15)}$$

Recalling the definitions of $\boldsymbol{F}_k$ and g_k we at once deduce

$$\|\boldsymbol{F}_{k_2}\|_{q,U_{k_1-1}} + \|g_k\|_{1,q,U_{k_1-1}} \le c_2 \left(\|\boldsymbol{u}\|_{1,q,U_{k_1-1}} + \|\tau\|_{q,U_{k_1-1}} \right)$$

which, possibly modifying τ by adding a suitable constant, in view of Lemma IV.1.1 in turn gives

$$\|\boldsymbol{F}_{k_2}\|_{q,U_{k_1-1}} + \|g_{k_2}\|_{1,q,U_{k_1-1}} \le c_3 \|\boldsymbol{u}\|_{1,q,U_{k_1-1}}.$$

Replacing this estimate back into (VI.1.15) furnishes

$$\|\boldsymbol{u}\|_{2,q,U_{k_2}} \le c_4 \left(\|\boldsymbol{f}\|_{q,U_{k_1-1}} + \|\boldsymbol{u}\|_{1,q,U_{k_1-1}} \right) \qquad \text{(VI.1.16)}$$

and so, in particular,

$$\|\boldsymbol{u}\|_{2,q,\omega_1} \le c_4 \left(\|\boldsymbol{f}\|_{q,\omega_{1,\delta}} + \|\boldsymbol{u}\|_{1,q,\omega_{1,\delta}} \right),$$

which, by virtue of $(\text{VI.1.11})_1$, proves (VI.1.13) for $m = 0$. We next choose $k_3 = k_2 + 1$ and apply to solutions to (VI.1.14) the results contained in Theorem IV.6.1 and Exercise IV.6.3 with $m = 1$, to deduce

$$\|\boldsymbol{u}\|_{2,q,U_{k_3}} \le c_5 \left(\|\boldsymbol{f}\|_{1,q,U_{k_2}} + \|\boldsymbol{F}_{k_3}\|_{1,q,U_{k_2}} + \|g_{k_3}\|_{2,q,U_{k_2}} \right). \qquad \text{(VI.1.17)}$$

Reasoning as before, we replace the obvious inequality

$$\|\boldsymbol{F}_{k_3}\|_{q,U_{k_2}} + \|g_{k_3}\|_{1,q,U_{k_2}} \le c_6 \|\boldsymbol{u}\|_{2,q,U_{k_2}}$$

into (VI.1.17) and use (VI.1.16) to recover

$$\|\boldsymbol{u}\|_{3,q,U_{k_3}} \le c_7 \left(\|\boldsymbol{f}\|_{1,q,U_{k_1-1}} + \|\boldsymbol{u}\|_{1,q,U_{k_1-1}} \right)$$

and so, in particular,

$$\|\boldsymbol{u}\|_{3,q,\omega_1} \le c_7 \left(\|\boldsymbol{f}\|_{1,q,\omega_{1,\delta}} + \|\boldsymbol{u}\|_{1,q,\omega_{1,\delta}} \right),$$

which, by $(\text{VI.1.12})_1$, proves (VI.1.13) for $m = 2$. Iterating this procedure as many times as we please, we prove (VI.1.13) for all $m \ge 0$. $\square$

Let us now come back to the asymptotic estimate for $\boldsymbol{v}$ and p. Recalling that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}$, from (VI.1.1)–(VI.1.3) we have

$$\left.\begin{array}{r}\Delta \boldsymbol{u} = \nabla \tau \\[2ex] \nabla \cdot \boldsymbol{u} = 0\end{array}\right\} \text{ in } \Omega_2^R$$
$$\boldsymbol{u} = 0 \text{ at } \partial\Omega_2^R - \Sigma_2^R \tag{VI.1.18}$$
$$\int_\Sigma \boldsymbol{u} \cdot \boldsymbol{n} = 0,$$

where

$$\tau = p - C_2 x_n, \quad \Sigma_2^R = \{x \in \Omega_2 : x_n = R\}.$$

(A system analogous to (VI.1.18) is verified in Ω_1^R.) Employing (VI.1.13) with $\delta = 1$, $s = R + j$, $j = 1, 2, \ldots$, $q = 2$, $\boldsymbol{f} \equiv 0$, and summing from $j = 1$ to $j = \infty$ it follows that

$$\|\boldsymbol{u}\|_{m+2,2,\Omega_2^{R+1}} + \|\nabla\tau\|_{m,2,\Omega_2^{R+1}} \leq 3c\|\boldsymbol{u}\|_{1,2,\Omega_2^R} \tag{VI.1.19}$$

for all $m \geq 0$. Since an analogous estimate holds with Ω_1 in place of Ω_2 and since $\boldsymbol{u}$, $\tau \in C^\infty(\Omega_{0R})$, for all $R > 0$ (Ω_{0R} defined in (VI.1.8)), we deduce

$$\boldsymbol{u} \in W^{m,2}(\Omega), \quad \text{for all } m \geq 0. \tag{VI.1.20}$$

By using the embedding Theorem II.3.4 along with (VI.1.20) it is then easily established that for each multi-index α with $|\alpha| \geq 0$, it holds that (see Exercise VI.1.2)

$$|D^\alpha \boldsymbol{u}(x)| \to 0 \text{ as } |x| \to \infty \text{ in } \Omega_i. \tag{VI.1.21}$$

Furthermore, by $(\text{VI.1.18})_1$ and (VI.1.20) we deduce $\nabla\tau \in W^{m,2}(\Omega)$ for all $m \geq 0$ and so

$$|D^\alpha \nabla\tau(x)| \to 0 \text{ as } |x| \to \infty \text{ in } \Omega_i, \tag{VI.1.22}$$

which completes the study of the asymptotic behavior.

The results obtained in this section can be summarized in

Theorem VI.1.2 *Let Ω satisfy the assumptions stated at the beginning of this section. Then, for every prescribed flux $\Phi \in \mathbb{R}$, Leray's problem admits one and only one generalized solution $\boldsymbol{v}, p$. This solution is in fact infinitely differentiable in the closure of every bounded subset of Ω and satisfies* (VI.1.1)–(VI.1.2) *in the ordinary sense. Furthermore, $\boldsymbol{v}$, together with all its derivatives of arbitrary order, tends to the corresponding Poiseuille velocity field in Ω_i as $|x| \to \infty$ and the same property holds for ∇p.*

Exercise VI.1.2 Let $\mathcal{C}$ be a semi-infinite cylinder of type Ω_2. Show that Theorem II.3.4 holds for $W^{m,q}(\mathcal{C})$. *Hint*: Let $\mathcal{C}_s = \{x \in \mathcal{C} : s < x_n < s+1\}$, $s = 0, 1, 2 \ldots$, and apply Theorem II.3.4 to $W^{m,q}(\mathcal{C}_s)$. The general case follows by noticing that the constants c_1, c_2, and c_3 entering the inequalities (II.3.17), (II.3.18) do not depend on s.

Exercise VI.1.3 Assume that instead of two exits to infinity, Ω_1 and Ω_2, the domain Ω has $m \geq 3$ exits $\Omega'_1, \dots, \Omega'_\ell$, where $\Omega'_1, \dots, \Omega'_j$ can be represented as Ω_1 ("upstream" exits) and $\Omega'_{j+1}, \dots, \Omega'_\ell$ as Ω_2 ("downstream" exits). Assume also that

$$\Omega - \cup_{i=1}^{\ell} \Omega'_i$$

is bounded and that Ω is of class C^∞. Denote by Φ_i the fluxes in Ω'_i. Then show that, for every choice of Φ_i satisfying the compatibility condition of zero total flux

$$\sum_{i=1}^{j} \Phi_i = \sum_{i=j+1}^{\ell} \Phi_i,$$

Leray's problem is solvable in Ω.

Remark VI.1.1 As already noticed at the beginning of this chapter, when the exits Ω_i have a uniformly *bounded but not necessarily constant* cross section, one does not know, in general, the explicit form of the limiting velocity field $\boldsymbol{v}_{\infty i}$, as $|x| \to \infty$ in Ω_i. However, in such a case, one can alternatively prescribe "growth" conditions at large distances (Ladyzhenskaya & Solonnikov 1980, Problem 1). For this type of question we wish to mention the following result, whose proof can be found in the paper of Ladyzhenskaya & Solonnikov. ■

Theorem VI.1.3 *Let*

$$\Omega = \{x \in \mathbb{R}^n : x_n \in \mathbb{R}, x' \in \Sigma(x_n)\},$$

with $\Sigma = \Sigma(x_n)$ a simply connected domain of $\mathbb{R}^{n-1}$, possibly varying with x_n. Assume there exist two constants Σ_1 and Σ_2 such that

$$0 < \Sigma_1 \leq |\Sigma(x_n)| < \Sigma_2 < \infty,$$

and that, in addition, there exists $\boldsymbol{a} \in W^{1,2}_{loc}(\Omega)$ such that

(i) $\nabla \cdot \boldsymbol{a} = 0$ *in* Ω*;*

(ii) $\displaystyle\int_\Sigma \boldsymbol{a} \cdot \boldsymbol{n} = 1$*;*

(iii) $|\boldsymbol{a}|_{1,2,\Omega_{t,t+1}} + |\boldsymbol{a}|_{1,2,\Omega_{-t,-t+1}} \leq c$, *for all* $t \geq 1$,

where, for $s \in \mathbb{R}$,

$$\Omega_{s,s+1} = \Omega \cap \{x \in \mathbb{R}^n : s < x_n < s+1\} \tag{VI.1.23}$$

and c is a constant independent of t. (Such a field certainly exists if the boundary $\partial\Omega$ is sufficiently smooth.) Then, for any $\Phi \in \mathbb{R}$, there exists a pair

$$\boldsymbol{v}, p \in C^\infty(\Omega) \cap W^{1,2}_{loc}(\overline{\Omega})^3$$

solving the Stokes problem

[3] Clearly, reasoning as in the case of a constant cross section, if Ω is of class C^∞, then $\boldsymbol{v}, p \in C^\infty(\overline{\Omega}')$, for all bounded $\Omega' \subset \Omega$.

$$\left.\begin{aligned} \Delta \boldsymbol{v} &= \nabla p \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}\right\} \quad \textit{in } \Omega$$

$$\boldsymbol{v} = 0 \quad \textit{at } \partial\Omega$$

$$\int_\Sigma \boldsymbol{v} \cdot \boldsymbol{n} = \Phi$$

Furthermore, the velocity field $\boldsymbol{v}$ *satisfies for all* $t \geq 1$ *and all* $s \in \mathbb{R}$ *the estimates*

$$\begin{aligned} &\int_{\Omega_t} \nabla \boldsymbol{v} : \nabla \boldsymbol{v} \leq c_1 t \\ &\int_{\Omega_{s,s+1}} \nabla \boldsymbol{v} : \nabla \boldsymbol{v} \leq c_2, \end{aligned} \tag{VI.1.24}$$

where

$$\Omega_t = \Omega \cap \{x \in \mathbb{R}^n : |x_n| < t\}$$

and c_1, c_2 *are constants independent of* t *and* s, *respectively. Finally, if* $\boldsymbol{w}, \pi$ *is another solution corresponding to the same flux* Φ *and satisfying a growth condition of the type* $(\mathrm{VI.1.24})_1$, *then* $\boldsymbol{w} \equiv \boldsymbol{v}$, $\nabla \pi \equiv \nabla p$.

For a more general uniqueness result related to the above solutions, we refer the reader to Exercise VI.2.2.

VI.2 Decay Estimates for Flow in a Semi-infinite Straight Channel

The next objective is to establish the rate at which solutions determined in the previous section decay to the corresponding Poiseuille flow. We shall show that they decay exponentially fast as $|x| \to \infty$. This result will be achieved as a corollary to a more general one holding for a large class of motions that includes those determined in Theorem VI.1.2.

We shall restrict our attention to flows occurring in a straight cylinder $\Omega = \{x_n > 0\} \times \Sigma$, where the cross section Σ is a C^∞-smooth, bounded and simply connected domain in $\mathbb{R}^{n-1}$, even though some of the results can be extended to a more general class of domains; see Exercise VI.2.1. The cross section at distance a from the origin is denoted by $\Sigma(a)$, despite all cross sections having the same shape and size. Denote by $\boldsymbol{u}, \tau$ a solution to the problem

$$\begin{aligned} &\left.\begin{aligned} \Delta \boldsymbol{u} &= \nabla \tau \\ \nabla \cdot \boldsymbol{u} &= 0 \end{aligned}\right\} \quad \text{in } \Omega \\ &\boldsymbol{u} = 0 \quad \text{at } \partial\Omega - \Sigma(0) \\ &\int_\Sigma \boldsymbol{u} \cdot \boldsymbol{n} = 0. \end{aligned} \tag{VI.2.1}$$

For simplicity, we assume $\boldsymbol{u}, \tau$ regular, that is, infinitely differentiable in the closure of any bounded subset of Ω. We also note, however, that the same conclusions may be reached merely assuming $\boldsymbol{u}$ and τ to possess the same regularity of generalized solutions to Leray's problem. Our first goal is to show that every regular solution to (VI.2.1) with $\boldsymbol{u}$ satisfying a general "growth" condition as $|x| \to \infty$ has, in fact, square summable gradients over the whole of Ω. Successively, we prove that these solutions decay exponentially fast in the Dirichlet integral, i.e.,

$$\|\boldsymbol{u}\|^2_{1,2,\Omega^R} \le c\|\boldsymbol{u}\|^2_{1,2,\Omega} \exp(-\sigma R), \tag{VI.2.2}$$

where

$$\Omega^a = \{x \in \Omega : x_n > a\}$$

and c, σ are constants depending on Σ. Once (VI.2.2) has been established, it is easy to prove that $\boldsymbol{u}$ and its derivatives decay exponentially fast. Actually, combining (VI.2.2) and (VI.1.19) (with $\Omega_2^R = \Omega^R$) gives

$$\|\boldsymbol{u}\|_{m+2,2,\Omega^{R+1}} + \|\nabla\tau\|_{m,2,\Omega^{R+1}} \le c_1\|\boldsymbol{u}\|_{1,2,\Omega} \exp(-\sigma R/2) \tag{VI.2.3}$$

and so, using the results of Exercise VI.1.2 into (VI.2.3), we obtain

$$|D^\alpha \boldsymbol{u}(x)| + |D^\alpha \nabla\tau(x)| \le c_2\|\boldsymbol{u}(x)\|_{1,2,\Omega} \exp(-\sigma x_n/2) \tag{VI.2.4}$$

for every $x \in \Omega$ with $x_n \ge 1$ and every $|\alpha| \ge 0$.

Remark VI.2.1 Estimate (VI.2.4) implies, in particular, that as $|x| \to \infty$, $\tau(x)$ tends to some constant, exponentially fast. Actually, denoting by $x = (x', x_n)$, $y = (y', y_n)$ two arbitrary points in $\overline{\Omega}^1$ and applying the mean value theorem, we have for some $\eta' \in \Sigma(y_n)$

$$|\tau(x) - \tau(y)| \le \left| \int_{x_n}^{y_n} \frac{\partial\tau(x', \xi)}{\partial\xi} d\xi \right| + \sum_{i=1}^{n-1} |D_i\tau(\eta', y_n)||x_i - y_i|$$

which, by (VI.2.4), implies that $\tau(x)$ tends to a constant τ_1 (say). Then, the stated property follows from the identity

$$\tau(x', x_n) = \tau_1 + \int_{x_n}^{\infty} \frac{\partial\tau(x', \xi)}{\partial\xi} d\xi$$

and again from the estimate (VI.2.4). ■

To recover the fundamental estimate (VI.2.2) we need some results concerning differential inequalities which we are going to show.

Lemma VI.2.1 *Let $y \in C^1(\mathbb{R}_+)$ be a non-negative function satisfying the inequality*

$$ay(t) \le b + y'(t), \quad \textit{for all } t \ge 0, \tag{VI.2.5}$$

where $a > 0$, $b \geq 0$. *Then, if* [1]

$$\liminf_{t\to\infty} y(t)e^{-at} = 0, \tag{VI.2.6}$$

it follows that $y(t)$ *is uniformly bounded and we have*

$$\sup_{t\geq 0} y(t) \leq b/a. \tag{VI.2.7}$$

Proof. From (VI.2.5) it follows that

$$-\frac{d}{dt}[y(t)e^{-at}] \leq be^{-at}$$

which, once integrated from t to t_1 ($> t$), furnishes

$$-y(t_1)e^{-at_1} + y(t)e^{-at} \leq \frac{b}{a}[e^{-at} - e^{-at_1}].$$

If we take the inferior limit of both sides of this relation as $t_1 \to \infty$ and use (VI.2.6), we then deduce (VI.2.7). □

Lemma VI.2.2 *Let* $\beta \leq \infty$ *and let* y *be a real, non-negative continuous function in* $[0, \beta)$ *such that*

$$y \in C^1(0, \beta),$$

$$\lim_{t\to\beta} y(t) = 0.$$

Then, if y *satisfies the integro-differential inequality*

$$y'(t) + a\int_t^\beta y(s)ds \leq by(t), \quad \textit{for all } t \in (0, \beta) \tag{VI.2.8}$$

with $a > 0$ *and* $b \in \mathbb{R}$,[2] *it follows that*

$$y(t) \leq ky(0)\exp(-\sigma t), \quad \textit{for all } t \in (0, \beta)\ , \tag{VI.2.9}$$

where

$$k = \frac{\sqrt{b^2 + 4a}}{\sigma}, \quad \sigma = \tfrac{1}{2}\left(\sqrt{b^2 + 4a} - b\right).$$

[1] Notice that the assumption $b \geq 0$ is necessary for (VI.2.6) to hold.
[2] Notice that if $b < 0$, (VI.2.8) at once implies (VI.2.9) with $k = 1$ and $\sigma = -b$.

Proof. Making the change of variable

$$\psi(t) = y(t)e^{-bt},$$

(VI.2.8) gives

$$\psi'(t) + a\int_t^\beta e^{-b(t-s)}\psi(s)ds \le 0.$$

From this relation, setting

$$F(t) = \psi(t) + \delta\int_t^\beta e^{-b(t-s)}\psi(s)ds, \quad \delta > 0,$$

we recover

$$\begin{aligned} F'(t) + \delta F(t) = \psi'(t) + a&\int_t^\beta e^{-b(t-s)}\psi(s)ds \\ &+(\delta^2 - \delta b - a)\int_t^\beta e^{-b(t-s)}\psi(s)ds \le 0 \end{aligned} \tag{VI.2.10}$$

provided we choose δ as the positive root to the equation

$$\delta^2 - \delta b - a = 0,$$

that is,

$$2\delta = b + \sqrt{b^2 + 4a}.$$

Integrating the differential inequality in (VI.2.10) furnishes

$$F(t) \le F(0)e^{-\delta t},$$

which can be equivalently rewritten as

$$y(t) + \delta\int_t^\beta y(s)ds \le F(0)e^{-(\delta-b)t}. \tag{VI.2.11}$$

We now estimate $F(0)$ in terms of $y(0)$. From (VI.2.11), setting

$$\sigma_1 = 2\delta - b,$$

it follows that

$$-\frac{d}{dt}\left[e^{\delta t}\int_t^\beta y(s)ds\right] \le F(0)e^{-\sigma_1 t}$$

which, upon integration from zero to β, gives

$$\int_0^\beta y(s)ds \le F(0)\frac{1-e^{-\sigma_1\beta}}{\sigma_1}.$$

If we substitute the value of $F(0)$ into this last inequality we deduce

$$\int_0^\beta y(s)ds \le y(0)\frac{1-e^{-\sigma_1\beta}}{\sigma_1-\delta(1-e^{-\sigma_1\beta})}$$

and so we obtain

$$F(0)=y(0)+\delta\int_0^\beta y(s)ds \le \frac{y(0)\sigma_1}{\sigma},$$

which along with (VI.2.11) completes the proof of the lemma. □

We are now ready to show the main results of this section.

Theorem VI.2.1 *Let* $\boldsymbol{u},\tau$ *be a regular solution to* (VI.2.1) *with*

$$\liminf_{x_n\to\infty}\left(\int_0^{x_n}\left[\int_{\Sigma(\xi)}\nabla\boldsymbol{u}:\nabla\boldsymbol{u}d\Sigma\right]d\xi\right)e^{-ax_n}=0, \tag{VI.2.12}$$

where

$$a^{-1}\equiv\left(\tfrac{1}{2}+c_0\right)\sqrt{\mu},$$

c_0 *is the constant specified in* (VI.2.14), *and* μ *is the Poincaré constant for* Σ (*see* (II.5.3) *and* (VI.2.15)). *Then*

$$|\boldsymbol{u}|_{1,2}<\infty.$$

Proof. Multiplying both sides of (VI.2.1) by $\boldsymbol{u}$ and integrating by parts in $(0,x_n)\times\Sigma$ we obtain

$$\begin{aligned}G(x_n)&\equiv\int_0^{x_n}\left[\int_{\Sigma(\xi)}\nabla\boldsymbol{u}:\nabla\boldsymbol{u}d\Sigma\right]dx_n\\&=\int_{\Sigma(x_n)}\left(\tau u_n-\tfrac{1}{2}\frac{\partial u^2}{\partial x_n}\right)-\int_{\Sigma(0)}\left(\tau u_n-\tfrac{1}{2}\frac{\partial u^2}{\partial x_n}\right).\end{aligned}$$

If we integrate this relation from t to $t+1$, $t\ge 0$, we have

$$\int_t^{t+1}G(x_n)dx_n=\int_{\Omega_{t,t+1}}\left(\tau u_n-\tfrac{1}{2}\frac{\partial u^2}{\partial x_n}\right)+B, \tag{VI.2.13}$$

where $\Omega_{t,t+1}$ is defined in (VI.1.23) and

$$B\equiv-\int_{\Sigma(0)}\left(\tau u_n-\tfrac{1}{2}\frac{\partial u^2}{\partial x_n}\right).$$

Let us consider the problem

$$\begin{aligned}&\nabla \cdot \boldsymbol{\omega} = u_n \quad \text{in } \Omega_{t,t+1}\\&\boldsymbol{\omega} \in W_0^{1,2}(\Omega_{t,t+1})\\&|\boldsymbol{\omega}|_{1,2,\Omega_{t,t+1}} \le c_0 \|u_n\|_{2,\Omega_{t,t+1}}.\end{aligned} \tag{VI.2.14}$$

Since

$$\int_{\Omega_{t,t+1}} u_n = 0,$$

from Theorem III.3.1 and Lemma III.3.3 problem (VI.2.14) admits a solution with a constant c_0 *independent of* t. Thus, from (VI.2.13) and (VI.2.1) it follows that

$$\begin{aligned}\int_t^{t+1} G(x_n)dx_n &= \int_{\Omega_{t,t+1}} \left(-\nabla\tau \cdot \boldsymbol{\omega} - \tfrac{1}{2}\frac{\partial u^2}{\partial x_n}\right) + B\\&= \int_{\Omega_{t,t+1}} \left(-\nabla \boldsymbol{u} : \nabla\boldsymbol{\omega} - \tfrac{1}{2}\frac{\partial u^2}{\partial x_n}\right)\\&\le \left(c_0 + \tfrac{1}{2}\right) \|\boldsymbol{u}\|_{2,\Omega_{t,t+1}} |\boldsymbol{u}|_{1,2,\Omega_{t,t+1}} + B.\end{aligned}$$

We next observe that, since $\boldsymbol{u}$ vanishes at $\partial\Omega$, $\mu = \mu(\Sigma) > 0$ (the *Poincaré constant for* Σ) exists such that

$$\|\boldsymbol{u}\|_{2,\Sigma}^2 \le \mu \|\nabla \boldsymbol{u}\|_{2,\Sigma}^2; \tag{VI.2.15}$$

see (II.5.3). From (II.5.5) and Exercise II.5.2 we may give the following estimate for μ:

$$\mu \le \begin{cases} \tfrac{1}{2}|\Sigma| & \text{if } \ n = 3\\[4pt] \dfrac{(2d)^2}{\pi^2} & \text{if } \ n = 2.\end{cases}$$

By using this inequality, we obtain

$$y(t) \equiv \int_t^{t+1} G(x_n)dx_n \le \sqrt{\mu}\left(c_0 + \tfrac{1}{2}\right) |\boldsymbol{u}|_{1,2,\Omega_{t,t+1}}^2 + B.$$

Since

$$|\boldsymbol{u}|_{1,2,\Omega_{t,t+1}}^2 = \frac{dy}{dt},$$

the preceding inequality furnishes

$$ay(t) \le b + \frac{dy(t)}{dt},$$

where a is defined in the statement of the theorem. Thus, we recover that $y(t)$ satisfies (VI.2.5) with $b = |B|$. Furthermore, it is readily shown that, in view of (VI.2.12), $y(t)$ also satisfies (VI.2.6) and consequently Lemma VI.2.1 implies

$$\int_t^{t+1} G(x_n)dx_n \le \frac{|B|}{a}, \quad \text{for all } t > 1. \tag{VI.2.16}$$

This inequality yields

$$\ell \equiv \lim_{x_n \to \infty} G(x_n) = |\boldsymbol{u}|_{1,2,\Omega} < \infty.$$

Actually, since $G(x_n)$ is monotonically increasing in x_n, ℓ exists (either finite or infinite) and (VI.2.16) then implies $\ell < \infty$. The theorem is proved. □

Remark VI.2.2 The previous theorem furnishes, in particular, that all regular solutions to (VI.2.1) satisfying (VI.1.12) must decay to zero uniformly, according to (VI.1.21) and (VI.1.22). This follows from (VI.1.19) and Exercise VI.1.2. ■

Exercise VI.2.1 Let

$$\Omega = \left\{ x \in \mathbb{R}^n : x_n > 0, x' \in \Sigma(x_n) \right\},$$

with $\Sigma(x_n)$ a smooth, simply connected domain of $\mathbb{R}^{n-1}$, possibly varying with x_n and satisfying the assumptions of Theorem VI.1.3. Assume Ω uniformly Lipschitz, i.e., for every $x_0 \in \partial\Omega$ there is $B_r(x_0)$ with r independent of x_0 such that $\partial\Omega \cap B_r(x_0)$ is a boundary portion of class $C^{0,1}$, with a Lipschitz constant *independent* of x_0. Show that Theorem VI.2.1 can be extended to such a domain Ω. *Hint*: It suffices to show that problem (VI.2.14) is solvable with a constant c_0 independent of t. This fact, however, can be established via the hypotheses on Ω and with the aid of estimate (III.3.13).

Exercise VI.2.2 (Ladyzhenskaya & Solonnikov 1980). Let

$$\Omega = \left\{ x \in \mathbb{R}^n : x_n \in \mathbb{R}, x' \in \Sigma(x_n) \right\},$$

and suppose that Ω and Σ satisfy the same assumptions of Exercise VI.2.1. Show that if $\boldsymbol{u}, \tau$ is a regular solution to $(\text{VI.2.1})_{1,2}$ vanishing at $\partial\Omega$, having zero flux through Σ and satisfying (VI.2.12), then $\boldsymbol{u} \equiv \nabla\tau \equiv 0$.

In the next theorem we establish the fundamental inequality (VI.2.2).

Theorem VI.2.2 *Let $\boldsymbol{u}, \tau$ be a regular solution to* (VI.2.1) *satisfying* (VI.2.12). *Then*

$$|\boldsymbol{u}|_{1,2} < \infty$$

and, for all $R > 0$, the following inequality holds:

$$\|\boldsymbol{u}\|^2_{1,2,\Omega^R} \le c\|\boldsymbol{u}\|^2_{1,2,\Omega} \exp(-\sigma R), \tag{VI.2.17}$$

with

$$c = \frac{2(c_0^2+2)^{1/2}}{(c_0^2+2)^{1/2} - c_0}$$

$$\sigma = \frac{1}{\sqrt{\mu}} \left[(c_0^2+2)^{1/2} - c_0 \right],$$

where c_0 is the constant specified in (VI.2.14) *and μ is the Poincaré constant for Σ. Moreover, for all $|\alpha| \ge 0$, $\boldsymbol{u}, \tau$ satisfy the pointwise estimate* (VI.2.4).

Proof. By Theorem VI.2.1 and what we have observed at the beginning of this section, we have to show only the validity of estimate (VI.2.17). For the sake of simplicity we shall restrict ourselves to treat the case $n = 3$; also, Cartesian coordinates will be denoted by x_1, x_2, x_3 and x, y, z, indifferently. Proceeding as in the proof of the previous theorem we may write the identity

$$-\int_z^{z_1}\left(\int_{\Sigma(\zeta)}\nabla \boldsymbol{u} : \ \nabla \boldsymbol{u} d\Sigma\right)d\zeta$$
$$= \int_{\Sigma(z_1)}\left(\tau u_3 - \tfrac{1}{2}\frac{\partial u^2}{\partial z}\right) - \int_{\Sigma(z)}\left(\tau u_3 - \tfrac{1}{2}\frac{\partial u^2}{\partial z}\right). \tag{VI.2.18}$$

From Theorem VI.2.1 and Lemma VI.1.2 we know that $\boldsymbol{u}, \nabla\tau \in W^{m,2}(\Omega)$ for all $m \geq 0$ and so, in particular, it easily follows that

$$\imath(z_1) \equiv \int_{\Sigma(z_1)} \tau u_3(x', z_1)dx' = o(1) \ \text{ as } \ z_1 \to \infty. \tag{VI.2.19}$$

In fact, setting

$$\overline{\tau}(z_1) = \frac{1}{|\Sigma|}\int_\Sigma \tau(x', z_1)dx',$$

from $(\text{VI.2.1})_4$ and Poincaré's inequality (II.5.10) it follows that

$$|\imath(z_1)| = \left|\int_{\Sigma(z_1)}(\tau - \overline{\tau})u_3(x', z_1)dx'\right| \leq c|\tau|_{1,2,\Sigma}\|\boldsymbol{u}\|_{2,\Sigma}$$

and (VI.2.19) becomes a consequence of (VI.1.21) and (VI.1.22). Thus, from (VI.2.19) and (VI.1.21), by letting $z_1 \to \infty$ into (VI.2.18), we find

$$H(z) \equiv \int_z^\infty\left(\int_{\Sigma(\zeta)}\nabla\boldsymbol{u} : \nabla\boldsymbol{u} d\Sigma\right)d\zeta = \int_{\Sigma(z)}\left(\tau u_3 - \tfrac{1}{2}\frac{\partial u^2}{\partial z}\right). \tag{VI.2.20}$$

We next integrate both sides of (VI.2.20) between $t + \ell$ and $t + \ell + 1$ with ℓ a non-negative integer to obtain

$$\int_{t+\ell}^{t+\ell+1} H(z) = \int_{t+\ell}^{t+\ell+1}\int_{\Sigma(z)}\tau u_3 - \tfrac{1}{2}\int_{\Sigma(t+\ell+1)}u^2 + \tfrac{1}{2}\int_{\Sigma(t+\ell)}u^2. \tag{VI.2.21}$$

By writing $u_3 = \nabla \cdot \boldsymbol{\omega}$, with $\boldsymbol{\omega}$ a solution to (VI.2.14) and by arguing as in the proof of Theorem VI.2.1, from (VI.2.21) it follows that

$$\int_{t+\ell}^{t+\ell+1} H(z) \leq c_0\sqrt{\mu}|\boldsymbol{u}|^2_{1,2,\Omega_{t+\ell,t+\ell+1}} - \tfrac{1}{2}\int_{\Sigma(t+\ell+1)}u^2 + \tfrac{1}{2}\int_{\Sigma(t+\ell)}u^2.$$

Summing both sides of this relation from $\ell = 0$ to $\ell = \infty$ and observing that, by Remark VI.2.1,

$$\lim_{z\to\infty}\int_{\Sigma(z)} u^2(x', z) = 0,$$

leads to

$$\int_t^\infty H(z) \le c_0\sqrt{\mu}H(t) + \tfrac{1}{2}\int_{\Sigma(t)} u^2. \qquad \text{(VI.2.22)}$$

Since, by (VI.2.15), we have

$$\int_{\Sigma(t)} u^2 \le \mu \int_{\Sigma(t)} \nabla\boldsymbol{u} : \nabla\boldsymbol{u} = -\mu H'(t),$$

inequality (VI.2.22) finally gives

$$H'(t) + \frac{2}{\mu}\int_t^\infty H \le \frac{2c_0}{\sqrt{\mu}}H(t)$$

which, by Lemma VI.2.2, completes the proof of the theorem. □

VI.3 Flow in Unbounded Channels with Unbounded Cross Sections. Existence, Uniqueness, and Regularity

In the present section we shall investigate existence and uniqueness of steady, slow motions of a viscous liquid in a domain $\Omega \subset \mathbb{R}^n, n = 2, 3$, with $m \ge 2$ "exits" to infinity Ω_i, $i = 1, \dots, m$, whose cross sections become suitably unbounded at large distances. To start, we shall assume $m = 2$ and that the domains Ω_i are bodies of rotation (see Section III.4.3). Several generalizations will be considered in Exercise VI.3.1-Exercise VI.3.4. Specifically, we take

$$\Omega = \bigcup_{i=0}^{2} \Omega_i$$

with Ω_0 a compact subset of $\mathbb{R}^n$ and Ω_i, $i = 1, 2$, disjoint domains which, in possibly different coordinate systems, are given by

$$\Omega_i = \{x \in \mathbb{R}^n : x_n > 0, |x'| < f_i(x_n)\},$$

where the functions f_i satisfy, for all $t, t_1, t_2 > 0$ and $i = 1, 2$,

(i) $f_i(t) \ge f_0 = \text{const.} > 0$;
(ii) $|f_i(t_2) - f_i(t_1)| \le M|t_2 - t_1|$

with M a positive constant. Furthermore, we shall assume f_i (and therefore Ω) of class C^∞.[1] We shall also suppose that the planar cross section $\Sigma_i = \Sigma_i(x_n)$

[1] See footnote 1 in Section VI.1.

of Ω_i, $i = 1, 2$, perpendicular to the axis $x' = 0$ and passing through the point $(0, x_n)$ tends to infinity as $|x| \to \infty$ in such a way that

$$\int_0^\infty f_i^{-(n+1)}(t)dt < \infty, \quad i = 1, 2. \tag{VI.3.1}$$

This condition is equivalent to condition (i) stated in the introduction to the chapter.

We denote, as usual, by Σ a *cross section* of Ω, that is, any bounded intersection of Ω with an $(n-1)$-dimensional plane that in Ω_i, $i = 1, 2$, reduces to Σ_i and by $\boldsymbol{n}$ a unit vector normal to Σ and oriented from Ω_1 toward Ω_2 (say). We want to study the following problem: *given* $\Phi \in \mathbb{R}$, *to determine a pair* $\boldsymbol{v}, p$ *such that*

$$\left.\begin{aligned} \Delta \boldsymbol{v} &= \nabla p \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}\right\} \quad \text{in } \Omega$$

$$\boldsymbol{v} = 0 \quad \text{at } \partial\Omega$$

$$\int_\Sigma \boldsymbol{v} \cdot \boldsymbol{n} = \Phi \tag{VI.3.2}$$

$$\lim_{|x| \to \infty} \boldsymbol{v}(x) = 0 \quad \text{in } \Omega_i, \ i = 1, 2.$$

To solve this problem, analogously to what we did in Section VI.1, we shall put it into an equivalent form that can be handled by the technique of generalized solutions. To this end, by multiplying (VI.3.1) by $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and integrating by parts we obtain formally

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) = 0, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega). \tag{VI.3.3}$$

We then give the following.

Definition VI.3.1. A field $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ is called a *weak* (or *generalized*) *solution to problem* (VI.3.2) if and only if

(i) $\boldsymbol{v} \in \widehat{\mathcal{D}}_0^{1,2}(\Omega)$;
(ii) $\boldsymbol{v}$ satisfies (VI.3.3);
(iii) $\boldsymbol{v}$ satisfies $(\text{VI.3.2})_4$ in the trace sense.

The meaning of conditions (ii) and (iii) is quite obvious. Let us comment on condition (i). First, we observe that it requires $\boldsymbol{v} \in \widehat{\mathcal{D}}_0^{1,2}(\Omega)$ and *not* $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega)$. As already remarked, this allows for a nonzero flux Φ of $\boldsymbol{v}$ through Σ; actually, if we required, instead, $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega)$ we would automatically impose $\Phi = 0$. Moreover, (i) ensures a certain degree of regularity for $\boldsymbol{v}$ and that $\boldsymbol{v}$ vanishes on the boundary in the trace sense. Finally, as we are going to show, condition (i) implies

$$\frac{1}{|\Sigma(x_n)|}\int_{\Sigma(x_n)} v^2(x) \to 0 \quad \text{as } |x| \to \infty \text{ in } \Omega_i,\ i=1,2, \tag{VI.3.4}$$

which represents the generalized form of $(\mathrm{VI.3.2})_5$. Property (VI.3.4) is a consequence of the general inequality

$$\int_{\Sigma(t)} |w|^q(x',t)dx' \le c f^{q-1}(t)\int_{\Omega(t)} |\nabla w|^q, \tag{VI.3.5}$$

where

$$\Omega(t) = \{x \in \mathbb{R}^n : x_n > t \ge 0, |x'| < f(x_n)\}$$
$$\Sigma(t) = \left\{x' \in \mathbb{R}^{n-1} : |x'| \le f(t)\right\},$$

f satisfies assumptions of the type (i), (ii) stated for f_i and, finally, w is an arbitrary member from $D^{1,q}(\Omega(t))$, $1 < q < \infty$, vanishing at the lateral surface $|x'| = f(x_n)$, $x_n > t$. To show (VI.3.5) we observe that (II.5.5) furnishes for almost all $t \ge 0$

$$\int_{\Sigma(t)} |w|^q(x',t)dx' \le c_1 f^q(t)\int_{\Sigma(t)} |\nabla w(x',t)|^q dx', \quad 1 < q < \infty. \tag{VI.3.6}$$

Consider, next, the function $g(x_n) \equiv f^{-q+1}(x_n)\boldsymbol{e}_n$ with $\boldsymbol{e}_n$ unit vector in the direction x_n. Integrating the identity

$$\nabla\cdot(\boldsymbol{g}|w|^q) = |w|^q\nabla\cdot\boldsymbol{g} + \boldsymbol{g}\cdot\nabla|w|^q$$

over Ω_{t,t_1}, in virtue of assumption (ii) on f we deduce for all $t_1 > t$

$$\begin{aligned}\frac{1}{f^{q-1}(t)}\int_{\Sigma(t)}|w|^q \le{}& \frac{1}{f^{q-1}(t_1)}\int_{\Sigma(t_1)}|w|^q\\ &+q\int_t^{t_1}\frac{1}{f^{q-1}(\tau)}\left(\int_{\Sigma(\tau)}|w|^{q-1}|\nabla w|d\Sigma\right)d\tau\\ &+M\int_{\Omega_{t,t_1}}\frac{|w|^q}{f^q}\end{aligned}$$

and so, from (VI.3.6) and the Hölder inequality, it follows that

$$\frac{1}{f^{q-1}(t)}\int_{\Sigma(t)}|w|^q \le \frac{1}{f^{q-1}(t_1)}\int_{\Sigma(t_1)}|w|^q + c_2\int_{\Omega(t)}|\nabla w|^q.$$

However, again by (VI.3.6) and assumption (ii) on f, since $w \in D^{1,q}(\Omega(t))$, we have that the first term on the right-hand side of this latter inequality must tend to zero as t_1 tends to infinity, at least along a sequence of values. Therefore, (VI.3.5) is proved.

Remark VI.3.1 By using Lemma IV.1.1 and Theorem VI.1.1, one can show that to every weak solution there corresponds a pressure field p satisfying the identity (VI.1.5) and that the pair $\boldsymbol{v}, p$ is infinitely differentiable up to the boundary. ■

Our next objective is to show existence and uniqueness of a generalized solution. To this end, we observe that in view of hypothesis (i) on f_i, from the results of the first part of Section III.5 and, in particular, from Theorem III.5.2, a vector field $\boldsymbol{a} \in C^\infty(\Omega)$ vanishing near $\partial\Omega$ exists such that

$$\begin{aligned} &\boldsymbol{a} \in \widehat{\mathcal{D}}_0^{1,2}(\Omega) \\ &\int_\Sigma \boldsymbol{a} \cdot \boldsymbol{n} = 1. \end{aligned} \tag{VI.3.7}$$

We then look for a generalized solution to (VI.3.2) of the form $\boldsymbol{v} = \boldsymbol{u} + \Phi\boldsymbol{a}$, with $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ obeying the identity

$$(\nabla\boldsymbol{u}, \nabla\boldsymbol{\varphi}) = -\Phi(\nabla\boldsymbol{a}, \nabla\boldsymbol{\varphi}), \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega). \tag{VI.3.8}$$

Clearly, if such a $\boldsymbol{u}$ exists, the vector field $\boldsymbol{v}$ satisfies all requirements of Definition VI.3.1. On the other hand, the existence of $\boldsymbol{u}$ is at once established by means of the Riesz theorem for, evidently, the right-hand side of (VI.3.8) defines a linear functional in $\mathcal{D}_0^{1,2}(\Omega)$. Let us next establish uniqueness. Let $\boldsymbol{v}_1$ be another generalized solution to (VI.3.2) corresponding to the same flux Φ. Letting $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v}_1$ we have $\boldsymbol{w} \in \widehat{\mathcal{D}}_0^1(\Omega)$ and the flux of $\boldsymbol{w}$ through Σ is zero. Therefore, by Theorem III.5.2, $\boldsymbol{w} \in \mathcal{D}_0^{1,2}(\Omega)$. However, from (VI.3.3) we also have

$$(\nabla\boldsymbol{w}, \nabla\boldsymbol{\varphi}) = 0, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega),$$

so that we conclude $\boldsymbol{w} \equiv 0$.

Finally, we wish to give an estimate for weak solutions. We recall that, by Theorem III.5.2, any $\boldsymbol{v} \in \widehat{\mathcal{D}}_0^{1,2}(\Omega)$ can be written as $\boldsymbol{v} = \boldsymbol{u} + \Phi\boldsymbol{a}$, where $\boldsymbol{a}$ is a given vector in $\widehat{\mathcal{D}}_0^{1,2}(\Omega)$ (independent of $\boldsymbol{v}$) and $\boldsymbol{u}$ is a vector in $\mathcal{D}_0^{1,2}(\Omega)$ uniquely related to $\boldsymbol{v}$. Writing (VI.3.3) for $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$ and taking $\boldsymbol{\varphi} = \boldsymbol{u}$ we thus readily recover

$$|\boldsymbol{v}|_{1,2} \le |\Phi||\boldsymbol{a}|_{1,2} \le c|\Phi|,$$

where c depends only on Ω and n.

Results obtained so far can then be summarized in the following.

Theorem VI.3.1 *For any $\Phi \in \mathbb{R}$ there exists one and only one generalized solution $\boldsymbol{v}$ to problem* (VI.3.2). *Such a solution satisfies the estimate*

$$|\boldsymbol{v}|_{1,2} \le c|\Phi|,$$

where c depends only on Ω and n. Furthermore, denoting by p the corresponding pressure field, $\boldsymbol{v}, p$ are infinitely differentiable in $\overline{\Omega'}$, with Ω' any bounded domain contained in Ω , and satisfy (VI.3.1)–(VI.3.4) *in the ordinary sense.*

Remark VI.3.2 If the cross sections widen at such a rate that condition (VI.3.1) on f_i is violated, then the unique solvability of problem (VI.3.2) becomes much more complicated. In this respect, we quote the results of Amick & Fraenkel (1980), when Ω is two-dimensional with two exits to infinity, and the more recent ones of Pileckas (1996a, 1996b, 1996c) valid for two-dimensional and three-dimensional domains as well, under the assumption that f_i satisfies (i), (ii) and the following condition

$$\int_0^\infty f_i^{-(n-1)(q-1)-q}(t)dt < \infty, \quad \text{for some } q \in (1, \infty). \tag{VI.3.9}$$

A typical result proved there is the following one.

Theorem VI.3.2 *Let* $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, *with* f_i *satisfying* (i), (ii) *and* (VI.3.9). *Then, for any* $\Phi \in \mathbb{R}$ *there exists a unique solution* $\boldsymbol{v}, p$ *to* (VI.3.2) *with* $\boldsymbol{v} \in \widehat{\mathcal{D}}_0^{1,q}(\Omega)$, *and satisfying the estimate*

$$|\boldsymbol{v}|_{1,q} \leq c|\Phi|$$

where $c = c(\Omega, n, q)$.

■

Remark VI.3.3 Another approach to unique solvability that holds in both two and three dimensions and for cross sections that need not verify (VI.3.1) is that proposed by Ladyzhenskaya and Solonnikov (1980) and Solonnikov (1983). However, it is not known if their solutions, which have a finite Dirichlet integral only on bounded subdomains of Ω, verify the condition at infinity $(\text{VI.3.2})_5$. ■

Exercise VI.3.1 Assume that a body force is acting on the liquid and add its contribution $\boldsymbol{f}$ to the right-hand side of $(\text{VI.3.2})_1$. Denoting by $[\mathcal{F}, \boldsymbol{\varphi}]$ the value of a linear functional $\mathcal{F}$ on $D_0^{1,2}(\Omega)$ at $\boldsymbol{\varphi}$, show that, given any bounded linear functional $\boldsymbol{f}$ on $D_0^{1,2}(\Omega)$ and any $\Phi \in \mathbb{R}$, there exists one and only one vector field $\boldsymbol{v}$ satisfying (i) and (ii) of Definition VI.3.1 and the identity

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) = -[\boldsymbol{f}, \boldsymbol{\varphi}], \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega). \tag{VI.3.10}$$

Moreover, prove that $\boldsymbol{v}$ satisfies the estimate

$$|\boldsymbol{v}|_{1,2} \leq c\left(|\Phi| + |\boldsymbol{f}|_{-1,2}\right) \tag{VI.3.11}$$

with $|\boldsymbol{f}|_{-1,2}$ denoting the norm of $\boldsymbol{f}$. Finally, show that if $\boldsymbol{f}$ is infinitely differentiable up to the boundary, the same holds for $\boldsymbol{v}$ and for the corresponding pressure p.

Exercise VI.3.2 Assume Ω has $m > 2$ exits Ω_i, $i = 1, \ldots, m$, all of the form specified at the beginning of this section and verifying (VI.3.1). Given m real numbers Φ_i subject to the restriction $\sum_{i=1}^m \Phi_i = 0$, we shall say that $\boldsymbol{v}$ is a generalized solution to the Stokes problem in Ω corresponding to the fluxes Φ_i if $\boldsymbol{v}$ satisfies (i) and (ii) of Definition VI.3.1 and if

$$\int_{\Sigma_i} \boldsymbol{v} \cdot \boldsymbol{n} = \Phi_i, \quad \text{in the trace sense.}$$

Show existence and uniqueness of this generalized solution. *Hint*: Use Lemma III.4.3.

Exercise VI.3.3 Extend the results of Exercise VI.3.2 to the case when Ω_i are not necessarily bodies of rotation but rather verify the more general conditions:

(a) $D_i^1 \subset \Omega_i \subset D_i^2$ where

$$D_i^1 = \{x \in \mathbb{R}^n : x_n > 0, |x'| < f_i(x_n)\}$$
$$D_i^2 = \{x \in \mathbb{R}^n : x_n > 0, |x'| < a_i f_i(x_n), a_i > 1\}$$

and

(b) In the domains

$$\{x \in \Omega : R_i < x_n < R + f(R_i)\}, \quad i = 1, .., m,$$

problem (III.4.19) is solvable with a constant c independent of R. *Hint*: See Remark III.5.1.

Exercise VI.3.4 Let Ω be a C^∞-smooth domain with $m \geq 2$ exits to infinity Ω_i. Suppose that the first ℓ ($\leq m$) exits satisfy the condition stated in Exercise VI.3.3, while the remaining $m - \ell$ are cylindrical. Show that, given m real numbers Φ_i subject to the condition $\sum_{i=1}^m \Phi_i = 0$, there exists one and only one pair $\boldsymbol{v}, p$ infinitely differentiable up to the boundary such that

$$\left.\begin{aligned} \Delta \boldsymbol{v} &= \nabla p \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}\right\} \quad \text{in} \quad \Omega$$

$$\boldsymbol{v} = 0 \quad \text{at} \quad \partial\Omega$$

$$\int_{\Sigma_i} \boldsymbol{v} \cdot \boldsymbol{n} = \Phi_i, \quad i = 1, \ldots m,$$

$$\boldsymbol{v} \in D_0^{1,2}(\Omega_i), \quad i = 1, \ldots \ell,$$

$$\boldsymbol{v} - \boldsymbol{v}_0^{(i)} \in W^{1,2}(\Omega_i), \quad i = \ell + 1 \ldots m,$$

where $\boldsymbol{v}_0^{(i)}$ are the Poiseuille velocity fields associated to Φ_i. *Hint*: Construct suitable extensions of the Poiseuille velocity fields and vectors having prescribed flux in Ω_i by means of the method used in Section VI.2 and Lemma III.4.3.

Exercise VI.3.5 (Flow through an aperture, Heywood 1976). Let Ω be the domain (VI.0.7), with S containing the unit disk $\{|x'| < 1\}$. Show that, given any bounded linear functional $\boldsymbol{f}$ on $D_0^{1,2}(\Omega)$ and any $\Phi \in \mathbb{R}$, there exists one and only one vector field $\boldsymbol{v}$ satisfying (i) and (ii) of Definition VI.3.1 and identity (VI.3.10) where, in this case, Φ is the flux of $\boldsymbol{v}$ through S. Furthermore, show that such a solution satisfies inequality (VI.3.11).

VI.4 Pointwise Decay of Flows in Channels with Unbounded Cross Section

To complete the study of problem (VI.3.2), it remains to investigate pointwise decay to zero of the velocity and pressure fields at large distances in the exits Ω_i. Seemingly, this study is not easily performed by a simple modification of the methods used in the case of channels with bounded cross sections and we shall employ a different technique.[1]

Let $\boldsymbol{v}, p$ be a pair of smooth functions[2] satisfying the system

$$\left.\begin{aligned} \Delta \boldsymbol{v} &= \nabla p \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\boldsymbol{v} = 0 \quad \text{at } \Gamma \tag{VI.4.1}$$
$$\int_{\Sigma} \boldsymbol{v} \cdot \boldsymbol{n} = \Phi,$$

where Ω is a semi-infinite channel with unbounded cross section, Γ its lateral surface, Σ its cross section, and Φ a prescribed number. For simplicity, we shall assume hereafter that Ω is a body of rotation. However, the proofs we give apply unchanged to the more general case where Ω contains a body of rotation $\widetilde{\Omega}$. In such a case, the results we find remain valid in $\widetilde{\Omega}$. We thus take for $n = 2, 3$,[3]

$$\begin{aligned} \Omega &= \{x \in \mathbb{R}^n : x_n > 0, |x'| < f(x_n)\} \\ \Sigma &= \Sigma(x_n) = \{x' \in \Omega : |x'| = f(x_n)\} \end{aligned} \tag{VI.4.2}$$

with $f \in C^\infty(\mathbb{R}_+)$ verifying the assumptions (i) and (ii) of the previous section, i.e.,

$$\begin{aligned} f(t) &\geq f_0 = \text{const.} > 0 \\ |f(t_1) - f(t_2)| &\leq M|t_1 - t_2| \end{aligned} \tag{VI.4.3}$$

for all $t, t_1, t_2 > 0$ and with M a positive constant. The aim of this section is to investigate decay as $|x| \to \infty$ of solutions to (VI.4.1) having $\boldsymbol{v} \in D^{1,q}(\Omega)$. Specifically, we show that if $1 < q \leq n$ then $\boldsymbol{v}$ and all its derivatives of arbitrary order tend to zero pointwise; moreover, if $1 < q < n$, we are also able to give the decay rate. Of course, for such solutions to exist, by the conservation of the flow through Σ, it is necessary that f satisfies the condition

$$\int_0^\infty f^{-(n-1)(q-1)-q}(t)dt < \infty \tag{VI.4.4}$$

[1] Alternatively, as suggested by Pileckas (1996a, 1996b, 1996c), one may use a "weighted de Saint-Venant principle" in conjunction with local estimates for the Stokes problem, to obtain sharper results than those obtained here.

[2] For example, $\boldsymbol{v} \in C^2(\overline{\Omega'})$, $p \in C^1(\overline{\Omega'})$ for all bounded subdomains $\Omega' \subset \Omega$.

[3] Some of the results we show, such as those of Theorem VI.4.1, hold in any space dimension $n \geq 2$.

These results furnish, in the particular case where $q = 2$, pointwise decay of solutions whose existence has been established in Theorem VI.3.1. Also, in such a case, we prove that the pressure field tends to a certain constant value at large distances.

To show all the above, we need some preliminary considerations. For $\gamma \in (0, 1]$, set

$$\Omega_\gamma = \{x \in \Omega : |x'| < \gamma f(x_n)\}, \quad \Omega_1 \equiv \Omega.$$

The following result gives a basic a priori estimate for solutions to (VI.4.1).

Lemma VI.4.1 *Let $\Omega \subset \mathbb{R}^3$. Assume that for some $\gamma \in (0, 1]$, $q \geq 2$ and $r \geq 0$* [4] *the following conditions hold:*

(i) $f^r \nabla \boldsymbol{v} \in L^q(\Omega_\gamma)$,
(ii) $f^{r-1} \boldsymbol{v} \in L^q(\Omega_\gamma)$,

where assumption (ii) is needed only if $\gamma < 1$. Then, for every $|\alpha| \geq 2$ we have

$$f^{|\alpha|+r-1} D^\alpha \boldsymbol{v} \in L^q(\Omega_{\gamma_\alpha}),$$

where γ_α is any positive number less than $\gamma/2$. If $\Omega \subset \mathbb{R}^2$ the same conclusion holds with $q = 2$.

Proof. First of all we notice that hypothesis (ii) follows from (i) if $\gamma = 1$, as a consequence of inequality (VI.3.6). To show the theorem we need a suitable"cut-off" function. Denote by $\psi \in C^\infty(\mathbb{R})$ a function such that $\psi(t) = 1$ if $t \leq 1$ and $\psi(t) = 0$ if $t \geq 2$ and set for $\beta \in (0, \gamma)$, $R_0 \geq 0$ and all $R > 2R_0$

$$\chi_{\beta,\gamma}(x) = \psi\left[\frac{1}{\gamma^2 - \beta^2}\left(\frac{|x'|^2}{\delta^2(x_n)} + \gamma^2 - 2\beta^2\right)\right]$$
$$\varphi_{R,R_0}(x) = \psi\left(\frac{|x|}{R_0}\right)\left[1 - \psi\left(\frac{|x|}{R_0}\right)\right],$$

where the function $\delta(t)$ has been introduced in Lemma III.4.2. Clearly, the function

$$u_{\beta,\gamma,R,R_0} \equiv \chi_{\beta,\gamma}\varphi_{R,R_0}$$

vanishes in the set

$$\{x \in \Omega : |x'| \geq \gamma f(x_n)\}$$

and for $|x| \geq 2R$, while

$$u_{\beta,\gamma,R,R_0}(x) = 1 \ \text{ for } x \in \left[(\omega_\beta \cap B_R) - \overline{B}_{2R_0}\right],$$

[4] Evidently, if $\gamma = 1$, for such solutions to exist the conservation of flux requires

$$\int_0^\infty f^{-(n-1)(q-1)+q(r-1)}(t)dt < \infty.$$

where

$$\omega_\beta = \{x \in \Omega : |x'| \le \beta\delta(x_n)\} .$$

By the properties of the function $\delta(t)$, ω_β can be taken arbitrarily close to $\Omega_{\gamma/2}$ by picking β sufficiently close to γ. Using Lemma III.4.2, it is not difficult to see that

$$|D^\ell u_{\beta,R,R_0}(x)| \le \frac{c}{f^{|\ell|}(x_n)}, \quad \text{for all } |\ell| \ge 0, \tag{VI.4.5}$$

with $c = c(\ell, \beta, \psi, R_0)$. We begin to show the lemma for $|\alpha| = 2$, the general case being obtained by iteration. To this end, letting

$$\Psi(x) = \delta^{1+r}(x_n) u_{\beta,R,R_0}(x) \tag{VI.4.6}$$

taking the curl of both sides of $(\text{VI.4.1})_1$, we obtain with $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$[5]

$$\Delta(\Psi\boldsymbol{\omega}) = \nabla\Psi \cdot \nabla\boldsymbol{\omega} + \nabla \cdot [\nabla\Psi \otimes \boldsymbol{\omega}]. \tag{VI.4.7}$$

Multiplying both sides of (VI.4.7) by $\Psi\boldsymbol{\omega}$ furnishes

$$\begin{aligned}(\nabla(\Psi\boldsymbol{\omega}), \nabla(\Psi\boldsymbol{\omega})) &= \tfrac{1}{2}\int_\Omega (\Psi\Delta\Psi + |\nabla\Psi|^2)\omega^2 + (\nabla\Psi \otimes \omega, \nabla(\boldsymbol{\Psi}\omega)) \\ &\equiv I_1 + I_2.\end{aligned} \tag{VI.4.8}$$

From (VI.4.6), (VI.4.5), and Lemma III.4.2 it follows that

$$\begin{aligned}|\nabla\Psi(x)| &\le c_1 f^r(x_n) \\ |D^2\Psi(x)| &\le c_2 f^{r-1}(x_n)\end{aligned} \tag{VI.4.9}$$

and, consequently, by $(\text{VI.4.3})_1$ and the Young inequality (II.2.5),

$$\begin{aligned}|I_1| &\le c_3 \|f^r\boldsymbol{\omega}\|^2_{2,\Omega_\gamma} \\ |I_2| &\le c_4 \|f^r\boldsymbol{\omega}\|^2_{2,\Omega_\gamma} + \tfrac{1}{2}\|\nabla(\Psi\boldsymbol{\omega})\|_{2,\Omega_{2,\gamma}}.\end{aligned} \tag{VI.4.10}$$

Thus, (VI.4.8) and (VI.4.10) along with the assumption on $\nabla\boldsymbol{v}$ give

$$\|\nabla(\Psi\boldsymbol{\omega})\|_{2,\Omega} \le c_5 \tag{VI.4.11}$$

with c_5 independent of R. Letting $R \to \infty$ into (VI.4.11) yields

$$\|\nabla(\zeta\boldsymbol{\omega})\|_{2,\Omega} \le c_5 \tag{VI.4.12}$$

with

$$\zeta(x) = \delta^{1+r}(x_n)\chi_{\beta,\gamma}(x)\psi\left(\frac{|x|}{R_0}\right).$$

[5] In two space dimension ω has only one nonzero component given by $\partial v_2/\partial x_1 - \partial v_1/\partial x_2$.

Clearly, $\zeta(x)$ satisfies estimates of the type (VI.4.9) and so, using the identity

$$\nabla(\zeta \boldsymbol{v}) = \nabla\zeta \times \boldsymbol{\omega} + \zeta\nabla \times \boldsymbol{\omega} + \boldsymbol{v}\Delta\zeta$$ [6]

together with (VI.4.12) it follows that

$$\|\Delta(\zeta\boldsymbol{v})\|_{2,\Omega} \le c_6 \left(\|f^r\boldsymbol{\omega}\|_{2,\Omega_\gamma} + \|f^{r-1}\boldsymbol{v}\|_{2,\Omega_\gamma}\right) \le c_7 \tag{VI.4.13}$$

where, in the last step, we made use of the assumption on $\boldsymbol{v}$. By considering $\zeta\boldsymbol{v}$ as a function defined in the whole of $\mathbb{R}^n$, (VI.4.13) furnishes $\Delta(\zeta\boldsymbol{v}) \in L^2(\mathbb{R}^n)$ and, by the Calderón–Zygmund Theorem II.11.4 (see Exercise II.11.9 and Exercise II.11.11) we have $D^2(\zeta\boldsymbol{v}) \in \mathbb{R}^n$. Since

$$\zeta D^2_{ij}\boldsymbol{v} = -D_i\zeta D_j\boldsymbol{v} - D_j\zeta D_i\boldsymbol{v} - \boldsymbol{v}D^2_{ij}\zeta + D^2(\zeta\boldsymbol{v})$$

we prove, as before,

$$\zeta D^2\boldsymbol{v} \in L^2(\mathbb{R}^n).$$

Recalling the relations

$$\chi_{\beta,\gamma}(x) = 1, \quad \delta(x_n) = f(x_n) \ \text{ in } \omega_\beta,$$

and the properties of ω_β, from this latter information we conclude

$$f^{1+r}D^2\boldsymbol{v} \in L^2(\Omega_{\gamma_2}), \tag{VI.4.14}$$

where γ_2 can be taken arbitrarily close to $\gamma/2$. The lemma is thus shown for $|\alpha| = 2$ and $q = 2$. Still assuming $q = 2$, the case $|\alpha| > 2$ can be easily proved by iteration. Actually, setting $\boldsymbol{\omega}_j \equiv D_j\boldsymbol{\omega}$, $j = 1, \dots, n$, we obtain that $\boldsymbol{\omega}_j$ satisfies (VI.4.7) with $\boldsymbol{\omega}_j$ in lieu of $\boldsymbol{\omega}$. Replace now Ψ with

$$\Psi_1 = \delta^{2+r}(x_n)u_{\beta,\gamma_2,R,R_0}(x),$$

so that

$$|\nabla\Psi_1(x)| \le c_1 f^{1+r}(x_n)$$
$$|D^2\Psi_1(x)| \le c_2 f^r(x_n)$$

and, consequently,

$$\left|\frac{1}{2}\int_\Omega (\Psi_1\Delta\Psi_1 + |\nabla\Psi_1|^2)\boldsymbol{\omega}_j^2 + \quad (\nabla\Psi_1 \otimes \boldsymbol{\omega}_j, \nabla(\Psi_1\boldsymbol{\omega}_j))\right|$$
$$\le c_3\|f^{1+r}D^2\boldsymbol{v}\|^2_{2,\Omega_{\gamma_2}} + (1/2)\|\nabla(\Psi_1\boldsymbol{\omega}_j)\|^2_{2,\Omega}.$$

[6] In two space dimension this identity is replaced by

$$\Delta(\zeta\boldsymbol{v}) = 2\nabla\zeta \cdot \nabla\boldsymbol{v} + (1/2)\zeta\vartheta + \boldsymbol{v}\Delta\zeta$$

with $\vartheta = (-\partial\omega/\partial x_2, \partial\omega/\partial x_1)$.

Therefore, from (VI.4.7) and (VI.4.8) with $\boldsymbol{\omega} \equiv \boldsymbol{\omega}_j$, $\Psi \equiv \Psi_1$ and from (VI.4.14) we deduce

$$\|\nabla(\Psi_1 \boldsymbol{\omega}_j)\|_{2,\Omega} \leq c_4.$$

Repeating step by step the arguments previously used for the case $|\alpha| = 2$ we may thus conclude for all $|\ell| = 3$

$$f^{2+r} D^\ell \boldsymbol{v} \in L^2(\Omega_{\gamma_3}), \tag{VI.4.15}$$

where $\gamma_3 < \gamma_2$ can be taken arbitrarily close to γ_2, *i.e.*, to $\gamma/2$. Iterating this procedure as many times as we please we finally prove the lemma for $q = 2$. In showing the result for arbitrary $q > 2$ we shall confine ourselves to proving it for $|\alpha| = 2$; the general case $|\alpha| > 2$ can be obtained by the same iterating procedure just used. Consider (VI.4.7) in three space dimensions. By the L^q-theory for the Laplace operator in the whole space (see Exercise II.11.9) it follows that

$$\|\nabla(\Psi \boldsymbol{\omega})\|_{q,\mathbb{R}^3} \leq c|\boldsymbol{f}|_{-1,q,\mathbb{R}^3}, \tag{VI.4.16}$$

where $\boldsymbol{f}$ denotes the right-hand side of (VI.4.7). If $\boldsymbol{\varphi}$ indicates an arbitrary vector function from $C_0^\infty(\mathbb{R}^3)$ we have

$$|(\boldsymbol{f}, \boldsymbol{\varphi})| = |(\nabla\Psi \cdot \nabla\boldsymbol{\omega}, \boldsymbol{\varphi}) - (\nabla\Psi \otimes \boldsymbol{\omega}, \nabla\boldsymbol{\varphi})| \equiv |F_1 + F_2|.$$

Use of (VI.4.9) then gives

$$|F_2| \leq c_1 \|f^r \boldsymbol{\omega}\|_{q,\Omega_\gamma} \|\nabla\boldsymbol{\varphi}\|_{q',\mathbb{R}^3}; \tag{VI.4.17}$$

furthermore,

$$|F_1| = |(\boldsymbol{\omega}\Delta\Psi, \boldsymbol{\varphi}) - (\nabla\Psi \cdot \nabla\boldsymbol{\varphi}, \boldsymbol{\omega})|. \tag{VI.4.18}$$

Since $\boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^2)$ and $q > 2$, we find that $\boldsymbol{\varphi}$ obeys inequality (II.6.10), i.e.,

$$\int_{\mathbb{R}^2} \frac{|\boldsymbol{\varphi}|^{q'}}{(x_1^2 + x_2^2)^{q'/2}} \leq \frac{2}{2-q'} \int_{\mathbb{R}^2} |\nabla\boldsymbol{\varphi}|^{q'} \tag{VI.4.19}$$

and so, by (VI.4.9) and (VI.4.19), by recalling that $|x'| \leq f(x_3)$, for all $x \in \Omega$, we deduce

$$\begin{aligned} |(\boldsymbol{\omega}\Delta\Psi, \boldsymbol{\varphi})| &\leq c_2 \int_{\Omega_\gamma} \frac{f^r|\boldsymbol{\omega}||\boldsymbol{\varphi}|}{|x'|} \\ &\leq c_2 \|f^r\boldsymbol{\omega}\|_{q,\Omega_\gamma} \left\{ \int_0^\infty \left(\int_{\mathbb{R}^2} \frac{|\boldsymbol{\varphi}|^{q'}}{(x_1^2 + x_2^2)^{q'/2}} dx' \right) dx_3 \right\} \\ &\leq c_3 \|f^r\boldsymbol{\omega}\|_{q,\Omega_\gamma} \|\nabla\boldsymbol{\varphi}\|_{q',\mathbb{R}^3}. \end{aligned} \tag{VI.4.20}$$

On the other hand, again by (VI.4.19), it is clear that

$$|(\nabla\Psi \cdot \nabla\boldsymbol{\varphi}, \boldsymbol{\omega})| \leq c_4 \|f^r\boldsymbol{\omega}\|_{q,\Omega_\gamma} \|\nabla\boldsymbol{\varphi}\|_{q',\mathbb{R}^3} \tag{VI.4.21}$$

and therefore, collecting (VI.4.16)–(VI.4.18), (VI.4.20), and (VI.4.21) we recover

$$\|\nabla(\Psi\boldsymbol{\omega})\|_{q,\Omega} \le c_5. \tag{VI.4.22}$$

By reasoning as in the case where $q = 2$, from (VI.4.22) we obtain

$$f^{1+r} D^2 \boldsymbol{v} \in L^q(\Omega_{\gamma_2})$$

and the proof of the lemma is then completed. □

Remark VI.4.1 The only point in the proof just given where we need the assumption $q > 2$ is in increasing the first integral on the right-hand side of (VI.4.18), by means of (VI.4.19) (see (VI.4.20)). However, if

$$f(t) = f_0 + Mt, \tag{VI.4.23}$$

i.e., Ω is an infinite portion of a straight cone, we have $|x| \le cf(x_n)$ for all $x \in \Omega$ and so if $q' < n$, that is, $q > n/(n-1)$, using (II.6.10) we have

$$|(\Delta\Psi\omega, \boldsymbol{\varphi})| \le c_2 \|f^r\omega\|_{q,\Omega} \|\boldsymbol{\varphi}/|x|\|_{q',\mathbb{R}^n} \le c_3 \|f^r\omega\|_{q,\Omega_\gamma} \|\nabla\boldsymbol{\varphi}\|_{q',\mathbb{R}^n}.$$

Therefore, if f satisfies (VI.4.23), Lemma VI.4.1 is valid for all $q \ge 3/2$ if $n = 3$ and for all $q \ge 2$ if $n = 2$. ■

Remark VI.4.2 If f satisfies $|f''(t)f(t)| \le c$, Lemma VI.4.1 can be proved with γ_α arbitrarily close to one. This is easily seen by using, throughout the proof, $f(t)$ in place of $\delta(t)$. ■

The theorem to follow gives pointwise decay for a solution $\boldsymbol{v} \in D^{1,q}(\Omega)$.

Theorem VI.4.1 *Assume $\boldsymbol{v}$ is a regular solution to* (VI.4.1) *such that $\boldsymbol{v} \in D^{1,q}(\Omega)$, for some $q > 1$. Then for all $|\alpha| \ge 0$ and all $\gamma \in (0, 1)$*

$$\lim_{|x|\to\infty} |D^\alpha \nabla \boldsymbol{v}(x)| = 0, \quad \textit{uniformly in } \Omega_\gamma \tag{VI.4.24}$$

and, if $1 < q \le n$,

$$\lim_{|x|\to\infty} |\boldsymbol{v}(x)| = 0, \quad \textit{uniformly in } \Omega_{\gamma'}, \tag{VI.4.25}$$

where $\gamma' \in (0, 1)$ [respectively, $\gamma' \in (0, 1/2)$] can be taken arbitrarily close to 1 [respectively, to $1/2$] if $1 < q < n$ [respectively, $q = n$].

Proof. From Lemma V.3.1 we have for all $x \in \Omega$

$$v_j(x) = \int_{\mathbb{R}^n} H^{(d)}_{ij}(x-y) v_i(y) dy, \tag{VI.4.26}$$

where $d < (1-\gamma)f_0$. Differentiating once (VI.4.26), and then differentiating it again $|\alpha|$ times, after using the Hölder inequality we obtain

$$|D^\alpha D_k v_j(x)| \le \|D^\alpha H_{ij}^{(d)}(x-y)\|_{q',\mathbb{R}^n} \|D_k v_i\|_{q,B_d} \le c\|D_k v_i\|_{q,B_d},$$

which proves (VI.4.24). Assume now $1 < q < n$. By the Sobolev inequality we have $\|\boldsymbol{v}\|_{s,\Omega} < \infty$, $s = nq/(n-q)$ (see Exercise VI.4.1) and (VI.4.26) gives

$$|v_j(x)| \le \|H_{ij}^{(d)}(x-y)\|_{s',\mathbb{R}^n} \|v_i\|_{s,B_d}$$

showing (VI.4.25) if $1 < q < n$. To prove the theorem completely, it remains to prove (VI.4.25) for $q = n$. To this end, setting for $a \in (0,1]$

$$\Sigma_a(x_n) = \{x \in \Sigma(x_n) : |x'| < af(x_n)\} \tag{VI.4.27}$$

we denote by $\psi = \psi(|x'|)$ a smooth function that is one in $\Sigma_\sigma(x_n)$ and zero outside $\Sigma_{\sigma+\varepsilon}(x_n)$, $\varepsilon > 0$. Moreover, we take

$$|\nabla\psi(|x'|)| \le \frac{c_1}{f(x_n)}$$

with c independent of x. Since $q > n-1$ and $\psi\boldsymbol{v} \in W_0^{1,n}(\Sigma)$, we may apply inequality (II.11.14) to $\psi\boldsymbol{v}$ and use the latter to obtain for all $x = (x', x_n) \in \Omega_\sigma$

$$\begin{aligned} |\boldsymbol{v}(x',x_n)|^n \le c \Bigg[& \frac{1}{f^{n-1}(x_n)} \int_{\Sigma_{\sigma+\varepsilon}(x_n)} v^n(\xi', x_n) d\xi' \\ & + f(x_n) \int_{\Sigma_{\sigma+\varepsilon}(x_n)} |\nabla \boldsymbol{v}(\xi', x_n)|^n d\xi' \Bigg] \\ \equiv {} & I_1(x_n) + I_2(x_n). \end{aligned}$$

From (VI.3.5) and (VI.4.24) it follows that

$$\lim_{|x|\to\infty} I_1(x_n) = 0 \ \text{ in } \Omega_\sigma$$

and so it remains to show

$$\lim_{|x|\to\infty} I_2(x_n) = 0 \ \text{ in } \Omega_\sigma. \tag{VI.4.28}$$

By a simple calculation we derive

$$\begin{aligned} \left| \frac{d}{dt} \|\nabla\boldsymbol{v}\|^n_{n,\Sigma_{\sigma+\varepsilon}(t)} \right| \le c_3 \Big\{ & f^{-1}(t) \|\nabla\boldsymbol{v}\|^n_{n,\Sigma_{\sigma+\varepsilon}(t)} + \|\nabla\boldsymbol{v}\|^{n-1}_{n,\Sigma_{\sigma+\varepsilon}(t)} \\ & + \|D^2\boldsymbol{v}\|_{n,\Sigma_{\sigma+\varepsilon}(t)} \Big\} \end{aligned}$$

and so, by employing inequality (II.2.7), it follows that

$$\left|\frac{dI_2}{dt}\right| \le c_4 \left[\|\nabla \boldsymbol{v}\|^n_{n,\Sigma_{\sigma+\varepsilon}(t)} + f^n(t)\|D^2\boldsymbol{v}\|^n_{n,\Sigma_{\sigma+\varepsilon}(t)}\right].$$

By assumption and Lemma VI.4.1 with $|\alpha| = 2$, $r = 0$ we have, for all $\sigma < 1/2$,

$$\frac{dI_2}{dt} \in L^1(0,\infty)$$

and therefore $\ell \in \mathbb{R}_+$ exists such that

$$\lim_{|t|\to\infty} I_2(t) = \ell.$$

However, again by assumption, there exists at least a sequence $\{t_k\}$ tending to infinity along which it holds

$$|\boldsymbol{v}|^n_{1,n,\Sigma_{\sigma+\varepsilon}(t_k)} = o(1/t_k)$$

while, by $(\text{VI.4.3})_2$,

$$f(t_k) = O(t_k).$$

As a consequence,

$$\lim_{t_k\to\infty} I(t_k) = 0,$$

which shows (VI.4.28). The proof of the theorem is therefore completed. □

In the following theorem we establish the decay rate.

Theorem VI.4.2 *Let* $\boldsymbol{v}$ *satisfy the hypotheses of* Theorem VI.4.1. *If* $n = 3$, *then for all* $x \in \Omega_{\gamma_\alpha}$ *and all* $|\alpha| \ge 0$

$$\begin{aligned} |D^\alpha \boldsymbol{v}(x)| &\le \frac{c_1}{f^{|\alpha|+\vartheta}(x_3)}, \quad 1 < q < 3 \\ |D^\alpha \nabla \boldsymbol{v}(x)| &\le \frac{c_2}{f^{|\alpha|}(x_3)}, \quad q \ge 3 \end{aligned} \tag{VI.4.29}$$

where $\vartheta = 1/2$ *if* $1 < q \le 2$, *while if* $2 < q < 3$, $\vartheta = (3-q)/q$ *for* $\alpha = 0$ *and* $\vartheta = 0$ *for* $|\alpha| \ge 1$. *If* $n = 2$, *then for all* $x \in \Omega_{\gamma_\alpha}$ *and* $|\alpha| \ge 0$

$$|D^\alpha \nabla \boldsymbol{v}(x)| \le \frac{c_3}{f^{|\alpha|}(x_2)}, \quad 1 < q \le 2. \tag{VI.4.30}$$

Here γ_α *is any number less than* $1/4$; *moreover, the constants* c_i *depend on* n, q, α, M, *and on the norm* $|\boldsymbol{v}|_{1,q}$.

Proof. We begin to prove $(\text{VI.4.29})_1$. By assumption

$$\boldsymbol{v} \in D^{1,q}(\Omega_\gamma), \quad \gamma < 1,$$

and, by (VI.3.6), $\boldsymbol{v}/f \in L^q(\Omega_\gamma)$. Without loss we may assume $q \ge 2$ since, by Theorem VI.4.1 and the regularity hypothesis on $\boldsymbol{v}$, if

$$\boldsymbol{v} \in D^{1,q}(\Omega), \quad \text{for some } q > 1,$$

then

$$\boldsymbol{v} \in D^{1,r}(\Omega_\gamma), \quad \boldsymbol{v}/f \in L^r(\Omega_\gamma) \ \text{ for all } r \geq q.$$

Set

$$\boldsymbol{w} = \delta u_{\beta,\gamma,R,R_0}\boldsymbol{v},$$

where δ is the function constructed in Lemma III.4.2 and u_{β,R,R_0} is the "cut-off" function introduced in the proof of Lemma VI.4.1. From (VI.4.5) and the properties of δ we derive

$$\|D^2\boldsymbol{w}\|_{q,\Omega} \leq c_1 \left(\|\boldsymbol{v}/f\|_{q,\Omega_\gamma} + \|\nabla \boldsymbol{v}\|_{q,\Omega_\gamma} + \|fD^2\boldsymbol{v}\|_{q,\Omega_\gamma}\right)$$

with c_1 independent of R and $\gamma < 1/2$ arbitrarily close to $1/2$. From Lemma VI.4.1 it follows that

$$\|D^2\boldsymbol{w}\|_{q,\Omega} \leq c_2. \tag{VI.4.31}$$

Considering the function $\boldsymbol{w}$ in the whole of $\mathbb{R}^3$ and recalling that $q < 3$, from the Sobolev inequality and (VI.4.31) we have

$$\|\nabla\boldsymbol{w}\|_{s,\mathbb{R}^3} \leq \kappa\|D^2\boldsymbol{w}\|_{q,\mathbb{R}^3} \leq c_3, \quad s = 3q/(3-q)$$

with c_3 independent of R. By taking into account the properties of the function δ it is not hard to show that the preceding inequality in the limit $R \to \infty$ furnishes

$$\|f\nabla\boldsymbol{v}\|_{s,\Omega_{\gamma_1}} \leq c_4, \tag{VI.4.32}$$

where $\gamma_1 < \gamma$ can be taken arbitrarily close to $\gamma/2$, i.e., to $1/4$. Since $q < 3$ we may apply to $\boldsymbol{v}$ the Sobolev inequality to deduce $\boldsymbol{v} \in L^s(\Omega)$ (see Exercise VI.4.1). Thus, from (VI.4.32) it follows that $\boldsymbol{v}$ satisfies the assumption of Lemma VI.4.1 and for all $|\alpha| \geq 1$ we conclude

$$\|f^{|\alpha|}D^\alpha\boldsymbol{v}\|_{s,\Omega_{\gamma_\alpha}} \leq c_5, \tag{VI.4.33}$$

where γ_α can be taken close to $1/4$. Take $x \in \Omega_{\gamma_\alpha}$, $d < \operatorname{dist}(x, \partial\Omega_{\gamma_\alpha})$ and set

$$\boldsymbol{u}_\alpha = \delta^{|\alpha|}D^\alpha\boldsymbol{v}$$

with $\delta \equiv f$ in Ω_{γ_α}. Applying Theorem II.3.2 to $\theta\boldsymbol{u}_\alpha$ with $\theta = 1$ in $B_{d/2}(x)$ and $\theta = 0$ outside $B_{(3/4)d}(x)$ we obtain

$$|f^{|\alpha|}(x_3)D^\alpha\boldsymbol{v}(x)| \leq c\|\boldsymbol{u}_\alpha\|_{m,s,B_d(x)} \tag{VI.4.34}$$

whenever $m \geq s/3$. However, from the properties of the function δ we easily deduce

$$\|\boldsymbol{u}_\alpha\|_{m,s,B_d(x)} \leq \|f^{|\alpha|}D^\alpha\boldsymbol{v}\|_{s,B_d(x)} + \sum_{|\ell|=1}^{m} \|f^{|\alpha|}D^\ell D^\alpha\boldsymbol{v}\|_{s,B_d(x)} \tag{VI.4.35}$$

and so by (VI.4.33)–(VI.4.35) we recover $(\text{VI.4.29})_1$. Using Theorem II.11.2 and inequality (II.11.11), and proceeding as in the proof of Theorem VI.4.1, we establish the estimate

$$\begin{aligned} f^3(x_3)|\boldsymbol{v}(x',x_3)|^s &\le c_2\left\{f(x_3)\|\boldsymbol{v}\|^s_{s,\Sigma_{\sigma+\varepsilon}(x_3)} + f^{s+1}(x_3)\|\nabla\boldsymbol{v}\|^s_{s,\Sigma_{\sigma+\varepsilon}(x_3)}\right\} \\ &\equiv I(x_3). \end{aligned} \tag{VI.4.36}$$

Denoting, temporarily, x_3 by t, we want to show that the function $I = I(t)$ is bounded for all sufficiently large t. To this end, it is *sufficient* to show that

$$dI/dt \in L^1(0,\infty).$$

We have

$$\begin{aligned} \left|\frac{dI}{dt}\right| &\le c_3\left\{\|\boldsymbol{v}\|^s_{s,\Sigma_{\sigma+\varepsilon}(t)} + f(t)\|\boldsymbol{v}\|^{s-1}_{s,\Sigma_{\sigma+\varepsilon}(t)}\|\nabla\boldsymbol{v}\|_{s,\Sigma_{\sigma+\varepsilon}(t)}\right. \\ &\qquad \left. + f^s(t)\|\nabla\boldsymbol{v}\|^s_{s,\Sigma_{\sigma+\varepsilon}(t)} + f^{s+1}(t)\|\nabla\boldsymbol{v}\|^{s-1}_{s,\Sigma_{\sigma+\varepsilon}(t)}\|D^2\boldsymbol{v}\|_{s,\Sigma_{\sigma+\varepsilon}(t)}\right\} \\ &\le c_4\left\{\|\boldsymbol{v}\|^s_{s,\Sigma_{\sigma+\varepsilon}(t)} + f^s(t)\|\nabla\boldsymbol{v}\|^s_{s,\Sigma_{\sigma+\varepsilon}(t)} + f^{2s}(t)\|D^2\boldsymbol{v}\|^s_{s,\Sigma_{\sigma+\varepsilon}(t)}\right\} \end{aligned} \tag{VI.4.37}$$

and since, by the Sobolev inequality (see Exercise VI.4.1)

$$\|\nabla\boldsymbol{v}\|^s_{s,\Sigma(t)} \in L^1(0,\infty),$$

from (VI.4.33), (VI.4.36) and (VI.4.37) we obtain $(\text{VI.4.29})_1$ also in the case where $\alpha = 0$. In order to show $(\text{VI.4.29})_2$ we set

$$\boldsymbol{u}_\alpha = \delta^{|\alpha|}D^\alpha\nabla\boldsymbol{v},$$

with $\delta \equiv f$ in Ω_{γ_α}. Proceeding as in the proof of $(\text{VI.4.29})_1$ we may establish the inequality (see (VI.4.34))

$$|f^{|\alpha|}(x_3)D^\alpha\nabla\boldsymbol{v}(x)| \le c\|\boldsymbol{u}_\alpha\|_{2,q,B_d(x)}, \tag{VI.4.38}$$

where $x \in \Omega_{\gamma_\alpha}$, $d < \operatorname{dist}(x, \partial\Omega_{\gamma_\alpha})$. Therefore, since

$$\|\boldsymbol{u}_\alpha\|_{2,q,B_d(x)} \le c\left(\|f^{|\alpha|}D^\alpha\nabla\boldsymbol{v}\|_{q,B_d(x)} + \sum_{|\ell|=1}^{2}\|f^{|\alpha|}D^\ell D^\alpha\nabla\boldsymbol{v}\|_{q,B_d(x)}\right), \tag{VI.4.39}$$

$(\text{VI.4.29})_2$ follows from (VI.4.38), (VI.4.39) and Lemma VI.4.1. The proof of (VI.4.30) is entirely identical to the one just given for (VI.4.29), if one recalls that, as already observed, $\boldsymbol{v} \in D^{1,q}(\Omega)$ for some $q < 2$ implies

$$\boldsymbol{v} \in D^{1,2}(\Omega_\sigma), \quad \boldsymbol{v}/f \in L^2(\Omega_\sigma) \ \text{ for all } \sigma < 1.$$

The proof of the theorem is therefore completed. □

Remark VI.4.3 If $f(t)$ verifies (VI.4.23), then estimate (VI.4.30) holds for any $q \in (1, \infty)$. This is a consequence of Remark VI.4.1. ■

Remark VI.4.4 If $|f''(t)f(t)| \leq c$, then all conclusions in Theorem VI.4.2 remain valid for $\gamma, \gamma_\alpha < 1$. ■

Exercise VI.4.1 Let Ω be a domain of the type introduced at the beginning of this section and let u be a smooth function in Ω, vanishing on Γ and with $u \in D^{1,q}(\Omega)$, $1 < q < n$. Show that $\|u\|_s < \infty$, $s = nq/(n-q)$ (Sobolev inequality). *Hint*: Extend u to zero outside Ω. The function $\psi(x/R_0)u(x)$, with ψ given in the proof of Lemma VI.4.1 belongs to $D^{1,q}(\mathbb{R}^n)$.

We now turn our attention to the behavior of the pressure. First of all, from Theorem VI.4.1, Theorem VI.4.2, and $(\text{VI.4.1})_1$ we at once obtain

Theorem VI.4.3 *Let $\boldsymbol{v}$ satisfy the assumptions of* Theorem VI.4.1. *Then for all $|\alpha| \geq 0$*

$$\lim_{|x|\to\infty} D^\alpha \nabla p(x) = 0$$

uniformly in Ω_γ, $\gamma < 1$. Moreover if $n = 3$, then for all $x \in \Omega_{\gamma_\alpha}$

$$|D^\alpha \nabla p(x)| \leq \frac{c_1}{f^{|\alpha|+1}(x_3)} \quad 1 < q < 3$$

$$|D^\alpha \nabla p(x)| \leq \frac{c_2}{f^{|\alpha|}(x_3)} \quad q \geq 3$$

while, if $n = 2$,

$$|D^\alpha \nabla p(x)| \leq \frac{c_3}{f^{|\alpha|}(x_2)} \quad 1 < q \leq 2,$$

where $\gamma_\alpha < 1/4$ and, for fixed α, it can be taken arbitrarily close to $1/4$.

Furthermore, using Theorem VI.4.2 we can prove the following result.

Theorem VI.4.4 *Let $\boldsymbol{v} \in D^{1,2}(\Omega)$. Then, there exists a constant $p_0 \in \mathbb{R}$ such that*

$$\lim_{|x|\to\infty} p(x) = p_0$$

uniformly in Ω_γ, for any $\gamma < 1/4$.

Proof. Denote by $\overline{p}(x_n)$ the mean of the pressure p over $\Sigma(x_n)$, i.e.,

$$\overline{p}(x_n) = \frac{1}{|\Sigma_\sigma(x_n)|} \int_{\Sigma_\sigma(x_n)} p(x', x_n)dx',$$

where Σ_σ is defined in (VI.4.27). The following inequality holds:

$$|p(x', x_n) - \overline{p}(x_n)|^n \le cf(x_n) \int_{\Sigma_\sigma(x_n)} |\nabla p(x', x_n)|^n dx', \tag{VI.4.40}$$

with $x' \in$, and γ an arbitrary number less than σ. Actually, applying inequality (II.11.14) to $\psi(p - \overline{p})$, with ψ the "cut-off" function introduced in the proof of Theorem VI.4.1, we readily deduce

$$\begin{aligned} &|p(x', x_n) - \overline{p}(x_n)|^n \\ &\le c \left[f^{1-n}(x_n) \int_{\Sigma_\sigma(x_n)} |p(x', x_n) - \overline{p}(x_n)|^n dx' \right. \\ &\qquad \left. + f(x_n) \int_{\Sigma_\sigma(x_n)} |\nabla p(x', x_n)|^n dx' \right]. \end{aligned} \tag{VI.4.41}$$

Increasing the first integral on the right-hand side of (VI.4.41) by means of inequality (II.5.10), we recover (VI.4.40). Denoting temporarily x_n by t and the right-hand side of (VI.4.40) by $I = I(t)$, we also have

$$\left| \frac{dI}{dt} \right| \le c_1 \left\{ \|\nabla p\|^n_{n,\Sigma_\sigma(t)} + f(t) \|\nabla p\|_{n,\Sigma_\sigma(t)} \|D^2 p\|_{n,\Sigma_\sigma(t)} \right\}$$

which, by Lemma VI.4.1 and $(\text{VI.4.1})_1$, in turn implies $dI/dt \in L^1(0, \infty)$ for all $\sigma < 1/4$. Thus,

$$\lim_{t\to\infty} I(t)$$

exists and since

$$\lim_{t_k\to\infty} I(t_k) = 0$$

at least along a suitable sequence $\{t_k\}$ (as a consequence of the summability of ∇p in $L^n(\Omega_\gamma)$, which follows from $(\text{VI.4.1})_1$, Lemma VI.4.1 and Theorem VI.4.3, and of the fact that $f(t) = O(t)$) we may conclude

$$\lim_{|x|\to\infty} |p(x', x_n) - \overline{p}(x_n)| = 0 \tag{VI.4.42}$$

uniformly in Ω_γ. Next, we shall show that $\overline{p}$ tends to a prescribed limit as $|x| \to \infty$. From the nth component of (VI.4.1) we derive

$$\frac{\partial}{\partial x_n} \overline{p}(x_n) = \frac{1}{|\Sigma_\sigma(x_n)|} \int_{\Sigma_\sigma(x_n)} \Delta v_n$$

which in turn furnishes

$$|\overline{p}(t_2) - \overline{p}(t_1)| \le c_1 \int_{t_1}^{t_2} f^{1-n}(t) \left(\int_{\Sigma_\sigma(t)} |D^2 \boldsymbol{v}(x', x_n)| dx' \right) dt$$

for arbitrary t_2, $t_1 > 0$. Use of the Schwarz inequality gives

$$
\begin{aligned}
|\overline{p}(t_2)-\overline{p}(t_1)| &\le c_2\int_{t_1}^{t_2} f^{-(n+1)/2}(t)\left(f(t)\|D^2\boldsymbol{v}\|_{2,\Sigma_\sigma(t)}\right)\\
&\le c_3\int_{t_1}^{t_2}\left(f^{-n-1}(t)+f^2(t)\|D^2\boldsymbol{v}\|^2_{2,\Sigma_\sigma(t)}\right)dt\\
&\equiv I_1(t)+I_2(t).
\end{aligned}
$$

Now, as $t_2, t_1 \to \infty$, $I_1(t)$ tends to zero because the assumption $\boldsymbol{v} \in D^{1,2}(\Omega)$ implies (VI.4.4) with $q = 2$, while I_2 tends to zero by Lemma VI.4.1 and we conclude

$$
\lim_{x_n\to\infty}\overline{p}(x_n)=p_0
$$

for some constant p_0. The theorem then follows from this latter condition and (VI.4.42). □

Remark VI.4.5 If $|f''(t)f(t)| \le c$, in the previous theorem we can take $\gamma < 1$. ■

Remark VI.4.6 If f does not satisfy (VI.4.4) with $q \le 2$ and, consequently, if $\boldsymbol{v} \notin D^{1,2}(\Omega)$, it is not expected that the pressure field tends to a constant at large distances in the exit. Actually, if $n = 2$, Amick & Fraenkel (1980) prove that p diverges. However, using the following heuristic argument, it is easy to convince oneself that the same property should hold also for $n = 3$. Since $f(x_n)$ is the only "natural length" of the problem, by the incompressibility condition $\nabla\cdot\boldsymbol{v} = 0$ the longitudinal velocity component v_3 has to have the asymptotic form $g(|x'|/f(x_n))/f^2(x_n)$. Now, if we take the third component of the Stokes equations we find $\partial p/\partial x_3 = \partial^2 v_3/\partial x_1^2 + \partial^2 v_3/\partial x_2^2 \approx f^{-4}(x_3)$. As a consequence, if f does not verify (VI.4.4) with some $q \le 2$, p diverges; see also Pileckas (1996a, 1996b, 1996c). ■

Remark VI.4.7 The physical meaning of the constant to which the pressure tends in the exits is very important and it is tightly related to the flux Φ. Specifically, for solutions whose existence has been established in Theorem VI.3.1, *to prescribe the flux is equivalent to prescribing the difference p_0 between the constant values p_{0i} to which the pressure field $p(x)$ tends in the exits $\Omega_i (i = 1, 2)$.* This property, which was originally discovered for a particular region of flow by Heywood (1976, Section 6), is simply proved in our case as follows. Since p is defined up to a constant, we can always adjust it in such a way that

$$
\begin{aligned}
&\lim_{|x|\to\infty} p(x)=0 \quad\text{in } (\Omega_1)_\gamma\equiv\Omega_{1\gamma}\\
&\lim_{|x|\to\infty} p(x)=p_0 \text{ in } (\Omega_2)_\gamma\equiv\Omega_{2\gamma}.
\end{aligned}
$$

By the linearity of the problem (VI.4.1), it is sufficient to show that if $p_0 = 0$ then the solution $\boldsymbol{v}$ determined in Theorem VI.3.1 is identically zero. On the other hand, this will follow if we show

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{a}) = 0, \tag{VI.4.43}$$

where $\boldsymbol{a}$ is the basis of $\widehat{\mathcal{D}}_0^{1,2}(\Omega) \,/\, \mathcal{D}_0^{1,2}(\Omega)$ determined in Theorem III.5.2. To prove this we observe that $\boldsymbol{v} \in \widehat{\mathcal{D}}_0^{1,2}(\Omega)$ and $\boldsymbol{v}$ is orthogonal to all $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$, see (VI.3.3). Therefore, since every $\boldsymbol{\psi} \in \widehat{\mathcal{D}}_0^{1,2}(\Omega)$ can be written as $\boldsymbol{\psi} = \boldsymbol{\varphi} + \lambda \boldsymbol{a}$, $\lambda \in \mathbb{R}$, from (VI.4.43) it follows that $\boldsymbol{v}$ is orthogonal to *all* functions of $\widehat{\mathcal{D}}_0^{1,2}(\Omega)$ and thus $\boldsymbol{v} \equiv 0$. To show (VI.4.43) we multiply (VI.4.1)$_1$ by $\boldsymbol{a}$ and integrate by parts to obtain

$$\begin{aligned} -\int_{\omega(t)} \nabla \boldsymbol{v} : \nabla \boldsymbol{a} = & -\int_{\Sigma_1(t_1)\cup\Sigma_2(t_1)} \boldsymbol{n}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{a} + p_0 \int_{\Sigma_2(t_1)} \boldsymbol{a}\cdot\boldsymbol{n} \\ & + \int_{\Sigma_2(t_1)} (p - p_0)\boldsymbol{a}\cdot\boldsymbol{n} + \int_{\Sigma_2(t_1)} p\boldsymbol{a}\cdot\boldsymbol{n}, \end{aligned} \tag{VI.4.44}$$

where

$$\omega(t) = \Omega_0 \cup \{x \in \Omega_1 : x_n < t\} \cup \{x \in \Omega_2 : x_n < t\}\,.$$

From Theorem III.5.2 we know that $\boldsymbol{a}$ can be taken to be zero in a neighborhood of the lateral surfaces of $\Omega_{i\gamma}$, $\gamma < 1/4$, and furthermore, $\boldsymbol{a}(x) \le c f^{-n-1}(x_3)$. Therefore, if $p_0 = 0$, we can pass to the limit as $t \to \infty$ in (VI.4.44) and use Theorem VI.4.1 and Theorem VI.4.4 to obtain (VI.4.43). ■

Remark VI.4.8 The decay results of Theorem VI.4.1–Theorem VI.4.4 can be fairly improved. In this respect, we refer the reader to Pileckas (1996a, 1996b, 1996c). A typical result proved there is the following one.

Theorem VI.4.5 *Let Ω and $\boldsymbol{v}$ be as in* Theorem VI.3.2 *and let p be the corresponding pressure given by* Lemma IV.1.1. *Assume, further, that Ω is of class $C^{l+2,\delta}$, $l \ge 0$, $\delta \in (0,1)$. Then the following decay estimate hold*

$$|D^\alpha \boldsymbol{v}(x)| \le C|\phi| f_i^{-n+1-|\alpha|}(x_n)$$

$$|D^\alpha \nabla p(x)| \le C|\phi| f_i^{-n+1-|\alpha|}(x_n),$$

for $x \in \Omega_i$, $l \ge |\alpha| \ge 0$. Moreover, there exists $p_0 \in \mathbb{R}$ such that for $x \in \Omega_i$

$$|p(x)| \le C|\phi| \int_0^{x_n} f_i^{-n+1}(t)dt + p_0.$$

In these relations, $C = C(\Omega, q, n)$.

■

VI.5 Existence, Uniqueness, and Asymptotic Behavior of Flow Through an Aperture

Let us consider the "aperture domain" (VI.0.7), *i.e.*,

$$\Omega = \{x \in \mathbb{R}^n : x_n \neq 0 \text{ or } x' \in S\}, \tag{VI.5.1}$$

where S is a bounded domain of $\mathbb{R}^{n-1}$. As we have seen in Section VI.3 (see Exercise VI.3.5), given any continuous linear functional $\boldsymbol{f}$ on $D_0^{1,2}(\Omega)$ and any $\Phi \in \mathbb{R}$, there exists one and only one vector field $\boldsymbol{v} \in \widehat{\mathcal{D}}_0^{1,2}(\Omega)$ such that

$$\begin{gathered}(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) = -[\boldsymbol{f}, \boldsymbol{\varphi}], \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega) \\ \int_S v_n = \Phi,\end{gathered} \tag{VI.5.2}$$

where $[\boldsymbol{f}, \boldsymbol{\varphi}]$ denotes the value of $\boldsymbol{f}$ at $\boldsymbol{\varphi}$. The aim of this section is to provide further existence results for problem (VI.5.2) with $\boldsymbol{v}$ in $D^{1,q}(\Omega)$, $q \neq 2$, together with corresponding estimates. Moreover, we shall also derive the asymptotic structure of such solutions.

In what follows, we shall denote by $D_0^{-1,q}(\Omega)$ the dual space of $D_0^{1,q'}(\Omega)$ and by $|\boldsymbol{f}|_{-1,q}$ the norm of $\boldsymbol{f} \in D_0^{-1,q}(\Omega)$. We also set

$$\Sigma = S \cup_{i=1}^m \omega_i$$

where ω_i, $i = 1, \dots m$, are the connected components of $\mathbb{R}^{n-1} - S$, and

$$\delta_S = \delta(\Sigma).$$

Finally, with the origin of coordinates in Σ, we put

$$\Omega_\pm^\delta = \Omega^{2\delta_S} \cap \mathbb{R}^n_\pm.$$

The following theorem holds.

Theorem VI.5.1 *Let Ω be given in* (VI.5.1), *$n \geq 2$, with S a bounded, locally Lipschitz domain of $\mathbb{R}^{n-1}$. Given*

$$\boldsymbol{f} \in D_0^{-1,2}(\Omega) \cap D_0^{-1,q}(\Omega), \quad \Phi \in \mathbb{R},$$

where $q \in (1, 2n/(n-2)]$ if $n \geq 3$, and $q \in (1, \infty)$ if $n = 2$, there exists one and only one field $\boldsymbol{v}$ satisfying (VI.5.2) *with*

$$\boldsymbol{v} \in D_0^{1,2}(\Omega) \cap D_0^{1,q}(\Omega_\pm^\delta).$$

Moreover, denoting by p the pressure field associated to $\boldsymbol{v}$ by Lemma IV.1.1, *there exist constants p_+, $p_- \in \mathbb{R}$ such that*

$$p - p_\pm \in L^2(\mathbb{R}^n_\pm) \cap L^q((\Omega^\delta_\pm),$$

and the following estimate holds:

$$\begin{aligned}|\boldsymbol{v}|_{1,2} + |\boldsymbol{v}|_{1,q,\Omega^\delta_\pm} + \|p - p_+\|_{2,\mathbb{R}^n_+} + \|p - p_-\|_{2,\mathbb{R}^n_-} \\ +\|p - p_+\|_{q,\Omega^\delta_+} + \|p - p_-\|_{q,\Omega^\delta_-} \\ \le c\,(|\boldsymbol{f}|_{-1,2} + |\boldsymbol{f}|_{-1,q} + |\Phi|)\,.\end{aligned}$$

Proof. The uniqueness part is proved exactly as in Theorem VI.3.1, and it is left to the reader as an exercise. To show existence, we look for a solution of the form

$$\boldsymbol{v} = \boldsymbol{u} + \Phi\boldsymbol{b}$$

where the field $\boldsymbol{b}$ has been given at the beginning of Section III.4.3. We have (see (III.4.6)–(III.4.8))

$$\begin{aligned}&\boldsymbol{b} \in C^\infty(\Omega)\\ &\nabla\cdot\boldsymbol{b}(x) = 0\\ &|\boldsymbol{b}(x)| \le c|x|^{-n+1}\\ &|\nabla\boldsymbol{b}(x)| \le c|x|^{-n}\\ &\int_S b_n = 1\end{aligned}$$

while $\boldsymbol{u}$ obeys

$$\begin{aligned}(\nabla\boldsymbol{u}, \nabla\boldsymbol{\varphi}) = -(\nabla\boldsymbol{b}, \nabla\boldsymbol{\varphi}) - [\boldsymbol{f}, \boldsymbol{\varphi}]\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega)\\ \int_S u_n = 0.\end{aligned} \tag{VI.5.3}$$

Since the right-hand side of $(\text{VI.5.3})_1$ defines a bounded linear functional in $D_0^{-1,2}(\Omega)$, we deduce the existence of $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ satisfying (VI.5.3). Furthermore, putting $\boldsymbol{u}$ in place of $\boldsymbol{\varphi}$ into $(\text{VI.5.3})_1$ (this can be done by a simplest density procedure) furnishes

$$|\boldsymbol{u}|^2_{1,2} \le (|\Phi||\boldsymbol{b}|_{1,2} + |\boldsymbol{f}|_{-1,2})\,|\boldsymbol{u}|_{1,2}$$

implying

$$|\boldsymbol{v}|_{1,2} \le c_1\,(|\Phi| + |\boldsymbol{f}|_{-1,2}) \tag{VI.5.4}$$

with

$$c_1 = (1 + 2|\boldsymbol{b}|_{1,2}).$$

In view of Lemma IV.1.1 we can associate to $\boldsymbol{v}$ a pressure field $p \in L^2_{loc}(\Omega)$ such that

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) = (p, \nabla \cdot \boldsymbol{\psi}) - [\boldsymbol{f}, \boldsymbol{\psi}], \quad \text{for all } \boldsymbol{\psi} \in C_0^\infty(\Omega). \tag{VI.5.5}$$

Let us show that there exist constants $p_+, p_- \in \mathbb{R}$ such that

$$\begin{aligned} p - p_+ &\in L^2(\mathbb{R}^n_+) \\ p - p_- &\in L^2(\mathbb{R}^n_-) \end{aligned} \tag{VI.5.6}$$

and that the following inequality holds

$$\|p - p_+\|_{2,\mathbb{R}^n_+} + \|p - p_-\|_{2,\mathbb{R}^n_-} \le c_2 \left(|\boldsymbol{f}|_{-1,2} + |\boldsymbol{v}|_{1,2}\right). \tag{VI.5.7}$$

Actually, consider the functional

$$\mathcal{F}(\psi) = (\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) + [\boldsymbol{f}, \psi], \quad \boldsymbol{\psi} \in D_0^{1,2}(\mathbb{R}^n_+).$$

Clearly, $\mathcal{F}$ is linear and bounded in $\boldsymbol{\psi} \in D_0^{1,2}(\mathbb{R}^n_+)$ and, in view of (VI.5.2), it vanishes identically on $\mathcal{D}_0^{1,2}(\mathbb{R}^n_+)$. Thus, by Corollary III.5.1, there exists $\pi_+ \in L^2(\mathbb{R}^n_+)$ such that

$$\mathcal{F}(\boldsymbol{\psi}) = (\pi_+, \nabla \cdot \boldsymbol{\psi})$$

which, once compared with (VI.5.5), proves $(\text{VI.5.6})_1$. In a completely analogous way one shows $(\text{VI.5.6})_2$. We now write (VI.5.5) with $p - p_+$ in place of p. By density, we deduce that (VI.5.5) continues to hold for all $\boldsymbol{\psi} \in D_0^{1,2}(\mathbb{R}^n_+)$. Choosing $\boldsymbol{\psi}$ as a solution to the problem

$$\begin{aligned} &\nabla \cdot \boldsymbol{\psi} = p - p_+ \\ &\boldsymbol{\psi} \in D_0^{1,2}(\mathbb{R}^n_+) \\ &|\boldsymbol{\psi}|_{1,2} \le c\|p - p_+\|_{2,\mathbb{R}^n_+} \end{aligned} \tag{VI.5.8}$$

(such a solution certainly exists in view of Corollary IV.3.1), from (VI.5.5) and (VI.5.8) we obtain

$$\|p - p_+\|^2_{2,\mathbb{R}^n_+} = (\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) + [\boldsymbol{f}, \boldsymbol{\psi}] \le c\left(|\boldsymbol{v}|_{1,2} + |\boldsymbol{f}|_{-1,2}\right) \|p - p_+\|_{2,\mathbb{R}^n_+}.$$

Analogously,

$$\|p - p_+\|_{2,\mathbb{R}^n_+} \le c\left(|\boldsymbol{v}|_{1,2} + |\boldsymbol{f}|_{-1,2}\right)$$

and (VI.5.7) follows from these latter inequalities. We shall next derive estimates for $\boldsymbol{v}$ in $D^{1,q}$. For $R \ge 2\delta_S$, we let $\zeta = \zeta(|x|)$ denote a non-decreasing, smooth function with $\zeta(|x|) = 0$ if $|x| \le R/2$ and $\zeta(|x|) = 1$ if $|x| \ge R$. Setting

$$\boldsymbol{w} = \zeta \boldsymbol{v}, \quad \tau = \zeta(p - p_\pm),$$

we easily obtain that $\boldsymbol{w}$ is a generalized solution to the following problem (see Section IV.3):

$$\left.\begin{aligned} \Delta \boldsymbol{w} &= \nabla \tau + \boldsymbol{F}_{\pm} \\ \nabla \cdot \boldsymbol{w} &= g \end{aligned}\right\} \text{ in } \mathbb{R}^n_{\pm} \tag{VI.5.9}$$
$$\boldsymbol{w} = 0 \ \text{ on } \Sigma \equiv \mathbb{R}^{n-1} \times \{0\},$$

where

$$\begin{aligned} \boldsymbol{F}_{\pm} &= 2\nabla\zeta \cdot \nabla \boldsymbol{v} - \Delta\zeta \boldsymbol{v} - (p - p_{\pm})\nabla\zeta + \zeta \boldsymbol{f} \\ g &= \nabla\zeta \cdot \boldsymbol{v}. \end{aligned}$$

For $R > 0$, we put

$$\begin{aligned} \Omega^{\pm}_{R,2R} &= \left\{x \in \mathbb{R}^n_{\pm} : R < |x| < 2R\right\} \\ \Omega^{\pm}_{R} &= \left\{x \in \mathbb{R}^n_{\pm} : |x| < R\right\}, \\ \Omega^{R} &= \mathbb{R}^n - \overline{B}_R, \\ \Omega^{R}_{\pm} &= \mathbb{R}^n_{\pm} \cap \Omega^R. \end{aligned}$$

Evidently,

$$\begin{aligned} |\boldsymbol{F}_{\pm}|_{-1,q} &\le c_3 \left(\|\boldsymbol{v}\|_{q,\Omega^{\pm}_{R,2R}} + \|p - p_{\pm}\|_{-1,q,\Omega^{\pm}_{R,2R}} + |\boldsymbol{f}|_{-1,q} \right) \\ \|g\|_q &\le \|\boldsymbol{v}\|_{q,\Omega^{\pm}_{R,2R}}. \end{aligned} \tag{VI.5.10}$$

Employing the embedding Theorem II.3.4 and inequality (II.5.18), from the assumptions made on q we easily find

$$\|\boldsymbol{v}\|_{q,\Omega^{\pm}_{2R}} + \|p - p_{\pm}\|_{-1,q,\Omega^{\pm}_{2R}} \le c\left(|\boldsymbol{v}|_{1,2} + \|p - p_{\pm}\|_2\right). \tag{VI.5.11}$$

Thus, recalling that $\zeta(|x|)$ is equal to one for $|x| \ge R$, from Theorem IV.3.3 together with (VI.5.4), (VI.5.7), and (VI.5.11) we find

$$|\boldsymbol{v}|_{1,q,\Omega^R} + \|p - p_+\|_{q,\Omega^R_+} + \|p - p_-\|_{q,\Omega^R_-} \le c\left(|\boldsymbol{f}|_{-1,2} + |\boldsymbol{f}|_{-1,q}\right) \tag{VI.5.12}$$

and the theorem follows from (VI.5.12), (VI.5.11), (VI.5.4), and (VI.5.7). □

Remark VI.5.1 The fact that, in the theorem just shown, we must require that $\boldsymbol{f}$ belong simultaneously to $D_0^{-1,2}(\Omega)$ and to $D_0^{-1,q}(\Omega)$, that q be suitably restricted, and that $\nabla \boldsymbol{v}, p$ belong to $L^q(\Omega \cap B_R)$, only for sufficiently large R, is due to the circumstance that Ω has no regularity near the boundary of S, since S is $(n-1)$-dimensional. However, if we assume that the "hole" S has "thickness," becoming a domain of $\mathbb{R}^n$ of class C^2, in such a way that Ω becomes likewise of class C^2, then one can show that, for any $\boldsymbol{f} \in D_0^{-1,q}(\Omega)$, $1 < q < \infty$, there is one and only one solution $\boldsymbol{v} \in D_0^{1,q}(\Omega)$ to (VI.5.2) and that the associated pressure field p satisfies $p - p_+ \in L^q(\mathbb{R}^n_+)$, $p - p_- \in L^q(\mathbb{R}^n_-)$ for some constants p_+, p_-. Moreover, $\boldsymbol{v}$ and p obey the corresponding estimates. Finally, we would like to observe that, if S has no "thickness," and $n = 2$, one can still prove that $\boldsymbol{v} \in L^\infty(\Omega)$, see Solonnikov (1988) and Galdi, Padula & Solonnikov (1996). ■

Remark VI.5.2 Arguing as in Remark VI.4.7, one can show that, for $\boldsymbol{f} \equiv 0$, $p_+ = p_-$ if and only if the flux Φ is zero. Actually, if $\boldsymbol{f} \not\equiv 0$, one cannot deduce $p_+ = p_-$, even if $\Phi = 0$. In fact, taking, for instance, $\boldsymbol{f} \in C_0^\infty(\Omega)$ we could prove that, for $\phi = 0$, the following relation holds:

$$p_- - p_+ = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w},$$

where $\boldsymbol{w}$ is a solution to (VI.5.2) with $\boldsymbol{f} \equiv 0$ and $\Phi = 1$. ■

In the last part of this section we shall analyze the asymptotic structure of the generalized solutions just obtained. We begin to show a general representation formula in the case when $\boldsymbol{f}$ is in divergence form.

Lemma VI.5.1 *Let $\boldsymbol{v} \in \widehat{\mathcal{D}}_0^{1,q}(\Omega)$, $1 < q < \infty$, satisfy* $(\text{VI.5.2})_1$. *The following assertions hold true:*

(i) *Suppose*

$$[\boldsymbol{f}, \boldsymbol{\varphi}] = -(\boldsymbol{F}, \nabla\boldsymbol{\varphi}),$$

where

$$\boldsymbol{F} \in L^q(\Omega) \cap L^r(\Omega) \quad , \text{ for some } r \in (1, n).$$

Then, for a.a. $x \in \mathbb{R}^n_\pm$:

$$v_j(x) = -\int_{\mathbb{R}^n_\pm} D_\ell G^\pm_{ij}(x,y) F_{\ell i}(y) dy - \int_S v_i(y) T_{i\ell}(\boldsymbol{G}^\pm_j, g^\pm_j)(x,y) n_\ell(y) d\sigma_y, \tag{VI.5.13}$$

where $\boldsymbol{G}^\pm = \{G^\pm_{ij}\}$, $\boldsymbol{g}^\pm = \{g^\pm_i\}$ is the Green's tensor for the Stokes problem in $\mathbb{R}^n_\pm$ (see (IV.3.46)–(IV.3.51)*), and $\boldsymbol{G}^\pm_j \equiv (G^\pm_{1j}, G^\pm_{2j}, \dots, G^\pm_{nj})$.*

(ii) *Suppose that*

$$\boldsymbol{f} \in L^q(\Omega), \quad \text{with bounded support.}$$

Then, there exist $p_+, p_- \in \mathbb{R}$ such that for a.a. $x \in \mathbb{R}^n_\pm$:

$$\begin{aligned} v_j(x) &= \int_{\mathbb{R}^n_\pm} G^\pm_{ij}(x,y) f_i(y) dy - \int_S v_i(y) T_{i\ell}(\boldsymbol{G}^\pm_j, g^\pm_j)(x,y) n_\ell(y) d\sigma_y \\ p(x) &= p_\pm - \int_{\mathbb{R}^n_\pm} g^\pm_i(x,y) f_i(y) dy - 2\int_S v_i(y) \frac{\partial g^\pm_i(x-y)}{\partial x_\ell} n_\ell(y) d\sigma_y, \end{aligned} \tag{VI.5.14}$$

where p is the pressure field associated to $\boldsymbol{v}$ by Lemma IV.1.1.

Proof. Since the proof is exactly the same for $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$, we shall show the validity of (i) and (ii) for $\mathbb{R}^n_+$. Moreover, for simplicity, the Green's tensor in $\mathbb{R}^n_+$ will be denoted by $\boldsymbol{G}, \boldsymbol{g}$. Let us commence to show (i). We begin to observe that, reasoning as in the proof of Theorem VI.5.1, we can associate to

$\boldsymbol{v}$ a pressure field p in the sense of Lemma IV.1.1 such that $p - p_+ \in L^q(\mathbb{R}^n_+)$. We next notice that from inequality (II.5.18) it follows that $\boldsymbol{v} \in W^{1,q}(C)$, for every cube C with a side at $x_n = 0$ and so, setting $\Sigma = \mathbb{R}^{n-1} \times \{0\}$, the trace of $\boldsymbol{v}$ at Σ belongs $W^{1-1/q,q}(S) \subset D^{1-1/q,q}(\Sigma)$ (see Sections II.3 and II.6) with support contained in S. Therefore, in view of Theorem II.4.1 and Theorem II.3.1, there exist two sequences

$$\{\boldsymbol{F}^{(k)}\} \subset C_0^\infty(\overline{\mathbb{R}}^n_+), \quad \{\eta^{(k)}\} \subset C_0^\infty(\Sigma)$$

approximating $\boldsymbol{F}$ and $\boldsymbol{v}$ in the norms of $L^q(\mathbb{R}^n_+) \cap L^r(\mathbb{R}^n_+)$ and $D^{1-1/q,q}(\Sigma)$, respectively. Let us denote by $\boldsymbol{v}^{(k)}$ and $p^{(k)}$, $k \in \mathbb{N}$, velocity and pressure fields of the Stokes problem in $\mathbb{R}^n_+$, corresponding to the force $-\nabla \cdot \boldsymbol{F}^{(k)}$ and to the velocity $\eta^{(k)}$ at Σ . From Theorem IV.3.3, we obtain that, as $k \to \infty$, $\boldsymbol{v}^{(k)}, p^{(k)}$ converge to $\boldsymbol{v}, p - p_+$ in the norm of $D^{1,q}(\mathbb{R}^n_+) \times L^q(\mathbb{R}^n_+)$. Therefore, in particular, we may select a subsequence, denoted again by $\boldsymbol{v}^{(k)}, p^{(k)}$, such that

$$\nabla \boldsymbol{v}^{(k)}(x) \to \nabla \boldsymbol{v}(x), \quad p^{(k)}(x) \to p(x) \quad a.e. \text{ in } \mathbb{R}^n_+. \tag{VI.5.15}$$

Since $\boldsymbol{v}$ and $\boldsymbol{v}_k$ are identically vanishing on $\Sigma - \overline{S}$, from (II.5.18) we readily obtain

$$\|\boldsymbol{v} - \boldsymbol{v}^{(k)}\|_{q,C} \le c(C)|\boldsymbol{v} - \boldsymbol{v}^{(k)}|_{1,q,C}, \tag{VI.5.16}$$

for any cube C with a side at $x_n = 0$ strictly containing $\overline{S}$. Choosing an increasing sequence of cubes of type C invading $\mathbb{R}^n_+$ and employing the Cantor diagonalization method, from (VI.5.15) and (VI.5.16) we finally deduce the existence of a sequence, which will be denoted again by $\boldsymbol{v}^{(k)}$, satisfying

$$\lim_{k\to\infty} \boldsymbol{v}^{(k)}(x) = \boldsymbol{v}(x), \quad \text{for a.a. } x \in \mathbb{R}^n_+. \tag{VI.5.17}$$

Now, because of the results of Exercise IV.8.2, the following representation holds for $\boldsymbol{v}^{(k)}$:

$$v_j^{(k)}(x) = \int_{\mathbb{R}^n_+} D_\ell G_{ij}(x,y) F_{\ell i}^{(k)}(y)dy + \int_S \eta_i^{(k)}(y) T_{i\ell}(\boldsymbol{G}_j, g_j)(x,y) n_\ell(y) d\sigma_y. \tag{VI.5.18}$$

We wish to let $k \to \infty$ into this relation. Set

$$\mathcal{F}^{(k)}(x) = \int_{\mathbb{R}^n_+} D_\ell G_{ij}(x,y)[F_{\ell i}^{(k)}(y) - F_{\ell i}(y)]dy. \tag{VI.5.19}$$

Since

$$|D_\ell G_{ij}(x,y)| \le c|x-y|^{-n+1}$$

(see (IV.3.50), (IV.3.52)), from the assumption made on $\boldsymbol{F}$ and the Sobolev Theorem II.11.3, we derive, along a subsequence at least,

$$\lim_{k\to\infty} \mathcal{F}^{(k)}(x) = 0 \quad \text{for a.a. } x \in \mathbb{R}^n_+. \tag{VI.5.20}$$

It remains to prove the convergence of the last term in (VI.5.18). Taking into account

$$|T_{i\ell}(\boldsymbol{G}_j, g_j)(x,y)| \le c|x-y|^{-n+1}$$

(see (IV.3.50), (IV.3.52)), and setting

$$\mathcal{V}^{(k)}(x) = \int_S [\eta_i^{(k)}(y) - v_i(y)] T_{i\ell}(\boldsymbol{G}_j, g_j)(x,y) n_\ell(y) d\sigma_y, \tag{VI.5.21}$$

from the Hölder inequality we obtain

$$|\mathcal{V}^{(k)}(x)| \le c\|\eta^{(k)} - \boldsymbol{v}\|_{q,S},$$

where $c = c(S,d)$, $d = \mathrm{dist}\,(x,S)$. Thus,

$$\mathcal{V}^{(k)}(x) \to 0 \ \text{ in } \mathbb{R}^n_+. \tag{VI.5.22}$$

Representation (VI.5.13) is then a consequence of (VI.5.17)–(VI.5.22). The proof of (VI.5.14) is similar. Actually, we now start with a sequence of functions $\{\boldsymbol{f}^{(k)}\}, \{\eta^{(k)}\}$ where $\eta^{(k)}$ is the same as before, while $\boldsymbol{f}^{(k)} \in C_0^\infty(\omega)$, with $\omega = \mathrm{supp}\,(\boldsymbol{f})$, converge to $\boldsymbol{f}$ in $L^q(\Omega) \cap L^r(\Omega)$. By the same technique used before, we then show the validity of $(\text{VI.5.14})_1$, with the only change that, to prove the convergence of the term

$$\int_{\mathbb{R}^n_+} G_{ij}(x,y) f_i^{(k)}(y) dy = \int_\omega G_{ij}(x,y) f_i^{(k)}(y) dy,$$

we have to employ exactly the same reasoning used to show the convergence of the term V_j in the proof ofTheorem IV.8.1. This is made possible by the fact that $\boldsymbol{G}$ and $\boldsymbol{U}$ obey pointwise estimates of the same type. Concerning the representation of the pressure, we easily establish, as before, the *a.e.* pointwise convergence of $p^{(k)}$ to $p - p_+$. Moreover, the *a.e.* pointwise convergence of the term

$$\int_\omega g_i(x,y) f_i^{(k)}(y) dy$$

is acquired by taking into account the estimate

$$|\boldsymbol{g}(x,y)| \le c|x-y|^{-n+1}$$

(see (IV.3.50), (IV.3.51)) and Exercise IV.3.3, and by using a reasoning similar to that adopted to show the convergence of the term P in the proof of Theorem IV.8.1 (details are left to the reader). Finally, observing that again from (IV.3.50), (IV.3.51), and Exercise IV.3.3,

$$|\nabla \boldsymbol{g}(x,y)| \le c|x-y|^{-n},$$

we have

$$\left| \int_S [\eta_i^{(k)}(y) - v_i(y)] n_\ell(y) d\sigma_y \right| \le c\|\eta^{(k)} - \boldsymbol{v}\|_{q,S}$$

which also implies the pointwise convergence of the boundary integral. The lemma is therefore completely proved. □

The result just shown furnishes the following one as a simple corollary.

Theorem VI.5.2 *Let $\boldsymbol{v} \in \widehat{\mathcal{D}}_0^{1,q}(\Omega)$, $1 < q < \infty$, satisfy* (VI.5.2)$_1$ *corresponding to $\boldsymbol{f} \in L^q(\Omega)$ of bounded support. Then, there exist constants p_+, p_- such that $\boldsymbol{v}$ and the corresponding pressure field p admit the following asymptotic expansion as $|x| \to \infty$ in $\mathbb{R}^n_\pm$:*

$$\begin{aligned} v_j(x) &= b_i^\pm T_{in}(\boldsymbol{G}_j^\pm, \boldsymbol{g}_j^\pm)(x,0) + \varphi_j^\pm(x) \\ p(x) &= p_\pm + 2b_i^\pm D_n(g_i^\pm)(x,0) + \beta^\pm(x), \end{aligned} \tag{VI.5.23}$$

where

$$b_i^\pm = \pm \int_S v_i \tag{VI.5.24}$$

and, for all $|\alpha| \geq 0$,

$$\begin{aligned} D^\alpha \varphi_j^\pm &= O(|x|^{-n+1-|\alpha|}) \\ D^\alpha \beta^\pm &= O(|x|^{-n-|\alpha|}). \end{aligned} \tag{VI.5.25}$$

In particular, if $\boldsymbol{f} \equiv 0$, then

$$\begin{aligned} D^\alpha \varphi_j^\pm &= O(|x|^{-n-|\alpha|}) \\ D^\alpha \beta^\pm &= O(|x|^{-n-1-|\alpha|}). \end{aligned} \tag{VI.5.26}$$

Proof. By the fundamental property of the Green's tensor we have that $\boldsymbol{G}^\pm(x,0) = 0$ for all $x \in \mathbb{R}^n_\pm$. Therefore, from (VI.5.14)$_1$ we find

$$\begin{aligned} v_j(x) = b_i^\pm T_{in}(\boldsymbol{G}_j^\pm, \boldsymbol{g}_j^\pm)(x,0) &+ \int_{\mathbb{R}^n_\pm} [G_{ij}^\pm(x,y) - G_{ij}^\pm(x,0)] f_i(y) dy \\ &- \int_S v_i(y)[T_{i\ell}(\boldsymbol{G}_j^\pm, g_j^\pm)(x,y) - T_{i\ell}(\boldsymbol{G}_j^\pm, g_j^\pm)(x,0)] n_\ell(y) d\sigma_y. \end{aligned} \tag{VI.5.27}$$

Applying the Lagrange theorem in the integrands of (VI.5.27) and using (IV.3.50), (IV.3.51), and Exercise IV.3.3 we may proceed as in the proof of Theorem V.3.2 to show the validity of (VI.5.23)$_1$, (VI.5.24), and (VI.5.25)$_1$ and, for $\boldsymbol{f} \equiv 0$, of ((VI.5.26)$_1$. Observing that, by (IV.3.46)–(IV.3.49), it also follows that $\boldsymbol{g}(x,0) = 0$, we may establish in a completely analogous way (VI.5.23)$_2$, (VI.5.24), (VI.5.25)$_2$, and, for $\boldsymbol{f} \equiv \boldsymbol{0}$, (VI.5.26)$_2$. The proof of the theorem is acquired. □

Remark VI.5.3 In view of the estimates on $\boldsymbol{G}$ and $\boldsymbol{g}$ given in (IV.3.50), (IV.3.51) and Exercise IV.3.3, Theorem VI.5.2 implies, in particular, that at large distances, $\boldsymbol{v}$ behaves as $|x|^{-n+1}$ for $n \geq 2$. ■

VI.6 Notes for the Chapter

Section VI.1. Although differing in details, the material presented here is based on the treatment of Amick (1977). In particular, Theorem VI.1.2 can be deduced from the work of this author.

Section VI.2. The main result of this section, Theorem VI.2.2, is due to me. It has been obtained by coupling the ideas of Horgan & Wheeler (1978) with those of Amick (1977,1978) and of Ladyzhenskaya & Solonnikov (1980). In particular, Lemma VI.2.1 and Theorem VI.2.1 are due to Ladyzhenskaya and Solonnikov, while Lemma VI.2.2 is proved by Horgan and Wheeler. Somewhat weaker results than those of Theorem VI.2.2 can be deduced from the papers of Horgan (1978) and Ames & Payne (1989). Extension of these results to compressible fluids has been recently proved by Padula & Pileckas (1992, §7).

Section VI.3. The guiding ideas are essentially taken from the works of Heywood (1976, §6) and Solonnikov & Pileckas (1977).

Concerning domains with varying cross-sections (not necessarily unbounded), we refer the reader to the papers of Fraenkel (1973), Iosif'jan (1978), Pileckas (1981), and Nazarov & Pileckas (1983).

Section VI.4. The approach proposed here is due to me. The proof of Theorem VI.4.3 is inspired by the work of Gilbarg & Weinberger (1978, §4).

The study of certain asymptotic behavior in domains with outlets containing a semi-infinite cone has been performed by Pileckas (1980a).

Results on existence, uniqueness and asymptotic decay of solutions in domains that become "layer-like" at infinity are provided by Nazarov & Pileckas (1999a, 2001). It is interesting to observe that, for $n = 3$, solutions show only a power-like decay, and not an exponential one.

Section VI.5. The "flow through an aperture" (or "flow through a slit" in the two-dimensional case) is a well studied problem in classical fluid dynamics, mostly, for its applications to resonance phenomena in narrow-mouthed harbors; see, e.g. Miles & Lee (1975). As a matter of fact, in absence of body forces, explicit solutions can be exhibited in the two-dimensional inviscid case (Lamb 1932, p. 73; Milne-Thomson 1938, §§6.10, 11.53) and in the viscous case as well (Stokes problem), when the aperture is a circle (Milne-Thomson 1938, §15.56). In the mathematical community, seemingly, this type of flow became popular and thoroughly investigated in its viscous formulation, only after the publication of the fundamental paper of Heywood (1976).

The theory described in this section is due to Galdi & Sohr (1992); see also Farwig and Sohr (1994b). Similar results have been obtained, independently and by different tools, by Borchers & Pileckas (1992). However, the asymptotic estimates given in Theorem VI.5.2 are somewhat more detailed than those provided by the latter authors.

VII

Steady Oseen Flow in Exterior Domains

> e vidi le fiammelle andare avante,
> lasciando retro a sè l'aer dipinto.
>
> DANTE, Purgatorio XXIX, vv. 73-74

Introduction

As we emphasized in the Introduction to Chapter V, the Stokes approximation may fail to describe the physical properties of a system constituted by an object $\mathcal{B}$ moving by assigned rigid motion with "small" translational ($\boldsymbol{v}_0$) and angular ($\boldsymbol{\omega}$) velocities in a viscous liquid, at least at "large" distances from $\mathcal{B}$, where the viscous effects become less important.

In particular, for $\mathcal{B}$ a ball translating without rotating (that is, $\boldsymbol{\omega} = \mathbf{0}$), the explicit solution one finds (see (V.0.4)) exhibits no wake behind the body and is, therefore, unacceptable from the physical viewpoint. Moreover, for $\mathcal{B}$ a circle (plane motion), the analogous problem admits no solution except for the trivial one, thus leading to the *Stokes paradox*. It is interesting to remark that a sort of similar paradox also arises in the three-dimensional case, the moment one tries to evaluate the first-order (in the Reynolds number) correction to the zero-*th* order solution (V.0.4); see Whitehead (1888). In addition to all the above, as observed by Oseen (1927, p.165), for the solution (V.0.4) we obtain, after a simple calculation,

$$\left|\frac{\boldsymbol{v}\cdot\nabla\boldsymbol{v}}{\Delta\boldsymbol{v}}\right| \to \infty \quad \text{as } |x| \to \infty,$$

no matter how small $|\boldsymbol{v}_0|$ is, thus violating the assumption under which the Stokes equations are derived (see the Introduction to Chapter IV).

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DOI 10.1007/978-0-387-09620-9_7,

As we already remarked, it is reasonable to argue that these "anomalous behaviors" must be chiefly ascribed to the fact that the Stokes approximation completely disregards the inertia of the liquid or, in equivalent mathematical terms, it ignores possibly significant information arising from the nonlinear term in the Navier-Stokes equation $(\mathrm{I.0.1})_1$. One is thus naturally lead to introduce other linearizations of $(\mathrm{I.0.1})_1$ that may, somehow, take into account this feature.

With this in mind, C.W. Oseen proposed in 1910 (see also Oseen 1927, §15) a linearization of the Navier-Stokes equations with the main objective of avoiding the paradoxes and the incongruities related to the Stokes approximation.[1] The original equations introduced by Oseen (which we will refer to as *Oseen approximation*) are formally obtained by linearizing the Navier-Stokes equations around a *nonzero* purely translational motion $\boldsymbol{v} = \boldsymbol{v}_0$, $p = p_0$, where $\boldsymbol{v}_0$ and p_0 are given constant vector and scalar quantities, respectively. However, for reasons that are mainly dictated by a considerable number of significant applications (see Galdi 2002, and the references therein), we shall analyze a more general approximation (which we will refer to as *generalized Oseen approximation*), consisting of linearizing $(\mathrm{I.2.2})_1$ around the (nonzero) *rigid motion*, $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{x} \equiv \boldsymbol{V}(x)$, $p = p_0$, where $\boldsymbol{v}_0$ and $\boldsymbol{\omega}$ are prescribed (constant) vectors and p_0 is a given scalar quantity. We recall that, from a physical viewpoint, $\boldsymbol{v}_0$ and $\boldsymbol{\omega}$ represent the (constant) translational and the angular velocity of the body $\mathcal{B}$, respectively, when the motion of the liquid is referred to a frame attached to $\mathcal{B}$.[2]

Thus, denoting by Ω the exterior region occupied by the liquid, from $(\mathrm{I.2.3})_1$ we obtain the following *generalized Oseen system*[3]

$$\left.\begin{aligned} \nu\Delta\boldsymbol{v} + \boldsymbol{V}\cdot\nabla\boldsymbol{v} - \boldsymbol{\omega}\times\boldsymbol{v} &= \nabla p + \boldsymbol{f} \\ \nabla\cdot\boldsymbol{v} &= 0 \end{aligned}\right\}\ \text{in } \Omega$$
$$\boldsymbol{v} = \boldsymbol{v}_* \ \text{ at } \partial\Omega, \tag{VII.0.1}$$

where $\boldsymbol{v}_*$ is a prescribed field at the boundary wall. To (VII.0.1) we append the condition at infinity[4]

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = \boldsymbol{0}\,. \tag{VII.0.2}$$

[1] As kindly pointed out to me by Professor Josef Bemelmans, an independent analysis of these questions, mostly motivated by the study of the range of validity of Stokes formula for the drag, was performed by Fritz Noether (1911).

[2] See also the introductions to Chapter X and Chapter XI.

[3] Sometimes, the system (VII.0.1) with $\boldsymbol{V} = \boldsymbol{0}$ is also referred to as *Sobolev system*; see, e.g., Maslennikova (1973).

[4] The Oseen approximation is typical for a flow occurring in an exterior region. In a bounded region it loses its physical meaning, while, from the mathematical point of view, it presents no difficulties and can be handled as a corollary to the theory developed for the Stokes problem in Section IV.4–Section IV.6; see Theorem VII.1.1.

In the current chapter we begin to investigate the properties of solutions to problem (VII.0.1)–(VII.0.2) in the simpler case when $\boldsymbol{\omega} = \mathbf{0}$, that is, to the problem originally formulated by Oseen, whereas in the next chapter we shall study it in its full generality, namely, with both $\boldsymbol{v}_0$ and $\boldsymbol{\omega}$ being non-zero.

It should be stressed that results from the original Oseen approximation have long been recognized to be much more successful than that of Stokes. As a matter of fact, at least in the particular case of the translational motion of a ball into a liquid, Oseen found a paraboloidal wake region behind the body (Oseen 1910, 1927 §16; Goldstein 1929). Furthermore, in the two-dimensional analogue, i.e., an infinite circular cylinder moving steadily in a viscous liquid, Lamb (1911) first proved the existence of a solution to (VII.0.1), (VII.0.2) with $\boldsymbol{V} \equiv \boldsymbol{v}_0 \neq \mathbf{0}$, that exhibit a wake region, thus removing the paradox coming from the Stokes approximation.

The aim of this chapter is to investigate existence, uniqueness, and the validity of corresponding estimates in homogeneous Sobolev spaces $D^{m,q}$ for solutions to (VII.0.1), (VII.0.2) with $\boldsymbol{V} \equiv \boldsymbol{v}_0 \neq \mathbf{0}$, in an arbitrary exterior domain Ω. All main ideas are taken from Galdi (1992).

The lines we shall follow are essentially the same we followed in Chapter V for the exterior Stokes problem, even though the study is here somehow complicated by the more involved form of the fundamental solution to $(\text{VII.0.1})_{1,2}$ in the whole space $\mathbb{R}^n$. However, because of the different structure of the equations, the results we shall obtain are substantially different from those proved for the Stokes problem. In this respect, we will show that problem (VII.0.1), (VII.0.2) (with $\boldsymbol{V} \equiv \boldsymbol{v}_0 \neq \mathbf{0}$ and with sufficiently smooth data) is solvable in three dimensions and two dimensions and that, if $\boldsymbol{f}$ is of bounded support, the corresponding solutions exhibit a paraboloidal "wake region" in a direction opposite to $\boldsymbol{v}_0$. This fact implies, in particular, that for problem (VII.0.1), (VII.0.2) with $\boldsymbol{V} \equiv \boldsymbol{v}_0 \neq \mathbf{0}$, no "Stokes paradox" arises and that the Oseen approximation is, in this respect, better than that proposed by Stokes.[5] Also, as in the Stokes problem, the existence of q-generalized (in $D^{1,q}$) and "strong" solutions (in $D^{m,q}$, $m > 1$) is proved only for q in a certain range $\mathcal{R}_n$ depending on the space dimension n; however, we find that $\mathcal{R}_n$ is *larger* than the analogous range $\mathcal{R}'_n$ for the Stokes problem. Precisely, we show that, formally, $\mathcal{R}_n = \mathcal{R}'_{n+1}$. This circumstance will lead to important consequences in the *nonlinear* context, when treating the motion of an object translating with constant velocity into a viscous liquid; see Chapters X, and XII.

Finally, we shall consider the behavior of solutions to (VII.0.1), (VII.0.2) with $\boldsymbol{V} \equiv \boldsymbol{v}_0$ in the limit of vanishing $\boldsymbol{v}_0$, with special emphasis on the case of plane motion. In this latter circumstance, we find that such solutions tend to those of the analogous Stokes *system*, i.e., (VII.0.1) with $\boldsymbol{v}_0 = 0$. However, as

[5] It should be observed, however, that the Oseen approximation leads to other paradoxical consequences in disagreement with the actual slow motion of a body into a viscous liquid; see Filon (1928), Imai (1951), Olmstead & Hector (1966), Olmstead & Gautesen (1968), and Olmstead (1968); see also Exercise VII.6.5.

expected in view of the Stokes paradox, the limiting process does *not* preserve condition (VII.0.2), which is, in fact, satisfied *if and only if* the data obey the compatibility condition determined in Section V.8.

For later purposes, we shall find it convenient to put (VII.0.1), (VII.0.2) with $\boldsymbol{V} \equiv \boldsymbol{v}_0$ into a suitable dimensionless form, and so we need comparison length d and velocity U. Without loss, we set $\boldsymbol{v}_0 = v_0\,\boldsymbol{e}_1$, $v_0 > 0$, and take $U = v_0$. Moreover, if $|\Omega^c| \neq \emptyset$, we can take $d = \delta(\Omega^c)$, and so, introducing the *Reynolds number*

$$\mathcal{R} = \frac{Ud}{\nu},$$

system (VII.0.1) becomes

$$\left.\begin{aligned} \Delta\boldsymbol{v} + \mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1} &= \nabla p + \mathcal{R}\boldsymbol{f} \\ \nabla\cdot\boldsymbol{v} &= 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\boldsymbol{v} = \boldsymbol{v}_* \ \text{ at } \partial\Omega, \tag{VII.0.3}$$

where $\boldsymbol{v}$, $\boldsymbol{v}_*$, p and $\boldsymbol{f}$ are now nondimensional quantities. If $\Omega \equiv \mathbb{R}^n$ the above choice of d is no longer possible, even though we can still give a meaning to (VII.0.3), which is what we shall do hereafter.

VII.1 Generalized Solutions. Regularity and Uniqueness

In analogy with similar questions treated for the Stokes approximation, we shall begin to give a generalized formulation of the Oseen problem. To this end, let us multiply $(\text{VII.0.3})_1$ by $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and integrate by parts to obtain formally

$$(\nabla\boldsymbol{v}, \nabla\boldsymbol{\varphi}) - \mathcal{R}(\frac{\partial \boldsymbol{v}}{\partial x_1}, \boldsymbol{\varphi}) = -\mathcal{R}[\boldsymbol{f}, \boldsymbol{\varphi}]. \tag{VII.1.1}$$

Definition VII.1.1. A vector field $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ is called a *q-weak* (or *q-generalized*) *solution* to (VII.0.2), (VII.0.3) if for some $q \in (1, \infty)$

(i) $\boldsymbol{v} \in D^{1,q}(\Omega)$;
(ii) $\boldsymbol{v}$ is (weakly) divergence-free in Ω;
(iii) $\boldsymbol{v}$ assumes the value $\boldsymbol{v}_*$ at $\partial\Omega$ (in the trace sense) or, if the velocity at the boundary is zero, $\boldsymbol{v} \in D_0^{1,q}(\Omega)$;
(iv) $\displaystyle\lim_{|x|\to\infty} \int_{S^{n-1}} |\boldsymbol{v}(x)| = 0$;
(v) $\boldsymbol{v}$ verifies (VII.1.1) for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$.

If $q = 2$, $\boldsymbol{v}$ will be simply called a *weak* (or *generalized*) *solution* to (VII.0.2), (VII.0.3).

Remark VII.1.1 If $\boldsymbol{v}$ is a q-weak solution, then, by Lemma II.6.1, we have that $\boldsymbol{v} \in W^{1,q}_{loc}(\Omega)$, and, if Ω is locally Lipschitz, $\boldsymbol{v} \in W^{1,q}_{loc}(\overline{\Omega})$. Regarding (iii), see Remark V.1.1. ■

If the function $\boldsymbol{f}$ has some mild degree of regularity, to each q-weak solution we can associate a corresponding pressure field in the usual way. Specifically, we have the following lemma whose proof, being entirely analogous to that of Lemma IV.1.1, will be omitted.

Lemma VII.1.1 *Let Ω be an exterior domain in $\mathbb{R}^n$, $n \geq 2$. Suppose $\boldsymbol{f} \in W_0^{-1,q}(\Omega')$, $1 < q < \infty$, for any bounded subdomain Ω', with $\overline{\Omega'} \subset \Omega$. Then, to every q-weak solution $\boldsymbol{v}$ we can associate a pressure field $p \in L^q_{loc}(\Omega)$ such that*

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) - \mathcal{R}(\frac{\partial \boldsymbol{v}}{\partial x_1}, \boldsymbol{\psi}) = (p, \nabla \cdot \boldsymbol{\psi}) - \mathcal{R}[\boldsymbol{f}, \boldsymbol{\psi}] \tag{VII.1.2}$$

for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. Furthermore, if Ω is locally Lipschitz and $\boldsymbol{f} \in W_0^{-1,q}(\Omega_R)$, $R > \delta(\Omega^c)$, then $p \in L^q(\Omega_R)$.

Remark VII.1.2 The last result stated in Lemma VII.1.1 is weaker than the analogous one proved for the Stokes problem in Lemma V.1.1, where, for Ω locally Lipschitz, one has $p \in L^q(\Omega)$ whenever $\boldsymbol{f} \in D_0^{-1,q}(\Omega)$. This is due to the fact that, in the case at hand, the functional

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) - \mathcal{R}(\frac{\partial \boldsymbol{v}}{\partial x_1}, \boldsymbol{\psi}) + \mathcal{R}[\boldsymbol{f}, \boldsymbol{\psi}]$$

is *not* continuous in $\boldsymbol{\psi} \in D_0^{1,q'}(\Omega)$ if $\boldsymbol{v} \in D^{1,q}(\Omega)$ *only*, because a priori we can *not* find a constant $c = c(\boldsymbol{v})$ such that

$$|(\frac{\partial \boldsymbol{v}}{\partial x_1}, \boldsymbol{\psi})| \leq c\, |\boldsymbol{\psi}|_{1,q'}\,, \text{ for all } \boldsymbol{\psi} \in C_0^\infty(\Omega)\ .$$

Consequently, we cannot apply Corollary III.5.1 but only the weaker version, Corollary III.5.2. Notice, however, that, by the very definition of q-weak solution, if $\boldsymbol{f} \in \mathcal{D}_0^{-1,q}(\Omega)$, then we can find $C > 0$ such that

$$|(\frac{\partial \boldsymbol{v}}{\partial x_1}, \boldsymbol{\varphi})| \leq C\, |\boldsymbol{\varphi}|_{1,q'}\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega)\,.$$

In any case, by using a completely different approach, in Section VII.7 (see Theorem VII.7.2), we shall show that if the region of motion is of class C^2 and the exponent q ranges in the interval $(n/(n-1), n+1)$, the pressure field p belongs to $L^q(\Omega)$, provided, of course, that $\boldsymbol{f} \in D_0^{-1,q}(\Omega)$. Furthermore, in Theorem VII.7.3 it will be proved that the same property continues to hold for $q \geq n+1$. It seems therefore an *open question* to ascertain whether or not for q-weak solutions with q in the interval $(1, n/(n-1)]$ the corresponding pressure p has a suitable degree of summability at large distances. ■

The next result establishes the regularity of q-weak solutions.

Theorem VII.1.1 *Let* $\boldsymbol{f} \in W^{m,q}_{loc}(\Omega)$, $m \geq 0$, $1 < q < \infty$, *and let*

$$\boldsymbol{v} \in W^{1,q}_{loc}(\Omega), \quad p \in L^q_{loc}(\Omega),\,[1]$$

with $\boldsymbol{v}$ *weakly divergence-free, satisfy* (VII.1.2) *for all* $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. *Then*

$$\boldsymbol{v} \in W^{m+2,q}_{loc}(\Omega), \quad p \in W^{m+1,q}_{loc}(\Omega).$$

In particular, if $\boldsymbol{f} \in C^\infty(\Omega)$, *then* $\boldsymbol{v}, p \in C^\infty(\Omega)$. *Furthermore, if* Ω *is of class* C^{m+2} *and*

$$\boldsymbol{f} \in W^{m,q}_{loc}(\overline{\Omega}), \quad \boldsymbol{v}_* \in W^{m+2-1/q,q}(\partial\Omega),$$

then

$$\boldsymbol{v} \in W^{m+2,q}_{loc}(\overline{\Omega}), \quad p \in W^{m+1,q}_{loc}(\overline{\Omega})\,,$$

provided $\boldsymbol{v} \in W^{1,q}_{loc}(\Omega)$.[2] *In particular, if* Ω *is of class* C^∞ *and* $\boldsymbol{f} \in C^\infty(\Omega)$, $\boldsymbol{v}_* \in C^\infty(\partial\Omega)$ *then* $\boldsymbol{v}, p \in C^\infty(\overline{\Omega'})$, *for all bounded* $\Omega' \subset \Omega$.

Proof. The proof is an easy consequence of Theorem IV.4.1, and Theorem IV.5.1, if one bears in mind that (VII.1.2) can be viewed as a weak form of the Stokes equation with $\boldsymbol{f}$ replaced by $\mathcal{R}(\boldsymbol{f} - \dfrac{\partial \boldsymbol{v}}{\partial x_1})$. $\square$

In the remaining part of this section we shall be concerned with the uniqueness of generalized solutions. Such a study is slightly more complicated than the analogous one for the Stokes problem. To see why, let $\boldsymbol{v}$ and $\boldsymbol{w}$ denote two generalized solutions corresponding to the same data. Setting $\boldsymbol{u} = \boldsymbol{w} - \boldsymbol{v}$, from (VII.1.1) we obtain that $\boldsymbol{u}$ obeys the identity

$$\mathcal{F}(\boldsymbol{\varphi}) \equiv (\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) - \mathcal{R}(\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{\varphi}) = 0, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega). \tag{VII.1.3}$$

Assuming Ω locally Lipschitz, as in the case of Stokes problem, we easily show that $\boldsymbol{u} \in \mathcal{D}^{1,2}_0(\Omega)$. However, it is not obvious that we can replace in (VII.1.3), $\boldsymbol{\varphi}$ with $\boldsymbol{u}$, nor is it obvious that, even if this replacement is permitted, we can conclude that

$$(\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{u}) = 0\,. \tag{VII.1.4}$$

[1] We observe that these assumptions are definitely satisfied by any q-weak solution. Actually, they are implied by the following one:

$$\boldsymbol{v} \in L^1_{loc}(\Omega)\,,\ \nabla \boldsymbol{v} \in L^q_{loc}(\Omega), \quad \text{with } \boldsymbol{v} \text{ satisfying (VII.1.1) for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega).$$

In fact, under these conditions, by Lemma II.6.1, we have $\boldsymbol{v} \in W^{1,q}_{loc}(\Omega)$ and then, by Lemma VII.1.1, it follows $p \in L^q_{loc}(\Omega)$; see also Remark VII.1.2.

[2] By Remark VII.1.2, this latter condition is certainly satisfied by any q-weak solution under the stated assumption on Ω.

Notwithstanding, if $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$, one has that the functional

$$\boldsymbol{\varphi} \in \mathcal{D}(\Omega) \to (\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{\varphi}) \in \mathbb{R}$$

can be extended to a (bounded) linear functional, $\delta_1 \boldsymbol{u}$, on $\mathcal{D}_0^{1,2}(\Omega)$; see Remark VII.1.2 and Theorem II.1.7. Since, clearly, $(\nabla \boldsymbol{u}, \cdot)$ defines a (bounded) linear functional, $\boldsymbol{A}(\boldsymbol{u})$, on $\mathcal{D}_0^{1,2}(\Omega)$ one can then equivalently rewrite (VII.1.3) in the following abstract form

$$\boldsymbol{A}(\boldsymbol{u}) - \mathcal{R}\delta_1 \boldsymbol{u} = \boldsymbol{0} \quad \text{in } \mathcal{D}_0^{-1,2}(\Omega). \tag{VII.1.5}$$

Now, Galdi (2007, Proposition 1.2) shows, in a different context, the "abstract" counterpart of (VII.1.4), namely, $[\delta_1 \boldsymbol{u}, \boldsymbol{u}] = 0$. Once we employ this information back in (VII.1.5), we immediately find $[\boldsymbol{A}(\boldsymbol{u}), \boldsymbol{u}] \equiv |\boldsymbol{u}|_{1,2}^2 = 0$, which, in turn, implies $\boldsymbol{u}(x) = \boldsymbol{0}$, for all $x \in \Omega$.

Here, in order to show uniqueness, we will use a different argument, based on the asymptotic behavior of solutions to (VII.1.3), that will be completely justified in Section VII.6. Another, still different, approach will be presented in Section VIII.2 for the more general case of generalized Oseen problem. We begin to observe that, from Theorem VII.1.1 it follows that $\boldsymbol{u}$ and the corresponding pressure field π, say, are infinitely differentiable in Ω so that (VII.1.3) can be written pointwise:

$$\begin{aligned} &\Delta \boldsymbol{u} + \mathcal{R}\frac{\partial \boldsymbol{u}}{\partial x_1} = \nabla \pi \\ &\nabla \cdot \boldsymbol{u} = 0. \end{aligned} \tag{VII.1.6}$$

Furthermore, for any $R > \delta(\Omega^c)$, from Theorem II.4.2 we find the existence of a sequence $\{\boldsymbol{u}_k^R\} \subset C^\infty(\overline{\Omega_R})$ vanishing near $\partial\Omega$ for all $k \in \mathbb{N}$ and approximating $\boldsymbol{u}$ in the norm of the space $W^{1,2}(\Omega_R)$. Multiplying (VII.1.6) by $\boldsymbol{u}_k^R$ and integrating by parts over Ω_R we easily deduce

$$\int_{\Omega_R} \left\{ \nabla \boldsymbol{u} : \nabla \boldsymbol{u}_k^R - \mathcal{R}\frac{\partial \boldsymbol{u}}{\partial x_1} \cdot \boldsymbol{u}_k^R \right\} = \int_{\partial B_R} \boldsymbol{n} \cdot \left\{ \nabla \boldsymbol{u} \cdot \boldsymbol{u}_k^R - \pi \boldsymbol{u}_k^R \right\},$$

where $\boldsymbol{n}$ is the outer normal to ∂B_R. We now let $k \to \infty$ into this relation, and recalling that $\boldsymbol{u}, \pi \in C^\infty(\Omega)$, with the aid of Theorem II.4.1 we deduce

$$|\boldsymbol{u}|_{1,2,\Omega_R}^2 - \frac{\mathcal{R}}{2}\int_{\Omega_R} \nabla \cdot (u^2 \boldsymbol{e}_1) = \int_{\partial B_R} \boldsymbol{n} \cdot \left\{ \nabla \boldsymbol{u} \cdot \boldsymbol{u} - \pi \boldsymbol{u} \right\}.$$

We next apply the results of Exercise II.4.3 to the second integral on the left-hand side of this identity and recall that $\boldsymbol{u}$ has zero trace at $\partial\Omega$ to recover

$$|\boldsymbol{u}|_{1,2,\Omega_R}^2 = \int_{\partial B_R} \boldsymbol{n} \cdot \left\{ \nabla \boldsymbol{u} \cdot \boldsymbol{u} + \frac{\mathcal{R}}{2} u^2 \boldsymbol{e}_1 - \pi \boldsymbol{u} \right\}. \tag{VII.1.7}$$

In Theorem VII.6.2 of Section VII.6 it will be proved that every sufficiently smooth solution to the Oseen system corresponding to a body force of compact support and having a certain degree of summability at infinity must decay there in a suitable way. In particular, such a theorem ensures for $\boldsymbol{u}$ and π the following estimates for every large R (see Exercise VII.6.1)

$$\begin{aligned}&\int_{\partial B_R} \left(\nabla \boldsymbol{u} : \nabla \boldsymbol{u} + u^2\right) \leq cR^{-(n-1)/2}\\ &\int_{\partial B_R} \pi^2 \leq cR^{-(n-1)}.\end{aligned} \tag{VII.1.8}$$

Then, employing the Schwarz inequality on the right-hand side of (VII.1.7), using (VII.1.8), and letting $R \to \infty$ we conclude $\boldsymbol{u} \equiv \boldsymbol{0}$.

We have thus proved

Theorem VII.1.2 *Let Ω be locally Lipschitz and let $\boldsymbol{v}$ be a generalized solution to* (VII.0.2), (VII.0.3) *corresponding to $\boldsymbol{f} \in W_0^{-1,2}(\Omega')$, Ω' any bounded subdomain with $\overline{\Omega}' \subset \Omega$, and $\boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega)$. Then, if $\boldsymbol{w}$ is another generalized solution corresponding to the same data, it is $\boldsymbol{v} \equiv \boldsymbol{w}$.*

Remark VII.1.3 Theorem VII.1.2 will be extended to the case of arbitrary q-generalized solutions ($q \neq 2$) in Exercise VII.6.2. ■

VII.2 Existence of Generalized Solutions for Three-Dimensional Flow

This section is devoted to proving existence of generalized solutions when Ω is a three-dimensional domain, the two-dimensional case being postponed to Section VII.5; see also Remark VII.2.1. To reach this goal, we begin to observe that, unlike for the Stokes problem, we can no longer employ the Riesz representation theorem, since the left-hand side of (VII.1.1) does not define a symmetric form for all $\mathcal{R} \neq 0$. We shall then use another method which, interestingly enough, though introduced by B.G. Galerkin in 1915 for studying *linear* problems, was used in the fluid dynamical context directly in the *nonlinear* case at the beginning of the fifties and sixties by E. Hopf and by H. Fujita, respectively, and only in 1965 was it used by R. Finn in linearized approximations of the Navier–Stokes equations. To apply this method, however, we need a preliminary result concerning the existence of a special complete set in $\mathcal{D}_0^{1,2}(\Omega)$.

Lemma VII.2.1 *Let Ω be an arbitrary domain of $\mathbb{R}^n$, $n \geq 2$. Then, there exists a denumerable set of functions $\{\boldsymbol{\varphi}_k\}$ whose linear hull is dense in $\mathcal{D}_0^{1,2}(\Omega)$ and has the following properties*

(i) *$\boldsymbol{\varphi}_k \in \mathcal{D}(\Omega)$, for all $k \in \mathbb{N}$;*

(ii) $(\nabla\boldsymbol{\varphi}_k, \nabla\boldsymbol{\varphi}_j) = \delta_{kj}$ *or* $(\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_j) = \delta_{kj}$, *for all* $k, j \in \mathbb{N}$;

(iii) *Given* $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$, *and* $\kappa \in \mathbb{N}$, *for any* $\varepsilon > 0$ *there exist* $m = m(\varepsilon) \in \mathbb{N}$ *and* $\gamma_1, ..., \gamma_m \in \mathbb{R}$, *such that*

$$\|\nabla\boldsymbol{\varphi} - \sum_{i=1}^{m} \gamma_i \nabla\boldsymbol{\varphi}_i\|_s + \|\rho(\boldsymbol{\varphi} - \sum_{i=1}^{m} \gamma_i \boldsymbol{\varphi}_i)\|_s < \varepsilon\,,$$

for all $s \geq 2$, *where* $\rho = (|x| + 1)^{\kappa/s}$.

Proof. Let $H^{\ell}_{0,\rho}(\Omega)$, with $\ell > n/2+1$, be the completion of $\mathcal{D}(\Omega)$ in the norm

$$\|\boldsymbol{\varphi}\|_{\ell,2,\rho} \equiv \|\rho\boldsymbol{\varphi}\|_2 + \|\boldsymbol{\varphi}\|_{\ell,2}\,.$$

Clearly, $H^{\ell}_{0,\rho}(\Omega)$ is a subspace of $W^{\ell,2}(\Omega)$. Moreover, it is also isomorphic to a closed subspace of $[L^2(\Omega)]^N$, for suitable $N = N(\ell, n)$, via the map

$$\boldsymbol{\varphi} \in H^{\ell}_{0,\rho}(\Omega) \to \big(\rho\,\varphi_1, \ldots, \rho\,\varphi_n\,;\, (D^{\alpha}\varphi_1)_{1\leq|\alpha|\leq\ell}\,; \cdots ;\, (D^{\alpha}\varphi_n)_{1\leq|\alpha|\leq\ell}\big) \in [L^2(\Omega)]^N.$$

Thus, in particular, $H^{\ell}_{0,\rho}(\Omega)$ is separable (see Theorem II.1.5), and so is its subset $\mathcal{D}(\Omega)$ (see Theorem II.1.1). As a consequence, there exists a basis in $H^{\ell}_{0,\rho}(\Omega)$ of functions from $\mathcal{D}(\Omega)$, which we will denote by $\{\boldsymbol{\psi}_k\}$. Since $H^{\ell}_{0,\rho}(\Omega) \hookrightarrow \mathcal{D}^{1,2}_0(\Omega)$, the linear hull of $\{\boldsymbol{\psi}_k\}$ must be dense in $\mathcal{D}^{1,2}_0(\Omega)$ as well. Take $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and fix $\varepsilon > 0$; there exist $N = N(\varepsilon) \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$ such that

$$\|\boldsymbol{\varphi} - \sum_{i=1}^{N} \alpha_i \boldsymbol{\psi}_i\|_{\ell,2,\rho} < \varepsilon.$$

By the embedding Theorem II.3.2, it follows that

$$\|\boldsymbol{\varphi} - \sum_{i=1}^{N} \alpha_i \boldsymbol{\psi}_i\|_{C^1} < c\,\varepsilon$$

with $c = c(\Omega, n, \ell)$. We may orthonormalize $\{\boldsymbol{\psi}_k\}$ in $\mathcal{D}^{1,2}_0(\Omega)$ or in $L^2(\Omega)$ by the Schmidt procedure, to obtain another denumerable set $\{\boldsymbol{\varphi}_k\}$ whose linear hull is still dense in $\mathcal{D}^{1,2}_0(\Omega)$. Since every $\boldsymbol{\varphi}_r$ is a linear combination of $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_r$ and, conversely, every $\boldsymbol{\psi}_r$ is a linear combination of $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_r$, it is easy to check that the system $\{\boldsymbol{\varphi}_k\}$ satisfies all the statements in the lemma which is thus completely proved. □

We are now in a position to prove the following.

Theorem VII.2.1 *Let* Ω *be a three-dimensional exterior, locally Lipschitz domain. Given*

$$\boldsymbol{f} \in D^{-1,2}_0(\Omega), \quad \boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega)\,,$$

there exists one and only one generalized solution to (VII.0.2), (VII.0.3). *This solution satisfies the estimates* [1]

$$\begin{aligned}
&\|\boldsymbol{v}\|_{2,\Omega_R} + |\boldsymbol{v}|_{1,2} \le c_1 \left\{\mathcal{R}|\boldsymbol{f}|_{-1,2} + (1+\mathcal{R})\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\right\} \\
&\int_{S^2} |\boldsymbol{v}(x)| = o(1/\sqrt{|x|}) \ \ \text{as } |x| \to \infty \qquad \text{(VII.2.1)} \\
&\|p\|_{2,\Omega_R/\mathbb{R}} \le c_2 \left\{\mathcal{R}|\boldsymbol{f}|_{-1,2} + (1+\mathcal{R})|\boldsymbol{v}|_{1,2}\right\}
\end{aligned}$$

for all $R > \delta(\Omega^c)$. *In* (VII.2.1) p *is the pressure field associated to* $\boldsymbol{v}$ *by* Lemma VII.1.1, *while* $c_i = c_i(R, \Omega)$ ($c_i \to \infty$ *as* $R \to \infty$).

Proof. We look for a solution of the form

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{V}_1 + \boldsymbol{\sigma}, \qquad \text{(VII.2.2)}$$

where

$$\boldsymbol{\sigma} = \frac{\Phi}{4\pi}\nabla\left(\frac{1}{|x|}\right)$$

$$\Phi = \int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n}$$

(the origin of coordinates has been taken in $\dot{\Omega}^c$). Further, $\boldsymbol{V}_1 \in W^{1,2}(\Omega)$ denotes the solenoidal extension of $\boldsymbol{v}_* - \sigma|_{\partial\Omega}$, of bounded support in Ω, that was constructed in the proof of Theorem V.1.1. We have

$$\begin{aligned}
&|\boldsymbol{V}_1|_{1,2} \le c\,\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \\
&D^\alpha \boldsymbol{\sigma} = O(1/|x|^{2+|\alpha|}), \ |\alpha| = 0, 1, \ \ \text{as } |x| \to \infty.
\end{aligned} \qquad \text{(VII.2.3)}$$

Finally, $\boldsymbol{w}$ is requested to be a member of $\mathcal{D}_0^{1,2}(\Omega)$ and to satisfy the identity

$$\begin{aligned}
&(\nabla\boldsymbol{w}, \nabla\boldsymbol{\varphi}) - \mathcal{R}(\frac{\partial\boldsymbol{w}}{\partial x_1}, \boldsymbol{\varphi}) \\
&= -\mathcal{R}[\boldsymbol{f}, \boldsymbol{\varphi}] - (\nabla\boldsymbol{V}_1, \nabla\boldsymbol{\varphi}) + \mathcal{R}(\boldsymbol{V}_1 + \boldsymbol{\sigma}, \frac{\partial\boldsymbol{\varphi}}{\partial x_1}),
\end{aligned} \qquad \text{(VII.2.4)}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. It is clear that, provided we show the existence of such a function $\boldsymbol{w}$, the field (VII.2.2) satisfies all requirements of generalized solution to (VII.0.2), (VII.0.3) given in Definition VII.1.1. Actually, from $(\text{VII.2.3})_2$ and the properties of $\boldsymbol{V}_1$ and $\boldsymbol{w}$, we have $\boldsymbol{v} \in D^{1,2}(\Omega)$; also, $\boldsymbol{v}$ is divergence-free and assumes the value $\boldsymbol{v}_*$ at the boundary. Finally, in view of $(\text{VII.2.3})_2$, we have, as $|x| \to \infty$,

$$\int_{S^2} |\boldsymbol{v}(x)| \le \int_{S^2} |\boldsymbol{w}(x)| + O(1/|x|^2) \qquad \text{(VII.2.5)}$$

[1] See (IV.6.1) for the definition of the norm involving p.

and by Theorem II.7.6 and Lemma II.6.2 we obtain $(\text{VII.2.1})_2$. Thus, to show the theorem it remains to prove the existence of the field $\boldsymbol{w}$ and the validity of estimates $(\text{VII.2.1})_{1,3}$. To this end, let $\{\boldsymbol{\varphi}_k\}$ be the base of $\mathcal{D}_0^{1,2}(\Omega)$ determined in Lemma VII.2.1. We shall construct an "approximate solution" $\boldsymbol{w}_m$ to (VII.2.4) in the following way:

$$\boldsymbol{w}_m = \sum_{\ell=1}^{m} \xi_{\ell m} \boldsymbol{\varphi}_\ell$$

$$\begin{aligned} &(\nabla \boldsymbol{w}_m, \nabla \boldsymbol{\varphi}_k) - \mathcal{R}(\boldsymbol{w}_m, \frac{\partial \boldsymbol{\varphi}_k}{\partial x_1}) \\ &= -\mathcal{R}[\boldsymbol{f}, \boldsymbol{\varphi}_k] - (\nabla \boldsymbol{V}_1, \nabla \boldsymbol{\varphi}_k) - \mathcal{R}(\boldsymbol{V}_1 + \boldsymbol{\sigma}, \frac{\partial \boldsymbol{\varphi}_k}{\partial x_1}) \equiv F_k, \\ &k = 1, 2, \ldots, m. \end{aligned} \tag{VII.2.6}$$

Using (ii) of Lemma VII.2.1 we obtain

$$\sum_{\ell=1}^{m} (\xi_{\ell m} \delta_{\ell k} - \mathcal{R} \xi_{\ell m} A_{\ell k}) = F_k, \quad k = 1, 2, \ldots, m \tag{VII.2.7}$$

where

$$A_{\ell k} \equiv (\frac{\partial \boldsymbol{\varphi}_\ell}{\partial x_1}, \boldsymbol{\varphi}_k).$$

System (VII.2.7) is linear in the unknowns $\xi_{\ell m}$, $\ell = 1, \ldots, m$, and since $A_{\ell k} = -A_{k\ell}$ it is readily seen that the determinant of the coefficients is non-zero. As a consequence, for each $m \in \mathbb{N}$, system (VII.2.6) admits a uniquely determined solution. Let us multiply $(\text{VII.2.6})_2$ by ξ_{km} and sum over k from 1 to m. We obtain

$$|\boldsymbol{w}_m|_{1,2}^2 = -\mathcal{R}[\boldsymbol{f}, \boldsymbol{w}_m] - (\nabla \boldsymbol{V}_1, \nabla \boldsymbol{w}_m) - \mathcal{R}(\boldsymbol{V}_1 + \boldsymbol{\sigma}, \frac{\partial \boldsymbol{w}_m}{\partial x_1}). \tag{VII.2.8}$$

Using (VII.2.3) and recalling that $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$, we easily show

$$\begin{aligned} -[\boldsymbol{f}, \boldsymbol{w}_m] &\le |\boldsymbol{f}|_{-1,2} |\boldsymbol{w}_m|_{1,2} \\ -(\nabla \boldsymbol{V}_1, \nabla \boldsymbol{w}_m) &\le c_1 \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} |\boldsymbol{w}_m|_{1,2} \\ -(\boldsymbol{V}_1 + \boldsymbol{\sigma}, \frac{\partial \boldsymbol{w}_m}{\partial x_1}) &\le c_2 \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} |\boldsymbol{w}_m|_{1,2} \end{aligned}$$

and (VII.2.7) furnishes

$$|\boldsymbol{w}_m|_{1,2} \le c \left\{ \mathcal{R} |\boldsymbol{f}|_{-1,2} + (1 + \mathcal{R}) \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \right\}. \tag{VII.2.9}$$

Therefore, the sequence $\{\boldsymbol{w}_m\}$ remains uniformly bounded in $D_0^{1,2}(\Omega)$ and, by Exercise II.6.2, there exist a subsequence, denoted again by $\{\boldsymbol{w}_m\}$, and a function $\boldsymbol{w} \in \mathcal{D}_0^{1,2}(\Omega)$ such that in the limit $m \to \infty$

$$(\nabla \boldsymbol{w}_m, \nabla \boldsymbol{\varphi}) \to (\nabla \boldsymbol{w}, \nabla \boldsymbol{\varphi}), \quad \text{for all } \boldsymbol{\varphi} \in D_0^{1,2}(\Omega).$$

Also, by (VII.2.9) and Theorem II.2.4 we infer

$$|\boldsymbol{w}|_{1,2} \le c \left\{ \mathcal{R} |\boldsymbol{f}|_{-1,2} + (1+\mathcal{R}) \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \right\} \tag{VII.2.10}$$

with $c = c(\Omega)$. For fixed k, we then pass to the limit $m \to \infty$ into $(\text{VII.2.6})_2$ to deduce with no difficulty that $\boldsymbol{v}$ satisfies (VII.1.1) for all $\boldsymbol{\varphi}_k$. Since, by Lemma VII.2.1, every $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ can be approximated in the $W^{1,2}$-norm by a linear combination of $\boldsymbol{\varphi}_k$, we establish the validity of (VII.1.1) for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. Let us next prove estimates $(\text{VII.2.3})_{1,3}$. We observe that, in view $(\text{VII.2.3})_2$, Theorem II.6.1 and the Hölder inequality, we deduce

$$\|\boldsymbol{v}\|_{2,\Omega_R} \le |\Omega_R|^{1/3} \|\boldsymbol{v}\|_{6,\Omega_R} \le c\, |\Omega_R|^{1/3} |\boldsymbol{v}|_{1,2} \tag{VII.2.11}$$

where $c = c(\Omega)$, and so, since

$$|\boldsymbol{v}|_{1,2} \le |\boldsymbol{w}|_{1,2} + |\boldsymbol{V}_1|_{1,2} + |\sigma|_{1,2},$$

inequality $(\text{VII.2.1})_1$ follows from (VII.2.10), (VII.2.11), and the properties of $\boldsymbol{V}_1$ and $\boldsymbol{\sigma}$. Let us finally show $(\text{VII.2.1})_3$. For fixed $R > \delta(\Omega^c)$, we add to the pressure p (defined through Lemma VII.1.1) the constant

$$\mathcal{C}(R) = -\frac{1}{|\Omega_R|} \int_{\Omega_R} p$$

so that

$$\int_{\Omega_R} (p + \mathcal{C}) = 0.$$

Successively, we take $\boldsymbol{\psi}$ into (VII.1.2) as a solution to the problem

$$\begin{aligned} &\nabla \boldsymbol{\psi} = p + \mathcal{C} \quad \text{in } \Omega_R \\ &\boldsymbol{\psi} \in W_0^{1,2}(\Omega_R) \\ &\|\boldsymbol{\psi}\|_{1,2} \le c_1 \|p + \mathcal{C}\|_{2,\Omega_R} \end{aligned}$$

for some $c_1 = c_1(\Omega_R)$. This problem is resolvable in virtue of Theorem II.4.1 and so from (VII.1.2) and the Schwarz inequality we have

$$\|p + \mathcal{C}\|_{2,\Omega_R} \le c_1 \left(|\boldsymbol{v}|_{1,2} + \mathcal{R} \|\boldsymbol{v}\|_{2,\Omega_R} + \mathcal{R} |\boldsymbol{f}|_{-1,2} \right) \tag{VII.2.12}$$

which, by (VII.2.11), in turn implies $(\text{VII.2.1})_3$. The solution $\boldsymbol{v}$ just constructed is unique in view of Theorem VII.1.2 and therefore the proof of the theorem is accomplished. □

Remark VII.2.1 The methods of Theorem VII.2.1 apply with no change to show existence of solutions in arbitrary space dimension $n \ge 3$, the only difference resulting in the asymptotic estimate $(\text{VII.2.1})_2$, which has to be replaced by

$$\int_{S_n} |\boldsymbol{v}(x)| = o(1/|x|^{n/2-1}).$$

If $n = 2$, by the same technique we can still establish the existence of a vector field $\boldsymbol{v}$ satisfying (i), (ii), (iii), and (v) of Definition VII.1.1, for $q = 2$; however, by this technique we are not able to show the validity of condition (iv) since, as we know, functions in $D^{1,2}(\Omega)$ for $n = 2$ need not tend to a prescribed value at infinity. Nevertheless, unlike the Stokes approximation, for the problem at hand we *can* prove existence of generalized solutions by means of more complicated tools, as will be shown in Theorem VII.5.1. ■

Remark VII.2.2 The observations made in Remark V.2.1 apply equally to the present situation. In particular, if $\boldsymbol{v}_* = \boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{x}$, for some $\boldsymbol{v}_0, \boldsymbol{\omega} \in \mathbb{R}^3$, the existence of a generalized solution is proved without regularity assumptions on Ω. ■

Exercise VII.2.1 Theorem VII.2.1 can be generalized to the case when $\nabla \cdot \boldsymbol{v} = g \not\equiv 0$, where g is a suitably prescribed function. Specifically, show that for Ω, $\boldsymbol{f}$ and $\boldsymbol{v}_*$ satisfying the same assumptions of Theorem VII.2.1 and for all $g \in L^2(\Omega) \cap D_0^{-1,2}(\Omega)$ there exists one and only one generalized solution to the nonhomogeneous Oseen problem, that is, a field $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ satisfying (i) (with $q = 2$), (iii), (iv), and (v) of Definition VII.1.1 together with $\nabla \cdot \boldsymbol{v} = g$ in the weak sense. Show, in addition, that, in such a case, estimate $(\text{VII.2.1})_1$ is modified by adding to its right-hand side the term

$$\|g\|_2 + \mathcal{R}|g|_{-1,2}.$$

VII.3 The Oseen Fundamental Solution and the Associated Volume Potentials

In order to derive further properties of solutions to the Oseen problem in exterior domains, we shall introduce a suitable singular solution to equations $(\text{VII.0.3})_{1,2}$ in the whole space. Though such a solution can be considered, for the problem at hand, the analogue of the Stokes fundamental solution (IV.2.3), (IV.2.4), it differs from this latter in several respects; the main difference is the behavior at large distances. Specifically, the Oseen fundamental solution has a "nonsymmetric" structure, presenting a "wake region" which, as we know, does not appear in the Stokes approximation.

Following Oseen (1927, §4), we denote by $\boldsymbol{E}$ and $\boldsymbol{e}$ tensor and vector fields, respectively, defined by

$$\begin{aligned} E_{ij}(x,y) &= \left(\delta_{ij}\Delta - \frac{\partial^2}{\partial y_i \partial y_j}\right)\Phi(x,y) \\ e_j(x,y) &= -\frac{\partial}{\partial y_j}\left(\Delta - 2\lambda\frac{\partial}{\partial y_1}\right)\Phi(x,y). \end{aligned} \tag{VII.3.1}$$

Here $i,j = 1,\ldots,n$, $\lambda = \mathcal{R}/2$, while $\Phi(x,y)$, with $x,y \in \mathbb{R}^n$, is any real function that is smooth for $x \neq y$. Moreover, the Laplace operator acts on the y-variables. Observe that if $\lambda = 0$, $\boldsymbol{E}$ and $\boldsymbol{e}$ formally coincide with $\boldsymbol{U}$ and $\boldsymbol{q}$ introduced in (IV.2.1). It is at once checked that fields (VII.3.1) satisfy the following relations for all $x \neq y$ and all $i,j = 1,\ldots,n$

$$\begin{aligned} \left(\Delta - 2\lambda\frac{\partial}{\partial y_1}\right)E_{ij} - \frac{\partial}{\partial y_i}e_j &= \delta_{ij}\Delta\left(\Delta - 2\lambda\frac{\partial}{\partial y_1}\right)\Phi \\ \frac{\partial}{\partial y_\ell}E_{\ell j} &= 0. \end{aligned} \tag{VII.3.2}$$

In order to render (VII.3.1) a *singular solution* to $(\text{VII.0.3})_{1,2}$, as in the case of the Stokes system, we choose the function Φ such that

$$\Delta\left(\Delta - 2\lambda\frac{\partial}{\partial x_1}\right)\Phi(x,y) = \Delta\mathcal{E}(|x-y|), \tag{VII.3.3}$$

where, we recall, $\mathcal{E}(x)$ is the fundamental solution (II.9.1) to the Laplace equation.[1] A solution to (VII.3.3) is now sought into the form

$$\begin{aligned} \Phi(x,y) = \frac{1}{2\lambda}\int^{x_1-y_1} &[\Phi_2(\tau, x_2-y_2,\ldots,x_n-y_n) \\ &-\Phi_1(\tau, x_2-y_2,\ldots,x_n-y_n)]d\tau \end{aligned} \tag{VII.3.4}$$

[1] Let L be a (spatial) differential operator and let $h(x,y)$ be a smooth function of $x,y \in \mathbb{R}^n$ except at $x=y$. By the notation

$$Lh(x,y) = \Delta\mathcal{E}(|x-y|) \tag{$*$}$$

we mean, as customary, $Lh(x,y) = 0$ for all $x \neq y$, while, at $x=y$, $h(x,y)$ becomes singular in such a way that for any $\psi \in C_0^\infty(\mathbb{R}^n)$ it holds that

$$\int_{\mathbb{R}^n} h(x,y)L^*\psi(y)dy = \psi(x),$$

where L^* denotes the formal adjoint of L. Another usually adopted way of writing $(*)$ is

$$Lh(x,y) = \delta(x-y),$$

where $\delta(x)$ is the symbolic Dirac function, i.e., a distribution defined by the *formal* relation

$$\int_{\mathbb{R}^n} \delta(x)\psi(x) = \psi(0)$$

for all $\psi \in C_0^\infty(\mathbb{R}^n)$.

with Φ_1 and Φ_2 to be selected appropriately. Replacing formally (VII.3.4) into (VII.3.3) we obtain that $\Phi_2 - \Phi_1$ must obey

$$\Delta\left(\Delta - 2\lambda\frac{\partial}{\partial y_1}\right)(\Phi_2 - \Phi_1) = -2\lambda\Delta(\frac{\partial \mathcal{E}}{\partial y_1}). \tag{VII.3.5}$$

Choosing

$$\Phi_2(x,y) = \mathcal{E}(|x-y|), \tag{VII.3.6}$$

for Φ_1 to be a solution to (VII.3.5) it is sufficient to take

$$\left(\Delta - 2\lambda\frac{\partial}{\partial y_1}\right)\Phi_1 = \Delta\mathcal{E}. \tag{VII.3.7}$$

We notice, in passing, that with the above choice of Φ, from $(\text{VII.3.1})_2$ we may take

$$e_j(x,y) = -\frac{\partial}{\partial y_j}\mathcal{E}(|x-y|), \tag{VII.3.8}$$

which shows that the "pressure" e_j coincides with the "pressure" q_j of the fundamental Stokes solution, see $(\text{IV.2.3})_2$, $(\text{IV.2.4})_2$. Writing

$$\Phi_1 = \frac{e^{-\lambda(x_1-y_1)}}{|x-y|^{(n-2)/2}} f(\lambda|x-y|), \tag{VII.3.9}$$

by a direct computation we deduce

$$\begin{aligned}\left(\Delta - 2\lambda\frac{\partial}{\partial y_1}\right)\Phi_1 &= \frac{e^{-\lambda(x_1-y_1)}}{|x-y|^{(n+2)/2}}\left\{z^2 f'' + z f' - \left[\left(\frac{n-2}{2}\right)^2 + z^2\right] f\right\}\\ &\equiv \frac{e^{-\lambda(x_1-y_1)}}{|x-y|^{(n+2)/2}}\mathcal{L}(f),\end{aligned}$$

where $z = \lambda|x-y|$ and the prime denotes differentiation with respect to z. Now the equation $\mathcal{L}(f) = 0$ is the well-known Bessel's modified equation which admits two independent solutions,

$$I_{(n-2)/2}(z) \quad \text{and} \quad K_{(n-2)/2}(z),$$

called *modified Bessel functions of the first and of the second kind*, respectively (MacRobert 1966, §100). However, $I_{(n-2)/2}(z)$ is regular for all values of the argument, while $K_{(n-2)/2}(z)$ is singular at $z = 0$ in such a way that

$$\begin{aligned}K_{(n-2)/2}(z) &= \log\frac{1}{z} + \log 2 - \gamma + \sigma_1(z), \quad \text{if } n = 2\\ K_{(n-2)/2}(z) &= \frac{2^{(n-2)/2}\Gamma(n/2)}{n-2}\frac{1}{z^{(n-2)/2}} + \sigma_2(z), \quad \text{if } n > 2,\end{aligned} \tag{VII.3.10}$$

where γ is the Euler constant, Γ is the gamma function, and the remainders σ_i satisfy

$$\sigma_1(z) = o(1), \quad \frac{d^k\sigma_1}{dz^k} = o(z^{-k}), \quad k \geq 1 \ \text{ as } z \to 0$$
$$\frac{d^k\sigma_2}{dz^k} = o(z^{(2-n)/2-k}), \quad k \geq 0 \ \text{ as } z \to 0$$

(Watson 1962, p. 80). Since Φ_1 must satisfy (VII.3.7) in the neighborhood of $z = 0$, it has to behave there like the fundamental solution $\mathcal{E}$. Thus, with a view to (II.9.1) and taking into account that $\omega_n = 2\pi^{n/2}/n\Gamma(n/2)$, from (VII.3.9) and (VII.3.10) it follows that we must take

$$\Phi_1 = -\frac{1}{2\pi}\left(\frac{\lambda}{2\pi|x-y|}\right)^{(n-2)/2} K_{(n-2)/2}(\lambda|x-y|)e^{\lambda(y_1-x_1)}. \tag{VII.3.11}$$

For $n = 3, K_{1/2}(z)$ takes the following simple form (Watson 1962, p. 80):

$$K_{1/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}, \tag{VII.3.12}$$

and consequently, from (VII.3.4), (VII.3.6), and (VII.3.11) we obtain

$$\Phi(x-y) = \frac{1}{8\pi\lambda}\int^{x_1-y_1} \frac{1-\exp\left\{-\lambda\left[\sqrt{\tau^2+(x_2-y_2)^2+(x_3-y_3)^2}-\tau\right]\right\}}{\sqrt{\tau^2+(x_2-y_2)^2+(x_3-y_3)^2}}d\tau.$$

Therefore, fixing the constant up to which Φ is defined by requiring $\Phi(0) = 0$, it follows that

$$\Phi(x-y) = \frac{1}{8\pi\lambda}\int_0^{\lambda(|x-y|+(x_1-y_1))} \frac{1-e^{-\tau}}{\tau}d\tau. \tag{VII.3.13}$$

Correspondingly, the pressure field $\boldsymbol{e}$ given in (VII.3.8) takes the form

$$e_j(x-y) = \frac{1}{4\pi}\frac{x_j-y_j}{|x-y|^3}. \tag{VII.3.14}$$

Let us now consider the case $n = 2$. From (VII.3.4), (VII.3.6) and (VII.3.11) we find

$$\Phi(x-y) = \frac{1}{2\lambda}\int_0^{x_1-y_1} \left[\Phi_2(\tau, x_2-y_2) - \Phi_1(\tau, x_2-y_2)\right] d\tau + \Phi_0(x_2-y_2)$$

where

$$\Phi_2(x-y) = \frac{1}{2\pi}\log|x-y|, \quad \Phi_1(x-y) = -\frac{1}{2\pi}K_0(\lambda|x-y|)e^{\lambda(y_1-x_1)}$$

and Φ_0 is a function of $x_2 - y_2$ only, to be fixed appropriately. The function $K_0(z)$ cannot be expressed in terms of elementary functions; however, we can provide an asymptotic expansion for large z:

$$\begin{aligned} K_0(z) &= \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left[\sum_{k=0}^{\nu-1}\frac{\Gamma(k+1/2)}{k!\Gamma(1/2-k)}(2z)^{-k}+\sigma_\nu(z)\right] \\ &= \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left[1-\frac{1}{8z}+\frac{9}{2!(8z)^2}+\ldots+\sigma_\nu(z)\right], \end{aligned} \tag{VII.3.15}$$

where

$$\frac{d^k\sigma_\nu}{dz^k} = O(z^{-k-\nu}) \ \text{ as } z\to\infty,\, k\geq 0$$

(Watson 1962, p. 202). In order to choose Φ_0, we observe that from (VII.3.6), (VII.3.7) it follows that

$$\frac{\partial^2}{\partial y_2^2}(\Phi_2-\Phi_1) = -\frac{\partial}{\partial y_1}\left[\frac{\partial}{\partial y_1}(\Phi_2-\Phi_1)+2\lambda\Phi_1\right],$$

and so

$$\begin{aligned} \frac{\partial^2\Phi}{\partial y_2^2} &= -\frac{1}{2\lambda}\left[\frac{\partial}{\partial y_1}(\Phi_2-\Phi_1)+2\lambda\Phi_1\right] \\ &\qquad +\frac{1}{2\lambda}\left[\frac{\partial}{\partial y_1}(\Phi_2-\Phi_1)+2\lambda\Phi_1\right]_{y_1=x_1}+\frac{\partial^2\Phi_0}{\partial y_2^2}. \end{aligned}$$

Since

$$\frac{\partial}{\partial y_1}\log|x-y|\bigg|_{y_1=x_1} = \frac{\partial}{\partial y_1}K_0(\lambda|x-y|)\bigg|_{y_1=x_1} = 0,$$

we conclude

$$\frac{\partial^2\Phi}{\partial y_2^2} = -\frac{1}{2\lambda}\left[\frac{\partial}{\partial y_1}(\Phi_2-\Phi_1)+2\lambda\Phi_1\right]+\frac{1}{4\pi}K_0(\lambda|x_2-y_2|)+\frac{\partial^2\Phi_0}{\partial y_2^2}.$$

Now, letting $x_2-y_2\to 0$ in this relation, $K_0(\lambda|x_2-y_2|)$ diverges logarithmically fast, while the first three terms on the right-hand side remain bounded. This would lead, by (VII.3.1), to an unacceptable singularity for E_{22}, unless we choose Φ_0 in such a way that

$$\Phi_0''(t) = -\frac{1}{4\pi}K_0(\lambda|t|).$$

If we do this and impose $\Phi_0(0)=\Phi'(0)=0$, we find

$$\Phi_0(x_2-y_2) = -\frac{1}{4\pi}\int_0^{y_2-x_2}(y_2-x_2-\tau)K_0(\lambda|\tau|)d\tau,$$

which, in turn, furnishes the following expression for Φ:

$$\Phi(x-y) = \frac{1}{4\pi\lambda}\int_0^{x_1-y_1}\left\{\log\sqrt{\tau^2+(x_2-y_2)^2}\right.$$
$$\left.+K_0\left(\lambda\sqrt{\tau^2+(x_2-y_2)^2}\right)e^{-\lambda\tau}\right\}d\tau \qquad \text{(VII.3.16)}$$
$$-\frac{1}{4\pi}\int_0^{y_2-x_2}(y_2-x_2-\tau)K_0(\lambda|\tau|)d\tau.$$

Correspondingly, the pressure field $\boldsymbol{e}$ (VII.3.8) becomes

$$e_j(x-y) = \frac{1}{2\pi}\frac{x_j-y_j}{|x-y|^2}. \qquad \text{(VII.3.17)}$$

The pair $\boldsymbol{E}, \boldsymbol{e}$ defined by (VII.3.1), (VII.3.13), and (VII.3.14) for $n=3$ and by (VII.3.1), (VII.3.16), and (VII.3.17) for $n=2$ is called the *Oseen fundamental solution*. In arbitrary dimensions $n>3$, the Oseen fundamental solution is defined by (VII.3.1), (VII.3.4), (VII.3.8), and (VII.3.11). In view of (VII.3.2) and (VII.3.3) this solution satisfies

$$\left.\begin{aligned}\left(\Delta-2\lambda\frac{\partial}{\partial y_1}\right)E_{ij}(x-y) &= \frac{\partial}{\partial y_i}e_j(x-y)\\ \frac{\partial}{\partial y_\ell}E_{\ell j}(x-y) &= 0\end{aligned}\right\}\quad \text{for } x\neq y; \qquad \text{(VII.3.18)}$$

this is the system *adjoint* to (VII.0.3). However, since

$$\frac{\partial E_{ij}}{\partial x_1} = -\frac{\partial E_{ij}}{\partial y_1} \quad \frac{\partial e_j}{\partial x_1} = -\frac{\partial e_j}{\partial y_1}$$

we also have

$$\left.\begin{aligned}\left(\Delta+2\lambda\frac{\partial}{\partial x_1}\right)E_{ij}(x-y) &= \frac{\partial}{\partial x_i}e_j(x-y)\\ \frac{\partial}{\partial x_\ell}E_{\ell j}(x-y) &= 0\end{aligned}\right\}\quad \text{for } x\neq y \qquad \text{(VII.3.19)}$$

with Δ operating now on the x-variables.

We wish to investigate the properties of $\boldsymbol{E}(x)$ and $\boldsymbol{e}(x)$ for large $|x|$. While those of $\boldsymbol{e}$ are quite obvious, those of $\boldsymbol{E}$ require a little more care. Let us begin to consider the case where $n=3$. Setting $r=|x-y|$, $s=\lambda(r+x_1-y_1)$ from $(\text{VII.3.1})_1$ and (VII.3.13) we derive the following expression for the nine components of the tensor $\boldsymbol{E}$

$$
\begin{aligned}
E_{11}(x-y) =& \frac{1}{4\pi r}\left\{-e^{-s}+\frac{1}{2\lambda r}\left[\frac{x_1-y_1}{r}(1-e^{-s})+se^{-s}\right]\right\}\\
E_{22}(x-y) =& -\frac{e^{-s}}{4\pi r}+\frac{1}{8\pi}\left\{\left[\frac{1}{r}-\frac{(x_2-y_2)^2}{r^3}\right]\frac{1-e^{-s}}{s}\right.\\
&\left. +\lambda\frac{se^{-s}-1+e^{-s}}{s^2}\frac{(x_2-y_2)^2}{r^2}\right\}\\
E_{33}(x-y) =& -\frac{e^{-s}}{4\pi r}+\frac{1}{8\pi}\left\{\left[\frac{1}{r}-\frac{(x_3-y_3)^2}{r^3}\right]\frac{1-e^{-s}}{s}\right.\\
&\left. +\lambda\frac{se^{-s}-1+e^{-s}}{s^2}\frac{(x_3-y_3)^2}{r^2}\right\}\\
E_{12}(x-y) =& E_{21}(x-y)=\frac{1}{8\pi r}\left\{\frac{y_2-x_2}{r}\left[e^{-s}-\frac{1-e^{-s}}{\lambda r}\right]\right\}\\
E_{13}(x-y) =& E_{31}(x-y)=\frac{1}{8\pi r}\left\{\frac{y_3-x_3}{r}\left[e^{-s}-\frac{1-e^{-s}}{\lambda r}\right]\right\}\\
E_{23}(x-y) =& E_{32}(x-y)=\frac{1}{8\pi}\left\{\frac{(x_2-y_2)(x_3-y_3)}{r^3}\left[\frac{e^{-s}-1}{s}\right.\right.\\
&\left.\left. +\lambda r\frac{se^{-s}-1+e^{-s}}{s^2}\right]\right\}.
\end{aligned}
\tag{VII.3.20}
$$

From (VII.3.20) it readily follows that in the limit of vanishing λr the tensor $\boldsymbol{E}$ reduces to the tensor $\boldsymbol{U}$ (IV.2.3)$_1$ associated to the Stokes fundamental solution. Specifically, we have

$$
E_{ij}(x-y)=U_{ij}(x-y)+o(1), \quad \text{as } \lambda r\to 0. \tag{VII.3.21}
$$

In view of (VII.3.21) and (VII.3.14), from the calculations leading to (IV.8.15) we can then show that the Oseen fundamental solution $\boldsymbol{E}, \boldsymbol{e}$ becomes singular at $x=y$ in such a way that, for any vector field $\boldsymbol{v}$ continuous at x and all $j=1,2,3$, it holds that

$$
\lim_{\varepsilon\to 0}\int_{|x-y|=\varepsilon}\boldsymbol{v}\cdot\boldsymbol{T}(\boldsymbol{w}_j,e_j)\cdot\boldsymbol{n}d\sigma_y=-v_j(x), \tag{VII.3.22}
$$

where $\boldsymbol{n}$ is the outer normal to $\partial B_\varepsilon(x)$ and

$$
\boldsymbol{w}_j\equiv(E_{1j},E_{2j},E_{3j}).
$$

The estimates of $\boldsymbol{E}(x-y)$ at infinity are, however, completely different from those of $\boldsymbol{U}(x-y)$. Denote by φ the polar angle made by a ray that starts from x and is directed toward y with the positively directed x_1-axis. We present the estimates for $\boldsymbol{E}(x-y)$ as a function of y for fixed x. *Considered as a function of* x, *all estimates remain true if* φ *is replaced by* $\pi - \varphi$, see Remark VII.3.1. Taking x as the origin of coordinates (this produces no loss of generality since $\boldsymbol{E}$ is a function of $x - y$ only) and noticing that

$$e^{-s} \leq (1-e^{-s})/s, \quad s > 0, \qquad |(y_j - x_j)/s| \leq 2/\lambda, \quad j = 2, 3,$$

from (VII.3.20) we obtain

$$|\boldsymbol{E}(y)| \leq \frac{c_1}{|y|}\frac{1-e^{-s}}{s}, \tag{VII.3.23}$$

where $c_1 = c_1(\lambda)$ and

$$s = \lambda(|y| - y_1) = \lambda|y|(1-\cos\varphi).$$

The bound (VII.3.23) furnishes, in particular, the uniform estimate

$$|\boldsymbol{E}(y)| \leq \frac{c_1}{|y|}, \tag{VII.3.24}$$

which coincides with that given for $\boldsymbol{U}$ in (IV.2.6). However, improved bounds can be derived from (VII.3.23) as a function of φ. Specifically, if

$$(1-\cos\varphi) \geq |y|^{-1+2\sigma} \quad \text{for some } \sigma \in [0, 1/2], \tag{VII.3.25}$$

then (VII.3.23) implies

$$|\boldsymbol{E}(y)| \leq \frac{c_2}{|y|^{1+2\sigma}}, \tag{VII.3.26}$$

with $c_2 = c_2(\lambda)$. From (VII.3.23)–(VII.3.26) it follows that if y belongs to the region defined by

$$|y|(1-\cos\varphi) \leq 1 \tag{VII.3.27}$$

then (VII.3.24) holds. This region represents a paraboloid with the axis in the direction of the negative x_1-axis and can be interpreted as a "wake."

Remark VII.3.1 The tensor $\boldsymbol{E}(x-y)$ considered as a function of y thus exhibits a "wake" region in the direction opposite to what is expected for a body moving in a liquid with the velocity $\boldsymbol{v}_0$ directed in the positive x_1-axis (as we have assumed at the beginning of the chapter). This is due to the fact that, as a function of y, $\boldsymbol{E}$ satisfies the *adjoint* system (VII.3.18). However, if we consider $\boldsymbol{E}$ as a function of x, then φ should be changed in $\pi - \varphi$ and the paraboloidal wake region becomes appropriately located with its axis in the direction of the negative x_1-axis. This remark is important in the context of the asymptotic structure of solutions to the Oseen system; see Theorem VII.6.2 and Remark VII.6.1. ■

Starting from (VII.3.23) we may also derive the summability properties of $\boldsymbol{E}(y)$ in the exterior $\mathcal{A}$ of a ball of unit (say) radius. In particular, by a straightforward calculation we show

$$\boldsymbol{E}(y) \in L^q(\mathcal{A}) \quad \text{for all } q > 2. \tag{VII.3.28}$$

This estimate is sharp; actually, from $(\text{VII.3.20})_1$ we have with $a = 1/\lambda r$

$$|E_{11}(y)| = \left(\frac{1}{8\pi r}\right) |e^{-s}[1 + (1-a)\cos\varphi] + a\cos\varphi|$$

Thus, for all $r \geq \lambda$ and all $\varphi \in [0, \pi/2]$, we find

$$|E_{11}(y)| \geq \frac{1}{8\pi r} e^{-\lambda r(1-\cos\varphi)} .$$

From this relation it follows that

$$\begin{aligned}
\|E_{11}\|_{q,\mathcal{A}}^q &\geq c\int_\lambda^\infty r^{-q+2}e^{-\lambda qr}\left(\int_0^{\pi/2} e^{\lambda qr\cos\varphi}\sin\varphi\, d\varphi\right) dr \\
&= c\int_\lambda^\infty r^{-q+2}e^{-\lambda qr}\left(\int_0^1 e^{\lambda qrx}\, dx\right) dr \\
&= c_1\int_\lambda^\infty r^{-q+1}(1 - e^{-\lambda qr})\, dr .
\end{aligned} \tag{VII.3.29}$$

Therefore, setting

$$R = \max\left\{\lambda, -\frac{1}{\lambda q}\ln\left(\frac{1}{2}\right)\right\} ,$$

from (VII.3.29) we obtain

$$\|E_{11}\|_{q,\mathcal{A}}^q \geq c_1 \int_R^\infty r^{-q+1}\, dr,$$

showing that E_{11} does not belong to $L^q(\mathcal{A})$ for all $q \in [1, 2]$. A similar conclusion can be reached for the other components of $\boldsymbol{E}$, by means of the same technique, which allows us to conclude

$$\boldsymbol{E}(y) \notin L^q(\mathcal{A}), \quad \text{for all } q \in [1, 2]. \tag{VII.3.30}$$

The summability properties (VII.3.28), (VII.3.30) should be contrasted with the analogous properties of the Stokes fundamental tensor $\boldsymbol{U}$, for which (in dimension 3) we have that $\boldsymbol{U}$ belongs to $L^q(\mathcal{A})$ for all and only all $q > 3$.

Estimates similar to (VII.3.23) can be derived for first derivatives. Specifically, from (VII.3.20) the following inequalities are directly obtained

$$\begin{aligned} \left|\frac{\partial \boldsymbol{E}(y)}{\partial y_i}\right| &\leq \frac{c}{|y|^{3/2}}\left[\frac{1-e^{-s}-se^{-s}}{s^{3/2}}+\frac{1}{|y|^{1/2}}\frac{1-e^{-s}}{s}\right], \quad i=2,3 \\ \left|\frac{\partial \boldsymbol{E}(y)}{\partial y_1}\right| &\leq \frac{c}{|y|^2}\frac{1-e^{-s}}{s} \end{aligned} \tag{VII.3.31}$$

with $c=c(\lambda)$. These formulas imply, in particular, the uniform bound for $|y|$ greater than any fixed $R_0>0$:

$$|\nabla \boldsymbol{E}(y)| \leq \frac{c_1}{|y|^{3/2}} \tag{VII.3.32}$$

with $c_1 = c_1(\lambda, R_0)$. However, better estimates can be derived outside the "wake region" (VII.3.27), in a way completely analogous to that used previously, and we leave them to the reader as an exercise. Concerning the summability of $\nabla\boldsymbol{E}$ in a neighborhood of infinity, we notice that by (VII.3.31) we have

$$\begin{aligned} \frac{\partial \boldsymbol{E}(y)}{\partial y_i} &\in L^q(\mathcal{A}), \quad \text{for all } q>4/3,\ i=2,3 \\ \frac{\partial \boldsymbol{E}}{\partial y_1} &\in L^q(\mathcal{A}), \quad \text{for all } q>1, \end{aligned} \tag{VII.3.33}$$

where $\mathcal{A}$, as before, denotes the exterior of a unit ball. Condition $(\text{VII.3.33})_1$ is sharp in the sense that

$$\frac{\partial \boldsymbol{E}(y)}{\partial y_i} \notin L^q(\mathcal{A}), \quad \text{for all } q\in[1,4/3],\ i=2,3 \tag{VII.3.34}$$

as a result of a calculation similar to that used to show (VII.3.30). It is interesting to observe that, even though the uniform bound (VII.3.32) is weaker than the analogous one for the Stokes fundamental tensor $\boldsymbol{U}$ (see (IV.2.6)), the summability properties (VII.3.33) are stronger than those for $\boldsymbol{U}$, where (in dimension 3) $\nabla\boldsymbol{U}\in L^q(\mathcal{A})$ for all and only all $q>3/2$.

Further estimates can be obtained for derivatives of order higher than one. Here, we shall limit ourselves to presenting some bounds of particular interest, leaving their proofs to the reader.

$$\begin{aligned} \left|\frac{\partial^2 \boldsymbol{E}(y)}{\partial y_i \partial y_j}\right| &\leq \frac{c}{|y|^2}\left[\frac{1-e^{-s}-se^{-s}}{4s^2|y|}+\frac{1-e^{-s}}{2s}\right], \quad y\in\mathbb{R}^3-\{0\} \\ |D^\alpha \boldsymbol{E}(y)| &\leq c|y|^{-1-|\alpha|/2}, \quad |\alpha|\geq 2\,, \quad \text{sufficiently large } |y|. \end{aligned} \tag{VII.3.35}$$

Exercise VII.3.1 Show that (in three dimensions) the tensor $\boldsymbol{E}$ satisfies the following estimates for all $R>0$:

$$\int_{\partial B_R} |\boldsymbol{E}(y)|^2 \le cR^{-1},$$

$$\int_{\partial B_R} |\nabla \boldsymbol{E}(y)| \le cR^{-1/2},$$

$$\int_{\partial B_R} |\nabla\nabla \boldsymbol{E}(y)| \le cR^{-1}.$$

Hint: Use estimates (VII.3.23), (VII.3.33), and (VII.3.35); see also Solonnikov (1996).

We next derive the form and corresponding estimates for $\boldsymbol{E}(x-y)$ in two space dimensions. From (VII.3.1) and (VII.3.16) we find

$$\begin{aligned}
E_{11}(x-y) &= \frac{\partial}{\partial y_1}\Psi(x-y) - \frac{1}{2\pi}K_0(\lambda|x-y|)e^{-\lambda(x_1-y_1)} \\
E_{12}(x-y) &= E_{21}(x-y) = -\frac{\partial}{\partial y_2}\Psi(x-y) \\
E_{22}(x-y) &= -\frac{\partial}{\partial y_1}\Psi(x-y)
\end{aligned}$$

where

$$\Psi(x-y) = \frac{1}{4\pi\lambda}\left(\log|x-y| + K_0(\lambda|x-y|)e^{-\lambda(x_1-y_1)}\right).$$

From (VII.3.16) and the properties (VII.3.10) of the function $K_0(z)$ near $z = 0$, we easily obtain, with $r = |x-y|$

$$\begin{aligned}
E_{ij} &= -\frac{1}{4\pi}\left[\delta_{ij}\log\frac{1}{2\lambda r} + \frac{(x_i-y_i)(x_j-y_j)}{r^2}\right] + o(1) \\
&= U_{ij}(x-y) - \frac{1}{4\pi}\delta_{ij}\log\frac{1}{2\lambda} + o(1), \quad \text{as } \lambda r \to 0,
\end{aligned} \tag{VII.3.36}$$

where $\boldsymbol{U}$ is the Stokes fundamental tensor (IV.2.4). Using (VII.3.36) and (VII.3.17), we then show the validity of (VII.3.22) also in two dimensions. It should be observed that, unlike the three-dimensional case (see (VII.3.21)), relation (VII.3.36) for *arbitrarily fixed* x, y becomes singular as $\lambda \to 0$. This is to be expected for, as we are going to show, the tensor $\boldsymbol{E}$ vanishes at large spatial distances, while $\boldsymbol{U}$ grows there logarithmically. Denote by φ the angle made by a ray that starts from x and is directed toward y, with the direction of the positive x_1-axis. From (VII.3.1), (VII.3.15) and (VII.3.16) we derive in the limit $\lambda r \to \infty$

$$
\begin{aligned}
E_{11}(x-y) &= -\frac{\cos\varphi}{4\pi\lambda r} + \frac{e^{-s}}{4\sqrt{2\lambda\pi r}}\left(1+\cos\varphi - \frac{1-3\cos\varphi}{8\lambda r} + \mathcal{R}(\lambda r)\right) \\
E_{12}(x-y) &= E_{21}(x-y) = \frac{\sin\varphi}{4\pi\lambda r} - \frac{e^{-s}\sin\varphi}{4\sqrt{2\lambda\pi r}}\left(1+\frac{3}{8\lambda r} + \mathcal{R}(\lambda r)\right) \\
E_{22}(x-y) &= \frac{\cos\varphi}{4\pi\lambda r} + \frac{e^{-s}}{4\sqrt{2\pi}(\lambda r)^{3/2}}\left(s - \frac{1+3\cos\varphi}{8} + \mathcal{R}(\lambda r)\right),
\end{aligned}
\tag{VII.3.37}
$$

where

$$
s = \lambda r(1-\cos\varphi)
$$

and the remainder $\mathcal{R}(t)$ satisfies

$$
\frac{d^k\mathcal{R}}{dt^k} = O(t^{-2-k}), \quad \text{as } t \to \infty,\ k \geq 0.
$$

From (VII.3.37) it follows that unlike the Stokes fundamental tensor, which grows logarithmically rapid for $r \to \infty$, the Oseen fundamental tensor vanishes at large distances. It is this difference that allowed Oseen to remove the discrepancy generating the Stokes paradox. Relations (VII.3.37) produce some useful estimates. We shall present them as a function of y, fixing the origin of coordinates at $x = 0$. Analogous estimates of $\boldsymbol{E}$ as a function of x remain true if φ is replaced by $\pi - \varphi$, and the considerations made in Remark VII.3.1 apply here.

From $(\text{VII.3.37})_1$ it readily follows that $E_{11}(y)$ exhibits the same parabolic "wake" region that was obtained in the three-dimensional case. In fact, if y is interior to the parabola[2]

$$
|y|(1-\cos\varphi) = 1
$$

we deduce the uniform bound

$$
|E_{11}(y)| \leq \frac{c}{|y|^{1/2}}, \quad \text{as } |y| \to \infty, \tag{VII.3.38}
$$

while if

$$
(1-\cos\varphi) \geq |y|^{-1+2\sigma} \quad \text{for some } \sigma \in [0, 1/2] \tag{VII.3.39}
$$

we have

$$
|E_{11}(y)| \leq \frac{c}{|y|^{1/2+\sigma}}, \quad \text{as } |y| \to \infty. \tag{VII.3.40}
$$

However, unlike the three-dimensional case, the remaining components of $\boldsymbol{E}$ *do not present such a nonuniform behavior.* In fact, observing that

[2] Setting $\theta = \pi - \varphi$, the parabolic region can be approximately described, for large $|y|$, by

$$
|\theta| \leq (2)^{1/2}|y|^{-1/2}.
$$

$$e^{-2s}\lambda r \sin^2\varphi = s e^{-2s}(1+\cos\varphi) \le e^{-1}$$

from $(\text{VII.3.37})_{2,3}$ we recover the *uniform* estimates

$$|E_{i2}(y)| \le \frac{c}{|y|}, \quad i=1,2, \quad \text{as } |y|\to\infty. \tag{VII.3.41}$$

Concerning the summability properties of $\boldsymbol{E}$, denoting by $\mathcal{A}$ the exterior of a unit circle, from (VII.3.41) we recover at once

$$E_{i2}(y) \in L^q(\mathcal{A}), \quad \text{for all } q>2,\ i=1,2. \tag{VII.3.42}$$

Moreover, setting $\cos\varphi = \mp(y-\lambda qr)/\lambda qr$, where "$-$" is taken if $\varphi\in[0,\pi]$ and "$+$" otherwise, by a direct calculation one shows for any $q\ge 1$

$$\begin{aligned}\int_0^{2\pi} e^{-qs}(1+\cos\varphi)^q d\varphi &\le \frac{2}{(4\lambda qr)^q}\int_0^{2\lambda qr} e^{-y}(2\lambda qr - y)^{q-1/2}y^{-1/2}dy \\ &\le \frac{2(4\lambda qr)^{q-1/2}}{(2\lambda qr)^q}\int_0^\infty e^{-y}y^{-1/2}dy = c_1/r^{1/2},\end{aligned}$$

with c_1 independent of r. As a consequence, from $(\text{VII.3.37})_1$ we obtain

$$E_{11}(y) \in L^q(\mathcal{A}), \quad \text{for all } q>3. \tag{VII.3.43}$$

Likewise, setting

$$f(\varphi,r) \equiv -\frac{\cos\varphi}{\sqrt{\pi\lambda r}} + \frac{e^{-s}}{\sqrt{2\lambda\pi}}\left(1+\cos\varphi - \frac{1-3\cos\varphi}{8\lambda r}\right)$$

one has

$$\begin{aligned}\int_0^{2\pi}|f(\varphi,r)|^q d\varphi &\ge \int_0^{\pi}|f(\varphi,r)|^q d\varphi \\ &= \frac{1}{(\lambda qr)^q}\int_0^{2\lambda qr}\left|\frac{y-\lambda qr}{\sqrt{\lambda\pi r}} + \frac{e^{-y/q}}{\sqrt{2\lambda\pi}}\left[2\lambda qr - y \right.\right. \\ &\qquad \left.\left. - \frac{3y-\lambda qr}{8\lambda qr}\right]\right|^q \frac{y^{-1/2}}{(2\lambda qr - y)^{1/2}}dy,\end{aligned}$$

and so, for all r sufficiently large

$$\begin{aligned}\int_0^{2\pi}|f(\varphi,r)|^q d\varphi &\ge \frac{c_1}{r^q}\int_0^1 e^{-y}(2\lambda qr - y)^{q-1/2}y^{-1/2}dy \\ &\ge \frac{c_2}{r^{1/2}}\int_0^1 e^{-y}y^{-1/2}dy = c_3/r^{1/2}\end{aligned}$$

with c_3 independent of r, and from (VII.3.37)_1 it follows that

$$E_{11}(y) \notin L^q(\mathcal{A}), \quad \text{for all } q \in [1,3]. \tag{VII.3.44}$$

So far as the behavior of the first derivatives of $\boldsymbol{E}$ is concerned, differentiating (VII.3.37) we derive the following bounds as $|y| \to \infty$

$$\begin{aligned} \left|\frac{\partial E_{11}(y)}{\partial y_2}\right| \leq \frac{c}{|y|}, \quad \left|\frac{\partial E_{12}(y)}{\partial y_1}\right| \leq \frac{c}{|y|^2}, \\ \left|\frac{\partial E_{1i}(y)}{\partial y_i}\right| \leq \frac{c}{|y|^{3/2}}, \quad \left|\frac{\partial E_{22}(y)}{\partial y_i}\right| \leq \frac{c}{|y|^2}, \quad i = 1,2. \end{aligned} \tag{VII.3.45}$$

Sharper estimates can be obtained for the derivatives of E_{11} and the derivative of E_{12} with respect to y_2 whenever y is exterior to the "wake" region, see Exercise VII.3.2. Moreover, besides the obvious summability properties obtainable from (VII.3.45) one can show (Exercise VII.3.2)

$$\begin{aligned} &\frac{\partial \boldsymbol{E}(y)}{\partial y_1} \in L^q(\mathcal{A}) \;\text{ for all } q > 1 \\ &\frac{\partial \boldsymbol{E}(y)}{\partial y_2} \in L^q(\mathcal{A}) \;\text{ for all } q > 3/2 \end{aligned} \tag{VII.3.46}$$

while

$$\frac{\partial \boldsymbol{E}(y)}{\partial y_2} \notin L^q(\mathcal{A}) \;\text{ for all } q \in [1, 3/2]. \tag{VII.3.47}$$

Further asymptotic bounds can be analogously derived for derivatives of order higher than two. For instance, one shows the validity of the following properties (see Exercise VII.3.3):

$$\begin{aligned} &D^2\boldsymbol{E}(y) \in L^q(\mathcal{A}), \quad \text{for all } q > 1, \\ &|D^\alpha \boldsymbol{E}(y)| \leq c|y|^{-(1+|\alpha|)/2}, \quad |\alpha| \geq 2, \quad \text{as } |x| \to \infty. \end{aligned} \tag{VII.3.48}$$

Exercise VII.3.2 Let $\boldsymbol{E}$ be given as in (VII.3.37). Show that for all $\sigma \in [0, 1/2]$ and all sufficiently large $|y|$,

$$\left|\frac{\partial E_{11}(y)}{\partial y_2}\right| \leq \frac{c}{|y|^{1+2\sigma}}$$

$$\left|\frac{\partial E_{11}(y)}{\partial y_1}\right|, \quad \left|\frac{\partial E_{12}(y)}{\partial y_2}\right| \leq \frac{c}{|y|^{3/2+\sigma}}$$

in region (VII.3.39). Furthermore, show the validity of (VII.3.46) and (VII.3.47).

Exercise VII.3.3 Show estimate (VII.3.48).

Exercise VII.3.4 Prove that (in two dimensions) the tensor $\boldsymbol{E}$ obeys the following estimates for all $R > 0$:

$$\int_{\partial B_R} |\boldsymbol{E}(y)|^2, \quad \int_{\partial B_R} |\nabla \boldsymbol{E}(y)| \le cR^{-1/2}.$$

Hint: Use (VII.3.39) and (VII.3.41).

Exercise VII.3.5 Let $\boldsymbol{E}(x-y) \equiv \boldsymbol{E}(x-y;2\lambda)$ denote the Oseen tensor corresponding to 2λ. Show the following homogeneity properties

$$\begin{aligned} \boldsymbol{E}(x-y;2\lambda) &= 2\lambda \boldsymbol{E}(2\lambda(x-y);1) \ \text{ for } n=3, \\ \boldsymbol{E}(x-y;2\lambda) &= \boldsymbol{E}(2\lambda(x-y);1) \ \text{ for } n=2. \end{aligned}$$

Remark VII.3.2 Estimates analogous to those presented for space dimension $n = 2, 3$ can be derived for all $n \ge 4$. Actually, from $(\text{VII.3.10})_2$ and (VII.3.11) it follows that the n-dimensional Oseen fundamental tensor $\boldsymbol{E}(x-y)$ (defined by (VII.3.1), (VII.3.4), and (VII.3.8)) satisfies (VII.3.21) and, consequently, becomes singular at $x = y$ in such a way that (VII.3.22) is verified. Furthermore, using the asymptotic expansion for large z:

$$\begin{aligned} K_{(n-2)/2} &= \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \left[\sum_{k=0}^{\nu-1} \frac{\Gamma(n/2+k-1/2)}{k!\Gamma(n/2-k-1/2)}(2z)^{-k} + \sigma_\nu(z)\right] \\ &= \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \left[1 + \frac{4(n-2)^2-1}{8z}\right. \\ &\qquad\qquad \left. + \frac{[4(n-2)^2-1][4(n-2)^2-3^2]}{2!(8z)^2} + \ldots + \sigma_\nu(z)\right] \end{aligned}$$

with

$$\frac{d^k \sigma_\nu}{dz^k} = O(z^{-k-\nu}) \ \text{ as } z \to \infty,\ k \ge 0,$$

see Watson (1962, p. 202), one can obtain estimates at large distances. For instance, we can show

$$\begin{aligned} &|D^\alpha \boldsymbol{E}(y)| \le c|y|^{-(n-1+|\alpha|)/2}, \quad |y| \to \infty, \ |\alpha| \ge 0, \\ &\boldsymbol{E}(y) \in L^q(\mathcal{A}), \quad q > \frac{n+1}{n-1} \\ &\nabla \boldsymbol{E}(y) \in L^r(\mathcal{A}), \quad r > \frac{n+1}{n} \\ &D^2 \boldsymbol{E}(y) \in L^s(\mathcal{A}), \quad s > 1. \end{aligned} \tag{VII.3.49}$$

■

In analogy with what we did for the Stokes problem, we now introduce the *Oseen volume potentials*:

$$\begin{aligned} \boldsymbol{u}(x) &= 2\lambda \int_{\mathbb{R}^n} \boldsymbol{E}(x-y) \cdot \boldsymbol{F}(y) dy \\ \pi(x) &= -2\lambda \int_{\mathbb{R}^n} \boldsymbol{e}(x-y) \cdot \boldsymbol{F}(y) dy, \end{aligned} \tag{VII.3.50}$$

where $\boldsymbol{F} \in C_0^\infty(\mathbb{R}^n)$. Since

$$\int_{\mathbb{R}^n} \boldsymbol{E}(x-y) \cdot \boldsymbol{F}(y) dy = \int_{\mathbb{R}^n} \boldsymbol{E}(z) \cdot \boldsymbol{F}(x+z) dz,$$

one has $\boldsymbol{u} \in C^\infty(\mathbb{R}^n)$ and, by the same token, $\pi \in C^\infty(\mathbb{R}^n)$. Moreover, it is easy to show that $\boldsymbol{u}$, π satisfy the Oseen system in $\mathbb{R}^n$. Actually, it is obvious that $\nabla \cdot \boldsymbol{u} = 0$. Also, using integration by parts and (VII.3.2) we deduce for all $x \in \mathbb{R}^n$

$$\Delta \boldsymbol{u}(x) + 2\lambda \frac{\partial \boldsymbol{u}(x)}{\partial x_1} - \nabla \pi = \Delta(\mathcal{E} * \boldsymbol{F}) = 2\lambda \boldsymbol{F}(x).$$

Moreover, it is easy to show that the solution $\boldsymbol{u}$, π behaves at large distances exactly as the fundamental solution $\boldsymbol{E}$, $\boldsymbol{e}$. This immediately follows by observing that from (VII.3.49) we have

$$\begin{aligned} \boldsymbol{u}(x) &= 2\lambda \boldsymbol{E}(x) \cdot \int_{\mathbb{R}^n} \boldsymbol{F}(y) dy + \boldsymbol{\sigma}(x) \\ \pi(x) &= -2\lambda \boldsymbol{e}(x) \cdot \int_{\mathbb{R}^n} \boldsymbol{F}(y) dy + \eta(x) \end{aligned} \tag{VII.3.51}$$

with

$$\boldsymbol{\sigma}(x) = 2\lambda \int_{\mathbb{R}^n} (\boldsymbol{E}(x-y) - \boldsymbol{E}(x)) \cdot \boldsymbol{F}(y) dy$$

$$\pi(x) = -2\lambda \int_{\mathbb{R}^n} (\boldsymbol{e}(x-y) - \boldsymbol{e}(x)) \cdot \boldsymbol{F}(y) dy$$

and that, using the mean-value theorem and the assumption $\boldsymbol{F} \in C_0^\infty(\mathbb{R}^n)$,

$$\left. \begin{aligned} D^\alpha \boldsymbol{\sigma}(x) &= O(D^\alpha \nabla \boldsymbol{E}(x)) \\ D^\alpha \eta(x) &= O(D^\alpha \nabla \boldsymbol{e}(x)) \end{aligned} \right. \quad |\alpha| \ge 0, \quad |x| \to \infty. \tag{VII.3.52}$$

Remark VII.3.3 Starting with $(\text{VII.3.50})_1$ and using Young's theorem on convolution (see Theorem II.11.1) one can prove at once L^q-estimates for $\boldsymbol{u}, \pi$ and their first derivatives. This is due to the circumstance that, unlike the Stokes tensor $\boldsymbol{U}$, the Oseen tensor possesses *global* summability properties in the *whole* of $\mathbb{R}^n$. For instance, from (VII.3.49) and (VII.3.21) we see that

$$\boldsymbol{E}(y) \in L^q(\mathbb{R}^n) \quad \text{for all } q \in ((n+1)/(n-1), n) \quad \text{if } n > 2,$$

$$\boldsymbol{E}(y) \in L^q(\mathbb{R}^n) \quad \text{for all } q \in (3, \infty) \quad \text{if } n = 2.$$

However, estimates obtained in such a way would not be sharp and, therefore, we shall not derive them here. Derivation of sharp estimates for the Oseen potentials, by different tools, will be the object of the next section. ■

VII.4 Existence, Uniqueness, and L^q-Estimates in the Whole Space

The objective of this section is to prove existence, uniqueness, and corresponding estimates of solutions $\boldsymbol{v}, p$ to the *nonhomogeneous Oseen system*

$$\left.\begin{array}{l} \Delta \boldsymbol{v} + \mathcal{R}\dfrac{\partial \boldsymbol{v}}{\partial x_1} - \nabla p = \mathcal{R}\boldsymbol{f} \\ \nabla \cdot \boldsymbol{v} = g \end{array}\right\} \quad \text{in } \mathbb{R}^n \tag{VII.4.1}$$

in homogeneous Sobolev spaces $D^{m,q}(\mathbb{R}^n)$. These results, though sharing some similarity with the analogous ones established for the Stokes system in Chapter IV, will differ from these latter in some crucial features that essentially mirror the basic differences existing between the two fundamental tensors $\boldsymbol{U}$ and $\boldsymbol{E}$.

In establishing estimates for (VII.4.1), it is important to single out the dependence of the constants entering the estimates on the dimensionless parameter $\mathcal{R}$. We shall therefore consider the problem

$$\left.\begin{array}{l} \Delta \boldsymbol{v} + \dfrac{\partial \boldsymbol{v}}{\partial x_1} - \nabla p = \boldsymbol{f} \\ \nabla \cdot \boldsymbol{v} = g \end{array}\right\} \quad \text{in } \mathbb{R}^n \tag{VII.4.2}$$

and establish corresponding estimates for its solutions. The analogous ones for solutions to (VII.4.1) will then be obtained if we make the replacements

$$\begin{aligned} &\boldsymbol{f} \to \boldsymbol{f}/\mathcal{R} \\ &g \to g/\mathcal{R} \\ &p \to p/\mathcal{R} \\ &x_i \to \mathcal{R}x_i. \end{aligned} \tag{VII.4.3}$$

Unlike the corresponding estimates for the Stokes system, here we cannot employ the Calderón–Zygmund theorem because the kernel $D^2_{ij}E_{ks}(x-y)$ does not satisfy the assumption (II.11.15) of that theorem, that is,

$$\alpha^{-n} D^2_{ij}E_{ks}(\xi) \neq D^2_{ij}E_{ks}(\alpha\xi), \quad \alpha > 0.$$

Rather, we shall make use of a more appropriate tool due to P.I. Lizorkin (1963, 1967), which we are going to describe.

Denote by $\mathcal{S}(\mathbb{R}^n)$ the space of *functions of rapid decrease* consisting of elements u from $C^\infty(\mathbb{R}^n)$ such that

$$\sup_{x\in\mathbb{R}^n}\left(|x_1|^{\alpha_1}\cdot\ldots\cdot|x_n|^{\alpha_n}|D^\beta u(x)|\right)<\infty$$

for all $\alpha_1,\ldots,\alpha_n\geq 0$ and $|\beta|\geq 0$. For $u\in\mathcal{S}(\mathbb{R}^n)$ we denote by $\widehat{u}$ its *Fourier transform*:

$$\widehat{u}(\xi)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}}u(x)dx,$$

where i stands for the imaginary unit. It is well known that $\widehat{u}\in\mathcal{S}(\mathbb{R}^n)$ and that, moreover,

$$u(x)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}}\widehat{u}(\xi)d\xi,$$

see, e.g., Reed & Simon (1975, Lemma on p. 2). Given a function $\Phi:\mathbb{R}^n\to\mathbb{R}$, let us consider the *integral transform*

$$Tu\equiv h(x)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}}\Phi(\xi)\widehat{u}(\xi)d\xi,\quad u\in\mathcal{S}(\mathbb{R}^n).\tag{VII.4.4}$$

Generalizing the works of Marcinkiewicz (1939) and Mikhlin (1957), Lizorkin (1963, 1967) has proved the following result, which we state without proof.

Lemma VII.4.1 *Let $\Phi:\mathbb{R}^n\to\mathbb{R}$ be continuous together with the derivative*

$$\frac{\partial^n\Phi}{\partial\xi_1\ldots\partial\xi_n}$$

and all preceding derivatives for $|\xi_i|>0$, $i=1,\ldots,n$. Then, if for some $\beta\in[0,1)$ and $M>0$

$$|\xi_1|^{\kappa_1+\beta}\cdot\ldots\cdot|\xi_n|^{\kappa_n+\beta}\left|\frac{\partial^\kappa\Phi}{\partial\xi_1^{\kappa_1}\ldots\partial\xi_n^{\kappa_n}}\right|\leq M,$$

where κ_i is zero or one and $\kappa=\sum_{i=1}^n\kappa_i=0,1,\ldots n$, the integral transform (VII.4.4) *defines a bounded linear operator from $L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$, $1<q<\infty$, $1/r=1/q-\beta$, and we have*

$$\|Tu\|_r\leq c\|u\|_q,$$

with $c=c_0(q,\beta)M$.

With this result in hand, we shall now look for a solution to (VII.4.2) corresponding to $\boldsymbol{f}$, $g\in C_0^\infty(\mathbb{R}^n)$ of the form

$$
\begin{aligned}
\boldsymbol{v}(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}} \boldsymbol{V}(\xi) d\xi \\
p(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}} P(\xi) d\xi .
\end{aligned}
\tag{VII.4.5}
$$

Replacing (VII.4.5) into (VII.4.2) furnishes the following algebraic system for $\boldsymbol{V}$ and P:

$$
\begin{aligned}
(\xi^2 + i\xi_1)V_m(\xi) + i\xi_m P(\xi) &= \widehat{f}_m(\xi) \\
i\xi_m V_m(\xi) &= \widehat{g},
\end{aligned}
\tag{VII.4.6}
$$

where $m = 1, \ldots, n$. Solving (VII.4.6) for $\boldsymbol{V}$ and P delivers

$$
\begin{aligned}
V_m(\xi) &= U_m(\xi) + W_m(\xi) \\
P(\xi) &= \Pi(\xi) + T(\xi)
\end{aligned}
\tag{VII.4.7}
$$

with

$$
\begin{aligned}
U_m(\xi) &= \frac{\xi_m\xi_k - \xi^2\delta_{mk}}{\xi^2(\xi^2 + i\xi_1)} \widehat{f}_k(\xi) \\
W_m(\xi) &= -i\frac{\xi_m \widehat{g}(\xi)}{\xi^2} \\
\Pi(\xi) &= i\frac{\xi_k \widehat{f}_k(\xi)}{\xi^2} \\
T(\xi) &= \left(i\frac{\xi_1}{\xi^2} + 1 \right) \widehat{g}(\xi).
\end{aligned}
\tag{VII.4.8}
$$

From (VII.4.5) and (VII.4.7), (VII.4.8) we have that a solution to (VII.4.2) is given by

$$
\begin{aligned}
\boldsymbol{v}(x) &= \boldsymbol{u}(x) + \boldsymbol{w}(x) \\
p(x) &= \pi(x) + \tau(x)
\end{aligned}
\tag{VII.4.9}
$$

with

$$
\begin{aligned}
\boldsymbol{u}(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}} \boldsymbol{U}(\xi) d\xi \\
\boldsymbol{w}(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}} \boldsymbol{W}(\xi) d\xi \\
\pi(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}} \Pi(\xi) d\xi \\
\tau(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}} T(\xi) d\xi \, .
\end{aligned}
\tag{VII.4.10}
$$

Observe that $\boldsymbol{u}, \pi$ and $\boldsymbol{w}, \tau$ are solutions of (VII.4.2) corresponding to $\boldsymbol{f} \neq \boldsymbol{0}$, $g = 0$ and to $\boldsymbol{f} = 0$, $g \neq 0$, respectively. Since $\widehat{\boldsymbol{f}}$ and $\widehat{g}$ are in $\mathcal{S}(\mathbb{R}^n)$, it is not hard to show that (VII.4.9) defines a C^∞-solution to (VII.4.2). Let us now determine some L^q-estimates for $\boldsymbol{v}$ and p. This will be done with the aid of Lemma VII.4.1. In this respect, an important role will be played by the function

$$\phi_{mk}(\xi) = \frac{\xi_m\xi_k - \xi^2\delta_{mk}}{\xi^2(\xi^2 + i\xi_1)}, \tag{VII.4.11}$$

whose properties will be investigated. Specifically, we have

Lemma VII.4.2 *Let $n \geq 2$ and let ϕ_{mk} be given by* (VII.4.11) *with m, k ranging in $\{1,\dots,n\}$. Then, the assumptions of* Lemma VII.4.1 *are satisfied:*

(a) *by ϕ_{mk} with $\beta = 2/(n+1)$;*
(b) *by $\xi_\ell\phi_{mk}$ with $\beta = 1/(n+1)$ and $\ell \in \{1,\dots,n\}$;*
(c) *by $\xi_1\phi_{mk}$ with $\beta = 0$;*
(d) *by $\xi_s\xi_\ell\phi_{mk}$ with $\beta = 0$ and $s,\ell \in \{1,\dots,n\}$.*

Finally, if $n = 2$, for all $\ell, k \in \{1,2\}$ the assumptions of Lemma VII.4.1 *are satisfied:*

(e) *by ϕ_{2k} with $\beta = 1/2$;*
(f) *by $\xi_\ell\phi_{2k}$ with $\beta = 0$.*

Proof. Clearly, ϕ_{mk} and the product of ϕ_{mk} with any product of the variables ξ_k satisfy the regularity assumptions of Lemma VII.4.1. Moreover, for all $\ell, m, k = 1,\dots,n$ it is immediately seen that

$$|\xi_1|^{\kappa_1}\cdot\ldots\cdot|\xi_n|^{\kappa_n}\left|\frac{\partial^\kappa\phi_{mk}}{\partial\xi_1^{\kappa_1}\ldots\partial\xi_n^{\kappa_n}}\right| \leq c_1\frac{1}{|\xi|^2 + |\xi_1|}$$

for some $c_1 = c_1(n)$ where κ_i is zero or one, $\kappa = \sum_{i=1}^n \kappa_i = 0, 1, \dots, n$, and therefore to show assertion (a) it is enough to show that

$$\frac{(|\xi_1|\cdot\ldots\cdot|\xi_n|)^{2/(n+1)}}{|\xi|^2 + |\xi_1|} \leq c_2 \tag{VII.4.12}$$

with $c_2 = c_2(n)$. Now, by a repeated use of Young's inequality (II.2.7)

$$(|\xi_1|\cdot\ldots\cdot|\xi_n|)^{2/(n+1)} \leq a_1[|\xi_1|^{1/2} + (|\xi_2|\cdot\ldots\cdot|\xi_n|)^{1/(n-1)}]^2 \leq a_2(|\xi_1| + |\xi|^2),$$

with $a_i = a_i(n)$, $i = 1, 2$, and so (VII.4.12) follows. Likewise, we can show assertions (b) and (d). Furthermore, observing that

$$|\xi_1|^{\kappa_1}\cdot\ldots\cdot|\xi_n|^{\kappa_n}\left|\frac{\partial^\kappa(\xi_1\phi_{mk})}{\partial\xi_1^{\kappa_1}\ldots\partial\xi_n^{\kappa_n}}\right| \leq c_3\frac{|\xi_1|}{|\xi|^2 + |\xi_1|} \leq c_3,$$

with $c_3 = c_3(n)$, property (c) follows. To prove the last part of the lemma we notice that if $n = 2$ from (VII.4.11),

$$\phi_{21}(\xi) = \frac{\xi_2\xi_1}{\xi^2(\xi^2 + i\xi_1)}$$

$$\phi_{22}(\xi) = \frac{-\xi_1^2}{\xi^2(\xi^2 + i\xi_1)},$$

and therefore, for $\kappa_i = 0, 1$, $i = 1, 2$ and $k = 1, 2$, we have

$$|\xi_1|^{\kappa_1}|\xi_2|^{\kappa_2}\left|\frac{\partial^\kappa \phi_{2k}}{\partial\xi_1^{\kappa_1}\partial\xi_2^{\kappa_2}}\right| \le c_4 \frac{|\xi_1|}{|\xi|(|\xi|^2 + |\xi_1|)}, \quad \kappa = \kappa_1 + \kappa_2.$$

Since

$$\frac{(|\xi_1||\xi_2|)^{1/2}|\xi_1|}{|\xi|(|\xi|^2 + |\xi_1|)} \le 1,$$

assertion (e) follows. Finally, from the inequality

$$|\xi_1|^{\kappa_1}|\xi_2|^{\kappa_2}\left|\frac{\partial^\kappa (\xi_\ell\phi_{2k})}{\partial\xi_\ell^{\kappa_1}\partial\xi_2^{\kappa_2}}\right| \le c_5 \frac{|\xi_1|}{|\xi|^2 + |\xi_1|} \le c_5,$$

(f) is proved and the proof of the lemma . □

Let us begin to estimate $\boldsymbol{u}$ and π. From (VII.4.8)$_1$ and (VII.4.10), with the help of Lemma VII.4.1 and Lemma VII.4.2 (c), (d), it follows at once that

$$\left\|\frac{\partial \boldsymbol{u}}{\partial x_1}\right\|_q \le c\|\boldsymbol{f}\|_q \tag{VII.4.13}$$

and

$$|\boldsymbol{u}|_{2,q} \le c\|\boldsymbol{f}\|_q. \tag{VII.4.14}$$

Also, observing that for all $s, k = 1, \ldots, n$ the function $\xi_s\xi_k/\xi^2$ satisfies the assumptions of Lemma VII.4.1 with $\beta = 0$, we have

$$|\pi|_{1,q} \le c\|\boldsymbol{f}\|_q.$$

From (VII.4.13), (VII.4.14) and from this last inequality we conclude

$$\left\|\frac{\partial \boldsymbol{u}}{\partial x_1}\right\|_q + |\boldsymbol{u}|_{2,q} + |\pi|_{1,q} \le c\|\boldsymbol{f}\|_q, \quad 1 < q < \infty, \tag{VII.4.15}$$

with $c = c(n, q)$. In the case of plane flow ($n = 2$), we are able to obtain a sharper estimate on the component u_2 of the velocity field. Specifically, from (VII.4.9), (VII.4.7), Lemma VII.4.1, and Lemma VII.4.2(f), we recover

$$|u_2|_{1,q} \le c\|\boldsymbol{f}\|_q, \quad 1 < q < \infty, \tag{VII.4.16}$$

which, along with (VII.4.14), then furnishes

$$|u_2|_{1,q} + \left\| \frac{\partial u_1}{\partial x_1} \right\|_q + |\boldsymbol{u}|_{2,q} + |\pi|_{1,q} \le c\|\boldsymbol{f}\|_q, \quad 1 < q < \infty. \tag{VII.4.17}$$

Other estimates can be obtained by suitably restricting the range of values of q. To this end, assume $1 < q < n+1$; we then obtain from Lemma VII.4.1 and Lemma VII.4.2(b)

$$|\boldsymbol{u}|_{1,s_1} \le c\|\boldsymbol{f}\|_q, \quad s_1 = \frac{(n+1)q}{n+1-q}, \quad 1 < q < n+1, \tag{VII.4.18}$$

where $c = c(n,q)$. In the case of plane flow, $n = 2$, in addition to (VII.4.18), from Lemma VII.4.2(e) we derive

$$\|u_2\|_{2q/(2-q)} \le c\|\boldsymbol{f}\|_q, \quad 1 < q < 2, \quad (n = 2). \tag{VII.4.19}$$

Finally, assuming $1 < q < (n+1)/2$, from Lemma VII.4.1 and Lemma VII.4.2(a) we find

$$\|\boldsymbol{u}\|_{s_2} \le c\|\boldsymbol{f}\|_q, \quad s_2 = \frac{(n+1)q}{n+1-2q}, \quad 1 < q < \frac{n+1}{2}. \tag{VII.4.20}$$

Let us now estimate the pair $\boldsymbol{w}, \tau$. Observing that

$$\widehat{D_\ell h} = i\xi_\ell \widehat{h}, \tag{VII.4.21}$$

and recalling that the function $\xi_s\xi_k/\xi^2$ satisfies the assumptions of Lemma VII.4.1 with $\beta = 0$, we at once obtain

$$\begin{aligned} |\boldsymbol{w}|_{1,r} &\le c\|g\|_r, \quad 1 < r < \infty \\ |\boldsymbol{w}|_{2,r} &\le c|g|_{1.r}, \quad 1 < r < \infty. \end{aligned} \tag{VII.4.22}$$

Likewise,

$$|\tau|_{1,r} \le c\|g\|_{1,r}, \quad 1 < r < \infty. \tag{VII.4.23}$$

Moreover, it is simple to show that ξ_k/ξ^2, $k = 1, \ldots, n$, satisfies the assumptions of Lemma VII.4.1 with $\beta = 1/n$ and so

$$\|\boldsymbol{w}\|_{nr/(n-r)} \le c\|g\|_r, \quad 1 < r < n. \tag{VII.4.24}$$

Thus, from (VII.4.15), (VII.4.22), and (VII.4.23), it follows that

$$\left\| \frac{\partial \boldsymbol{v}}{\partial x_1} \right\|_q + |\boldsymbol{v}|_{2,q} + |p|_{1,q} \le c(\|\boldsymbol{f}\|_q + \|g\|_{1,q}), \quad 1 < q < \infty, \tag{VII.4.25}$$

and, from (VII.4.16), (VII.4.22), and (VII.4.23),

$$|v_2|_{1,q} + \left\| \frac{\partial \boldsymbol{v}}{\partial x_1} \right\|_q + |\boldsymbol{v}|_{2,q} + |p|_{1,q} \le c(\|\boldsymbol{f}\|_q + \|g\|_{1,q}), \quad 1 < q < \infty, \quad (n = 2). \tag{VII.4.26}$$

Remark VII.4.1 Estimates (VII.4.25) and (VII.4.26) *have no analogue in the Stokes problem*, because of the presence of the terms in $\frac{\partial \boldsymbol{v}}{\partial x_1}$ and v_2. Just these estimates, from the point of view of the L^q-approach, make the difference between Oseen and Stokes approximations. ■

If we restrict the values of q we can obtain other estimates for $\boldsymbol{v}$. Specifically, from (VII.4.18) and (VII.4.22) we obtain

$$|\boldsymbol{v}|_{1,s_1} \le c(\|\boldsymbol{f}\|_q + \|g\|_{s_1}), \quad s_1 = \frac{(n+1)q}{n+1-q}, \quad 1 < q < n+1.$$

On the other hand, by the embedding Theorem II.3.1, it easily follows that

$$\|g\|_{s_1} \le c\|g\|_{1,q}, \quad s_1 = \frac{(n+1)q}{n+1-q}, \quad 1 < q < n+1,$$

and so, in particular,

$$|\boldsymbol{v}|_{1,s_1} \le c(\|\boldsymbol{f}\|_q + \|g\|_{1,q}), \quad s_1 = \frac{(n+1)q}{n+1-q}, \quad 1 < q < n+1. \tag{VII.4.27}$$

If $n = 2$, from (VII.4.19) and (VII.4.24) and from the embedding TTheorem II.3.1, we have

$$\|v_2\|_{2q/(2-q)} \le c(\|\boldsymbol{f}\|_q + \|g\|_{1,q}), \quad 1 < q < 2, \quad (n=2). \tag{VII.4.28}$$

Finally, if $1 < q < (n+1)/2$, we choose in (VII.4.24) the exponent r such that $nr/(n-r) = q(n+1)/(n+1-2q)$ to obtain

$$\|\boldsymbol{w}\|_{s_2} \le c\|g\|_{r_1}, \quad r_1 = \frac{n(n+1)q}{n(n+1-q)+q}, \quad 1 < q < \frac{(n+1)}{2}.$$

(Notice that $r_1 < n$, since $q < (n+1)/2$.) Again using the embedding Theorem II.3.1 on the right-hand side of this relation yields

$$\|\boldsymbol{w}\|_{s_2} \le c\|g\|_{1,q},$$

which together with (VII.4.20), in turn, implies

$$\|\boldsymbol{v}\|_{s_2} \le c(\|\boldsymbol{f}\|_q + \|g\|_{1,q}), \quad s_2 = \frac{(n+1)q}{n+1-2q}, \quad 1 < q < \frac{(n+1)}{2}. \tag{VII.4.29}$$

The results obtained so far can be immediately extended along the following two directions. First of all, estimates (VII.4.25)–(VII.4.29) can be generalized to derivatives of arbitrary order, by operating with D_ℓ on both sides of (VII.4.10) and then using (VII.4.21). Secondly, $\boldsymbol{f}$ and g can be merely assumed to be in $W^{m,q}(\mathbb{R}^n)$ and $W^{m+1,q}(\mathbb{R}^n)$, respectively. In fact, we may use a standard argument of the type employed in Section IV.2 for the Stokes problem along with the inequalities just derived (and those for higher-order derivatives) to establish existence of solutions to (VII.4.2) and related estimates under the above-stated larger assumptions on $\boldsymbol{f}$ and g. Corresponding results for system (VII.4.1) can then be obtained via transformation (VII.4.3). The first part of the following theorem is then acquired.

Theorem VII.4.1 *Given*

$$\boldsymbol{f} \in W^{m,q}(\mathbb{R}^n), \quad g \in W^{m+1,q}(\mathbb{R}^n), \quad m \geq 0, \quad 1 < q < \infty,$$

there exists a pair of functions $\boldsymbol{v}, p$ *with*

$$\boldsymbol{v} \in W^{m+2,q}(B_R), \quad p \in W^{m+1,q}(B_R) \quad , \textit{for any } R > 0,$$

and satisfying a.e. the nonhomogeneous Oseen system (VII.4.1). *Moreover, for all integers* $\ell \in [0, m]$, *the quantities*

$$\left|\frac{\partial \boldsymbol{v}}{\partial x_1}\right|_{\ell,q}, \quad |\boldsymbol{v}|_{\ell+2,q}, \quad |p|_{\ell+1,q}$$

are finite and satisfy the estimate

$$\mathcal{R}\left|\frac{\partial \boldsymbol{v}}{\partial x_1}\right|_{\ell,q} + |\boldsymbol{v}|_{\ell+2,q} + |p|_{\ell+1,q} \leq c\,(\mathcal{R}|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q} + \mathcal{R}|g|_{\ell,q})\,. \quad \text{(VII.4.30)}$$

If $n = 2$,

$$|v_2|_{\ell+1,q}$$

is also finite and it holds that

$$\begin{aligned} \mathcal{R}|v_2|_{\ell+1,q} + \mathcal{R}\left|\frac{\partial v_1}{\partial x_1}\right|_{\ell,q} + |\boldsymbol{v}|_{\ell+2,q} + |p|_{\ell+1,q} \\ \leq c\,(\mathcal{R}|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q} + \mathcal{R}|g|_{\ell,q})\,. \end{aligned} \quad \text{(VII.4.31)}$$

If $1 < q < n+1$, $|\boldsymbol{v}|_{\ell+1,s_1}$ *is finite,* $s_1 = (n+1)q/(n+1-q)$, *and*

$$\begin{aligned} \mathcal{R}^{1/(n+1)}|\boldsymbol{v}|_{\ell+1,s_1} + \mathcal{R}\left|\frac{\partial \boldsymbol{v}}{\partial x_1}\right|_{\ell,q} + |\boldsymbol{v}|_{\ell+2,q} + |p|_{\ell+1,q} \\ \leq c\,(\mathcal{R}|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q} + \mathcal{R}|g|_{\ell,q})\,. \end{aligned} \quad \text{(VII.4.32)}$$

If $n = 2$ *and* $1 < q < 2$, $|v_2|_{\ell,2q/(2-q)}$ *is also finite and we have*

$$\begin{aligned} \mathcal{R}|v_2|_{\ell,2q/(2-q)} + \mathcal{R}|v_2|_{\ell+1,q} + \mathcal{R}^{1/3}|\boldsymbol{v}|_{\ell+1,3q/(3-q)} + \mathcal{R}\left|\frac{\partial v_1}{\partial x_1}\right|_{\ell,q} \\ + |\boldsymbol{v}|_{\ell+2,q} + |p|_{\ell+1,q} \leq c\,(\mathcal{R}|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q} + \mathcal{R}|g|_{\ell,q})\,. \end{aligned} \quad \text{(VII.4.33)}$$

Furthermore, if $1 < q < (n+1)/2$, $|\boldsymbol{v}|_{\ell,s_2}$ *is finite where* $s_2 = (n+1)q/(n+1-2q)$, *and*

$$\begin{aligned} \mathcal{R}^{2/(n+1)}|\boldsymbol{v}|_{\ell,s_2} + \mathcal{R}^{1/(n+1)}|\boldsymbol{v}|_{\ell+1,s_1} + \mathcal{R}\left|\frac{\partial \boldsymbol{v}}{\partial x_1}\right|_{\ell,q} \\ + |\boldsymbol{v}|_{\ell+2,q} + |p|_{\ell+1,q} \leq c\,(\mathcal{R}|\boldsymbol{f}|_{\ell,q} + |g|_{\ell+1,q} + \mathcal{R}|g|_{\ell,q})\,, \end{aligned} \quad \text{(VII.4.34)}$$

so that, in particular, if $n=2$, *from* (VII.4.33) *and* (VII.4.34) *it follows that*

$$\begin{aligned}&\mathcal{R}|v_2|_{\ell,2q/(2-q)}+\mathcal{R}|v_2|_{\ell+1,q}+\mathcal{R}^{2/3}|\boldsymbol{v}|_{\ell,3q/(3-2q)}\\&+\mathcal{R}^{1/3}|\boldsymbol{v}|_{\ell+1,3q/(3-q)}+\mathcal{R}\left|\frac{\partial v_1}{\partial x_1}\right|_{\ell,q}+|\boldsymbol{v}|_{\ell+2,q}+|p|_{\ell+1,q}\\&\le c\left(\mathcal{R}|\boldsymbol{f}|_{\ell,q}+|g|_{\ell+1,q}+\mathcal{R}|g|_{\ell,q}\right).\end{aligned}\tag{VII.4.35}$$

Here the constants c_i, $i=1,2,3$, *depend only on* n,ℓ, *and* q. *Finally, if* $\boldsymbol{w},\tau$ *is another solution to* (VII.4.1) *corresponding to the same data* $\boldsymbol{f},g$ *with*[1]

$$\left|\frac{\partial \boldsymbol{w}}{\partial x_1}\right|_{\ell,q},\quad |\boldsymbol{w}|_{\ell+2,q}$$

finite, for some $\ell\in[0,m]$, *then*

$$\left|\frac{\partial}{\partial x_1}(\boldsymbol{w}-\boldsymbol{v})\right|_{\ell,q}\equiv|\boldsymbol{w}-\boldsymbol{v}|_{\ell+2,q}\equiv|\tau-p|_{\ell+1,q}\equiv 0.$$

Proof. We have to show the uniqueness part only. In view of Theorem VII.1.1, we have $(\boldsymbol{w}-\boldsymbol{v})$, $(\tau-p)\in C^\infty(\mathbb{R}^n)$. Thus, letting $\boldsymbol{z}=D^\alpha(\boldsymbol{w}-\boldsymbol{v})$, $s=D^\alpha(\tau-p)$, with $|\alpha|=\ell$, we derive, in particular, that $\boldsymbol{z},s$ is a smooth solution to the following system in $\mathbb{R}^n$:

$$\begin{aligned}&\Delta\boldsymbol{z}+\mathcal{R}\frac{\partial\boldsymbol{z}}{\partial x_1}=\nabla s\\&\nabla\cdot\boldsymbol{z}=0.\end{aligned}\tag{VII.4.36}$$

Using $(\text{VII.4.36})_2$ in $(\text{VII.4.36})_1$ we find that s is harmonic in the whole space and, since $\nabla s\in L^q(\mathbb{R}^n)$, from Exercise II.11.11 we have $\nabla s\equiv 0$, namely,

$$|\nabla s|_{\ell,q}\equiv|\tau-p|_{\ell+1,q}\equiv 0.\tag{VII.4.37}$$

From this and $(\text{VII.4.36})_1$ we infer

$$\Delta z+\mathcal{R}\frac{\partial z}{\partial x_1}=0,\tag{VII.4.38}$$

where z is any component of $\boldsymbol{z}$. Since $D^2z\in L^q(\mathbb{R}^n)$, from the following Exercise VII.4.1 we conclude

$$|D^2\boldsymbol{z}|_{\ell,q}\equiv|\boldsymbol{w}-\boldsymbol{v}|_{\ell+2,q}\equiv 0.\tag{VII.4.39}$$

Relations (VII.4.37)–(VII.4.39) complete the proof of the theorem. □

[1] The assumptions on $\boldsymbol{w}$ can be weakened. Weaker assumptions, however, would be unessential for our aims.

Exercise VII.4.1 Let z be a function from $C^2(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$, $r \in (1,\infty)$, satisfying (VII.4.38). Show $z \equiv 0$. *Hint* (Rionero & Galdi 1979): Multiply both sides of (VII.4.38) by $\varphi_R z|z|^{r-2}$, where φ_R is a positive C^∞-function in $\mathbb{R}^n$ that is one in B_R and zero in B_{2R}^c, and satisfies $|D^\alpha \varphi_R| \le cR^{-|\alpha|}$, $|\alpha| = 1,2$. Integrating by parts one arrives at

$$\int_{\mathbb{R}^n} \varphi_R |z|^{r-2} |\nabla z|^2 \le c_1 \int_{\mathbb{R}^n} (|\Delta \varphi_R| + |\nabla \varphi_R|)\, |z|^r, \tag{$*$}$$

for some c_1 independent of R. The result then follows by letting $R \to \infty$ in $(*)$.

The last part of this section is devoted to show existence and uniqueness of a q-weak solution to (VII.4.2) (or, what amounts to the same thing, to (VII.4.1)). By a *q-weak solution to* (VII.4.2) we mean a field $\boldsymbol{v}$ satisfying the conditions

$$\begin{aligned} &\text{(i) } \boldsymbol{v} \in D_0^{1,q}(\mathbb{R}^n); \\ &\text{(ii) } (\nabla \boldsymbol{v}, \nabla \boldsymbol{\phi}) - (\frac{\partial \boldsymbol{v}}{\partial x_1}, \boldsymbol{\phi}) = -[\boldsymbol{f}, \boldsymbol{\phi}], \quad \text{for all } \boldsymbol{\phi} \in \mathcal{D}(\Omega); \\ &\text{(iii) } (\boldsymbol{v}, \nabla \varphi) = -(g, \varphi), \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^n). \end{aligned} \tag{VII.4.40}$$

In view of Lemma VII.1.1, we can associate to $\boldsymbol{v}$ a pressure field $p \in L^q_{loc}(\mathbb{R})$ such that

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) - (\frac{\partial \boldsymbol{v}}{\partial x_1}, \boldsymbol{\psi}) - (p, \nabla \cdot \boldsymbol{\psi}) = -[\boldsymbol{f}, \boldsymbol{\psi}], \quad \text{for all } \boldsymbol{\psi} \in C_0^\infty(\mathbb{R}^n).$$

As before, we begin to take $\boldsymbol{f}, g \in C_0^\infty(\mathbb{R}^n)$. Consequently, (VII.4.8), (VII.4.9), (VII.4.10) is a C^∞-solution to (VII.4.2). We now observe that $\boldsymbol{f}$ and g can be written in a divergence form; that is,

$$\begin{aligned} f_j(x) &= D_\ell F_{\ell j}(x) \\ g(x) &= D_\ell G_\ell(x), \end{aligned} \tag{VII.4.41}$$

where $\boldsymbol{F} = \boldsymbol{F}(x)$ and $\boldsymbol{G} = \boldsymbol{G}(x)$ are second-order tensor and vector fields, respectively, satisfying

$$\begin{aligned} &|\boldsymbol{f}|_{-1,q} \le \|\boldsymbol{F}\|_q \le c_1 |\boldsymbol{f}|_{-1,q} \\ &|g|_{-1,q} \le \|\boldsymbol{G}\|_q \le c_1 |g|_{-1,q} \qquad \text{for all } q \in (1,\infty) \\ &|\boldsymbol{G}|_{1,q} \le c_1 \|g\|_q. \end{aligned} \tag{VII.4.42}$$

Actually, we may choose

$$\begin{aligned} F_{\ell j} &= D_\ell(\mathcal{E} * f_j) \\ G_\ell &= D_\ell(\mathcal{E} * g) \end{aligned}$$

with $\mathcal{E}$ the Laplace fundamental solution (II.9.1), and then use the result of Exercise II.11.9. Notice that

$$\begin{aligned} \widehat{D_\ell F_{\ell j}} &= i\xi_\ell \widehat{F}_{\ell j} \\ \widehat{D_\ell G_\ell} &= i\xi_\ell \widehat{G}_\ell . \end{aligned} \tag{VII.4.43}$$

We begin to give an estimate for the pair $\boldsymbol{u}, \pi$. From (VII.4.10), (VII.4.8)$_{1,3}$, and (VII.4.43), with the aid of Lemma VII.4.1 and Lemma VII.4.2(d), it follows that

$$|\boldsymbol{u}|_{1,q} + \|\pi\|_q \le c_2 \|\boldsymbol{F}\|_q, \quad 1 < q < \infty,$$

and (VII.4.42)$_1$ implies

$$|\boldsymbol{u}|_{1,q} + \|\pi\|_q \le c_3 |\boldsymbol{f}|_{-1,q}, \quad 1 < q < \infty. \tag{VII.4.44}$$

In the case of plane flow ($n = 2$), we can use Lemma VII.4.2(f) to obtain a further estimate for the component u_2, namely,

$$\|u_2\|_q \le c_4 \|\boldsymbol{F}\|_q\,, \quad 1 < q < \infty.$$

Therefore, in such a case, this last relation, along with (VII.4.42)$_1$ and (VII.4.44), furnishes

$$\|u_2\|_{1,q} + |u_1|_{1,q} + \|\pi\|_q \le c_5 |\boldsymbol{f}|_{-1,q}\,, \quad 1 < q < \infty. \tag{VII.4.45}$$

If we restrict the range of values of q, we may obtain another estimate. Specifically, from (VII.4.10), (VII.4.8)$_1$, Lemma VII.4.1, and Lemma VII.4.2(b) we find

$$\|\boldsymbol{u}\|_{s_1} \le c_6 \|\boldsymbol{F}\|_q, \quad s_1 = \frac{(n+1)q}{n+1-q}, \quad 1 < q < n+1,$$

so that (VII.4.42)$_1$ and (VII.4.44) imply

$$\|\boldsymbol{u}\|_{s_1} + |\boldsymbol{u}|_{1,q} + \|\pi\|_q \le c_7 |\boldsymbol{f}|_{-1,q}, \quad s_1 = \frac{(n+1)q}{n+1-q}, \quad 1 < q < n+1. \tag{VII.4.46}$$

Moreover, if $n = 2$, inequalities (VII.4.45) and (VII.4.46) deliver

$$\|u_2\|_{1,q} + \|\boldsymbol{u}\|_{3q/(3-q)} + |u_1|_{1,q} + \|\pi\|_q \le c_8 |\boldsymbol{f}|_{-1,q}, \quad 1 < q < 3. \tag{VII.4.47}$$

We shall next estimate the pair $\boldsymbol{w}, \tau$ defined by (VII.4.10)$_{2,4}$ and (VII.4.8)$_{2,4}$, respectively, with g given by (VII.4.41)$_2$. Actually, recalling that $\xi_k \xi_s / \xi^2$, $k, s = 1, \ldots, n$, satisfies the assumptions of Lemma VII.4.1 with $\beta = 0$ we at once obtain

$$\begin{aligned} \|\boldsymbol{w}\|_r &\le c_9 |g|_{-1,r} \\ |\boldsymbol{w}|_{1,r} &\le c_9 \|g\|_r \qquad\qquad 1 < r < \infty \\ \|\tau\|_r &\le c_9 \left(\|g\|_r + |g|_{-1,r}\right). \end{aligned} \tag{VII.4.48}$$

On the other hand, from the embedding Theorem II.3.1 we find

$$\|\boldsymbol{G}\|_{s_1} \le c\|\boldsymbol{G}\|_{1,q}, \quad s_1 = \frac{(n+1)q}{n+1-q}, \quad 1 < q < n+1, \tag{VII.4.49}$$

and so, from $(\text{VII.4.41})_{2,3}$, (VII.4.48), and (VII.4.49) it follows that

$$\begin{aligned} &\|\boldsymbol{w}\|_{1,q} + \|\tau\|_q \le c_{10}\left(\|g\|_q + |g|_{-1,q}\right), \quad 1 < q < \infty, \\ &\|\boldsymbol{w}\|_{s_1} \le c_{11}\left(\|g\|_q + |g|_{-1,q}\right), \quad s_1 = \frac{(n+1)q}{n+1-q}, \quad 1 < q < n+1. \end{aligned} \tag{VII.4.50}$$

In view of (VII.4.9), (VII.4.44)–(VII.4.47), (VII.4.50), and using the replacements (VII.4.3) we may conclude that, for given $\boldsymbol{f}$ and g from $C_0^\infty(\mathbb{R}^n)$ properties stated previously, there exists a q-weak solution to (VII.4.1) satisfying together with the corresponding pressure p, the following estimates: if $1 < q < \infty$

$$\begin{aligned} |\boldsymbol{v}|_{1,q} + \|p\|_q &\le c\left(\mathcal{R}|\boldsymbol{f}|_{-1,q} + \mathcal{R}|g|_{-1,q} + \|g\|_q\right) \\ \mathcal{R}\|v_2\|_q + |\boldsymbol{v}|_{1,q} + \|p\|_q &\le c\left(\mathcal{R}|\boldsymbol{f}|_{-1,q} + \mathcal{R}|g|_{-1,q} + \|g\|_q\right) \quad (\text{ if } n=2) \end{aligned} \tag{VII.4.51}$$

and if $1 < q < n+1$

$$\begin{aligned} &\mathcal{R}^{1/(n+1)}\,\|\boldsymbol{v}\|_{s_1} + |\boldsymbol{v}|_{1,q} + \|p\|_q \\ &\qquad \le c\left(\mathcal{R}|\boldsymbol{f}|_{-1,q} + \mathcal{R}|g|_{-1,q} + \|g\|_q\right), \quad s_1 = \frac{(n+1)q}{n+1-q} \\ &\mathcal{R}\|v_2\|_q + \mathcal{R}^{1/3}\|\boldsymbol{v}\|_{3q/(3-q)} + |\boldsymbol{v}|_{1,q} + \|p\|_q \\ &\qquad \le c\left(\mathcal{R}|\boldsymbol{f}|_{-1,q} + \mathcal{R}|g|_{-1,q} + \|g\|_q\right) \quad (\text{ if } n=2) \end{aligned} \tag{VII.4.52}$$

with $c = c(n,q)$.

Starting from (VII.4.51) and (VII.4.52), we may use a now standard density argument to extend the above results to the case when $\boldsymbol{f}$ and g merely satisfy the assumptions

$$\boldsymbol{f} \in D_0^{-1,q}(\mathbb{R}^n), \quad g \in L^q(\mathbb{R}^n) \cap D_0^{-1,q}(\mathbb{R}^n).$$

However, to do this we need a result that ensures us that we can approximate g by functions from $C_0^\infty(\mathbb{R}^n)$ in the norm of $L^q(\mathbb{R}^n) \cap D_0^{-1,q}(\mathbb{R}^n)$. This is the content of the following.

Lemma VII.4.3 *Let*

$$g \in L^q(\mathbb{R}^n) \cap D_0^{-1,q}(\mathbb{R}^n), \quad 1 < q < \infty.$$

Then, for any $\eta > 0$ there exists $g^{(\eta)} \in C_0^\infty(\mathbb{R}^n)$ such that

$$|g^{(\eta)} - g|_{-1,q} + \|g^{(\eta)} - g\|_q < \eta.$$

Proof. Consider the problem

$$\Delta\Psi = g \quad \text{in } \mathbb{R}^n. \tag{VII.4.53}$$

In view of the assumption made on g, by Exercise II.11.9 there exist two (a priori distinct) solutions Ψ_1 and Ψ_2 to (VII.4.53) such that

$$\begin{aligned} |\Psi_1|_{2,q} &\le c\|g\|_q \\ |\Psi_2|_{1,q} &\le c|g|_{-1,q}. \end{aligned} \tag{VII.4.54}$$

It is not difficult to show that

$$D^2(\Psi_1 - \Psi_2) \equiv 0. \tag{VII.4.55}$$

In fact, the difference $\Psi = \Psi_1 - \Psi_2$ satisfies

$$\Delta\Psi = 0 \quad \text{in } \mathbb{R}^n. \tag{VII.4.56}$$

We may now represent Ψ by means of (V.3.14). Taking into account (VII.4.56) and that $H^{(R)}$ is of bounded support, this latter leads, in particular, to the following

$$D_iD_j\Psi(x) = -\int_{\mathbb{R}^n} H^{(R)}(x-y)D_iD_j\Psi(y)dy\,. \tag{VII.4.57}$$

From (VII.4.57) we infer

$$\begin{aligned} D_iD_j\Psi(x) = &-\int_{\mathbb{R}^n} H^{(R)}(x-y)D_iD_j\Psi_1(y)dy \\ &+\int_{\mathbb{R}^n} D_iH^{(R)}(x-y)D_j\Psi_2(y)dy, \end{aligned}$$

and so, by the Hölder inequality, we obtain

$$|D_iD_j\Psi(x)| \le \|H^{(R)}\|_{q'}|\Psi_1|_{2,q} + |H^{(R)}|_{1,q'}|\Psi_2|_{1,q}.$$

Letting $R \to \infty$ into this relation and using (VII.4.54) and (V.3.13) proves (VII.4.55). We may then state that problem (VII.4.53) with g verifying the assumptions of the lemma admits a unique solution $\Psi \in D_0^{1,q}(\mathbb{R}^n) \cap D^{2,q}(\mathbb{R}^n)$ and that this solution satisfies

$$\begin{aligned} |\Psi|_{2,q} &\le c\|g\|_q \\ |\Psi|_{1,q} &\le c|g|_{-1,q}. \end{aligned} \tag{VII.4.58}$$

Given $\rho > 0$, let us denote by $\zeta_\rho = \zeta_\rho(x)$ a nonincreasing, smooth function that equals 1 for $|x| \le \rho$ and 0 for $|x| \ge 2\rho$ and

$$|\nabla\zeta_\rho| \le c/\rho. \tag{VII.4.59}$$

Set

$$\boldsymbol{u} = \nabla\Psi, \quad \boldsymbol{u}_{\rho,\varepsilon} = \zeta_\rho \nabla\Psi_\varepsilon, \quad g_{\rho,\varepsilon} = \nabla\cdot\boldsymbol{u}_{\rho,\varepsilon}, \tag{VII.4.60}$$

where Ψ_ε is the regularizer of Ψ. Evidently

$$g_{\rho,\varepsilon} \in C_0^\infty(\mathbb{R}^n).$$

Let us next show that $g_{\rho,\varepsilon}$ approaches g simultaneously in $L^q(\mathbb{R}^n)$ and $D_0^{-1,q}(\mathbb{R}^n)$ as $\rho \to \infty$ and $\varepsilon \to 0$. By (VII.4.53), (VII.4.59), and (VII.4.60), by the Minkowski inequality, property $(\text{II}.2.9)_2$ of the regularizer and by Exercise II.3.2 we find

$$\begin{aligned}\|g_{\rho,\varepsilon} - g\|_q = \|\nabla\cdot\boldsymbol{u}_{\rho,\varepsilon} - \nabla\cdot\boldsymbol{u}\|_q \leq & \|\nabla\zeta_\rho\cdot\nabla\Psi_\varepsilon\|_q + \|\zeta_\rho g_\varepsilon - g\|_q \\ \leq & \frac{C}{\rho}|g|_{-1,q} + \|(\zeta_\rho - 1)g\|_q + \|g_\varepsilon - g\|_q.\end{aligned}$$

This inequality, along with $(\text{II}.2.9)_2$, establishes that for all $\eta > 0$ we may find $\rho_1 = \rho_1(\eta, g) > 0$ and $\varepsilon_1 = \varepsilon_1(\eta, g) > 0$ such that

$$\|g_{\rho,\varepsilon} - g\|_q < \eta/2, \quad \text{for all } \rho > \rho_1,\ \varepsilon > \varepsilon_1. \tag{VII.4.61}$$

Furthermore, from (VII.4.53), (VII.4.60) and the Minkowski inequality, we have

$$|g_{\rho,\varepsilon} - g|_{-1,q} \leq \|\boldsymbol{u}_{\rho,\varepsilon} - \boldsymbol{u}\|_q \leq |\Psi_\varepsilon - \Psi|_{1,q} + \|(\zeta_\rho - 1)\nabla\Psi\|_q$$

and so, again by Exercise II.3.2 and $(\text{VII}.4.58)_2$, it follows that for all $\eta > 0$ we may find $\rho_2 = \rho_2(\eta, g) > 0$ and $\varepsilon_2 = \varepsilon_2(\eta, g) > 0$ such that

$$|g_{\rho,\varepsilon} - g|_{-1,q} < \eta/2, \quad \text{for all } \rho > \rho_2,\ \varepsilon > \varepsilon_2. \tag{VII.4.62}$$

The lemma then follows from (VII.4.61) and (VII.4.62). □

We shall finally show that the q-generalized solutions just obtained are in fact unique among their class of existence.[2] Actually, suppose $\boldsymbol{w}, \tau$ is another solution corresponding to the same data, with

$$\boldsymbol{w} \in D_0^{1,q}(\mathbb{R}^n), \quad \tau \in L^q_{loc}(\mathbb{R}^n).^3$$

The differences

$$\boldsymbol{z} = \boldsymbol{w} - \boldsymbol{v} \in \mathcal{D}_0^{1,q}(\mathbb{R}^n), \quad s = \tau - p \in L^q_{loc}(\mathbb{R}^n)$$

satisfy the identity

[2] We observe that uniqueness is an immediate consequence of Theorem VII.6.2, which will be proved in Section VII.6. Here, we prefer to give another proof, which makes all treatment self-contained.

[3] This assumption on τ is a consequence of that made on $\boldsymbol{w}$, see Lemma VII.1.1. Notice that *we need no global summability assumption on τ*.

$$(\nabla \boldsymbol{z}, \nabla \boldsymbol{\psi}) - \mathcal{R}(\frac{\partial \boldsymbol{z}}{\partial x_1}, \boldsymbol{\psi}) = (s, \nabla \cdot \boldsymbol{\psi}) \tag{VII.4.63}$$

for all $\boldsymbol{\psi} \in C_0^\infty(\mathbb{R}^n)$. From Theorem VII.1.1 we deduce $\boldsymbol{z}, s \in C^\infty(\mathbb{R}^n)$ and so, in particular, using (VII.4.63), we find

$$\left.\begin{aligned} &\Delta \boldsymbol{z} + \mathcal{R}\frac{\partial \boldsymbol{z}}{\partial x_1} - \nabla s = 0 \\ &\nabla \cdot \boldsymbol{z} = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^n \tag{VII.4.64}$$
$$\boldsymbol{z} \in D_0^{1,q}(\mathbb{R}^n).$$

Using Theorem V.5.3 (with $\boldsymbol{f} \equiv -\mathcal{R}(\partial \boldsymbol{z}/\partial x_1)$) it follows that

$$\boldsymbol{z} \in D^{2,q}(\mathbb{R}^n), \quad s \in D^{1,q}(\mathbb{R}^n).$$

Operating with $\nabla\cdot$ on both sides of (VII.4.64)$_1$ we find that s is harmonic throughout $\mathbb{R}^n$ and, therefore, by Exercise II.11.11 we deduce $\nabla s \equiv 0$ in $\mathbb{R}^n$. Relation (VII.4.64) then furnishes that each component z of $\boldsymbol{z}$ satisfies (VII.4.38) and since $z \in D_0^{1,q}(\mathbb{R}^n)$ we may use Exercise VII.4.1 to deduce $z = 0$ in $D_0^{1,q}(\mathbb{R}^n)$ and uniqueness is proved.

The results just shown are summarized in the following.

Theorem VII.4.2 *Given*

$$\boldsymbol{f} \in D_0^{-1,q}(\mathbb{R}^n), \quad g \in L^q(\mathbb{R}^n) \cap D_0^{-1,q}(\mathbb{R}^n), \quad 1 < q < \infty,$$

there exists at least one q-generalized solution $\boldsymbol{v}$ to (VII.4.1). *Moreover, the pressure field p associated to $\boldsymbol{v}$ satisfies*

$$p \in L^q(\mathbb{R}^n)$$

and $\boldsymbol{v}, p$ verify the estimate (VII.4.51). *Also, if $1 < q < n+1$, then*

$$\boldsymbol{v} \in L^{s_1}(\mathbb{R}^n), \quad s_1 = \frac{(n+1)q}{n+1-q},$$

with

$$v_2 \in L^q(\mathbb{R}^2), \quad \text{if } n = 2,$$

and estimate (VII.4.52) *holds. Finally, if $\boldsymbol{w}$ is another q-generalized solution to* (VII.4.1) *corresponding to the same data $\boldsymbol{f}$ and g, then $\boldsymbol{w} \equiv \boldsymbol{v} + c_1$, $\tau \equiv p + c_2$, for some constants c_i, $i = 1, 2$, where τ is the pressure field associated to $\boldsymbol{w}$ by* Lemma VII.1.1. *If $q < n+1$, we may take $c_1 = 0$.*

VII.5 Existence of Generalized Solutions for Plane Flows in Exterior Domains

In the previous section we proved, among other things, existence of plane flow in the whole space tending to a prescribed value at infinity. In this section we shall establish the same result in the more general case when the relevant region of flow is an exterior domain. This is in sharp contrast with the Stokes approximation, where we know that the problem is resolvable if and only if certain restrictions are imposed on the data, see Section V.7. However, as we already observed in Remark VII.2.1, to show existence we cannot use *sic et simpliciter* the technique of Theorem VII.1.1, for then we would not be able to control the behavior of the solution at large distances. Consequently, we have to employ a different approach.

To this end, following Finn & Smith (1967a), we begin to consider the modified problem

$$\left.\begin{aligned} &\Delta \boldsymbol{v} + \mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1} - \varepsilon \boldsymbol{v} = \nabla p + \mathcal{R}\boldsymbol{f} \\ &\nabla \cdot \boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\boldsymbol{v} = \boldsymbol{v}_* \ \text{ at } \partial\Omega \tag{VII.5.1}$$
$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = 0$$

with $\varepsilon \in (0, 1]$ and prove for it existence of solutions and suitable L^q-estimates for any value of ε. Successively, using such estimates, we show that in the limit $\varepsilon \to 0$ these solutions converge in a well-defined sense to a generalized solution of the original problem (VII.0.2), (VII.0.3).

The above assertions will be proved through several steps. First of all we consider existence of a solution to the following nonhomogeneous approximating system in $\mathbb{R}^2$:

$$\begin{aligned} &\Delta \boldsymbol{u} + \mathcal{R}\frac{\partial \boldsymbol{u}}{\partial x_1} - \varepsilon \boldsymbol{u} = \nabla \pi + \mathcal{R}\boldsymbol{F} \\ &\nabla \cdot \boldsymbol{u} = g \end{aligned} \tag{VII.5.2}$$

Specifically, we have

Lemma VII.5.1 *Let $\boldsymbol{F} \in L^q(\mathbb{R}^2)$, $g \in W^{1,q}(\mathbb{R}^2)$, $1 < q < 3/2$. Then for all $\varepsilon \in (0, 1]$ there exists a solution $\boldsymbol{u}^\varepsilon$, π^ε to* (VII.5.2) *such that*

$$\begin{aligned} &\boldsymbol{u}^\varepsilon \in W^{2,q}(\mathbb{R}^2) \cap D_0^{1,3q/(3-q)}(\mathbb{R}^2) \cap L^{3q/(3-2q)}(\mathbb{R}^2) \\ &\pi^\varepsilon \in D^{1,q}(\mathbb{R}^2) \\ &u_2^\varepsilon \in L^{2q/(2-q)}(\mathbb{R}^2) \cap D_0^{1,q}(\mathbb{R}^2) \\ &\frac{\partial u_1}{\partial x_1} \in L^q(\mathbb{R}^2) \end{aligned}$$

and satisfying the estimate

$$\begin{aligned}\mathcal{R}\|u_2^\varepsilon\|_{2q/(2-q)} + &\mathcal{R}|u_2^\varepsilon|_{1,q} + \mathcal{R}^{2/3}\|\boldsymbol{u}^\varepsilon\|_{3q/(3-2q)} + \mathcal{R}^{1/3}|\boldsymbol{u}^\varepsilon|_{1,3q/(3-q)}\\ &+\mathcal{R}\left\|\frac{\partial u_1^\varepsilon}{\partial x_1}\right\|_q + |\boldsymbol{u}^\varepsilon|_{1,q} + |\pi^\varepsilon|_{1,q} \qquad\qquad \text{(VII.5.3)}\\ &\le c\left(\mathcal{R}|\boldsymbol{f}|_q + |g|_{1,q} + \mathcal{R}\|g\|_q\right),\end{aligned}$$

where $c = c(q)$. Moreover, if $\boldsymbol{w}, \tau$ are such that

(a) $\boldsymbol{w} \in W^{1,2}(\mathbb{R}^2)$, $\quad \tau \in L_{loc}(\mathbb{R}^2)$;

(b) $(\nabla \boldsymbol{w}, \nabla \boldsymbol{\psi}) - \mathcal{R}(\dfrac{\partial \boldsymbol{w}}{\partial x_1}, \boldsymbol{\psi}) + \varepsilon(\boldsymbol{w}, \boldsymbol{\psi}) = (\tau, \nabla\cdot\boldsymbol{\psi}) - \mathcal{R}[\boldsymbol{F}, \boldsymbol{\psi}]$,

$(\boldsymbol{w}, \nabla\cdot\phi) = -(g, \phi)$, *for all* $\boldsymbol{\psi}, \phi \in C_0^\infty(\mathbb{R}^2)$,

necessarily $\boldsymbol{w} = \boldsymbol{u}^\varepsilon$ and $\tau = \pi^\varepsilon + \text{const}$ a.e. in $\mathbb{R}^2$.

Proof. The proof of the existence of $\boldsymbol{u}^\varepsilon, \pi^\varepsilon$ satisfying (VII.5.3) is entirely analogous to that of Theorem VII.4.1. The only point that deserves a little care is to show that the constant c in (VII.5.3) can be taken independent of ε. To see how this can be done, we shall sketch the proof of (VII.5.3) when $\boldsymbol{F} \in C_0^\infty(\mathbb{R}^2)$ and $g \equiv 0$, leaving to the reader the simple tasks of giving details and extending the result to the general case. As in Section VII.4, we first make a change of variable of the type (VII.4.3), which formally brings (VII.5.2) into a similar system having $\mathcal{R} = 1$ and $\varepsilon\mathcal{R}$ in place of ε. Let us denote by (VII.5.2′) this latter system. We next construct a solution to (VII.5.2′) by Fourier transform, that is,

$$\boldsymbol{u}^\varepsilon(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}}\boldsymbol{U}^\varepsilon(\xi)d\xi$$

$$\pi^\varepsilon(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}}\Pi^\varepsilon(\xi)d\xi,$$

where

$$U_m^\varepsilon(\xi) = \frac{\xi_m\xi_k - \xi^2\delta_{mk}}{\xi^2(\xi^2 + \varepsilon\mathcal{R} + i\xi_1)}\widehat{F}_k(\xi), \quad m = 1, 2$$

$$\Pi^\varepsilon(\xi) = i\frac{\widehat{F}_k(\xi)\xi_k}{\xi^2},$$

and $\widehat{\boldsymbol{F}}$ is the Fourier transform of $\boldsymbol{F}$. Setting

$$\Psi_{mk}(\xi) = \frac{\xi_m\xi_k - \xi^2\delta_{mk}}{\xi^2(\xi^2 + \varepsilon\mathcal{R} + i\xi_1)}, \tag{VII.5.4}$$

using the same reasonings as in the proof of Lemma VII.4.2 one shows with no difficulty that the functions

$$\Psi_{mk}(\xi), \quad \xi_\ell\Psi_{mk}(\xi), \quad \xi_\ell\xi_s\Psi_{mk}(\xi), \quad \ell, s, m, k = 1, 2,$$

satisfy the assumption of Lemma VII.4.1 with $\beta = 2/3$, $\beta = 1/3$ and $\beta = 0$, respectively, with a constant M *independent of* $\varepsilon\mathcal{R}$. The same is true for

$\Psi_{2k}(\xi)$, with $\beta = 1/2$, and for $\xi_\ell \Psi_{2k}(\xi)$, with $\beta = 0$, $k = 1, 2$. As a consequence, arguing exactly as in the proof of Theorem VII.4.1, we obtain (VII.5.3) with a constant c independent of ε. Let us next show

$$\boldsymbol{u}^\varepsilon \in W^{1,q}(\mathbb{R}^2). \tag{VII.5.5}$$

By Lemma VII.4.1, it suffices to prove, for some $C = C(\varepsilon, \mathcal{R})$ and all $\ell, m, k = 1, 2$

$$|\xi_1^{\kappa_1}|\ |\xi_2^{\kappa_2}| \left| \frac{\partial^\kappa \Psi_{mk}}{\partial \xi_1^{\kappa_1} \partial \xi_2^{\kappa_2}} \right| \leq C$$

$$|\xi_1^{\kappa_1}|\ |\xi_2^{\kappa_2}| \left| \frac{\partial^\kappa (\xi_\ell \Psi_{mk})}{\partial \xi_1^{\kappa_1} \partial \xi_2^{\kappa_2}} \right| \leq C$$

with $\kappa_1, \kappa_2 = 0, 1$ and $\kappa_1 + \kappa_2 = \kappa$. We now have, with $c = c(\kappa_1, \kappa_2)$,

$$|\xi_1^{\kappa_1}|\ |\xi_2^{\kappa_2}| \left| \frac{\partial^\kappa \Psi_{mk}}{\partial \xi_1^{\kappa_1} \partial \xi_2^{\kappa_2}} \right| \leq c \frac{1}{\xi^2 + \varepsilon \mathcal{R} + |\xi_1|} \leq \frac{c}{\varepsilon \mathcal{R}}$$

and

$$|\xi_1^{\kappa_1}|\ |\xi_2^{\kappa_2}| \left| \frac{\partial^\kappa (\xi_\ell \Psi_{mk})}{\partial \xi_1^{\kappa_1} \partial \xi_2^{\kappa_2}} \right| \leq c \frac{|\xi|}{\xi^2 + \varepsilon \mathcal{R} + |\xi_1|} \leq \frac{c}{(\varepsilon \mathcal{R})^{1/2}}$$

and (VII.5.5) follows. Finally, let $\boldsymbol{w}, \tau$ satisfy conditions (a) and (b) stated in the lemma and set $\boldsymbol{v} = \boldsymbol{w} - \boldsymbol{u}^\varepsilon$, $p = \tau - \pi^\varepsilon$. We then obtain that $\boldsymbol{v}$ obeys the identity

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) - \mathcal{R}(\boldsymbol{v}, \frac{\partial \boldsymbol{\varphi}}{\partial x_1}) + \varepsilon(\boldsymbol{v}, \boldsymbol{\varphi}) = 0, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega). \tag{VII.5.6}$$

However, by embedding Theorem II.3.2, it follows that $\boldsymbol{u}^\varepsilon \in W^{1,2}(\mathbb{R}^2)$ and since $\nabla \cdot \boldsymbol{v} = 0$ we conclude $\boldsymbol{v} \in H^1(\mathbb{R}^2)$. By continuity, we may then extend identity (VII.5.6) to all $\boldsymbol{\varphi} \in H^1(\mathbb{R}^2)$ and, in particular, we may take $\boldsymbol{\varphi} = \boldsymbol{v}$ to deduce

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{v}) + \varepsilon(\boldsymbol{v}, \boldsymbol{v}) = \mathcal{R}(\boldsymbol{v}, \frac{\partial \boldsymbol{v}}{\partial x_1}). \tag{VII.5.7}$$

Since

$$(\boldsymbol{v}, \frac{\partial \boldsymbol{v}}{\partial x_1}) = 0,$$

(VII.5.7) yields $\boldsymbol{v} = \boldsymbol{0}$ *a.e.* in $\mathbb{R}^2$ and, consequently, from (b) and the analogous identity written for $\boldsymbol{u}^\varepsilon$, we have

$$(p, \nabla \cdot \psi) = 0, \quad \text{for all } \psi \in C_0^\infty(\mathbb{R}^2),$$

implying $\tau = \pi^\varepsilon + \text{const}$ a.e. in $\mathbb{R}^2$. The proof of the lemma is therefore complete. □

The second step consists in proving the existence of a generalized solution to (VII.5.1), that is, of a field $\boldsymbol{v} : \Omega \to \mathbb{R}^2$ satisfying the requirements (i)-(iv) of Definition VII.1.1 with $q = 2$, along with the identity

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) - \mathcal{R}(\frac{\partial \boldsymbol{v}}{\partial x_1}, \boldsymbol{\varphi}) + \varepsilon(\boldsymbol{v}, \boldsymbol{\varphi}) = -\mathcal{R}[\boldsymbol{f}, \boldsymbol{\varphi}], \quad \text{for all } \varphi \in \mathcal{D}(\Omega). \quad \text{(VII.5.8)}$$

We have

Lemma VII.5.2 *Let Ω be a locally Lipschitz, exterior domain of $\mathbb{R}^2$, and let*

$$\boldsymbol{f} \in D_0^{-1,2}(\Omega), \quad \boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega).$$

Then, for all $\varepsilon \in (0, 1]$ there exists a generalized solution $\boldsymbol{v}^\varepsilon$ to (VII.5.1)*. This solution verifies the estimate*

$$\varepsilon\|\boldsymbol{v}^\varepsilon\|_2 + \|\boldsymbol{v}^\varepsilon\|_{2,\Omega_R} + |\boldsymbol{v}^\varepsilon|_{1,2} \le c_1 \left\{\mathcal{R}|\boldsymbol{f}|_{1,2} + (1 + \mathcal{R})\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\right\} \quad \text{(VII.5.9)}$$

for all $R > \delta(\Omega^c)$ and with a constant $c_1 = c_1(R, \Omega)$. Moreover, denoting by $p^\varepsilon \in L^2_{loc}(\Omega)$ the corresponding pressure field,[1] *we have*

$$\|p^\varepsilon\|_{2,\Omega_R/\mathbb{R}} \le c_2 \{\mathcal{R}|\boldsymbol{f}|_{-1,2} + (1 + \mathcal{R})|\boldsymbol{v}_\varepsilon|_{1,2}\} \qquad \text{(VII.5.10)}$$

for all $R > \delta(\Omega^c)$ and with a constant $c_2 = c_2(R, \Omega)$.

Proof. Again, the proof is completely analogous to that given in Theorem VII.2.1. Actually, we look for a solution $\boldsymbol{v}^\varepsilon$ of the form

$$\boldsymbol{v}^\varepsilon = \boldsymbol{w}^\varepsilon + \boldsymbol{V}_1 + \sigma,$$

where $\boldsymbol{V}_1$ is a suitable extension of $\boldsymbol{v}_*$ constructed exactly as in the proof of Theorem VII.2.1. In addition,

$$\boldsymbol{\sigma} = -\frac{1}{4\pi}\nabla\left(\log\frac{1}{|x|}\right)\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n}$$

(the origin of coordinates having been taken in $\dot{\Omega}^c$) and $\boldsymbol{w}^\varepsilon \in H^1(\Omega)$ verifies the identity

$$\begin{aligned}(\nabla \boldsymbol{w}^\varepsilon, \nabla \boldsymbol{\varphi}) &-\mathcal{R}(\frac{\partial \boldsymbol{w}^\varepsilon}{\partial x_1}, \boldsymbol{\varphi}) + \varepsilon(\boldsymbol{w}^\varepsilon, \boldsymbol{\varphi}) \\ &= -\mathcal{R}[\boldsymbol{f}, \boldsymbol{\varphi}] - (\nabla \boldsymbol{V}_1, \nabla \boldsymbol{\varphi}) - \mathcal{R}(\frac{\partial \boldsymbol{V}_1}{\partial x_1} + \frac{\partial \boldsymbol{\sigma}}{\partial x_1}, \boldsymbol{\varphi}),\end{aligned}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. Using the Galerkin method and exploiting the properties of $\boldsymbol{V}_1$, $\boldsymbol{\sigma}$, and $\boldsymbol{a}$, we easily show the existence of $\boldsymbol{w}^\varepsilon$ obeying the previous identity and the following inequality

[1] Associated to $\boldsymbol{v}^\varepsilon$ in a way completely analogous to that used in Lemma VII.1.1 for generalized solutions with $\varepsilon = 0$.

$$\varepsilon\|\boldsymbol{w}^\varepsilon\|_2 + |\boldsymbol{w}^\varepsilon|_{1,2} \le c_3\left\{\mathcal{R}|\boldsymbol{f}|_{-1,2} + (1+\mathcal{R})\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\right\}$$

(see the proof of Theorem VII.2.1). From this we deduce the existence of a generalized solution $\boldsymbol{v}^\varepsilon$ satisfying (VII.5.9). The estimate for p^ε is obtained again as in the proof of Theorem VII.2.1. Notice that this time, since $\varepsilon \neq 0$, we are able to control the behavior of $\boldsymbol{v}^\varepsilon$ at infinity and to show, in particular, the validity of condition (iv) of Definition VII.1.1. Actually, for $\varepsilon > 0$, we have $\boldsymbol{v}^\varepsilon \in W^{1,2}(\Omega)$ and so, putting $|x| = r$ and

$$\mathcal{I}(r) = \int_0^{2\pi} |\boldsymbol{v}^\varepsilon(r,\theta)|^2 d\theta, \quad r > \delta(\Omega^c),$$

we recover

$$\mathcal{I}(r) \in L^1(0,\infty), \quad \frac{d\mathcal{I}}{dr} \in L^1(0,\infty),$$

which implies $\mathcal{I}(r) = o(1)$ as $r \to \infty$. □

The third step is to prove that if, in addition to the assumptions made in Lemma VII.5.2, $\boldsymbol{f}$ belongs to $L^q(\Omega)$ for some $q \in (1, 3/2)$, then $\boldsymbol{v}^\varepsilon$ and its derivatives belong to suitable Lebesgue spaces and satisfy there an estimate in terms of the data uniformly in $\varepsilon \in (0,1]$. Specifically, we have

Lemma VII.5.3 *Let Ω, $\boldsymbol{f}$, and $\boldsymbol{v}_*$ satisfy the hypotheses of* Lemma VII.5.2. *Suppose, further, $\boldsymbol{f} \in L^q(\Omega)$, $1 < q < 3/2$. Then, the generalized solution $\boldsymbol{v}^\varepsilon$ determined in* Lemma VII.5.2 *and the corresponding pressure field p^ε verify, in addition, for all $R > \delta(\Omega^c)$*

$$\begin{aligned}
&\boldsymbol{v}^\varepsilon \in D^{2,q}(\Omega^R) \cap D^{1,3q/(3-q)}(\Omega^R) \cap L^{3q/(3-2q)}(\Omega)\\
&v_2^\varepsilon \in L^{2q/(2-q)}(\Omega) \cap D^{1,q}(\Omega)\\
&\frac{\partial v_1^\varepsilon}{\partial x_1} \in L^q(\Omega)\\
&p^\varepsilon \in D^{1,q}(\Omega^R)
\end{aligned}$$

along with the estimate

$$\begin{aligned}
&\mathcal{R}\left(\|v_2^\varepsilon\|_{2q/(2-q)} + |v_2^\varepsilon|_{1,q} + \left\|\frac{\partial v_1^\varepsilon}{\partial x_1}\right\|_q\right)\\
&+b\|\boldsymbol{v}^\varepsilon\|_{3q/(3-2q)} + \mathcal{R}^{1/3}|\boldsymbol{v}^\varepsilon|_{1,3q/(3-q),\Omega^R} + |\boldsymbol{v}^\varepsilon|_{2,q,\Omega^R} + |p^\varepsilon|_{1,q,\Omega^R}\\
&\quad \le c\left\{\mathcal{R}\left(\|\boldsymbol{f}\|_q + (1+\mathcal{R})|\boldsymbol{f}|_{-1,2}\right) + (1+\mathcal{R})^2\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\right\},
\end{aligned} \tag{VII.5.11}$$

where $b = \min\{1, \mathcal{R}^{2/3}\}$ and $c = c(q,\Omega,R)$.

Proof. Let $\chi \in C^\infty(\mathbb{R}^2)$ be zero in $B_{R/2}$ and one in B_R^c, for some arbitrarily fixed $R > \delta(\Omega^c)$. Setting $\boldsymbol{u}^\varepsilon = \chi\boldsymbol{v}^\varepsilon$, $\pi^\varepsilon = \chi p^\varepsilon$ it is easy to show that $\boldsymbol{u}^\varepsilon$, π^ε satisfy (VII.5.2) [2] with

[2] In the generalized sense.

$$\begin{aligned}
\mathcal{R}\boldsymbol{F} &= \Delta\chi\boldsymbol{v}^\varepsilon + 2\nabla\chi\cdot\nabla\boldsymbol{v}^\varepsilon + \mathcal{R}\boldsymbol{v}\frac{\partial\chi}{\partial x_1} + p^\varepsilon\nabla\chi + \mathcal{R}\chi\boldsymbol{f} \\
g &= \nabla\chi\cdot\boldsymbol{v}^\varepsilon .
\end{aligned}$$

By the properties of χ, we readily deduce the estimates

$$\begin{aligned}
\mathcal{R}\|\boldsymbol{F}\|_q &\le c_1\left[\mathcal{R}\|\boldsymbol{f}\|_q + (1+\mathcal{R})\|\boldsymbol{v}^\varepsilon\|_{1,2,\Omega_R} + \|p^\varepsilon\|_{2,\Omega_R}\right] \\
\|g\|_q &\le c_2\|\boldsymbol{v}^\varepsilon\|_{2,\Omega_R} \\
|g|_{1,q} &\le c_3\|\boldsymbol{v}^\varepsilon\|_{1,2,\Omega_R}
\end{aligned} \tag{VII.5.12}$$

with $c_i = c_i(\chi)$, $i = 1, 2$. Using these inequalities along with Lemma VII.5.2 we deduce that

$$\boldsymbol{F}\in L^q(\mathbb{R}^2), \quad g\in W^{1,q}(\mathbb{R}^2), \quad 1<q<3/2$$

and that

$$\mathcal{R}\|\boldsymbol{F}\|_q + |g|_{1,q} + \mathcal{R}\|g\|_q$$

is increased through the right-hand side of (VII.5.11). From Lemma VII.5.1 we then deduce

$$\begin{aligned}
&\boldsymbol{u}^\varepsilon\in W^{2,q}(\mathbb{R}^2)\cap D_0^{1,3q/(3-q)}(\mathbb{R}^2)\cap L^{3q/(3-2q)}(\mathbb{R}^2) \\
&\pi^\varepsilon\in D^{1,q}(\mathbb{R}^2) \\
&u_2^\varepsilon\in L^{2q/(2-q)}(\mathbb{R}^2)\cap D_0^{1,q}(\mathbb{R}^2) \\
&\frac{\partial u_1^\varepsilon}{\partial x_1}\in L^q(\mathbb{R}^2)
\end{aligned}$$

and that $\boldsymbol{u}^\varepsilon, \pi^\varepsilon$ satisfy (VII.5.3). The proof then follows from (VII.5.9) and (VII.5.12), and by recalling that $\chi = 1$ in B_R^c and that

$$\|v_2^\varepsilon\|_{2q/(2-q),\Omega_R} + |v_2^\varepsilon|_{1,q,\Omega_R} + \left\|\frac{\partial v_1^\varepsilon}{\partial x_1}\right\|_{q,\Omega_R} + \|\boldsymbol{v}^\varepsilon\|_{3q/(3-2q),\Omega_R} \le c\|\boldsymbol{v}^\varepsilon\|_{1,2,\Omega_R}.$$

□

We are now in a position to prove the following.

Theorem VII.5.1 *Let Ω be a two-dimensional, locally Lipschitz exterior domain. Then, given*

$$\begin{gathered}
\boldsymbol{f}\in D_0^{-1,2}(\Omega)\cap L^q(\Omega), \quad 1<q<3/2, \\
\boldsymbol{v}_*\in W^{1/2,2}(\partial\Omega),
\end{gathered}$$

there exists a unique generalized solution $\boldsymbol{v}$ to (VII.0.2) *and* (VII.0.3). *Moreover, for all $R>\delta(\Omega^c)$ this solution verifies*

$$
\begin{aligned}
&\boldsymbol{v} \in D^{2,q}(\Omega^R) \cap D^{1,3q/(3-q)}(\Omega^R) \cap L^{3q/(3-2q)}(\Omega) \\
&v_2 \in L^{2q/(2-q)}(\Omega) \cap D^{1,q}(\Omega) \\
&\frac{\partial v_1}{\partial x_1} \in L^q(\Omega) \\
&p \in D^{1,q}(\Omega^R),
\end{aligned}
\tag{VII.5.13}
$$

where p is the pressure field associated to $\boldsymbol{v}$ by Lemma VII.1.1. *Finally, the following estimate holds:*

$$
\begin{aligned}
&\|\boldsymbol{v}\|_{2,\Omega_R} + |\boldsymbol{v}|_{1,2} + \mathcal{R}\left(\|v_2\|_{2q/(2-q)} + |v_2|_{1,q} + \left\|\frac{\partial v_1}{\partial x_1}\right\|_q\right) \\
&+b\|\boldsymbol{v}\|_{3q/(3-2q)} + \mathcal{R}^{1/3}|\boldsymbol{v}|_{1,3q/(3-q),\Omega^R} + |\boldsymbol{v}|_{2,q,\Omega^R} + |p|_{1,q,\Omega^R} \\
&\le c\left\{\mathcal{R}\left(\|\boldsymbol{f}\|_q + (1+\mathcal{R})|\boldsymbol{f}|_{-1,2}\right) + (1+\mathcal{R})^2\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\right\},
\end{aligned}
\tag{VII.5.14}
$$

where $b = \min\{1, \mathcal{R}^{2/3}\}$ *and* $c = c(q, \Omega, R)$.

Proof. Uniqueness is already known from Theorem VII.1.2. Concerning existence, we proceed as follows. We take $\varepsilon = 1/m$, $m \in \mathbb{N}$, in (VII.5.1) and denote by $\boldsymbol{v}_m, p_m$ the corresponding generalized solution and the associated pressure field which, by Lemmas 5.2 and 5.3 exist and satisfy inequalities (VII.5.9)–(VII.5.11) with constants c_1, c_2 and c *independent* of m. In particular, such inequalities for any fixed $R > \delta(\Omega^c)$ lead to the uniform bound

$$
\begin{aligned}
&\|v_{m2}\|_{2q/(2-q)} + |v_{m2}|_{1,q} + \left\|\frac{\partial v_{m1}}{\partial x_1}\right\|_q + |\boldsymbol{v}_m|_{1,3q/(3-q),\Omega^R} \\
&+\|\boldsymbol{v}_m\|_{3q/(3-2q)} + |\boldsymbol{v}_m|_{1,2} + |\boldsymbol{v}|_{2,q,\Omega^R} + |p_m|_{1,q} \le M
\end{aligned}
\tag{VII.5.15}
$$

with M independent of m. Using the weak compactness of the spaces L^r and $\dot{D}^{m,r}$, $1 < r < \infty$ (Theorem II.2.4 and Theorem II.3.1 and Exercise II.6.2), together with the strong compactness results of Exercise II.5.8, from (VII.5.9), (VII.5.11) and (VII.5.15) we then deduce the existence of a subsequence, denoted again by $\{\boldsymbol{v}_m, p_m\}$, and of fields $\boldsymbol{v}, p$ verifying (VII.5.13) and (VII.5.14) and, moreover, as $m \to \infty$,

$$
\boldsymbol{v}_m \to \boldsymbol{v}, \quad \text{weakly in } W^{1,2}(\Omega_R) \text{ and strongly in } L^2(\Omega_R), \tag{VII.5.16}
$$

for any $R > \delta(\Omega^c)$. It is simple to show that $\boldsymbol{v}$ satisfies (VII.1.1) for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. To see this, we notice that $\boldsymbol{v}_m$ satisfies (VII.5.8) with $\varepsilon = 1/m$, which in view of (VII.5.16) reduces to (VII.1.1) in the limit $m \to \infty$. Clearly, $\boldsymbol{v}$ is weakly divergence-free and, by (VII.5.16) and Theorem II.4.1, $\boldsymbol{v} = \boldsymbol{v}_*$ at $\partial\Omega$ in the trace sense. To prove the theorem completely, it remains to show condition (i) of Definition VII.1.1. Actually, we shall prove something more; that is,

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = 0. \tag{VII.5.17}$$

In fact, from the property $\boldsymbol{v} \in D^{2,q}(\Omega^R)$, $1 < q < 3/2$, and from Theorem II.6.1 we obtain $\boldsymbol{v} \in D^{1,2q/(2-q)}(\Omega^R)$. Thus

$$\boldsymbol{v} \in D^{1,2q/(2-q)}(\Omega^R) \cap L^{3q/(3-2q)}(\Omega^R)$$

and (VII.5.17) follows from Theorem II.9.1. □

VII.6 Representation of Solutions. Behavior at Large Distances and Related Results

We shall presently investigate the behavior at infinity of solutions to the Oseen system and, in particular, we shall determine its asymptotic structure. To reach this goal, we will pattern essentially the same ideas and techniques used for analogous questions within the Stokes approximation in Section IV.8 and Section V.3 and therefore, here and there, we may leave details to the reader.

Let us begin to show a representation formula for smooth solutions in a *bounded* domain of class C^1. Denoting, as usual, by $\boldsymbol{T}$ the stress tensor of a given flow, for $\boldsymbol{v}, p$ and $\boldsymbol{u}, \pi$ enough regular fields the following identities hold:

$$\begin{aligned}
\int_\Omega \left(\nabla\cdot\boldsymbol{T}(\boldsymbol{v},p) + \mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1}\right)\cdot\boldsymbol{u} &= -\int_\Omega \left(\boldsymbol{T}(\boldsymbol{v},p) : \nabla\boldsymbol{u} + \mathcal{R}\boldsymbol{v}\cdot\frac{\partial \boldsymbol{u}}{\partial x_1}\right) \\
&\quad + \int_{\partial\Omega}\left(\boldsymbol{u}\cdot\boldsymbol{T}(\boldsymbol{v},p)\cdot\boldsymbol{n} + \mathcal{R}\boldsymbol{v}\cdot\boldsymbol{u}\ \boldsymbol{e}_1\cdot\boldsymbol{n}\right) \\
\int_\Omega \left(\nabla\cdot\boldsymbol{T}(\boldsymbol{u},\pi) + \mathcal{R}\frac{\partial \boldsymbol{u}}{\partial x_1}\right)\cdot\boldsymbol{v} &= -\int_\Omega \left(\boldsymbol{T}(\boldsymbol{u},\pi) : \nabla\boldsymbol{v} + \mathcal{R}\boldsymbol{v}\cdot\frac{\partial \boldsymbol{u}}{\partial x_1}\right) \\
&\quad + \int_{\partial\Omega}\left(\boldsymbol{v}\cdot\boldsymbol{T}(\boldsymbol{u},\pi)\cdot\boldsymbol{n}\right),
\end{aligned} \tag{VII.6.1}$$

where, as usual, $\boldsymbol{e}_1$ denotes the unit vector along the positive x_1-axis. Assuming $\boldsymbol{u}$ and $\boldsymbol{v}$ solenoidal implies

$$\int_\Omega \boldsymbol{T}(\boldsymbol{u},\pi) : \nabla\boldsymbol{v} = \int_\Omega \boldsymbol{T}(\boldsymbol{v},p) : \nabla\boldsymbol{u}$$

and so, subtracting $(\text{VII.6.1})_2$ from $(\text{VII.6.1})_1$, we obtain

$$\begin{aligned}
&\int_\Omega \left\{\left(\nabla\cdot\boldsymbol{T}(\boldsymbol{v},p) + \mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1}\right)\cdot\boldsymbol{u} - \left(\nabla\cdot\boldsymbol{T}(\boldsymbol{u},\pi) + \mathcal{R}\frac{\partial \boldsymbol{u}}{\partial x_1}\right)\cdot\boldsymbol{v}\right\} \\
&= \int_{\partial\Omega}\left(\boldsymbol{u}\cdot\boldsymbol{T}(\boldsymbol{v},p) - \boldsymbol{v}\cdot\boldsymbol{T}(\boldsymbol{u},\pi) + \mathcal{R}\boldsymbol{v}\cdot\boldsymbol{u}\ \boldsymbol{e}_1\right)\cdot\boldsymbol{n}.
\end{aligned} \tag{VII.6.2}$$

Identity (VII.6.2) is the *Green's formula for the Oseen system.* Proceeding as in Section IV.8, it is easy to derive from (VII.6.2) a representation formula for $\boldsymbol{v}$ and p satisfying the Oseen system. Actually, for fixed $j = 1, \ldots, n$ and $x \in \Omega$ we choose

$$\begin{aligned} \boldsymbol{u}(y) &= \boldsymbol{w}_j(x-y) \equiv (E_{1j}, E_{2j}, \ldots, E_{nj}) \\ \pi(y) &= e_j(x-y), \end{aligned} \tag{VII.6.3}$$

where $\boldsymbol{E}, \boldsymbol{e}$ is the fundamental solution introduced in Section VII.3. Replacing (VII.6.3) into (VII.6.2) with $\Omega_\varepsilon \equiv \Omega - \{|x-y| \le \varepsilon\}$ in place of Ω and then letting $\varepsilon \to 0$, in view of (VII.3.18) and (VII.3.22) we recover

$$\begin{aligned} v_j(x) = &\mathcal{R}\int_\Omega E_{ij}(x-y) f_i(y) dy + \int_{\partial\Omega} [v_i(y) T_{i\ell}(\boldsymbol{w}_j, e_j)(x-y) \\ &- E_{ij}(x-y) T_{i\ell}(\boldsymbol{v}, p)(y) - \mathcal{R} v_i(y) E_{ij}(x-y)\delta_{1\ell}] n_\ell d\sigma_y, \end{aligned} \tag{VII.6.4}$$

where

$$\mathcal{R}\boldsymbol{f} = \Delta\boldsymbol{v} + \mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1} - \nabla p. \tag{VII.6.5}$$

We now turn to the representation for the pressure. By means of classical potential theory one can show that if $\boldsymbol{f}$ is Hölder continuous, the volume potentials

$$W_j(x) = \mathcal{R}\int_\Omega E_{ij}(x-y) f_i(y) dy$$

$$S(x) = -\mathcal{R}\int_\Omega e_i(x-y) f_i(y) dy$$

are (at least) of class C^2 and C^1, respectively, and, moreover,

$$L W_j = \frac{\partial S}{\partial x_j} + \mathcal{R} f_j \quad \text{in } \Omega, \tag{VII.6.6}$$

where

$$Lu \equiv \Delta u + \mathcal{R}\frac{\partial u}{\partial x_1}.$$

From (VII.6.4) and (VII.6.5) we have

$$\begin{aligned} \frac{\partial p}{\partial x_j} + \mathcal{R} f_j = L v_j = L W_j + &\int_{\partial\Omega} [v_i L T_{i\ell}(\boldsymbol{w}_j, e_j) \\ &- (L E_{ij}) T_{i\ell}(\boldsymbol{v}, p) - \mathcal{R} v_i L E_{ij} \delta_{1\ell}] n_\ell \end{aligned} \tag{VII.6.7}$$

and observing that e_j is harmonic (for $x \ne y$) and that $\partial e_i/\partial x_j = \partial e_j/\partial x_i$, from (VII.3.19) and from the definition of $\boldsymbol{T}$ it also follows that

$$
\begin{aligned}
LT_{i\ell}(\boldsymbol{w}_j, e_j) &= Le_j\delta_{i\ell} + \frac{\partial}{\partial x_\ell}(LE_{ij}) + \frac{\partial}{\partial x_i}(LE_{\ell j}) \\
&= -\mathcal{R}\frac{\partial e_j}{\partial x_1}\delta_{i\ell} - 2\frac{\partial^2 e_i}{\partial x_i \partial x_\ell}.
\end{aligned}
\tag{VII.6.8}
$$

Using (VII.6.5) and (VII.6.8) in (VII.6.7) we conclude the validity (up to a constant) of the following formula for all $x \in \Omega$:

$$
\begin{aligned}
p(x) = -\mathcal{R}\int_\Omega e_i(x-y)f_i(y)dy + \int_{\partial\Omega} \Big\{ & e_i(x-y)T_{i\ell}(\boldsymbol{v}, p)(y) \\
& -2v_i(y)\frac{\partial}{\partial x_\ell}e_i(x-y) - \mathcal{R}[e_1(x-y)v_\ell(y) \\
& -v_i(y)e_i(x-y)\delta_{1\ell}]\Big\} n_\ell d\sigma_y.
\end{aligned}
\tag{VII.6.9}
$$

Formulas (VII.6.4) and (VII.6.9) can be generalized toward the following two directions:

(i) To derive analogous formulas for derivatives of $\boldsymbol{v}$ and p of arbitrary order;
(ii) To show their validity with $\boldsymbol{v}$ and p only belonging to suitable Sobolev spaces.

The first issue is trivially achieved (provided $\boldsymbol{v}, p$, and $\boldsymbol{f}$ are sufficiently smooth) by replacing in (VII.6.4) and (VII.6.9) $\boldsymbol{v}$, p and $\boldsymbol{f}$ with $D^\alpha\boldsymbol{v}$, $D^\alpha p$, and $D^\alpha\boldsymbol{f}$, respectively. The second one can be proved along the same lines of Theorem IV.8.1. However, we need the following result, whose validity is established by means of Theorem IV.4.1 and Theorem IV.5.1.

Lemma VII.6.1 *Let Ω be a bounded domain of class C^{m+2}, $m \geq 0$. For any $\boldsymbol{f} \in W^{m,q}(\Omega)$, $\boldsymbol{v}_* \in W^{m+2-1/q,q}(\partial\Omega)$, $1 < q < \infty$, with*

$$
\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} = 0,
$$

there exists one and only one function $\boldsymbol{v} \in W^{m+2,q}(\Omega)$, $p \in W^{m+1,q}(\Omega)$ satisfying a.e. the Oseen problem

$$
\left.
\begin{aligned}
\Delta\boldsymbol{v} + \mathcal{R}\frac{\partial\boldsymbol{v}}{\partial x_1} &= \nabla p + \mathcal{R}\boldsymbol{f} \\
\nabla\cdot\boldsymbol{v} &= 0
\end{aligned}
\right\} \ \text{in } \Omega
\\
\boldsymbol{v} = \boldsymbol{v}_* \ \ \text{at } \partial\Omega.
\tag{VII.6.10}
$$

Proof. The existence of a generalized solution is at once established with the Galerkin method used in the proof of Theorem VII.2.1. Employing Theorem IV.4.1 and Theorem IV.5.1 we then show that such a solution satisfies all requirements stated in the theorem. Uniqueness is a simple exercise. □

Lemma VII.6.1, together with an argument entirely analogous to that of Theorem IV.8.1, implies the following result.

Lemma VII.6.2 *Let Ω satisfy the assumption of* Lemma VII.6.1. *Let $\boldsymbol{v} \in W^{m+2,q}(\Omega)$, $p \in W^{m+1,q}(\Omega)$ be a solution to* $(\text{VII.6.10})_1$ *corresponding to $\boldsymbol{f} \in W^{m,q}(\Omega)$, $m \geq 0$, $1 < q < \infty$. Then, for all $|\alpha| \in [0, m]$ and almost all $x \in \Omega$,*

$$\begin{aligned} D^\alpha v_j(x) = {} & \mathcal{R}\int_\Omega E_{ij}(x-y) D^\alpha f_i(y) dy + \int_{\partial\Omega} [D^\alpha v_i(y) T_{i\ell}(\boldsymbol{w}_j, e_j)(x-y) \\ & - E_{ij}(x-y) T_{i\ell}(D^\alpha \boldsymbol{v}, D^\alpha p)(y) - \mathcal{R} D^\alpha v_i(y) E_{ij}(x-y)\delta_{1\ell}] n_\ell d\sigma_y, \\ D^\alpha p(x) = {} & -\mathcal{R}\int_\Omega e_i(x-y) D^\alpha f_i(y) dy + \int_{\partial\Omega} \Big\{ e_i(x-y) T_{i\ell}(D^\alpha \boldsymbol{v}, D^\alpha p)(y) \\ & -2 D^\alpha v_i(y) \frac{\partial}{\partial x_\ell} e_i(x-y) - \mathcal{R}[e_1(x-y) D^\alpha v_\ell(y) \\ & - D^\alpha v_i(y) e_i(x-y)\delta_{1\ell}] \Big\} n_\ell d\sigma_y. \end{aligned}$$

Our next task is to extend the above results to the case when Ω is an exterior domain. As in the case of the Stokes approximation, we shall use a suitable "truncation" of the Oseen fundamental tensor, along the lines suggested by Fujita (1961) in the nonlinear context. Thus, the *Oseen-Fujita truncated fundamental solution* $E_{ij}^{(R)}$, $e_j^{(R)}$ is defined by (VII.3.1), (VII.3.4), (VII.3.8), and (VII.3.11) with Φ replaced by $\psi_R \Phi$, where ψ_R is the "cut-off" function introduced in Section V.3. Clearly,

$$E_{ij}^{(R)}(x-y) = E_{ij}(x-y), \quad e_j^{(R)}(x-y) = e_j(x-y), \quad \text{if } |x-y| \leq R/2,$$

while $E_{ij}^{(R)}(x-y)$ and $e_j^{(R)}(x-y)$ vanish identically for $|x-y| \geq R$. Furthermore, from (VII.3.1) and (VII.3.2) one immediately obtains

$$\left.\begin{aligned} \left(\Delta - \mathcal{R}\frac{\partial}{\partial y_1}\right) E_{ij}^{(R)}(x-y) - \frac{\partial}{\partial y_i} e_j^{(R)}(x-y) &= \mathcal{H}_{ij}^{(R)}(x-y) \\ \frac{\partial}{\partial y_\ell} E_{\ell j}^{(R)}(x-y) &= 0, \end{aligned}\right\} \text{for } x \neq y \tag{VII.6.11}$$

where $\mathcal{H}_{ij}^{(R)}(x-y)$ is defined by $\mathcal{H}_{ij}^{(R)}(0) = 0$ and

$$\mathcal{H}_{ij}^{(R)}(x-y) = \delta_{ij}\Delta\left(\Delta - \mathcal{R}\frac{\partial}{\partial y_1}\right)(\psi_R \Phi).$$

Similar to the function $H_{ij}^{(R)}(x-y)$ introduced for the Stokes-Fujita truncated fundamental solution, $\mathcal{H}_{ij}^{(R)}(x-y)$ is also an infinitely differentiable function vanishing unless $R/2 < |x-y| < R$. However, due to the inhomogeneity of the Oseen differential operator, the uniform asymptotic properties of $\mathcal{H}_{ij}^{(R)}$ as $R \to \infty$ are somewhat different from those of $H_{ij}^{(R)}$. In fact, we have the following estimate, whose proof is left to the reader[1]:

$$|D^\alpha \mathcal{H}_{ij}^{(R)}(x-y)| = O(R^{-(n+1+|\alpha|)/2}), \quad |\alpha| \geq 0. \tag{VII.6.12}$$

The next result is proved exactly as in Lemma V.3.1.

Lemma VII.6.3 *Let Ω be an arbitrary domain of $\mathbb{R}^n$. Let $\boldsymbol{v} \in W^{1,2}_{loc}(\Omega)$ be weakly divergence-free and satisfy* (VII.1.1) *for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. Then, if $\boldsymbol{f} \in W^{m,q}_{loc}(\Omega)$ it follows that $\boldsymbol{v} \in W^{m+2,q}_{loc}(\Omega)$ and, moreover, for all fixed $d > 0$ and all $|\alpha| \in [0,m]$, $\boldsymbol{v}$ obeys the identity*

$$D^\alpha v_j(x) = \int_{B_d(x)} E_{ij}^{(d)}(x-y) D^\alpha f_i(y)dy - \int_{\beta(x)} \mathcal{H}_{ij}^{(d)}(x-y) D^\alpha v_i(y)dy \tag{VII.6.13}$$

for almost all $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) > d$, *where $\beta(x) = B_d(x) - B_{d/2}(x)$.*

Lemma VII.6.3 allows us to argue as in Theorem V.3.1, to show the following.

Theorem VII.6.1 *Let $\boldsymbol{v}$ be a q-generalized solution to the Oseen problem in an exterior domain Ω with $\boldsymbol{v} \in L^s(\Omega^R)$, for some $s \in (1,\infty)$ and some $R > \delta(\Omega^c)$. Then, if $\boldsymbol{f} \in W^{m,r}(\Omega)$, $m \geq 0$, $n/2 < r < \infty$,[2] it follows that*

$$\lim_{|x|\to\infty} D^\alpha \boldsymbol{v}(x) = \mathbf{0}, \quad 0 \leq |\alpha| \leq m\,.$$

To conclude this section it remains to investigate the structure of the solution at infinity and the corresponding rate of decay. Again, as in the Stokes approximation, we shall use the truncated fundamental solution. Starting with the Green's identity (VII.6.2) and choosing as $\boldsymbol{u}, \pi$ this latter solution we may readily show, by means of the same procedure adopted in Chapter V, the validity of the identities

$$\begin{aligned} v_j(x) = &\mathcal{R}\int_\Omega E_{ij}(x-y) f_i(y)dy + \int_{\partial\Omega} [v_i(y) T_{i\ell}(\boldsymbol{w}_j, e_j)(x-y) \\ &\qquad - E_{ij}(x-y) T_{i\ell}(\boldsymbol{v}, p)(y) - \mathcal{R} v_i(y) E_{ij}(x-y)\delta_{1\ell}] n_\ell d\sigma_y \\ &\qquad - \int_\Omega \mathcal{H}_{ij}^{(R)}(x-y) v_i(y)dy \end{aligned} \tag{VII.6.14}$$

[1] Bounds more accurate than those given in (VII.6.12) can be obtained according to whether we are inside or outside the "wake" region. However, (VII.6.12) will suffice for our purposes.

[2] See footnote 1 of Section V.3.

and

$$\frac{\partial}{\partial x_j}p(x) = \frac{\partial}{\partial x_j}\left\{-\mathcal{R}\int_\Omega e_i(x-y)f_i(y)dy + \int_{\partial\Omega}\{e_i(x-y)T_{i\ell}(\boldsymbol{v},p)(y)\right.$$
$$-2v_i(y)\frac{\partial}{\partial x_\ell}e_i(x-y)$$
$$\left.-\mathcal{R}[e_1(x-y)v_\ell(y)-v_i(y)e_i(x-y)\delta_{1\ell}]\}n_\ell d\sigma_y\right\}$$
$$-\int_\Omega\left(\Delta-\mathcal{R}\frac{\partial}{\partial x_1}\right)\mathcal{H}_{ij}^{(R)}(x-y)v_i(y)dy. \tag{VII.6.15}$$

Relations (VII.6.14) and (VII.6.15), which hold for almost all $x\in\Omega$, are valid for Ω of class C^2, $\boldsymbol{v}\in W^{2,q}_{loc}(\overline{\Omega})$, $q\in(1,\infty)$, and $\boldsymbol{f}$ belonging to $L^q(\Omega)$ and with compact support in Ω. Moreover, R is so large that $B_R(x)$ contains Ω^c and the support of $\boldsymbol{f}$.[3] Denote by $\mathcal{I}(x)$ the last integral on the right-hand side of (VII.6.14). It is easy to show that if

$$\int_{\Omega_{R/2,R}}|\boldsymbol{v}| = O(R^k),$$

$\mathcal{I}(x)$ is a polynomial whose degree depends on k and n. In fact, observing that $\mathcal{I}(x)$ *is independent of* R, we have

$$D^\alpha\mathcal{I}(x) = -\int_\Omega D^\alpha\mathcal{H}_{ij}^{(R)}(x-y)v_i(y)dy$$

and so, by (VII.6.12),

$$|D^\alpha\mathcal{I}(x)|\le cR^{-(n+1+|\alpha|)/2}\int_{\Omega_{R/2,R}}|\boldsymbol{v}|\le c_1R^{-(n+1+|\alpha|-2k)/2}.$$

Thus, choosing $|\alpha|=2k-n$ (say), we deduce $D^\alpha\mathcal{I}(x)=0$. Evidently, since as $|x|\to\infty$

$$\mathcal{I}(x)=\boldsymbol{v}(x)+o(1),$$

$\mathcal{I}(x)$ must reduce to a constant whenever $\boldsymbol{v}$ does not "grow" too fast at large distances. Also, if $\mathcal{I}(x)$ is a constant, the last integral on the right-hand side of (VII.6.15) is identically zero. Bearing this in mind, reasoning in complete analogy with Theorem V.3.2 and recalling the estimate for the Oseen fundamental solution given in Section VII.3, we obtain

Theorem VII.6.2 *Let Ω be a C^2-smooth, exterior domain and let $\boldsymbol{v}\in W^{2,q}_{loc}(\overline{\Omega})$, $q\in(1,\infty)$, be weakly divergence-free and satisfy* (VII.1.1) *for all $\boldsymbol{\varphi}\in\mathcal{D}(\Omega)$ with $\boldsymbol{f}\in L^q(\Omega)$. Assume further that the support of $\boldsymbol{f}$ is bounded. Then, if at least one of the following conditions is satisfied as $|x|\to\infty$:*

[3] Recall that in $B_R(x)$ the fundamental solution and the truncated fundamental solution coincide

(i) $\int_{S_n} |\boldsymbol{v}(x)| = o(|x|)$

(ii) $\int_{|x|\le r} \frac{|\boldsymbol{v}(x)|^t}{(1+|x|)^{n+t}} dx = o(\log r)$, *some* $t \in (1,\infty)$,

there exist vector and scalar constants $\boldsymbol{v}_0$, p_0 *such that for almost all* $x \in \Omega$ *we have*

$$\begin{aligned} v_j(x) = v_{0j} + \mathcal{R}\int_\Omega E_{ij}(x-y) f_i(y) dy + \int_{\partial\Omega} [v_i(y) T_{i\ell}(\boldsymbol{w}_j, e_j)(x-y) \\ - E_{ij}(x-y) T_{i\ell}(\boldsymbol{v}, p)(y) - \mathcal{R} v_i(y) E_{ij}(x-y)\delta_{1\ell}] n_\ell d\sigma_y \\ \equiv v_{0j} + v_j^{(1)}(x), \end{aligned} \tag{VII.6.16}$$

and

$$\begin{aligned} p(x) = p_0 - \mathcal{R}\int_\Omega e_i(x-y) f_i(y) dy + \int_{\partial\Omega} \{e_i(x-y) T_{i\ell}(\boldsymbol{v}, p)(y) \\ -2v_i(y) \frac{\partial}{\partial x_\ell} e_i(x-y) \\ -\mathcal{R}[e_1(x-y) v_\ell(y) - v_i(y) e_i(x-y)\delta_{1\ell}]\} n_\ell d\sigma_y \\ \equiv p_0 + p^{(1)}(x). \end{aligned} \tag{VII.6.17}$$

Moreover, as $|x| \to \infty$, $\boldsymbol{v}^{(1)}(x)$ *and* $p^{(1)}(x)$ *are infinitely differentiable and there the following asymptotic representations hold:*

$$\begin{aligned} v_j^{(1)}(x) = E_{ij}(x)\mathcal{M}_i + \sigma_j(x) \\ p^{(1)}(x) = -e_i(x)\mathcal{M}_i^* + \eta(x), \end{aligned} \tag{VII.6.18}$$

where

$$\begin{aligned} \mathcal{M}_i = -\int_{\partial\Omega} [T_{i\ell}(\boldsymbol{v}, p) + \mathcal{R}\delta_{1\ell} v_i] n_\ell + \mathcal{R}\int_\Omega f_i \\ \mathcal{M}_i^* = -\int_{\partial\Omega} \{T_{i\ell}(\boldsymbol{v}, p) + \mathcal{R}[\delta_{1\ell} v_i - \delta_{1i} v_\ell]\} n_\ell + \mathcal{R}\int_\Omega f_i \end{aligned} \tag{VII.6.19}$$

and, for all $|\alpha| \ge 0$,

$$\begin{aligned} D^\alpha \boldsymbol{\sigma}(x) = O(|x|^{-(n+|\alpha|)/2}) \\ D^\alpha \eta(x) = O(|x|^{-n-|\alpha|}). \end{aligned} \tag{VII.6.20}$$

Remark VII.6.1 Theorem VII.6.2 asserts, among other things, that every q-weak solution to (VII.0.2) and (VII.0.3) behaves asymptotically as the Oseen fundamental solution. In particular, taking into account the properties of this solution, every q-weak solution exhibits a paraboloidal wake region in the direction of the *positive* x_1-axis; see Remark VII.3.1. ■

Some interesting consequences of Theorem VII.6.2 are left to the reader in the following exercises.

Exercise VII.6.1 Let $\boldsymbol{v}$ satisfy the assumption of Theorem VII.6.2. Show that, for all sufficiently large R,

$$\int_{\partial B_R} \left(v^2 + \nabla \boldsymbol{v} : \nabla \boldsymbol{v}\right) \leq c_1 R^{-(n-1)/2}$$

$$\int_{\partial B_R} |p - p_0|^2 \leq c_1 R^{-(n-1)}.$$

Hint: Use Theorem VII.6.2 together with Exercise VII.3.1 and Exercise VII.3.4.

Exercise VII.6.2 The following result generalizes uniqueness Theorem VII.1.2. Let $\boldsymbol{v}, p$ be a q-generalized solution to the Oseen problem (VII.0.3), (VII.0.2) in an exterior domain Ω of class C^2. Show that if $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv \boldsymbol{0}$ then $\boldsymbol{v} \equiv 0$, $p \equiv$const. Under these latter assumptions on the data, show that if $\boldsymbol{v}$, p is a corresponding smooth solution with $\boldsymbol{v} = o(1)$ as $|x| \to \infty$, then $\boldsymbol{v} \equiv 0$, $p \equiv$ const.

Exercise VII.6.3 Show that the remainder $\boldsymbol{\sigma}$ in $(\text{VII.6.18})_1$ has the following summability properties:

$$\left.\begin{array}{l}\sigma \in L^q(\Omega^R), \quad \text{for all } q > n/(n-1) \\ \sigma \in L^q(\Omega^R), \quad \text{for all } q > (n+1)/n, \text{ if } \Phi \equiv \int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} = 0\end{array}\right. \qquad R > \delta(\Omega^c).$$

Hint: As $|x| \to \infty$, it is $\sigma(x) = O(\boldsymbol{e}(x))$ if $\Phi \neq 0$, and $O(\nabla \boldsymbol{E}(x))$ if $\Phi = 0$. Then use the summability properties of $\boldsymbol{e}$ and $\nabla \boldsymbol{E}$.

Exercise VII.6.4 Let $\boldsymbol{v}, p$ be a smooth solution to the Oseen problem in an exterior domain $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$, with $\boldsymbol{v} = 0$ at $\partial\Omega$. Show that if $\boldsymbol{v} \not\equiv 0$, necessarily $\|\boldsymbol{v} - \boldsymbol{v}_\infty\|_{2,\Omega} = \infty$, for any choice of $\boldsymbol{v}_\infty \in \mathbb{R}^n$.[4] *Hint*: Recall that $\boldsymbol{E} \notin L^2(\Omega^R)$.

Exercise VII.6.5 (Olmstead & Gautesen 1968) Show the following "drag paradox" for the Oseen approximation. Let $\mathcal{B}$ be a (smooth) body moving in a viscous liquid that fills the whole space, with no spin and translational velocity $\lambda \boldsymbol{e}_1$. Set $\Omega := \mathbb{R}^3 - \mathcal{B}$, and denote by $\boldsymbol{\mathcal{D}}(\boldsymbol{e}_1) := \boldsymbol{e}_1 \int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{T}$, the drag exerted by the liquid on $\mathcal{B}$ in the Oseen approximation, where $\boldsymbol{n}$ is the inner unit normal to $\partial\Omega$ ($\equiv \partial\mathcal{B}$). Prove that the drag is the same if the direction of the translational velocity is reversed, namely, $\boldsymbol{\mathcal{D}}(\boldsymbol{e}_1) = -\boldsymbol{\mathcal{D}}(-\boldsymbol{e}_1)$. *Hint:* Use (appropriately!) (VII.6.1) on the domain Ω_R, then let $R \to \infty$ and employ the asymptotic properties of Theorem VII.6.2.

The representation formula (VII.6.16) allows us to obtain an interesting asymptotic estimate for the *vorticity field* $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ in three dimensions. To this end, we observe that, setting

[4] If $n \geq 4$ this statement no longer holds since $\boldsymbol{E} \in L^2(\Omega^R)$.

$$\mathsf{f}(x-y) \equiv \frac{e^{-\rho}}{4\pi\mathcal{R}|x-y|}, \quad \rho = \frac{\mathcal{R}}{2}(|x-y| - (x_1 - y_1)), \tag{VII.6.21}$$

by a direct calculation one shows

$$\nabla_x \times (\boldsymbol{E}(x-y) \cdot \boldsymbol{G}(y)) = \nabla_x \mathsf{f}(x-y) \times \boldsymbol{G}(y)$$

and, consequently, (VII.6.16) furnishes, for all sufficiently large $|x|$

$$\begin{aligned}\boldsymbol{\omega}(x) = \mathcal{R}\int_\Omega \nabla_x \mathsf{f}(x-y) \times \boldsymbol{f}(y)dy + \int_{\partial\Omega} [\nabla_x(\nabla_x \mathsf{f}(x-y) \cdot \boldsymbol{n}) \times \boldsymbol{v}(y) \\ + \nabla_x \mathsf{f}(x-y) \times (-\boldsymbol{T}(\boldsymbol{v},p)(y) \cdot \boldsymbol{n} - \mathcal{R}\boldsymbol{v}(y) \cdot \boldsymbol{n})]d\sigma_y.\end{aligned} \tag{VII.6.22}$$

Applying the mean value theorem in the integrands in (VII.6.22), we easily deduce

$$\boldsymbol{\omega}(x) = \nabla \mathsf{f}(x) \times \mathcal{M} + O(D^2 \mathsf{f}(x)), \quad \text{as } |x| \to \infty, \tag{VII.6.23}$$

where the vector $\mathcal{M}$ is defined in $(\text{VII.6.19})_1$. From (VII.6.21) and (VII.6.23) it is apparent that in the region R situated outside the wake region (VII.3.27) and sufficiently far from $\partial\Omega$, the vorticity decays exponentially fast. This means, essentially, that the flow is potential in R, as expected from the physical point of view. The reader will prove with no difficulty that an analogous conclusion holds in the case of a plane flow with $\omega = \partial v_2/\partial x_1 - \partial v_1/\partial x_2$; see Clark (1971, §§2.2 and 3.2).

VII.7 Existence, Uniqueness, and L^q-Estimates in Exterior Domains

The aim of this section is to investigate to what extent the theorems proved in Section VII.4 in the whole space can be extended to the more general situation when the region of flow is an exterior domain. Of course, the results we shall prove rely heavily on those of Section VII.4 and, like those, they will be similar to those derived for the Stokes problem in Chapter V; nevertheless, they will differ from these in some crucial features that resemble the difference existing between the fundamental tensors $\boldsymbol{U}$ and $\boldsymbol{E}$.

Let us begin to consider the Oseen problem (VII.0.2), (VII.0.3) in an (exterior) domain Ω of class $C^{m+2}, m \geq 0$, with data

$$\boldsymbol{f} \in C_0^\infty(\overline{\Omega}), \quad \boldsymbol{v}_* \in W^{m+2-1/q}\ \ (\partial\Omega), \quad 1 < q < \infty\,.$$

By Theorem VII.2.1 and Theorem VII.5.1 we may then construct a solution $\boldsymbol{v}, p$ such that

$$\boldsymbol{v} \in W^{m+2,q}_{loc}(\overline{\Omega}) \cap C^\infty(\Omega), \quad p \in W^{m+1,q}_{loc}(\overline{\Omega}) \cap C^\infty(\Omega)$$

and which at large distances has the asymptotic structure of the type proved in Theorem VII.6.2. Denote next by ψ a "cut-off" function that equals one in $\Omega^{R/2}$ and zero in Ω_ρ, where $R/2 > \rho > \delta(\Omega^c)$. Putting $\boldsymbol{u} = \psi \boldsymbol{v}$, $\pi = \psi p$, from (VII.0.3) it follows that $\boldsymbol{u}, \pi$ satisfies the following Oseen problem in $\mathbb{R}^n$:

$$\begin{aligned} &\Delta \boldsymbol{u} + \mathcal{R}\frac{\partial \boldsymbol{u}}{\partial x_1} = \nabla \pi + \mathcal{R}\boldsymbol{F} \\ &\nabla \cdot \boldsymbol{u} = g, \end{aligned} \tag{VII.7.1}$$

where

$$\begin{aligned} &\mathcal{R}\boldsymbol{F} = \mathcal{R}\psi \boldsymbol{f} - \mathcal{R}\frac{\partial \psi}{\partial x_1}\boldsymbol{v} + (2\nabla\psi \cdot \boldsymbol{v} + \Delta\psi \boldsymbol{v} - p\nabla\psi) \\ &g = \nabla\psi \cdot \boldsymbol{v}. \end{aligned} \tag{VII.7.2}$$

Employing Theorem VII.4.1 we deduce the existence of a solution $\boldsymbol{w}, \tau$ to (VII.7.1), (VII.7.2) satisfying, in particular, the properties

$$\begin{aligned} &\boldsymbol{w} \in \bigcap_{\ell=0}^{m} D^{\ell+2,q}(\mathbb{R}^n), \quad \tau \in \bigcap_{\ell=0}^{m} D^{\ell+1,q}(\mathbb{R}^n), \quad 1 < q < \infty \\ &\boldsymbol{w} \in \bigcap_{\ell=0}^{m} D^{\ell+1,s_1}(\mathbb{R}^n), \quad s_1 = \frac{(n+1)q}{n+1-q}, \quad 1 < q < n+1 \\ &\boldsymbol{w} \in \bigcap_{\ell=0}^{m} D^{\ell,s_2}(\mathbb{R}^n) \quad s_2 = \frac{(n+1)q}{n+1-2q}, \quad 1 < q < \frac{n+1}{2} \end{aligned} \tag{VII.7.3}$$

together with inequalities (VII.4.30)–(VII.4.35). We then apply Theorem VII.6.2 to $\boldsymbol{w}$ in the domain $\Omega^{R/2}$, which does not contain the support of g. Because of $(\text{VII.7.3})_3$, $\boldsymbol{w}$ satisfies assumption (ii) of that theorem and, consequently, it has the asymptotic structure $(\text{VII.6.18})_1$, from which we conclude $\boldsymbol{u} \equiv \boldsymbol{w}$, $\pi \equiv p+$const; see Exercise VII.6.2. Recalling (VII.7.2) and that $\boldsymbol{v} = \boldsymbol{u}$, $p = \pi$ in $\Omega^{R/2}$, from (VII.4.34) and (VII.4.35) it follows, in particular, that for all $\ell \in [0, m]$ and all $q \in (1, (n+1)/2)$, the pair $\boldsymbol{v}, p$ obeys the inequality

$$\begin{aligned} &\mathcal{R}^{2/(n+1)}|\boldsymbol{v}|_{\ell,s_2,\Omega^{R/2}} + \mathcal{R}^{1/(n+1)}|\boldsymbol{v}|_{\ell+1,s_1,\Omega^{R/2}} \\ &+\mathcal{R}\left|\frac{\partial \boldsymbol{v}}{\partial x_1}\right|_{\ell,q,\Omega^{R/2}} + |\boldsymbol{v}|_{\ell+2,q,\Omega^{R/2}} + |p|_{\ell+1,q,\Omega^{R/2}} \\ &\le c_1 \left(\mathcal{R}|\boldsymbol{f}|_{\ell,q} + (1+\mathcal{R})|\boldsymbol{v}|_{\ell+1,q,\Omega_R} + |p|_{\ell,q,\Omega_R}\right), \end{aligned} \tag{VII.7.4}$$

where $s_1 = \frac{(n+1)q}{n+1-q}$, $s_2 = \frac{(n+1)q}{n+1-2q}$ and, for $n = 2$,

$$
\begin{aligned}
&\mathcal{R}|v_2|_{\ell,2q/(2-q),\Omega^{R/2}} + \mathcal{R}|v_2|_{\ell+1,q,\Omega^{R/2}} + \mathcal{R}^{2/3}|\boldsymbol{v}|_{\ell,3q/(3-2q),\Omega^{R/2}} \\
&+\mathcal{R}^{1/3}|\boldsymbol{v}|_{\ell+1,3q/(3-q),\Omega^{R/2}} + \mathcal{R}\left|\frac{\partial \boldsymbol{v}}{\partial x_1}\right|_{\ell,q,\Omega^{R/2}} + |\boldsymbol{v}|_{\ell+2,q,\Omega^{R/2}} + |p|_{\ell+1,q,\Omega^{R/2}} \\
&\le c_1\left(\mathcal{R}|\boldsymbol{f}|_{\ell,q} + (1+\mathcal{R})|\boldsymbol{v}|_{\ell+1,q,\Omega_R} + |p|_{\ell,q,\Omega_R}\right).
\end{aligned}
\tag{VII.7.5}
$$

Let us next derive analogous inequalities in Ω_R. From (IV.6.3) we have

$$
\begin{aligned}
&\|\boldsymbol{v}\|_{m+2,q,\Omega_R} + \|p\|_{m+1,q,\Omega_R} \\
&\le c_2\left\{\mathcal{R}\|\boldsymbol{f}\|_{m,q,\Omega_R} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)} + \mathcal{R}\left\|\frac{\partial \boldsymbol{v}}{\partial x_1}\right\|_{m,q,\Omega^R}\right. \\
&\left. + \|\boldsymbol{v}\|_{m+2-1/q,q(\partial B_R)} + \|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{q,\Omega_R}\right\},
\end{aligned}
\tag{VII.7.6}
$$

where, as usual, the origin of coordinates has been taken in the interior of Ω^c. By the trace Theorem II.4.4 we have

$$
\|\boldsymbol{v}\|_{m+2-1/q,q(\partial B_R)} \le c_3\left(|\boldsymbol{v}|_{m+2,q,\Omega^{R/2}} + \|\boldsymbol{v}\|_{m+1,q,\Omega_R}\right). \tag{VII.7.7}
$$

Furthermore, by the embedding Theorem II.3.4,

$$
\|\boldsymbol{v}\|_{m,s,\Omega_{R/2}} + \sum_{\ell=0}^{m}|\boldsymbol{v}|_{\ell+1,s_1,\Omega_{R/2}} \le c_4\|\boldsymbol{v}\|_{m+2,q,\Omega_R} \tag{VII.7.8}
$$

and

$$
\|\boldsymbol{v}\|_{m,2q/(2-q),\Omega_{R/2}} \le c_4\|\boldsymbol{v}\|_{m+1,q,\Omega_R}, \quad \text{if } n=2, \tag{VII.7.9}
$$

and so, collecting (VII.7.4), (VII.7.5)–(VII.7.9), we derive, in particular, for some $c = c(n, q, \Omega, m)$ and all $q \in (1, (n+1)/2)$

$$
\begin{aligned}
&a_1\|\boldsymbol{v}\|_{m,s_2,\Omega} + \mathcal{R}\left\|\frac{\partial \boldsymbol{v}}{\partial x_1}\right\|_{m,q,\Omega} \\
&+\sum_{\ell=0}^{m}\left\{a_2|\boldsymbol{v}|_{\ell+1,s_1,\Omega} + |\boldsymbol{v}|_{\ell+2,q,\Omega} + |p|_{\ell+1,q,\Omega}\right\} \\
&\le c_5\left(\mathcal{R}\|\boldsymbol{f}\|_{m,q,\Omega} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)} + (1+\mathcal{R})\|\boldsymbol{v}\|_{m+1,q,\Omega_R} + \|p\|_{m,q,\Omega_R}\right)
\end{aligned}
\tag{VII.7.10}
$$

and, if $n = 2$,

$$
\begin{aligned}
&\mathcal{R}\left(\|v_2\|_{m,2q/(2-q),\Omega}+\|\nabla v_2\|_{m+1,q,\Omega}\right)\\
&+a_1\|\boldsymbol{v}\|_{m,3q/(3-2q),\Omega}+\mathcal{R}\left\|\frac{\partial \boldsymbol{v}}{\partial x_1}\right\|_{m,q,\Omega}\\
&+\sum_{\ell=0}^{m}\left\{a_2|\boldsymbol{v}|_{\ell+1,3q/(3-q),\Omega}+|\boldsymbol{v}|_{\ell+2,q,\Omega}+|p|_{\ell+1,q,\Omega}\right\}\\
&\le c_5\left(\mathcal{R}\|\boldsymbol{f}\|_{m,q,\Omega}+\|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)}+(1+\mathcal{R})\|\boldsymbol{v}\|_{m+1,q,\Omega_R}+\|p\|_{m,q,\Omega_R}\right)
\end{aligned}
\tag{VII.7.11}
$$

where

$$
\begin{aligned}
&a_1=\min\{1,\mathcal{R}^{2/(n+1)}\},\quad a_2=\min\{1,\mathcal{R}^{1/(n+1)}\},\\
&s_1=\frac{(n+1)q}{n+1-q},\quad s_2=\frac{(n+1)q}{n+1-2q}.
\end{aligned}
\tag{VII.7.12}
$$

By a repeated use of Ehrling's inequality (see Exercise II.5.16), for all $\varepsilon>0$ it follows that

$$
\|p\|_{m,q,\Omega_R}\le\varepsilon\|p\|_{m+1,q,\Omega_R}+c_6\|p\|_{q,\Omega_R}
\tag{VII.7.13}
$$

with $c_6=c_6(\varepsilon,n,m,q,\Omega_R)$. In addition, possibly modifying p by a suitable constant (which causes no loss of generality), from Lemma IV.4.1 we derive

$$
\|p\|_{q,\Omega_R}\le c_7[(1+\mathcal{R})\|\boldsymbol{v}\|_{1,q,\Omega_R}+\mathcal{R}\|\boldsymbol{f}\|_{q,\Omega}].
\tag{VII.7.14}
$$

Inequalities (VII.7.10), (VII.7.11), (VII.7.13), and (VII.7.14) then yield

$$
\begin{aligned}
&a_1\|\boldsymbol{v}\|_{m,s_2,\Omega}+\mathcal{R}\left\|\frac{\partial \boldsymbol{v}}{\partial x_1}\right\|_{m,q,\Omega}\\
&+\sum_{\ell=0}^{m}\{a_2|\boldsymbol{v}|_{\ell+1,s_1,\Omega}+|\boldsymbol{v}|_{\ell+2,q,\Omega}+|p|_{\ell+1,q,\Omega}\}\\
&\le c_8\left(\mathcal{R}\|\boldsymbol{f}\|_{m,q,\Omega}+\|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)}+(1+\mathcal{R})\|\boldsymbol{v}\|_{m+1,q,\Omega_R}\right)
\end{aligned}
\tag{VII.7.15}
$$

and, if $n=2$,

$$
\begin{aligned}
&\mathcal{R}\left(\|v_2\|_{m,2q/(2-q),\Omega}+\|\nabla v_2\|_{m+1,q,\Omega}\right)\\
&+a_1\|\boldsymbol{v}\|_{m,3q/(3-2q),\Omega}+\mathcal{R}\left\|\frac{\partial \boldsymbol{v}}{\partial x_1}\right\|_{m,q,\Omega}\\
&+\sum_{\ell=0}^{m}\left\{a_2|\boldsymbol{v}|_{\ell+1,3q/(3-q),\Omega}+|\boldsymbol{v}|_{\ell+2,q,\Omega}+|p|_{\ell+1,q,\Omega}\right\}\\
&\le c_8\left(\mathcal{R}\|\boldsymbol{f}\|_{m,q,\Omega}+\|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)}+(1+\mathcal{R})\|\boldsymbol{v}\|_{m+1,q,\Omega_R}\right).
\end{aligned}
\tag{VII.7.16}
$$

We now look for an inequality of the type

$$\|\boldsymbol{v}\|_{m+1,q,\Omega} \leq c_9 \left(\mathcal{R}\|\boldsymbol{f}\|_{m,q,\Omega} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)}\right) \tag{VII.7.17}$$

for a suitable constant independent of $\boldsymbol{v}$, $\boldsymbol{f}$ and $\boldsymbol{v}_*$. The proof of (VII.7.17) can be obtained, as in the case of the Stokes problem, by a contradiction argument. Actually, admitting the invalidity of (VII.7.17) means to assume the existence of two sequences

$$\{\boldsymbol{f}_k\} \subset C_0^\infty(\Omega), \quad \{\boldsymbol{v}_{*k}\} \subset W^{m+2-1/q,q}(\partial\Omega)$$

such that, by denoting by $\{\boldsymbol{v}_k, p_k\}$ the corresponding solutions,

$$\begin{aligned} &\mathcal{R}\|\boldsymbol{f}_k\|_{m,q,\Omega} + \|\boldsymbol{v}_{*k}\|_{m+2-1/q,q(\partial\Omega)} \leq 1/k \\ &\|\boldsymbol{v}_k\|_{m+1,q,\Omega_R} = 1. \end{aligned} \tag{VII.7.18}$$

However, reasoning as in the proof of Lemma V.4.5, we can show that, as $k \to \infty$, $\boldsymbol{v}_k$ converges to a solution $\boldsymbol{v}$ of the Oseen problem (VII.0.2), (VII.0.3) with $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv \boldsymbol{v}_\infty \equiv \mathbf{0}$. Furthermore, by (VII.7.15), it follows that $\boldsymbol{v} \in L^{s_2}(\Omega)$ and therefore by Theorem VII.6.2 and Exercise VII.6.2 we have $\boldsymbol{v} \equiv 0$. The point to discuss now is that a priori the constant c depends also on $\mathcal{R}$ and, consequently, we lose the dependence of inequality (VII.7.15) + $\frac{c_8}{c_7}$(VII.7.17) on the Reynolds number $\mathcal{R}$. We may thus wonder if, at least in some cases, this undesired feature can be avoided. We shall presently show that if $n > 2$ and $q \in (1, n/2)$ we may take the constant c_9 independent of $\mathcal{R} \in (0, B]$ for any positive, arbitrarily fixed B. Actually assume (VII.7.17) does not hold, then there exist sequences

$$\{\boldsymbol{f}_k\} \subset C_0^\infty(\Omega), \quad \{\boldsymbol{v}_{*k}\} \subset W^{m+2-1/q,q}(\partial\Omega) \tag{VII.7.19}$$

and

$$\{\mathcal{R}_k\} \subset (0, B] \tag{VII.7.20}$$

such that, denoting by $\{\boldsymbol{v}_k, p_k\}$ the solutions to the Oseen problems

$$\left.\begin{aligned} &\Delta\boldsymbol{v}_k + \mathcal{R}_k\frac{\partial\boldsymbol{v}_k}{\partial x_1} - \nabla p_k = \mathcal{R}_k\boldsymbol{f}_k \\ &\nabla\cdot\boldsymbol{v}_k = 0 \end{aligned}\right\} \tag{VII.7.21}$$

$$\boldsymbol{v}_k = \boldsymbol{v}_{*k} \quad \text{at } \partial\Omega,$$

the following condition holds

$$\begin{aligned} &\mathcal{R}_k\|\boldsymbol{f}_k\|_{m,q,\Omega} + \|\boldsymbol{v}_{*k}\|_{m+2-1/q,q(\partial\Omega)} \leq 1/k \\ &\|\boldsymbol{v}_k\|_{m+1,q,\Omega_R} = 1. \end{aligned} \tag{VII.7.22}$$

In view of (VII.7.20) there is a subsequence, indicated again by $\{\mathcal{R}_k\}$, and a number $\mathcal{R} \geq 0$ to which $\mathcal{R}_k$ converges as $k \to \infty$. Furthermore, by (VII.7.6), (VII.7.14), (VII.7.15), and (VII.7.18), for all $k \in \mathbb{N}$ we have for all fixed R

$$\|\boldsymbol{v}_k\|_{1,q,\Omega_R} + \|D^2\boldsymbol{v}_k\|_{m,q} + \|\nabla p_k\|_{m,q} \leq M \tag{VII.7.23}$$

for some constant M independent of k. The results of Exercise II.6.2 on weak compactness of $\dot{D}^{m,q}$-spaces along with Exercise II.5.8 on strong compactness (on bounded domains) imply the existence of a subsequence, still denoted by $\{\boldsymbol{v}_k, p_k\}$, and of two functions $\boldsymbol{v} \in D^{2,q}(\Omega), p \in D^{1,q}(\Omega)$ such that as $k \to \infty$

$$\begin{aligned} &D^2\boldsymbol{v}_k \overset{w}{\to} D^2\boldsymbol{v}, \quad \nabla p_k \overset{w}{\to} \nabla p, \quad \text{in } L^q(\Omega) \\ &\boldsymbol{v}_k \to \boldsymbol{v}, \quad \text{in } W^{m+1,q}(\Omega_R). \end{aligned} \tag{VII.7.24}$$

From (VII.7.22), (VII.7.21), and (VII.7.24) it immediately follows that $\boldsymbol{v}, p$ is a solution to the homogeneous Oseen problem

$$\left.\begin{aligned} &\Delta\boldsymbol{v} + \mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1} - \nabla p = 0 \\ &\nabla \cdot \boldsymbol{v} = 0 \end{aligned}\right\} \tag{VII.7.25}$$
$$\boldsymbol{v} = 0 \ \text{ at } \partial\Omega,$$

satisfying

$$\|\boldsymbol{v}\|_{m+1,q,\Omega_R} = 1. \tag{VII.7.26}$$

Let us now distinguish the following two cases: (i) $\mathcal{R} > 0$, (ii) $\mathcal{R} = 0$. In case (i) from (VII.7.15) (written along the subsequences) and (VII.7.22) we obtain

$$\boldsymbol{v} \in L^{s_2}(\Omega) \tag{VII.7.27}$$

and, since $\boldsymbol{v}$ solves (VII.7.25), from Theorem VII.6.2 and Exercise VII.6.2 we deduce $\boldsymbol{v} \equiv \boldsymbol{0}$, contradicting (VII.7.26). If the limiting value $\mathcal{R}$ is zero, we can no longer deduce (VII.7.27). Nevertheless, if $q \in (1, n/2)$, we can still deduce that $\boldsymbol{v}$ belongs to some space $L^r(\Omega)$. Actually, by a double application of inequality (II.6.22), and recalling that, for each fixed k, $\boldsymbol{v}_k(x)$ and $\nabla\boldsymbol{v}_k(x)$ tend to zero uniformly as $|x|$ tends to infinity, from (VII.7.23) we have

$$\|\boldsymbol{v}_k\|_{nq/(n-2q)} \leq c\|D^2\boldsymbol{v}_k\|_q \leq M$$

and therefore

$$\boldsymbol{v} \in L^{nq/(n-q)}(\Omega).$$

Replacing this information into (VII.7.25) with $\mathcal{R} = 0$ and using this time Theorem V.3.2 and Theorem V.3.4 we conclude $\boldsymbol{v} \equiv \boldsymbol{0}$, which contradicts (VII.7.26).

Once (VII.7.15) and (VII.7.17) have been established, we can prove the following.

Theorem VII.7.1 *Let Ω be an exterior domain in $\mathbb{R}^n$ of class C^{m+2}, $m \geq 0$. Given*

$$\boldsymbol{f} \in W^{m,q}(\Omega), \quad \boldsymbol{v}_* \in W^{m+2-1/q,q}(\partial\Omega), \quad 1 < q < (n+1)/2\,,$$

there exists one and only one corresponding solution $\boldsymbol{v}, p$ to the Oseen problem (VII.0.2), (VII.0.3) *such that*

$$\boldsymbol{v} \in W^{m,s_2}(\Omega) \cap \left\{ \bigcap_{\ell=0}^{m} \left[D^{\ell+1,s_1}(\Omega) \cap D^{\ell+2,q}(\Omega) \right] \right\}$$

$$p \in \bigcap_{\ell=0}^{m} D^{\ell+1,q}(\Omega)$$

with $s_1 = \frac{(n+1)q}{n+1-q}$, $s_2 = \frac{(n+1)q}{n+1-2q}$. If $n = 2$, we also have

$$v_2 \in W^{m,2q/(2-q)}(\Omega) \cap \left(\bigcap_{\ell=0}^{m} D^{\ell+1,q}(\Omega) \right).$$

Moreover, $\boldsymbol{v}, p$ verify

$$\begin{aligned} a_1 \|\boldsymbol{v}\|_{m,s_2} + \mathcal{R} \left\| \frac{\partial \boldsymbol{v}}{\partial x_1} \right\|_{m,q} + \sum_{\ell=0}^{m} \{ a_2 |\boldsymbol{v}|_{\ell+1,s_1} + |\boldsymbol{v}|_{\ell+2,q} + |p|_{\ell+1,q} \} \\ \leq c \left(\mathcal{R} \|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)} \right) \end{aligned} \tag{VII.7.28}$$

and, if $n = 2$,

$$\begin{aligned} \mathcal{R} \left(\|v_2\|_{m,2q/(2-q)} + \|\nabla v_2\|_{m+1,q} \right) + a_1 \|\boldsymbol{v}\|_{m,3q/(3-2q)} + \mathcal{R} \left\| \frac{\partial \boldsymbol{v}}{\partial x_1} \right\|_{m,q} \\ + \sum_{\ell=0}^{m} \left\{ a_2 |\boldsymbol{v}|_{\ell+1,3q/(3-q)} + |\boldsymbol{v}|_{\ell+2,q} + |p|_{\ell+1,q} \right\} \\ \leq c \left(\mathcal{R} \|\boldsymbol{f}\|_{m,q,} + \|\boldsymbol{v}_*\|_{m+2-1/q,q(\partial\Omega)} \right) \end{aligned} \tag{VII.7.29}$$

with a_1 and a_2 given in (VII.7.12). *The constant c depends on m, q, n, Ω, and $\mathcal{R}$. However, if $q \in (1, n/2)$ and $\mathcal{R} \in (0, B]$ for some $B > 0$, c depends solely on m, q, n, Ω, and B.*

Proof. The existence part, together with the validity of (VII.7.28), has been already established for $\boldsymbol{f} \in C_0^\infty(\overline{\Omega})$. However, from (VII.7.28) and from a now standard density argument we can extend existence to all $\boldsymbol{f} \in W^{m,q}(\Omega)$. Finally, uniqueness of solutions is most easily discussed if we take into account that, indicating by $\boldsymbol{u}, \pi$ the difference between two solutions corresponding to the same data, we have $\boldsymbol{u} \in L^{s_2}(\Omega)$. Therefore, Theorem VII.6.2 and Exercise VII.6.2 ensure $\boldsymbol{u} \equiv 0$. The proof is thus completed. $\square$

Remark VII.7.1 Theorem VII.7.1 leaves out the question of existence and uniqueness for $q > (n+1)/2$. However, by using a treatment analogous to that employed in Section V.3 for the Stokes problem, one could show existence, uniqueness, and validity of corresponding L^q-estimates in suitable quotient spaces. We shall not treat this here. For related results for q-generalized solutions, we refer the reader to Remark VII.7.3. ■

Remark VII.7.2 The validity of inequalities of the type (VII.7.29) with c independent of $\mathcal{R} \in [0, B]$ is of fundamental importance for treating nonlinear, plane-steady flow with nonzero velocity at infinity. However, because of the Stokes paradox, one expects that the constant c in (VII.7.29) may become unbounded as $\mathcal{R}$ approaches zero. On the other hand, if $1 < q < 6/5$, in SectionXII.4, we shall prove the validity of an inequality *weaker* than (VII.7.29) for a constant c which, in general, can be rendered independent of $\mathcal{R}$ for $\mathcal{R}$ ranging in $[0, B]$. ■

Exercise VII.7.1 Extend the results of Theorem VII.7.1 to the case $\nabla \cdot \boldsymbol{v} = g \not\equiv 0$, with g a prescribed function from $W^{m+1,q}(\Omega)$. Show further that, in this case, inequalities (VII.7.28) and (VII.7.29) are modified by adding the term $(1+\mathcal{R})\|g\|_{m+1,q}$ to its right-hand side.

Our subsequent objective is to extend Theorem VII.4.2 to the case of an exterior domain Ω of class C^2. We start with $\boldsymbol{f} \in C_0^\infty(\overline{\Omega})$, and $\boldsymbol{v}_* \in W^{1-1/q,q}(\partial\Omega)$. As in Theorem VII.7.1, corresponding to these data there exists a solution $\boldsymbol{v}, p$ to (VII.0.2), (VII.0.3) with

$$\boldsymbol{v} \in W^{1,q}_{loc}(\overline{\Omega}) \cap C^\infty(\Omega), \quad p \in L^q_{loc}(\overline{\Omega}) \cap C^\infty(\Omega)$$

satisfying the asymptotic behavior of the type described in Theorem VII.7.2. Furthermore, $\boldsymbol{u} \equiv \psi \boldsymbol{v}$ and $\pi \equiv \psi p$ satisfy problem (VII.7.1), (VII.7.2) in $\mathbb{R}^n$, to which we apply the results stated in Theorem VII.4.2. We thus deduce the existence of a solution $\boldsymbol{w}, \tau$ to (VII.7.1), (VII.7.2) enjoying, in particular, the properties

$$\begin{aligned} &\boldsymbol{w} \in D^{1,q}(\mathbb{R}^n) \cap L^{s_1}(\mathbb{R}^n), \quad \tau \in L^q(\mathbb{R}^n) \\ &s_1 = \frac{(n+1)q}{(n+1-q)}, \quad 1 < q < n+1 \\ &w_2 \in L^q(\mathbb{R}^2), \quad \text{if } n = 2, \end{aligned} \tag{VII.7.30}$$

together with the estimates

$$\begin{aligned}&\mathcal{R}^{1/(n+1)}\|\boldsymbol{w}\|_{s_1}+|\boldsymbol{w}|_{1,q}+\|\tau\|_q\\&\qquad\le c_1\left(\mathcal{R}|\boldsymbol{F}|_{-1,q}+\mathcal{R}|g|_{-1,q}+\|g\|_q\right),\end{aligned}\qquad \text{(if } n>2)$$

$$\begin{aligned}&\mathcal{R}\|w_2\|_q+\mathcal{R}^{1/3}\|\boldsymbol{w}\|_{3q/(3-q)}+|\boldsymbol{w}|_{1,q}+\|\tau\|_q\\&\qquad\le c_1\left(\mathcal{R}|\boldsymbol{F}|_{-1,q}+\mathcal{R}|g|_{-1,q}+\|g\|_q\right).\end{aligned}\qquad \text{(if } n=2)$$

(VII.7.31)

As in Theorem VII.7.1, we prove $\boldsymbol{w}\equiv\boldsymbol{u}$, $\tau\equiv p$.[1] Assuming $q>n/(n-1)$ and reasoning exactly as we did in Theorem V.5.1, we have

$$\begin{aligned}&\mathcal{R}|\boldsymbol{F}|_{-1,q}+\mathcal{R}|g|_{-1,q}+\|g\|_q\\&\qquad\le c_2\left(\mathcal{R}|\boldsymbol{f}|_{-1,q}+(1+\mathcal{R})\|\boldsymbol{v}\|_{q,\Omega_R}+\|p\|_{-1,q,\Omega_R}\right)\end{aligned}\tag{VII.7.32}$$

and so, recalling that $\boldsymbol{u}=\boldsymbol{v}$, $\pi=p$ in $\Omega^{R/2}$, from (VII.7.30)–(VII.7.32) we deduce if $n>2$:

$$\begin{aligned}&\mathcal{R}^{1/(n+1)}\|\boldsymbol{v}\|_{s_1,\Omega^{R/2}}+|\boldsymbol{v}|_{1,q,\Omega^{R/2}}+\|p\|_{q,\Omega^{R/2}}\\&\qquad\le c_3\left(\mathcal{R}|\boldsymbol{F}|_{-1,q}+(1+\mathcal{R})\|\boldsymbol{v}\|_{q,\Omega_R}+\|p\|_{-1,q,\Omega_R}\right),\end{aligned}\tag{VII.7.33}$$

and if $n=2$:

$$\begin{aligned}&\mathcal{R}\|v_2\|_{q,\Omega^{R/2}}+\mathcal{R}^{1/3}\|\boldsymbol{v}\|_{3q/(3-q),\Omega^{R/2}}+|\boldsymbol{v}|_{1,q,\Omega^{R/2}}+\|p\|_{q,\Omega^{R/2}}\\&\qquad\le c_3\left(\mathcal{R}|\boldsymbol{F}|_{-1,q}+(1+\mathcal{R})\|\boldsymbol{v}\|_{q,\Omega_R}+\|p\|_{-1,q,\Omega_R}\right).\end{aligned}\tag{VII.7.34}$$

To obtain an estimate "near" the boundary, we apply inequality (IV.6.7), that is,

$$\begin{aligned}\|\boldsymbol{v}\|_{1,q,\Omega_R}+\|p\|_{q,\Omega_R}&\le c_4\Big(\mathcal{R}|\boldsymbol{f}|_{-1,q}+\|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)}\\&\quad+(1+\mathcal{R})\|\boldsymbol{v}\|_{q,\Omega_R}+\|p\|_{-1,q,\Omega_R}+\|\boldsymbol{v}\|_{1-1/q,q(\partial B_R)}\Big),\end{aligned}\tag{VII.7.35}$$

where we used the obvious inequality

$$\|\boldsymbol{f}\|_{-1,q,\Omega_R}\le|\boldsymbol{f}|_{-1,q}.$$

Employing the trace Theorem II.4.4 at the boundary term on ∂B_R we find

$$\|\boldsymbol{v}\|_{1-1/q,q(\partial B_R)}\le c\|\boldsymbol{v}\|_{1,q,\Omega_{R,2R}}$$

and so, by (VII.7.33), (VII.7.34), and (VII.7.35) it follows that

$$\begin{aligned}\|\boldsymbol{v}\|_{1,q,\Omega_R}+\|p\|_{q,\Omega_R}&\le c_5\Big(\mathcal{R}|\boldsymbol{f}|_{-1,q}+\|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)}\\&\quad+(1+\mathcal{R})\|\boldsymbol{v}\|_{q,\Omega_R}+\|p\|_{-1,q,\Omega_R}\Big).\end{aligned}\tag{VII.7.36}$$

[1] *A priori*, $\tau=p+\text{const.}$ but we can choose the constant up to which p is defined in such a way that $\tau=p$.

Taking into account that, by Theorem II.3.4,

$$\|\boldsymbol{v}\|_{s_1} \le c_6 \|\boldsymbol{v}\|_{1,q,\Omega_{R/2}}, \tag{VII.7.37}$$

combining (VII.7.33), (VII.7.36), and (VII.7.37) we conclude, for all $q \in (n/(n-1), n+1)$, if $n > 2$:

$$\begin{aligned} &a_2\|\boldsymbol{v}\|_{s_1,\Omega} + |\boldsymbol{v}|_{1,q,\Omega} + \|p\|_{q,\Omega} \\ &\le c_7 \left(\mathcal{R}|\boldsymbol{f}|_{-1,q} + \|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)} + (1+\mathcal{R})\|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R}\right), \end{aligned} \tag{VII.7.38}$$

and if $n = 2$:

$$\begin{aligned} &\mathcal{R}\|v_2\|_{q,\Omega} + a_2\|\boldsymbol{v}\|_{3q/(3-q),\Omega} + |\boldsymbol{v}|_{1,q,\Omega} + \|p\|_{q,\Omega} \\ &\le c_7 \left(\mathcal{R}|\boldsymbol{f}|_{-1,q} + \|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)} + (1+\mathcal{R})\|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R}\right), \end{aligned} \tag{VII.7.39}$$

where a_2 is defined in (VII.7.12). By a reasoning totally similar to that developed in the proof of Theorem VII.7.1 we can prove the existence of a constant c_8 independent of $\boldsymbol{v}, p, \boldsymbol{f}$, and $\boldsymbol{v}_*$ such that

$$\|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R} \le c_8\{\mathcal{R}|\boldsymbol{f}|_{-1,q} + \|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)}\}. \tag{VII.7.40}$$

The constant c_8 will also depend, in general, on $\mathcal{R}$. However, if $q \in (n/(n-1), n)$, by employing inequality (II.6.22), namely,

$$\|\boldsymbol{v}\|_{nq/(n-q)} \le c|\boldsymbol{v}|_{1,q},$$

one shows, as in Theorem VII.7.1, that c_8 can be chosen independent of $\mathcal{R}$ ranging in $(0, B]$, with an arbitrarily fixed B.

We are thus in a position to prove the following.

Theorem VII.7.2 *Let Ω be an exterior domain in $\mathbb{R}^n$ of class C^2. Given*

$$\boldsymbol{f} \in D_0^{-1,q}(\Omega), \quad \boldsymbol{v}_* \in W^{1-1/q,q}(\partial\Omega), \quad n/(n-1) < q < n+1\,.$$

there exists one and only one q-generalized solution $\boldsymbol{v}$ to (VII.0.2)–(VII.0.3). *Furthermore, it holds that*

$$\begin{aligned} &\boldsymbol{v} \in L^{s_1}(\Omega), \quad s_1 = \frac{(n+1)q}{(n+1-q)}, \\ &p \in L^q(\Omega) \end{aligned}$$

with p pressure field associated to $\boldsymbol{v}$ by Lemma VII.1.1, *and if $n = 2$,*

$$v_2 \in L^q(\Omega).$$

Finally, $\boldsymbol{v}$ and p obey the estimate

$$a_2\|\boldsymbol{v}\|_{s_1} + |\boldsymbol{v}|_{1,q} + |p|_q \le c\left(\mathcal{R}|\boldsymbol{f}|_{-1,q} + \|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)}\right) \tag{VII.7.41}$$

and, if $n=2$,

$$\begin{aligned}\mathcal{R}\|v_2\|_q + a_2\|\boldsymbol{v}\|_{3q/(3-q)} + |\boldsymbol{v}|_{1,q} + \|p\|_q \\ \le c\left(\mathcal{R}|\boldsymbol{f}|_{-1,q} + \|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)}\right),\end{aligned} \tag{VII.7.42}$$

where $c = c(n,q,\Omega,\mathcal{R})$ *and* a_2 *is defined in* (VII.7.12). *If* $n>2$ *and* $q \in (n/(n-1),n)$, *for* $\mathcal{R}\in(0,B]$, *with* B *arbitrarily fixed number,* c *depends solely on* n,q,Ω, *and* B.

Proof. The existence part follows from (VII.7.38) and (VII.7.40) whenever $\boldsymbol{f} \in C_0^\infty(\overline{\Omega})$. By a density argument, based on Theorem II.8.1 and on inequality (VII.7.41), we easily extend the result to all $\boldsymbol{f}\in D_0^{-1,q}(\Omega)$. Finally, uniqueness follows from Exercise VII.6.2.

□

Exercise VII.7.2 Extend the results of Theorem VII.7.2 to the case where $\nabla\cdot\boldsymbol{v} = g \not\equiv 0$, with g a prescribed function from $L^q(\Omega)\cap D_0^{-1,q}(\Omega)$. Show further that, in this case, inequality (VII.7.41) is modified by adding to its right-hand side the term

$$\|g\|_q + \mathcal{R}|g|_{-1,q}.$$

Remark VII.7.3 For future reference, we would like to point out that, *as far as existence goes*, Theorem VII.7.2 admits a suitable extension to the case $q\in[n+1,\infty)$, $n\ge 3$. More precisely, assuming Ω as in that theorem, for any given

$$\boldsymbol{f}\in D_0^{-1,q}(\Omega),\quad \boldsymbol{v}_*\in W^{1-1/q,q}(\partial\Omega),\quad n+1\le q<\infty\,.$$

there exists at least a solenoidal vector field $\boldsymbol{v}\in D^{1,q}(\Omega)$, with $\boldsymbol{v}=\boldsymbol{v}_*$ at $\partial\Omega$ (in the sense of trace), and a scalar field $p\in L^q(\Omega)$ satisfying (VII.1.2). The proof of this result is similar to that of Theorem V.5.1 (when $q\ge n$) and we shall sketch it here. Let $(\widetilde{\boldsymbol{v}}_i,\widetilde{\pi}_i)$ be the solutions to the Oseen problem (VII.0.3) with $\boldsymbol{f}\equiv\boldsymbol{0}$ and $\boldsymbol{v}_* = -\boldsymbol{e}_i$, $i=1,\dots,n$. From Theorem VII.1.1, Theorem VII.2.1 and Theorem VII.6.2, we deduce that these solutions exist, are of class $C^\infty(\Omega)$ and, moreover, they have the asymptotic behavior given in (VII.6.16). From this latter and (VII.3.49) we obtain $\widetilde{\boldsymbol{h}}_i := (\widetilde{\boldsymbol{v}}_i+\boldsymbol{e}_i)\in D^{1,q}(\Omega)$, $q\in((n+1)/n,\infty)$, with $\widetilde{\boldsymbol{h}}_i=\boldsymbol{0}$ at $\partial\Omega$, $i=1,\dots,n$. Thus, from the characterization given in Theorem II.7.6(ii), it follows that $\widetilde{\boldsymbol{h}}_i\in\mathcal{D}_0^{1,q}(\Omega)$, $q\in[n+1,\infty)$. As in Theorem V.5.1, we easily show that the set $\{\widetilde{\boldsymbol{h}}_i,\widetilde{\pi}_i\}$ is linearly independent and its linear span constitutes an n-dimensional subspace, $\widetilde{\mathcal{S}}_q$, of $\mathcal{D}_0^{1,q}(\Omega)\times L^q(\Omega)$, $q\in[n+1,\infty)$. Arguing as in the proof of Theorem VII.7.2, corresponding to smooth data $\boldsymbol{f}$ and $\boldsymbol{v}_*$, we can find a solution $(\boldsymbol{v},p)$ to the Oseen problem satisfying the following estimate, for *any* (fixed) $q\in[n+1,\infty)$:

$$|\boldsymbol{v}|_{1,q,\Omega} + \|p\|_{q,\Omega} \le c\left(|\boldsymbol{f}|_{-1,q} + \|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)} + \|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R}\right),$$

with $c = c(\mathcal{R}, q, n\Omega)$, which, in turn, implies

$$\inf_{(\widetilde{\boldsymbol{h}},\widetilde{\pi})\in\widetilde{\mathcal{S}}_q} \left\{|\boldsymbol{v} - \widetilde{\boldsymbol{h}}|_{1,q,\Omega} + \|p - \widetilde{\pi}\|_{q,\Omega}\right\}$$

$$\le c\left(|\boldsymbol{f}|_{-1,q} + \|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)} + \inf_{(\widetilde{\boldsymbol{h}},\widetilde{\pi})\in\widetilde{\mathcal{S}}_q} \|\boldsymbol{v} - \widetilde{\boldsymbol{h}}\|_{q,\Omega_R} + \|p - \widetilde{\pi}\|_{-1,q,\Omega_R}\right).$$

With this inequality in hand, one can proceed exactly like in the proof of Theorem V.5.1 and show the desired result. ■

We end this section with a result similar to that shown in Theorem V.5.3 that will furnish, in particular, summability properties at large distances of the pressure field p associated to a q-weak solution, when $q \ge n+1$. To this end we observe that if $\boldsymbol{v}$ is a q-weak solution corresponding to $\boldsymbol{f} \in D_0^{-1,r}(\Omega)$, where a priori $r \neq q$, by Lemma VII.1.1 one can always associate to $\boldsymbol{v}$ a pressure field p satisfying (VII.1.2) with $p \in L^{\mu}_{loc}(\Omega)$, $\mu = \min(r,q)$.

Theorem VII.7.3 *Let Ω be an exterior domain of $\mathbb{R}^n$ and let $\boldsymbol{v}$ be a q-generalized solution to* (VII.0.2)–(VII.0.3) *in Ω. Then, if $\boldsymbol{f} \in D_0^{-1,r}(\Omega)$, $r > n/(n-1)$, for all $R > \delta(\Omega^c)$ it holds that*

$$\boldsymbol{v} \in D^{1,r}(\Omega^R), \quad p \in L^r(\Omega^R),$$

where p is the pressure field associated to $\boldsymbol{v}$ by Lemma VII.1.1.

Proof. By a reasoning analogous to that employed in Theorem V.5.3, one obtains

$$\boldsymbol{v} \in W^{1,r}_{loc}(\Omega), \quad p \in L^r_{loc}(\Omega). \tag{VII.7.43}$$

(We leave details to the reader.) Afterward, we recall that $\boldsymbol{u} = \varphi\boldsymbol{v}$ and $\pi = \varphi p$ (with φ "cut-off" function defined in Theorem V.5.3) obey problem (VII.7.1), (VII.7.2) in $\mathbb{R}^n$. Using (VII.7.43) and proceeding again as in the proof of Theorem V.5.2, one shows that if $r > n/(n-1)$,

$$\boldsymbol{F} \in D_0^{-1,r}(\mathbb{R}^n), \quad g \in L^r(\mathbb{R}^n). \tag{VII.7.44}$$

Furthermore, denoting by ϕ an arbitrary element from $C_0^\infty(\mathbb{R}^n)$, we deduce

$$|(g,\phi)| \le c\|\boldsymbol{v}\|_{r,\Omega_R}\|\phi\|_{r',\Omega_R} \le c_1\|\boldsymbol{v}\|_{r,\Omega_R}\|\phi\|_{nr'/(n-r'),\Omega_R},$$

and therefore, by the Sobolev inequality,

$$g \in D_0^{-1,r}(\mathbb{R}^n). \tag{VII.7.45}$$

From (VII.7.44), (VII.7.45), and Theorem VII.4.2 we infer the existence of a solution $\boldsymbol{w}, \tau$ to (VII.7.1), (VII.7.2) with

$$\boldsymbol{w} \in D^{1,r}(\mathbb{R}^n), \quad \tau \in L^r(\mathbb{R}^n).$$

The theorem will be therefore proved if we show $\nabla \boldsymbol{u} \equiv \nabla \boldsymbol{w}$. To this end, if we set $\boldsymbol{z} = \boldsymbol{w} - \boldsymbol{u}$ and $s = \tau - \pi$, it follows that $\boldsymbol{z}, s$ is a solution of class C^∞ to the homogeneous Oseen system in $\mathbb{R}^n$ and, by Lemma VII.6.3, we have, for all multi-index α, all $x \in \Omega$ and all $d > 0$

$$D^\alpha z_j(x) = -\int_{B_d(x)} \mathcal{H}_{ij}^{(d)}(x-y) D^\alpha z_i(y) dy.$$

Choosing $|\alpha| = \ell + 1$, and using the asymptotic properties (VII.6.12) of the function $\mathcal{H}_{ij}^{(d)}$, from this identity, with the help of the Hölder inequality, we derive

$$|D^\alpha z_j(x)| \le c \left(d^{-(n+1+\ell)/2} d^{n(1-1/q)} |\boldsymbol{u}|_{1,q} + d^{-(n+1+\ell)/2} d^{n(1-1/r)} |\boldsymbol{w}|_{1,r} \right).$$

In this relation we take

$$\ell > \max\{-1 + n(q-2)/q, -1 + n(r-2)/2\}$$

and then let $d \to \infty$ to obtain

$$D^\alpha z_j(x) = 0, \quad \text{for all } x \in \mathbb{R}^n, \ |\alpha| = \ell + 1.$$

As a consequence, $\nabla z_j(x) = \nabla(w_j - u_j)(x)$ must be a polynomial $\mathcal{P}(x)$ of degree $\ell - 1$. However, since

$$|\nabla \boldsymbol{w}|^r + |\nabla \boldsymbol{u}|^q \in L^1(\mathbb{R}^n),$$

there exists at least a sequence R_k such that

$$\lim_{R_k \to \infty} \int_{S_n} \left(|\nabla \boldsymbol{w}(R_k, \omega)| + |\nabla \boldsymbol{u}(R_k, \omega)| \right) d\omega = 0,$$

implying $\mathcal{P}(x) \equiv 0$, which completes the proof of the theorem. □

VII.8 Limit of Vanishing Reynolds Number. Transition to the Stokes Problem

In this last section we shall consider the behavior of solutions to the Oseen problem in the limit $\mathcal{R} \to 0$. Although most of the results we find apply (even in a stronger form) to three-dimensional flow, here we shall be mainly interested in plane motions. This is because, in such a case, the limiting process is fairly more interesting, giving rise to a *singular* perturbation problem, which throws additional light on the Stokes paradox. Concerning three-dimensional flow, we thus refer the reader to Chapter IX, directly in the nonlinear context.

Though differing in the treatment, the basic ideas presented in this section are due to Finn & Smith (1967a).

Let us consider the following Oseen problem[1]:

$$\left.\begin{aligned} \Delta \boldsymbol{v} + \mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1} &= \nabla p \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\boldsymbol{v} = \boldsymbol{v}_* \ \text{ at } \partial\Omega \tag{VII.8.1}$$
$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = \boldsymbol{0},$$

where Ω is a smooth exterior domain of $\mathbb{R}^2$ and $\boldsymbol{v}_*$ is a prescribed regular function on the boundary.[2] By virtue of Theorem VII.5.1, we know that there is one and only one solution $\boldsymbol{v}, p$ to (VII.8.1) which, by Theorem VII.1.1, is of class $C^\infty(\Omega)$. Moreover, this solution satisfies the uniform bound

$$|\boldsymbol{v}|_{1,2} \le c_1(1+B)\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \tag{VII.8.2}$$

for all $\mathcal{R} \in (0, B]$ and with c_1 independent of $\mathcal{R}$. Fix $R_1 > \delta(\Omega^c)$. From Theorem IV.4.1 and Theorem IV.5.1 we obtain the following estimates for $\boldsymbol{v}$:

$$\begin{aligned} &\|\boldsymbol{v}\|_{2,2,\Omega_{R_2}} + \|p\|_{1,2,\Omega_{R_2}} \\ &\quad\le c_2\left(\mathcal{R}\left\|\frac{\partial \boldsymbol{v}}{\partial x_1}\right\|_{2,\Omega_{R_1}} + \|\boldsymbol{v}\|_{1,2,\Omega_{R_1}} + \|p\|_{2,\Omega_{R_1}} + \|\boldsymbol{v}_*\|_{3/2,2(\partial\Omega)}\right), \end{aligned} \tag{VII.8.3}$$

where $\delta(\Omega^c) < R_2 < R_1$. Using (VII.7.14), with $q = 2$, (VII.8.2), and (VII.8.3) we thus recover the inequality

$$\|\boldsymbol{v}\|_{2,2,\Omega_{R_2}} + \|p\|_{1,2,\Omega_{R_2}} \le c_3(B, \boldsymbol{v}_*), \tag{VII.8.4}$$

where p has been possibly modified by adding a suitable constant depending on Ω_{R_1}. Again applying Theorem IV.4.1 and Theorem IV.5.1, for $\delta(\Omega^c) < R_3 < R_2$ we deduce

$$\begin{aligned} &\|\boldsymbol{v}\|_{3,2,\Omega_{R_3}} + \|p\|_{2,2,\Omega_{R_3}} \\ &\quad\le c_4\left(\mathcal{R}\left\|\frac{\partial \boldsymbol{v}}{\partial x_1}\right\|_{1,2,\Omega_{R_2}} + \|\boldsymbol{v}\|_{1,2,\Omega_{R_2}} + \|p\|_{2,\Omega_{R_2}} + \|\boldsymbol{v}_*\|_{5/2,3(\partial\Omega)}\right) \end{aligned} \tag{VII.8.5}$$

[1] For simplicity, we shall restrict ourselves to the case of zero body force. Extension of the results to nonzero $\boldsymbol{f}$ presents no conceptual difficulty and is therefore left to the reader as an exercise.

[2] It will become clear from the context how smooth Ω and $\boldsymbol{v}$ must be.

and so, using (VII.8.4), we obtain

$$\|\boldsymbol{v}\|_{3,2,\Omega_{R_3}} + \|p\|_{2,2,\Omega_{R_3}} \le c_5(B, \boldsymbol{v}_*).$$

Iterating this procedure we therefore establish the following inequalities for all $m \ge 0$:

$$\|\boldsymbol{v}\|_{m+2,2,\Omega_{R_{m+2}}} + \|p\|_{m+1,2,\Omega_{R_{m+2}}} \le C_m(B, \boldsymbol{v}_*), \tag{VII.8.6}$$

with $\delta(\Omega^c) < R_{m+2} < R_{m+1}$. We now let $\mathcal{R} \to 0$ along a sequence $\{\mathcal{R}_k\}$, say, and denote by $\{\boldsymbol{v}_k, p_k\}$ the corresponding solutions. In view of (VII.8.2), from the weak compactness of the space $\dot{D}^{1,2}$ (see Exercise II.6.2) it follows that, at least along a subsequence,

$$\nabla \boldsymbol{v}_k \overset{w}{\to} \nabla \boldsymbol{w} \;\text{ in } L^2, \tag{VII.8.7}$$

for some $\boldsymbol{w} \in D^{1,2}(\Omega)$. In addition, given arbitrary $\rho > \delta(\Omega^c)$, from the embedding Theorem II.3.4 and the compactness results of Exercise II.5.8, we infer that $\boldsymbol{w} \in C^2(\overline{\Omega}_\rho)$ and that, for some $\pi \in C^1(\overline{\Omega}_\rho)$, along a subsequence of the previous one, it holds that

$$\begin{aligned} &\boldsymbol{v}_k \to \boldsymbol{w} \;\text{ in } C^2(\overline{\Omega}_\rho) \\ &p_k \to \pi \;\text{ in } C^1(\overline{\Omega}_\rho). \end{aligned} \tag{VII.8.8}$$

From (VII.8.1) (written for $\boldsymbol{v} = \boldsymbol{v}_k, p = p_k$), (VII.8.7), and (VII.8.8) we conclude that the limit functions $\boldsymbol{w}, \pi$ obey the following Stokes system:

$$\left.\begin{aligned} \Delta \boldsymbol{w} &= \nabla \pi \\ \nabla \cdot \boldsymbol{w} &= 0 \end{aligned}\right\} \text{ in } \Omega \qquad \begin{aligned} &\boldsymbol{w} = \boldsymbol{v}_* \;\text{ at } \partial\Omega \\ &|\boldsymbol{w}|_{1,2} < \infty. \end{aligned} \tag{VII.8.9}$$

Because of Theorem V.2.2, $\boldsymbol{w}$ is uniquely determined since it is the *only* solution to $(\text{VII.8.9})_{1,2,3}$ verifying $(\text{VII.8.9})_4$. Therefore, (VII.8.7) and (VII.8.8) are verified not only along a subsequence but as long as $\mathcal{R} \to 0$. We shall next prove that in the limiting process the continuity of the datum at infinity $(\text{VII.8.1})_4$ is generally lost. Actually, setting

$$\boldsymbol{\mathcal{I}}(\boldsymbol{a}) \equiv \int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{a}, s) \cdot \boldsymbol{n}$$

with s the pressure field associated to the velocity field $\boldsymbol{a}$, by the results of Theorem V.3.2 and Exercise V.3.2, we know that the solution $\boldsymbol{w}$ verifies the following conditions:

$$\boldsymbol{\mathcal{I}}(\boldsymbol{w}) = 0$$

$$w_j(x) = w_{0j} + \int_{\partial\Omega} [\boldsymbol{v}_*(y) T_{i\ell}(\boldsymbol{u}_j, q_j)(x-y) \\ - U_{ij}(x-y) T_{i\ell}(\boldsymbol{w}, \pi)(y)] n_\ell(y) d\sigma_y, \tag{VII.8.10}$$

for all $x \in \Omega$ and for some $\boldsymbol{w}_0 \in \mathbb{R}^2$ *that is in general not zero.* The next step is to investigate how relations (VII.8.10) come out from the limit process and, in particular, the meaning of the vector $\boldsymbol{w}_0$. As we shall see, this vector which within the Stokes approximation has apparently no clear meaning, is due to the fact that, as $\mathcal{R} \to 0$, problem (VII.8.1) becomes singular in the sense that the value at infinity is in general not preserved. Actually, by Theorem VII.6.2 we have the representation

$$v_j(x) = \int_{\partial\Omega} [\boldsymbol{v}_*(y) T_{i\ell}(\boldsymbol{w}_j, e_j)(x-y) \\ - E_{ij}(x-y) T_{i\ell}(\boldsymbol{v}, p)(y) - \mathcal{R} v_{*i} E_{ij}(x-y)\delta_{1\ell}] n_\ell(y) d\sigma_y. \tag{VII.8.11}$$

From (VII.3.36) and (VII.8.11), it follows that

$$v_j(x) = \frac{1}{4\pi} \mathcal{I}_i(\boldsymbol{v}) \log \frac{1}{\mathcal{R}} + \int_{\partial\Omega} [v_{*i}(y) T_{i\ell}(\boldsymbol{u}_j, q_j)(x-y) \\ - U_{ij}(x-y) T_{i\ell}(\boldsymbol{v}, p)(y)] n_\ell(y) d\sigma_y + o(1) \quad \text{as } \mathcal{R}|x-y| \to 0. \tag{VII.8.12}$$

Since, by (VII.8.8), for any fixed $x \in \Omega$ all terms in this relation tend to finite limits as $\mathcal{R} \to 0$, this must be the case also for the first term on the right-hand side of (VII.8.12). Thus, in particular,

$$\mathcal{I}_i(\boldsymbol{v}) \to 0, \quad \text{as } \mathcal{R} \to 0,$$

and we recover $(\text{VII.8.10})_1$. By the same token, from (VII.8.12) we deduce $(\text{VII.8.10})_2$, where

$$\boldsymbol{w}_0 = \frac{1}{4\pi} \lim_{\mathcal{R}\to 0} \mathcal{I}(\boldsymbol{v}) \log \frac{1}{\mathcal{R}}, \tag{VII.8.13}$$

which furnishes the desired characterization of the field $\boldsymbol{w}_0$. It is interesting to observe that, according to the results of Section V.7, the vector $\boldsymbol{w}_0$ is in general not zero and that it *is* zero if and only if the restriction (V.7.2) on $\boldsymbol{v}_*$ is satisfied. Therefore, in such a case and only in such a case the limiting process preserves the condition at infinity.

In the final part of this section we wish to derive a fundamental estimate for the integral $\boldsymbol{\mathcal{I}}(\boldsymbol{v})$ that will play an essential role in the existence of solutions to the nonlinear exterior plane problem. Specifically, we have

Theorem VII.8.1 *Let Ω be a two-dimensional exterior domain of class C^2. Assume for some $q \in (1, 2]$*

$$\boldsymbol{v}_* \in W^{2-1/q,q}(\partial\Omega)$$

and denote by $\boldsymbol{v}$, p *the corresponding solution to* (VII.8.1). *Then, there exist* $\overline{B} > 0$ *and* $c = c(\Omega, q, \overline{B}) > 0$ *such that*

$$\left| \int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{v}, p) \cdot \boldsymbol{n} \right| \le c |\log \mathcal{R}|^{-1} \|\boldsymbol{v}_*\|_{2-1/q,q(\partial\Omega)},$$

for all $\mathcal{R} \in (0, \overline{B}]$.

Proof. Fix $R_1 > R_2 > \delta(\Omega^c)$. Using Theorem IV.4.1 and Theorem IV.5.1 into $(\text{VII.8.1})_{1,2}$ we find

$$\|\boldsymbol{v}\|_{2,q,\Omega_{R_2}} + \|p\|_{2,q,\Omega_{R_2}} \le c_1 \big(\|\boldsymbol{v}\|_{1,q,\Omega_{R_1}} + \|p\|_{q,\Omega_{R_1}} + \|\boldsymbol{v}_*\|_{2-1/q,q(\partial\Omega)} \big). \tag{VII.8.14}$$

By the trace Theorem II.4.4 it is

$$\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \le c_2 \|\boldsymbol{v}_*\|_{2-1/q,q(\partial\Omega)}$$

and, therefore, (VII.8.2) yields

$$|\boldsymbol{v}|_{1,2} \le c_3 (1 + B) \|\boldsymbol{v}_*\|_{2-1/q,q(\partial\Omega)}. \tag{VII.8.15}$$

Moreover, from Theorem II.3.4 and inequality (II.5.18) we have

$$\|\boldsymbol{v}\|_{q,\Omega_{R_1}} \le c_4 \big(|\boldsymbol{v}|_{1,2} + \|\boldsymbol{v}_*\|_{2-1/q,q(\partial\Omega)} \big)$$

which, in turn, with the help of (VII.8.2), furnishes

$$\|\boldsymbol{v}\|_{q,\Omega_{R_1}} \le c_5 \|\boldsymbol{v}_*\|_{2-1/q,q(\partial\Omega)}. \tag{VII.8.16}$$

Also, using the estimate (VII.7.14) for the pressure field together with (VII.8.15) and (VII.8.16), it follows that

$$\|p\|_{q,\Omega_{R_1}} \le c_6 \|\boldsymbol{v}_*\|_{2-1/q,q(\partial\Omega)}, \tag{VII.8.17}$$

with $c_6 = c_6(\Omega, q, B)$. Collecting (VII.8.14)–(VII.8.17), we then conclude

$$\|\boldsymbol{v}\|_{2,q,\Omega_{R_2}} + \|p\|_{2,q,\Omega_{R_2}} \le c_7 \|\boldsymbol{v}_*\|_{2-1/q,q(\partial\Omega)} \tag{VII.8.18}$$

with $c_7 = c_7(\Omega, q, B)$. With the help of (VII.8.18) we can now show the desired estimate. Actually, from (VII.3.36) follows the existence of $B_1 > 0$ such that

$$\begin{aligned} |\boldsymbol{E}(x-y)| &\le |\boldsymbol{U}(x-y)| + \frac{1}{4\pi} |\log \mathcal{R}| + c_8 \\ |D_k \boldsymbol{E}(x-y)| &\le |D_k \boldsymbol{U}(x-y)| + c_9 \end{aligned} \tag{VII.8.19}$$

for all $x \in \Omega_{R_1,R_2}$, all $y \in \partial\Omega$, and all $\mathcal{R} \in (0, B_1]$, and with c_8 and c_9 depending on Ω_{R_1,R_2}, $\partial\Omega$, and B_1 but otherwise independent of $\mathcal{R}$. Since, clearly,

$$\begin{aligned} &|\boldsymbol{U}(x-y)| + |D_k \boldsymbol{U}(x-y)| \le c_9 \\ &|\boldsymbol{e}_j(x-y)| \le c_{10} \end{aligned} \qquad x \in \Omega_{R_1,R_2}, \ \ y \in \partial\Omega, \tag{VII.8.20}$$

from (VII.8.11), (VII.8.19), and (VII.8.20) we derive for all $x \in \Omega_{R_1,R_2}$ and all $\mathcal{R} \in (0, B_1]$

$$\mathcal{J}(\mathcal{R}) \equiv |\log \mathcal{R}| \left| \int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{v},p)\cdot\boldsymbol{n} \right| \le |\boldsymbol{v}(x)| + c_{11} \int_{\partial\Omega} [|\boldsymbol{T}(\boldsymbol{v},p)\cdot\boldsymbol{n}| + |\boldsymbol{v}_*|] \tag{VII.8.21}$$

with c_{11} independent of $\mathcal{R}$. Employing trace Theorem II.4.1 in the integral on the right-hand side of (VII.8.21) in conjunction with (VII.8.18) we find

$$\mathcal{J}(\mathcal{R}) \le |\boldsymbol{v}(x)| + c_{12} \|\boldsymbol{v}_*\|_{2-1/q,q(\partial\Omega)}.$$

Integrating both sides of this relation over Ω_{R_1,R_2} and using the Hölder inequality, we deduce

$$\mathcal{J}(\mathcal{R}) \le c_{13} \left(\|\boldsymbol{v}\|_{q,\Omega_{R_1}} + \|\boldsymbol{v}_*\|_{2-1/q,q(\partial\Omega)} \right). \tag{VII.8.22}$$

The desired estimate is then a consequence of (VII.8.16) and (VII.8.22). □

VII.9 Notes for the Chapter

Section VII.1. The first complete treatment of existence and uniqueness of the Oseen problem in exterior domains is due to Faxén (1928/1929), who generalized the method introduced by Odqvist in his thesis for the Stokes problem (Odqvist 1930).

The variational formulation (VII.1.1) is taken from Finn (1965a). Theorem VII.1.2 is due to me.

Section VII.2. Theorem VII.2.1 generalizes an analogous result of Finn (1965a, Theorem 2.5).

Section VII.4. Theorem VII.4.1 is a detailed and expanded version of an analogous one given by Galdi (1992). The case where $m = 0$, $n = 3$, $q \in (1, 4)$, and $g \equiv 0$ was first proved by Babenko (1973). Other L^q-estimates for $n = 3$ can be found in Salvi (1991, Theorem 4).

Lemma VII.4.2 and Theorem VII.4.2 are due to me.

Section VII.5. Theorem VII.5.1 is due to me.

Section VII.6. Lemma VII.6.3 is essentially due to Fujita (1961), while Theorem VII.6.2 is an extension of a classical result of Chang & Finn (1961).

Section VII.7. Most of the results of this section are an expanded version of those given by Galdi (1992). Theorem VII.7.3 was, however, proved by me in the first edition of this book.

Existence and uniqueness results for three-dimensional flows in weighted anisotropic Sobolev spaces with weights reflecting the decay properties of the fundamental solution have been proved by Farwig (1992a, 1992b) and Shibata (1999). Generalization of results obtained by these authors are given by Kračmar, Novotný & Pokorný (2001). These authors also provide very detailed estimates for the Oseen fundamental solution in dimensions 2 and 3. Estimates for the Oseen volume potentials in weighted Hölder spaces have been studied by Solonnikov (1996), and in weighted anisotropic Lebesgue spaces by Kračmar, Novotný & Pokorný (2001); see also Kračmar, Novotný & Pokorný (1999).

A modified Oseen problem that contains an "anisotropic" second derivative and that is relevant in the study of certain non-Newtonian liquid models has been investigated by Farwig, Novotný, & Pokorný (2000).

A boundary integral approach to the existence and uniqueness of solution to the Oseen problem is addressed by Fischer, Hsiao & Wendland (1985).

A detailed analysis of different problems and results related to two-dimensional flows can be found in the review article of Olmstead & Gautesen (1976) and in the references included therein.

VIII

Steady Generalized Oseen Flow in Exterior Domains

. . . e quelle anime liete,
si fero spere sopra fissi poli,
fiammando, volte, a guisa di comete.

DANTE, Paradiso XIV, vv. 10-12

Introduction

The Oseen approximation, which we analyzed to a large extent in the previous chapter, aims at describing the motion of a Navier–Stokes liquid around a rigid body, $\mathcal{B}$, that moves with a constant and "sufficiently small" purely translational velocity. However, assuming this kind of motion for $\mathcal{B}$ might be restrictive in several significant applied problems, occurring on both large and small scales, where $\mathcal{B}$ is allowed to translate *and to rotate*. Typical examples of these problems are furnished by the orientation of rigid bodies in the stream of a viscous liquid, and by the self-propulsion of microorganisms in a viscous liquid; see Galdi (2002) for a review of these and related questions.

In order to appropriately describe situations in which $\mathcal{B}$ moves by a generic, but "small," rigid motion, one needs the more general approximation that we introduced in (VII.0.1), which we will call *generalized Oseen approximation.* For the reader's sake, we reproduce the relevant equations here:

$$\left.\begin{aligned}\nu\Delta\boldsymbol{v}+\boldsymbol{v}_0\cdot\nabla\boldsymbol{v}+\boldsymbol{\omega}\times\boldsymbol{x}\cdot\nabla\boldsymbol{v}-\boldsymbol{\omega}\times\boldsymbol{v}&=\nabla p+\boldsymbol{f}\\ \nabla\cdot\boldsymbol{v}&=0\end{aligned}\right\}\ \text{in }\Omega\,,\qquad \boldsymbol{v}=\boldsymbol{v}_*\ \text{ at }\partial\Omega, \tag{VIII.0.1}$$

along with the condition at infinity

G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems*, Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-09620-9_8,

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = \boldsymbol{0}\,. \tag{VIII.0.2}$$

We recall that in (VIII.0.1), $\boldsymbol{v}_0$ and $\boldsymbol{\omega}$ are given constant vectors representing the translational and angular velocity, respectively, of the rigid motion of $\mathcal{B}$.

We shall *assume that Ω is an exterior domain of* $\mathbb{R}^3$, and refer the reader to the Notes for this chapter for some remarks and relevant bibliography related to the two-dimensional case.

In order to describe problems and results, it is convenient to rewrite (VIII.0.1) and (VIII.0.2) in a suitable dimensionless form. To this end, we assume, without loss, that $\boldsymbol{\omega}$ is directed along the positive x_1-axis, that is, $\boldsymbol{\omega} = \omega\,\boldsymbol{e}_1$, while $\boldsymbol{v}_0 = v_0\boldsymbol{e}$, $v_0 \geq 0$.[1] Moreover, we scale the length with $d = \delta(\Omega^c)$, and the velocity with v_0, if $v_0 \neq 0$, and with $\omega\, d$ otherwise. Therefore, introducing the dimensionless numbers

$$\mathcal{R}' = \frac{v_0 d}{\nu} \;\;\text{(Reynolds number)}\quad \mathcal{T} = \frac{\omega d^2}{\nu} \;\;\text{(Taylor number)}\,, \tag{VIII.0.3}$$

the system (VIII.0.1) assumes the following form

$$\left.\begin{aligned} \Delta\boldsymbol{v} + \mathcal{R}'\boldsymbol{e}\cdot\nabla\boldsymbol{v} + \mathcal{T}\,(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{v} - \boldsymbol{e}_1\times\boldsymbol{v}) = \nabla p + \boldsymbol{f} \\ \nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega\,,$$
$$\boldsymbol{v} = \boldsymbol{v}_* \;\text{ at } \partial\Omega, \tag{VIII.0.4}$$

where now $\boldsymbol{v}$, $\boldsymbol{v}_*$, p, and $\boldsymbol{f}$ are nondimensional quantities.[2] If $\Omega \equiv \mathbb{R}^3$ the above choice of d is no longer possible, but we can still give a meaning to (VIII.0.4), which is what we shall do hereinafter.

At this point we observe that, in general, $\boldsymbol{\omega}$ and $\boldsymbol{v}_0$, that is, $\boldsymbol{e}_1$ and $\boldsymbol{e}$, have different directions. However, by shifting the coordinate system by a constant quantity, we can always reduce the original equations to new ones where $\boldsymbol{e} = \boldsymbol{e}_1$. This change of coordinates, known as Mozzi–Chasles transformation (see Mozzi 1763, Chasles 1830), reads as follows:

$$\boldsymbol{x}^* = \boldsymbol{x} - \lambda\,\boldsymbol{e}_1\times\boldsymbol{e}\,,\quad \lambda := \frac{\mathcal{R}'}{\mathcal{T}} \equiv \frac{v_0}{\omega\, d}. \tag{VIII.0.5}$$

Thus, defining

$$\begin{aligned} &\Omega^* = \left\{\boldsymbol{x}^*\in\mathbb{R}^3 : \boldsymbol{x}^* = \boldsymbol{x} - \lambda\boldsymbol{e}_1\times\boldsymbol{e}\,,\text{ for some } \boldsymbol{x}\in\Omega\right\}\,,\\ &\boldsymbol{v}^*(x^*) = \boldsymbol{v}(\boldsymbol{x}^* + \lambda\boldsymbol{e}_1\times\boldsymbol{e})\,,\quad p^*(x^*) = p(\boldsymbol{x}^* + \lambda\boldsymbol{e}_1\times\boldsymbol{e})\,,\\ &\boldsymbol{f}^*(x^*) = \boldsymbol{f}(\boldsymbol{x}^* + \lambda\boldsymbol{e}_1\times\boldsymbol{e})\,,\\ &\mathcal{R} = \mathcal{R}'\boldsymbol{e}\cdot\boldsymbol{e}_1\,, \end{aligned} \tag{VIII.0.6}$$

[1] Of course, we suppose $\boldsymbol{\omega} \neq \boldsymbol{0}$; otherwise, the analysis reduces to that already performed in Chapter V, when $\boldsymbol{v}_0 = \boldsymbol{0}$, and in Chapter VII, when $\boldsymbol{v}_0 \neq \boldsymbol{0}$.

[2] Notice that, in contrast to the Oseen problem studied in the previous chapter, here the body force has been scaled in such a way that dimensionless body force is $-\boldsymbol{f}$ instead of $-\mathcal{R}'\boldsymbol{f}$. This is because we may allow $\mathcal{R}' = 0$, while keeping a nonzero force term.

the system (VIII.0.4) becomes (with stars omitted)

$$\left.\begin{aligned}\Delta \boldsymbol{v} + \mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1} + \mathcal{T}\left(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}\right) = \nabla p + \boldsymbol{f} \\ \nabla \cdot \boldsymbol{v} = 0\end{aligned}\right\} \text{ in } \Omega \qquad \text{(VIII.0.7)}$$

$$\boldsymbol{v} = \boldsymbol{v}_* \ \text{ at } \partial\Omega,$$

Throughout this chapter, we shall therefore focus on the resolution of problem (VIII.0.7), (VIII.0.2).

We would like to point out the following interesting feature of (VIII.0.7). In view of what we have found in the previous chapter, we may guess that the wake-like behavior of the velocity field $\boldsymbol{v}$ at large distances is produced by the term $\mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1}$. Now, this term is zero whenever the "effective" Reynolds number $\mathcal{R}$ vanishes. In view of the Mozzi-Chasles transformation, this happens not only when (as intuitively expected) $\boldsymbol{v}_0 = 0$, but, more generally, when $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0$, namely, when the translational velocity of the body is perpendicular to its angular velocity. In fact, as we shall prove later on, the formation of a "wake" is possible, in a suitable sense, if and only if $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq 0$; see Section VIII.6. For a physical interpretation of this circumstance, we refer to the Introduction to Chapter XI.

The study of the mathematical properties of the solutions to (VIII.0.7), (VIII.0.2) is, in principle, much more challenging than the analogous study performed for the Oseen problem (VII.0.1), (VII.0.2), the main reason being the presence, in $(\text{VIII.0.7})_1$, of the term $\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v}$, whose coefficient becomes unbounded as $|x| \to \infty$. One important consequence of this fact is that (VIII.0.7) can by no means be viewed as a perturbation of (VII.0.1), even for "small" $\mathcal{T}$ ("small" angular velocities, that is).

Despite this difficulty, one is able to prove, with relative ease, the existence of at least one generalized solution to (VIII.0.7), (VIII.0.2) and to show that such a solution is smooth, provided the data are equally smooth; see Section VIII.1. As in the case of the Oseen approximation, the generalized solution is constructed via the Galerkin method, with the help of a suitable basis and of an appropriate a priori estimate for the Dirichlet norm of $\boldsymbol{v}$; see Theorem VIII.1.2. This estimate can be established thanks to the *crucial* fact that (as the reader will immediately verify)

$$\int_\Omega \left(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{\varphi} \cdot \boldsymbol{\varphi} - \boldsymbol{e}_1 \times \boldsymbol{\varphi} \cdot \boldsymbol{\varphi}\right) = 0\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega)\,.$$

With a view to applications to the full nonlinear problem, as in the case of the Oseen approximation, also in the case at hand the next important question to investigate is the behavior at large distances of generalized solutions. This problem already arises, naturally, in the study of the uniqueness of generalized solutions, which is here established by a method different from those used in Theorem V.2.1 and Theorem VII.1.2 for the Stokes and Oseen approximation, respectively. In fact, this method relies chiefly on a sharp result concerning the asymptotic behavior of the pressure field associated with a generalized solution corresponding to a body force that is square-summable in Ω^R, for sufficiently large R; see Section VIII.2.

Unlike the analogous problems for the Stokes and Oseen approximations analyzed in Section V.3 and Section VII.6, the study of the asymptotic behavior of generalized solutions to (VIII.0.7), (VIII.0.2) does not appear to be feasible by means of the classical method based on volume potential representations along with asymptotic estimates of the fundamental solution. Actually, based on heuristic considerations, we are expecting that the velocity field decays like $|x|^{-1}$, uniformly for large $|x|$. However, as shown by Farwig, Hishida & Müller (2004, Proposition 2.1), see also Hishida (2006, Proposition 4.1), the fundamental tensor solution, $\mathfrak{G} = \mathfrak{G}(x, y)$, associated with the equations (VIII.0.1)$_1$ (with $\Omega \equiv \mathbb{R}^3$) does *not* satisfy a uniform estimate of the type

$$|\mathfrak{G}(x, y)| \leq C|x - y|^{-1}, \quad \text{for } all\ x, y \in \mathbb{R}^3 ,$$

with C independent of x, y, at least in the case $\mathcal{R} = 0$, which is, in fact, the most complicated.

Therefore, we will argue in a different way.

The approach we shall follow to study the asymptotic behavior of generalized solutions and to show the corresponding estimates is treated in Section VIII.3 through Section VIII.6. It was first introduced by Galdi (2003) and then further generalized and improved by Galdi & Silvestre (2007a, 2007b). It develops according to the following steps. In the first step, by means of a "cut-off" technique that we have already used in previous chapters, we reduce the original problem to an analogous one in the whole space. At this stage, the above-mentioned uniqueness property plays a fundamental role, because we can then identify the generalized solution in the whole space problem with the original one, for sufficiently large $|x|$, $x \in \Omega$. In the second step, we consider the *time-dependent* version of (VIII.0.7) in the whole-space (Cauchy problem) corresponding to the same "body force" and to zero initial data; see (VIII.5.8). We thus show that at each time $t \geq 0$, the velocity field, $\boldsymbol{u} = \boldsymbol{u}(x, t)$, of the corresponding solution decays at least like $|x|^{-1}$ for large $|x|$, and that it is *uniformly* bounded in time by a function that exhibits the same spatial decay properties as the Stokes or Oseen fundamental tensor, depending on whether $\mathcal{R}$ is zero or not zero, respectively. Therefore, in this sense, $\boldsymbol{u}$ shows a "wake-like" feature whenever $\mathcal{R} \neq 0$. Successively, thanks to these pointwise, uniform (in time) estimates, we may thus pass to the limit $t \to \infty$ and show, on the one hand, that $\boldsymbol{u}(x, t)$ converges, uniformly pointwise, to the generalized solution $\boldsymbol{v} = \boldsymbol{v}(x)$ of the steady-state problem, and that on the other hand, $\boldsymbol{v}$ has the same asymptotic properties and that it obeys the same estimates as $\boldsymbol{u}$ does.

In the last two sections of the chapter, we will investigate the summability properties of generalized solutions in homogeneous Sobolev spaces, together with corresponding estimates, when $\boldsymbol{f}$ belongs to the Lebesgue space L^q, for suitable values of q, and $\boldsymbol{v}_*$ is in the appropriate trace space. As we shall see, these results are, formally, completely analogous to those shown in Theorem VII.7.1 for the Oseen problem with $m = 0$ and $n = 3$.

Unless otherwise stated, throughout this chapter we shall always assume $\mathcal{T} > 0$,[3] *whereas we take* $\mathcal{R} \ge 0$.

VIII.1 Generalized Solutions. Regularity and Existence

We begin with a weak formulation of the generalized Oseen problem through a, by now, familiar procedure. Thus, multiplying $(\text{VIII.0.7})_1$ by $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and integrating by parts, we obtain

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) - \mathcal{R}\left(\frac{\partial \boldsymbol{v}}{\partial x_1}, \boldsymbol{\varphi}\right) - \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}, \boldsymbol{\varphi}) = -[\boldsymbol{f}, \boldsymbol{\varphi}]. \quad \text{(VIII.1.1)}$$

Definition VIII.1.1. A vector field $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ is called a *q-weak* (or *q-generalized*) *solution* to (VIII.0.2), (VIII.0.7) if for some $q \in (1, \infty)$,

(i) $\boldsymbol{v} \in D^{1,q}(\Omega)$;
(ii) $\boldsymbol{v}$ is (weakly) divergence-free in Ω;
(iii) $\boldsymbol{v}$ assumes the value $\boldsymbol{v}_*$ at $\partial\Omega$ (in the trace sense) or, if the velocity at the boundary is zero, $\boldsymbol{v} \in D_0^{1,q}(\Omega)$;
(iv) $\displaystyle\lim_{|x|\to\infty} \int_{S^2} |\boldsymbol{v}(x)| = 0$;
(v) $\boldsymbol{v}$ satisfies (VIII.1.1) for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$.

As usual, if $q = 2$, $\boldsymbol{v}$ will be called simply a *weak* (or *generalized*) *solution* to (VIII.0.2), (VIII.0.7).

Remark VIII.1.1 If $\boldsymbol{v}$ is a q-weak solution then by Lemma II.6.1, $\boldsymbol{v} \in W^{1,q}_{loc}(\Omega)$, and if Ω is locally Lipschitz, then $\boldsymbol{v} \in W^{1,q}_{loc}(\overline{\Omega})$. Concerning (iii), see Remark V.1.1. Moreover, if $q \in (1, 3)$, then, by (iv) and Theorem II.6.1(i), it follows that $\boldsymbol{v} \in L^{3q/(3-q)}(\Omega)$, and that

$$\|\boldsymbol{v}\|_{3q/(3-q)} \le c\,|\boldsymbol{v}|_{1,q}\,,$$

where $c = c(q, \Omega)$. Finally, in regards to (iv), we notice that, if $\boldsymbol{f}$ is in $L^2(\Omega^\rho)$, for a sufficiently large ρ then it can be shown that every generalized solution tends to zero as $|x| \to \infty$, uniformly pointwise; see Exercise VIII.2.1. ■

If the function $\boldsymbol{f}$ has a sufficient degree of regularity, to each q-weak solution we can associate a corresponding pressure field in a way completely analogous to that used in Lemma IV.1.1. Specifically, we have the following result, whose proof we leave to the reader as an exercise.

[3] See footnote 1.

Lemma VIII.1.1 *Let Ω be an exterior domain in $\mathbb{R}^n$, $n \geq 2$. Suppose $\boldsymbol{f} \in W_0^{-1,q}(\Omega')$, $1 < q < \infty$, for any bounded subdomain Ω', with $\overline{\Omega'} \subset \Omega$. Then to every q-weak solution $\boldsymbol{v}$ we can associate a pressure field $p \in L^q_{loc}(\Omega)$ such that*

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) - \mathcal{R}\left(\frac{\partial \boldsymbol{v}}{\partial x_1}, \boldsymbol{\psi}\right) - \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}, \boldsymbol{\psi}) = (p, \nabla \cdot \boldsymbol{\psi}) - [\boldsymbol{f}, \boldsymbol{\psi}] \tag{VIII.1.2}$$

for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. Furthermore, if Ω is locally Lipschitz and $\boldsymbol{f} \in W_0^{-1,q}(\Omega_R)$, $R > \delta(\Omega^c)$, then $p \in L^q(\Omega_R)$.

Remark VIII.1.2 The result stated in Lemma VIII.1.1 for Ω locally Lipschitz is weaker than its counterpart for the Stokes problem, proved in Lemma V.1.1, for reasons that are analogous to that explained in Remark VII.1.2 for the Oseen problem. Basically, this is due to the fact that the functional

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) - \mathcal{R}\left(\frac{\partial \boldsymbol{v}}{\partial x_1}, \boldsymbol{\psi}\right) - \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}, \boldsymbol{\psi}) + [\boldsymbol{f}, \boldsymbol{\psi}]$$

is *not* continuous in $\boldsymbol{\psi} \in D_0^{1,q'}(\Omega)$ if we *merely* require $\boldsymbol{v} \in D^{1,q}(\Omega)$. In other words, under this assumption alone on $\boldsymbol{v}$ we cannot guarantee the existence of a constant $c = c(\boldsymbol{v}, \mathcal{R}, \mathcal{T})$ such that

$$\left| \mathcal{R}\left(\frac{\partial \boldsymbol{v}}{\partial x_1}, \boldsymbol{\psi}\right) + \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}, \boldsymbol{\psi}) \right| \leq c\,|\boldsymbol{\psi}|_{1,q'}\,, \text{ for all } \boldsymbol{\psi} \in C_0^\infty(\Omega)\ ,$$

and so, we cannot apply Corollary III.5.1, but only its weaker version given in Corollary III.5.2. ■

The following result furnishes regularity of q-weak solutions.

Theorem VIII.1.1 *Let $\boldsymbol{f} \in W^{m,q}_{loc}(\Omega)$, $m \geq 0$, $1 < q < \infty$, and let*

$$\boldsymbol{v} \in W^{1,q}_{loc}(\Omega), \quad p \in L^q_{loc}(\Omega),$$ [1]

with $\boldsymbol{v}$ weakly divergence-free, satisfy (VIII.1.2) *for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. Then*

$$\boldsymbol{v} \in W^{m+2,q}_{loc}(\Omega), \quad p \in W^{m+1,q}_{loc}(\Omega).$$

In particular, if $\boldsymbol{f} \in C^\infty(\Omega)$, then $\boldsymbol{v}, p \in C^\infty(\Omega)$. Furthermore, if Ω is of class C^{m+2} and

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[1] Actually, these assumptions are satisfied by any q-weak solution. In fact, they are implied by the following one:

$$\boldsymbol{v} \in L^1_{loc}(\Omega)\,, \;\; \nabla \boldsymbol{v} \in L^q_{loc}(\Omega), \;\; \text{with } \boldsymbol{v} \text{ satisfying (VIII.1.1) for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega).$$

For under this hypothesis, by Lemma II.6.1, $\boldsymbol{v} \in W^{1,q}_{loc}(\Omega)$ and then, by Lemma VIII.1.1, we deduce $p \in L^q_{loc}(\Omega)$; see also Remark VIII.1.2.

$$\boldsymbol{f} \in W^{m,q}_{loc}(\overline{\Omega}), \quad \boldsymbol{v}_* \in W^{m+2-1/q,q}(\partial\Omega),$$

then

$$\boldsymbol{v} \in W^{m+2,q}_{loc}(\overline{\Omega}), \quad p \in W^{m+1,q}_{loc}(\overline{\Omega}),$$

provided $\boldsymbol{v} \in W^{1,q}_{loc}(\overline{\Omega})$.[2] *In particular, if* Ω *is of class* C^∞, *and* $\boldsymbol{f} \in C^\infty(\Omega)$, $\boldsymbol{v}_* \in C^\infty(\partial\Omega)$, *then* $\boldsymbol{v}, p \in C^\infty(\overline{\Omega'})$, *for all bounded* $\Omega' \subset \Omega$.

Proof. The proof follows at once from Theorem IV.4.1 and Theorem IV.5.1 if one bears in mind that (VIII.1.2) can be viewed as a weak form of the Stokes equation with $\boldsymbol{f}$ replaced by $\boldsymbol{f} - \mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1} - \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v})$. □

We shall next establish the existence of generalized solutions. Precisely, we have the following.

Theorem VIII.1.2 *Let* Ω *be a three-dimensional exterior, locally Lipschitzian domain. Given*

$$\boldsymbol{f} \in D_0^{-1,2}(\Omega), \quad \boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega),$$

there exists at least one generalized solution to (VIII.0.2), (VIII.0.7). *This solution satisfies the estimates*[3]

$$\begin{aligned}
&\|\boldsymbol{v}\|_{2,\Omega_R} + |\boldsymbol{v}|_{1,2} \le c_1 \left\{ |\boldsymbol{f}|_{-1,2} + (1 + \mathcal{R} + \mathcal{T})\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \right\}, \\
&\int_{S^2} |\boldsymbol{v}(x)| = o\left(1/\sqrt{|x|}\right) \quad \text{as } |x| \to \infty, \qquad \text{(VIII.1.3)} \\
&\|p\|_{2,\Omega_R/\mathbb{R}} \le c_2 \left\{ |\boldsymbol{f}|_{-1,2} + (1 + \mathcal{R} + \mathcal{T})|\boldsymbol{v}|_{1,2} \right\},
\end{aligned}$$

for all $R > \delta(\Omega^c)$. *In* (VIII.1.3) p *is the pressure field associated to* $\boldsymbol{v}$ *by* Lemma VIII.1.1, *while* $c_i = c_i(R, \Omega)$ ($c_i \to \infty$ *as* $R \to \infty$).

Proof. The method of proof is very close to that of Theorem VII.2.1. We look for a solution of the form

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{V}_1 + \boldsymbol{\sigma}, \qquad \text{(VIII.1.4)}$$

where $\boldsymbol{V}_1$ and $\boldsymbol{\sigma}$ are solenoidal fields introduced in the proof of Theorem VII.2.1, which we will recall here for the reader's sake. Specifically, $\boldsymbol{V}_1 \in W^{1,2}(\Omega)$ is the solenoidal extension of $\boldsymbol{v}_* - \sigma|_{\partial\Omega}$ of bounded support in Ω constructed in the proof of Theorem V.1.1, while with the origin of coordinates in $\mathring{\Omega}^c$,

$$\begin{aligned}
\boldsymbol{\sigma} &= \frac{\Phi}{4\pi} \nabla \left(\frac{1}{|x|} \right), \\
\Phi &= \int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n}.
\end{aligned} \qquad \text{(VIII.1.5)}$$

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[2] Notice that any q-generalized solution enjoys this requirement, as a consequence of the stated assumptions on Ω and Remark VIII.1.2.

[3] See (IV.6.1) for the definition of the norm involving p.

Thus,

$$\begin{aligned}&|\boldsymbol{V}_1|_{1,2} \le c\,\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\\&D^\alpha\boldsymbol{\sigma} = O(1/|x|^{2+|\alpha|})\,,\ |\alpha| = 0,1, \ \text{ as } |x|\to\infty.\end{aligned} \tag{VIII.1.6}$$

As a consequence, $\boldsymbol{w}$ is required to be a member of $\mathcal{D}_0^{1,2}(\Omega)$ and to satisfy, for all $\boldsymbol{\varphi}\in\mathcal{D}(\Omega)$, the equation

$$\begin{aligned}&(\nabla\boldsymbol{w},\nabla\boldsymbol{\varphi}) - \mathcal{R}\left(\frac{\partial\boldsymbol{w}}{\partial x_1},\boldsymbol{\varphi}\right) - \mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{w} - \boldsymbol{e}_1\times\boldsymbol{w},\boldsymbol{\varphi})\\&= -[\boldsymbol{f},\boldsymbol{\varphi}] - (\nabla\boldsymbol{V}_1,\nabla\boldsymbol{\varphi}) + \mathcal{R}\left(\frac{\partial\boldsymbol{V}_\sigma}{\partial x_1},\boldsymbol{\varphi}\right) + \mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{V}_\sigma - \boldsymbol{e}_1\times\boldsymbol{V}_\sigma,\boldsymbol{\varphi}),\end{aligned} \tag{VIII.1.7}$$

where we set

$$\boldsymbol{V}_\sigma \equiv \boldsymbol{V}_1 + \boldsymbol{\sigma}.$$

However, we have the identity

$$\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{\sigma} - \boldsymbol{e}_1\times\boldsymbol{\sigma} = \boldsymbol{0}\,. \tag{VIII.1.8}$$

In fact, denoting by ε_{ijk} the *alternating symbol*[4] and by $\boldsymbol{L}$ the left-hand side of (VIII.1.8) times $4\pi/\Phi$, we obtain (with $r\equiv|x|$ and $i=1,2,3$)

$$L_i = \varepsilon_{k1\ell}\,x_\ell\,(3\,\frac{x_ix_k}{r^5} - \frac{\delta_{ik}}{r^3}) + \varepsilon_{i1m}\frac{x_m}{r^3} = 3\,\varepsilon_{213}\,\frac{x_ix_2x_3}{r^5} + 3\,\varepsilon_{312}\,\frac{x_ix_3x_2}{r^5} = 0\,.$$

Using (VIII.1.8) in (VIII.1.7), we deduce

$$\begin{aligned}&(\nabla\boldsymbol{w},\nabla\boldsymbol{\varphi}) - \mathcal{R}\left(\frac{\partial\boldsymbol{w}}{\partial x_1},\boldsymbol{\varphi}\right) - \mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{w} - \boldsymbol{e}_1\times\boldsymbol{w},\boldsymbol{\varphi})\\&= -[\boldsymbol{f},\boldsymbol{\varphi}] - (\nabla\boldsymbol{V}_1,\nabla\boldsymbol{\varphi}) + \mathcal{R}\left(\frac{\partial\boldsymbol{V}_1}{\partial x_1},\boldsymbol{\varphi}\right) + \mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{V}_1 - \boldsymbol{e}_1\times\boldsymbol{V}_1,\boldsymbol{\varphi})\,.\end{aligned} \tag{VIII.1.9}$$

It is clear that, provided we show the existence of a function $\boldsymbol{w}\in\mathcal{D}_0^{1,2}(\Omega)$ satisfying (VIII.1.9) for all $\boldsymbol{\varphi}\in\mathcal{D}(\Omega)$, the field (VIII.1.4) satisfies all requirements of a generalized solution to (VIII.0.2), (VIII.0.7) as given in Definition VIII.1.1. In fact, (VIII.1.6)$_2$, and the properties of $\boldsymbol{V}_1$ and $\boldsymbol{w}$, imply $\boldsymbol{v}\in D^{1,2}(\Omega)$. In addition, $\boldsymbol{v}$ is solenoidal and assumes the value $\boldsymbol{v}_*$ at the boundary. Finally, in view of (VIII.1.6)$_2$, we obtain

$$\int_{S^2}|\boldsymbol{v}(x)| \le \int_{S^2}|\boldsymbol{w}(x)| + O(1/|x|^2)\,, \tag{VIII.1.10}$$

[4] We recall that the alternating symbol is defined as follows: $\varepsilon_{ijk}=1$ if (i,j,k) is an even permutation of (1,2,3); $\varepsilon_{ijk}=-1$ if (i,j,k) is an odd permutation of (1,2,3), and $\varepsilon_{ijk}=0$ if any two of i,j,k are equal.

which, by Lemma II.6.2, delivers $(\mathrm{VIII.1.3})_2$. Thus, to establish the theorem it remains to prove the existence of the field $\boldsymbol{w}$ and the validity of estimates $(\mathrm{VIII.1.3})_{1,3}$. To this end, let $\{\boldsymbol{\varphi}_k\}$ be the base of $\mathcal{D}_0^{1,2}(\Omega)$ given in Lemma VII.2.1. We shall construct an "approximate solution" $\boldsymbol{w}_m$ to (VIII.1.9) in the following way:

$$\boldsymbol{w}_m = \sum_{\ell=1}^{m} \xi_{\ell m} \boldsymbol{\varphi}_\ell \,,$$

$$\begin{aligned}
&(\nabla \boldsymbol{w}_m, \nabla \boldsymbol{\varphi}_k) - \mathcal{R}\left(\frac{\partial \boldsymbol{w}_m}{\partial x_1}, \boldsymbol{\varphi}_k\right) - \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{w}_m - \boldsymbol{e}_1 \times \boldsymbol{w}_m, \boldsymbol{\varphi}_k) \\
&= -[\boldsymbol{f}, \boldsymbol{\varphi}_k] - (\nabla \boldsymbol{V}_1, \nabla \boldsymbol{\varphi}_k) + \mathcal{R}\left(\frac{\partial \boldsymbol{V}_\sigma}{\partial x_1}, \boldsymbol{\varphi}_k\right) + \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{V}_1 - \boldsymbol{e}_1 \times \boldsymbol{V}_1, \boldsymbol{\varphi}_k) \\
&\equiv F_k \,, \quad k = 1, 2, \ldots, m.
\end{aligned} \tag{VIII.1.11}$$

Using (ii) of Lemma VII.2.1 we obtain

$$\sum_{\ell=1}^{m} (\xi_{\ell m} \delta_{\ell k} - \xi_{\ell m} A_{\ell k}) = F_k, \quad k = 1, 2, \ldots, m \tag{VIII.1.12}$$

where

$$A_{\ell k} \equiv \mathcal{R}\left(\frac{\partial \boldsymbol{\varphi}_\ell}{\partial x_1}, \boldsymbol{\varphi}_k\right) + \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{\varphi}_\ell - \boldsymbol{e}_1 \times \boldsymbol{\varphi}_\ell, \boldsymbol{\varphi}_k).$$

System (VIII.1.12) is linear in the unknowns $\xi_{\ell m}$, $\ell = 1, \ldots, m$, and since $A_{\ell k} = -A_{k\ell}$, we deduce that the determinant of the coefficients is nonzero. As a consequence, for each $m \in \mathbb{N}$, system (VIII.1.12) admits a uniquely determined solution. Let us multiply (VIII.1.12) by ξ_{km} and sum over k from 1 to m. We obtain

$$\begin{aligned}
|\boldsymbol{w}_m|_{1,2}^2 &= -[\boldsymbol{f}, \boldsymbol{w}_m] \\
&\quad - (\nabla \boldsymbol{V}_1, \nabla \boldsymbol{w}_m) + \mathcal{R}\left(\frac{\partial \boldsymbol{V}_\sigma}{\partial x_1}, \boldsymbol{w}_m\right) + \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{V}_1 - \boldsymbol{e}_1 \times \boldsymbol{V}_1, \boldsymbol{w}_m).
\end{aligned} \tag{VIII.1.13}$$

By $(\mathrm{VIII.1.6})_1$, and the fact that $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$, we readily show that

$$\begin{aligned}
-[\boldsymbol{f}, \boldsymbol{w}_m] &\le |\boldsymbol{f}|_{-1,2} |\boldsymbol{w}_m|_{1,2} \,, \\
-(\nabla \boldsymbol{V}_1, \nabla \boldsymbol{w}_m) &\le c_1 \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} |\boldsymbol{w}_m|_{1,2} \,, \\
-\left(\boldsymbol{V}_1 + \boldsymbol{\sigma}, \frac{\partial \boldsymbol{w}_m}{\partial x_1}\right) &\le c_2 \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} |\boldsymbol{w}_m|_{1,2} \,, \\
(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{V}_1 - \boldsymbol{e}_1 \times \boldsymbol{V}_1, \boldsymbol{w}_m) &\le c_3 \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} |\boldsymbol{w}_m|_{1,2} \,.
\end{aligned} \tag{VIII.1.14}$$

Recalling (VIII.1.14) and using (VIII.1.13) we obtain

$$|\boldsymbol{w}_m|_{1,2} \le c \left\{ |\boldsymbol{f}|_{-1,2} + (1 + \mathcal{R} + \mathcal{T}) \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \right\} . \tag{VIII.1.15}$$

Therefore, the sequence $\{\boldsymbol{w}_m\}$ remains uniformly bounded in $D_0^{1,2}(\Omega)$, and by Exercise II.6.2, there exist a subsequence, denoted again by $\{\boldsymbol{w}_m\}$, and a function $\boldsymbol{w} \in \mathcal{D}_0^{1,2}(\Omega)$ such that in the limit $m \to \infty$,

$$(\nabla \boldsymbol{w}_m, \nabla \boldsymbol{\varphi}) \to (\nabla \boldsymbol{w}, \nabla \boldsymbol{\varphi}), \quad \text{for all } \boldsymbol{\varphi} \in D_0^{1,2}(\Omega). \tag{VIII.1.16}$$

Also, by (VIII.1.15) and Theorem II.2.4 we infer

$$|\boldsymbol{w}|_{1,2} \leq c\left\{|\boldsymbol{f}|_{-1,2} + (1+\mathcal{R}+\mathcal{T})\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\right\} \tag{VIII.1.17}$$

with $c = c(\Omega)$. Furthermore, from Exercise II.5.8 and by a simple diagonalization procedure, we can select another subsequence, again denoted by $\{\boldsymbol{w}_m\}$ such that

$$\boldsymbol{w}_m \to \boldsymbol{w} \ \text{ in } L^2(\Omega_R), \text{ for all } R > \delta(\Omega^c). \tag{VIII.1.18}$$

For fixed k, we then pass to the limit $m \to \infty$ into $(\text{VIII.1.11})_2$ to deduce with the help of (VIII.1.16), (VIII.1.18) that $\boldsymbol{v}$ satisfies (VIII.1.1) for all $\boldsymbol{\varphi}_k$. Since, by Lemma VII.2.1, every $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ can be approximated by a linear combination of $\boldsymbol{\varphi}_k$ in the $W^{1,2}$-norm and in the norm $\|\rho \cdot\|_2$, $\rho = (1+|x|)$, we easily establish the validity of (VIII.1.1) for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. Let us next prove estimates $(\text{VIII.1.3})_{1,3}$. From $(\text{VIII.1.3})_2$, Theorem II.6.1, and the Hölder inequality, we deduce

$$\|\boldsymbol{v}\|_{2,\Omega_R} \leq |\Omega_R|^{1/3}\|\boldsymbol{v}\|_{6,\Omega_R} \leq c\,|\Omega_R|^{1/3}|\boldsymbol{v}|_{1,2}\,, \tag{VIII.1.19}$$

where $c = c(\Omega)$, and so, since

$$|\boldsymbol{v}|_{1,2} \leq |\boldsymbol{w}|_{1,2} + |\boldsymbol{V}_1|_{1,2} + |\sigma|_{1,2}\,,$$

inequality $(\text{VIII.1.3})_1$ follows from (VIII.1.17), (VIII.1.19), and (VIII.1.6). Let us finally prove $(\text{VIII.1.3})_3$. For fixed $R > \delta(\Omega^c)$, we add to the pressure p (defined through Lemma VIII.1.1) the constant

$$\mathcal{C}(R) = -\frac{1}{|\Omega_R|}\int_{\Omega_R} p\,,$$

so that

$$\int_{\Omega_R}(p+\mathcal{C}) = 0.$$

Successively, we take $\boldsymbol{\psi}$ into (VII.1.2) as a solution to the problem

$$\begin{aligned}
&\nabla\cdot\boldsymbol{\psi} = p+\mathcal{C} \ \text{ in } \Omega_R\,,\\
&\boldsymbol{\psi} \in W_0^{1,2}(\Omega_R)\,,\\
&\|\boldsymbol{\psi}\|_{1,2} \leq c_1\|p+\mathcal{C}\|_{2,\Omega_R}\,,
\end{aligned}$$

for some $c_1 = c_1(\Omega_R)$. This problem is resolvable by virtue of Theorem II.4.1 and so from (VIII.1.2) and the Schwarz inequality we have

$$\|p+\mathcal{C}\|_{2,\Omega_R} \le c_1\left(|\boldsymbol{v}|_{1,2} + (\mathcal{R}+\mathcal{T})\|\boldsymbol{v}\|_{2,\Omega_R} + |\boldsymbol{f}|_{-1,2}\right) \tag{VIII.1.20}$$

which, by (VIII.1.3)$_1$, in turn implies (VIII.1.3)$_3$. The theorem is thus completely proved. □

VIII.2 Generalized Solutions. Uniqueness

The objective of this section is to prove that the generalized solution determined in Theorem VIII.1.2 is unique in the class of all generalized solutions corresponding to the same data. In view of the linearity of the problem, this amounts to showing that the only generalized solution to (VIII.0.2), (VIII.0.7) corresponding to $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv \boldsymbol{0}$ is identically zero. As in the case of the Oseen problem, in the case at hand, a fortiori, we are not allowed to replace $\boldsymbol{v}$ for $\boldsymbol{\varphi}$ in (VIII.1.1) if $\boldsymbol{v}$ merely belongs to $D^{1,2}(\Omega)$; see Remark VIII.1.2. Moreover, even if we could, it would not be obvious to conclude that the contribution from the second and third term on the left-hand side of (VIII.1.1) with $\boldsymbol{\varphi} \equiv \boldsymbol{v}$ is zero. We will, therefore, argue differently. To this end, we need to prove a number of preparatory results.

Lemma VIII.2.1 *Let $\boldsymbol{v}$ be a generalized solution to* (VIII.0.2), (VIII.0.7) *corresponding to $\boldsymbol{f} \in L^2(\Omega^\rho)$, for some $\rho > \delta(\Omega^c)$. Then $\boldsymbol{v} \in D^{2,2}(\Omega^r)$, for all $r > \rho$. Moreover, there is $c = c(r, B)$ (with $c(r) \to \infty$ as $r \to \rho^+$) such that for all $\mathcal{R}, \mathcal{T} \in [0, B]$,*

$$\|D^2\boldsymbol{v}\|_{2,\Omega^r} \le c\left(\|\boldsymbol{f}\|_{2,\Omega^\rho} + |\boldsymbol{v}|_{1,2}\right). \tag{VIII.2.1}$$

Finally, if $\Omega = \mathbb{R}^3$ and $\boldsymbol{f} \in L^2(\mathbb{R}^3)$, then $\boldsymbol{v} \in D^{2,2}(\mathbb{R}^3)$ and the following estimate holds:

$$\|D^2\boldsymbol{v}\|_{2,\mathbb{R}^3} \le c_1\left(\|\boldsymbol{f}\|_{2,\mathbb{R}^3} + |\boldsymbol{v}|_{1,2}\right), \tag{VIII.2.2}$$

with $c_1 = c_1(B)$.

Proof. In view of the assumption on $\boldsymbol{f}$ and Theorem VIII.1.1, we know that $\boldsymbol{v} \in W^{2,2}_{loc}(\Omega^\rho)$, $p \in W^{1,2}_{loc}(\Omega^\rho)$ and that they satisfy the following equations:

$$\left.\begin{aligned} \Delta\boldsymbol{v} + \mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1} + \mathcal{T}\left(\boldsymbol{e}_1 \times \boldsymbol{x}\cdot\nabla\boldsymbol{v} - \boldsymbol{e}_1\times\boldsymbol{v}\right) &= \nabla p + \boldsymbol{f} \\ \nabla\cdot\boldsymbol{v} &= 0 \end{aligned}\right\} \text{ a.e. in } \Omega^\rho. \tag{VIII.2.3}$$

Let $R > r > \rho$, and denote by $\psi_R = \psi_R(|x|)$ a smooth, non-negative "cut-off" function that is 0 for $|x| \le \rho$ and $|x| \ge 2R$, while it is 1 for $|x| \in [r, R]$. Moreover, we may take $|D^\alpha\psi_R|, |D^\alpha(\sqrt{\psi_R})| \le M\, R^{-|\alpha|}$, all $|\alpha| \ge 0$, with M independent of R. We next dot-multiply both sides of (VIII.2.3)$_1$ by

$-\nabla \times (\psi_R \nabla \times \boldsymbol{v})$, and integrate over Ω. Since[1]

$$\begin{aligned} -\nabla \times (\psi_R \nabla \times \boldsymbol{v}) &= -\psi_R \nabla \times \nabla \times \boldsymbol{v} + (\nabla \times \boldsymbol{v}) \times \nabla \psi_R \\ &= \psi_R \Delta \boldsymbol{v} + (\nabla \times \boldsymbol{v}) \times \nabla \psi_R \,, \end{aligned} \tag{VIII.2.4}$$

we obtain[2]

$$\begin{aligned} \|\sqrt{\psi_R}\Delta \boldsymbol{v}\|_2^2 = &-\left(\mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1} + \boldsymbol{f}, \psi_R \Delta \boldsymbol{v} + (\nabla \times \boldsymbol{v}) \times \nabla \psi_R\right) \\ &-\tfrac{1}{2}((\nabla \times \boldsymbol{v}) \times \nabla(\sqrt{\psi_R}), \sqrt{\psi_R}\Delta \boldsymbol{v}) \\ &+\mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{v}, \nabla \times (\psi_R \nabla \times \boldsymbol{v}) - (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v}, \nabla \times (\psi_R \nabla \times \boldsymbol{v}))\,. \end{aligned} \tag{VIII.2.5}$$

By the Schwarz inequality, the Cauchy inequality (II.2.5), and the properties of ψ_R, we obtain

$$-\,((\nabla \times \boldsymbol{v}) \times \nabla(\sqrt{\psi_R}), \sqrt{\psi_R}\Delta \boldsymbol{v}) \le \tfrac{1}{4}\|\sqrt{\psi_R}\Delta \boldsymbol{v}\|_2^2 + M^2\,|\boldsymbol{v}|_{1,2}^2 \tag{VIII.2.6}$$

and

$$-\left(\mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1} + \boldsymbol{f}, \psi_R \Delta \boldsymbol{v} + (\nabla \times \boldsymbol{v}) \times \nabla \psi_R\right) \le \tfrac{1}{4}\|\sqrt{\psi_R}\Delta \boldsymbol{v}\|_2^2 + c\,(|\boldsymbol{v}|_{1,2}^2 + \|\boldsymbol{f}\|_2^2)\,, \tag{VIII.2.7}$$

where $c = c(B, \rho, r)$. We need now to estimate the last two terms on the right-hand side of (VIII.2.5). Employing the well-known identities

$$\begin{aligned} \nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) &= \boldsymbol{B} \cdot \nabla \times \boldsymbol{A} - \boldsymbol{A} \cdot \nabla \times \boldsymbol{B}\,, \\ \nabla \times (\boldsymbol{a} \times \boldsymbol{B}) &= \boldsymbol{a}\,\nabla \cdot \boldsymbol{B} - \boldsymbol{a} \cdot \nabla \boldsymbol{B}\,, \end{aligned} \tag{VIII.2.8}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$ are vector fields and $\boldsymbol{a}$ is a constant vector, along with the properties of ψ_R, we obtain

$$\begin{aligned} (\boldsymbol{e}_1 \times \boldsymbol{v}, \nabla \times (\psi_R \nabla \times \boldsymbol{v})) &= -\int_\Omega \nabla \cdot [\psi_R(\boldsymbol{e}_1 \times \boldsymbol{v}) \times (\nabla \times \boldsymbol{v})] \\ &\quad + (\psi_R \nabla \times \boldsymbol{v}, \nabla \times (\boldsymbol{e}_1 \times \boldsymbol{v})) \\ &= -(\psi_R \boldsymbol{e}_1 \cdot \nabla \boldsymbol{v}, \nabla \times \boldsymbol{v}) \le 2|\boldsymbol{v}|_{1,2}^2\,. \end{aligned} \tag{VIII.2.9}$$

Furthermore, again with the help of $(\text{VIII.2.8})_1$, and by the properties of ψ_R, we deduce

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[1] This vector identity as well as the others used throughout the proof is known for "smooth" vector fields. However, by a simple approximating procedure based on Theorem II.3.1, we can show that they continue to hold a.e. in Ω^ρ also if the fields merely belong to $W^{2,2}_{loc}(\Omega^\rho)$.

[2] Throughout the proof, for simplicity, we set $\|\cdot\|_{2,\Omega^\rho} \equiv \|\cdot\|_2$, and $(\cdot,\cdot)_{\Omega^\rho} \equiv (\cdot,\cdot)$.

$$-(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v}, \nabla \times (\psi_R \nabla \times \boldsymbol{v})) = \int_\Omega \nabla \cdot [\psi_R(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v}) \times (\nabla \times \boldsymbol{v})]$$
$$- (\psi_R \nabla \times \boldsymbol{v}, \nabla \times (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v}))$$
$$= - (\psi_R \nabla \times \boldsymbol{v}, \nabla \times (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v}))\,. \tag{VIII.2.10}$$

In order to estimate the term on the right-hand side of (VIII.2.10), we recall the following classical form of $\nabla \times \boldsymbol{A}$, and that of the cross product of two vectors, $\boldsymbol{a}$, $\boldsymbol{b}$ in terms of their components:

$$\nabla \times \boldsymbol{A} = \varepsilon_{ijk} D_j A_k \boldsymbol{e}_i\,, \quad \boldsymbol{a} \times \boldsymbol{b} = \varepsilon_{ijk} a_j b_k \boldsymbol{e}_i\,,$$

where ε_{ijk} is the alternating symbol.[3] We thus have, for $i = 1, 2, 3$,

$$[\nabla \times (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v})]_i = \varepsilon_{ijk} D_j (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla v_k)$$
$$= [\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla(\nabla \times \boldsymbol{v})]_i + \varepsilon_{ijk} \varepsilon_{lmj} e_m D_l v_k\,,$$

and so, employing the well-known identities (see, e.g., Evett (1966))

$$\varepsilon_{ijk} \varepsilon_{lmj} = \delta_{im} \delta_{kl} - \delta_{il} \delta_{km}\,,$$

and recalling (VIII.2.3)$_2$, we deduce

$$\nabla \times (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v}) = \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla(\nabla \times \boldsymbol{v}) - \nabla \boldsymbol{v} \cdot \boldsymbol{e}_1\,.$$

Taking into account that $\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \psi_R(|x|) = 0$, for all $x \in \mathbb{R}^3$, from this latter relation and again from the properties of ψ_R, we obtain

$$- (\psi_R \nabla \times \boldsymbol{v}, \nabla \times (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v})) = -\tfrac{1}{2} \int_\Omega \nabla \cdot [\psi_R\, (\nabla \times \boldsymbol{v})^2 \boldsymbol{e}_1 \times \boldsymbol{x}]$$
$$+ (\psi_R \nabla \times \boldsymbol{v}, \nabla \boldsymbol{v} \cdot \boldsymbol{e}_1)$$
$$= (\psi_R \nabla \times \boldsymbol{v}, \nabla \boldsymbol{v} \cdot \boldsymbol{e}_1) \le 2 |\boldsymbol{v}|_{1,2}^2\,. \tag{VIII.2.11}$$

Collecting (VIII.2.5)–(VIII.2.7), (VIII.2.9), and (VIII.2.11), we thus obtain

$$\| \sqrt{\psi_R} \Delta \boldsymbol{v} \|_2^2 \le c\, (|\boldsymbol{v}|_{1,2}^2 + \|\boldsymbol{f}\|_2^2)\,, \tag{VIII.2.12}$$

and so, recalling that $\psi_R(|x|) = 1$, for $x \in \Omega_{r,R}$, we conclude, in particular, that

$$\|\Delta \boldsymbol{v}\|_{2,\Omega_{r,R}}^2 \le c\, (|\boldsymbol{v}|_{1,2}^2 + \|\boldsymbol{f}\|_2^2)$$

where c is independent of R. Letting $R \to \infty$ in this relation gives $\Delta \boldsymbol{v} \in L^2(\Omega^r)$, for all $r > \rho$, and moreover,

$$\|\Delta \boldsymbol{v}\|_{2,\Omega^r} \le c\, (|\boldsymbol{v}|_{1,2} + \|\boldsymbol{f}\|_{2,\Omega^\rho})\,. \tag{VIII.2.13}$$

[3] See footnote 4 in Section VIII.1.

From (VIII.2.13) and from its proof, it immediately follows that in the particular case $\Omega = \mathbb{R}^3$ and $\boldsymbol{f} \in L^2(\mathbb{R}^3)$, we have $\Delta \boldsymbol{v} \in L^2(\mathbb{R}^3)$ and

$$\|\Delta \boldsymbol{v}\|_{2,\mathbb{R}^3} \le c_1 \left(|\boldsymbol{v}|_{1,2} + \|\boldsymbol{f}\|_{2,\mathbb{R}^3}\right), \tag{VIII.2.14}$$

with $c_1 = c_1(B)$. We now observe that as a consequence of this property and the Helmholtz–Weyl decomposition theorem, Theorem III.1.1, $\boldsymbol{v}$ satisfies the following Stokes system:

$$\left.\begin{aligned} \Delta \boldsymbol{v} &= \boldsymbol{F} + \nabla \zeta \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}\right\} \text{ in } \Omega^r ,$$

where $\boldsymbol{F} \in H(\Omega^r) \subset L^2(\Omega^r)$ and $\zeta \in D^{1,2}(\Omega^r)$. Moreover, $\boldsymbol{v} \in D^{1,2}(\Omega^r)$ by assumption, and so from Theorem V.5.3, we readily obtain $\boldsymbol{v} \in D^{2,2}(\Omega^r)$, and the property stated in the lemma follows. Clearly, if $\Omega = \mathbb{R}^3$ and $\boldsymbol{f} \in L^2(\mathbb{R}^3)$, this argument furnishes $D^2\boldsymbol{v} \in L^2(\mathbb{R}^3)$. Therefore, since

$$\boldsymbol{v} \in D^{2,2}(\mathbb{R}^3) \cap D_0^{1,2}(\mathbb{R}^3),$$

by Theorem II.7.6 it easily follows that $\boldsymbol{v} \in D_0^{2,2}(\mathbb{R}^3)$. Thus, by Exercise II.7.4, we have

$$\|D^2 \boldsymbol{v}\|_2 = \|\Delta \boldsymbol{v}\|_2 ,$$

and from (VIII.2.14) we also prove (VIII.2.2). The case $\Omega \neq \mathbb{R}^3$ can be treated by similar considerations. In fact, let $\zeta = \zeta(x)$ a smooth function that is 1 for $|x| \ge r$ and is 0 for $|x| \le \rho$, and set $\boldsymbol{w} = \zeta\, \boldsymbol{v}$. Then, by the property of generalized solutions and with the help of Theorem II.7.6 we prove that $\boldsymbol{w} \in D_0^{2,2}(\mathbb{R}^3)$, and so, as before, we find that $\|D^2\boldsymbol{w}\|_2 = \|\Delta \boldsymbol{w}\|_2$. However, by a simple calculation, we show that

$$\begin{aligned} \|D^2 \boldsymbol{v}\|_{2,\Omega^r} &\le \|\Delta \boldsymbol{w}\|_2 \le c_1(r) \left(\|\Delta \boldsymbol{v}\|_{2,\Omega^r} + \|\boldsymbol{v}\|_{1,2,\Omega_{\rho,r}}\right) \\ &\le c_2(r) \left(\|\Delta \boldsymbol{v}\|_{2,\Omega^r} + |\boldsymbol{v}|_{1,2}\right), \end{aligned} \tag{VIII.2.15}$$

where in the last step, we have used the Hölder inequality together with inequality $\|\boldsymbol{v}\|_6 \le c|\boldsymbol{v}|_{1,2}$. The relation (VIII.2.1) is then a consequence of (VIII.2.13) and (VIII.2.15).

□

The next result furnishes a representation of the pressure field under very general assumptions on $\boldsymbol{f}$.

Lemma VIII.2.2 *Let $\boldsymbol{v}$ be a generalized solution to* (VIII.0.2), (VIII.0.7) *corresponding to $\boldsymbol{f} = \widetilde{\boldsymbol{f}} + \nabla \cdot \boldsymbol{F}$, where*

$$\widetilde{\boldsymbol{f}} \in L^2(\Omega^\rho) \cap L^q(\Omega^\rho), \quad \textit{for some } \rho > \delta(\Omega^c) \textit{ and } q \in (1, \infty),$$

and $\boldsymbol{F}$ is a second-order tensor field such that

$$\nabla \cdot \boldsymbol{F} \in L^2(\Omega^\rho)\,, \quad \boldsymbol{F} \in L^t(\Omega^\rho) \cap L^2(\Omega_r) \cap L^q(\Omega_r)\,, \quad \textit{for some } t \in (1,\infty).$$

Then, there exists $p_\infty \in \mathbb{R}$ *such that the pressure* p *associated to* $\boldsymbol{v}$ *by* Lemma VIII.1.1 *admits the following decomposition:*

$$p = p_\infty + p_1 + p_2\,, \tag{VIII.2.16}$$

where

$$\begin{aligned} &p_1 \in L^6(\Omega^r) \cap D^{1,2}(\Omega^r) \cap D^{1,q}(\Omega^r)\,,\\ &p_2 \in L^6(\Omega^r) \cap L^t(\Omega^r) \cap D^{1,2}(\Omega^r)\,. \end{aligned} \tag{VIII.2.17}$$

Moreover, if $q \in (1,3)$*, we have also*

$$p_1 \in \cap L^{3q/(3-q)}(\Omega^r)\,. \tag{VIII.2.18}$$

Finally, if

$$(\boldsymbol{f}, \nabla\psi) = 0 \quad \textit{for all } \psi \in C_0^\infty(\Omega^\rho)\,, \tag{VIII.2.19}$$

namely, $\nabla \cdot \boldsymbol{f} = 0$ *in the generalized sense, and* $q \in (1, 3/2]$*,* $t \in (1,3]$*, then*

$$D^\alpha(p(x) - p_\infty) = \chi_\alpha(x)\,, \quad \textit{for all } |\alpha| \geq 0, \;\; |x| > r\,, \tag{VIII.2.20}$$

where

$$\chi_\alpha(x) = O(|x|^{-2-|\alpha|})\,.$$

Proof. By the Helmholtz–Weyl decomposition theorem, Theorem III.1.2, we may write

$$\widetilde{\boldsymbol{f}} = \boldsymbol{f}^* + \nabla p^*,$$

where $\boldsymbol{f}^*, \nabla p^* \in L^2(\Omega^\rho) \cap L^q(\Omega^\rho)$. By Theorem II.6.1, we can add a constant to p^* in such a way that the modified function, which we continue to denote by p^*, belongs to $L^6(\Omega^\rho) \cap L^{3q/(3-q)}(\Omega^\rho)$. We set $\widetilde{p} = p + p^*$, so that (VIII.2.3) becomes

$$\left.\begin{array}{r} \Delta\boldsymbol{v} + \mathcal{R}\dfrac{\partial \boldsymbol{v}}{\partial x_1} + \mathcal{T}\left(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla\boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}\right) = \nabla\widetilde{p} + \boldsymbol{f}^* + \nabla \cdot \boldsymbol{F} \\ \nabla \cdot \boldsymbol{v} = 0 \end{array}\right\} \text{ a.e. in } \Omega^\rho\,. \tag{VIII.2.21}$$

We begin by proving the properties (VIII.2.16)–(VIII.2.18). Fix $r > \rho$, and let ψ be a smooth "cut-off" function that is 1 if $|x| \geq r$, and 0 if $|x| \leq \rho$. Moreover, let

$$\boldsymbol{\sigma}(x) := -\left(\int_{\partial B_r} \boldsymbol{v} \cdot \boldsymbol{n}\right) \nabla\mathcal{E}\,, \tag{VIII.2.22}$$

where $\mathcal{E}$ is the (three-dimensional) fundamental Laplace solution (II.9.1), and consider the following problem:

$$\begin{aligned} &\nabla \cdot \boldsymbol{H} = \nabla \cdot [\psi(\boldsymbol{v}+\boldsymbol{\sigma})] \ \text{ in } B_r\,, \\ &\boldsymbol{H} \in W^{3,2}(\mathbb{R}^3)\,, \quad \operatorname{supp}(\boldsymbol{H}) \subset B_r\,. \end{aligned} \tag{VIII.2.23}$$

Since

$$\int_{B_r} \nabla \cdot [\psi(\boldsymbol{v}+\boldsymbol{\sigma})] = \int_{\partial B_r} \psi(\boldsymbol{v}+\boldsymbol{\sigma}) \cdot \boldsymbol{n} = 0\,,$$

we know, by the properties of $\boldsymbol{v}$ and Theorem III.3.1, that (VIII.2.23) has at least one solution. Put

$$\boldsymbol{w} = \psi\boldsymbol{v} + \psi\boldsymbol{\sigma} - \boldsymbol{H}\,, \quad \pi = \psi\,\widetilde{p}.$$

Taking into account (VIII.1.8), from (VIII.2.21) by a direct calculation we obtain[4]

$$\left.\begin{aligned} \Delta\boldsymbol{w} + \frac{\partial \boldsymbol{w}}{\partial x_1} + (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla\boldsymbol{w} - \boldsymbol{e}_1 \times \boldsymbol{w}) = \nabla\pi + \psi\boldsymbol{f}^* + \psi\nabla\cdot\boldsymbol{F} + \boldsymbol{G} \\ \nabla \cdot \boldsymbol{w} = 0 \end{aligned}\right\} \text{a.e. in } \mathbb{R}^3, \tag{VIII.2.24}$$

where $\boldsymbol{G} \in L^2(\mathbb{R}^3)$, with $\operatorname{supp}(\boldsymbol{G}) \subset B_r$. Now, for all $\psi \in C_0^\infty(\Omega^\rho)$,

$$(\Delta\boldsymbol{w}, \nabla\psi) = -(\nabla\cdot\boldsymbol{w}, \Delta\psi) = 0\,, \quad \left(\frac{\partial \boldsymbol{w}}{\partial x_1}, \nabla\psi\right) = \left(\nabla\cdot\boldsymbol{w}, \frac{\partial\psi}{\partial x_1}\right) = 0\,. \tag{VIII.2.25}$$

Also, since

$$\nabla\cdot(\boldsymbol{e}_1 \times \boldsymbol{x}\cdot\nabla\boldsymbol{\Phi} - \boldsymbol{e}_1 \times \boldsymbol{\Phi}) = \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla(\nabla\cdot\boldsymbol{\Phi}) = 0\,, \qquad \text{for all } \boldsymbol{\Phi} \in W^{2,2}_{loc}(\Omega) \text{ with } \nabla\cdot\boldsymbol{\Phi} = 0\,, \tag{VIII.2.26}$$

we obtain, in particular,

$$(\boldsymbol{e}_1 \times \boldsymbol{x}\cdot\nabla\boldsymbol{w} - \boldsymbol{e}_1 \times \boldsymbol{w}, \nabla\psi) = -(\nabla\cdot(\boldsymbol{e}_1 \times \boldsymbol{x}\cdot\nabla\boldsymbol{w} - \boldsymbol{e}_1 \times \boldsymbol{w}), \psi) = 0\,. \tag{VIII.2.27}$$

Using (VIII.2.25) and (VIII.2.27) from $(\text{VIII.2.24})_1$, we deduce that π is a generalized solution to the following problem:

$$-\Delta\pi = \nabla\cdot(\psi\boldsymbol{f}^*) + \nabla\cdot(\psi\nabla\cdot\boldsymbol{F}) + \nabla\cdot\boldsymbol{G} \ \text{ in } \mathbb{R}^3\,. \tag{VIII.2.28}$$

A solution to (VIII.2.28) is given by

$$\begin{aligned} \overline{\pi} &= -\nabla\mathcal{E} * (\psi\boldsymbol{f}^*) - \nabla\mathcal{E} * \boldsymbol{G} + \nabla\mathcal{E} * (\nabla\psi\cdot\boldsymbol{F}) - \nabla\mathcal{E} * [\nabla\cdot(\psi\boldsymbol{F})] \\ &\equiv \sum_{i=1}^4 \overline{\pi}_i\,, \end{aligned} \tag{VIII.2.29}$$

[4] Since their values are irrelevant to the proof, we set throughout $\mathcal{R} = \mathcal{T} = 1$.

where $\mathcal{E}$ is the fundamental solution of Laplace's equation (II.9.1). We recall that $|\nabla\mathcal{E}(\xi)| \le c|\xi|^{-2}$, and that $D_{ij}\mathcal{E}$ is a singular kernel, for all $i, j = 1, 2, 3$ (see Exercise II.11.7). Therefore, taking into account that $\psi \boldsymbol{f}^*, \boldsymbol{G}, \nabla\psi \cdot \boldsymbol{F} \in L^q(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, $q \in (1, 3)$, from Theorem II.11.3 and Theorem II.11.4 we obtain

$$\left.\begin{array}{l} \overline{\pi}_i \in L^6(\mathbb{R}^3) \cap D^{1,q}(\mathbb{R}^3) \cap D^{1,2}(\mathbb{R}^3), \quad q \in (1, \infty) \\ \overline{\pi}_i \in L^{3q/(3-q)}(\mathbb{R}^3) \cap L^6(\mathbb{R}^3) \cap D^{1,q}(\mathbb{R}^3) \cap D^{1,2}(\mathbb{R}^3), \quad q \in (1, 3) \end{array}\right\} \; i = 1, 2, 3, \tag{VIII.2.30}$$

and by the same token,

$$\overline{\pi}_4 \in L^t(\mathbb{R}^3) \cap L^6(\mathbb{R}^3) \cap D^{1,2}(\mathbb{R}^3)\,; \tag{VIII.2.31}$$

see Exercise VIII.2.2. Let us show that (up to an additive constant) $\pi = \overline{\pi}$ in Ω^r. To this end, we observe that clearly, $\Psi := \pi - \overline{\pi}$ is harmonic in $\mathbb{R}^3$, for which the following local representation holds (see (VII.4.57)):

$$\Psi(x) = -\int_{\mathbb{R}^3} H^{(R)}(x-y)\Psi(y)dy \quad x \in \mathbb{R}^3\,,$$

with $H^{(R)}$ satisfying (V.3.11)–(V.3.13). Therefore, by differentiating both sides of this relation and using the Schwarz and Hölder inequalities, we deduce

$$\begin{aligned} |\nabla\Psi(x)| \le \left(\int_{\mathbb{R}^3} |y|^2\,|H^{(R)}(x-y)|^2 dy\right)^{1/2} \left\|\frac{\nabla\pi}{y}\right\|_2 \\ + \|\nabla H^{(R)}\|_{3q/(4q-3)} \|\overline{\pi}\|_{3q/(3q-3)}\,. \end{aligned} \tag{VIII.2.32}$$

Also, recalling the properties (V.3.11), (V.3.13) of the function $H^{(R)}$ and noticing that $|y|^2 \le 2(|x-y|^2 + |x|^2)$, from (VIII.2.32) we obtain

$$|\nabla\Psi(x)| \le c_1\,\frac{1+|x|}{\sqrt{R}} \left\|\frac{\nabla\pi}{y}\right\|_2 + c_2 R^{-5+4q/3} \|\overline{\pi}\|_{3q/(3q-3)}\,, \tag{VIII.2.33}$$

for all fixed $x \in \mathbb{R}^3$ and all $R > 0$. On the other hand, dividing both sides of (VIII.2.21)$_1$ by $|x|$, squaring the resulting equation and integrating over Ω^r, $r > \rho$, we deduce

$$\left\|\frac{\nabla\widetilde{p}}{|x|}\right\|_{2,\Omega^r} \le c\left[\|\Delta\boldsymbol{v}\|_{2,\Omega^r} + |\boldsymbol{v}|_{1,2,\Omega^r} + \|\boldsymbol{f}\|_{2,\Omega^r} + \left\|\frac{\boldsymbol{v}}{|x|}\right\|_{2,\Omega^r}\right], \tag{VIII.2.34}$$

where $c = c(r)$. From Lemma VIII.2.1 and the assumption on $\boldsymbol{f}$, we infer that $\Delta\boldsymbol{v} \in L^2(\Omega^r)$. Therefore, taking into account that $\boldsymbol{v} \in D^{1,2}(\Omega)$, we deduce that the first three terms on the right-hand side of (VIII.2.34) are finite. Moreover, by Theorem II.6.1(i) and property (iv) in Definition VIII.1.1, we have also that the last term on the right-hand side of (VIII.2.34) is finite, so that we conclude, in particular, that

$$\int_{\Omega^r} \left| \frac{\nabla \widetilde{p}}{x} \right|^2 < \infty \,,$$

namely, since $\pi = \psi \widetilde{p}$ and $\widetilde{p} \in L^2_{loc}(\Omega^\rho)$,

$$\int_{\mathbb{R}^3} \left| \frac{\nabla \pi}{x} \right|^2 < \infty \,.$$

If we take into account this condition along with (VIII.2.30), and pass to the limit $R \to \infty$ in (VIII.2.33), we conclude that $\nabla \Psi(x) = 0$, for all $x \in \mathbb{R}^3$, that is, $\pi = \overline{\pi} + \text{const}$. Therefore, since $\pi(x) = \widetilde{p}(x) \equiv p(x) + p^*(x)$, $x \in \Omega^r$, the desired summability property (VIII.2.16)–(VIII.2.17) for p follows from this latter, the analogous property of p^*, and (VIII.2.29)–(VIII.2.31). We shall now complete the proof of the theorem. Using (VIII.2.25), (VIII.2.27) (with $\boldsymbol{w} \equiv \boldsymbol{v}$), and bearing in mind (VIII.2.19), from (VIII.2.21) we obtain $(\nabla p, \nabla \psi) = 0$, for all $\psi \in C_0^\infty(\Omega^\rho)$, which, in turn, by well-known results of Caccioppoli (1937), Cimmino (1938a, 1938b), and Weyl (1940), implies that p is harmonic in Ω^ρ and belongs to $C^\infty(\Omega^\rho)$. Recalling that p satisfies (VIII.2.16)–(VIII.2.17), we may use the results of Exercise V.3.6(i) to obtain the following representation for $\widetilde{p}(x)$, for all $x \in \Omega^r$:

$$D^\alpha p(x) = -\left(\int_{\partial B_r} \frac{\partial \widetilde{p}}{\partial n_\ell} n_\ell \right) D^\alpha \mathcal{E}(x) + \chi_\alpha(x) \,, \quad \text{for all } |\alpha| \geq 0. \qquad \text{(VIII.2.35)}$$

Take $\alpha = 0$ in this relation. Then, if $q \in (1, 3/2]$, $t \in (1, 3]$, by (VIII.2.16), (VIII.2.17) both p and χ_0 are summable in Ω^r with some exponent $s \in (3/2, 3]$, and since $\mathcal{E} \notin L^s(\Omega^r)$, this implies that the surface integral in (VIII.2.35) must vanish, which concludes the proof of the lemma. □

Remark VIII.2.1 As the reader may have noticed, the proof of the previous lemma remains unaltered if we take $\mathcal{T} = 0$. ■

We are now in a position to prove the following.

Lemma VIII.2.3 *Let Ω be locally Lipschitz and let $\boldsymbol{u}$ be a generalized solution to* (VIII.0.2), (VIII.0.7) *corresponding to $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv \boldsymbol{0}$. Moreover, denote by p the associated pressure field from* Lemma VIII.1.1. *Then $\boldsymbol{u} \equiv \boldsymbol{0}$, $p \equiv p_1 + \text{const}$.*

Proof. By assumption, we have

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{\psi}) - \mathcal{R}\left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{\psi} \right) - \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{u}, \boldsymbol{\psi}) = (\pi, \nabla \cdot \boldsymbol{\psi}) \,, \quad \text{(VIII.2.36)}$$

for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. By Theorem III.5.1, we obtain that $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$, while by Theorem VIII.1.1, it also follows that $\boldsymbol{u}, p \in C^\infty(\Omega)$. Employing this latter property and (VIII.2.36), we deduce that

$$\left.\begin{array}{r}\Delta \boldsymbol{u}+\mathcal{R}\dfrac{\partial \boldsymbol{u}}{\partial x_1}+\mathcal{T}\left(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{u}-\boldsymbol{e}_1\times\boldsymbol{u}\right)=\nabla\pi\\[6pt] \nabla\cdot\boldsymbol{u}=0\end{array}\right\}\ \text{ in } \Omega\,. \tag{VIII.2.37}$$

Now by Lemma VIII.2.2, we have

$$\pi(x)-\pi_\infty=O(|x|^{-2})\,,\quad \text{as } |x|\to\infty, \tag{VIII.2.38}$$

for some $\pi_\infty\in\mathbb{R}$. Since (VIII.2.36) remains unchanged if we replace π with $\pi-\pi_\infty$, we will take, without loss, $\pi_\infty=0$. Our next objective is, basically, to replace $\boldsymbol{\psi}$ in (VIII.2.36) with $\boldsymbol{u}$. In order to reach this goal, we begin to notice that since $\mathcal{D}_0^{1,2}(\Omega)\subset W^{1,2}(\Omega_R)$, for all $R>\delta(\Omega^c)$, one can show, by a simple density argument, that (VIII.2.36) continues to hold for $\boldsymbol{\psi}\in W_0^{1,2}(\Omega_R)$, for all $R>\delta(\Omega^c)$. We next choose $\boldsymbol{\psi}=\psi_{4,R}\,\boldsymbol{u}$, where $\psi_{\alpha,R}$ is the "cut-off" function determined in Lemma II.6.4. We recall that by that lemma, we have, in particular,

$$\begin{aligned}
&\operatorname{supp}(\psi_{4,R})\subset\Omega^{\frac{R}{\sqrt2}}\,,\\
&\lim_{R\to\infty}\psi_{4,R}(x)=1\ \text{ uniformly pointwise}\,,\\
&\left\|\frac{\partial\psi_{4,R}}{\partial x_1}\right\|_{3/2}\le C_1\,,\quad \|\boldsymbol{u}\,|\nabla\psi_{4,R}|\,\|_2\le C_2|\boldsymbol{u}|_{1,2,\Omega^{\frac{R}{\sqrt2}}}\,,\\
&(\boldsymbol{e}_1\times\boldsymbol{x})\cdot\nabla\psi_{4,R}(x)=0\ \text{ for all } x\in\mathbb{R}^3\,.
\end{aligned} \tag{VIII.2.39}$$

Thus, replacing $\psi_{4,R}\boldsymbol{u}$ for $\boldsymbol{\psi}$ in (VIII.2.36), we obtain after a few integrations by parts (for notational simplicity, we drop the subscript "$_4$"),

$$\begin{aligned}
0&=(\psi_R\nabla\boldsymbol{u},\nabla\boldsymbol{u})+(\nabla\psi_R\cdot\nabla\boldsymbol{u},\boldsymbol{u})+\frac{\mathcal{R}}{2}\left(\boldsymbol{u}^2,\frac{\partial\psi_R}{\partial x_1}\right)\\
&\quad+\frac{\mathcal{T}}{2}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\psi_R,\boldsymbol{u}^2)-(\pi,\nabla\psi_R\cdot\boldsymbol{u})\\
&=(\psi_R\nabla\boldsymbol{u},\nabla\boldsymbol{u})+(\nabla\psi_R\cdot\nabla\boldsymbol{u},\boldsymbol{u})+\frac{\mathcal{R}}{2}\left(\boldsymbol{u}^2,\frac{\partial\psi_R}{\partial x_1}\right)-(\pi,\nabla\psi_R\cdot\boldsymbol{u})\,.
\end{aligned}$$

From this relation, with the help of (VIII.2.39) and the Schwarz and Hölder inequalities, we derive

$$\|\sqrt{\psi_R}\nabla\boldsymbol{u}\|_2^2\le c\left(|\boldsymbol{u}|^2_{1,2,\Omega^{\frac{R}{\sqrt2}}}+\mathcal{R}\|\boldsymbol{u}\|^2_{6,\Omega^{\frac{R}{\sqrt2}}}+\|\pi\|_2|\boldsymbol{u}|_{1,2,\Omega^{\frac{R}{\sqrt2}}}\right)\,, \tag{VIII.2.40}$$

with c independent of R. From (VIII.2.38) we know that $\pi\in L^2(\Omega)$, whereas by Theorem II.7.5, we have $\boldsymbol{u}\in L^6(\Omega)$. Therefore, in view of these properties, we may let $R\to\infty$ in (VIII.2.40) to deduce

$$\lim_{R\to\infty}\|\sqrt{\psi_R}\nabla\boldsymbol{u}\|_2=0\,,$$

which, in turn, by (VIII.2.39)$_2$ and by the Lebesgue dominated convergence theorem of Lemma II.2.1, shows that $\|\nabla \boldsymbol{u}\|_2 = 0$, that is, $\boldsymbol{u} \equiv 0$ in Ω. The proof of the theorem is thus completed. $\square$

From the previous lemma, we at once deduce the following uniqueness theorem, which constitutes the main accomplishment of this section.

Theorem VIII.2.1 *Let Ω be locally Lipschitz and let $\boldsymbol{v}$ be a generalized solution to* (VIII.0.2), (VIII.0.7) *corresponding to $\boldsymbol{f} \in W_0^{-1,2}(\Omega')$, Ω' any bounded subdomain with $\overline{\Omega'} \subset \Omega$, and to $\boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega)$. Moreover, denote by p the associated pressure field from* Lemma VIII.1.1. *Then, if $\boldsymbol{w}$ is another generalized solution corresponding to the same data, with associated pressure field p_1, we have $\boldsymbol{v} \equiv \boldsymbol{w}$, $p \equiv p_1 + \text{const}$.*

Exercise VIII.2.1 Let $\boldsymbol{v}$ be a generalized solution to (VIII.0.2), (VIII.0.7), and suppose that $\boldsymbol{f}$ satisfies the assumption of Lemma VIII.2.1. Show that

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = \boldsymbol{0}\,,$$

uniformly pointwise. *Hint*: Couple the results of Lemma VIII.2.1 with those of Theorem II.9.1.

Exercise VIII.2.2 Prove properties (VIII.2.30), (VIII.2.31).

VIII.3 The Fundamental Solution to the Time-Dependent Oseen Problem and Related Properties

As we observed in the Introduction to this chapter, the asymptotic properties of generalized solutions to (VIII.0.7), (VIII.0.2) will be established by first proving similar properties for the solutions to an associated *initial-value* problem, and then by showing that in the limit as $t \to \infty$, the latter converge to the former, uniformly in suitable norms.

Actually, as we shall show later on, in order to achieve this goal, it suffices to investigate the above properties for solutions to the following *Oseen initial-value problem*:

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{u}}{\partial t} = \Delta \boldsymbol{u} + \mathcal{R}\frac{\partial \boldsymbol{u}}{\partial x_1} - \nabla\phi + \boldsymbol{f} \\ &\nabla\cdot\boldsymbol{u} = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3\times(0,\infty) \tag{VIII.3.1}$$
$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0\,,$$

where $\boldsymbol{f} = \boldsymbol{f}(x,t)$ and $\boldsymbol{u}_0 = \boldsymbol{u}_0(x)$ are given vector fields, with $\boldsymbol{u}_0$ solenoidal, satisfying appropriate assumptions. In turn, the study of the asymptotic properties (in space and time) of solutions to problem (VIII.3.1) is more conveniently done by means of the integral representation of the solutions through the fundamental tensor solution of the Oseen system $(\mathrm{VIII.3.1})_{1,2}$. Such a solution is well known, and was originally introduced by Oseen (1927, §5).

The objective of this section will thus be to recall some of the relevant properties of the Oseen fundamental solution, and to establish certain *uniform* properties concerning its asymptotic behavior in space and time.

Since the study of all principal properties of this solution will be performed in great depth in Volume II,[1] we shall restrict ourselves to state those we need here, without detailed proofs, referring the reader to the existing literature for all the missing details.

Following Oseen (1927, §5), for all $\mathcal{R} \geq 0$ we introduce tensor and vector fields, $\boldsymbol{\Gamma}$ and $\boldsymbol{\gamma}$, respectively, defined by[2]

$$\begin{aligned}\Gamma_{ij}(x-y,t-\tau;\mathcal{R}) &= -\delta_{ij}\Delta\Psi(|x+\mathcal{R}(t-\tau)\boldsymbol{e}_1-y|,t-\tau)\\ &\qquad+\frac{\partial^2}{\partial y_i\partial y_j}\Psi(|x+\mathcal{R}(t-\tau)\boldsymbol{e}_1-y|,t-\tau)\\ \gamma_j(x-y,t-\tau) &= \frac{\partial}{\partial y_j}(\Delta+\frac{\partial}{\partial\tau})\Psi(|x+\mathcal{R}(t-\tau)\boldsymbol{e}_1-y|,t-\tau)\,,\end{aligned} \tag{VIII.3.2}$$

where $\Psi = \Psi(r,s)$ is any real function that is defined and smooth for all $r \geq 0$ and all $s > 0$. By a straightforward calculation, we show that the fields (VIII.3.2) satisfy the following equations:

$$\begin{aligned}\frac{\partial\Gamma_{ij}}{\partial\tau} - \mathcal{R}\frac{\partial\Gamma_{ij}}{\partial y_1} + \Delta\Gamma_{ij} - \frac{\partial\gamma_j}{\partial y_i} &= -\delta_{ij}\left(\frac{\partial}{\partial\tau}+\Delta\right)\Delta\Psi\,,\\ \frac{\partial\Gamma_{ij}}{\partial y_i} &= 0\,,\end{aligned} \tag{VIII.3.3}$$

where "Δ" operates on the y-variables. The idea is now to choose Ψ in such a way that

$$\Delta\Psi(r,s) = -W(r,s)\,, \tag{VIII.3.4}$$

where $W = W(r,s)$ is the three-dimensional *Weierstrass kernel*, namely, the fundamental solution to the three-dimensional heat equation:

$$W = \begin{cases}(4\pi\, s)^{-3/2}e^{-\frac{r^2}{4s}}\,, & s>0\,,\\ 0\,, \quad s\leq 0\,.\end{cases} \tag{VIII.3.5}$$

This choice will then ensure that the right-hand side of $(\mathrm{VIII.3.3})_1$ vanishes for $x, y \in \mathbb{R}^3$ and $t > \tau$, and moreover, that when $\tau \to t^-$, it becomes "appropriately singular," in a way that will be clarified later on, in Lemma VIII.3.1. The requirement (VIII.3.4) means that

[1] Actually, in any space dimensions $n \geq 2$.

[2] The properties that we will state in this section continue to hold also for $\mathcal{R} < 0$. However, for the application we have in mind, we suppose that $\mathcal{R}$ is nonnegative.

$$\Delta\Psi \equiv \frac{1}{r}\frac{\partial^2(r\Psi)}{\partial r^2} = -(4\pi\, s)^{-3/2}e^{-\frac{r^2}{4s}}\,,$$

which gives, after a simple calculation and by fixing the integration constants suitably,

$$\Psi(r,s) = \frac{1}{4\pi^{3/2}r\, s^{1/2}}\int_0^r e^{-\frac{\rho^2}{4s}}\,d\rho\,. \tag{VIII.3.6}$$

Moreover, from (VIII.3.6) it also follows that (Oseen 1927, pp. 40–41)

$$\left(\frac{\partial}{\partial s} + \Delta_\xi\right)\Psi(|\xi|,s) = 0\,,\quad s>0\,,$$

so that from (VIII.3.2) we deduce $\gamma_j(x-y,t-\tau)=0$ for $t>\tau$.

Collecting all the above information, we then conclude that the fundamental tensor $\boldsymbol{\Gamma}$ has the following form, for $t>\tau$:

$$\begin{aligned}\Gamma_{ij}(x-y,t-\tau;\mathcal{R}) &= -\delta_{ij}\Delta\Psi(|x+\mathcal{R}(t-\tau)\boldsymbol{e}_1-y|,t-\tau)\\ &\quad+\frac{\partial^2}{\partial y_i\partial y_j}\Psi(|x+\mathcal{R}(t-\tau)\boldsymbol{e}_1-y|,t-\tau)\,,\\ \Psi(r,t-\tau) &= \frac{1}{4\pi^{3/2}r}\int_0^r \frac{e^{-\frac{\rho^2}{4(t-\tau)}}}{(t-\tau)^{1/2}}d\rho\,,\end{aligned} \tag{VIII.3.7}$$

while $\boldsymbol{\gamma}=\mathbf{0}$.

When $\mathcal{R}=0$, we set, for simplicity, $\boldsymbol{\Gamma}(\xi,s;0)=\boldsymbol{\Gamma}(\xi,s)$, $\xi\in\mathbb{R}^3$, $s>0$.

Clearly, for $t>\tau$, we have

$$\begin{aligned}\frac{\partial\Gamma_{ij}}{\partial\tau} - \mathcal{R}\frac{\partial\Gamma_{ij}}{\partial y_1} + \Delta\Gamma_{ij} &= 0\,,\\ \frac{\partial\Gamma_{ij}}{\partial y_i} &= 0\,,\end{aligned} \tag{VIII.3.8}$$

with "Δ" operating on the y-variables, whereas, taking into account that

$$\frac{\partial\Gamma_{ij}}{\partial\tau} = -\frac{\partial\Gamma_{ij}}{\partial t}\,,\quad \frac{\partial\Gamma_{ij}}{\partial y_k} = -\frac{\partial\Gamma_{ij}}{\partial x_k}\,,\quad \frac{\partial^2\Gamma_{ij}}{\partial y_k\partial y_\ell} = \frac{\partial^2\Gamma_{ij}}{\partial x_k\partial x_\ell}\,,\quad i,j,k,\ell=1,2,3\,,$$

we have

$$\begin{aligned}\frac{\partial\Gamma_{ij}}{\partial t} - \mathcal{R}\frac{\partial\Gamma_{ij}}{\partial x_1} - \Delta\Gamma_{ij} &= 0\,,\\ \frac{\partial\Gamma_{ij}}{\partial x_i} &= 0\,,\end{aligned} \tag{VIII.3.9}$$

where now "Δ" operates on the x-variables.

The next result shows the way in which the tensor $\boldsymbol{\Gamma}=\boldsymbol{\Gamma}(x-y,t-\tau;\mathcal{R})$ becomes singular at $(x,t)=(y,\tau)$. For its proof we refer the reader to Oseen (1927, §53).

Lemma VIII.3.1 *Let A be a bounded, locally Lipschitz domain and let $\boldsymbol{u} = \boldsymbol{u}(x,t)$ be a solenoidal vector field in $C(\overline{A} \times [t-\delta, t])$, for some $\delta > 0$. Then, for all $\mathcal{R} \geq 0$,*

$$\lim_{\tau \to t^-} \int_A u_i(y,\tau) \Gamma_{ij}(x-y, t-\tau; \mathcal{R}) dy = u_j(x,t) - \frac{1}{4\pi} \int_{\partial A} \frac{(x_j - y_j)}{|x-y|^3} u_i(y,t) N_i d\sigma_y \,,$$

where $\boldsymbol{N}$ is the unit outer normal at ∂A.

An issue that is of basic importance to our aims is the study of suitable asymptotic properties in space of the fundamental solution $\boldsymbol{\Gamma}$ and its spatial derivatives. In order to reach this goal, we begin to recall the following result, for whose proof we refer to Oseen (1927, §55, §73) and Solonnikov (1964, Corollary in §5).

Lemma VIII.3.2 *There exists a constant $C > 0$ such that the following estimates hold, for all $(\tau, \xi) \in [0, \infty) \times \mathbb{R}^3 - \{(0,0)\}$:*

$$\begin{aligned}
|D^2_\xi \Psi(\xi,\tau)| &\leq \frac{C}{(\tau + |\xi|^2)^{3/2}} \,, \\
|\boldsymbol{\Gamma}(\xi,\tau;\mathcal{R})| &\leq \frac{C}{(\tau + |\xi + \mathcal{R}\,\tau\, \boldsymbol{e}_1|^2)^{3/2}} \,, \\
|D_\xi \boldsymbol{\Gamma}(\xi,\tau;\mathcal{R})| &\leq \frac{C}{(\tau + |\xi + \mathcal{R}\,\tau\, \boldsymbol{e}_1|^2)^{2}} \,, \\
|D^2_\xi \boldsymbol{\Gamma}(\xi,\tau;\mathcal{R})| &\leq \frac{C}{(\tau + |\xi + \mathcal{R}\,\tau\, \boldsymbol{e}_1|^2)^{5/2}} \,.
\end{aligned}$$

With this result in hand, we can then prove the next one.

Lemma VIII.3.3 *Let*

$$\Gamma_\ell(\xi;\mathcal{R}) = \int_0^\infty |D^\ell \boldsymbol{\Gamma}(\xi,\tau;\mathcal{R})|\, d\tau \,, \quad \ell = 0,1,2 \,,$$

and set $\Gamma_\ell(\xi) := \Gamma_\ell(\xi, 0)$, $\ell = 0,1,2$. The following properties hold:

(i) *If $\mathcal{R} = 0$, we have, for all $\xi \in \mathbb{R}^3 - \{0\}$,*

$$\Gamma_\ell(\xi) \leq \frac{C}{|\xi|^{\ell+1}} \,, \quad \ell = 0,1,2 \,, \tag{VIII.3.10}$$

with $C = C(\ell) > 0$.

(ii) *If $\mathcal{R} > 0$, let*

$$s(\xi) = |\xi| + \xi_1, \quad \xi \in \mathbb{R}^3 \,, \tag{VIII.3.11}$$

and $0 < \beta < \frac{1}{2}$. Then, for all $\xi \in \mathbb{R}^3 - \{0\}$,

$$\begin{aligned}
&\Gamma_0(\xi;\mathcal{R}) \le \frac{2}{|\xi|(1+2\mathcal{R}\,s(\xi))}\,,\\
&\Gamma_1(\xi;\mathcal{R}) \le C\begin{cases}\mathcal{R}^{1/2}|\xi|^{-3/2}(1+2\mathcal{R}\,s(\xi))^{-3/2}, & \textit{if } |\xi|\ge\beta/\mathcal{R},\\ |\xi|^{-2}, \quad \textit{if } |\xi|\in(0,\beta/\mathcal{R})\,, \end{cases}\\
&\Gamma_2(\xi;\mathcal{R}) \le C\begin{cases}\mathcal{R}|\xi|^{-2}(1+2\mathcal{R}\,s(\xi))^{-2}, & \textit{if } |\xi|\ge\beta/\mathcal{R},\\ |\xi|^{-3}, \quad \textit{if } |\xi|\in(0,\beta/\mathcal{R})\,, \end{cases}
\end{aligned} \tag{VIII.3.12}$$

where $C = C(\beta)$.

Proof. We begin by recalling the following noteworthy formulas (see, e.g., Gradshteyn & Ryzhik 1980, §§3.241, 3.249, 3.252):

$$\begin{aligned}
&\int_0^\infty \frac{dx}{(ax^2+bx+c)^{3/2}} = \frac{2}{(b+2\sqrt{ac})\sqrt{c}}\,, \quad \text{if } a\ge 0,\ b,c>0\,,\\
&\int_0^\infty \frac{dx}{(a\,x^2+b\,x+c)^2} = \begin{cases}\dfrac{\pi}{4c\sqrt{ac}}\,, & \text{if } b=0,\ a,c>0\,,\\[2mm] \dfrac{1}{b\,c}\,, & \text{if } a=0,\ b,c>0\,,\end{cases}\\
&\int_0^\infty \frac{dx}{(ax^2+bx+c)^{5/2}} = \frac{4}{3}\frac{4\sqrt{ac}+b}{c^{3/2}(\sqrt{ac}+b)^2}\,, \quad \text{if } a\ge 0,\ b,c>0\,.
\end{aligned} \tag{VIII.3.13}$$

The inequalities in part (i) are an immediate consequence of the estimates given in Lemma VIII.3.2 and of (VIII.3.13) evaluated for $a=0$, $b=1$ and $c=|\xi|^2$. In order to prove part (ii), we notice that in view of the estimates given in Lemma VIII.3.2, it is enough to show that the integral

$$I(\xi,\theta) \equiv \int_0^\infty \frac{1}{(\tau+|\xi+\tau\mathcal{R}\,e_1|^2)^\theta}d\tau = \int_0^\infty \frac{1}{(\mathcal{R}^2\tau^2+(1+2\mathcal{R}\,\xi_1)\tau+|\xi|^2)^\theta}d\tau\,,$$

for $\theta = 3/2, 2, 5/2$, can be increased by the right-hand side of $(\text{VIII.3.12})_1$ if $\theta = 3/2$, and by that of $(\text{VIII.3.12})_2$ if $\theta = 2$, for the specified values of $|\xi|$. To this end, we observe that $I(\xi,\theta)$ is convergent for the values of θ we are considering, provided $\xi \ne 0$. Now, for $\theta = 3/2$, by a direct calculation that uses $(\text{VIII.3.13})_1$ with $a=\mathcal{R}^2$, $b=1+2\mathcal{R}\,\xi_1$, and $c=|\xi|^2$, we get

$$I(\xi,3/2) = \frac{2}{|\xi|(1+2\mathcal{R}\,s(\xi))},$$

for all $\xi\ne 0$, which proves the first inequality in (VIII.3.12). Consider next the cases $\theta = 2, 5/2$. For all $|\xi| < \beta/\mathcal{R}$, we have

$$\tau + |\xi+\tau\mathcal{R}\,e_1|^2 \ge (1-2\beta)\tau + |\xi|^2 \tag{VIII.3.14}$$

and therefore by $(\text{VIII.3.13})_{2,3}$ with $a=0$, $b=(1-2\beta)$, and $c=|\xi|^2$,

$$I(\xi,\theta) \le \int_0^\infty \frac{1}{((1-2\beta)\tau+|\xi|^2)^\theta}d\tau = \frac{c_{\theta,\beta}}{|\xi|^{2(\theta-1)}} \quad \text{for all } \xi \neq 0\,,$$

which proves the second part of the second and third inequalities in (VIII.3.12). It remains to discuss the case $|\xi| \ge \beta/\mathcal{R}$. We consider separately the cases (a) $1+2\mathcal{R}\,\xi_1 \ge 0$ and (b) $1+2\mathcal{R}\,\xi_1 < 0$. In case (a), we have $\tau + |\xi+\tau\mathcal{R}\,e_1|^2 \ge \mathcal{R}^2\tau^2+|\xi|^2$ for all $\tau \ge 0$, and consequently from (VIII.3.13)$_{2,3}$ with $b=0$, $a=\mathcal{R}^2$, and $c=|\xi|^2$, we have

$$I(\xi,\theta) \le \int_0^\infty \frac{1}{(\mathcal{R}^2\tau^2+|\xi|^2)^\theta}d\tau = \frac{C_\theta}{\mathcal{R}|\xi|^{2\theta-1}} \quad \text{for all } \xi \neq 0.$$

We next observe that $|\xi| \ge \beta/\mathcal{R}$ implies

$$1+2\mathcal{R}\;s(\xi) \le 1+4\mathcal{R}|\xi| \le \frac{(1+4\beta)\mathcal{R}}{\beta}|\xi|\,,$$

from which it follows that

$$I(\xi,\theta) \le C_{\theta,\beta}\frac{\mathcal{R}^{\theta-3/2}}{(1+2\mathcal{R}\;s(\xi))^{\theta-1/2}|\xi|^{\theta-1/2}}.$$

In case (b), since

$$\begin{aligned}4\mathcal{R}^2|\xi|^2-(1+2\mathcal{R}\,\xi_1)^2 &= (2\mathcal{R}\,|\xi|-2\mathcal{R}\,\xi_1-1)(2\mathcal{R}\,|\xi|+2\mathcal{R}\,\xi_1+1)\\ &= (2\mathcal{R}\;s(\xi)+1)(2\mathcal{R}\,|\xi|-(2\mathcal{R}\,\xi_1+1))>0,\end{aligned}$$

we obtain

$$\begin{aligned}I(\xi,\theta) &= \int_0^\infty \frac{d\xi}{(|\xi|^2+(1+2\mathcal{R}x_1)\tau+\mathcal{R}^2\tau^2)^\theta}\\ &= \int_0^\infty \frac{d\xi}{\left[(\mathcal{R}\tau+(1+2\mathcal{R}x_1)/2\mathcal{R})^2+\left(4\mathcal{R}^2|\xi|^2-(1+2\mathcal{R}\xi_1)^2\right)/(2\mathcal{R})^2\right]^\theta}\\ &\le \frac{1}{\mathcal{R}}\int_0^\infty \frac{dr}{\left[r^2+(2\mathcal{R}\;s(\xi)+1)(2\mathcal{R}\,|\xi|-(2\mathcal{R}\,\xi_1+1))/(2\mathcal{R})^2\right]^\theta}\\ &\le \frac{1}{\mathcal{R}}\int_0^\infty \frac{dr}{\left[r^2+(\mathcal{R}s(\xi)+1)|\xi|/(2\mathcal{R})\right]^\theta}\\ &\le c_\theta\frac{\mathcal{R}^{\theta-3/2}}{(1+2\mathcal{R}\;s(\xi))^{\theta-1/2}|\xi|^{\theta-1/2}}\,.\end{aligned}$$

The proof of the lemma is complete. □

Remark VIII.3.1 The reader might have recognized that the estimates proved in Lemma VIII.3.3 are of the same type as those furnished for the Stokes fundamental solution $\boldsymbol{U}(\xi)$ in (IV.2.3), in the case $\mathcal{R}=0$, and for the (steady-state) Oseen fundamental solution, $\boldsymbol{E}(\xi;\mathcal{R}/2)$, in (VII.3.23) and (VII.3.31), in the case $\mathcal{R}>0$. These properties–which are at the foundation of our approach to the study of the asymptotic behavior of weak solutions to (VIII.0.7), (VIII.0.2)–are not surprising, in that they can be expected from the fact that for all $\xi\in\mathbb{R}^3-\{0\}$,

$$\lim_{t\to\infty}\int_0^t \boldsymbol{\Gamma}(\xi,\tau)d\tau=\boldsymbol{U}(\xi)\,,$$
$$\lim_{t\to\infty}\int_0^t \boldsymbol{\Gamma}(\xi,\tau;\mathcal{R})d\tau=\boldsymbol{E}(\xi;\mathcal{R}/2)\,;$$

see Thomann & Guenther (2006), Deuring, Kračmar, & Nečasová (2009). ■

Exercise VIII.3.1 (Generalization of Lemma VIII.3.3(i)) Let α be a multi-index, and set

$$\Gamma_\alpha(\xi)=\int_0^\infty |D^\alpha\boldsymbol{\Gamma}(\xi,t;0)|\,dt\,.$$

Employing the pointwise inequality (Solonnikov 1964; Corollary in §5)

$$|D^\alpha_\xi\boldsymbol{\Gamma}(\xi,t;0)|\le\frac{C}{(t+|\xi|^2)^{(|\alpha|+3)/2}}\,,\quad |\alpha|=0,1,2,\dots\,,$$

show that

$$\Gamma_\alpha(\xi)\le\frac{C_\alpha}{|\xi|^{|\alpha|+1}}\,,\quad\text{for all }\xi\in\mathbb{R}^3-\{0\}\,.$$

Exercise VIII.3.2 (Mizumachi, 1984, Lemma 3) Prove the following inequalities. Let $1\le q_1<\frac{3}{2}$ and $\frac{3}{2}<q_2$. Then

$$\int_0^\infty\left(\int_{\mathbb{R}^3\cap\{x_2^2+x_3^2<1\}}(t+|x+t\boldsymbol{e}_1|^2)^{-2q_1}\,dx\right)^{\frac{1}{q_1}}dt<\infty,$$
$$\int_0^\infty\left(\int_{\mathbb{R}^3\cap\{x_2^2+x_3^2\ge1\}}(t+|x+t\boldsymbol{e}_1|^2)^{-2q_2}\,dx\right)^{\frac{1}{q_2}}dt<\infty\,.$$

Hint: Use polar coordinates with respect to (x_2,x_3).

Our next task is to give pointwise estimates for the following convolution integrals:

$$\begin{aligned}\mathcal{K}(x)&=\int_{\mathbb{R}^3}\frac{\Gamma_1(x-y)}{(1+|y|)^2}\,dy\,,\\ \mathcal{J}(x;\mathcal{R})&=\int_{\mathbb{R}^3}\frac{\Gamma_1(x-y;\mathcal{R})}{(1+|y|)^2(1+2\mathcal{R}\,s(y))^2}\,dy\,,\quad\mathcal{R}>0\,.\end{aligned}\tag{VIII.3.15}$$

For $\mathcal{K}$, we have the following result, of simple proof.

Lemma VIII.3.4 *There exists a constant $C_1 > 0$ such that*

$$\mathcal{K}(x) \le \frac{C_1}{1+|x|}\,. \tag{VIII.3.16}$$

Proof. By Lemma VIII.3.3(i), we have

$$\mathcal{K}(x) \le c\int_{\mathbb{R}^3} \frac{1}{|x-y|^2(1+|y|)^2}\,dy\,,$$

and so, by Lemma II.9.2, it follows that

$$\mathcal{K}(x) \le c|x|^{-1}\,, \quad \text{for all } x \in \mathbb{R}^3 - \{0\}. \tag{VIII.3.17}$$

Furthermore, by the Hölder inequality with exponent $q \in (3/2, 3)$, we obtain

$$\begin{aligned}\mathcal{K}(x) &\le c_1\int_{|x-y|\le 1} |x-y|^{-2}dy \\ &\quad + c_2\left(\int_{\mathbb{R}^3}(1+|x-y|)^{-2q'}dy\right)^{1/q'}\left(\int_{\mathbb{R}^3}(1+|y|)^{-2q}dy\right)^{1/q} \\ &\le c_3\,.\end{aligned} \tag{VIII.3.18}$$

The proof of (VIII.3.16) then follows from (VIII.3.17) and (VIII.3.18). The lemma is completely proved.

□

In order to evaluate the other convolution in (VIII.3.15), we need the following lemma, which is a particular case of a more general result due to Farwig (1992b, Lemma 3.1), and to which we refer the reader for the proof (see also Kračmar, Novotný, & Pokorný 2001, Theorem 3.2).

Lemma VIII.3.5 *Let*

$$\mathcal{G}(x) = \int_{\mathbb{R}^3} \eta_1(y)\eta_2(x-y)\,dy\,,$$

where η_1 satisfies at least one of the conditions

$$\eta_1(y) \le \begin{cases} c_1(1+|y|)^{-3/2}(1+2s(y))^{-3/2} & \textit{if } |y| \ge c_0\,, \\ c_2|y|^{-2} & \textit{if } |y| \le c_0\,, \end{cases} \tag{A}$$

$$\eta_1(y) \le c_3(1+|y|)^{-3/2}(1+2s(y))^{-3/2}\,, \quad \textit{for all } y \in \mathbb{R}^3\,, \tag{B}$$

whereas η_2 satisfies

$$\eta_2(y) \le c_4(1+|y|)^{-2}(1+2s(y))^{-2}\,, \quad \textit{for all } y \in \mathbb{R}^3\,,$$

for some (positive) constants c_i, $i = 0, \dots, 4$. In case (B), *this assumption may be weakened to the following one:*

$$\eta_2(y) \le \begin{cases} c_5(1+|y|)^{-2}(1+2s(y))^{-2} & \text{if } |y| \ge c_6\,, \\ c_7|y|^{-2} & \text{if } |y| \le c_6\,, \end{cases}$$

for some $c_5, c_6, c_7 > 0$. Then, there is $C > 0$ such that[3]

$$\mathcal{G}(x) \le \frac{C}{(1+|x|)(1+2s(x))}\,.$$

The previous lemma allows us to prove the following one.

Lemma VIII.3.6 *Let $K = \max\{1, \mathcal{R}\}$. There exists a constant $C > 0$, depending only on β, such that*

$$\mathcal{J}(x;\mathcal{R}) \le \frac{K\,C}{(1+|x|)(1+2\mathcal{R}s(x))}\,. \tag{VIII.3.19}$$

Proof. The only difficulty of the proof consists in a careful evaluation of the dependence on the Reynolds number $\mathcal{R}$ of the constant entering the numerator of the fraction in the inequality (VIII.3.19). For a given $x \in \mathbb{R}^3$, we consider the following partition of $\mathbb{R}^3$:

$$\mathbb{R}^3 = \{y \in \mathbb{R}^3 : |x-y| \le \beta/\mathcal{R}\} \cup \{y \in \mathbb{R}^3 : |x-y| > \beta/\mathcal{R}\}\,.$$

From (VIII.3.12) and (VIII.3.15)$_2$, we thus have, accordingly,

$$\begin{aligned} \mathcal{J}(x;\mathcal{R}) \le C(\beta)\Big(&\int_{\mathcal{R}|x-y|\le\beta} |x-y|^{-2}(1+|y|)^{-2}(1+2\mathcal{R}\,s(y))^{-2}dy \\ &+\mathcal{R}^{\frac12}\int_{\mathcal{R}|x-y|>\beta}[|x-y|(1+2\mathcal{R}\,s(x-y))]^{-\frac32}(1+|y|)^{-2}(1+2\mathcal{R}\,s(y))^{-2}dy\Big) \\ &\equiv C(\beta)\left(\mathcal{J}_1(x)+\mathcal{J}_2(x)\right)\,. \end{aligned} \tag{VIII.3.20}$$

We begin by estimating the function $\mathcal{J}_2(x)$. With the substitution $\xi = \mathcal{R}y$, we obtain

$$\mathcal{J}_2(x) = \frac{1}{\mathcal{R}}\int_{|\mathcal{R}\,x-\xi|>\beta}[|\mathcal{R}x-\xi|(1+2s(\mathcal{R}\,x-\xi))]^{-\frac32}(1+|\xi|/\mathcal{R})^{-2}(1+2s(\xi))^{-2}d\xi\,, \tag{VIII.3.21}$$

[3] Actually, as shown by Farwig (1992b), under the given assumption, the function $\mathcal{G}$ possesses a better decay rate than that stated here. However, this improvement will not be necessary to our purposes. For related inequalities, see also Lemma XI.6.2

and so

$$\begin{aligned}\mathcal{J}_2(x) &= \frac{1}{\mathcal{R}}\int_{\{|\mathcal{R}\,x-\xi|>\beta\}\cap\{|\xi|<1\}} j(x,y)dy + \frac{1}{\mathcal{R}}\int_{\{|\mathcal{R}\,x-\xi|>\beta\}\cap\{|\xi|\geq 1\}} j(x,y)dy \\ &\equiv \mathcal{J}_2^{(1)}(x) + \mathcal{J}_2^{(2)}(x)\,,\end{aligned} \tag{VIII.3.22}$$

where $j(x,y)$ denotes the integrand function in (VIII.3.21). Observing that

$$\xi \in \{\zeta \in \mathbb{R}^3 : |\mathcal{R}x - \zeta| \geq \beta\} \Longrightarrow \frac{1}{|\mathcal{R}x-\xi|} \geq \frac{4}{1+|\mathcal{R}x-\xi|}\,, \tag{VIII.3.23}$$

we obtain

$$\begin{aligned}\mathcal{J}_2^{(1)}(x) &\leq \mathcal{R}\int_{|\xi|<1}[(1+|\mathcal{R}x-\xi|)(1+2s(\mathcal{R}\,x-\xi))]^{-\frac{3}{2}}|\xi|^{-2}(1+2s(\xi))^{-2}d\xi \\ &= \mathcal{R}\int_{\mathbb{R}^3}[(1+|\mathcal{R}x-\xi|)(1+2s(\mathcal{R}\,x-\xi))]^{-\frac{3}{2}}\eta(\xi)d\xi\,,\end{aligned}$$

where

$$\eta(z) = \begin{cases} |z|^{-2} & \text{if } |z|<1\,, \\ 0 & \text{if } |z|\geq 1\,. \end{cases} \tag{VIII.3.24}$$

From Lemma VIII.3.5 we thus obtain

$$\mathcal{J}_2^{(1)}(x) \leq c\frac{\mathcal{R}}{(1+\mathcal{R}|x|)(1+2\mathcal{R}\,s(x))} \leq \frac{c\,K}{(1+|x|)(1+2\mathcal{R}\,s(x))}\,, \tag{VIII.3.25}$$

where we have used the elementary inequality

$$\frac{t}{1+t\,a} \leq \frac{\max\{1,t\}}{1+a}\,, \quad t,a\geq 0\,. \tag{VIII.3.26}$$

In a similar fashion, again by (VIII.3.23) and by Lemma VIII.3.5,

$$\begin{aligned}\mathcal{J}_2^{(2)}(x) &\leq \mathcal{R}\int_{|\xi|\geq 1}[(1+|\mathcal{R}x-\xi|)(1+2s(\mathcal{R}\,x-\xi))]^{-\frac{3}{2}}[(1+|\xi|)(1+2s(\xi))]^{-2}d\xi \\ &\leq \mathcal{R}\int_{\mathbb{R}^3}[(1+|\mathcal{R}x-\xi|)(1+2s(\mathcal{R}\,x-\xi))]^{-\frac{3}{2}}[(1+|\xi|)(1+2s(\xi))]^{-2}d\xi \\ &\leq \frac{c\,K}{(1+|x|)(1+2\mathcal{R}\,s(x))}\,.\end{aligned}$$

From this latter relation and from (VIII.3.21)–(VIII.3.25), we conclude that

$$\mathcal{J}_2(x) \leq \frac{c\,K}{(1+|x|)(1+2\mathcal{R}\,s(x))}\,. \tag{VIII.3.27}$$

In order to estimate $\mathcal{J}_1(x)$ we distinguish the following cases: (a) $|x|\leq\beta/\mathcal{R}$, and (b) $|x|>\beta/\mathcal{R}$. In case (a), from the obvious inequality

$$\frac{1}{1+4\beta} \le \frac{1}{1+2\mathcal{R}\,s(x)} \le 1 \quad \text{for all } x \in \overline{B_{\beta/\mathcal{R}}}\,,$$

and from Lemma II.9.2 and (VIII.3.18), we obtain

$$\begin{aligned}\mathcal{J}_1(x) &\le \int_{\mathbb{R}^3} |x-y|^{-2}(1+|y|)^{-2}dy \le \frac{c_1}{1+|x|}\\ &\le \frac{c_2(\beta)}{(1+|x|)(1+2\mathcal{R}\,s(x))}\,, \text{ for } |x| \le \beta/\mathcal{R}\,.\end{aligned} \tag{VIII.3.28}$$

We next assume $|x| > \beta/\mathcal{R}$. From (VIII.3.20), we have

$$\begin{aligned}\mathcal{J}_1(x) \le C(\beta)\Big(&\int_{\mathcal{R}|x-y|\le\frac{\beta}{2}} |x-y|^{-2}(1+|y|)^{-2}(1+2\mathcal{R}\,s(y))^{-2}dy\\ &+\int_{\frac{\beta}{2}<\mathcal{R}|x-y|\le\beta} |x-y|^{-2}(1+|y|)^{-2}(1+2\mathcal{R}\,s(y))^{-2}dy\Big)\\ &\equiv C(\beta)\left(\mathcal{J}_3(x)+\mathcal{J}_4(x)\right)\,.\end{aligned} \tag{VIII.3.29}$$

With the change of variable $\xi = \mathcal{R}y$, we obtain

$$\begin{aligned}\mathcal{J}_3(x) &= \frac{1}{\mathcal{R}}\int_{|\mathcal{R}x-\xi|\le\frac{\beta}{2}} |\mathcal{R}x-\xi|^{-2}(1+|\xi|/\mathcal{R})^{-2}(1+2\,s(\xi))^{-2}d\xi\\ &\le \mathcal{R}\int_{|\mathcal{R}x-\xi|\le\frac{\beta}{2}} |\mathcal{R}x-\xi|^{-2}|\xi|^{-2}(1+2\,s(\xi))^{-2}d\xi\,.\end{aligned}$$

Since $\mathcal{R}\,|x| > \beta$, we have, by the triangle inequality,

$$|\xi| \ge \mathcal{R}\,|x| - |\mathcal{R}\,x-\xi| > \beta-\beta/2 = \beta/2\,,$$

for all ξ such that $|\mathcal{R}x-\xi| \le \beta/2$. Therefore,

$$\begin{aligned}\mathcal{J}_3(x) &\le c(\beta)\mathcal{R}\int_{|\mathcal{R}x-\xi|\le\frac{\beta}{2}} |\mathcal{R}x-\xi|^{-2}(1+|\xi|)^{-2}(1+2\,s(\xi))^{-2}d\xi\\ &= c(\beta)\mathcal{R}\int_{\mathbb{R}^3} \eta(\xi)(1+|\mathcal{R}x-\xi|)^{-2}(1+2\,s(\mathcal{R}x-\xi))^{-2}d\xi\,,\end{aligned}$$

where the function η is defined in (VIII.3.24). As a consequence, from Lemma VIII.3.5 and (VIII.3.26) we deduce

$$\mathcal{J}_3(x) \le \frac{c\,K}{(1+|x|)(1+2\mathcal{R}\,s(x))}\,. \tag{VIII.3.30}$$

Concerning $\mathcal{J}_4(x)$, we obtain

$$\begin{aligned}\mathcal{J}_4(x) &= \frac{1}{\mathcal{R}}\int_{\frac{\beta}{2}<|\mathcal{R}x-\xi|\le\beta} |\mathcal{R}x-\xi|^{-2}(1+|\xi|/\mathcal{R})^{-2}(1+2\,s(\xi))^{-2}d\xi\\ &\le \mathcal{R}\int_{\frac{\beta}{2}<|\mathcal{R}x-\xi|\le\beta} |\mathcal{R}x-\xi|^{-2}|\xi|^{-2}(1+2\,s(\xi))^{-2}d\xi\,,\end{aligned} \tag{VIII.3.31}$$

which implies

$$\begin{aligned}\mathcal{J}_4(x) &\leq \mathcal{R}\int_{\{\frac{\beta}{2}<|\mathcal{R}\,x-\xi|\leq\beta\}\cap\{|\xi|<1\}}\widetilde{j}(x,\xi)d\xi \\ &\quad+\mathcal{R}\int_{\{\frac{\beta}{2}<|\mathcal{R}\,x-\xi|\leq\beta\}\cap\{|\xi|\geq 1\}}\widetilde{j}(x,\xi)d\xi \\ &\equiv \mathcal{J}_4^{(1)}(x)+\mathcal{J}_4^{(2)}(x)\,,\end{aligned} \tag{VIII.3.32}$$

where $\widetilde{j}=\widetilde{j}(x,\xi)$ is the integrand function in the last integral in (VIII.3.31). We observe that obviously,

$$\begin{gathered}\frac{1}{|\mathcal{R}x-\xi|^2}\leq\frac{c(\beta)}{(1+|\mathcal{R}x-\xi|)^{3/2}(1+2s(\mathcal{R}x-\xi))^{3/2}}\,, \\ \text{for all } \xi \text{ such that } \tfrac{\beta}{2}<|\mathcal{R}\,x-\xi|\leq\beta\,.\end{gathered} \tag{VIII.3.33}$$

Consequently,

$$\mathcal{J}_4^{(1)}(x)\leq\mathcal{R}\int_{\mathbb{R}^3}[(1+|\mathcal{R}x-\xi|)(1+2s(\mathcal{R}x-\xi))]^{-3/2}\eta(\xi)d\xi\,,$$

where η is defined in (VIII.3.24). So, from Lemma VIII.3.5 and (VIII.3.26) we obtain

$$\mathcal{J}_4^{(1)}(x)\leq\frac{c\,K}{(1+|x|)(1+2\mathcal{R}\,s(x))}\,. \tag{VIII.3.34}$$

Likewise, using (VIII.3.33), we obtain

$$\mathcal{J}_4^{(2)}(x)\leq\mathcal{R}\int_{\mathbb{R}^3}[(1+|\mathcal{R}x-\xi|)(1+2s(\mathcal{R}x-\xi))]^{-3/2}[(1+|\xi|)(1+2\,s(\xi))]^{-2}d\xi\,,$$

and again by Lemma VIII.3.5 and (VIII.3.26), we obtain

$$\mathcal{J}_4^{(2)}(x)\leq\frac{c\,K}{(1+|x|)(1+2\mathcal{R}\,s(x))}\,. \tag{VIII.3.35}$$

By collecting (VIII.3.29)–(VIII.3.32), and (VIII.3.34), (VIII.3.35), we deduce

$$\mathcal{J}_1(x)\leq\frac{c\,K}{(1+|x|)(1+2\mathcal{R}\,s(x))}\,,\quad |x|>\beta/\mathcal{R}\,. \tag{VIII.3.36}$$

Inequality (VIII.3.19) follows from (VIII.3.20), (VIII.3.27), (VIII.3.29), and (VIII.3.36), and the proof of the lemma is complete.

□

VIII.4 On the Unique Solvability of the Oseen Initial-Value Problem

The objective of this section is to study existence and uniqueness of solutions to the Cauchy problem (VIII.3.1), under suitable assumptions on the data $\boldsymbol{f}$ and $\boldsymbol{u}_0$. Since the problem is linear, we can split it into the following two separate problems:

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{w}}{\partial t} = \Delta \boldsymbol{w} + \mathcal{R}\frac{\partial \boldsymbol{w}}{\partial x_1} - \nabla \phi + \boldsymbol{f} \\ &\nabla \cdot \boldsymbol{w} = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3 \times (0,T) \qquad \text{(VIII.4.1)}$$
$$\boldsymbol{w}(x,0) = \boldsymbol{0}\,,$$

and

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{v}}{\partial t} = \Delta \boldsymbol{v} + \mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1} \\ &\nabla \cdot \boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3 \times (0,T) \qquad \text{(VIII.4.2)}$$
$$\boldsymbol{v}(x,0) = \boldsymbol{u}_0\,,$$

where T is an arbitrary positive number.

It is useful to introduce some notation. If A is a domain in $\mathbb{R}^3$, and $\mathfrak{t} \in (0,\infty]$, we define

$$A_{\mathfrak{t}} = A \times (0,\mathfrak{t})\,.$$

Moreover, for $q \in (1,\infty)$, we set

$$\mathcal{L}^q(A_{\mathfrak{t}}) = \Big\{ \boldsymbol{w} : A_{\mathfrak{t}} \to \mathbb{R}^3, \text{ and } \phi : A_{\mathfrak{t}} \to \mathbb{R} : \\ \boldsymbol{w}\,,\ \frac{\partial \boldsymbol{w}}{\partial t}\,,\ \nabla \boldsymbol{w}\,,\ D^2\boldsymbol{w} \in L^q(A_{\mathfrak{t}})\,,\ \ \phi \in L^q_{loc}(A_{\mathfrak{t}})\,,\ \nabla\phi \in L^q(A_{\mathfrak{t}}) \Big\}\,.$$

We also set

$$\|u\|_{(q,r),A,\mathfrak{t}} \equiv \begin{cases} \left(\displaystyle\int_0^{\mathfrak{t}} \|u(t)\|^r_{q,A}\, dt \right)^{1/r} & \text{if } r \in [1,\infty),\ q \in [1,\infty]\,, \\ \operatorname*{ess\,sup}_{t\in[0,\mathfrak{t}]} \|u(t)\|_{q,A} & \text{if } r = \infty,\ q \in [1,\infty]\,, \end{cases}$$

and define

$$L^{r,q}(A_{\mathfrak{t}}) = \{u : A_{\mathfrak{t}} \to \mathbb{R}\,, \text{with } \|u\|_{(q,r),A,\mathfrak{t}} < \infty\}\,.$$

As is customary, whenever confusion does not arise, we shall omit the subscript A. Clearly, $L^{q,q}(A_{\mathfrak{t}}) = L^q(A_{\mathfrak{t}})$.

Exercise VIII.4.1 Show that if $\boldsymbol{w} \in \mathcal{L}^q(A_{\mathfrak{t}})$, then, possibly redefined on a set of zero measure in $(0, \mathfrak{t})$, $\boldsymbol{w}$ is continuous at all $t \in (0, \mathfrak{t})$ in the norm $\|\cdot\|_q$. *Hint*: Mollify $\boldsymbol{w}$ in t, and then use the inequality

$$\|\boldsymbol{w}(t_1) - \boldsymbol{w}(t_2)\|_q \leq q \int_{t_1}^{t_2} \|\boldsymbol{w}(s) - \boldsymbol{w}(t_1)\|_q^{q-1} \left\| \frac{\partial \boldsymbol{w}}{\partial s} \right\|_q ds\,.$$

We now investigate the solvability of problem (VIII.4.1).

Theorem VIII.4.1 *Let* $\boldsymbol{f} \in L^q(\mathbb{R}^3_T)$, $1 < q < \infty$. *Then there exists a pair* $(\boldsymbol{w}, \phi) \in \mathcal{L}^q(\mathbb{R}^3_T)$ *satisfying* (VIII.4.1)$_{1,2}$ *a.e. in* $\mathbb{R}^3_T$. *If* $q \in (1,3)$, *then* ϕ *is also in* $L^{q,3q/(3-q)}(\mathbb{R}^3_T)$. *Furthermore,* $\boldsymbol{w}$ *obeys* (VIII.4.1)$_3$ *in the following sense:*

$$\lim_{t\to 0^+} \|\boldsymbol{w}(t)\|_q = 0\,. \qquad \text{(VIII.4.3)}$$

In addition, there is $C_1 = C_1(q) > 0$ *such that*

$$\int_{\mathbb{R}^3_T} \left(|\boldsymbol{w}(t)|^q_{2,q} + \|\nabla\phi(t)\|^q_q\right) dt \leq C_1 \int_{\mathbb{R}^3_T} \|\boldsymbol{f}(t)\|^q_q dt\,, \qquad \text{(VIII.4.4)}$$

so that if $\mathcal{R} = 0$, *we also obtain*

$$\int_{\mathbb{R}^3_T} \left\| \frac{\partial \boldsymbol{w}}{\partial t} \right\|^q_q \leq 3^{q-1} C_1 \int_{\mathbb{R}^3_T} \|\boldsymbol{f}(t)\|^q_q dt\,. \qquad \text{(VIII.4.5)}$$

In addition, if $q \in (1,3)$, *we have*

$$\int_{\mathbb{R}^3_T} \|\phi(t)\|^q_{3q/(3-q)} dt \leq C_2 \int_{\mathbb{R}^3_T} \|\boldsymbol{f}(t)\|^q_q dt\,, \qquad \text{(VIII.4.6)}$$

for some $C_2 = C_2(q) > 0$. *Finally, let* $(\boldsymbol{w}_1, \phi_1) \in \mathcal{L}^r(\mathbb{R}^3_T)$, *for some* $r \in (1,\infty)$, *be another solution corresponding to the same* $\boldsymbol{f}$. *Then* $\boldsymbol{w} = \boldsymbol{w}_1$ *and* $\phi = \phi_1 + h(t)$, *a.e. in* $\mathbb{R}^3_T$, *where* h *is a function of time only.*[1]

Proof. It suffices to prove the result when $\mathcal{R} = 0$, in that the case $\mathcal{R} \neq 0$ follows by the change of variable $\boldsymbol{x} \to \boldsymbol{x} + \mathcal{R}\, t\boldsymbol{e}_1$. To show existence, we observe that we need to consider only the case $\boldsymbol{f} \in C_0^\infty(\mathbb{R}^3_T)$, since, as the reader may wish to show, the general case will then be a consequence of an elementary density argument. With all the above in mind, we set $\boldsymbol{w}$ and ϕ identically zero for $t < 0$, and define for a fixed $\lambda > 0$,

$$\boldsymbol{u}(x,t) = \boldsymbol{w}(x,t)\, e^{-\lambda t}\,, \quad \Phi(x,t) = \phi(x,t)\, e^{-\lambda t}\,, \quad \boldsymbol{F}(x,t) = \boldsymbol{f}(x,t)\, e^{-\lambda t}\,, \qquad \text{(VIII.4.7)}$$

so that $\boldsymbol{u}$, Φ, and $\boldsymbol{F}$ satisfy the following problem

[1] A more general uniqueness theorem will be given in Lemma VIII.4.2.

$$\left.\begin{aligned}\frac{\partial \boldsymbol{u}}{\partial t} &= \Delta \boldsymbol{u} - \lambda \boldsymbol{u} - \nabla \Phi + \boldsymbol{F} \\ \nabla \cdot \boldsymbol{u} &= 0\end{aligned}\right\} \text{ in } \mathbb{R}^3 \times (0,T) \tag{VIII.4.8}$$
$$\boldsymbol{u}(x,0) = \boldsymbol{0}\,.$$

A solution to (VIII.4.8) will be found by the Fourier transform method. Thus, letting

$$\begin{aligned}\boldsymbol{u}(x,t) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}+it\,\xi_0}\boldsymbol{U}(\xi,\xi_0)d\xi d\xi_0\,,\\ \Phi(x,t) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}+it\,\xi_0}\Psi(\xi,\xi_0)d\xi d\xi_0\,,\\ \boldsymbol{F}(x,t) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}+it\,\xi_0}\boldsymbol{G}(\xi,\xi_0)d\xi d\xi_0\,,\end{aligned} \tag{VIII.4.9}$$

and replacing these expressions back in $(\text{VIII.4.8})_{1,2}$, we obtain

$$\begin{aligned}(i\,\xi_0 + \xi^2 + \lambda)U_j &= -i\,\xi_j\Psi + G_j\,, \quad j=1,2,3\,,\\ \xi_k U_k &= 0\,.\end{aligned}$$

Solving these equations for $\boldsymbol{U}$ and Ψ furnishes

$$\begin{aligned}U_j &= \frac{\delta_{ij}\xi^2 - \xi_j\xi_k}{(i\,\xi_0+\xi^2+\lambda)\xi^2}\,G_j \equiv \mathcal{M}_{jk}G_k\,, \quad j=1,2,3\,,\\ \Psi &= -i\,\frac{\xi_k}{\xi^2}G_k\,.\end{aligned} \tag{VIII.4.10}$$

It is easy to check that for any fixed $j,k,l,m = 1,2,3$, the functions $\mathcal{M}_{jk}$, $\xi_0\mathcal{M}_{jk}$, $\xi_l\mathcal{M}_{jk}$, satisfy the assumptions of Lizorkin's theorem, Theorem VII.4.1 with $\beta=0$ and $M=M(q,\lambda)$, while $i\,\xi_j/\xi^2$and $\xi_l\xi_m\mathcal{M}_{jk}$ satisfy the same assumption, but with a constant M *independent of* λ. Consequently, on the one hand, $(\boldsymbol{u},\Phi)$ is in the class $\mathcal{L}^q(A_t)$, for all $q\in(1,\infty)$, and on the other hand, we can find a constant $C_1=C_1(q)>0$ such that $\boldsymbol{u}$ and Φ satisfy inequality (VIII.4.4) with $\boldsymbol{f}\equiv\boldsymbol{F}$, which in turn, recalling (VIII.4.7), implies

$$\int_{\mathbb{R}^3_T}\left(|\boldsymbol{w}(t)|^q_{2,q} + \|\nabla\phi(t)\|^q_q\right)dt \le C_1\,e^{q\lambda T}\int_{\mathbb{R}^3_T}\|\boldsymbol{f}(t)\|^q_q dt\,. \tag{VIII.4.11}$$

Also, if $q\in(1,3)$, then it is easy to check that the multiplier $i\,\xi_j/\xi^2$ satisfies Theorem VII.4.1 with $\beta=1/3$ and M independent of λ. Therefore, by that theorem, $\Phi(t)\in L^{3q/(3-q)}(\mathbb{R}^3)$, for a.a. $t\in[0,T]$, and

$$\|\Phi\|_{3q/(3-q)} \le C_2\|\boldsymbol{F}\|_q\,,$$

with $C_2=C_2(q)>0$, namely, again by (VIII.4.7),

$$\|\phi\|_{3q/(3-q)} \le C_2 e^{\lambda T}\|\boldsymbol{f}\|_q\,. \tag{VIII.4.12}$$

Since λ is an arbitrary positive number and C_1, C_2 do not depend on it, from (VIII.4.11) and (VIII.4.12) we recover (VIII.4.4) and (VIII.4.6). Moreover, recalling that $\boldsymbol{G} \in \mathcal{S}(\mathbb{R}^4)$ (see Section VII.4), we prove immediately that $\boldsymbol{u}$, Φ are smooth functions that satisfy (VIII.4.8)$_{1,2}$. Summarizing all the above, we thus obtain the existence of a smooth solution $(\boldsymbol{w}, \phi)$ to (VIII.4.1)$_{1,2}$ in the class $\mathcal{L}^q(A_T)$, with $\phi \in L^{q,3q/(3-q)}(\mathbb{R}^3_T)$ if $q \in (1,3)$, and satisfying (VIII.4.4), and (VIII.4.6) as well, if $q \in (1,3)$. We shall next analyze the way in which $\boldsymbol{u}$ (and hence $\boldsymbol{w}$) tends to zero as $t \to 0^+$. To this end, we observe that from (VIII.4.9) and (VIII.4.10), we have

$$u_j(x,t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} e^{i\boldsymbol{x}\cdot\boldsymbol{\xi} + it\,\xi_0} \frac{\delta_{ij}\xi^2 - \xi_j\xi_k}{(i\,\xi_0 + \xi^2 + \lambda)\xi^2} G_j(\xi,\xi_0)d\xi d\xi_0\,. \quad \text{(VIII.4.13)}$$

Now, in view of the assumptions on $\boldsymbol{F}$, it follows that $\boldsymbol{G}$ is analytic in ξ_0. Therefore, by a simple application of the Cauchy integral theorem (MacRobert 1966, §27), we can show that the value of the integral

$$\int_{-\infty}^{\infty} e^{it\,\xi_0} \frac{G_j(\xi,\xi_0)}{(i\,\xi_0 + \xi^2 + \lambda)} d\xi_0\,, \quad t \in \mathbb{R}^3\,,$$

does not change (and therefore the value of the integral in (VIII.4.13) does not change) if we perform the integration on a line $\{\alpha_0\}$ in the complex plane parallel to $\{\xi_0\}$, provided $i\,\xi_0 + \xi^2 + \lambda$ does not vanish along points of $\{\alpha_0\}$. We thus choose $\alpha_0 = \xi_0 + i\,\delta$, $\delta > 0$, so that (VIII.4.13) furnishes

$$u_j(x,t) = \frac{e^{\delta\,t}}{(2\pi)^2} \int_{\mathbb{R}^4} e^{i\boldsymbol{x}\cdot\boldsymbol{\xi} + it\,\alpha_0} \frac{(\delta_{ij}\xi^2 - \xi_j\xi_k)\,G_j(\xi,\alpha_0 - i\,\delta)}{(i\,\alpha_0 + \delta + \xi^2 + \lambda)\xi^2}\, d\xi d\alpha_0\,. \quad \text{(VIII.4.14)}$$

Moreover, we obtain

$$\boldsymbol{G}(\xi,\alpha_0 - i\,\delta) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \boldsymbol{F}(x,t) e^{-\delta\,t}\, e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi} - it\,\alpha_0} dx\,dt\,,$$

which, by Parseval's theorem, furnishes

$$\int_{\mathbb{R}^4} |\boldsymbol{G}(\xi,\alpha_0 - i\,\delta)|^2 d\xi\, d\alpha_0 = \int_0^T dt \int_{\mathbb{R}^3} |\boldsymbol{F}(x,t)|^2 e^{-\delta t} dx\,,$$

where we have also employed the fact that $\boldsymbol{F}$ is of compact support in $(0,T)$. Using this information in (VIII.4.14) along with the Schwarz inequality, we deduce

$$|\boldsymbol{u}(x,t)| \le c_1\, e^{\delta\,t} \left(\int_{\mathbb{R}^4} \frac{d\zeta\, d\xi}{\zeta^2 + \xi^4 + \delta^2 + \lambda^2} \right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{R}^3} |\boldsymbol{F}(x,t)|^2 e^{-\delta t} \right)^{\frac{1}{2}} \le c_2\, e^{\delta\,t},$$

with c_2 independent of δ. If we take $t < 0$ in this latter relation and let $\delta \to \infty$, we conclude that $\boldsymbol{u}(x,t) = 0$ for all $t < 0$ and $x \in \mathbb{R}^3$, and so, by the continuity of $\boldsymbol{u}$, $\boldsymbol{u}(x,0)=\boldsymbol{w}(x,0)=\boldsymbol{0}$, for all $x\in\mathbb{R}^3$. Once this property is established, we notice that for all $t\in(0,T]$,

$$\|\boldsymbol{w}(t)\|_q^q=\int_0^t\frac{d}{ds}\|\boldsymbol{w}(s)\|_q^q ds\le q\left(\int_0^t\|\boldsymbol{w}(s)\|_q^q ds\right)^{\frac{1}{q'}}\left(\int_0^t\left\|\frac{\partial\boldsymbol{w}}{\partial s}\right\|_q^q ds\right)^{\frac{1}{q}},$$

which, in turn, together with (VIII.4.4), proves (VIII.4.3). It remains to show uniqueness. This amounts to proving that the problem

$$\left.\begin{aligned}&\frac{\partial\boldsymbol{w}}{\partial t}=\Delta\boldsymbol{w}-\nabla\phi\\&\nabla\cdot\boldsymbol{w}=0\end{aligned}\right\}\ \text{in }\mathbb{R}^3\times(0,T)\qquad\boldsymbol{w}(x,0)=\boldsymbol{0}\,,\tag{VIII.4.15}$$

with $(\boldsymbol{w},\phi)\equiv(\boldsymbol{w}_1+\boldsymbol{w}_2,\phi_1+\phi_2)$, $(\boldsymbol{w}_i,\phi_i)\in\mathcal{L}^{q_i}(\mathbb{R}^3_T)$, for some $q_i\in(1,\infty)$, $i=1,2$, has only the solution $\boldsymbol{w}\equiv\nabla\phi\equiv\boldsymbol{0}$. In order to prove the above, we begin by observing the following two obvious facts: (a) The method of proof just described leads to the existence of a solution $(\boldsymbol{v},\zeta)\in\mathcal{L}^q(\mathbb{R}^3_T)$ to problem (VIII.4.1) with $\mathcal{R}$ replaced by $-\mathcal{R}$; (b) if $\boldsymbol{f}\in C_0^\infty(\mathbb{R}^3_T)$, then $(\boldsymbol{v},\zeta)$ is in the class $\mathcal{L}^q(\mathbb{R}^3_T)$ for *all* $q\in(1,\infty)$. With these two remarks in hand, we set

$$\boldsymbol{V}(x,t)=\boldsymbol{v}(x,T-t)\,,\ Z(x,t)=\zeta(x,T-t)\,,\ \boldsymbol{H}=\boldsymbol{f}(x,T-t)\,,$$

with arbitrary $\boldsymbol{f}\in C_0^\infty(\mathbb{R}^3_T)$. Then the fields $\boldsymbol{V}$, Z, and $\boldsymbol{H}$ satisfy the following problem:

$$\left.\begin{aligned}&\frac{\partial\boldsymbol{V}}{\partial t}+\Delta\boldsymbol{V}=\nabla Z-\boldsymbol{H}\\&\nabla\cdot\boldsymbol{V}=0\end{aligned}\right\}\ \text{in }\mathbb{R}^3\times(0,T)\qquad\boldsymbol{V}(x,T)=\boldsymbol{0}\,,\tag{VIII.4.16}$$

and moreover,

$$\boldsymbol{V}\,,\ \frac{\partial\boldsymbol{V}}{\partial t}\,,\ \nabla\boldsymbol{V}\,,\ D^2\boldsymbol{V}\in L^q(\mathbb{R}^3_T)\,,\quad Z\in L^q_{loc}(\mathbb{R}^3\times[0,T])\,,\ \nabla Z\in L^q(\mathbb{R}^3_T)\quad\text{for } all\ q\in(1,\infty)\,.\tag{VIII.4.17}$$

We next multiply $(\text{VIII.4.15})_1$ by $\boldsymbol{V}$ and integrate by parts over $B_R\times[0,T]$. Taking into account the properties (VIII.4.16), (VIII.4.17) of $\boldsymbol{V}$, the assumption on $\boldsymbol{w},\phi$, and Exercise II.4.3, we deduce

$$\begin{aligned}\int_0^T\int_{B_R} \boldsymbol{w}(x,t)\cdot\boldsymbol{H}(x,t)dx\,dt & \\ &= \sum_{i=1}^2\int_0^T\int_{\partial B_R}\Big(\frac{\partial \boldsymbol{w}_i}{\partial \boldsymbol{n}}\cdot\boldsymbol{V}-\boldsymbol{w}_i\cdot\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{n}}+\phi_i\boldsymbol{V}\cdot\boldsymbol{n}-Z\boldsymbol{w}_i\cdot\boldsymbol{n}\Big)d\sigma\,dt \\ &= \sum_{i=1}^2\int_{\partial B_R}\int_0^T\Big(\frac{\partial \boldsymbol{w}_i}{\partial \boldsymbol{n}}\cdot\boldsymbol{V}-\boldsymbol{w}_i\cdot\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{n}}+\phi_i\boldsymbol{V}\cdot\boldsymbol{n}-Z\boldsymbol{w}_i\cdot\boldsymbol{n}\Big)dt\,d\sigma \\ &\equiv \mathcal{I}_1(R)+\mathcal{I}_2(R)\,,\end{aligned} \tag{VIII.4.18}$$

where $\boldsymbol{n}$ is the unit outer normal to ∂B_R. By choosing $q = q_i/(q_i-1)$ in $\mathcal{I}_i$, $i = 1, 2$, it is now readily seen that

$$\mathcal{I}_1+\mathcal{I}_2 \in L^1(1,\infty)\,,$$

and consequently, that there exists at least a sequence $\{R_m\}$, with $R_m \to \infty$ as $m \to \infty$, such that

$$\lim_{m\to\infty}\mathcal{I}_1(R_m)+\mathcal{I}_2(R_m)=0\,.$$

Plugging this information back into (VIII.4.18), and recalling the definition of $\boldsymbol{H}$, we deduce

$$\int_0^T\int_{B_R}\boldsymbol{w}(x,t)\cdot\boldsymbol{f}(x,T-t)dx\,dt=0\,,\quad\text{for all } \boldsymbol{f}\in C_0^\infty(\mathbb{R}^3_T)\,,$$

which shows that $\boldsymbol{w}(x,t) = \boldsymbol{0}$ a.e. in $\mathbb{R}^3_T$. This, in turn, with the help of (VIII.4.15), implies $\nabla\phi(x,t) = \boldsymbol{0}$ a.e. in $\mathbb{R}^3_T$, and the proof of uniqueness is complete.

□

The velocity field corresponding to the solution determined in the previous theorem admits a simple representation in terms of the Oseen fundamental solution, under suitable conditions on $\boldsymbol{f}$. To this end, we introduce the following lemma.

Lemma VIII.4.1 *Let A be a bounded, locally Lipschitz domain of $\mathbb{R}^3$, and let $\boldsymbol{u}_i$, p_i, $i = 1, 2$, be vector and scalar fields, respectively, with $\boldsymbol{u}_i$, $i = 1, 2$, solenoidal, and belonging to the class $\mathcal{L}^{q_i}(A_\mathfrak{t})$, $q_1 = q$, $q_2 = q'$. Then the following Green's identity holds, for all $0 < t_1 \le t_2 < \mathfrak{t}$:*

$$\int_{t_1}^{t_2}\int_A \Big[\boldsymbol{u}_1 \cdot \Big(\frac{\partial \boldsymbol{u}_2}{\partial \tau} + \Delta \boldsymbol{u}_2 - \mathcal{R}\frac{\partial \boldsymbol{u}_2}{\partial x_1} - \nabla p_2\Big) \\ - \boldsymbol{u}_2 \cdot \Big(-\frac{\partial \boldsymbol{u}_1}{\partial \tau} + \Delta \boldsymbol{u}_1 + \mathcal{R}\frac{\partial \boldsymbol{u}_1}{\partial x_1} - \nabla p_1\Big)\Big] dx\, d\tau$$

$$= \int_{t_1}^{t_2}\int_{\partial A} \Big[(\boldsymbol{u}_1 \cdot \boldsymbol{T}(\boldsymbol{u}_2, p_2) - \boldsymbol{u}_2 \cdot \boldsymbol{T}(\boldsymbol{u}_1, p_1) + \mathcal{R}(\boldsymbol{u}_1 \cdot \boldsymbol{u}_2)\, \boldsymbol{e}_1 \Big] \cdot \boldsymbol{N}\, d\sigma d\tau \\ + \int_A \boldsymbol{u}_1 \cdot \boldsymbol{u}_2 \Big|_{\tau=t_1}^{\tau=t_2},$$

where $\boldsymbol{N}$ *is the unit outer normal to* ∂A*, and, we recall,* $\boldsymbol{T}$ *is the Cauchy stress tensor defined in* (IV.8.6), (IV.8.7).

Proof. The proof is at once achieved by integration by parts, once we take into account Lemma II.4.1, Exercise II.4.3, (IV.8.9), Exercise VIII.4.1, and the identity $\nabla \cdot \boldsymbol{T}(\boldsymbol{u}, p) = \Delta \boldsymbol{u} - \nabla p$, valid for solenoidal fields $\boldsymbol{u}$. □

We can now prove the following result.

Theorem VIII.4.2 *Suppose that*

$$\boldsymbol{f} \in L^q(\mathbb{R}^3_T), \quad 1 < q < \infty,$$

and let $\boldsymbol{w}, \phi$ *be the solution to* (VIII.4.1) *in the class* $\mathcal{L}^q(\mathbb{R}^3_T)$*, corresponding to* $\boldsymbol{f}$ *and satisfying* (VIII.4.3)*. Then, for a.a.* $(x,t) \in \mathbb{R}^3_T$*, we have the following volume potential representation for* $\boldsymbol{w}$

$$\boldsymbol{w}(x,t) = \int_0^t \int_{\mathbb{R}^3} \boldsymbol{\Gamma}(x-y, t-\tau; \mathcal{R}) \cdot \boldsymbol{f}(y,\tau)\, dy\, d\tau. \tag{VIII.4.19}$$

Proof. Let $\{\boldsymbol{f}_k\}$ be a sequence of functions from $C_0^\infty(\mathbb{R}^3_T)$ such that

$$\lim_{k\to\infty} \|\boldsymbol{f}_k - \boldsymbol{f}\|_{(r,q),T} = 0.^2 \tag{VIII.4.20}$$

We denote by $\{(\boldsymbol{w}_k, \phi_k)\}$ the sequence of solutions corresponding to $\boldsymbol{f}_k$ constructed in Theorem VIII.4.1. By the uniqueness properties, the following facts are at once established:

(a) $(\boldsymbol{w}_k, \phi_k) \in \mathcal{L}^q(\mathbb{R}^3_T)$ for all $q \in (1,\infty)$, all $k \in \mathbb{N}$;

(b) $\displaystyle\lim_{k\to\infty}\int_{\mathbb{R}^3_T} \|\boldsymbol{w}_k(t) - \boldsymbol{w}(t)\|_q^q dt = 0.$

[2] The functions $\boldsymbol{f}_k$ can be easily constructed by multiplying $\boldsymbol{f}$ by a "cut-off" function that is 1 in B_{R_k} and 0 in B^{2R_k}, and then mollifying the obtained function with parameter $\varepsilon = 1/R_k$.

Moreover, since $\boldsymbol{w}_k$ is in $L^{q_1}(\mathbb{R}^3) \cap D^{1,q_2}(\mathbb{R}^3)$ for some $q_1 > 1$ and $q_2 > 3$ for a.a. $t \in [0,T]$, from Theorem II.9.1 we obtain, for all $k \in \mathbb{N}$,

(c) $\boldsymbol{w}_k(x,t), \to 0$ uniformly, as $|x| \to \infty$, for a.a. $t \in [0,T]$.

Finally, property (a), together with the embedding Theorem II.3.2, ensures that

(d) $\boldsymbol{w}_k(x,t) \in C(\mathbb{R}^3_T)\,,$ all $k \in \mathbb{N}\,.$

Set

$$\boldsymbol{\Gamma}_i = (\Gamma_{i1}, \Gamma_{i2}, \Gamma_{i3})\,,$$

for fixed $i \in \{1,2,3\}$. We then use Green's identity in Lemma VIII.4.1 with $A = B_R(x)$, $t_1 = \eta$, $t_2 = t - \varepsilon < T$, $\varepsilon, \eta > 0$, $\boldsymbol{u}_1 = \boldsymbol{w}_k$, and $\boldsymbol{u}_2 = \boldsymbol{\Gamma}_i$. Taking into account that $\boldsymbol{\Gamma}$ satisfies (VIII.3.8) for all $t > \tau$, and that $\boldsymbol{w}_k$ satisfies (VIII.4.1) with $\boldsymbol{f} \equiv \boldsymbol{f}_k$, we deduce[3]

$$\begin{aligned}
&\int_\eta^{t-\varepsilon}\int_{B_R(x)} \boldsymbol{\Gamma}_i(x-y,t-\tau)\cdot \boldsymbol{f}_k(y,\tau)\,dy\,d\tau \\
&\qquad = \int_\eta^{t-\varepsilon}\int_{\partial B_R(x)} \Big[\boldsymbol{w}_k(y,\tau)\cdot \boldsymbol{T}(\boldsymbol{\Gamma}_i,0)(x-y,t-\tau) \\
&\qquad\qquad\qquad -\boldsymbol{\Gamma}_i(x-y,t-\tau)\cdot \boldsymbol{T}(\boldsymbol{w}_k,\phi_k)(y,\tau) \\
&\qquad\qquad\qquad +\mathcal{R}(\boldsymbol{w}_k(y,\tau)\cdot \boldsymbol{\Gamma}_i(x-y,t-\tau))\,\boldsymbol{e}_1\Big]\cdot \boldsymbol{N}\,d\sigma_y d\tau \\
&\qquad\quad + \int_{B_R(x)} \boldsymbol{w}_k(y,\tau)\cdot \boldsymbol{\Gamma}_i(x-y,t-\tau)\Big|_{\tau=\eta}^{\tau=t-\varepsilon}\,.
\end{aligned}$$

We now let $\eta, \varepsilon \to 0$ in this relation. In view of property (d) above, Lemma VIII.3.1, and (VIII.4.3), we obtain for $i = 1,2,3$,

$$\begin{aligned}
w_{ki}(x,t) = &\int_0^t\int_{B_R(x)} \boldsymbol{\Gamma}_i(x-y,t-\tau)\cdot \boldsymbol{f}_k(y,\tau)\,dy\,d\tau \\
&+\frac{1}{4\pi}\int_{\partial B_R(x)} \frac{x_i - y_i}{|x-y|^3}\boldsymbol{w}_k(y,t)\cdot \boldsymbol{N}(y)d\sigma_y \\
&+\int_0^t\int_{\partial B_R(x)} \Big[\boldsymbol{w}_k(y,t-\tau)\cdot \boldsymbol{T}(\boldsymbol{\Gamma}_i,0)(x-y,\tau) \\
&\qquad -\boldsymbol{\Gamma}_i(x-y,\tau)\cdot \boldsymbol{T}(\boldsymbol{w}_k,\phi_k)(y,t-\tau) \\
&\qquad +\mathcal{R}(\boldsymbol{w}_k(y,\tau)\cdot \boldsymbol{\Gamma}_i(x-y,t-\tau))\,\boldsymbol{e}_1\Big]\cdot \boldsymbol{N}\,d\sigma_y d\tau\,.
\end{aligned} \tag{VIII.4.21}$$

[3] In order to simplify the notation, in the rest of the proof we omit the dependence of $\boldsymbol{\Gamma}$ on $\mathcal{R}$.

We next pass to the limit as $R \to \infty$ in this latter relation. Clearly, since $\boldsymbol{f}_k$ is of compact support,

$$\lim_{R\to\infty} \int_0^t\!\!\int_{B_R(x)} \boldsymbol{\Gamma}_i(x-y,t-\tau)\cdot \boldsymbol{f}_k(y,\tau)\,dy = \int_0^t\!\!\int_{\mathbb{R}^3} \boldsymbol{\Gamma}_i(x-y,t-\tau)\cdot \boldsymbol{f}_k(y,\tau)\,dy\,. \tag{VIII.4.22}$$

Moreover, in view of the property for $\boldsymbol{w}_k$ mentioned in (c) above,

$$\lim_{R\to\infty} \frac{1}{4\pi}\int_{\partial B_R(x)} \frac{\boldsymbol{x}-\boldsymbol{y}}{|x-y|^3}\boldsymbol{w}_k(y,t)\cdot \boldsymbol{N}(y) d\sigma_y = 0\,, \quad \text{for a.a. } t\in[0,T]. \tag{VIII.4.23}$$

Also, from Lemma VIII.3.2 we obtain, for $r=|x-y|$ sufficiently large and $\tau \in [0,T]$

$$|\boldsymbol{\Gamma}(r,\tau)| = O(r^{-3})\,, \quad |\nabla\boldsymbol{\Gamma}(r,\tau)| = O(r^{-4})\,.$$

Thus, denoting by $I_R(t)$ the last integral on the right-hand side of (VIII.4.21), we infer

$$\begin{aligned} I_R(t) \le c\int_0^t \Big[& R^{-1}\int_{S^2} |\boldsymbol{w}_k(y,t-\tau)| d\sigma_y \\ & + R^{-1}\int_{S^2}\big(|\nabla \boldsymbol{w}_k(y,t-\tau)| + |\phi_k(y,t-\tau)|\big) d\sigma_y\Big] d\tau \\ \le \frac{c_1}{R}\int_0^t & \Big[\|\boldsymbol{w}_k(t-\tau)\|_{q,S^2} + \|\nabla \boldsymbol{w}_k(t-\tau)\|_{q,S^2} + \|\phi_k(t-\tau)\|_{q,S^2}\Big] d\tau\,. \end{aligned} \tag{VIII.4.24}$$

We take $q<3$ in (VIII.4.24) (this is allowed by property (a) mentioned previously) and use Lemma II.6.3. Consequently, by possibly modifying ϕ_k by the addition of a function of time only, from (VIII.4.24) it follows that for all sufficiently large R,

$$\begin{aligned} I_R(t) &\le c_2 R^{-3/q}\int_0^t \left(\|\nabla \boldsymbol{w}_k(s)\|_{1,q} + \|\nabla\phi_k(s)\|_q\right) ds \\ &\le c_3 R^{-3/q}\left(\int_0^T \left(\|\nabla \boldsymbol{w}_k(s)\|_{1,q}^q + \|\nabla\phi_k(s)\|_q^q\right) ds\right)^{\frac{1}{q}}\,, \end{aligned}$$

where $c_3 = c_3(T)$. From this inequality and from (VIII.4.24) we obtain

$$\lim_{R\to\infty} I_R(t) = 0\,, \quad \text{for all } t\in[0,T]\,, \tag{VIII.4.25}$$

and so, collecting (VIII.4.21)–(VIII.4.25), we may conclude that

$$w_{ki}(x,t) = \int_0^t\!\!\int_{\mathbb{R}^3} \boldsymbol{\Gamma}_i(y,\tau)\cdot \boldsymbol{f}_k(x-y,t-\tau)\,dy\,d\tau\,. \tag{VIII.4.26}$$

We next observe that, by property (b) and Lemma II.2.2, we have

$$\lim_{k\to\infty} \boldsymbol{w}_k(x,t) = \boldsymbol{w}(x,t) \quad \text{for a.a. } (x,t) \in \mathbb{R}^3_T. \tag{VIII.4.27}$$

Furthermore, from Lemma VIII.3.2, we have, for all $r \in (1,\infty)$,

$$\|\boldsymbol{\Gamma}(t)\|_\sigma \leq c_4 \int_{\mathbb{R}^3} \frac{dy}{(t+|y|^2)^{3r/2}} = c_5 t^{\frac{3}{2}\left(1-\frac{1}{r}\right)}\,. \tag{VIII.4.28}$$

We also set

$$V[\boldsymbol{g}](x,t) := \int_0^t\!\!\int_{\mathbb{R}^3} \boldsymbol{\Gamma}_i(y,\tau)\cdot \boldsymbol{g}(x-y,t-\tau)\, dy\, d\tau\,.$$

Let $B \subset \mathbb{R}^3$ be an arbitrary ball and $s \in (1,q)$. Taking into account (VIII.4.26), with the help of the Minkowski inequality we obtain

$$\|\boldsymbol{w} - V[\boldsymbol{f}]\|_{s,B_T} \leq \|\boldsymbol{w} - \boldsymbol{w}_k\|_{s,B_T} + \|V[\boldsymbol{f}_k - \boldsymbol{f}]\|_{s,B_T}\,. \tag{VIII.4.29}$$

In view of (VIII.4.27), we have (along a subsequence, at least)

$$\lim_{k\to\infty} \|\boldsymbol{w}_k - \boldsymbol{w}\|_{s,B_T}\,. \tag{VIII.4.30}$$

We shall next prove the inequality

$$\|V[\boldsymbol{f}_k - \boldsymbol{f}]\|_{s,B_T} \leq c\, \|\boldsymbol{f}_k - \boldsymbol{f}\|_{q,\mathbb{R}^3_T}\,, \tag{VIII.4.31}$$

so that from (VIII.4.20) and (VIII.4.29)–(VIII.4.31), by the arbitrariness of B we deduce $\boldsymbol{w} = V[\boldsymbol{f}]$ a.e. in $\mathbb{R}_T$, which proves the theorem. In order to prove (VIII.4.31), we notice that by the Young inequality (II.11.2) and (VIII.4.28), it follows that

$$\|V[\boldsymbol{f}_k - \boldsymbol{f}](t)\|_{s,B} \leq c_6 \int_0^T (t-\tau)^{\frac{3}{2}\left(\frac{1}{q}-\frac{1}{s}\right)} \|\boldsymbol{f}_k - \boldsymbol{f}\|_{q,\mathbb{R}^3} d\tau\,.$$

Since for any given $q \in (1,\infty)$ we can find $s \in (1,q)$ such that $(3/2)(1/s - 1/q) \leq (1-1/q)$, from Theorem II.11.2 we deduce that the map

$$t \in (0,T) \to \int_0^T (t-\tau)^{\frac{3}{2}\left(\frac{1}{q}-\frac{1}{s}\right)} \|\boldsymbol{f}_k - \boldsymbol{f}\|_{q,\mathbb{R}^3} d\tau$$

is bounded from $L^s(0,T)$ into itself, and, consequently, since $s < q$, from $L^s(0,T)$ into $L^q(0,T)$. This shows the validity of (VIII.4.31), and the theorem is thus proved.

□

We now turn to the unique solvability of problem (VIII.4.2). In this respect, we begin by proving the following general uniqueness result.

Lemma VIII.4.2 *Suppose $\boldsymbol{u}, \phi$ are such that*

$$\frac{\partial \boldsymbol{u}}{\partial t}\,,\ D^2\boldsymbol{u}\,,\ \phi\,,\ \nabla\phi \in L^s_{loc}((0,T]\times\mathbb{R}^3)\,,\quad \textit{for some } s\in(1,\infty)\,,$$

and satisfy a.e. the system

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{u}}{\partial t} = \Delta\boldsymbol{u} + \mathcal{R}\frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla\phi \\ &\nabla\cdot\boldsymbol{u} = 0 \end{aligned}\right\}\ \textit{in } \mathbb{R}^3\times(0,T) \tag{VIII.4.32}$$

along with the initial condition

$$\lim_{t\to 0^+}\|\boldsymbol{u}(t)\|_{r,B_\rho} = 0\,,\quad \textit{for all } \rho>0 \ \textit{ and some } r\in(1,\infty)\,. \tag{VIII.4.33}$$

Then, if

$$\boldsymbol{u} = \sum_{i=1}^N \boldsymbol{u}_i\,,\ \boldsymbol{u}_i\in L^{q_i}(\mathbb{R}_T)\,,\quad \textit{for some } q_i\in(1,\infty)\,,\quad i=1,\dots,N\,,$$

it follows that $\boldsymbol{u}=\boldsymbol{0}$ and $\nabla\phi=0$ a.e. in $\mathbb{R}^3_T$.

Proof. For simplicity, we consider the case $N=2$, the general case being treated in an entirely similar way. Let $\psi_R=\psi_R(x)$ be a "cut-off" function that is 1 for $|x|<R$, is 0 for $|x|>2R$, and satisfies $|D^\alpha\psi_R|\le M\,R^{-|\alpha|}$, $|\alpha|=1,2$, for some M independent of R. Moreover, let $\boldsymbol{V}, Z$ be the solution to (VIII.4.16), (VIII.4.17), and denote by $\boldsymbol{w}=\boldsymbol{w}(x,t)$ a vector field satisfying the following properties for a.a. $t\in[0,T]$:

$$\begin{aligned} &\nabla\cdot\boldsymbol{w} = -\nabla\psi_R\cdot\boldsymbol{V}(x,t) \quad \text{in } B_{R,2R}\,,\\ &\boldsymbol{w}(t)\in W^{3,q}_0(B_{R,2R})\,,\quad \frac{\partial\boldsymbol{w}}{\partial t}\in W^{1,q}_0(B_{R,2R})\,,\\ &\|\nabla\boldsymbol{w}(t)\|_{q,B_{R,2R}} \le C_1\,(\|\nabla\psi_R\cdot\boldsymbol{V}(t)\|_{q,B_{R,2R}}\,,\\ &\|D^2\boldsymbol{w}(t)\|_{q,B_{R,2R}} \le C_2\,(\|\nabla\psi_R\cdot\boldsymbol{V}(t)\|_{q,B_{R,2R}} + \|\nabla(\nabla\psi_R\cdot\boldsymbol{V}(t))\|_{q,B_{R,2R}})\,,\\ &\left\|\nabla\left(\frac{\partial\boldsymbol{w}}{\partial t}\right)\right\|_q \le C_3\left\|\nabla\psi_R\cdot\frac{\partial\boldsymbol{V}(t)}{\partial t}\right\|_{q,B_{R,2R}}\,. \end{aligned} \tag{VIII.4.34}$$

Obviously, $f\equiv-\nabla\psi_R\cdot\boldsymbol{V}(x,t)$ belongs to $W^{2,q}_0(B_{R,2R})$, while $\frac{\partial f}{\partial t}\in L^q(B_{R,2r})$, for a.a. $t\in[0,T]$. Furthermore, taking into account that $\psi_R(x)=0$ for $|x|=2R$, and that $\nabla\cdot\boldsymbol{V}=0$ in $\mathbb{R}^3$, we have

$$\int_{B_{R,2R}} f = -\int_{B_{R,2R}}\nabla\cdot(\psi_R\boldsymbol{V}) = \frac{1}{R}\int_{\partial B_R}\boldsymbol{V}\cdot\boldsymbol{x} = 0\,.$$

Thus, from Exercise III.3.7 and from the fact that $\boldsymbol{V}$ is in the class (VIII.4.17), we deduce the existence of a field $\boldsymbol{w}$ satisfying all the properties listed in

(VIII.4.34), for *all* $q \in (1, \infty)$. Furthermore, again by Exercise III.3.7 and by the properties of $\boldsymbol{V}$, we find that $\boldsymbol{w}(t)$ is continuous, in the L^q-norm, for all $t \geq 0$, and in particular, $\boldsymbol{w}(\cdot, T) = \boldsymbol{0}$. We next observe that since $B_{R,2R}$ is homothetic to $B_{1,2}$ via the transformation $\Phi_i(x) = x_i/R$, $i = 1, 2, 3$, from Lemma III.3.3 we obtain that C_1 and C_3 are independent of R. By a similar argument, we can show that $C_2 \leq k(1 + 1/R)$, with k independent of R. We then conclude that for R large enough, all constants C_i, $i = 1, 2, 3$, can be taken independent of R. Consequently, bearing in mind the properties of ψ_R, from (VIII.4.34)$_{3,4}$, we deduce in particular

$$\|\nabla \boldsymbol{w}(t)\|_q + \|D^2 \boldsymbol{w}(t)\|_q \leq k_1 \|\boldsymbol{V}(t)\|_{1,q,B_{R,2R}}, \tag{VIII.4.35}$$

with k_1 independent of R. Next, from Exercise II.5.4 we obtain

$$\left\| \frac{\partial \boldsymbol{w}}{\partial t} \right\|_{q,B_{R,2R}} \leq C_4 \, R \left\| \nabla \left(\frac{\partial \boldsymbol{w}}{\partial t} \right) \right\|_{q,B_{R,2R}}$$

with C_4 independent of R. Consequently, again from the properties of ψ_R and (VIII.4.34)$_5$ we obtain

$$\left\| \frac{\partial \boldsymbol{w}}{\partial t} \right\|_{q,B_{R,2R}} \leq k_2 \left\| \frac{\partial \boldsymbol{V}}{\partial t} \right\|_{q,B_{R,2R}}, \tag{VIII.4.36}$$

with k_2 independent of R. Let us extend $\boldsymbol{w}$ to zero outside $B_{R,2R}$ and continue to denote by $\boldsymbol{w}$ the extension. Thus, if we dot-multiply (VIII.4.32)$_1$ by $V_R(x,t) \equiv \psi_R \boldsymbol{V}(x,t) + \boldsymbol{w}(x,t)$, then integrate over $[\eta, T] \times \mathbb{R}^3$, $\eta > 0$, and recall that $\boldsymbol{V}, Z$ satisfy (VIII.4.16), we obtain

$$\begin{aligned} &(\boldsymbol{u}(\eta)\,, \boldsymbol{V}_R(\eta)) \\ &= \int_\eta^T \int_{\mathbb{R}^3} \boldsymbol{u}. \Big[-2\nabla\psi_R \cdot \nabla \boldsymbol{V} - \boldsymbol{V}\, \Delta\psi_R + \mathcal{R}\boldsymbol{V} \frac{\partial \psi_R}{\partial x_1} - \psi_R \nabla Z \\ &\qquad + \psi_R \boldsymbol{H} - \frac{\partial \boldsymbol{w}}{\partial t} - \Delta \boldsymbol{w} + \mathcal{R} \frac{\partial \boldsymbol{w}}{\partial x_1} \Big] dx\, dt\,. \end{aligned} \tag{VIII.4.37}$$

We now pass to the limit $\eta \to 0$ and use (VIII.4.33) along with the continuity of $\boldsymbol{V}_R$ at $t = 0$ in the L^q-norm and the hypothesis on $\boldsymbol{u}$ to deduce

$$\begin{aligned} 0 = \int_0^T \int_{\mathbb{R}^3} (\boldsymbol{u}_1 + \boldsymbol{u}_2) \cdot \Big[&-2\nabla\psi_R \cdot \nabla \boldsymbol{V} - \boldsymbol{V}\, \Delta\psi_R + \mathcal{R}\boldsymbol{V} \frac{\partial \psi_R}{\partial x_1} - \psi_R \nabla Z \\ &+ \psi_R \boldsymbol{H} - \frac{\partial \boldsymbol{w}}{\partial t} - \Delta \boldsymbol{w} + \mathcal{R} \frac{\partial \boldsymbol{w}}{\partial x_1} \Big] dx\, dt\,. \end{aligned} \tag{VIII.4.38}$$

Next, we choose $q = q_i'$, $i = 1, 2$, and recall the properties of $\boldsymbol{V}$ given in (VIII.4.17) and those of $\boldsymbol{w}$ given in (VIII.4.35), (VIII.4.36). As a consequence, by means of the Hölder inequality, the properties of ψ_R, and the Lebesgue dominated converge theorem (Lemma II.2.1), it is easy to show that

$$\lim_{R\to\infty}\int_0^T\int_{\mathbb{R}^3}\boldsymbol{u}_i\cdot\Big[-2\nabla\psi_R\cdot\nabla\boldsymbol{V}-\boldsymbol{V}\,\Delta\psi_R+\mathcal{R}\boldsymbol{V}\frac{\partial\psi_R}{\partial x_1}$$
$$-\frac{\partial\boldsymbol{w}}{\partial t}-\Delta\boldsymbol{w}+\mathcal{R}\,\frac{\partial\boldsymbol{w}}{\partial x_1}\Big]dx\,dt=0\,,\quad i=1,2. \tag{VIII.4.39}$$

Furthermore, we observe that, for $i=1,2$,

$$\left|\int_0^T\!\!\int_{\mathbb{R}^3}\boldsymbol{u}_i\cdot\nabla Z-\int_0^T\!\!\int_{\mathbb{R}^3}\psi_R\boldsymbol{u}_i\cdot\nabla Z\right|\le\int_0^T\!\!\int_{\mathbb{R}^3}|1-\psi_R||\boldsymbol{u}_i||\nabla Z|\,,$$

and since $|\boldsymbol{u}_i|\,|\nabla Z|\in L^1(\mathbb{R}^3_T)$, we may use again the Lebesgue dominated convergence theorem along with the property of ψ_R to show that

$$\lim_{R\to\infty}\int_0^T\!\!\int_{\mathbb{R}^3}\psi_R\boldsymbol{u}_i\cdot\nabla Z=\int_0^T\!\!\int_{\mathbb{R}^3}\boldsymbol{u}_i\cdot\nabla Z\,.$$

However, from the assumption and (VIII.4.32)$_2$, we have that $\boldsymbol{u}_i(t)\in H_q(\mathbb{R}^3)$, $i=1,2$, for a.a. $t\in[0,T]$ (see Theorem III.2.3), and since $\nabla Z(t)\in L^{q'}(\mathbb{R}^3)$, for a.a. $t\in[0,T]$, by Lemma III.2.1 we conclude that

$$\lim_{R\to\infty}\int_0^T\!\!\int_{\mathbb{R}^3}\psi_R\boldsymbol{u}\cdot\nabla Z=0\,. \tag{VIII.4.40}$$

Passing to the limit $R\to\infty$ in (VIII.4.39), utilizing (VIII.4.39) and (VIII.4.40), and recalling that $\boldsymbol{H}(x,t)=\boldsymbol{f}(x,T-t)$, we finally deduce

$$\int_0^T\!\!\int_{\mathbb{R}^3}\boldsymbol{u}(x,t)\cdot\boldsymbol{f}(x,T-t)dx\,dt\,,\quad\text{for all }\boldsymbol{f}\in C_0^\infty(\mathbb{R}^3_T)\,,$$

which entails $\boldsymbol{u}=0$ a.e. in $\mathbb{R}^3_T$. The proof of the lemma is complete.

□

We now turn to the well-posedness of problem (VIII.4.2). Specifically, we have the following.

Theorem VIII.4.3 *Let $\boldsymbol{u}_0\in H_q(\mathbb{R}^3)$, $1\le q\le\infty$. Then there exists $\boldsymbol{v}$ such that*

$$\operatorname*{ess\,sup}_{t\in[0,T]}\|\boldsymbol{v}(t)\|_q<\infty\,,$$
$$\boldsymbol{v}\,,\ \frac{\partial\boldsymbol{v}}{\partial t}\,,\ D^2\boldsymbol{v}\in L^r([\varepsilon,T]\times\mathbb{R}^3)\,,\quad\textit{for all }\varepsilon>0\textit{ and all }r\ge q\,, \tag{VIII.4.41}$$

and satisfying (VIII.4.2)$_{1,2}$ *a.e. in* $\mathbb{R}^3_T$, *for any $T>0$. Moreover, for arbitrary $t>0$, we have*

$$\|D_t^jD_x^\alpha\boldsymbol{v}(t)\|_r\le c\,t^{-(\mu+(j+|\alpha|)/2)}\|\boldsymbol{u}_0\|_q\,,\ j=0,1\,;\ 0\le|\alpha|\le2\,,$$
$$\lim_{t\to0^+}\|\boldsymbol{v}(t)-\boldsymbol{u}_0\|_q=0\,, \tag{VIII.4.42}$$

where $\mu = 3(1/q - 1/r)/2$ *and* $c = c(r, q, j, \alpha, \mathcal{R})$. *The function* $\boldsymbol{v}$ *has the following representation*

$$\boldsymbol{v}(x,t) = \left(\frac{1}{4\pi\, t}\right)^{3/2} \int_{\mathbb{R}^3} e^{-|x+\mathcal{R}\, t\boldsymbol{e}_1 - y|^2/4t} \boldsymbol{u}_0(y)dy\,.$$

Finally, let $\boldsymbol{v}_1$ *be such that*

$$\boldsymbol{v}_1 \in L^q(\mathbb{R}^3_T)\,, \quad \frac{\partial \boldsymbol{v}_1}{\partial t}\,, \ D^2\boldsymbol{v}_1 \in L^s_{loc}((0,T]\times\mathbb{R}^3)\,, \quad \textit{some } s \in (1,\infty)\,,$$
$$\lim_{t\to 0^+} \|\boldsymbol{v}_1(t) - \boldsymbol{u}_0\|_q = 0\,,$$

and satisfying (VIII.4.2)$_{1,2}$ *a.e. in* $\mathbb{R}^3_T$. *Then* $\boldsymbol{v} = \boldsymbol{v}_1$, *a.e. in* $\mathbb{R}^3_T$.

Proof. The uniqueness part is a direct consequence of Lemma VIII.4.2. We shall now prove the existence part. Using the properties of the fundamental solution $\boldsymbol{\varGamma}$ (see, in particular, (VIII.3.9)), it is immediately verified that a solution to (VIII.4.2)$_{1,2}$ is given by the following volume potential:

$$v_i(x,t) = \int_{\mathbb{R}^3} \varGamma_{ij}(x-y,t;\mathcal{R}) u_{0j}(y)dy\,, \quad i=1,2,3\,, \quad t>0\,. \tag{VIII.4.43}$$

However, from (VIII.3.7), Lemma VIII.3.2, and the fact that $\boldsymbol{u}_0 \in H_q(\mathbb{R}^3)$, it follows that for $i = 1, 2, 3$ and all $t > 0$,

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{\partial^2}{\partial x_i \partial x_j}\, &\varPsi(|x+\mathcal{R}\, t\boldsymbol{e}_1 - y|, t) u_{0j}(y)dy \\ &= \int_{\mathbb{R}^3} \nabla\left(\frac{\partial}{\partial x_i}\varPsi(|x+\mathcal{R}\, t\boldsymbol{e}_1 - y|, t)\right)\cdot \boldsymbol{u}_0(y)dy = 0\,.\end{aligned}$$

As a consequence, taking into account (VIII.3.7) and (VIII.3.4), (VIII.3.5), the volume potential (VIII.4.43) reduces to

$$\boldsymbol{v}(x,t) = \left(\frac{1}{4\pi\, t}\right)^{3/2} \int_{\mathbb{R}^3} e^{-|x+\mathcal{R}\, t\boldsymbol{e}_1 - y|^2/4t} \boldsymbol{u}_0(y)dy\,.$$

From this relation we easily deduce, for any multi-index β with $|\beta| \ge 0$,

$$D^\beta_x \boldsymbol{v}(x,t) = \left(\frac{1}{\pi}\right)^{3/2} \left(\frac{1}{2\sqrt{t}}\right)^{|\beta|} \int_{\mathbb{R}^3} \left(D^\beta_z e^{-|z|^2}\right) \boldsymbol{u}_0(x - \mathcal{R}\, t\boldsymbol{e}_1 - 2z\sqrt{t})dz\,, \tag{VIII.4.44}$$

which, in turn, with the help of Young's inequality (II.11.2), furnishes, in particular, (VIII.4.42)$_1$ with $j = 0$. The case $j = 1$ follows from (VIII.4.44) and (VIII.4.2)$_1$. To complete the existence proof, it remains to check the validity of (VIII.4.42)$_2$. To this end, we observe that

$$\int_{\mathbb{R}^3} e^{-|y|^2} dy = \pi^{3/2} ,$$

and thus from (VIII.4.44) with $\beta = 0$, we obtain

$$\boldsymbol{v}(x,t) - \boldsymbol{u}_0(x) = \pi^{-3/2} \int_{\mathbb{R}^3} e^{-|z|^2} \left[\boldsymbol{u}_0(x - \mathcal{R}\, t\, \boldsymbol{e}_1 - 2z\sqrt{t}) - \boldsymbol{u}(x) \right] dz .$$

Therefore, using the generalized Minkowski inequality (II.2.8), we infer

$$\|\boldsymbol{v}(t) - \boldsymbol{u}_0\|_q \le \pi^{-3/2} \int_{\mathbb{R}^3} \left(e^{-|z|^2} \|\boldsymbol{u}_0(\cdot - \mathcal{R}\, t\, \boldsymbol{e}_1 - 2\sqrt{t}z) - \boldsymbol{u}_0\|_q \right) dz . \tag{VIII.4.45}$$

In view of Exercise II.2.8, we have, for each fixed $z \in \mathbb{R}^3$,

$$\lim_{t \to 0^+} \|\boldsymbol{u}_0(\cdot - \mathcal{R}\, t\, \boldsymbol{e}_1 - 2\sqrt{t}z) - \boldsymbol{u}_0\|_q = 0 ,$$

and consequently, passing to the limit $t \to 0^+$ on both sides of (VIII.4.45) and using the Lebesgue dominated convergence theorem (see Lemma II.2.1), we establish (VIII.4.42)$_2$. The proof of the theorem is complete. □

The last part of this section is devoted to the solvability of (VIII.4.1) in a space of functions that possess a suitable asymptotic behavior in space, uniformly in time. To this end, we introduce the following notation.

If $\boldsymbol{U}$ is a vector or a second-order tensor field, α a nonnegative integer, $\mathcal{A}$ a domain in $\mathbb{R}^3$, and $\mathcal{R} \ge 0$, we set

$$[\![\boldsymbol{U}]\!]_{\alpha,\mathcal{R},\mathcal{A}} = \sup_{x \in \mathcal{A}} [(1+|x|)^\alpha (1 + 2\mathcal{R}\, s(x))^\alpha |\boldsymbol{U}(x)|] , \tag{VIII.4.46}$$

where, we recall, $s(x)$ is defined in (VIII.3.11).

If $\mathcal{R} = 0$, we shall simply write $[\![\boldsymbol{U}]\!]_{\alpha,\mathcal{A}}$ instead of $[\![\boldsymbol{U}]\!]_{\alpha,0,\mathcal{A}}$. Furthermore, whenever confusion does not arise, we shall omit the subscript $\mathcal{A}$.

We have the following.

Theorem VIII.4.4 *Let $\boldsymbol{\mathcal{G}}$ be a second-order tensor field in $\mathbb{R}^3 \times (0,\infty)$ such that*

$$\operatorname*{ess\,sup}_{t \ge 0} [\![(\boldsymbol{\mathcal{G}}(t)]\!]_{2,\mathcal{R}} + \operatorname*{ess\,sup}_{t \ge 0} \|\nabla \cdot \boldsymbol{\mathcal{G}}(t)\|_2 < \infty ,$$

and let $\boldsymbol{h} \in L^{\infty,q}(\mathbb{R}^3 \times (0,\infty))$, $q \in (3,\infty)$, with spatial support contained in B_ρ, for some $\rho > 0$. Then, the problem (VIII.4.1) *with $\boldsymbol{f} \equiv \nabla \cdot \boldsymbol{\mathcal{G}} + \boldsymbol{h}$ has one and only one solution such that for all $T > 0$,*

$$(\boldsymbol{w}, \phi) \in \mathcal{L}^2(\mathbb{R}^3_T) , \quad \phi \in L^{2,6}(\mathbb{R}_T). \tag{VIII.4.47}$$

Moreover,

$$\operatorname*{ess\,sup}_{t \ge 0} [\![\boldsymbol{w}(t)]\!]_{1,\mathcal{R}} + \operatorname*{ess\,sup}_{t \ge 0} \|\phi(t)\|_r < \infty ,$$

for arbitrary $r > \frac{3}{2}$, *and the following inequality holds:*

$$\operatorname*{ess\,sup}_{t\geq 0} [\![\boldsymbol{w}(t)]\!]_{1,\mathcal{R}} + \operatorname*{ess\,sup}_{t\geq 0} \|\phi(t)\|_r \leq C \operatorname*{ess\,sup}_{t\geq 0} \left([\![\boldsymbol{\mathcal{G}}(t)]\!]_{2,\mathcal{R}} + \|\boldsymbol{h}(t)\|_q\right) \tag{VIII.4.48}$$

with $C = C(r, q, \rho, B)$, *whenever* $\mathcal{R} \in [0, B]$, *for some* $B > 0$.

Finally, assume $\boldsymbol{\mathcal{G}} = \boldsymbol{0}$ *and that* $\boldsymbol{h}$ *satisfies the further assumption*

$$\mathfrak{H} := \|\boldsymbol{h}\|_{\infty,\mathbb{R}^3_\infty} + \|\nabla\boldsymbol{h}\|_{\infty,\mathbb{R}^3_\infty} < \infty\,. \tag{VIII.4.49}$$

Then,

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\geq 0} \left([\![\nabla\boldsymbol{w}(t)]\!]_2 + [\![D^2\boldsymbol{w}(t)]\!]_3\right) < \infty\,, \quad \textit{if}\ \mathcal{R} = 0\,,\\
&\operatorname*{ess\,sup}_{t\geq 0} \left([\![\nabla\boldsymbol{w}(t)]\!]_{3/2,\mathcal{R}} + [\![D^2\boldsymbol{w}(t)]\!]_{2,\mathcal{R}}\right) < \infty\,, \quad \textit{if}\ \mathcal{R} \neq 0
\end{aligned} \tag{VIII.4.50}$$

and there exist constants $C1 = C_1(\rho)$ *and* $C_2 = C_2(\rho, B)$, *whenever* $\mathcal{R} \in (0, B]$, *such that*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\geq 0} \left([\![\nabla\boldsymbol{w}(t)]\!]_2 + [\![D^2\boldsymbol{w}(t)]\!]_3\right) \leq C\,\mathfrak{H}\,, \quad \textit{if}\ \mathcal{R} = 0\,,\\
&\operatorname*{ess\,sup}_{t\geq 0} \left([\![\nabla\boldsymbol{w}(t)]\!]_{3/2,\mathcal{R}} + [\![D^2\boldsymbol{w}(t)]\!]_{2,\mathcal{R}}\right) \leq C\,\mathfrak{H}\,, \quad \textit{if}\ \mathcal{R} \neq 0
\end{aligned} \tag{VIII.4.51}$$

Proof. The existence of a unique solution satisfying (VIII.4.47) is an immediate consequence of Theorem VIII.4.1, and the assumption on $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{h}$. In order to prove the remaining properties, we observe that, by Theorem VIII.4.2 the vector field $\boldsymbol{w}$ admits the following representation:[4]

$$\begin{aligned}
\boldsymbol{w}(x,t) &= \int_0^t \int_{\mathbb{R}^3} \boldsymbol{\Gamma}(x-y, t-\tau)\cdot[(\nabla\cdot\boldsymbol{\mathcal{G}})(y,\tau) + \boldsymbol{h}(y,\tau)]\,dy\,d\tau\\
&\equiv \boldsymbol{w}_1(x,t) + \boldsymbol{w}_2(x,t)\,.
\end{aligned}$$

By an integration by parts we find that

$$\boldsymbol{w}_1(x,t) = -\int_0^t \int_{\mathbb{R}^3} \frac{\partial \Gamma_{ij}}{\partial x_k}(x-y,\tau)\mathcal{G}_{kj}(y, t-\tau)\boldsymbol{e}_i dy\,d\tau,$$

and so by Fubini's theorem and by Lemma VIII.3.4, Lemma VIII.3.5, we deduce that

[4] To alleviate the notation, and unless otherwise specified, we temporarily suppress the dependence of $\boldsymbol{\Gamma}$ on $\mathcal{R}$.

$$
\begin{aligned}
|\boldsymbol{w}_1(x,t)| &\le \int_0^t \int_{\mathbb{R}^3} |\nabla \boldsymbol{\Gamma}(x-y,\tau)||\boldsymbol{\mathcal{G}}(y,t-\tau)|dyds \\
&= \int_{\mathbb{R}^3} \int_0^t |\nabla \boldsymbol{\Gamma}(x-y,\tau)||\boldsymbol{\mathcal{G}}(y,t-\tau)|d\tau dy \\
&\le \operatorname*{ess\,sup}_{t\ge 0} [\![\boldsymbol{\mathcal{G}}(t)]\!]_{2,\mathcal{R}} \int_{\mathbb{R}^3}\int_0^\infty \frac{|\nabla \boldsymbol{\Gamma}(\chi-y,\tau)|}{[(1+|y|)(1+\mathcal{R}\,s(y))]^2} d\tau dy \\
&\le C\, \max\{1,\mathcal{R}\} \frac{\operatorname*{ess\,sup}_{t\ge 0} [\![\mathcal{G}(t)]\!]_{2,\mathcal{R}}}{(1+|x|)(1+2\mathcal{R}\,s(x))}
\end{aligned}
$$

for all $x \in \mathbb{R}^3$ and all $t \ge 0$. We now prove an analogous estimate for $\boldsymbol{w}_2$. We begin by observing that from Exercise III.3.9 and the hypothesis on $\boldsymbol{h}$, it follows that there is a second-order tensor field $\boldsymbol{H} \in L^\infty(\mathbb{R}^3_\infty)$ with $\nabla \boldsymbol{H} \in L^q(\mathbb{R}^3_T)$, for all $T > 0$, such that

$$
\begin{gathered}
\nabla \cdot \boldsymbol{H}(t) = \boldsymbol{h}(t)\,, \quad \text{a.a. } t \in [0,\infty)\,, \\
\operatorname*{ess\,sup}_{t\ge 0} \|\boldsymbol{H}(t)\|_\infty \le c \operatorname*{ess\,sup}_{t\ge 0} \|\boldsymbol{h}(t)\|_q \equiv h_0\,.
\end{gathered}
\tag{VIII.4.52}
$$

Replacing $\nabla \cdot \boldsymbol{H}$ for $\boldsymbol{h}$ in the expression of $\boldsymbol{w}_2$, integrating by parts, and recalling that $\operatorname{supp}(\boldsymbol{h}(t)) \subset B_\rho$, for all $t \ge 0$, we obtain

$$
\begin{aligned}
w_{2i} &= \int_0^t \int_{B_\rho} \Gamma_{ij}(x-y,\tau) D_\ell H_{\ell j}(y,t-\tau)dy\,d\tau \\
&= -\int_0^t \int_{B_\rho} D_\ell \Gamma_{ij}(x-y,\tau)\, H_{\ell j}(y,t-\tau)dy\,d\tau \\
&\quad + \int_0^t \int_{\partial B_\rho} \Gamma_{ij}(x-y,\tau) H_{\ell j}(y,t-\tau) n_\ell(y) d\sigma_y\, d\tau \\
&\equiv I_1 + I_2\,.
\end{aligned}
\tag{VIII.4.53}
$$

From (VIII.4.52), Lemma VIII.3.6, and Fubini's theorem, we have

$$
\begin{aligned}
|I_1| &\le c\, h_0 \Big(\int_{B_\rho} |x-y|^{-2} dy + \mathcal{R}^{1/2} \int_{B_\rho} [|x-y|(1+2\mathcal{R}\,s(x-y))]^{-3/2}\, dy \Big) \\
&\equiv c\, h_0 \left(I_1^{(1)} + I_1^{(2)} \right)\,.
\end{aligned}
\tag{VIII.4.54}
$$

If $|x| \le 2\rho$, we obtain, obviously,

$$
I_1^{(1)} \le \int_0^{3\rho} dr \le \frac{c_1(\rho)\, K}{(1+|x|)(1+2\mathcal{R}\,s(x))}\,,
$$

whereas since

$$|x| \geq 2\rho \implies |x-y| \geq |x|/2 \ \text{ for any } y \in B_\rho\,, \tag{VIII.4.55}$$

we deduce

$$I_1^{(1)} \leq \frac{c_2(\rho)}{(1+|x|)^2} \leq \frac{c_3(\rho)\,K}{(1+|x|)(1+2\mathcal{R}\,s(x))}\,.$$

Thus, we obtain

$$I_1^{(1)} \leq \frac{c_4(\rho)\,K}{(1+|x|)(1+2\mathcal{R}\,s(x))}\,, \quad x \in \mathbb{R}^3. \tag{VIII.4.56}$$

We next observe that for $|x| \leq 2\rho$,

$$I_1^{(2)} \leq \int_0^{3\rho} r^{-1/2} dr \leq \frac{c_5(\rho)\,K}{(1+|x|)(1+2\mathcal{R}\,s(x))}\,, \tag{VIII.4.57}$$

whereas if $|x| \geq 2\rho$, from (VIII.4.55) and the fact that for any $y \in B_\rho$,

$$\begin{aligned} 1+2\mathcal{R}\,s(x) &\leq 1+2\mathcal{R}\,s(x-y)+2\mathcal{R}\,s(y) \\ &\leq (1+4\mathcal{R}\,\rho)(1+2\mathcal{R}\,s(x-y)) \\ &\leq c_6\,K\,(1+2\mathcal{R}\,s(x-y))\,, \end{aligned} \tag{VIII.4.58}$$

we infer

$$I_1^{(2)} \leq \frac{c_7(\rho)\,K}{(1+|x|)(1+2\mathcal{R}\,s(x))}\,. \tag{VIII.4.59}$$

By (VIII.4.54)–(VIII.4.59) we conclude that

$$I_1 \leq \frac{c_8(\rho)\,K}{(1+|x|)(1+2\mathcal{R}\,s(x))}\,, \quad x \in \mathbb{R}^3. \tag{VIII.4.60}$$

Furthermore, by (VIII.4.52), Lemma VIII.3.6, and Fubini's theorem, we have

$$\begin{aligned} I_2 &\leq c_9\,h_0 \int_0^\infty \int_{\partial B_\rho} |\boldsymbol{\Gamma}(x-y,s)|\,ds \\ &\leq c_{10}\,h_0 \int_{\partial B_\rho} \frac{dy}{(1+|x-y|)(1+2\mathcal{R}\,s(x-y))}\,. \end{aligned}$$

If we integrate both sides of this inequality over ρ between ρ_1 and $2\rho_1$, we deduce

$$I_2 \leq \frac{c_{10}\,h_0}{\rho_1} \int_{B_{2\rho_1}} \frac{dy}{(1+|x-y|)(1+2\mathcal{R}\,s(x-y))}\,.$$

Therefore, proceeding exactly as in the proof of the estimates (VIII.4.57) and (VIII.4.59), we conclude that

$$I_2 \leq \frac{c_{11}(\rho)\,K}{(1+|x|)(1+2\mathcal{R}\,s(x))}\,, \quad x \in \mathbb{R}^3. \tag{VIII.4.61}$$

The desired estimate for $\boldsymbol{w}_2$ is then a consequence of (VIII.4.53), (VIII.4.60), and (VIII.4.61). Concerning the stated properties of the pressure, we notice that from (VIII.4.47), (VIII.4.1) and from the Helmholtz–Weyl decomposition theorem, Theorem III.1.2, it follows that

$$(\nabla\phi(t),\nabla\chi) = (\nabla\cdot\boldsymbol{\mathcal{G}}(t),\nabla\chi) + (\boldsymbol{h}(t),\nabla\chi)\,, \qquad \text{for a.a. } t\in(0,\infty), \text{ and all } \chi\in D^{1,2}(\mathbb{R}^3). \tag{VIII.4.62}$$

We then choose in (VIII.4.62) the function χ as a solution to the Poisson problem $\Delta\chi=\psi$, where ψ is arbitrary from $C_0^\infty(\mathbb{R}^3)$. Recalling that from the representation $\chi=\mathcal{E}*\psi$, we have $D^\beta\chi = O(|x|^{1-|\beta|})$, $|\beta|\geq 0$, and using the properties of ϕ and $\boldsymbol{\mathcal{G}}$, by (VIII.4.62) we easily obtain

$$(\phi,\psi) = (\boldsymbol{\mathcal{G}},\nabla\nabla\chi) - (\boldsymbol{h},\nabla\chi)\,. \tag{VIII.4.63}$$

By Exercise II.11.9, we have $\|D^2\chi\|_{r'}\leq c\|\psi\|_{r'}$, for all $r'\in(1,\infty)$, and $\|\nabla\chi\|_{3r'/(3-r')}\leq c\|D^2\chi\|_{r'}$, for all $r'\in(1,3)$. Thus, with the help of the Hölder inequality, we obtain, on the one hand,

$$(\boldsymbol{\mathcal{G}}(t),\nabla\nabla\chi)\leq c\,[\![\boldsymbol{\mathcal{G}}(t)]\!]_{2,\mathcal{R}}\|\psi\|_{r'}\,, \quad \text{for a.a. } t\in(0,\infty), \text{ all } r'\in(1,\infty)\,. \tag{VIII.4.64}$$

On the other hand, since $q>3$, we have $q'<3r'/(3-r')$ for all $r'\in(1,3)$, and consequently,

$$\begin{aligned} -(\boldsymbol{h}(t),\nabla\chi) &\leq \|\boldsymbol{h}(t)\|_q\|\nabla\chi\|_{q,B_\rho}\leq c_1(\rho)\|\boldsymbol{h}(t)\|_q\|\nabla\chi\|_{3r'/(3-r')}\\ &\leq c_2(\rho)\|\boldsymbol{h}(t)\|_q\|D^2\chi\|_{r'}\leq c_3(\rho)\|\boldsymbol{h}(t)\|_q\|\psi\|_{r'}\,,\\ &\qquad \text{for a.a. } t\in(0,\infty), \text{ and all } r'\in(1,3)\,. \end{aligned} \tag{VIII.4.65}$$

Collecting (VIII.4.63)–(VIII.4.65) we conclude that

$$|(\phi(t),\psi)|\leq C_1\left([\![\boldsymbol{\mathcal{G}}(t)]\!]_{2,\mathcal{R}}+\|\boldsymbol{h}(t)\|_q\right)\|\psi\|_{r'}\,, \text{ for a.a. } t\in(0,\infty), \text{ all } r'\in(1,3)\,,$$

with C_1 depending on r, q, and ρ. However, by Exercise II.2.12, this latter inequality implies

$$\operatorname*{ess\,sup}_{t\geq 0}\|\phi(t)\|_r\leq C_2\operatorname*{ess\,sup}_{t\geq 0}\left([\![\boldsymbol{\mathcal{G}}(t)]\!]_{2,\mathcal{R}}+\|\boldsymbol{h}(t)\|_q\right)\,,$$

with $C=C(r,q,\rho)$, and the proof of the first part of the theorem is complete.
Assume next that $\boldsymbol{\mathcal{G}}=\boldsymbol{0}$ and $\boldsymbol{h}$ satisfies the further assumptions (VIII.4.49). We then have $\boldsymbol{w}(x,t)\equiv\boldsymbol{w}_2(x,t)$, where $\boldsymbol{w}_2$ satisfies (VIII.4.53), namely, recalling $(\text{VIII.4.52})_1$, we deduce

$$w_i(x,t)=\int_0^t\int_{B_\rho}\varGamma_{ij}(x-y,\tau;\mathcal{R})h_j(y,t-\tau)dy\,d\tau\,. \tag{VIII.4.66}$$

We first consider the case $\mathcal{R}=0$. Differentiating both sides with respect to x, and taking into account estimate (VIII.3.10) of Lemma VIII.3.3, we obtain

$$|D_k\boldsymbol{w}(x,t)| \le \|\boldsymbol{h}\|_{\infty,\mathbb{R}^3_\infty}\int_{B_\rho}\int_0^\infty |D_k\boldsymbol{\Gamma}(x-y,\tau)|d\tau\,dy$$
$$\le c_1\|\boldsymbol{h}\|_{\infty,\mathbb{R}^3_\infty}\int_{B_\rho}\frac{dy}{|x-y|^2}\,,$$

from which, by distinguishing the two cases $|x|\le 2\rho$ and $|x|>2\rho$, we easily prove the desired estimate for $\nabla\boldsymbol{w}$. We now differentiate (VIII.4.66) two times. Taking into account that

$$w_i(x,t)=\int_0^t\int_{\mathbb{R}^3}\Gamma_{ij}(y,\tau)h_j(x-y,t-\tau)dy\,d\tau$$

and the assumptions on $\boldsymbol{h}$, we readily prove that

$$D_kD_lw_i(x,t)=\int_0^t\int_{B_\rho}D_k\Gamma_{ij}(x-y,\tau)D_lh_j(y,t-\tau)dy\,d\tau\,, \tag{VIII.4.67}$$

so that again by Lemma VIII.3.3, we obtain

$$|D_kD_lw_i(x,t)|\le c_1\|\nabla\boldsymbol{h}\|_{\infty,\mathbb{R}^3_\infty}\int_{B_\rho}\frac{dy}{|x-y|^2}\,.$$

From this relation, it easily follows that

$$|D^2\boldsymbol{w}(x,t)|\le c_2\|\nabla\boldsymbol{h}\|_{\infty,\mathbb{R}^3_\infty}\,,\quad \text{for all } (x,t)\in B_{2\rho}\times\mathbb{R}_+\,. \tag{VIII.4.68}$$

Take now x outside $\overline{B_{2\rho}}$, and observe that, for such x and for $y\in B_\rho$, $\boldsymbol{\Gamma}(x-y,t)$ is a smooth function of x. Therefore, we may differentiate (VIII.4.66) twice with respect to x, and use the estimates given in part (i) of Lemma VIII.3.3 to obtain

$$\begin{aligned}|D^2w_i(x,t)| &\le c_3\|\boldsymbol{h}\|_{\infty,\mathbb{R}^3_\infty}\int_{B_\rho}\int_0^\infty|D^2\boldsymbol{\Gamma}(x-y,\tau)|dt\,dy\\ &\le c_4\|\boldsymbol{h}\|_{\infty,\mathbb{R}^3_\infty}\int_{B_\rho}\frac{dy}{|x-y|^3}\\ &\le c_5\frac{\|\boldsymbol{h}\|_{\infty,\mathbb{R}^3_\infty}}{|x|^3}\quad\text{for all } (x,t)\in B^{2\rho}\times\mathbb{R}_+\,.\end{aligned} \tag{VIII.4.69}$$

The claimed estimates on the second derivatives of $\boldsymbol{w}$ follow from (VIII.4.68), (VIII.4.69). The proof of (VIII.4.50) and (VIII.4.51) in the case $\mathcal{R}\ne 0$ is very similar, once we use the estimates (VIII.3.12), and will be only sketched here. We take first $|x|\ge 2\rho$, and in (VIII.3.12)$_2$ we choose $\beta=\rho\mathcal{R}$. Consequently, since $|x-y|\ge\rho=\beta/\mathcal{R}$, differentiating (VIII.4.66) once in x and then using (VIII.3.12)$_2$, we obtain

$$|D_k\boldsymbol{w}(x,t)|\le c_6\,\|\boldsymbol{h}\|_{\infty,\mathbb{R}^3_\infty}\int_{B_\rho}\frac{dy}{|x-y|^{3/2}(1+2\mathcal{R}\,s(x-y))^{3/2}}\,,\quad |x|\ge 2\rho\,,$$

with $c_6 = c_6(\rho, B)$, which, in turn, by virtue of (VIII.4.58), produces

$$\operatorname*{ess\,sup}_{t\geq 0} [\![\nabla \boldsymbol{w}(t)]\!]_{3/2,\mathcal{R},B^{2\rho}} \leq c_7 \, \|\boldsymbol{h}\|_{\infty,\mathbb{R}^3_\infty} \,, \tag{VIII.4.70}$$

with $c_7 = c_7(\rho, B)$. If $x \in B_{2\rho}$, we have $x - y| \leq 3\rho$, and so, choosing this time $\beta = 3\rho\mathcal{R}$ in (VIII.3.12)$_2$, we deduce

$$|D_k \boldsymbol{w}(x,t)| \leq c_8 \, \|\boldsymbol{h}\|_{\infty,\mathbb{R}^3_\infty} \int_{B_\rho} \frac{dy}{|x-y|^2} \leq c_9 \,, \quad |x| \leq 2\rho \,.$$

Combining this latter with (VIII.4.70) we obtain the desired result for $\nabla \boldsymbol{w}$. In order to estimate $D^2\boldsymbol{w}$, we start with the representation (VIII.4.67) and $|x| \leq 2\rho$. Choosing $\beta = 3\rho\mathcal{R}$ in (VIII.3.12)$_2$, we obtain, as before,

$$|D^2 \boldsymbol{w}(x,t)| \leq c_{10} \, \|\nabla \boldsymbol{h}\|_{\infty,\mathbb{R}^3_\infty} \int_{B_\rho} \frac{dy}{|x-y|^2} \leq c_{11} \,, \quad |x| \leq 2\rho \,. \tag{VIII.4.71}$$

If $|x| \geq 2\rho$, we can differentiate the fundamental tensor solution twice with respect to x and use (VIII.3.12)$_3$ with $\beta = \rho\mathcal{R}$. We thus obtain

$$|D^2 \boldsymbol{w}(x,t)| \leq c_{12} \, \|\boldsymbol{h}\|_{\infty,\mathbb{R}^3_\infty} \int_{B_\rho} \frac{dy}{|x-y|^2(1+2\mathcal{R}\, s(x-y))^2} \,, \quad |x| \geq 2\rho \,,$$

with $c_{12} = c_{12}(\rho, B)$, which, in turn, by (VIII.4.58), implies

$$\operatorname*{ess\,sup}_{t\geq 0} [\![D^2 \boldsymbol{w}(t)]\!]_{2,\mathcal{R},B^{2\rho}} \leq c_7 \, \|\boldsymbol{h}\|_{\infty,\mathbb{R}^3_\infty} \,.$$

The statement about $D^2\boldsymbol{w}$ then follows from this inequality and (VIII.4.71). The theorem is completely proved. $\square$

VIII.5 Existence, Uniqueness, and Pointwise Estimates of Solutions in the Whole Space

The objective of this section is to prove existence, uniqueness, and corresponding estimates of solutions $\boldsymbol{v}, p$ to the *inhomogeneous generalized Oseen problem*

$$\left.\begin{aligned} &\Delta \boldsymbol{v} + \mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1} + \mathcal{T}\left(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}\right) = \nabla p + \boldsymbol{f} \\ &\nabla \cdot \boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3$$
$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = \boldsymbol{0} \,, \tag{VIII.5.1}$$

under suitable assumptions on the datum $\boldsymbol{f}$.

We will keep the notation used in the previous section given, in particular, in (VIII.4.46).

Our main result is the following.

Theorem VIII.5.1 *Let $\boldsymbol{\mathcal{F}}$ be a second-order tensor field such that $\nabla\cdot\boldsymbol{\mathcal{F}}\in L^2(\mathbb{R}^3)$, with $[\![\boldsymbol{\mathcal{F}}]\!]_{2,\mathcal{R}}<\infty$, and let $\boldsymbol{g}\in L^q(\mathbb{R}^3)$, $q\in(3,\infty)$, with support contained in Ω_ρ, for some $\rho>0$. Then, the problem* (VIII.5.1) *with $\boldsymbol{f}\equiv\nabla\cdot\boldsymbol{\mathcal{F}}+\boldsymbol{g}$ has at least one solution such that*[1]

$$
\begin{aligned}
&\boldsymbol{v}\in W^{2,2}_{loc}(\mathbb{R}^3)\cap D^{2,2}(\mathbb{R}^3)\cap D^{1,2}(\mathbb{R}^3),\ [\![\boldsymbol{v}]\!]_{1,\mathcal{R}}<\infty,\\
&p\in W^{1,2}_{loc}(\mathbb{R}^3)\cap D^{1,2}(\mathbb{R}^3)\cap L^r(\mathbb{R}^3),\ r>3/2.
\end{aligned}
\tag{VIII.5.2}
$$

This solution satisfies the estimates

$$
\begin{aligned}
|\boldsymbol{v}|_{2,2}+|\boldsymbol{v}|_{1,2}+|p|_{1,2}&\le C_1\left(\|\nabla\cdot\boldsymbol{\mathcal{F}}\|_2+[\![\boldsymbol{\mathcal{F}}]\!]_{2,\mathcal{R}}+\|\boldsymbol{g}\|_q\right)\\
[\![\boldsymbol{v}]\!]_{1,\mathcal{R}}+\|p\|_r&\le C_2\left([\![\boldsymbol{\mathcal{F}}]\!]_{2,\mathcal{R}}+\|\boldsymbol{g}\|_q\right)
\end{aligned}
\tag{VIII.5.3}
$$

where $C_1=C_1(\rho,q,B)$, $C_2=C_2(\rho,r,q,B)$ whenever $\mathcal{R},\mathcal{T}\in[0,B]$, for some $B>0$.

Moreover, assume, in particular, that $\boldsymbol{\mathcal{F}}=\boldsymbol{0}$ and $\boldsymbol{g}$ satisfies the further assumption $\boldsymbol{g}\in W^{1,\infty}(\Omega_\rho)$. Then, $\boldsymbol{v}$ satisfies also

$$
[\![\nabla\boldsymbol{v}]\!]_2+[\![D^2\boldsymbol{v}]\!]_3<\infty\quad \text{if } \mathcal{R}=0\,,\tag{VIII.5.4}
$$

$$
[\![\nabla\boldsymbol{v}]\!]_{3/2,\mathcal{R}}+[\![D^2\boldsymbol{v}]\!]_{2,\mathcal{R}}<\infty\quad \text{if } \mathcal{R}\neq0\,,\tag{VIII.5.5}
$$

and there are constants $C_3=C_3(\rho)$, $C_4(\rho,B)$ if $\mathcal{R}\in(0,B]$ such that

$$
[\![\nabla\boldsymbol{v}]\!]_2+[\![D^2\boldsymbol{v}]\!]_3\le C_3\,\|\boldsymbol{g}\|_{1,\infty}\,,\quad \text{if } \mathcal{R}=0\,,\tag{VIII.5.6}
$$

$$
[\![\nabla\boldsymbol{v}]\!]_{3/2,\mathcal{R}}+[\![D^2\boldsymbol{v}]\!]_{2,\mathcal{R}}\le C_4\,\|\boldsymbol{g}\|_{1,\infty}\,,\quad \text{if } \mathcal{R}\neq0\,.\tag{VIII.5.7}
$$

Finally, if $(\boldsymbol{v}_1,p_1)$ is a generalized solution to (VIII.5.1) *corresponding to the same $\boldsymbol{f}$, then $\boldsymbol{v}=\boldsymbol{v}_1$ and $p=p_1+\mathrm{const}$.*

Proof. The uniqueness part is an immediate consequence of Theorem VIII.2.1. Moreover, from Theorem VIII.1.1, Theorem VIII.1.2, and Lemma VIII.3.2 it follows that the only property for $\boldsymbol{v}$ that remains to be proved is the asymptotic properties of $\boldsymbol{v}$, along with the corresponding estimates.[2] In order to prove these latter, we consider the following *unsteady* Cauchy problem associated with (VIII.5.1):

$$
\left.
\begin{aligned}
&\frac{\partial\boldsymbol{u}}{\partial t}=\Delta\boldsymbol{u}+\mathcal{R}\frac{\partial\boldsymbol{u}}{\partial x_1}-\mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{u}-\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{u})\\
&\qquad\qquad\qquad\qquad -\nabla q-\nabla\cdot\boldsymbol{\mathcal{F}}-\boldsymbol{g}\\
&\nabla\cdot\boldsymbol{u}=0
\end{aligned}
\right\}\ \text{in } \mathbb{R}^3\times(0,\infty)
$$

$$
\boldsymbol{u}(x,0)=\boldsymbol{0},\quad x\in\mathbb{R}^3\,.
\tag{VIII.5.8}
$$

[1] Recall the notation given in (VIII.4.46).

[2] Notice that $\nabla\cdot\boldsymbol{\mathcal{F}}\in D_0^{-1,2}(\mathbb{R}^3)$ and that $|\nabla\cdot\boldsymbol{\mathcal{F}}|_{-1,2}\le c\,[\![\boldsymbol{\mathcal{F}}]\!]_{2,\mathcal{R}}$. Likewise, since

$$
|(\boldsymbol{g},\boldsymbol{\varphi})|\le\|\boldsymbol{g}\|_{6/5}\|\boldsymbol{\varphi}\|_6\le c\,\|\boldsymbol{g}\|_q|\boldsymbol{\varphi}|_{1,2}\,,
$$

for all $\boldsymbol{\varphi}\in D_0^{1,2}(\Omega)$, we deduce $|\boldsymbol{g}|_{-1,2}\le c\,\|\boldsymbol{g}\|_q$.

We begin by proving the properties stated in (VIII.5.2) and (VIII.5.3). To this end we shall prove that (i) $[\![\boldsymbol{u}(t)]\!]_{1,\mathcal{R}}$ is uniformly bounded in time, and that (ii) $\boldsymbol{u}(\cdot,t)$ converges as $t \to \infty$ to the solution $\boldsymbol{v}$ of (VIII.5.1) in appropriate norms. As a consequence, the asymptotic (spatial) behavior of $\boldsymbol{v}$ will be shown to be the same as that of $\boldsymbol{u}$. To reach this goal, we make a change of variables that brings (VIII.5.8) into an appropriate (unsteady) Oseen problem. Specifically, we define

$$\begin{aligned}
&\boldsymbol{\chi} = \boldsymbol{Q}(t)\cdot\boldsymbol{x},\\
&\boldsymbol{w}(\chi,t) = \boldsymbol{Q}(t)\cdot\boldsymbol{u}(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{\chi},t), \quad \pi(\chi,t) = q(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{\chi},t)\,,\\
&\boldsymbol{\mathcal{G}}(\chi,t) = -\boldsymbol{Q}(t)\cdot\boldsymbol{\mathcal{F}}(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{\chi})\cdot\boldsymbol{Q}^\top(t)\,, \quad \boldsymbol{h}(\chi,t) = -\boldsymbol{Q}(t)\cdot\boldsymbol{g}(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{\chi})\,,
\end{aligned} \tag{VIII.5.9}$$

where $^\top$ denotes transpose, and $\boldsymbol{Q} = \boldsymbol{Q}(t)$, $t \geq 0$, is a one-parameter family of second-order tensors satisfying the following initial-value problem:

$$\begin{cases} \dfrac{d\boldsymbol{Q}}{dt} = \mathcal{T}\,\boldsymbol{Q}\cdot\boldsymbol{W}(\boldsymbol{e}_1)\,,\\ \boldsymbol{Q}(0) = \boldsymbol{I}\,,\end{cases} \tag{VIII.5.10}$$

with

$$\boldsymbol{W}(\boldsymbol{e}_1) = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}. \tag{VIII.5.11}$$

Since the matrix $\boldsymbol{W}$ is skew-symmetric, it follows that for each $t \geq 0$, $\boldsymbol{Q}(t)$ defines a proper orthogonal transformation, i.e.,

$$\boldsymbol{Q}(t)\cdot\boldsymbol{Q}(t)^\top = \boldsymbol{Q}^\top(t)\cdot\boldsymbol{Q}(t) = \boldsymbol{I} \quad (\boldsymbol{I} = \text{identity tensor})\,.$$

More precisely, by integrating (VIII.5.10))–(VIII.5.11), we obtain

$$\boldsymbol{Q}(t) = \begin{bmatrix} 1 & 0 & 0\\ 0 & \cos(\mathcal{T}t) & -\sin(\mathcal{T}t)\\ 0 & \sin(\mathcal{T}t) & \cos(\mathcal{T}t)\end{bmatrix}. \tag{VIII.5.12}$$

We also notice that

$$\boldsymbol{W}(\boldsymbol{e}_1)\cdot\boldsymbol{a} = \boldsymbol{e}_1\times\boldsymbol{a}\,, \quad \text{for all } \boldsymbol{a}\in\mathbb{R}^3\,, \tag{VIII.5.13}$$

and moreover, that

$$\boldsymbol{Q}(t)\cdot\boldsymbol{e}_1 = \boldsymbol{e}_1\,, \quad \text{for all } t \geq 0. \tag{VIII.5.14}$$

Now, using (VIII.5.9), (VIII.5.10), and (VIII.5.13) together with the identity

$$\frac{d\boldsymbol{Q}^\top}{dt}\cdot\boldsymbol{Q}(t) = -\boldsymbol{Q}^\top(t)\cdot\frac{d\boldsymbol{Q}}{dt}\,,$$

we obtain $\left(\bullet \equiv \dfrac{d}{dt}\right)$

$$
\begin{aligned}
\frac{\partial \boldsymbol{w}}{\partial t} &= \boldsymbol{Q}(t) \cdot \left(\frac{\partial \boldsymbol{u}}{\partial t} + \left(\dot{\boldsymbol{Q}}^{\top}(t) \cdot \boldsymbol{Q}(t) \cdot \boldsymbol{x} \right) \cdot \nabla \boldsymbol{u} + \boldsymbol{Q}^{\top}(t) \cdot \dot{\boldsymbol{Q}}(t) \cdot \boldsymbol{u} \right) \\
&= \boldsymbol{Q}(t) \cdot \left(\frac{\partial \boldsymbol{u}}{\partial t} - \mathcal{T} \left(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{u} \right) \right)
\end{aligned}
\tag{VIII.5.15}
$$

and

$$
\Delta_\chi \boldsymbol{w} = \boldsymbol{Q}(t) \cdot \Delta \boldsymbol{u}. \tag{VIII.5.16}
$$

Moreover, from (VIII.5.9) and (VIII.5.14) we also obtain

$$
\frac{\partial \boldsymbol{w}}{\partial \chi_1}(\chi, t) = \boldsymbol{Q}(t) \cdot \left((\boldsymbol{Q}^{\top}(t) \cdot \boldsymbol{e}_1) \cdot \nabla \boldsymbol{u}(x, t) \right) = \boldsymbol{Q}(t) \cdot \frac{\partial \boldsymbol{u}}{\partial x_1}(x, t). \tag{VIII.5.17}
$$

Therefore, collecting (VIII.5.15)–(VIII.5.17), we deduce that the Cauchy problem (VIII.5.8) can be equivalently rewritten as follows:

$$
\left.
\begin{aligned}
&\frac{\partial \boldsymbol{w}}{\partial t} = \Delta \boldsymbol{w} + \mathcal{R} \frac{\partial \boldsymbol{w}}{\partial \chi_1} - \nabla \pi + \nabla \cdot \boldsymbol{\mathcal{G}} + \boldsymbol{h} \\
&\nabla \cdot \boldsymbol{w} = 0
\end{aligned}
\right\} \text{ in } \mathbb{R}^3 \times (0, \infty),
\\
\boldsymbol{w}(\chi, 0) = \boldsymbol{0}, \quad \chi \in \mathbb{R}^3 .
\tag{VIII.5.18}
$$

At this point we observe that in view of the assumptions on $\boldsymbol{\mathcal{F}}$ and the orthogonality properties of the family $\boldsymbol{Q}(t)$, $t \ge 0$,

$$
\begin{gathered}
\operatorname*{ess\,sup}_{t \ge 0} [\![\boldsymbol{\mathcal{G}}(t)]\!]_{2,\mathcal{R}} = [\![\boldsymbol{\mathcal{F}}]\!]_{2,\mathcal{R}} , \\
\operatorname*{ess\,sup}_{t \ge 0} \| \nabla \cdot \boldsymbol{\mathcal{G}}(t) \|_2 = \| \nabla \cdot \boldsymbol{\mathcal{F}} \|_2 , \\
\operatorname*{ess\,sup}_{t \ge 0} \| \boldsymbol{h}(t) \|_q = \| \boldsymbol{g} \|_q .
\end{gathered}
$$

As a consequence, from Theorem VIII.4.4 it follows that problem (VIII.5.18) has one and only one solution $(\boldsymbol{w}, \phi)$ such that

$$
(\boldsymbol{w}, \nabla \phi) \in \mathcal{L}^2(\mathbb{R}^3_T), \ \phi \in L^{2,6}(\mathbb{R}^3_T), \tag{VIII.5.19}
$$

which, in addition, satisfies the following estimate, for all $r \in (3/2, \infty)$:

$$
\operatorname*{ess\,sup}_{t \ge 0} [\![\boldsymbol{w}(t)]\!]_{1,\mathcal{R}} + \operatorname*{ess\,sup}_{t \ge 0} \| \phi(t) \|_r \le C \left([\![\boldsymbol{\mathcal{F}}]\!]_{2,\mathcal{R}} + \| \boldsymbol{g} \|_q \right), \tag{VIII.5.20}
$$

with C depending on $(r, q$ and$)$ an upper bound for $\mathcal{R}$ and $\mathcal{T}$. We now go back to the original fields $\boldsymbol{u}, q$. Since $\boldsymbol{Q} = \boldsymbol{Q}(t)$ is orthogonal, we have the identities

$$
|\boldsymbol{x}| = |\boldsymbol{\chi}|, \ |\boldsymbol{u}(x, t)| = |\boldsymbol{w}(\chi, t)|, \ |q(x, t)| = |\phi(\chi, t)|, \tag{VIII.5.21}
$$

and so from (VIII.5.20), we obtain

$$\operatorname*{ess\,sup}_{t\geq 0} [\![\boldsymbol{u}(t)]\!]_{1,\mathcal{R}} + \operatorname*{ess\,sup}_{t\geq 0} \|q(t)\|_r \leq C\left([\![\boldsymbol{\mathcal{F}}]\!]_{2,\mathcal{R}} + \|\boldsymbol{g}\|_q\right). \tag{VIII.5.22}$$

It remains to prove the convergence of $\boldsymbol{u}$ to $\boldsymbol{v}$ when $t \to \infty$. Setting

$$\begin{aligned}
\boldsymbol{V}(\chi,t) &= \boldsymbol{Q}(t)\cdot\boldsymbol{v}(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{\chi}), \quad P(\chi,t) = p(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{\chi}),\\
\boldsymbol{U}(\chi,t) &= \boldsymbol{w}(\chi,t) - \boldsymbol{V}(\chi,t), \qquad \pi(\chi,t) = \phi(\chi,t) - P(\chi,t),
\end{aligned}$$

and observing that (see (VIII.5.15))

$$\frac{\partial \boldsymbol{V}}{\partial t} = -\mathcal{T}\,\boldsymbol{Q}(t)\cdot(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla_x\boldsymbol{v} - \boldsymbol{e}_1\times\boldsymbol{v}), \tag{VIII.5.23}$$

we find that the pair $(\boldsymbol{U},\pi)$ satisfies

$$\left.\begin{aligned}
&\frac{\partial \boldsymbol{U}}{\partial t} = \Delta\boldsymbol{U} + \mathcal{R}\frac{\partial \boldsymbol{U}}{\partial \chi_1} - \nabla\pi\\
&\nabla\cdot\boldsymbol{U} = 0
\end{aligned}\right\}\ \text{in } \mathbb{R}^3\times(0,\infty),$$
$$\boldsymbol{U}(\chi,0) = \boldsymbol{v}(\chi), \quad \chi\in\mathbb{R}^3. \tag{VIII.5.24}$$

Multiplying both sides of the first equation by $\nabla\psi$, with $\psi \in C_0^\infty(\mathbb{R}^3)$, we easily show, for a.a. $t\in[0,T]$, that

$$(\nabla\pi(t),\nabla\psi)_{\mathbb{R}^3} = 0, \quad \text{for all } \psi\in C_0^\infty(\mathbb{R}^3),$$

which implies that $\pi(t)$ is harmonic in the whole of $\mathbb{R}^3$, for a.a. $t\in[0,T]$. Furthermore, the result of Lemma VIII.2.2 and (VIII.5.20) implies $\pi(t)\in L^6(\mathbb{R}^3)$, for a.a. $t\in[0,T]$. Therefore, by Exercise II.11.11, we obtain $\pi(x,t) = 0$ for all $x\in\mathbb{R}^3$ and a.a. $t\in[0,T]$, and (VIII.5.24) becomes

$$\left.\begin{aligned}
&\frac{\partial \boldsymbol{U}}{\partial t} = \Delta\boldsymbol{U} + \mathcal{R}\frac{\partial \boldsymbol{U}}{\partial \chi_1}\\
&\nabla\cdot\boldsymbol{U} = 0
\end{aligned}\right\}\ \text{in } \mathbb{R}^3\times(0,\infty),$$
$$\boldsymbol{U}(\chi,0) = \boldsymbol{v}(\chi), \quad \chi\in\mathbb{R}^3. \tag{VIII.5.25}$$

We now observe that the initial datum $\boldsymbol{v}$ is in $L^6(\mathbb{R}^3)$ and that moreover, by the properties of $\boldsymbol{w}$ and $\boldsymbol{v}$ (see (VIII.5.20), (VIII.5.19), and (VIII.5.23)), we have

$$\begin{aligned}
&\boldsymbol{V}\in L^6(\mathbb{R}^3_T), \ \frac{\partial \boldsymbol{V}}{\partial t}\in L^2_{loc}((0,T]\times\mathbb{R}^3), \ D^2\boldsymbol{V}\in L^2(\mathbb{R}^3_T),\\
&\boldsymbol{w}\in L^6(\mathbb{R}^3_T), \ \frac{\partial \boldsymbol{w}}{\partial t}, \ D^2\boldsymbol{w}\in L^2(\mathbb{R}^3_T).
\end{aligned}$$

From these conditions we infer, in particular,

$$\boldsymbol{U}\in L^6(\mathbb{R}^3_T), \ \frac{\partial \boldsymbol{U}}{\partial t}, \ D^2\boldsymbol{U}\in L^2_{loc}((0,T]\times\mathbb{R}^3). \tag{VIII.5.26}$$

Thus, from Theorem VIII.4.3 we deduce

$$\begin{aligned} \|\boldsymbol{U}\|_\sigma &\le C t^{-1/4+3/(2\sigma)} \|\boldsymbol{v}\|_6 ,\ \sigma > 6 , \\ \|\nabla \boldsymbol{U}\|_6 &\le C t^{-1/2} \|\boldsymbol{v}\|_6 , \end{aligned}$$

for all $t > 0$. From these two latter relations with $\sigma = 12$, and with the help of inequality (II.9.7), we obtain

$$|\boldsymbol{U}(x,t)| \le C t^{-1/8} \|\boldsymbol{v}\|_6 , \quad \text{for all } t \ge 1 ,$$

namely

$$|\boldsymbol{u}(x,t) - \boldsymbol{v}(x,t)| \le C t^{-1/8} \|\boldsymbol{v}\|_6 , \quad \text{for all } t \ge 1 , \qquad \text{(VIII.5.27)}$$

since $|\boldsymbol{U}| = |\boldsymbol{u} - \boldsymbol{v}|$. We are now in a position to obtain the desired asymptotic behavior and the corresponding estimate for $\boldsymbol{v}$. In fact, by (VIII.5.22) and (VIII.5.27), we obtain, for all $t \ge 1$,

$$\begin{aligned} |\boldsymbol{v}(x)|(1+|x|)(1+2\mathcal{R}\, s(x)) &\le |\boldsymbol{u}(x,t) - \boldsymbol{v}(x)|(1+|x|)(1+2\mathcal{R}\, s(x)) \\ &\quad + |\boldsymbol{u}(x,t)|(1+|x|)(1+2\mathcal{R}\, s(x)) \\ &\le c_1 (1+|x|)(1+2\mathcal{R}\, s(x)) t^{-1/8} \|\boldsymbol{v}\|_6 + c_2 \, [\![\boldsymbol{\mathcal{F}}]\!]_{2,\mathcal{R}} . \end{aligned}$$

Passing to the limit $t \to \infty$, we finally deduce

$$|\boldsymbol{v}(x)|(1+|x|)(1+2\mathcal{R}\, s(x)) \le c_2 \, [\![\mathcal{F}]\!]_{2,\mathcal{R}} ,$$

which furnishes the desired result for $\boldsymbol{v}$. In order to complete the proof of the first part of the theorem, it remains to prove the stated properties for the pressure along with the estimate (VIII.5.3). From Theorem VIII.1.1 and the assumptions on $\boldsymbol{\mathcal{F}}$ we recover $p \in W^{1,2}_{loc}(\mathbb{R}^3)$. This property together with (VIII.2.5) implies, possibly by modifying p by the addition of a constant, $p \in D^{1,2}(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$. Then, from (VIII.5.1), (VIII.5.2)$_1$, we deduce that

$$(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}) \in L^2(\mathbb{R}^3) ,$$

which, coupled with (VIII.2.26), furnishes

$$(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}) \in H(\mathbb{R}^3) .$$

We next operate with the Helmholtz–Weyl projection operator P (see Remark III.1.1) on both sides of (VIII.5.1)$_1$. Observing that

$$\begin{aligned} & P\, \Delta \boldsymbol{v} = \Delta \boldsymbol{v} ,\ P \left(\frac{\partial \boldsymbol{v}}{\partial x_1} \right) = \frac{\partial \boldsymbol{v}}{\partial x_1} , \\ & P\, (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}) = \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v} , \end{aligned}$$

it follows that

$$\mathcal{T}\,(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}) = -\mathcal{R}\,\frac{\partial \boldsymbol{v}}{\partial x_1} - \Delta \boldsymbol{v} + P\,(\nabla \cdot \boldsymbol{\mathcal{F}} + \boldsymbol{g})\,.$$

Therefore, recalling that $\boldsymbol{v}$ satisfies (VIII.5.3), we obtain

$$\mathcal{T}\|\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}\|_2 \le c\,(\llbracket \boldsymbol{\mathcal{F}} \rrbracket_{2,\mathcal{R}} + \|\nabla \cdot \boldsymbol{\mathcal{F}}\|_2 + \|\boldsymbol{g}\|_q)\,, \qquad \text{(VIII.5.28)}$$

where c depends on B. Employing (VIII.5.28), (VIII.5.1)$_1$, and the estimate (VIII.5.3) for $\boldsymbol{v}$, we thus conclude that

$$|p|_{1,2} \le c_1\,(\llbracket \boldsymbol{\mathcal{F}} \rrbracket_{2,\mathcal{R}} + \|\nabla \cdot \boldsymbol{\mathcal{F}}\|_2 + \|\boldsymbol{g}\|_q)\,.$$

Next, we have to show that $p \in L^r(\mathbb{R}^3)$, $r \in (3/2, \infty)$, and prove the corresponding estimate given in (VIII.5.3). In fact, these properties can be shown in an entirely similar way to that employed at the end of the proof of Theorem VIII.4.4 to prove analogous properties for the pressure ϕ, and so the proof will be only sketched here. We multiply both sides of (VIII.5.1)$_1$ by $\nabla\chi$, $\chi \in D^{1,2}(\Omega)$, to obtain

$$(\nabla p, \nabla \chi) = -(\nabla \cdot \boldsymbol{\mathcal{F}} + \boldsymbol{g}, \nabla \chi)\,.$$

Choosing χ as the solution to the Poisson equation $\Delta\chi = \psi$, $\psi \in C_0^\infty(\mathbb{R}^3)$ and using the facts that $p \in L^6(\Omega)$ and $q > 3$, one shows that

$$|(p, \psi)| \le c\,(\llbracket \boldsymbol{\mathcal{F}} \rrbracket_{2,\mathcal{R}} + \|\boldsymbol{g}\|_q)\,\|\psi\|_{r'}\,, \quad \text{for all } \psi \in C_0^\infty(\mathbb{R}^3), \text{ and all } r' \in (1,3)\,.$$

From this relation it then follows that $p \in L^r(\mathbb{R}^3)$, for all $r \in (3/2, \infty)$ and that $\|p\|_r \le c\,(\llbracket \boldsymbol{\mathcal{F}} \rrbracket_{2,\mathcal{R}} + \|\boldsymbol{g}\|_q)$. This concludes the proof of the first part of the theorem. Now assume $\boldsymbol{\mathcal{F}} = \boldsymbol{0}$ and that $\boldsymbol{g}$ satisfies the further assumption $\boldsymbol{g} \in W^{1,\infty}(\Omega_\rho)$. By the orthogonality properties of the family $\boldsymbol{Q}(t)$, $t \ge 0$, we thus have

$$\operatorname*{ess\,sup}_{t\ge 0}\left(\|\boldsymbol{h}(t)\|_{\infty,\mathbb{R}^3_\infty} + \|\nabla \boldsymbol{h}(t)\|_{\infty,\mathbb{R}^3_\infty}\right) = \|\boldsymbol{g}\|_{1,\infty}\,. \qquad \text{(VIII.5.29)}$$

Moreover, from the results of Theorem VIII.4.4 applied to the Cauchy problem (VIII.5.18) with $\mathcal{R} = 0$ and $\boldsymbol{\mathcal{G}} = \boldsymbol{0}$, we obtain

$$\operatorname*{ess\,sup}_{t\ge 0}\left(\llbracket \nabla \boldsymbol{w}(t) \rrbracket_2 + \llbracket D^2 \boldsymbol{w}(t) \rrbracket_3\right) \le c_2 \operatorname*{ess\,sup}_{t\ge 0}\left(\|\boldsymbol{h}(t)\|_{\infty,\mathbb{R}^3_\infty} + \|\nabla \boldsymbol{h}(t)\|_{\infty,\mathbb{R}^3_\infty}\right),$$

that is, by (VIII.5.29),

$$\operatorname*{ess\,sup}_{t\ge 0}\left(\llbracket \nabla \boldsymbol{w}(t) \rrbracket_2 + \llbracket D^2 \boldsymbol{w}(t) \rrbracket_3\right) \le c_2 \|\boldsymbol{g}\|_{1,\infty}\,.$$

Bearing in mind (VIII.5.21), we infer that this inequality continues to hold with $\boldsymbol{u}$ in place of $\boldsymbol{w}$. Consequently, reasoning exactly as in the proof of the first part (precisely, the argument following (VIII.5.22)), we establish the properties (VIII.5.4) and (VIII.5.6) for $\boldsymbol{v}$. By completely analogous reasoning, we can prove (VIII.5.5) and (VIII.5.7) as well. The theorem is completely proved.

$\square$

Remark VIII.5.1 In connection with the assumptions of Theorem VIII.5.1, it may be of some interest to find conditions under which a vector field $\boldsymbol{f}$, defined on a generic exterior domain $\Omega \subseteq \mathbb{R}^3$, can be written in the form $\nabla \cdot \boldsymbol{\mathcal{F}}$, with $\boldsymbol{\mathcal{F}}$ satisfying the assumptions of Theorem VIII.5.1. For example, if $\boldsymbol{f}$ is of bounded support in Ω and satisfies some further assumptions, then $\boldsymbol{f}$ can be represented in the desired way; see Exercise III.3.9 and Exercise VIII.5.1. When $\mathcal{R} = 0$, a simple sufficient condition (but of course, one could formulate others as well) is that $\boldsymbol{f} \in L^1(\Omega)$, it is bounded on any bounded set, and decays to zero as $|x| \to \infty$ like $|x|^{-3}$, as shown in the following.

Lemma VIII.5.1 *Let Ω be an exterior domain of $\mathbb{R}^3$, and let $\boldsymbol{f} \in L^1(\Omega)$ with $[\![\boldsymbol{f}]\!]_3 < \infty$. Then, there exists a second-order tensor field $\boldsymbol{\mathcal{F}}$ such that $\boldsymbol{f} = \nabla \cdot \boldsymbol{\mathcal{F}}$ $(\in L^2(\Omega))$, and moreover, $[\![\boldsymbol{\mathcal{F}}]\!]_2 < \infty$.*

Proof. Extend $\boldsymbol{f}$ to zero outside Ω and set $\psi_i(x) = (\mathcal{E} * f_i)(x)$, $i = 1, 2, 3$, where $\mathcal{E}$ is the (three-dimensional) Laplace fundamental solution. With the help of Lemma II.9.1, Lemma II.9.2, and Theorem II.11.2 and employing the hypothesis made on $\boldsymbol{f}$, we can show that ψ_i is well defined and that it satisfies $\Delta \psi_i = f_i$, in $\mathbb{R}^3$, $i = 1, 2, 3$. We next want to estimate the quantity

$$D_k \psi_i(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x_k - y_k}{|x-y|^3} f_i(y)\, dy\,, \quad i, k = 1, 2, 3\,. \tag{VIII.5.30}$$

It is easy to see that $D_k \psi \in L^\infty(\mathbb{R}^3)$. This is because, by the properties of $\boldsymbol{f}$,

$$\begin{aligned} |D_k \psi_i(x)| &\le c\, [\![\boldsymbol{f}]\!]_3 \left(\int_{|x-y|<1} \frac{dy}{|x-y|^2} + \int_{|x-y|\ge 1} \frac{dy}{|x-y|^2 (1+|y|)^3} \right) \\ &\le c_1\, [\![\boldsymbol{f}]\!]_3\,. \end{aligned}$$

We now set $|x| = R > 0$, and write the integral in (VIII.5.30) as the sum of three integrals, over the domains $B_{R/2}$, $B_{R/2,2R}$, and B^{2R}, respectively. We denote these integrals, in order, by $I_1(R)$, $I_2(R)$, and $I_3(R)$. We have

$$|I_1(R)| \le \frac{c_1}{R^2} \int_{B_{R/2}} |f_i| \le c_2 \frac{\|\boldsymbol{f}\|_1}{R^2}\,.$$

Moreover,

$$|I_2(R)| \le c_3\, \frac{[\![\boldsymbol{f}]\!]_3}{R^3} \int_{B_{R/2,2R}} \frac{dy}{|x-y|^2} \le c_4\, \frac{[\![\boldsymbol{f}]\!]_3}{R^3} \int_0^{3R} dr \le c_5\, \frac{[\![\boldsymbol{f}]\!]_3}{R^2}\,.$$

Finally,

$$|I_3(R)| \le \frac{c_6}{R^2} \int_{B^{2R}} |\boldsymbol{f}| \le c_6 \frac{\|\boldsymbol{f}\|_1}{R^2}\,.$$

The lemma then follows with $\mathcal{F}_{ij} = D_i \psi_j$, $i, j = 1, 2, 3$. $\square$

$\blacksquare$

Exercise VIII.5.1 Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ and let $\boldsymbol{f} \in L^q(\Omega)$, $q \geq n$, with bounded support and $\overline{\boldsymbol{f}}_\Omega = \mathbf{0}$. Show that there exists a second-order tensor field $\boldsymbol{\mathcal{F}}$ defined in Ω, and with bounded support, such that

$$\boldsymbol{f} = \nabla \cdot \boldsymbol{\mathcal{F}} \ \text{ in } \Omega\,, \quad \|\boldsymbol{\mathcal{F}}\|_\infty \leq c\,\|\boldsymbol{f}\|_q\,.$$

Hint: Use Theorem III.3.1 and Theorem III.3.2.

VIII.6 On the Pointwise Asymptotic Behavior of Generalized Solutions

In this section we will investigate the behavior at large distances from the boundary of generalized solutions to (VIII.0.2), (VIII.0.7) in a generic (sufficiently smooth) exterior domain of $\mathbb{R}^3$. Precisely, we shall show that, under suitable assumptions on the body force, the magnitude of the velocity field can be pointwise bounded by the function $w(x) = (1 + |x|)(1 + 2\mathcal{R}\, s(x))$, where, we recall, $s(x) = |x| + x_1$. Thus, if $\mathcal{R} > 0$, the function w shows the same asymptotic behavior as the Oseen fundamental solution $\boldsymbol{E}$ (see (VII.3.23) and Remark VII.3.1), whereas if $\mathcal{R} = 0$, the behavior is the same as the Stokes fundamental solution (see (IV.2.6)). We also recall that by virtue of the Mozzi–Chasles transformation considered in the Introduction, $\mathcal{R} = 0$ if and only if the translational velocity $\boldsymbol{v}_0$ and the angular velocity $\boldsymbol{\omega}$ are orthogonal, or, in particular, $\boldsymbol{v}_0 = \mathbf{0}$.[1] Therefore, $w(x)$ does not present a wave-like behavior unless $\boldsymbol{v}_0$ has a nonzero component in the direction of the angular velocity.

Also in this section, we will keep the notation introduced in (VIII.4.46).

In order to prove the above-mentioned properties, we need, as usual, some preparatory results.

Lemma VIII.6.1 *Let Ω be an exterior domain of class C^2, and let $\mathcal{R}, \mathcal{T} \in [0, B]$, for some $B > 0$. Assume that $\boldsymbol{f} \in L^q(\Omega_R)$, $R > \delta(\Omega^c)$, and $\boldsymbol{v}_* \in W^{2-1/q,q}(\partial\Omega)$, $q \in (1, \infty)$. Then, the generalized solution $(\boldsymbol{v}, p)$ to* (VIII.0.2), (VIII.0.7) *satisfies*

$$\boldsymbol{v} \in W^{2,q}(\Omega_r), \quad p \in W^{1,q}(\Omega_r)\,, \tag{VIII.6.1}$$

for all $r \in (\delta(\Omega^c), R)$. Moreover,

$$\|\boldsymbol{v}\|_{2,q,\Omega_r} + \|p\|_{1,q,\Omega_r} \leq C(\|\boldsymbol{f}\|_{q,\Omega_R} + \|\boldsymbol{v}_*\|_{2-1/q,q,\partial\Omega} + \|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{q,\Omega_R}), \tag{VIII.6.2}$$

where $C = C(\Omega, r, R, B)$.

Proof. The proof of (VIII.6.1) is a consequence of Theorem VIII.1.1 and the assumptions made on the data $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{v}_*$. The proof of (VIII.6.2) goes as follows. We write (VIII.0.7) as a Stokes problem

[1] We recall that we are assuming $\boldsymbol{\omega} \neq \mathbf{0}$; otherwise, the analysis coincides with that performed in Chapter VII.

$$\left.\begin{array}{l} \Delta \boldsymbol{v} = \nabla p + \boldsymbol{H} \\ \nabla \cdot \boldsymbol{v} = 0 \end{array}\right\} \text{ in } \Omega\,,$$
$$\boldsymbol{v} = \boldsymbol{v}_*, \text{ at } \partial\Omega\,,$$

with

$$\boldsymbol{H} = -\mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1} - \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}) + \boldsymbol{f}.$$

Exploiting Theorem IV.5.1 and Theorem IV.5.3, and taking into account this expression for the function $\boldsymbol{H}$, we readily get

$$\|\boldsymbol{v}\|_{2,q,\Omega_r} + \|p\|_{1,q,\Omega_r} \le C\,(\|\boldsymbol{f}\|_{q,\Omega_R} + \|\boldsymbol{v}_*\|_{2-1/q,q,\partial\Omega} \\ + \|\boldsymbol{H}\|_{-1,q,\Omega_R} + \|\boldsymbol{v}\|_{q,\Omega_R} + \|p\|_{q,\Omega_R})$$

with C depending only on Ω, r, and R. Since

$$\|\boldsymbol{H}\|_{-1,q,\Omega_R} \le C(1+\mathcal{R}+\mathcal{T})\|\boldsymbol{v}\|_{q,\Omega_R} + \|\boldsymbol{f}\|_{q,\Omega_R},$$

we obtain the desired estimate. □

We are now in a position to establish the main result of this section.

Theorem VIII.6.1 *Let Ω be an exterior domain of class C^2. Assume that $\boldsymbol{\mathcal{F}}$ is a second-order tensor field on Ω such that $\nabla\cdot\boldsymbol{\mathcal{F}} \in L^2(\Omega)$ with $[\![\boldsymbol{\mathcal{F}}]\!]_{2,\mathcal{R}} < \infty$, and that $\boldsymbol{v}_* \in W^{3/2,2}(\partial\Omega)$. Let $\boldsymbol{v}$ be the corresponding generalized solution to* (VIII.0.2), (VIII.0.7) *with $\boldsymbol{f} \equiv \nabla\cdot\boldsymbol{\mathcal{F}}$. Then, denoting by p the pressure field associated to $\boldsymbol{v}$ by* Lemma VIII.1.1, *we have*

$$\boldsymbol{v} \in W^{2,2}_{loc}(\overline{\Omega}) \cap D^{1,2}(\Omega) \cap D^{2,2}(\Omega)\,, \quad [\![\boldsymbol{v}]\!]_{1,\mathcal{R}} < \infty\,,$$
$$p \in D^{1,2}(\Omega) \cap L^{q_1}(\Omega) \cap L^{q_2}(\Omega^\rho) \;\; \textit{for all } q_1 \in (3/2, 6], \textit{ and all } q_2 \in (6,\infty)\,, \tag{VIII.6.3}$$

where ρ is an arbitrary number greater than $\delta(\Omega^c)$. Moreover, $\boldsymbol{v}, p$ satisfy the following estimate

$$|\boldsymbol{v}|_{2,2} + |\boldsymbol{v}|_{1,2} + [\![\boldsymbol{v}]\!]_{1,\mathcal{R}} + |p|_{1,2} + \|p\|_{q_1} + \|p\|_{q_2,\Omega^\rho} \\ \le C(\|\nabla\cdot\boldsymbol{\mathcal{F}}\|_2 + [\![\boldsymbol{\mathcal{F}}]\!]_{2,\mathcal{R}} + \|\boldsymbol{v}_*\|_{3/2,2,\partial\Omega})\,, \tag{VIII.6.4}$$

where C depends only on Ω, B, q_1, and ρ, whenever $\mathcal{R}, \mathcal{T} \in [0,B]$.

Proof. In view of the stated assumptions, Lemma VIII.6.1, Lemma VIII.2.1, and Lemma VIII.2.2, we have only to show that $[\![\boldsymbol{v}]\!]_{1,\mathcal{R}} < \infty$, $p \in L^{q_1}(\Omega) \cap L^{q_2}(\Omega^\rho)$, along with the validity of the corresponding estimates. Notice that by virtue of Lemma VIII.6.1, Lemma VIII.2.2, and the assumptions on $\boldsymbol{\mathcal{F}}$, we can suppose $p \in L^6(\Omega)$.

To reach the above goal, for a *fixed* $R > \delta(\Omega^c)$, we denote by φ a smooth "cut-off" function that is 0 for $|x| \le R$ and is 1 for $|x| \ge 2R$, and set $\vartheta = 1-\varphi$. We then introduce the fields

$$\boldsymbol{u} = \varphi \boldsymbol{v} + \vartheta\, \Phi\, \nabla \mathcal{E} + \boldsymbol{w}\,, \quad q = \varphi\, p\,,$$

where $\mathcal{E}$ is the Laplace fundamental solution defined in (II.9.1),

$$\Phi = \int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n}\,,$$

with $\boldsymbol{n}$ the unit outer normal to $\partial\Omega$, and $\boldsymbol{w}$ satisfies

$$\begin{aligned} &\nabla \cdot \boldsymbol{w} = -\nabla\varphi \cdot (\boldsymbol{v} + \Phi\nabla\mathcal{E}) \quad \text{in } \Omega_{2R}\,, \\ &\boldsymbol{w} \in W_0^{3,2}(\Omega_{2R})\,, \\ &\|\boldsymbol{w}\|_{3,2} \le C\,(\|\boldsymbol{v}\|_{2,2,\Omega_{2R}} + |\Phi|)\,, \end{aligned} \tag{VIII.6.5}$$

where $C = C(R, \Omega)$. Clearly, $\nabla\varphi \cdot (\boldsymbol{v} + \Phi\nabla\mathcal{E}) \in W_0^{2,2}(\Omega_{2R})$. Moreover, since[2]

$$\int_{\partial\Omega} \boldsymbol{n} \cdot \nabla\mathcal{E} = -1\,,$$

it follows that

$$\begin{aligned} \int_{\Omega_{2R}} \nabla\varphi \cdot (\boldsymbol{v} + \Phi\nabla\mathcal{E}) &= \int_{\partial B_{2R}} (\boldsymbol{v} + \Phi\nabla\mathcal{E}) \cdot \boldsymbol{n} - \int_{\Omega_{2R}} \varphi\nabla \cdot (\boldsymbol{v} + \Phi\nabla\mathcal{E}) \\ &= \int_{\partial\Omega} (\boldsymbol{v} + \Phi\nabla\mathcal{E}) \cdot \boldsymbol{n} = 0\,. \end{aligned}$$

Thus, Theorem III.3.3 guarantees the existence of the field $\boldsymbol{w}$. Furthermore, the pair $(\boldsymbol{u}, q)$ obeys the following problem:

$$\left.\begin{aligned} &\Delta\boldsymbol{u} + \mathcal{R}\frac{\partial\boldsymbol{u}}{\partial x_1} + \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla\boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{v}) = \nabla q + \nabla \cdot \boldsymbol{\mathcal{G}} + \boldsymbol{g} \\ &\nabla \cdot \boldsymbol{u} = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3\,,$$
$$\lim_{|x|\to\infty} \boldsymbol{u}(x) = 0\,, \tag{VIII.6.6}$$

where

$$\begin{aligned} \boldsymbol{g} = {} & (\Delta\varphi)(\boldsymbol{v} + \Phi\nabla\mathcal{E}) + 2\nabla\varphi \cdot \nabla(\boldsymbol{v} + \Phi\nabla\mathcal{E}) + \mathcal{R}\frac{\partial\varphi}{\partial x_1}\boldsymbol{v} \\ & + \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla\varphi)(\boldsymbol{v} + \Phi\nabla\mathcal{E}) \\ & - p\nabla\varphi - \mathcal{R}(\nabla\varphi) \cdot \boldsymbol{\mathcal{F}} + \Delta\boldsymbol{w} + \mathcal{R}\frac{\partial\boldsymbol{w}}{\partial x_1} + \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla\boldsymbol{w} - \boldsymbol{e}_1 \times \boldsymbol{w}), \\ \boldsymbol{\mathcal{G}} = {} & \mathcal{R}\varphi\boldsymbol{\mathcal{F}} - \mathcal{R}\,\Phi\,\vartheta\,(\boldsymbol{e}_1 \otimes \nabla\mathcal{E})\,, \end{aligned}$$

[2] Recall that $\boldsymbol{n}$ is the unit *outer* normal to $\partial\Omega$.

and we have used the fact that $\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla(\nabla\mathcal{E}) - \boldsymbol{e}_1 \times (\nabla\mathcal{E}) = 0$. Taking into account that $\operatorname{supp}(\theta) \subset B_{3R}$, we find that the function $\boldsymbol{\mathcal{G}}$ satisfies

$$
\begin{aligned}
\|\nabla \cdot \boldsymbol{\mathcal{G}}\|_2 &\le c_1(R)\, \mathcal{R}\, \left(\|\nabla \cdot \boldsymbol{\mathcal{F}}\|_2 + [\![\boldsymbol{\mathcal{F}}]\!]_{2,\mathcal{R}} + |\Phi|\right), \\
[\![\boldsymbol{\mathcal{G}}]\!]_{2,\mathcal{R}} &\le c_1(R)\mathcal{R}\left([\![\boldsymbol{\mathcal{F}}]\!]_{2,\mathcal{R}} + |\Phi|\right).
\end{aligned}
\tag{VIII.6.7}
$$

We also observe that using several times the embedding Theorem II.3.3 and the properties of $\boldsymbol{w}$ given in (VIII.6.5), we deduce

$$
\|\boldsymbol{g}\|_6 \le c_2\left([\![\boldsymbol{\mathcal{F}}]\!]_{2,\mathcal{R}} + \|\boldsymbol{v}\|_{2,2,\Omega_{2R}} + \|q\|_{1,2,\Omega_{2R}} + \|\boldsymbol{v}_*\|_{3/2,2,\partial\Omega}\right), \tag{VIII.6.8}
$$

with $c_2 = c_2(R, B)$, whenever $\mathcal{R}, \mathcal{T} \in [0, B]$. Thus, in view of (VIII.6.7), (VIII.6.8), we check that the hypotheses of Theorem VIII.5.1 are satisfied, and consequently, there exists at least one solution $(\overline{\boldsymbol{u}}, \overline{q})$ to (VIII.6.6) satisfying all the properties listed in that theorem. Again by Theorem VIII.5.1, this solution is unique in the class of generalized solutions, and so we must have $\overline{\boldsymbol{u}} = \boldsymbol{u}$ and $\overline{q} = q + q_0$, where $q_0 \in \mathbb{R}$. However, $q \in L^6(\mathbb{R}^3)$ and $\overline{q} \in L^r(\mathbb{R}^3)$, for all $r > 3/2$, which implies $q_0 = 0$. Thus, recalling that $\boldsymbol{u} = \boldsymbol{v}$ and $q = p$ in Ω^{2R}, from Lemma VIII.6.1, (VIII.6.7), (VIII.6.8), (VIII.5.3), and the inequality

$$
\|p\|_{q,\Omega_{2R}} \le c_1(R, q)\|p\|_{6,\Omega} \le c_2(R, q)|p|_{1,2}, \quad q \in [1, 6], \tag{VIII.6.9}
$$

we find, on the one hand, that $\boldsymbol{v}, p$ satisfy (VIII.6.3), and on the other hand, that they satisfy the estimate

$$
\begin{aligned}
&\|D^2\boldsymbol{v}\|_{2,\Omega^{2R}} + \|\nabla\boldsymbol{v}\|_{2,\Omega^{2R}} + [\![\boldsymbol{v}]\!]_{1,\mathcal{R},\Omega^{2R}} + \|p\|_{r,\Omega^{2R}} + \|\nabla p\|_{2,\Omega^{2R}} \\
&\quad \le C([\![\boldsymbol{\mathcal{F}}]\!]_{2,\mathcal{R}} + \|\nabla \cdot \boldsymbol{\mathcal{F}}\|_2 + \|\boldsymbol{v}_*\|_{3/2,2,\partial\Omega} + \|\boldsymbol{v}\|_{2,2,\Omega_{2R}} + \|p\|_{1,2,\Omega_{2R}}).
\end{aligned}
$$

This latter, in turn, combined with Lemma VIII.6.1 and with (VIII.6.9), yields

$$
\begin{aligned}
&|\boldsymbol{v}|_{2,2} + |\boldsymbol{v}|_{1,2} + [\![\boldsymbol{v}]\!]_{1,\mathcal{R}} + |p|_{1,2} + \|p\|_{q_1} + \|p\|_{q_2,\Omega^{2R}} \\
&\qquad \le C\left([\![\boldsymbol{\mathcal{F}}]\!]_{2,\mathcal{R}} + \|\nabla \cdot \boldsymbol{\mathcal{F}}\|_2 + \|\boldsymbol{v}_*\|_{3/2,2,\partial\Omega} + \|\boldsymbol{v}\|_{2,\Omega_{3R}} + \|p\|_{2,\Omega_{3R}}\right),
\end{aligned}
\tag{VIII.6.10}
$$

where $C = C(\Omega, R, q_1, q_2, B)$. By means of a standard argument that we have already used several times, we will now show that

$$
\|\boldsymbol{v}\|_{2,\Omega_{3R}} + \|p\|_{2,\Omega_{3R}} \le C(\|\nabla \cdot \boldsymbol{\mathcal{F}}\|_2 + [\![\boldsymbol{\mathcal{F}}]\!]_{2,\mathcal{R}} + \|\boldsymbol{v}_*\|_{3/2,2,\partial\Omega}), \tag{VIII.6.11}
$$

for some constant C satisfying the property stated in the theorem. Then, combining (VIII.6.11) with (VIII.6.10), we will obtain (VIII.6.4), and this will conclude the proof of the theorem. Assume that (VIII.6.11) does not hold. Then, in view of the linearity of problem (VIII.0.2), (VIII.0.7), we can find a sequence $\{\boldsymbol{\mathcal{F}}_n, \boldsymbol{v}_{*n}, \mathcal{R}_n, \mathcal{T}_n\}$, with $\mathcal{R}_n, \mathcal{T}_n \in [0, B]$, and a sequence of corresponding solutions $\{\boldsymbol{v}_n, p_n\}$ such that

$$
\begin{aligned}
&[\![\boldsymbol{\mathcal{F}}_n]\!]_{2,\mathcal{R}_n} + \|\nabla \cdot \boldsymbol{\mathcal{F}}_n\|_2 + \|\boldsymbol{v}_{*n}\|_{2-1/q,q,\partial\Omega} \le \frac{1}{n}, \\
&\|\boldsymbol{v}_n\|_{q,\Omega_{3R}} + \|p_n\|_{q,\Omega_{3R}} = 1.
\end{aligned}
\tag{VIII.6.12}
$$

From (VIII.6.4), it follows that the sequence of solutions is bounded in the norm defined by the left-hand side of (VIII.6.4) and that therefore, it converges, in a suitable topology, to a pair $\{\boldsymbol{v}_0, p_0\}$ that belongs to the class defined by (VIII.6.3). Since, in particular,

$$\|\boldsymbol{v}_n\|_{1,2,\Omega_{3R}} + \|p_n\|_{1,2,\Omega_{3R}} \leq M$$

with M independent of n, by Rellich's compactness theorem, Theorem II.5.2, and by the second equation in (VIII.6.12) we infer

$$\|\boldsymbol{v}_0\|_{q,\Omega_{3R}} + \|p_0\|_{q,\Omega_{3R}} = 1. \tag{VIII.6.13}$$

Moreover, using (VIII.6.12)$_1$, it is easy to show that $\boldsymbol{v}_0, p_0$ is a solution of the following boundary-value problem:

$$\left.\begin{aligned} &\Delta \boldsymbol{v}_0 + \mathcal{R}_0 \frac{\partial \boldsymbol{v}_0}{\partial x_1} + \mathcal{T}_0 \left(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v}_0 - \boldsymbol{e}_1 \times \boldsymbol{v}_0\right) = \nabla p_0 \\ &\nabla \cdot \boldsymbol{v}_0 = 0 \end{aligned}\right\} \quad \text{in } \Omega \tag{VIII.6.14}$$
$$\boldsymbol{v}_0 = 0 \ \text{ at } \partial\Omega,$$

where $\mathcal{R}_0 = \lim_{n\to\infty} \mathcal{R}_n$, $\mathcal{T}_0 = \lim_{n\to\infty} \mathcal{T}_n$. However, $\boldsymbol{v}_0, p_0$ satisfy (VIII.6.3), so that in particular, $\boldsymbol{v}_0$ is a weak solution to (VIII.6.14). Thus, by the uniqueness Theorem VIII.2.1 we obtain $\boldsymbol{v}_0 = p_0 = 0$,[3] contradicting (VIII.6.13). This proves (VIII.6.11), and concludes the proof of the theorem.

□

Remark VIII.6.1 The methods used in the proof of the previous theorem provide pointwise asymptotic estimates for the velocity field. However, these methods can be further exploited to give similar results also for the derivatives of the velocity and of the associated pressure fields. In particular, one can prove the following two theorems, for whose proof we refer to Galdi (2003, Theorem 4.1) and Galdi & Silvestre (2007b, Theorem 3), respectively.

Theorem VIII.6.2 *Let Ω, $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{v}_*$ be as in* Theorem VIII.6.1, *and let $\boldsymbol{v}$ be the corresponding generalized solution to* (VIII.0.2), (VIII.0.7) *with* $\mathcal{R} = 0$. *Suppose also that*

$$[\![D_i \mathcal{F}_{ij} \boldsymbol{e}_j]\!]_3 + [\![D_j D_i \mathcal{F}_{ij}]\!]_4 < \infty$$

and $\boldsymbol{v}_* \in W^{2-1/q,q}(\partial\Omega)$ for all $q > 1$. Then, in addition to the properties stated in Theorem VIII.6.1, *$\boldsymbol{v}$ and the associated pressure p satisfy*

$$\begin{aligned} &\boldsymbol{v} \in W^{2,q}_{loc}(\overline{\Omega}), \ \text{all } q \geq 1, \quad [\![\nabla \boldsymbol{v}]\!]_2 < \infty, \\ &p \in W^{1,q}_{loc}(\overline{\Omega}), \ \text{all } q \geq 1, \quad [\![p]\!]_2 + [\![\nabla p]\!]_{3,\Omega^R} < \infty, \quad \text{all } R > \delta(\mathcal{B}). \end{aligned}$$

Moreover, the following estimate holds:

[3] Recall that, for instance, $p_0 \in L^6(\Omega)$.

$$\begin{aligned}&\|\boldsymbol{v}\|_{2,q,\Omega_R}+ [\![\nabla \boldsymbol{v}]\!]_2 + [\![p]\!]_2 + [\![\nabla p]\!]_{3,\Omega^R}\\ &\qquad \le c\left([\![\boldsymbol{\mathcal{F}}]\!]_2 + [\![D_i\mathcal{F}_{ij}\boldsymbol{e}_j]\!]_3 + [\![D_jD_i\mathcal{F}_{ij}]\!]_4 + \|\boldsymbol{v}_*\|_{2-1/q,q,\partial\Omega}\right),\end{aligned}$$

where the constant c depends only on Ω, q, R, and B, whenever $\mathcal{T} \in [0, B]$.

Theorem VIII.6.3 *Let Ω and $\boldsymbol{v}_*$ be as in* Theorem VIII.6.1, $\boldsymbol{f} \in L^\infty(\Omega)$ *with $[\![\boldsymbol{f}]\!]_{\frac{5}{2},\mathcal{R}} < \infty$, and let $\boldsymbol{v}$ be the generalized solution to* (VIII.0.2), (VIII.0.7) *corresponding to $\boldsymbol{f}$ and $\boldsymbol{v}_*$.*[4] *Then, $\boldsymbol{v}$ and the associated pressure field p satisfy*

$$\begin{aligned}&\boldsymbol{v} \in W^{2,2}_{loc}(\overline{\mathcal{D}}) \cap D^{1,2}(\mathcal{D}) \cap D^{2,2}(\mathcal{D}), \quad [\![\boldsymbol{v}]\!]_{1,\mathcal{R}} + [\![\nabla \boldsymbol{v}]\!]_{\frac{3}{2},\mathcal{R}} < \infty\\ &p \in W^{1,2}(\mathcal{D}),\end{aligned}$$

along with the estimate

$$|\boldsymbol{v}|_{2,2} + |\boldsymbol{v}|_{1,2} + [\![\boldsymbol{v}]\!]_{1,1} + [\![\nabla \boldsymbol{v}]\!]_{\frac{3}{2},\mathcal{R}} + \|p\|_{1,2} \le C\left(\|\boldsymbol{v}_*\|_{\frac{3}{2},2} + \mathcal{R}^{-\frac{1}{2}}[\![\boldsymbol{f}]\!]_{\frac{5}{2},\mathcal{R}}\right),$$

with $C = C(\Omega, B)$, whenever $\mathcal{R}, \mathcal{T} \in (0, B)$.

Notice that in both theorems, the gradient of the velocity field decays exactly as the gradient of the Stokes fundamental tensor (Theorem VIII.6.2) and that of the Oseen fundamental tensor (Theorem VIII.6.3). ■

Remark VIII.6.2 An interesting problem that can be naturally posed, is to determine the asymptotic *structure* of a generalized solution. To date, the investigation of this issue is still in progress. However, at least in the case $\mathcal{R} = 0$, Farwig & Hishida (2009) furnish a detailed picture, when the tensor field $\boldsymbol{\mathcal{F}}$ in Theorem VIII.6.1 has components in $C_0^\infty(\Omega)$, and the boundary condition reduces to a rigid rotation, that is, $\boldsymbol{v}_* = \boldsymbol{e}_1 \times \boldsymbol{x}$. In particular, these authors prove an asymptotic expansion, for large $|x|$, of the velocity field, and they show that the leading term of this expansion is given by

$$\boldsymbol{e}_1 \cdot \left(\int_{\partial\Omega} [\boldsymbol{T}(\boldsymbol{v}, p) + \boldsymbol{\mathcal{F}}] \cdot \boldsymbol{n}\right) \boldsymbol{u}_1(x),$$

where $\boldsymbol{u}_1(x) = (U_{11}(x), U_{21}(x), U_{31}(x))$, and $(\boldsymbol{U}(x), \boldsymbol{q}(x))$ is the Stokes fundamental solution. A crucial tool in the proof of this result is provided by the fact that the field $\boldsymbol{u}_1(x), q_1(x)$ is a solution to the generalized Oseen system with $\mathcal{R} = 0$:

$$\begin{aligned}&\Delta \boldsymbol{u}_1(x) + \mathcal{T}\left(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{u}_1(x) - \boldsymbol{e}_1 \times \boldsymbol{u}_1(x)\right) = \nabla q_1(x),\\ &\nabla \cdot \boldsymbol{u}_1(x) = 0,\\ &\qquad\qquad \text{for all } x \in \mathbb{R}^3 - \{0\},\end{aligned}$$

[4] Observe that the assumption on $\boldsymbol{f}$ implies that $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$, as the reader will easily prove. Furthermore, the statement of this result given in Galdi & Silvestre (2007b) requires the condition $\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} = 0$, which in fact, is not needed.

since, as the reader may wish to show,

$$\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{u}_1(x) - \boldsymbol{e}_1 \times \boldsymbol{u}_1(x) = \boldsymbol{0}\,, \quad \text{for all } x \in \mathbb{R}^3 - \{0\}.$$

■

VIII.7 Existence, Uniqueness, and L^q-Estimates. The case $\mathcal{R} = 0$

In this and the next section we will investigate existence, uniqueness, and associated estimates of solutions corresponding to right-hand side $\boldsymbol{f}$ from Lebesgue space L^q, and boundary data $\boldsymbol{v}_*$ in the trace space $W^{2-2/q,q}(\partial\Omega)$, for suitable values of q. Since the results for the cases $\mathcal{R} = 0$ and $\mathcal{R} \neq 0$ are quite different, we prefer to analyze them separately, beginning with the case $\mathcal{R} = 0$ in the current section, while deferring the other case to the following one.

The starting point of our analysis (regardless of whether $\mathcal{R} = 0$ or $\neq 0$) is an appropriate uniqueness result. Precisely, we have the following.

Lemma VIII.7.1 *Suppose that for some $r \in (1, \infty)$ and all $R > 0$, $(\boldsymbol{u}, q) \in W^{2,r}(B_R) \times W^{1,r}(B_R)$ is a solution to* (VIII.5.1)$_{1,2}$ *corresponding to $\boldsymbol{f} = \boldsymbol{0}$. The following properties hold:*

(a) *If*

$$\boldsymbol{u} = \sum_{i=1}^{N} \boldsymbol{u}_i\,, \quad \boldsymbol{u}_i \in L^{q_i}(\mathbb{R}^3)\,, \quad \textit{for some } q_i \in (1, \infty)\,, \quad i = 1, \dots, N\,, \tag{VIII.7.1}$$

then $\boldsymbol{u}(x) = \boldsymbol{0}$, $\nabla q(x) = 0$ for a.a. $x \in \mathbb{R}^3$.

(b) *If*

$$D^2\boldsymbol{u} = \sum_{i=1}^{M} \widehat{\boldsymbol{u}}_i\,, \quad \widehat{\boldsymbol{u}}_i \in L^{\widehat{q}_i}(\mathbb{R}^3)\,, \quad \textit{for some } \widehat{q}_i \in (1, \infty)\,, \quad i = 1, \dots, M\,, \tag{VIII.7.2}$$

then $D^2\boldsymbol{u}(x) = 0$ for a.a. $x \in \mathbb{R}^3$.

Proof. We begin by proving the property in (a). For simplicity, we shall consider the case $N = 2$, leaving to the reader the simple task of establishing the result in the general case. If we perform the change of variables (VIII.5.9)–(VIII.5.11), the problem (VIII.5.1) (with $\boldsymbol{f} = \boldsymbol{0}$) produces the following Cauchy problem:

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{w}}{\partial t} = \Delta \boldsymbol{w} + \mathcal{R}\frac{\partial \boldsymbol{w}}{\partial \chi_1} - \nabla \pi \\ &\nabla \cdot \boldsymbol{w} = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3 \times (0, \infty)\,, \tag{VIII.7.3}$$

$$\lim_{t \to 0+} \|\boldsymbol{w}(t) - \boldsymbol{u}\|_{\underline{q}, B_R} = 0\,,$$

where $\underline{q}=\min\{q_1,q_2\}$, and R is an arbitrary positive number. Now let $\boldsymbol{W}_i=\boldsymbol{W}_i(\chi,\overline{t})$, $i=1,2$, be the solution that we constructed in Theorem VIII.4.3 to the following Cauchy problem:

$$\left.\begin{aligned}&\frac{\partial \boldsymbol{W}_i}{\partial t}=\Delta\boldsymbol{W}_i+\mathcal{R}\frac{\partial \boldsymbol{W}_i}{\partial \chi_1}\\&\nabla\cdot\boldsymbol{W}_i=0\end{aligned}\right\}\ \text{in } \mathbb{R}^3\times(0,\infty)\,, \tag{VIII.7.4}$$

$$\lim_{t\to 0}\|\boldsymbol{W}_i(t)-\boldsymbol{u}_i\|_{q_i}=0\,.$$

As we know from that theorem, $\boldsymbol{W}_i$ satisfies, in particular, the following properties for $i=1,2$:

$$\begin{aligned}&\boldsymbol{W}_i\in L^{q_i}(\mathbb{R}^3_T)\,,\quad \text{all } T>0\,,\\&\frac{\partial \boldsymbol{W}_i}{\partial t}\,,\quad D^2\boldsymbol{W}_i\in L^{q_i}_{loc}((0,T]\times\mathbb{R}^3)\\&\|\boldsymbol{W}_i(t)\|_r\le c_i\,t^{-3(1/\overline{q}-1/r)/2}\|\boldsymbol{u}_i\|_{q_i}\,,\quad i=1,2\,,\quad \text{all } t>0\,,\end{aligned} \tag{VIII.7.5}$$

where $r>\overline{q}\equiv\max\{q_1,q_2\}$. Therefore, from (VIII.7.3) and (VIII.7.4) we infer that the vector field $\boldsymbol{W}\equiv\boldsymbol{w}-\boldsymbol{W}_1-\boldsymbol{W}_2$ satisfies

$$\left.\begin{aligned}&\frac{\partial \boldsymbol{W}}{\partial t}=\Delta\boldsymbol{W}+\mathcal{R}\frac{\partial \boldsymbol{W}}{\partial \chi_1}+\nabla\pi\\&\nabla\cdot\boldsymbol{W}=0\end{aligned}\right\}\ \text{in } \mathbb{R}^3\times(0,\infty)\,, \tag{VIII.7.6}$$

$$\lim_{t\to 0}\|\boldsymbol{W}(t)\|_{\underline{q},B_R}=0\,,$$

where R is an arbitrary positive number. Taking also into account that (see (VIII.5.15))

$$\frac{\partial \boldsymbol{w}}{\partial t}=-\mathcal{T}\,\boldsymbol{Q}(t)\cdot(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla u-\boldsymbol{e}_1\times\boldsymbol{u})$$

by the assumptions made on $\boldsymbol{u}$ and q, we easily deduce

$$\frac{\partial \boldsymbol{w}}{\partial t}\,,\quad D^2\boldsymbol{w}\,,\quad \nabla\pi\in L^r_{loc}(\mathbb{R}^3_T)\,,\quad \text{for all } T>0.$$

Moreover, obviously,

$$\boldsymbol{w}=\boldsymbol{w}_1+\boldsymbol{w}_2\,,\quad \boldsymbol{w}_i\in L^{q_i}(\mathbb{R}^3_T)\,,\quad i=1,2\,,\quad \text{for all } T>0.$$

Consequently, from these latter properties and from $(\text{VIII.7.5})_{1,2}$ we find that the fields $\boldsymbol{W},\pi$ satisfy the assumptions of Lemma VIII.4.2, so that we conclude

$$\boldsymbol{w}=\boldsymbol{W}_1+\boldsymbol{W}_2\,,\ \nabla\pi=0\ \text{ a.e. in } \mathbb{R}^3_T. \tag{VIII.7.7}$$

Recalling that $\pi(\chi,t)=q(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{\chi},t)$, $\boldsymbol{x}=\boldsymbol{Q}^\top\cdot\boldsymbol{\chi}$, the second relation in (VIII.8.11) delivers $\nabla q=0$ in $\mathbb{R}^3$. Furthermore, taking into account that $\boldsymbol{w}(\chi,t)=\boldsymbol{Q}(t)\cdot\boldsymbol{u}(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{\chi})$, from (VIII.7.7) and $(\text{VIII.7.5})_3$ we also have

$$
\begin{aligned}
\|\boldsymbol{u}\|_r = \|\boldsymbol{w}(t)\|_r &\le \|\boldsymbol{W}_1(t)\|_r + \|\boldsymbol{W}_2(t)\|_r \\
&\le c\,t^{-3(1/\overline{q}-1/r)/2}\left(\|\boldsymbol{u}_1\|_{q_1} + \|\boldsymbol{u}_2\|_{q_2}\right).
\end{aligned}
$$

Letting $t \to \infty$ in this relation gives $\|\boldsymbol{u}\|_r = 0$, which concludes the proof of part (a). We shall next prove part (b), again for the simplest case $M = 2$. We recall that by Theorem VIII.1.1, $\boldsymbol{u}, q \in C^\infty(\mathbb{R}^3)$, which implies that $(\boldsymbol{w}, \pi)$ is of class C^∞ in space and time. Thus, setting $\mathfrak{w} := \Delta \boldsymbol{w}, \mathfrak{q} := \Delta q$, from (VIII.7.3) and (VIII.5.16) we deduce

$$
\left.
\begin{aligned}
&\frac{\partial \mathfrak{w}}{\partial t} = \Delta \mathfrak{w} + \mathcal{R}\frac{\partial \mathfrak{w}}{\partial \chi_1} - \nabla \mathfrak{q} \\
&\nabla \cdot \mathfrak{w} = 0
\end{aligned}
\right\}\ \text{in } \mathbb{R}^3 \times (0,\infty)\,, \tag{VIII.7.8}
$$
$$
\lim_{t\to 0+} \|\mathfrak{w} - \Delta \boldsymbol{u}\|_{\underline{\widehat{q}},B_R} = 0\,,
$$

where $\underline{\widehat{q}} = \min\{\widehat{q}_1, \widehat{q}_2\}$, and $R > 0$ is arbitrary. We next repeat for the problem (VIII.7.8) exactly the same argument used for the proof of the property in (a), and show that $\Delta \boldsymbol{u} = 0$ a.e. in $\mathbb{R}^3_T$. In view of the assumption on $D^2\boldsymbol{u}$ and Exercise II.11.11, we thus conclude that $D^2\boldsymbol{u} = 0$ a.e. in $\mathbb{R}^3_T$, and the proof of the lemma is complete.

□

An important consequence of the previous result is the following general uniqueness result.

Theorem VIII.7.1 *Let Ω be locally Lipschitz, and let $(\boldsymbol{v}_i, p_i)$, $i = 1, 2$, be two solutions to* (VIII.0.7), (VIII.0.2) *corresponding to the same data $\boldsymbol{f}$ and $\boldsymbol{v}_*$ and such that for all $R > \delta(\Omega^c)$,*

$$
\begin{aligned}
&(\boldsymbol{v}_i, p_i) \in W^{2,q_i}(\Omega_R) \times W^{1,q_i}(\Omega_R)\,, \quad \boldsymbol{v}_i \in L^{r_i}(\Omega)\,, \\
&\qquad\qquad \textit{for some } q_i, r_i \in (1,\infty),\ i = 1, 2\,.
\end{aligned} \tag{VIII.7.9}
$$

Then $\boldsymbol{v}_1 = \boldsymbol{v}_2$, $\nabla(p_1 - p_2) = 0$, a.e. in Ω.

Proof. Set $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$, $p = p_1 - p_2$. From (VIII.0.7) we obtain

$$
\left.
\begin{aligned}
\Delta \boldsymbol{v} + \mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1} + \mathcal{T}\left(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}\right) &= \nabla p \\
\nabla \cdot \boldsymbol{v} &= 0
\end{aligned}
\right\}\ \text{in } \Omega \tag{VIII.7.10}
$$
$$
\boldsymbol{v} = \boldsymbol{0} \ \text{ at } \partial\Omega\,.
$$

In view of the assumptions and Theorem VIII.1.1, it follows, in particular, that

$$
(\boldsymbol{v}, p) \in [C^\infty(\Omega) \times C^\infty(\Omega)] \cap [W^{2,2}(\Omega_\rho) \times W^{1,2}(\Omega_\rho)]\,, \ \text{ for all } \rho > \delta(\Omega^c). \tag{VIII.7.11}
$$

For a fixed $\sigma > \delta(\Omega^c)$, let $\psi = \psi(x)$ be a smooth "cut-off" function that is 0 for $|x| \le \sigma$ and 1 for $|x| \ge 2\sigma$, and set $\boldsymbol{u} = \psi\,\boldsymbol{v} + \boldsymbol{w}$, $\Phi = \psi\,p$, where $\boldsymbol{w}$ satisfies the following properties:

$$\begin{aligned}&\nabla\cdot\boldsymbol{w} = -\boldsymbol{v}\cdot\nabla\psi \ \text{ in } \Omega_{\sigma,2\sigma}\,,\ \boldsymbol{w}\in C_0^\infty(\Omega_{\sigma,2\sigma})\,,\\ &\|\boldsymbol{w}\|_{2,q,\mathbb{R}^3} \le c\,\|\boldsymbol{v}\|_{1,q,\Omega_{\sigma,2\sigma}}\,.\end{aligned} \tag{VIII.7.12}$$

Since

$$\int_{\Omega_{\sigma,2\sigma}} \boldsymbol{v}\cdot\nabla\psi = \int_{\Omega_{2\sigma}} \nabla\cdot(\psi\boldsymbol{v}) = \int_{\partial B_{2\sigma}} \psi\boldsymbol{v}\cdot\boldsymbol{n} = -\int_{\partial\Omega} \boldsymbol{v}\cdot\boldsymbol{n} = 0\,, \tag{VIII.7.13}$$

from Theorem III.3.3 we know that problem (VIII.7.12) has at least one solution. By a direct computation that starts with (VIII.7.10), we thus find that the pair $(\boldsymbol{u},\Phi)$ is a solution to the following problem:

$$\left.\begin{aligned}\Delta\boldsymbol{u} + \mathcal{R}\frac{\partial\boldsymbol{u}}{\partial x_1} + \mathcal{T}\left(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{u} - \boldsymbol{e}_1\times\boldsymbol{u}\right) = \nabla\Phi + \boldsymbol{F}\\ \nabla\cdot\boldsymbol{u} = 0\end{aligned}\right\}\ \text{in } \mathbb{R}^3\,, \tag{VIII.7.14}$$

with

$$\begin{aligned}\boldsymbol{F} = \Delta\boldsymbol{w} + \mathcal{R}\frac{\partial\boldsymbol{w}}{\partial x_1} + \mathcal{T}\big(\boldsymbol{e}_1\times\boldsymbol{w} - \boldsymbol{e}_1\times\boldsymbol{w}\big)\\ +\mathcal{R}\boldsymbol{v}\,\frac{\partial\psi}{\partial x_1} - p\,\nabla\psi + 2\nabla\psi\cdot\nabla\boldsymbol{v} + \boldsymbol{v}\,\Delta\psi\,.\end{aligned} \tag{VIII.7.15}$$

In view of (VIII.7.11) and (VIII.7.12), we deduce $\boldsymbol{F}\in C_0^\infty(\Omega)$, and so, in particular, $\boldsymbol{F}\in D_0^{1,2}(\mathbb{R}^3)$. By Theorem VIII.1.1 and Theorem VIII.1.2, we then infer that problem (VIII.7.14) has at least one solution,

$$(\overline{\boldsymbol{u}},\overline{\Phi}) \in [D_0^{1,2}(\mathbb{R}^3)\cap L^6(\mathbb{R}^3)\cap C^\infty(\mathbb{R}^3)]\times C^\infty(\mathbb{R}^3)\,. \tag{VIII.7.16}$$

Since

$$\overline{\boldsymbol{u}} - \boldsymbol{u} = \overline{\boldsymbol{u}} + (\boldsymbol{w} - \psi\boldsymbol{v}_1) + \psi\boldsymbol{v}_2 \equiv \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3$$

is a solution to (VIII.7.14) with $\boldsymbol{F}\equiv\boldsymbol{0}$, and since from (VIII.7.16), (VIII.7.12) and the assumption (VIII.7.9), we have $\boldsymbol{u}_i\in L^{s_i}(\mathbb{R}^3)$, for suitable $s_i\in(1,\infty)$, $i = 1,2,3$, by Lemma VIII.7.1 we infer $\overline{\boldsymbol{u}} = \boldsymbol{u}$. This latter, in turn, by the properties of ψ and by (VIII.7.11) and (VIII.7.16), implies, in particular,

$$\boldsymbol{v}\in D^{1,2}(\Omega_{2\sigma})\,.$$

This information along with (VIII.7.11) furnishes that $\boldsymbol{v}$ is a generalized solution to (VIII.7.10), and so by Lemma VIII.2.3, we conclude that $\boldsymbol{v} = \boldsymbol{0}$ in Ω, which ends the proof of the lemma. □

We shall now begin our investigation of the summability properties of solutions to (VIII.0.7), (VIII.0.2), by considering first the case $\mathcal{R} = 0$ and $\Omega = \mathbb{R}^3$. The following lemma holds.

Lemma VIII.7.2 *Let $\boldsymbol{f} \in L^q(\mathbb{R}^3)$, for some $q \in (1,\infty)$. Then, the problem*

$$\left.\begin{aligned}&\Delta \boldsymbol{v} + \mathcal{T}\left(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}\right) = \nabla p + \boldsymbol{f}\\ &\nabla \cdot \boldsymbol{v} = 0\end{aligned}\right\} \quad \text{a.e. in } \mathbb{R}^3 \tag{VIII.7.17}$$

has at least one solution $\boldsymbol{v}, p$ such that

$$\begin{aligned}&(\boldsymbol{v}, p) \in D^{2,q}(\mathbb{R}^3) \cap D^{1,q}(\mathbb{R}^3)\,,\\ &(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}) \in L^q(\mathbb{R}^3)\,,\end{aligned} \tag{VIII.7.18}$$

satisfying

$$|\boldsymbol{v}|_{2,q} + |p|_{1,q} + \mathcal{T}\|\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}\|_q \le c_1\,\|\boldsymbol{f}\|_q\,, \tag{VIII.7.19}$$

where $c_1 = c_1(q) > 0$. If $1 < q < 3$, there is a solution that in addition to (VIII.7.19) *also satisfies*

$$(\boldsymbol{v}, p) \in D^{1,3q/(3-q)}(\mathbb{R}^3) \cap L^{3q/(3-q)}(\mathbb{R}^3) \tag{VIII.7.20}$$

along with the inequality

$$|\boldsymbol{v}|_{1,3q/(3-q)} + \|p\|_{3q/(3-q)} \le c_2\,\|\boldsymbol{f}\|_q \tag{VIII.7.21}$$

and $c_2 = c_2(q) > 0$. Furthermore, if $1 < q < 3/2$, we can find a solution that in addition to (VIII.7.19) *and* (VIII.7.19), *satisfies further,*

$$\boldsymbol{v} \in L^{3q/(3-2q)}(\mathbb{R}^3) \tag{VIII.7.22}$$

and

$$\|\boldsymbol{v}\|_{3q/(3-2q)} \le c_3\,\|\boldsymbol{f}\|_q\,, \tag{VIII.7.23}$$

with $c_3 = c_3(q) > 0$.

Finally, if $\boldsymbol{f} \in L^q(\mathbb{R}^3)$ for all $q \in (1,\infty)$, there exists a solution $(\boldsymbol{v}, p)$ to (VIII.7.17) *satisfying* (VIII.7.18)–(VIII.7.23) *for all specified values of q.*

Proof. It will be enough to prove the existence of a solution satisfying the stated estimates for the appropriate derivatives of $\boldsymbol{v}$ and p only, in that the inequality for the "rotational" term $(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v})$ becomes a consequence of (VIII.7.17)$_1$ and these estimates. Let us take first $\boldsymbol{f} \in C_0^\infty(\mathbb{R}^3)$. Then, from Theorem VIII.5.1, we know that problem (VIII.7.17) has one and only one smooth solution $(\boldsymbol{v}, p)$ such that

$$\boldsymbol{v} \in L^{s_1}(\mathbb{R}^3) \cap D^{1,s_2}(\mathbb{R}^3) \cap D^{2,s_3}(\mathbb{R}^3)\,, \quad \text{all } s_1 > 3,\ s_2 > 3/2, \text{ and } s_3 > 1. \tag{VIII.7.24}$$

Making the change of variables (VIII.5.9)–(VIII.5.11) (with $\boldsymbol{u} \equiv \boldsymbol{v}$), we showed in Section VIII.5 that problem (VIII.7.17) goes into the following Cauchy problem:

$$
\left.\begin{aligned}
&\frac{\partial \boldsymbol{w}}{\partial t} = \Delta \boldsymbol{w} - \nabla \pi + \boldsymbol{F} \\
&\nabla \cdot \boldsymbol{w} = 0
\end{aligned}\right\} \text{ in } \mathbb{R}^3 \times (0, \infty)\,, \tag{VIII.7.25}
$$

$$
\lim_{t \to 0+} \|\boldsymbol{w}(t) - \boldsymbol{v}\|_{s_1} = 0\,,
$$

where $\boldsymbol{F}(\chi, t) = -\boldsymbol{Q}(t) \cdot \boldsymbol{g}(\boldsymbol{Q}^\top(t) \cdot \boldsymbol{\chi})$. Then, from Theorem VIII.4.1, Theorem VIII.4.3, Lemma VIII.4.2, and (VIII.7.24), we at once obtain that we may write $\boldsymbol{w} = \boldsymbol{w}_1 + \boldsymbol{w}_2$, where $(\boldsymbol{w}_1, \pi)$ satisfy (VIII.7.25) with $\boldsymbol{v} \equiv \boldsymbol{0}$, while

$$
\boldsymbol{w}_2(x, t) = \left(\frac{1}{4\pi\, t}\right)^{3/2} \int_{\mathbb{R}^3} e^{-|x-y|^2/4t} \boldsymbol{v}(y)dy\,. \tag{VIII.7.26}
$$

Again by Theorem VIII.4.1, we obtain

$$
\int_0^T \left(|\boldsymbol{w}_1(t)|_{2,q}^q + |\pi(t)|_{1,q}^q\right)\, dt \le C_1 \int_0^T \|\boldsymbol{F}(t)\|_q^q dt\,, \tag{VIII.7.27}
$$

with a constant C_1 independent of $T > 0$. Furthermore, by differentiating twice both sides of (VIII.7.26) with respect to the x variable, and then applying Young's inequality (II.11.2), we deduce

$$
|\boldsymbol{w}_2(t)|_{2,q} \le C_2 t^{-\frac{3}{2}\left(\frac{1}{s_3} - \frac{1}{q}\right)} |\boldsymbol{v}|_{2,s_3}\,,
$$

with C_2 independent of $t > 0$, and $s_3 \in (1, q)$. From this latter it follows that

$$
\int_0^T |\boldsymbol{w}_2(t)|_{2,q}^q dt \le C_3 T^{-\frac{3}{2}\left(\frac{q}{s_3} - 1\right) + 1} |\boldsymbol{v}|_{2,s_3}^q\,, \tag{VIII.7.28}
$$

with C_3 independent of T. Consequently, recalling that

$$
|\boldsymbol{v}|_{2,q} = |\boldsymbol{w}(t)|_{2,q}\,, \quad |p|_{1,q} = |\pi(t)|_{1,q}\,, \quad \|\boldsymbol{f}\|_q = \|\boldsymbol{F}(t)\|_q\,, \quad \text{for all } t \ge 0\,,
$$

we obtain

$$
\begin{aligned}
T\, \left(|\boldsymbol{v}|_{2,q}^q + |p|_{1,q}^q\right) &\le C_4 \int_0^T \left(|\boldsymbol{w}_1(t)|_{2,q}^q + |\boldsymbol{w}_2(t)|_{2,q} + |\pi(t)|_{1,q}^q\right)\, dt \\
&\le C_5 \Big(\int_0^T \|\boldsymbol{F}(t)\|_q^q dt + T^{-\frac{3}{2}\left(\frac{q}{s_3} - 1\right) + 1} |\boldsymbol{v}|_{2,s_3}^q\Big) \\
&= C_5 \big(T \|\boldsymbol{f}\|_q^q + T^{-\frac{3}{2}\left(\frac{q}{s_3} - 1\right) + 1} |\boldsymbol{v}|_{2,s_3}^q\big)\,.
\end{aligned}
$$

Dividing both sides of this inequality by T, and passing to the limit $T \to \infty$, we then get

$$
|\boldsymbol{v}|_{2,q} + |p|_{1,q} \le C_6 \, \|\boldsymbol{f}\|_q\,, \tag{VIII.7.29}
$$

with $C_6 = C_6(q)$. Next, let $\boldsymbol{f}$ be merely in $L^q(\mathbb{R}^3)$, and denote by $\{\boldsymbol{f}_k\} \subset C_0^\infty(\mathbb{R}^3)$ a sequence converging to $\boldsymbol{f}$ in $L^q(\mathbb{R}^3)$. We shall show that the sequence of corresponding solutions $\{\boldsymbol{v}_k, p_k\}$ can be modified by adding to $\boldsymbol{v}_k$ a suitable linear function, in such a way that the new sequence converges to a solution $(\boldsymbol{v}, p)$ to (VIII.7.17) satisfying the estimates stated in the lemma. We begin by considering the case $q \in (1, \infty)$. From (VIII.7.29) and the uniqueness Theorem VIII.7.1, it follows that $\{\boldsymbol{v}_k, p_k\}$ is Cauchy in $\dot{D}^{2,q}(\mathbb{R}^3) \times \dot{D}^{1,q}(\mathbb{R}^3)$, and, consequently, so is $\{\widetilde{\boldsymbol{v}}_k, p_k\}$, where

$$\begin{aligned}
&\widetilde{\boldsymbol{v}}_k := \boldsymbol{v}_k - \boldsymbol{a}_k x_1 - \boldsymbol{b}_k x_2 - \boldsymbol{c}_k x_3 - \boldsymbol{d}_k \,, \\
&\boldsymbol{a}_k := \overline{(D_1 \boldsymbol{v}_k)}_{B_1} \,, \quad \boldsymbol{b}_k := \overline{(D_2 \boldsymbol{v}_k)}_{B_1} \,, \quad \boldsymbol{c}_k := \overline{(D_3 \boldsymbol{v}_k)}_{B_1} \,, \quad \boldsymbol{d}_k := \overline{\boldsymbol{v}_k}_{B_1} \,.
\end{aligned} \tag{VIII.7.30}$$

Thus, by Lemma II.6.2 and (VIII.7.29), there exists at least one $(\widetilde{\boldsymbol{v}}, p) \in D^{2,q}(\mathbb{R}^3) \times D^{1,q}(\mathbb{R}^3)$ such that

$$D^2 \widetilde{\boldsymbol{v}}_k \to D^2 \widetilde{\boldsymbol{v}} \,, \quad \nabla p_k \to \nabla p \ \text{ in } L^q(\mathbb{R}^3) \,, \tag{VIII.7.31}$$

which satisfies

$$|\widetilde{\boldsymbol{v}}|_{2,q} + |p|_{1,q} \le C_6 \, \|\boldsymbol{f}\|_q \,, \tag{VIII.7.32}$$

In addition, also by means of the first of the obvious properties

$$\int_{B_R} x_i = \int_{B_R} x_i x_j = 0 \,, \quad \text{for all } R > 0 \text{ and } i, j = 1, 2, 3 \,, \, i \ne j \,, \tag{VIII.7.33}$$

we obtain

$$\overline{(D_1 \widetilde{\boldsymbol{v}}_k)}_{B_1} = \overline{(D_2 \widetilde{\boldsymbol{v}}_k)}_{B_1} = \overline{(D_3 \widetilde{\boldsymbol{v}}_k)}_{B_1} = \overline{\widetilde{\boldsymbol{v}}_k}_{B_1} = \boldsymbol{0} \,,$$

and so, from a double application of Theorem II.5.4, Exercise II.6.1 and (VIII.7.31), we obtain

$$\widetilde{\boldsymbol{v}}_k \to \widetilde{\boldsymbol{v}} \ \text{ in } W^{1,q}(B_R), \text{ for all } R > 0 \,. \tag{VIII.7.34}$$

Thus, noticing that $\nabla \cdot \widetilde{\boldsymbol{v}}_k = 0$ for all $k \in \mathbb{N}$, we deduce, in particular,

$$\nabla \cdot \widetilde{\boldsymbol{v}} = 0 \,. \tag{VIII.7.35}$$

Since

$$\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla x_1 = 0 \,, \quad \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla x_2 = -x_3 \,, \quad \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla x_3 = x_2 \,, \tag{VIII.7.36}$$

we have

$$\begin{aligned}
&\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v}_k = \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \widetilde{\boldsymbol{v}}_k + x_2 \boldsymbol{c}_k - x_3 \boldsymbol{b}_k \,, \\
&\boldsymbol{e}_1 \times \boldsymbol{v}_k = \boldsymbol{e}_1 \times \widetilde{\boldsymbol{v}}_k + \boldsymbol{e}_1 \times (x_1 \boldsymbol{a}_k + x_2 \boldsymbol{b}_k + x_3 \boldsymbol{c}_k + \boldsymbol{d}_k) \,.
\end{aligned}$$

Consequently, from $(\text{VIII.7.17})_1$, which with $\boldsymbol{f} \equiv \boldsymbol{f}_k$ is satisfied by $(\boldsymbol{v}_k, p_k)$ for all $k \in \mathbb{N}$, we obtain

$$\begin{aligned}&\big(\Delta\widetilde{\boldsymbol{v}}_k+\mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\widetilde{\boldsymbol{v}}_k-\boldsymbol{e}_1\times\widetilde{\boldsymbol{v}}_k),\psi\big)-(\nabla p_k,\psi)-(\boldsymbol{f},\psi)\\&\qquad=\mathcal{T}\left(x_2\boldsymbol{B}_k+x_3\boldsymbol{C}_k+\boldsymbol{e}_1\times(x_1\boldsymbol{a}_k+\boldsymbol{d}_k),\psi\right)\,,\\&\boldsymbol{B}_k:=-\boldsymbol{c}_k+\boldsymbol{e}_1\times\boldsymbol{b}_k\,,\quad\boldsymbol{C}_k:=\boldsymbol{b}_k+\boldsymbol{e}_1\times\boldsymbol{c}_k\,,\end{aligned}\tag{VIII.7.37}$$

for all $\psi\in L^\infty(\mathbb{R}^3)$ with bounded support. In view of (VIII.7.31) and (VIII.7.34), it follows that as $k\to\infty$,

$$\begin{aligned}&\big(\Delta\widetilde{\boldsymbol{v}}_k+\mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\widetilde{\boldsymbol{v}}_k-\boldsymbol{e}_1\times\widetilde{\boldsymbol{v}}_k),\psi\big)-(\nabla p_k,\psi)-(\boldsymbol{f},\psi)\\&\qquad\to\big(\Delta\widetilde{\boldsymbol{v}}+\mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\widetilde{\boldsymbol{v}}-\boldsymbol{e}_1\times\widetilde{\boldsymbol{v}}),\psi\big)-(\nabla p,\psi)-(\boldsymbol{f},\psi)\,,\end{aligned}\tag{VIII.7.38}$$

for all functions ψ specified above. Denote by χ_R the characteristic function of B_R. Then, if we choose in (VIII.7.37), in order, $\psi=\chi_R$, $\psi=\chi_R x_i$, $i=1,2,3$, recall (VIII.7.33), and use (VIII.7.38), it is easily shown that there are $\boldsymbol{d},\boldsymbol{a},\boldsymbol{B},\boldsymbol{C}\in\mathbb{R}^3$ such that

$$\boldsymbol{e}_1\times\boldsymbol{d}_k\to\boldsymbol{e}_1\times\boldsymbol{d}\,,\ \boldsymbol{e}_1\times\boldsymbol{a}_k\to\boldsymbol{e}_1\times\boldsymbol{a}\,,\ \boldsymbol{B}_k\to\boldsymbol{B}\,,\ \boldsymbol{C}_k\to\boldsymbol{C}\,,\quad\text{as }k\to\infty\,,\tag{VIII.7.39}$$

where the vectors $\boldsymbol{a},\boldsymbol{B},\boldsymbol{C}$ may be taken to satisfy

$$\boldsymbol{a}\cdot\boldsymbol{e}_1=0\,,\quad\boldsymbol{B}=B_1\boldsymbol{e}_1+\boldsymbol{e}_1\times\boldsymbol{C}\,,\quad\boldsymbol{C}=C_1\boldsymbol{e}_1-\boldsymbol{e}_1\times\boldsymbol{B}\,.\tag{VIII.7.40}$$

Combining (VIII.7.38)–(VIII.7.40), by the arbitrariness of ψ we thus deduce the validity of the following equation, a.e. in $\mathbb{R}^3$:

$$\begin{aligned}&\Delta\widetilde{\boldsymbol{v}}+\mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\widetilde{\boldsymbol{v}}-\boldsymbol{e}_1\times\widetilde{\boldsymbol{v}})-\nabla p-\boldsymbol{f}\\&\qquad=\mathcal{T}\left(x_2(B_1\boldsymbol{e}_1+\boldsymbol{e}_1\times\boldsymbol{C})+x_3(C_1\boldsymbol{e}_1-\boldsymbol{e}_1\times\boldsymbol{B})+\boldsymbol{e}_1\times(x_1\boldsymbol{a}+\boldsymbol{d})\right)\,.\end{aligned}\tag{VIII.7.41}$$

Set

$$\boldsymbol{v}^*=\widetilde{\boldsymbol{v}}+x_1\boldsymbol{a}+\boldsymbol{d}+x_2C_1\boldsymbol{e}_1-x_3B_1\boldsymbol{e}_1\,.$$

Observing that with the help of (VIII.7.36), we have

$$\begin{aligned}\boldsymbol{e}_1\times\widetilde{\boldsymbol{v}}+x_1\boldsymbol{e}_1\times\boldsymbol{a}+\boldsymbol{e}_1\times\boldsymbol{d}&=\boldsymbol{e}_1\times(\widetilde{\boldsymbol{v}}+x_1\boldsymbol{a}+\boldsymbol{d}+x_2C_1\boldsymbol{e}_1-x_3B_1\boldsymbol{e}_1)\,,\\&=\boldsymbol{e}_1\times\boldsymbol{v}^*\\\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\widetilde{\boldsymbol{v}}-x_2B_1\boldsymbol{e}_1-x_3C_1\boldsymbol{e}_1&=\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla(\widetilde{\boldsymbol{v}}-x_3B_1\boldsymbol{e}_1+x_2C_1\boldsymbol{e}_1)\\&=\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla(\widetilde{\boldsymbol{v}}-x_3B_1\boldsymbol{e}_1+x_2C_1\boldsymbol{e}_1+x_1\boldsymbol{a}+\boldsymbol{d})\\&=\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{v}^*\,,\end{aligned}$$

equation (VIII.7.41) can be rewritten as follows:

$$\Delta\boldsymbol{v}^*+\mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{v}^*-\boldsymbol{e}_1\times\boldsymbol{v}^*)-\nabla p-\boldsymbol{f}=\mathcal{T}\left(x_2\boldsymbol{e}_1\times\boldsymbol{C}-x_3\boldsymbol{e}_1\times\boldsymbol{B}\right)\,.\tag{VIII.7.42}$$

Moreover, by (VIII.7.35) and $(\text{VIII.7.40})_1$, we obtain

$$\nabla \cdot \boldsymbol{v}^* = 0\,. \tag{VIII.7.43}$$

We next observe that setting $\boldsymbol{B}' = (0, B_2, B_3)$ and $\boldsymbol{C}' = (0, C_2, C_3)$, from $(\text{VIII.7.40})_{2,3}$, we deduce

$$\boldsymbol{B}' = \boldsymbol{e}_1 \times \boldsymbol{C} = \boldsymbol{e}_1 \times \boldsymbol{C}'\,, \quad \boldsymbol{C}' = -\boldsymbol{e}_1 \times \boldsymbol{B} = -\boldsymbol{e}_1 \times \boldsymbol{B}'\,. \tag{VIII.7.44}$$

Therefore, if we let $\boldsymbol{v} := \boldsymbol{v}^* + \frac{1}{2}(x_2\boldsymbol{C}' - x_3\boldsymbol{B}')$, equation (VIII.7.42) delivers

$$\Delta \boldsymbol{v} + \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v}^* - \tfrac{1}{2}(x_2\boldsymbol{B}' + x_3\boldsymbol{C}') - \boldsymbol{e}_1 \times \boldsymbol{v}) = \nabla p + \boldsymbol{f}\,. \tag{VIII.7.45}$$

However, employing the identities $(\text{VIII.7.36})_{2,3}$, we have

$$\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v}^* - \tfrac{1}{2}(x_2\boldsymbol{B}' + x_3\boldsymbol{C}') = \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla\big(\boldsymbol{v}^* + \tfrac{1}{2}(x_2\boldsymbol{C}' - x_3\boldsymbol{B}')\big) = \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v}\,,$$

so that from the latter and (VIII.7.45), we establish that $(\boldsymbol{v}, p)$ satisfies $(\text{VIII.7.17})_1$. Since $\boldsymbol{v}$ and $\widetilde{\boldsymbol{v}}$ differ by a linear function of $\boldsymbol{x}$, by (VIII.7.32), we infer that $(\boldsymbol{v}, p)$ also satisfies (VIII.7.19). Finally, noting that by (VIII.7.44) $C_3' = B_2'$, and using (VIII.7.43), we get $\nabla \cdot \boldsymbol{v} = 0$, and we conclude that $(\boldsymbol{v}, p)$ satisfies all the properties stated in the lemma, which is thus proved for a generic $q \in (1, \infty)$. If $q \in (1, 3)$, we define $\widetilde{\boldsymbol{v}}_k := \boldsymbol{v}_k - \overline{\boldsymbol{v}_k}_{B_1}$, and proceed as in the previous proof, by formally taking $\boldsymbol{a}_k = \boldsymbol{b}_k = \boldsymbol{c}_k = \boldsymbol{0}$. The only thing that we have to notice is that when $q \in (1, 3)$, the fields $\boldsymbol{v}_k$ and p_k converge (strongly) also in $\dot{D}^{3q/(3-q)}(\mathbb{R}^3)$, in view of the following inequalities (see Theorem II.6.3, and, in particular, (II.6.49)):

$$\left.\begin{aligned} |\boldsymbol{v}_k|_{3q/(3-q)} &\le c_1\, |\boldsymbol{v}_k|_{2,q} \\ |p_k|_{3q/(3-q)} &\le c_1\, |p_k|_{1,q} \end{aligned}\right\}, \quad \text{for all } k \in \mathbb{N}.$$

As a consequence, the fields $\widetilde{\boldsymbol{v}}$ (and therefore $\boldsymbol{v} := \widetilde{\boldsymbol{v}} + \boldsymbol{d}$; see (VIII.7.39)) and p will satisfy the same inequality, which, together with (VIII.7.32), completes the proof also when $q \in (1, 3)$. Finally, if $q \in (1, 3/2)$, again by Theorem II.6.3 and by (VIII.7.29) the approximating solution $\{\boldsymbol{v}_k, p_k\}$ satisfies the inequality

$$\|\boldsymbol{v}_k\|_{3q/(3-2q)} + |\boldsymbol{v}_k|_{3q/(3-q)} + |\boldsymbol{v}_k|_{2,q} + |p_k|_{1,q} \le c_2\, \|\boldsymbol{f}_k\|_q\,, \quad \text{for all } k \in \mathbb{N}. \tag{VIII.7.46}$$

Thus, from (VIII.7.46) and the uniqueness Theorem VIII.7.1, it follows that $\{\boldsymbol{v}_k, p_k\}$ is Cauchy in the (Banach) space of functions having finite norm on the left-hand side of (VIII.7.46). Therefore, the sequence converges to some $(\boldsymbol{v}, p)$ that, on the one hand, satisfies (VIII.7.46) with $\boldsymbol{f}$ in place of $\boldsymbol{f}_k$, and, on the other hand, as shown by a simple argument, satisfies (VIII.7.17). Finally, assume $\boldsymbol{f} \in L^q(\mathbb{R}^3)$ for *all* $q \in (1, \infty)$. By taking, in particular, $q \in (1, 3/2)$, from the existence result just shown, we obtain a solution $(\overline{\boldsymbol{v}}, \overline{p})$ satisfying (VIII.7.18)–(VIII.7.23) for this specific value of q. However, by the existence result and the uniqueness Lemma VIII.7.1, $(\overline{\boldsymbol{v}}, \overline{p})$ will satisfy (VIII.7.18)–(VIII.7.23) for all $q \in (1, 3/2)$. Next, let $q \in [3/2, 3)$ and let

$$(\boldsymbol{v}_1, p_1) \in [D^{2,q}(\mathbb{R}^3) \cap D^{1,3q/(3-q)}(\mathbb{R}^3)] \times [D^{1,q}(\mathbb{R}^3) \cap L^{3q/(3-q)}(\mathbb{R}^3)]$$

be the solution previously constructed. We claim that

$$D^2\boldsymbol{v}_1 = D^2\overline{\boldsymbol{v}} \quad \nabla\boldsymbol{v}_1 = \nabla\overline{\boldsymbol{v}}\,, \quad \text{and} \quad p_1 = \overline{p}\,, \text{ a.e. in } \mathbb{R}^3. \tag{VIII.7.47}$$

In fact, again by Lemma VIII.7.1, the first relation in (VIII.7.47) follows immediately. This latter implies $\nabla\boldsymbol{v}_1 = \nabla\overline{\boldsymbol{v}} + \boldsymbol{A}$, where $\boldsymbol{A}$ is a constant second-order tensor. However, since for $s \in (1, 3/2)$,

$$\int_{\mathbb{R}^3} \left(|\nabla\boldsymbol{v}_1|^{3q/(3-q)} + |\nabla\overline{\boldsymbol{v}}|^{3s/(3-s)}\right) < \infty\,,$$

there exists an unbounded sequence $\{\rho_k\} \subset (0, \infty)$ such that

$$\lim_{k\to\infty} \int_{S^2} \left(|\nabla\boldsymbol{v}_1(\rho_k, \omega)|^{3q/(3-q)} + |\nabla\overline{\boldsymbol{v}}(\rho_k, \omega)|^{3s/(3-s)}\right) d\omega = 0,$$

which implies

$$\lim_{k\to\infty} \int_{S^2} (|\nabla\boldsymbol{v}_1(\rho_k, \omega)| + |\nabla\overline{\boldsymbol{v}}(\rho_k, \omega)|)\, d\omega = 0,$$

and so we conclude that $\boldsymbol{A} = 0$, which proves the second relation in (VIII.7.47). As for the pressure, we observe that since both $(\boldsymbol{v}_1, p_1)$ and $(\boldsymbol{v}, p)$ obey (VIII.7.17), by a simple argument we show that, setting $\pi := p_1 - \overline{p}$, it is $(\nabla\pi, \nabla\psi) = 0$, for all $\psi \in C_0^\infty(\mathbb{R}^3)$. Thus, π is harmonic in $\mathbb{R}^3$, and since $p_1 \in L^{3q/(3-q)}(\mathbb{R}^3)$, $\overline{p} \in L^{3s/(3-s)}(\mathbb{R}^3)$, $s \in (1, 3/2)$, the third condition in (VIII.7.47) follows from Exercise II.11.11. Finally, suppose $q \in [3, \infty)$. Then, by the existence part of the lemma, we find a solution $(\boldsymbol{v}_2, p_2)$ such that

$$(\boldsymbol{v}_2, p_2) \in D^{2,q}(\mathbb{R}^3) \times [D^{1,q}(\mathbb{R}^3)\,.$$

We claim

$$D^2\boldsymbol{v}_2 = D^2\overline{\boldsymbol{v}} \quad \text{and} \quad \nabla p_2 = \nabla\overline{p}\,, \text{ a.e. in } \mathbb{R}^3. \tag{VIII.7.48}$$

In fact, again by the uniqueness Lemma VIII.7.1, the first condition in (VIII.7.48) is at once verified. Concerning the second one, we can show, as before, that $p_2 - \overline{p}$ is harmonic in $\mathbb{R}^3$ and since $p_2 \in D^{1,q}(\mathbb{R}^3)$, $\overline{p} \in D^{1,s}(\mathbb{R}^3)$ for $s \in (1, 3/2)$, from Exercise II.11.11 we infer the validity of the second relation in (VIII.7.48), and the proof of the lemma is complete. □

Exercise VIII.7.1 Let $\boldsymbol{v}$ be the velocity field of the solution determined in Lemma VIII.7.2 corresponding to $\boldsymbol{f} \in L^q(\mathbb{R}^3)$, $q \in (1, \infty)$, and let (x_1, r, θ) be a system of cylindrical coordinates. Show that $\partial v_1/\partial\theta \in L^q(\mathbb{R}^3)$. Show also that this latter implies $(v_1 - \overline{v_1}) \in L^q(\mathbb{R}^3)$, where $\overline{v_1} = \overline{v_1}(x_1, r) := \frac{1}{2\pi}\int_0^{2\pi} v_1(x_1, r, \theta)\, d\theta$. *Hint*: Use (VIII.7.19) and the Wirtinger inequality (II.5.17).

The next result concerns an extension of the previous lemma to the case of an exterior domain.

Theorem VIII.7.2 *Let Ω be an exterior domain of class C^2. Then, for any given*

$$\boldsymbol{f} \in L^q(\Omega)\,, \quad \boldsymbol{v}_* \in W^{2-1/q,q}(\partial\Omega)\,, \quad q \in (1,3/2)\,, \tag{VIII.7.49}$$

there exists at least one solution, $(\boldsymbol{v},p)$, to the generalized Oseen problem (VIII.0.7), (VIII.0.2) *with $\mathcal{R}=0$, such that*[1]

$$\boldsymbol{v} \in D^{2,q}(\Omega) \cap D^{1,3q/(3-q)}(\Omega \cap L^{3q/(3-2q)}(\Omega)\,, \quad p \in D^{1,q}(\Omega \cap L^{3q/(3-q)}(\Omega)\,. \tag{VIII.7.50}$$

Moreover, the following estimate holds:

$$\begin{aligned}\|\boldsymbol{v}\|_{3q/(3-2q)} + |\boldsymbol{v}|_{3q/(3-q)} + |\boldsymbol{v}|_{2,q} + |p|_{1,q} + \mathcal{T}\|\boldsymbol{e}_1 \times \boldsymbol{x}\cdot\nabla\boldsymbol{v} - \boldsymbol{e}_1\times\boldsymbol{v}\|_q \\ \le C\,\left(\|\boldsymbol{f}\|_q + \|\boldsymbol{v}_*\|_{2-2/q,q(\partial\Omega)}\right)\,,\end{aligned} \tag{VIII.7.51}$$

where $C = C(q,\Omega,B)$ whenever $\mathcal{T} \in [0,B]$. Finally, suppose that for some $r \in (1,\infty)$ and all $R > \delta(\Omega^c)$, $(\boldsymbol{v}_1,p_1) \in W^{2,r}(\Omega_R)\times W^{1,r}(\Omega_R)$ is another solution to (VIII.0.7) *with $\mathcal{R}=0$ corresponding to the same data and that*

$$\boldsymbol{v} \in L^{\overline{q}}(\Omega)\,, \quad \text{for some } \overline{q} \in (1,\infty)\,. \tag{VIII.7.52}$$

Then $\boldsymbol{v}(x) = \boldsymbol{v}_1(x)$, $p(x) = p_1(x) + p_0$ for a.a. $x \in \Omega$, and some constant p_0.

Proof. We begin by observing that the uniqueness part is an immediate consequence of Theorem VIII.7.1. We thus proceed to the proof of existence. For given $\boldsymbol{f}$ and $\boldsymbol{v}_*$ satisfying the assumptions of the theorem, let $\{\boldsymbol{f}_k\}$ and $\{\boldsymbol{v}_{*k}\}$ be sequences of smooth functions, with $\boldsymbol{f}_k$ of bounded support for each $k \in \mathbb{N}$, converging to $\boldsymbol{f}$ and $\boldsymbol{v}_*$ in the $L^q(\Omega)$- and $W^{2-1/q,q}(\partial\Omega)$-norms, respectively. We then denote by $\boldsymbol{v}_k$, $k \in \mathbb{N}$, the (unique) generalized solution corresponding to $\boldsymbol{f}_k$, $\boldsymbol{v}_{*k}$, and by $p_k \in W^{1,2}(\Omega_R)$ the associated pressure field. We also set

$$\boldsymbol{u}_k = \psi\,(\boldsymbol{v}_k + \Phi_k\boldsymbol{\sigma}) + \boldsymbol{w}_k\,, \quad \phi_k = \psi\, p_k\,,$$

where ψ is the "cut-off" function introduced in the proof of Theorem VIII.7.1,

$$\Phi_k := \int_{\partial\Omega} \boldsymbol{v}_{*k}\cdot\boldsymbol{n}\,,$$

$\boldsymbol{\sigma}$ is defined in (VIII.1.5), and finally, $\boldsymbol{w}_k$ satisfies (VIII.7.12) with $\boldsymbol{v} \equiv \boldsymbol{v}_k + \Phi_k\boldsymbol{\sigma}$. Using the properties of $\boldsymbol{\sigma}$, it is at once checked that (VIII.7.13) is satisfied. We next observe that $\boldsymbol{u}_k$ and ϕ_k satisfy (VIII.7.17) with $\boldsymbol{f} \equiv \boldsymbol{F}_k$, where

[1] See Remark VIII.7.1.

$$\boldsymbol{F}_k := \boldsymbol{f}_k + \Delta \boldsymbol{w}_k + \mathcal{T}\big(\boldsymbol{e}_1 \times \boldsymbol{w}_k - \boldsymbol{e}_1 \times \boldsymbol{w}_k\big) + 2\Phi_k \nabla\psi \cdot \nabla\boldsymbol{\sigma} + \Phi_k \boldsymbol{\sigma} \Delta\psi$$
$$-p_k\, \nabla\psi + 2\nabla\psi \cdot \nabla \boldsymbol{v}_k + \boldsymbol{v}_k\, \Delta\psi\,,$$

and where we have used (VIII.1.8). Therefore, recalling (VIII.7.12)$_2$, that for all $R > \delta(\Omega^c)$,

$$\boldsymbol{v}_k \in W^{2,2}(\Omega_R) \cap L^6(\Omega)\,, \quad p_k \in W^{1,2}(\Omega_R) \tag{VIII.7.53}$$

(see Remark VIII.1.1), and that moreover, by the trace Theorem II.4.1, $|\Phi_k| \le c\|\boldsymbol{v}_k\|_{1,r,\Omega_{2\sigma}}$, for any $r \ge 1$, we obtain, after a simple computation,

$$\|\boldsymbol{F}_k\|_q \le C\, \left(\|\boldsymbol{f}_k\|_q + \|\boldsymbol{v}_k\|_{1,q,\Omega_{2\sigma}} + \|p_k\|_{q,\Omega_{2\sigma}}\right)\,, \quad \text{for all } q \in (1, 3/2), \tag{VIII.7.54}$$

where $C = C(\sigma, \Omega, q, B)$. Now, again by (VIII.7.53), we have

$$\boldsymbol{u}_k \in L^6(\mathbb{R}^3) \cap W^{2,2}(B_R)\,, \quad \phi_k \in W^{1,2}(B_R)\,,$$

for all $R > 0$, and so by Lemma VIII.7.2, Lemma VIII.7.1, and (VIII.7.54), we obtain, on the one hand, that $(\boldsymbol{u}_k, \phi_k)$ is in the class (VIII.7.50), for all $k \in \mathbb{N}$, and that, on the other hand, it must satisfy (VIII.7.51) with $\boldsymbol{f}$ replaced by $\boldsymbol{F}_k$. Recalling that $\boldsymbol{u}_k(x) = \boldsymbol{v}_k(x)$, $\phi_k(x) = p_k(x)$ for all $x \in \Omega^{2\sigma}$, and taking into account (VIII.7.54), we thus deduce

$$\begin{aligned}\|\boldsymbol{v}_k\|_{3q/(3-2q),\Omega^{2\sigma}} &+ |\boldsymbol{v}_k|_{3q/(3-q),\Omega^{2\sigma}} + |\boldsymbol{v}_k|_{2,q,\Omega^{2\sigma}} + |p_k|_{1,q,\Omega^{2\sigma}} \\ &\le C\, \left(\|\boldsymbol{f}_k\|_q + \|\boldsymbol{v}_k\|_{1,q,\Omega_{2\sigma}} + \|p_k\|_{q,\Omega_{2\sigma}}\right)\,,\end{aligned} \tag{VIII.7.55}$$

with $C = C(q, \Omega, \sigma)$. However, by Lemma VIII.6.1, we also have

$$\begin{aligned}\|\boldsymbol{v}_k\|_{2,q,\Omega_{2\sigma}} &+ \|p_k\|_{1,q,\Omega_{2\sigma}} \\ &\le c_2\, (\|\boldsymbol{f}_k\|_{q,\Omega_{3\sigma}} + \|\boldsymbol{v}_{*k}\|_{2-1/q,q,\partial\Omega} + \|\boldsymbol{v}_k\|_{q,\Omega_{3\sigma}} + \|p_k\|_{q,\Omega_{3\sigma}}),\end{aligned} \tag{VIII.7.56}$$

where $c_2 = c_2(q, \sigma, \Omega, B)$. Therefore, combining (VIII.7.55) and (VIII.7.56) we deduce

$$\begin{aligned}\|\boldsymbol{v}_k\|_{3q/(3-2q)} &+ |\boldsymbol{v}_k|_{3q/(3-q)} + |\boldsymbol{v}_k|_{2,q} + |p_k|_{1,q} \\ &\le c_3\, \left(\|\boldsymbol{f}_k\|_q + \|\boldsymbol{v}_{*k}\|_{2-1/q,q(\partial\Omega)} + \|\boldsymbol{v}_k\|_{1,q\Omega_{3\sigma}} + \|p_k\|_{q,\Omega_{3\sigma}}\right)\,,\end{aligned} \tag{VIII.7.57}$$

where $c_3 = c_3(q, \Omega, \sigma, B)$. The next task is to prove the existence of a constant $c_4 = c_4(q, \Omega, \sigma, B)$ such that

$$\|\boldsymbol{v}_k\|_{1,q,\Omega_{3\sigma}} + \|p_k\|_{q,\Omega_{3\sigma}} \le c_4\, \left(\|\boldsymbol{f}_k\|_q + \|\boldsymbol{v}_{*k}\|_{2-1/q,q(\partial\Omega)}\right)\,. \tag{VIII.7.58}$$

The proof of (VIII.7.58) is based on a contradiction argument that uses (i) the embedding $W^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega_R)$ (see Exercise II.5.8), and (ii) the uniqueness of solutions to (VIII.0.7), (VIII.0.2) with $\mathcal{R} = 0$ *and* $\mathcal{T} \in [0, B]$, in the class of functions (VIII.7.50). Since the argument is entirely analogous to that employed in the proofs of Theorem V.5.1, Theorem V.8.1, and Theorem VII.7.1, it will be omitted. From (VIII.7.57) and (VIII.7.58), we thus infer

$$\|\boldsymbol{v}_k\|_{3q/(3-2q)}+|\boldsymbol{v}_k|_{3q/(3-q)}+|\boldsymbol{v}_k|_{2,q}+|p_k|_{1,q} \le c_5\,\left(\|\boldsymbol{f}_k\|_q+\|\boldsymbol{v}_{*k}\|_{2-1/q,q(\partial\Omega)}\right)\,, \tag{VIII.7.59}$$

with $c_5=c_5(q,\Omega,\sigma,B)$. We now let $k\to\infty$ in this relation and, again by a by now standard reasoning, we show that $\{(\boldsymbol{v}_k,p_k)\}$ tends, in the topology defined by the left-hand side of (VIII.7.59), to a solution $(\boldsymbol{v},p)$ to (VIII.0.7), (VIII.0.2), with $\mathcal{R}=0$, satisfying all the statements of the theorem, which is therefore completely proved. □

Remark VIII.7.1 We would like to specify the way in which solutions constructed in the previous theorem satisfy the condition at infinity (VIII.0.2). Since $q\in(1,3/2)$, we have $\boldsymbol{v}\in D^{1,s}(\Omega)$, where $s:=3q/(3-q)<3$. Consequently, taking also into account that $\boldsymbol{v}\in L^{3q/(3-2q)}(\Omega)$, from Lemma II.6.3 we have

$$\int_{S^2}|\boldsymbol{v}(|x|,\omega)|\,d\omega=o\left(|x|^{2-3/q}\right)\quad\text{as }|x|\to\infty\,.$$

■

Exercise VIII.7.2 Let the assumptions of Theorem VIII.7.2 on Ω, $\boldsymbol{f}$, and $\boldsymbol{v}_*$ be satisfied, and suppose, in addition, that $\boldsymbol{f}\in L^s(\Omega)$, $\boldsymbol{v}_*\in W^{2-1/s,s}(\partial\Omega)$, for some $s\in(1,\infty)$. Show that the corresponding solution $(\boldsymbol{v},p)$, besides the properties stated in that theorem, satisfies also the following ones:

$$\begin{aligned}&(\boldsymbol{v},p)\in D^{2,s}(\Omega)\times D^{1,s}(\Omega)\,,\\&|\boldsymbol{v}|_{2,s}+|p|_{1,s}\le C\,\left(\|\boldsymbol{f}\|_s+\|\boldsymbol{v}_*\|_{2-1/s(\partial\Omega)}\right)\,,\end{aligned}$$

with $C=C(q,\Omega,B)$, whenever $\mathcal{T}\in[0,B]$. Moreover, show that

$$\lim_{|x|\to\infty}\boldsymbol{v}(x)=\boldsymbol{0}\ \text{ if } s\in(1,3/2)\,;\quad \lim_{|x|\to\infty}\nabla\boldsymbol{v}(x)=\boldsymbol{0}\ \text{ if } s\in(1,3)\,.$$

Hint: Use the arguments of the proof of Theorem V.4.8.

VIII.8 Existence, Uniqueness, and L^q-Estimates. The Case $\mathcal{R}\neq 0$

We begin by proving existence of solutions, with corresponding estimates, to the system $(\text{VIII.5.1})_{1,2}$, when $\boldsymbol{f}\in L^q(\mathbb{R}^3)$. To this end, we will use the following result of Farwig (2006, Theorem 1.1) which we state without proof; see also Galdi & Kyed (2011b, Theorem 1.1) for a simple proof.

Lemma VIII.8.1 *Let $\boldsymbol{f}\in L^q(\mathbb{R}^3)$, for some $q\in(1,\infty)$, and $\mathcal{R}>0$. Suppose that the problem*

$$\left.\begin{array}{l} \Delta \boldsymbol{v} + \mathcal{R}\dfrac{\partial \boldsymbol{v}}{\partial x_1} + \mathcal{T}\left(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}\right) = \nabla p + \boldsymbol{f} \\ \nabla \cdot \boldsymbol{v} = 0 \end{array}\right\} \quad \text{a.e. in } \mathbb{R}^3 \tag{VIII.8.1}$$

has a solution $(\boldsymbol{v}, p) \in D^{2,s}(\mathbb{R}^3) \times D^{1,s}(\mathbb{R}^3)$, *for some* $s \in (1, \infty)$. *Then*

$$\frac{\partial \boldsymbol{v}}{\partial x_1}\,, \quad (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}) \in L^q(\mathbb{R}^3)\,,$$

and the following estimate holds:

$$\mathcal{R}\left\|\frac{\partial \boldsymbol{v}}{\partial x_1}\right\|_q + \mathcal{T}\|\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}\|_q \leq C_1 \left(1 + \frac{\mathcal{R}^4}{\mathcal{T}^2}\right) \|\boldsymbol{f}\|_q\,.$$

We are now in a position to prove the following.

Lemma VIII.8.2 *Let* $\boldsymbol{f} \in L^q(\mathbb{R}^3)$, *for some* $q \in (1, \infty)$, *and* $\mathcal{R} > 0$. *Then, problem* (VIII.8.1) *has at least one solution* $(\boldsymbol{v}, p)$ *such that*

$$\begin{aligned} &(\boldsymbol{v}, p) \in D^{2,q}(\mathbb{R}^3) \times D^{1,q}(\mathbb{R}^3)\,, \\ &\frac{\partial \boldsymbol{v}}{\partial x_1}\,, \quad (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}) \in L^q(\mathbb{R}^3)\,, \end{aligned} \tag{VIII.8.2}$$

and that satisfies the inequalities

$$|\boldsymbol{v}|_{2,q} + |p|_{1,q} \leq C_1 \|\boldsymbol{f}\|_q \tag{VIII.8.3}$$

and

$$\mathcal{R}\left\|\frac{\partial \boldsymbol{v}}{\partial x_1}\right\|_q + \mathcal{T}\|\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}\|_q \leq C_1 \left(1 + \frac{\mathcal{R}^4}{\mathcal{T}^2}\right) \|\boldsymbol{f}\|_q \tag{VIII.8.4}$$

with $C_1 = C_1(q)$. *If* $q \in (1, 4)$, *then there is a solution that, in addition to* (VIII.8.2)–(VIII.8.4), *satisfies also*

$$\begin{aligned} &\boldsymbol{v} \in D^{1,4q/(4-q)}(\mathbb{R}^3)\,, \\ &\mathcal{R}^{1/4}|\boldsymbol{v}|_{1,4q/(4-q)} \leq C_2 \left(1 + \frac{\mathcal{R}^4}{\mathcal{T}^2}\right) \|\boldsymbol{f}\|_q\,, \end{aligned} \tag{VIII.8.5}$$

with $C_2 = C_2(q)$. *Furthermore, if* $q \in (1, 2)$, *then we can find a solution that, in addition to* (VIII.8.2)–(VIII.8.5), *satisfies further,*[1]

$$\boldsymbol{v} \in L^{2q/(2-q)}(\mathbb{R}^3)\,, \quad p \in L^{3q/(3-q)}(\mathbb{R}^3) \tag{VIII.8.6}$$

along with the inequalities

[1] In fact, this summability property for p below requires only $q \in (1, 3)$.

$$
\begin{aligned}
&\mathcal{R}^{1/2}\|\boldsymbol{v}\|_{2q/(2-q)} \le C_3\left(1+\frac{\mathcal{R}^4}{\mathcal{T}^2}\right)\|\boldsymbol{f}\|_q\,, \\
&\|p\|_{3q/(3-q)} \le C_3\,\|\boldsymbol{f}\|_q\,,
\end{aligned}
\tag{VIII.8.7}
$$

where $C_3 = C_3(q)$.

Finally, if $\boldsymbol{f} \in L^q(\mathbb{R}^3)$ *for all* $q \in (1,\infty)$, *then there exists a solution* $(\boldsymbol{v}, p)$ *to* (VIII.8.1) *satisfying* (VIII.8.2)–(VIII.8.7) *for all specified values of* q.

Proof. The proof of this result follows the same main arguments used in that of Lemma VIII.7.2, and therefore will be only sketched here leaving to the reader the (straightforward) task of filling in the details. We take first $\boldsymbol{f} \in C_0^\infty(\mathbb{R}^3)$. Thus, from Theorem VIII.5.1, we know that problem (VIII.8.1) has one and only one smooth solution $(\boldsymbol{v}, p)$ such that

$$
\begin{aligned}
&[\![\boldsymbol{v}]\!]_{1,\mathcal{R}} + [\![\nabla\boldsymbol{v}]\!]_{2,\mathcal{R}} + [\![D^2\boldsymbol{v}]\!]_{2,\mathcal{R}} < \infty\,, \\
&|p|_{1,2} + \|p\|_r < \infty\,, \quad \text{all } r \in (3/2,\infty)\,.
\end{aligned}
$$

It is a simple exercise to show that

$$
\boldsymbol{v} \in L^{s_1}(\mathbb{R}^3) \cap D^{1,s_2}(\mathbb{R}^3) \cap D^{2,s_3}(\mathbb{R}^3)\,, \quad \text{all } s_1 > 2,\ s_2 > 4/3, \text{ and } s_3 > 1\,;
\tag{VIII.8.8}
$$

see also (VII.3.28) and (VII.3.33). Therefore, arguing exactly as in the part of the proof of Lemma VIII.7.2 leading to (VIII.7.29), we can show that $\boldsymbol{v}, p$ satisfy the estimate (VIII.8.3). From Lemma VIII.8.1 we then infer (VIII.8.2)$_2$, together with the validity of (VIII.8.4). Take now $\boldsymbol{f}$ just in $L^q(\mathbb{R}^3)$, and let $\{\boldsymbol{f}_k\} \subset C_0^\infty(\mathbb{R}^3)$ be a sequence converging to $\boldsymbol{f}$ in $L^q(\mathbb{R}^3)$, and $\{(\boldsymbol{v}_k, p_k)\}$ the sequence of corresponding solutions. From what we have shown, $(\boldsymbol{v}_k, p_k)$ satisfies (VIII.8.2)–(VIII.8.4), with $\boldsymbol{f} \equiv \boldsymbol{f}_k$, for all $k \in \mathbb{N}$ and all $q \in (1,\infty)$. We then consider the fields $\widetilde{\boldsymbol{v}}_k$ defined in (VIII.7.30) by formally setting $\boldsymbol{a}_k = \boldsymbol{0}$, and follow, step by step, the analogous proof given in Lemma VIII.7.2 for the case of a generic $q \in (1,\infty)$, to prove the existence of a solution for such a value of q. This solution satisfies the estimates (VIII.8.3) and that for $D_1\boldsymbol{v}$ in (VIII.8.4). Thus, from (VIII.8.1), we deduce that it satisfies (VIII.8.4). The lemma is then proved for $q \in (1,\infty)$. If $q \in (1,4)$, we formally rewrite the generalized Oseen system (VIII.8.1) for $(\boldsymbol{v}_k, p_k)$ as the following Oseen system:

$$
\left.
\begin{aligned}
&\Delta\boldsymbol{v}_k + \mathcal{R}\frac{\partial\boldsymbol{v}_k}{\partial x_1} = \nabla p_k + \boldsymbol{F}_k \\
&\nabla\cdot\boldsymbol{v}_k = 0
\end{aligned}
\right\}\ \text{in } \mathbb{R}^3\,,
\tag{VIII.8.9}
$$

where

$$
\boldsymbol{F}_k := \boldsymbol{f}_k - \mathcal{T}\,(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{v}_k - \boldsymbol{e}_1\times\boldsymbol{v}_k)\,.
$$

From Theorem VII.4.1 and the uniqueness property of the pair $(\boldsymbol{v}_k, p_k)$ we find that the sequence $\{\boldsymbol{v}_k\}$ is Cauchy (also) in $D^{4q/(4-q)}(\mathbb{R}^3)$, and thus, after modifying the $\boldsymbol{v}_k$ by a constant vector and proceeding as in the proof of Lemma VIII.7.2, we show that the modified sequence, together with the corresponding sequence $\{p_k\}$, converges to a solution to (VIII.8.1) with the properties (VIII.8.2)–(VIII.8.5). In the last case, $q \in (1,2)$, again from Theorem VII.4.1 applied to (VIII.8.9), we obtain that the sequence $\{(\boldsymbol{v}_k, p_k)\}$ satisfies (VIII.8.2)–(VIII.8.7) with $\boldsymbol{f} \equiv \boldsymbol{f}_k$. Using again the uniqueness property of $(\boldsymbol{v}_k, p_k)$ along with the estimates in (VIII.8.3)–(VIII.8.7), it is routine to show that the sequence $\{\boldsymbol{v}_k, p_k)\}$ converges to a solution $(\boldsymbol{v}, p)$ to (VIII.8.1) satisfying the desired properties. Finally, assume $\boldsymbol{f} \in L^q(\mathbb{R}^3)$ for all $q \in (1,\infty)$. Then, we may follow verbatim the argument adopted in the proof of Lemma VIII.7.2 to establish the existence of a solution satisfying the stated properties. Details are again left to the reader.

□

With Lemma VIII.8.2 in hand, we shall now proceed to prove the main result of this section. To this end, it is convenient to introduce a suitable function class. Let Ω be an exterior domain of $\mathbb{R}^3$, and let $\boldsymbol{v}, p$ be vector and scalar fields, respectively, defined on Ω. We shall say that $(\boldsymbol{v}, p)$ *is in the class* $\mathcal{C}_q(\Omega)$, $q \in (1,2)$, *if*

$$\begin{aligned}
&\boldsymbol{v} \in D^{2,q}(\Omega) \cap D^{1,4q/(4-q)}(\Omega) \cap L^{2q/(2-q)}(\Omega)\,,\\
&\frac{\partial \boldsymbol{v}}{\partial x_1}\,,\ (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}) \in L^q(\Omega)\,,\\
&p \in D^{1,q}(\Omega) \cap L^{3q/(3-q)}(\Omega)\,.
\end{aligned}$$

Remark VIII.8.1 We observe that by Lemma II.6.1, $\mathcal{C}_q(\Omega) \subset W^{2,q}(\Omega_R)$, for all $R > \delta(\Omega^c)$, if Ω is locally Lipschitz. ■

Our main result thus reads as follows.

Theorem VIII.8.1 *Let Ω be an exterior domain of $\mathbb{R}^3$ of class C^2, and assume $\mathcal{R} > 0$. Then, for any given*

$$\boldsymbol{f} \in L^q(\Omega)\,, \quad \boldsymbol{v}_* \in W^{2-1/q,q}(\partial\Omega)\,, \quad q \in (1,2)\,, \tag{VIII.8.10}$$

there exists at least one corresponding solution $\boldsymbol{v}, p$ to the generalized Oseen problem (VIII.0.7), (VIII.0.2) *in the class $\mathcal{C}_q(\Omega)$. Moreover, the following estimate holds:*

$$\begin{aligned}
|\boldsymbol{v}|_{2,q} + \mathcal{R}\left\|\frac{\partial \boldsymbol{v}}{\partial x_1}\right\|_q + \mathcal{T}|\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}\|_q + \mathcal{R}^{1/4}|\boldsymbol{v}|_{1,4q/(4-q)}&\\
+\mathcal{R}^{1/2}\|\boldsymbol{v}\|_{2q/(2-q)} + |p|_{1,q} + \|p\|_{3q/(3-q)} \le C\,\left(\|\boldsymbol{f}\|_q + \|\boldsymbol{v}_*\|_{2-1/q,q(\partial\Omega)}\right)&\,,
\end{aligned} \tag{VIII.8.11}$$

with $C = C(q, \Omega, \mathcal{R}, \mathcal{T})$. However, if $q \in (1, 3/2)$, we may take $C = C(q, \Omega, B, \mathcal{T})$ whenever $\mathcal{R} \in (0, B]$.

Finally, suppose that for some $r \in (1,\infty)$ *and all* $R > \delta(\Omega^c)$, $(\boldsymbol{v}_1, p_1) \in W^{2,r}(\Omega_R) \times W^{1,r}(\Omega_R)$ *is another solution to* (VIII.0.7) *corresponding to the same data and that in addition,*

$$\boldsymbol{v} \in L^{\overline{q}}(\Omega)\,, \quad \textit{for some } \overline{q} \in (1,\infty)\,, \tag{VIII.8.12}$$

Then $\boldsymbol{v}(x) = \boldsymbol{v}_1(x)\,,\, p(x) = p_1(x) + p_0$ *for a.a.* $x \in \Omega$, *and some constant* p_0.

Proof. We begin by observing that the uniqueness part is an immediate consequence of Theorem VIII.7.1. It remains to prove existence. For given $\boldsymbol{f}$ and $\boldsymbol{v}_*$ satisfying the assumptions of the theorem, we let $\{\boldsymbol{f}_k\}$ and $\{\boldsymbol{v}_{*k}\}$ be sequences of smooth functions, with $\boldsymbol{f}_k$ of bounded support for each $k \in \mathbb{N}$, converging to $\boldsymbol{f}$ and $\boldsymbol{v}_*$ in the L^q- and $W^{2-1/q,q}$-norms, respectively. We next denote by $\boldsymbol{v}_k$, $k \in \mathbb{N}$, the (unique) generalized solution corresponding to $\boldsymbol{f}_k$, $\boldsymbol{v}_{*k}$, and by $p_k \in W^{1,2}(\Omega_R)$ the associated pressure field. We also set

$$\boldsymbol{u}_k = \psi\,(\boldsymbol{v}_k + \Phi_k \boldsymbol{\sigma}) + \boldsymbol{w}_k\,, \quad \phi_k = \psi\, p_k\,,$$

where ψ, Φ_k, $\boldsymbol{\sigma}$, and $\boldsymbol{w}_k$ are the same quantities introduced in the proof of Theorem VIII.7.2. We next observe that $\boldsymbol{u}_k$ and ϕ_k satisfy $(\text{VIII.5.1})_{1,2}$ with $\boldsymbol{f} \equiv \boldsymbol{F}_k$, where

$$\begin{aligned}\boldsymbol{F}_k = \boldsymbol{f}_k + \Delta \boldsymbol{w}_k + \mathcal{R}\frac{\partial \boldsymbol{w}_k}{\partial x_1} + \mathcal{T}\big(\boldsymbol{e}_1 \times \boldsymbol{w}_k - \boldsymbol{e}_1 \times \boldsymbol{w}_k\big) \\ +\mathcal{R}\boldsymbol{v}_k\,\frac{\partial \psi}{\partial x_1} - p_k\,\nabla\psi + 2\nabla\psi\cdot\nabla\boldsymbol{v}_k + \boldsymbol{v}_k\,\Delta\psi \\ +2\Phi_k\nabla\psi\cdot\nabla\boldsymbol{\sigma} + \Phi_k\boldsymbol{\sigma}\Delta\psi + \mathcal{R}\Phi_k\boldsymbol{\sigma}\frac{\partial\psi}{\partial x_1}\,,\end{aligned}$$

where we have used (VIII.1.8). Therefore, recalling $(\text{VIII.7.12})_2$ and that for all $R > \delta(\Omega^c)$,

$$\boldsymbol{v}_k \in W^{2,2}(\Omega_R) \cap L^6(\Omega)\,, \quad p_k \in W^{1,2}(\Omega_R) \tag{VIII.8.13}$$

(see Remark VIII.1.1) we show, as in the proof of Theorem VIII.7.2, that

$$\|\boldsymbol{F}_k\|_q \le C\,\left(\|\boldsymbol{f}_k\|_q + \|\boldsymbol{v}_k\|_{1,q,\Omega_{2\sigma}} + \|p\|_{q,\Omega_{2\sigma}}\right)\,,\ \text{ for all } q \in (1,2), \tag{VIII.8.14}$$

where $C = C(\sigma, \Omega, q, B, \mathcal{T})$. Now again by (VIII.8.13), we have

$$\boldsymbol{u}_k \in L^6(\mathbb{R}^3) \cap W^{2,2}(B_R)\,, \quad \Phi_k \in W^{1,2}(B_R)\,,$$

for all $R > 0$, and so by Lemma VIII.8.2 and Lemma VIII.7.1 and (VIII.8.14), we obtain, on the one hand, that $(\boldsymbol{u}_k, \phi_k)$ is in the class $C_q(\mathbb{R}^3)$, for all $k \in \mathbb{N}$, and that on the other hand, it must satisfy (VIII.8.4)–(VIII.8.7) with $\boldsymbol{f}$ replaced by $\boldsymbol{F}_k$. Since $\boldsymbol{u}_k(x) = \boldsymbol{v}_k(x)$, $\phi_k(x) = p_k(x)$ for all $x \in \Omega^{2\sigma}$, by (VIII.8.14) we thus obtain

$$
\begin{aligned}
&|\boldsymbol{v}_k|_{2,q,\Omega^{2\sigma}} + \mathcal{R}\left\|\frac{\partial \boldsymbol{v}_k}{\partial x_1}\right\|_{q,\Omega^{2\sigma}} + \mathcal{T}\|\boldsymbol{e}_1 \times \boldsymbol{x}\cdot\nabla \boldsymbol{v}_k - \boldsymbol{e}_1 \times \boldsymbol{v}_k\|_{q,\Omega^{2\sigma}} \\
&+\mathcal{R}^{1/4}|\boldsymbol{v}_k|_{4q/(4-q),\Omega^{2\sigma}} + \mathcal{R}^{1/2}\|\boldsymbol{v}_k\|_{2q/(2-q),\Omega^{2\sigma}} + |p_k|_{1,q,\Omega^{2\sigma}} + \|p_k\|_{3q/(3-q),\Omega^{2\sigma}} \\
&\qquad \leq c_1\left(1+\frac{\mathcal{R}^4}{\mathcal{T}^2}\right)\left(\|\boldsymbol{f}_k\|_q + \|\boldsymbol{v}_k\|_{1,q,\Omega_{2\sigma}} + \|p_k\|_{q,\Omega_{2\sigma}}\right),
\end{aligned}
\tag{VIII.8.15}
$$

with $c_1 = c_1(\sigma, q, \Omega, B, \mathcal{T})$. However, by Lemma VIII.6.1, we also have

$$
\begin{aligned}
&\|\boldsymbol{v}_k\|_{2,q,\Omega_{2\sigma}} + \|p_k\|_{1,q,\Omega_{2\sigma}} \\
&\qquad \leq c_2\left(\|\boldsymbol{f}_k\|_{q,\Omega_{3\sigma}} + \|\boldsymbol{v}_{*k}\|_{2-1/q,q,\partial\Omega} + \|\boldsymbol{v}_k\|_{q,\Omega_{3\sigma}} + \|p_k\|_{q,\Omega_{3\sigma}}\right),
\end{aligned}
\tag{VIII.8.16}
$$

where $c_2 = c_2(\Omega, \sigma, B, \mathcal{T})$. Therefore, combining (VIII.8.15) and (VIII.8.16), we deduce

$$
\begin{aligned}
&|\boldsymbol{v}_k|_{2,q,\Omega} + \mathcal{R}\left\|\frac{\partial \boldsymbol{v}_k}{\partial x_1}\right\|_{q,\Omega} + \mathcal{T}\|\boldsymbol{e}_1 \times \boldsymbol{x}\cdot\nabla \boldsymbol{v}_k - \boldsymbol{e}_1 \times \boldsymbol{v}_k\|_{q,\Omega} \\
&+\mathcal{R}^{1/4}\,|\boldsymbol{v}_k|_{4q/(4-q),\Omega} + \mathcal{R}^{1/2}\|\boldsymbol{v}_k\|_{2q/(2-q),\Omega} + |p_k|_{1,q,\Omega} + \|p_k\|_{3q/(3-q),\Omega} \\
&\qquad \leq c_1\left(\|\boldsymbol{f}_k\|_q + \|\boldsymbol{v}_{*k}\|_{2-1/q,q,\partial\Omega} + \|\boldsymbol{v}_k\|_{1,q,\Omega_{3\sigma}} + \|p_k\|_{q,\Omega_{3\sigma}}\right),
\end{aligned}
\tag{VIII.8.17}
$$

Using a by now standard contradiction argument already employed several times previously, and which relies on the embedding $W^{1,q}(\Omega_R) \hookrightarrow\hookrightarrow L^q(\Omega_R)$ (see Exercise II.5.8) and the uniqueness Theorem VIII.7.1, we prove the existence of a constant c_3 depending on Ω, σ, $\mathcal{R}$, and $\mathcal{T}$ but otherwise independent of $k \in \mathbb{N}$ such that

$$
\|\boldsymbol{v}_k\|_{1,q,\Omega_{3\sigma}} + \|p_k\|_{q,\Omega_{3\sigma}} \leq c_3\left(\|\boldsymbol{f}_k\|_q + \|\boldsymbol{v}_{*k}\|_{2-1/q,q,\partial\Omega}\right).
$$

However, we observe that if $q \in (1, 3/2)$, the constant c_3 may be taken dependent only on q, Ω, B, and $\mathcal{T}$, whenever $\mathcal{R} \in (0, B]$. The argument that we need to prove this assertion is entirely analogous to that given in the proof of Theorem VII.7.1, and precisely in that part following (VII.7.19). We leave it to the reader. From this latter inequality and (VIII.8.17) we infer

$$
\begin{aligned}
&|\boldsymbol{v}_k|_{2,q} + \mathcal{R}\left\|\frac{\partial \boldsymbol{v}_k}{\partial x_1}\right\|_{q} + \mathcal{T}\|\boldsymbol{e}_1 \times \boldsymbol{x}\cdot\nabla \boldsymbol{v}_k - \boldsymbol{e}_1 \times \boldsymbol{v}_k\|_{q} \\
&\qquad +\mathcal{R}^{1/4}|\boldsymbol{v}_k|_{4q/(4-q)} + \mathcal{R}^{1/2}\|\boldsymbol{v}_k\|_{2q/(2-q)} + |p_k|_{1,q,\Omega} + \|p_k\|_{3q/(3-q)} \\
&\qquad\qquad \leq c_4\left(\|\boldsymbol{f}_k\|_q + \|\boldsymbol{v}_{*k}\|_{2-1/q,q,\partial\Omega}\right),
\end{aligned}
\tag{VIII.8.18}
$$

where for simplicity, we have omitted the subscript Ω. We now let $k \to \infty$ in (VIII.8.18), and using the fact that $\boldsymbol{f}_k \to \boldsymbol{f}$ in $L^q(\Omega)$ and $\boldsymbol{v}_{*k} \to \boldsymbol{v}_*$ in $W^{2-1/q,q}(\partial\Omega)$, we easily establish, again by a standard argument, that $\{(\boldsymbol{v}_k, p_k)\}$ tends, in the topology defined by the norms on the left-hand side of (VIII.8.18), to a pair $(\boldsymbol{v}, p) \in \mathcal{C}_q(\Omega)$ satisfying (VIII.0.7), (VIII.0.2). The proof of the theorem is thus complete. □

Remark VIII.8.2 We wish to clarify how the solutions of Theorem VIII.8.1 satisfy the condition at infinity (VIII.0.2). If $q \in (3/2, 2)$, since $\boldsymbol{v} \in D^{2,q}(\Omega)$, by the embedding Theorem II.6.1(i) it follows that $\boldsymbol{v} \in D^{1,q^*}(\Omega)$, for some $q^* > 3$. Therefore, since $\boldsymbol{v} \in L^{2q/(2-q)}(\Omega)$, from Theorem II.9.1 we deduce

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = \mathbf{0}\,, \quad \text{uniformly pointwise.}$$

If $q \in (1, 3/2]$, in view of the fact that $\boldsymbol{v} \in D^{1,4q/(4-q)}(\Omega)$, it follows that $\boldsymbol{v} \in D^{1,s}(\Omega)$, for some $s \in (4/3, 12/5]$. Consequently, from Lemma II.6.3 we have

$$\int_{S^2} |\boldsymbol{v}(|x|, \omega)|\, d\omega = o\left(|x|^{1-3/s}\right) \quad \text{as } |x| \to \infty\,.$$

■

Theorem VIII.8.1 immediately leads to the following corollary.

Corollary VIII.8.1 *Let Ω and $\mathcal{R}$ be as in* Theorem VIII.8.1. *Then, for any $\boldsymbol{f}, \boldsymbol{v}_*$ satisfying* (VIII.8.10), *problem* (VIII.0.7), (VIII.0.2) *has one and only one corresponding solution $(\boldsymbol{v}, p)$ in the class $\mathcal{C}_q(\Omega)$. Moreover, this solution satisfies the inequality* (VIII.8.11).

Finally, taking into account Remark VIII.1.1 and Theorem VIII.1.1, from Theorem VIII.8.1 we also at once deduce the following.

Corollary VIII.8.2 *Let Ω and $\mathcal{R}$ be as in* Theorem VIII.8.1, *and let $\boldsymbol{v}$ be a generalized solution to problem* (VIII.0.7), (VIII.0.2). *If $\boldsymbol{f}, \boldsymbol{v}_*$ satisfy* (VIII.8.10), *then $\boldsymbol{v}$ and its associated pressure p (by* Lemma VIII.1.1*) are in the class $\mathcal{C}_q(\Omega)$. Moreover, $(\boldsymbol{v}, p)$ satisfies the inequality* (VIII.8.11).

VIII.9 Notes for the Chapter

Section VIII.1. The first result on the existence of generalized solutions to (VIII.0.7), (VIII.0.2) can be traced back to the paper of Leray (1933, Chapter III); see also Weinberger (1973), Serre (1987), and Borchers (1992, Korollar 4.1).

Theorem VIII.1.2 is due to me.

Existence, uniqueness, and corresponding estimates of q-generalized solutions when $\mathcal{R} = 0$ were proved by Hishida (2006) when $\Omega = \mathbb{R}^3$ and, more generally, when Ω is a (smooth) exterior domain of $\mathbb{R}^3$. Hishida's results are formally analogous to those derived in Theorem IV.2.2 and Theorem V.5.1 for the Stokes problem, and in particular, if $\partial\Omega \neq \emptyset$, they require the restriction $q \in (3/2, 3)$. In the case $\Omega = \mathbb{R}^3$, Hishida's results have been extended to

$\mathcal{R} \neq 0$ by Kračmar, Nečasová, & Penel (2006). See also Kračmar, Nečasová, & Penel (2005, 2007, 2010).

Weak solutions in Lorentz spaces have been studied by Farwig & Hishida (2007) when $\mathcal{R} = 0$. Their results are the analogous counterpart, for $\mathcal{T} \neq 0$, of those obtained by Kozono & Yamazaki (1998) for the Stokes problem. In fact, they reduce to these latter when $\mathcal{T} = 0$.

As we mentioned in the Introduction, there has been very little contribution to the study of the generalized Oseen problem for $n = 2$. As a matter of fact, in two dimensions, the problem of existence of generalized solutions is either rather well known or very complicated. Actually, because of $n = 2$, we have to restrict ourselves to the case in which the angular velocity $\boldsymbol{\omega}$ is perpendicular to the plane, Π, that contains the relevant region of motion of the liquid. However, the translational velocity $\boldsymbol{v}_0$ must belong to Π and therefore is orthogonal to $\boldsymbol{\omega}$. Now, if $\boldsymbol{\omega} = \boldsymbol{0}$, we go back to the Oseen problem already treated in great detail and solved in the previous chapter. If, however, $\boldsymbol{\omega} \neq \boldsymbol{0}$, the Mozzi–Chasles transformation reduces the problem, formally, to the study of (VIII.0.7), (VIII.0.2) with $\mathcal{R} = 0$ and with $\boldsymbol{v} = (v_2(x_2, x_3), v_3(x_2, x_3))$, $p = p(x_2, x_3)$. For this latter, using the same method employed in the proof of Theorem VIII.1.2, one can prove the existence of a vector field $\boldsymbol{v} \in D^{1,2}(\Omega)$ that satisfies (VIII.1.1) (with $\mathcal{R} = 0$) along with properties (ii) and (iii) of Definition VIII.1.1. However, as in the analogous Stokes problem considered in Chapter V, with this information alone one cannot ensure that the velocity field tends (even in a weak sense) to a prescribed limit at infinity. In other words, one cannot exclude the occurrence of a "Stokes paradox." It is interesting to observe that, in the case that Ω is the exterior of a circle and $\boldsymbol{f} \equiv \boldsymbol{0}$, $\boldsymbol{v}_* = \boldsymbol{e}_1 \times \boldsymbol{x}$, the problem has the explicit solution given in (V.0.8) (with $\boldsymbol{\omega} \equiv \boldsymbol{e}_1$). This fact would suggest that a solution might exists also when Ω is a more general (sufficiently smooth) exterior domain and with more general data, possibly satisfying a suitable compatibility condition, but no proof is, to date, available. We end these considerations by mentioning the paper by Farwig, Krbec, & Nečasová (2008) in which existence is investigated in the case $\Omega = \mathbb{R}^2$.

Section VIII.2. All main results and, in particular, the uniqueness Theorem VIII.2.1 are due to me. Seemingly, this is the first (and only one, to my knowledge) uniqueness theorem of generalized solutions in their own class. The main tool is the proof that the pressure possesses suitable summability properties in a neighborhood of infinity (see Lemma VIII.2.2).

The "regularization" result for generalized solutions obtained in Lemma VIII.2.1 is also new. However, the *existence* of a generalized solution satisfying the properties stated in Lemma VIII.2.1, including the estimate (VIII.2.3), can also be proved by the method of "invading domains" adopted by Silvestre (2004).

Section VIII.3. The proofs of the main results (Lemma VIII.3.1, Lemma VIII.3.4, and Lemma VIII.3.6) are taken (and expanded) from the work of

Galdi & Silvestre (2007a, 2007b). A different proof of Lemma VIII.3.1 can also be found in Mizumachi (1984, Lemma 2).

We would like to emphasize that Lemma VIII.3.1 admits also of a two-dimensional counterpart. Specifically, one could show that for $\mathcal{R} = 0$, which, as we observed previously, is the only relevant case for two dimensions, the fundamental tensor solution, $\boldsymbol{\Gamma}(\xi, \tau)$, satisfies the following estimates

$$\left.\begin{aligned} \varGamma_0(\xi) &\equiv \int_0^\infty |\boldsymbol{\Gamma}(\xi,\tau)|\, d\tau \le c\,|\log|\xi|\,|\,, \\ \varGamma_1(\xi) &\equiv \int_0^\infty |\nabla_\xi \boldsymbol{\Gamma}(\xi,\tau)|\, d\tau \le c\,|\xi|^{-1}\,, \end{aligned}\right\} \quad \xi \neq 0\,, \tag{VIII.8.1}$$

which coincide with the estimates satisfied by the Stokes fundamental tensor solution $\boldsymbol{U}$ in dimension 2 (see (IV.2.6)).

Section VIII.4. Theorem VIII.4.1 is basically due to O.A. Ladyzhenskaya. Its proof appears for the first time in English in Chapter 4, §5, of the first edition, dated 1963, of Ladyzhenskaya (1969).

Lemma VIII.4.2, in its generality, seems to be new and is due to me.

Theorem VIII.4.4, which represents the main contribution of the section, is taken from Galdi & Silvestre (2007b). In this regard, and in connection with the remarks made in the Notes to Section VIII.1, we would like to emphasize that the method used in the proof of Theorem VIII.4.4 fails in dimension $n = 2$. In fact, we recall that for $n = 2$, we have to consider only the case $\mathcal{R} = 0$, and in such a case, according to (VIII.8.1), the function $\varGamma_0(\xi)$ is bounded above by $\log|\xi|$, for large $|\xi|$. This prevents us from proving the existence of solutions to (VIII.4.1) that are spatially decaying, even for a right-hand side $\boldsymbol{f}$ with compact support in the space variables.

Section VIII.5. The entire approach described in this section is due to Galdi (2002) and was further developed by Galdi & Silvestre (2007a, 2007b). In particular, Theorem VIII.5.1 is proved in Galdi & Silvestre (2007b).

Section VIII.6. Theorem VIII.6.1 and Theorem VIII.6.2 are improved versions of analogous results derived in Galdi & Silvestre (2007a, 2007b).

Section VIII.7. The uniqueness results proved in Lemma VIII.7.1 and Theorem VIII.7.1 are due to me. They also appear, in a slightly different form, in Galdi & Kyed (2011a).

Existence and associated estimates of solutions in homogeneous Sobolev spaces corresponding to $\boldsymbol{f}$ in L^q were first proved by Hishida (1999a, 1999b)) for $q = 2$. Specifically, Hishida (1999b, Proposition 3.1) shows, with $\varOmega = \mathbb{R}^3$, that for every $\boldsymbol{f} \in L^2(\mathbb{R}^3)$ there exists one solution to (VIII.0.7) belonging to $D^{2,2}(\mathbb{R}^3)$ and satisfying corresponding estimates. Hishida's result was extended to the case $\mathcal{R} \neq 0$ by Galdi (2002, Lemma 4.14), who also proves uniqueness.

The first complete treatment of the problem for arbitrary $q \in (1, \infty)$ and $\varOmega = \mathbb{R}^n$ is due to Farwig, Hishida, & Müller (2004) (for both cases $n = 2, 3$).

For $n = 3$ their existence result, based on multiplier and Littlewood–Paley theories, essentially coincides with that given in Lemma VIII.7.2, which is, however, established by a completely different approach.

Theorem VIII.7.2 is due to me.

Section VIII.8. Existence and corresponding L^q-estimates when $\Omega = \mathbb{R}^3$ were first derived by Farwig (2006). It should be noted that, as shown in the proof of Lemma VIII.8.2, the estimates associated with the first derivatives of the velocity field of solutions constructed by Farwig have the puzzling (but, apparently, necessary) feature that the constant involved depends on the (square of the) inverse of the Taylor number $\mathcal{T}$. Therefore, it becomes unbounded as $\mathcal{T} \to 0$, even though one can easily show that as $\mathcal{T} \to 0$, the corresponding solutions tend, in fact, in a suitable sense, to the solution of the problem with $\mathcal{T} = 0$. As proved in Farwig (2005), this feature can be removed if the first derivatives are estimated in suitable weighted homogeneous Sobolev spaces.

Theorem VIII.8.1 is due to me.

IX

Steady Navier–Stokes Flow in Bounded Domains

Καὶ ἠγάπησαν οἱ ἄνθρωποι
μᾶλλον τὸ φῶs η τὸ σκότος.

Introduction

The objective of this and the following chapters is the study of steady motions of a viscous incompressible fluid described by the full nonlinear Navier–Stokes system. In the present chapter we shall focus on the case where the region of flow, Ω, is *bounded.* More specifically, we shall analyze the boundary value problem obtained by coupling the following system

$$\left.\begin{aligned} &\nu \Delta \boldsymbol{v} = \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla p + \boldsymbol{f} \\ &\nabla \cdot \boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega \tag{IX.0.1}$$

with the adherence boundary condition:

$$\boldsymbol{v} = \boldsymbol{v}_* \text{ at } \partial\Omega, \tag{IX.0.2}$$

where $\boldsymbol{v}_*$ a prescribed field.

The first and fundamental contribution to the resolution of (IX.0.1), (IX.0.2) for Ω bounded, is due to F.K.G. Odqvist in 1930. The method used by this author articulates into several steps. First, he transforms (IX.0.1), (IX.0.2) into a (nonlinear) integral equation, by means of Green's tensor $\boldsymbol{G}$ associated to the corresponding linearized Stokes problem (see Section IV.6);

G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems*, Springer Monographs in Mathematics,
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then he derives suitable estimates for $\boldsymbol{G}$ and combines these with a successive approximation technique that, finally, produces existence. However, for this procedure to work it is necessary to *restrict appropriately the size of the data.* At the time when Odqvist derived his results, such a restriction was, in a sense, expected from both the physical and the mathematical points of view. In fact, on the one hand, it confirmed the idea that the agreement between the Navier–Stokes theory and experiment should hold only at "small" Reynolds numbers.[1] On the other hand, the highly nonlinear character of the problem enhanced the possibility of solving it only "locally." Therefore, when in his celebrated paper of 1933, Jean Leray proved existence of solutions to (IX.0.1), (IX.0.2) *without restrictions on the size of the data*, the result sounded absolutely remarkable, since it was not predictable from known experimental observation, nor was it obvious from the structure of the equations. Leray's proof was based on the discovery that every solution to (IX.0.1), (IX.0.2) formally obeys the following a priori estimate, whatever the size of the data may be:

$$\int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{v} \leq M,$$

where M depends only on the data $\boldsymbol{f}$, $\boldsymbol{v}_*$ and on Ω and ν (see Section IX.3 and Section IX.4). Such a uniform bound along with the Odqvist estimate for Green's tensor and with a new method for determining fixed points of nonlinear maps in Banach spaces[2] allowed Leray to show the stated result; see Leray (1933, 1936). This outstanding achievement, however, presents an undesired feature in the case when the boundary $\partial\Omega$ has more than one connected component, say, Γ_i, $i = 1, ..., m$. In such a case, the compatibility condition on the velocity $\boldsymbol{v}_*$ at the boundary, required by the incompressibility of the fluid, is

$$\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} = \sum_{i=1}^{m} \int_{\Gamma_i} \boldsymbol{v}_* \cdot \boldsymbol{n} = 0, \tag{IX.0.3}$$

with $\boldsymbol{n}$ unit outer normal to $\partial\Omega$, while Leray's argument works if the condition

$$\int_{\Gamma_i} \boldsymbol{v}_* \cdot \boldsymbol{n} = 0 \quad i = 1, ..., m \tag{IX.0.4}$$

is imposed (see also Hopf 1941, 1957). Clearly, if $m > 1$, (IX.0.4) *is* stronger than (IX.0.3) and, in particular, it does not allow for the presence of extended "sinks" and "sources" into the region of flow. The question of whether problem (IX.0.1), (IX.0.2) admits a solution only under the natural restriction

[1] This view relies on the experimental evidence that "laminar motions" such as Couette or Poiseuille flows are observed only for small values of the Reynolds number $\mathcal{R}$ ("large" viscosity ν) even though, as mathematical solutions, they exists for *all values of* $\mathcal{R}$. As is known, this view has been modified in light of the modern theory of bifurcation.

[2] This method will lead to the celebrated Leray-Schauder theorem, see Leray & Schauder (1934).

(IX.0.3) was left out by Leray and it still is a fundamental open question in the mathematical theory of the Navier–Stokes equations.

In this chapter we study existence, uniqueness, and regularity of solutions to problem (IX.0.1), (IX.0.2) with Ω bounded.

Following Ladyzhenskaya (1959b), we introduce a variational formulation of (IX.0.1), (IX.0.2) and corresponding generalized (weak) solutions. The technique we shall use for existence of generalized solutions is different (and simpler) than Leray's and is based on a variant of the Galerkin method we used for the linearized Oseen problem in Section VII.2. This approach is due to Fujita (1961) and, independently, to Vorovich & Youdovich (1961); see also Finn (1965b, §2.7). Moreover, in place of condition (IX.0.4), we prove existence under the following weaker assumption on the flux of $\boldsymbol{v}_*$ through Γ_i:

$$\sum_{i=1}^{m} c_i \left| \int_{\Gamma_i} \boldsymbol{v}_* \cdot \boldsymbol{n} \right| < \nu$$

where c_i, $i = 1, ..., m$, are computable constants depending on the domain Ω; see Galdi (1991).[3]

Concerning uniqueness, we obtain, as expected, that generalized solutions are unique only for "small" data (or, equivalently, for "large" viscosity ν). We also give examples of non-uniqueness for "large" data (or, equivalently, for "small" ν). These examples are based on ideas of Youdovich (1967) and, to the best of our knowledge, they are the only ones so far explicitly given for the steady-state Navier–Stokes problem in a bounded domain with adherence boundary conditions.[4]

Regularity of a generalized solution is investigated by means of a general technique which, in fact, allows us to study regularity of a much larger class of solutions (in arbitrary dimension $n \geq 2$).

Finally, we analyze the behavior of generalized solutions in the limit of large viscosity ν (small Reynolds number). Specifically, we shall show that, under suitable regularity assumptions on the data, such solutions tend uniformly pointwise, as $\nu \to \infty$, to solutions of the Stokes problem corresponding to the same data.

As in the linearized approximations, also for the full nonlinear equations we shall mainly be concerned with the physically interesting cases when Ω is a three-dimensional or (for plane flow) two-dimensional domain. However, we shall also explicitly remark if and in what form a result can be extended to spatial dimension $n > 3$. In this respect, we wish to notice that all main results we prove in this chapter for $n = 2, 3$ carry over (more or less simply)

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[3] See also Borchers and Pileckas (1994) and the more recent results of Kozono and Yanagisawa (2009a, 2009b). For other results under the general assumption (IX.0.3), we refer the reader to the Notes for this Chapter.

[4] For non-uniqueness, and even bifurcation, of steady-state solutions under periodic boundary conditions, we refer to Zeidler (1997, §§ 72.7–72.8), where a detailed analysis of these phenomena is performed for the so called Taylor-Couette flow.

to the case where $n = 4$. If $n \geq 5$ new difficulties appear that are not easily attacked by the known methods. This is because a generalized solution has a priori only the property of being in the Sobolev space $W^{1,2}(\Omega)$, and this is not enough, by use of the embedding theorems, to dominate appropriately the nonlinear term in $(\text{IX.0.1})_1$. For instance, whether or not *any* generalized solution in higher dimension corresponding to regular data of arbitrary size is regular, remains an open question.[5] However, Frehse and Růžička (1994a, 1994b, 1995), and Struwe (1995) have shown the existence of regular solutions in dimension $n \geq 5$ without restrictions on the size of the data. We refer the reader to these papers and to the review article by Frehse and Růžička (1996) for details and further related literature.

IX.1 Generalized Solutions. Preliminary Considerations

In the present section we begin to introduce the weak (or generalized) formulation of the steady Navier–Stokes problem in a bounded domain and to investigate some of the basic properties of the associated solutions. In doing this we will essentially pattern the same lines followed for the linear problem in Section IV.1.

Following Ladyzhenskaya (1959b), we give a *weak (or variational) formulation* of (IX.0.1). Let $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. Assuming $\boldsymbol{v} \in C^2(\Omega)$, $p \in C^1(\Omega)$, $\boldsymbol{f} \in C(\Omega)$, if we multiply $(\text{IX.0.1})_1$ by $\boldsymbol{\varphi}$ and integrate by parts over Ω we obtain

$$\nu \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{\varphi} + \int_\Omega \boldsymbol{v} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{\varphi} = - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}. \tag{IX.1.1}$$

Thus every classical solution to (IX.0.1) satisfies (IX.1.1). Conversely, if $\boldsymbol{v} \in C^2(\Omega)$ satisfies (IX.1.1) for some $\boldsymbol{f} \in C(\Omega)$ and for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$, then

$$\int_\Omega (\nu \Delta \boldsymbol{v} - \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \boldsymbol{f}) \cdot \boldsymbol{\varphi} = 0$$

and we use Lemma III.1.1 to show that $\boldsymbol{v}$ obeys $(\text{IX.0.1})_1$ for suitable $p \in C^1(\Omega)$. However, if $\boldsymbol{v}$ merely satisfies (IX.1.1) for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and is not sufficiently differentiable, we cannot go from (IX.1.1) to $(\text{IX.0.1})_1$, and therefore (IX.1.1) is to be interpreted as a generalized version of $(\text{IX.0.1})_1$.

As in the linear case, it is convenient to investigate the more general situation, where the right-hand side of (IX.1.1) is defined by a functional $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$.

[5] If $n = 5$, a *partial* regularity result for generalized solutions can be found in Struwe (1988). Extension of this result to arbitrary dimension is due to Tian & Xin (1999). It should be added that, if the size of the data is suitably restricted, any generalized solution satisfying the so-called *energy inequality* corresponding to smooth data is smooth as well, for any $n \geq 5$; see Remark IX.5.5.

With a view to all this and in analogy with Definition IV.1.1, we give the following definition of generalized solution to (IX.0.1), (IX.0.2).

Definition IX.1.1. Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 2$. A field $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ is called a *weak* (or *generalized*) *solution to the Navier–Stokes problem* (IX.0.1), (IX.0.2) if and only if

(i) $\boldsymbol{v} \in D^{1,2}(\Omega)$;
(ii) $\boldsymbol{v}$ is (weakly) divergence-free in Ω;
(iii) $\boldsymbol{v}$ satisfies boundary condition (IX.0.2) (in the trace sense) or, if $\boldsymbol{v}_* \equiv 0$, then $\boldsymbol{v} \in D_0^{1,2}(\Omega)$;
(iv) $\boldsymbol{v}$ obeys the identity

$$\nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) + (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi}) = -\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle \tag{IX.1.2}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$.

Remark IX.1.1 Since every function in $D^{1,2}(\Omega)$ is also in $W^{1,2}_{loc}(\Omega)$ (see Lemma II.6.1), from the inequality

$$|(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi})| \leq \sup_{\Omega_0} |\boldsymbol{\varphi}| \, \|\boldsymbol{v}\|^2_{1,2,\Omega_0}, \quad \Omega_0 \equiv \operatorname{supp}(\boldsymbol{\varphi}),$$

we see that identity (IX.1.2) is meaningful, whatever the regularity of Ω may be. ■

Remark IX.1.2 Remark IV.1.1 and Remark IV.1.2, with $q = 2$, equally apply to generalized solutions of the Navier–Stokes problem (IX.0.1), (IX.0.2). ■

Remark IX.1.3 In analogy with the Stokes problem, one may give the definition of *q-weak* (or *q-generalized*) *solution*, by replacing in (i) and (iii) $D^{1,2}(\Omega)$ with $D^{1,q}(\Omega), 1 < q < \infty$. Of course, q should be chosen in such a way that the nonlinear term in (IX.1.2) is meaningful. This happens whenever $q \geq 2n/(n+2)$. In fact, by an integration by parts, we have

$$|(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi})| = |(\boldsymbol{v} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{v})| \leq \sup_{\Omega_0} |\nabla \boldsymbol{\varphi}| \, \|\boldsymbol{v}\|_{2,\Omega_0}$$

with Ω_0 as in Remark IX.1.1. From Lemma II.6.1, $\boldsymbol{v} \in W^{1,q}(\Omega_0)$ and so, by Theorem II.3.4, it follows that $\boldsymbol{v} \in L^s(\Omega)$ for all $s \geq 1$, if $q \geq n$, and for all $s \in (1, nq/(n-q)]$, if $q < n$. Therefore, if $q \geq n$, then $\boldsymbol{v} \in L^2(\Omega_0)$, while if $q < n$ this latter circumstance happens if $2 \leq nq/(n-q)$, that is, $q \geq 2n/(n+2)$. ■

Existence and uniqueness of generalized solutions in a bounded domain will be the object of the next sections. In the remaining part of this section we wish to point out some notable facts relating to them, which we shall collect in the following lemmas.

Lemma IX.1.1 *Let Ω be an arbitrary bounded domain of $\mathbb{R}^n$, $n = 2, 3$. Then the trilinear form*

$$a(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \equiv (\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{w}) \tag{IX.1.3}$$

is continuous in the space

$$W_0^{1,2}(\Omega) \times W^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$$

and we have

$$|a(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le k |\boldsymbol{u}|_{1,2} |\boldsymbol{v}|_{1,2} |\boldsymbol{w}|_{1,2} \tag{IX.1.4}$$

where

$$k = \begin{cases} \dfrac{2\sqrt{2}|\Omega|^{1/6}}{3} & \textit{if } \ n = 3 \\ \dfrac{|\Omega|^{1/2}}{2} & \textit{if } \ n = 2. \end{cases} \tag{IX.1.5}$$

Furthermore, if Ω is bounded and locally Lipschitz, then a is continuous in the space

$$W^{1,2}(\Omega) \times W^{1,2}(\Omega) \times W_0^{1,2}(\Omega).$$

Thus, in particular, every weak solution corresponding to the data

$$\boldsymbol{f} \in D_0^{-1,2}(\Omega), \quad \boldsymbol{v}_* \equiv 0,$$

satisfies (IX.1.2) *for all $\boldsymbol{\varphi} \in H^1(\Omega)$. The same property holds when $\boldsymbol{v}_* \not\equiv 0$, if Ω is bounded and locally Lipschitz.*

Proof. The proof of the second part of the lemma concerning weak solutions is a consequence of the first, since $(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi})$ and $\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle$ are bounded linear functionals in $\boldsymbol{\varphi} \in H^1(\Omega)$ and, under the stated assumptions, $\boldsymbol{v} \in W^{1,2}(\Omega)$ so that also $(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi})$ is continuous in $\boldsymbol{\varphi} \in H^1(\Omega)$. In order to show the continuity of $a(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$, assume first that $n = 3$ and $\boldsymbol{u} \in W_0^{1,2}(\Omega)$. From the Hölder inequality

$$|a(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le \|\boldsymbol{u}\|_{2n/(n-2)} |\boldsymbol{v}|_{1,2} \|\boldsymbol{w}\|_n \tag{IX.1.6}$$

so that the Sobolev inequality (II.3.7) implies

$$|a(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le (2/\sqrt{3}) |\boldsymbol{u}|_{1,2} |\boldsymbol{v}|_{1,2} \|\boldsymbol{w}\|_n. \tag{IX.1.7}$$

However, since $n = 3$, Lemma II.3.2 and (II.5.5) furnish

$$\|\boldsymbol{w}\|_n \le k_1 |\boldsymbol{w}|_{1,2} \tag{IX.1.8}$$

with

$$k_1 = \sqrt{2}|\Omega|^{1/6}/\sqrt{3}.$$

Placing (IX.1.8) into (IX.1.7) gives (IX.1.4). Assuming $\boldsymbol{u} \in W^{1,2}(\Omega)$ with Ω locally Lipschitz, from the embedding Theorem II.3.4 we obtain

$$\|\boldsymbol{u}\|_{2n/(n-2)} \le c\|\boldsymbol{u}\|_{1,2} \tag{IX.1.9}$$

for some $c = c(\Omega, n)$; thus inequality (IX.1.7) becomes

$$|a(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le c\|\boldsymbol{u}\|_{1,2}|\boldsymbol{v}|_{1,2}\|\boldsymbol{w}\|_n$$

and the continuity of a follows from this last relation and from (IX.1.8). Consider, next, $n = 2$. By the Hölder inequality we have

$$|a(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le \|\boldsymbol{u}\|_r|\boldsymbol{v}|_{1,2}\|\boldsymbol{w}\|_s, \quad r^{-1} + s^{-1} = 2^{-1} \tag{IX.1.10}$$

and therefore, choosing, for instance, $r = s = 4$, since by Lemma II.3.1 and (II.5.6) we have

$$\|f\|_4 \le (1/\sqrt{2})|\Omega|^{1/4}|f|_{1,2}, \quad \text{for all } f \in W_0^{1,2}(\Omega),$$

(IX.1.4) becomes a consequence of this last relation and of (IX.1.10). Finally, if $\boldsymbol{u} \in W^{1,2}(\Omega)$, inequality (IX.1.4) with a suitable constant $k = k(\Omega, n)$ is secured from (IX.1.9) and from the following one

$$\|f\|_4 \le c\|f\|_{1,2},$$

which is proved in the embedding Theorem II.3.4, for some $c = c(\Omega, n)$. The proof of the lemma is thus complete. □

Remark IX.1.4 If $n = 4$, Lemma IX.1.1 continues to hold with $k = (3/4)^2$, since, in such a case, inequality (IX.1.8) remains valid with $k_1 = 3/4$. (Notice that $2n/(n-2) = n$, for $n = 4$). If $n \ge 5$ (IX.1.8) *no longer holds* and $a(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ is continuous in the space

$$W_0^{1,2}(\Omega) \times W^{1,2}(\Omega) \times [W_0^{1,2}(\Omega) \cap L^n(\Omega)]$$

[respectively in the space

$$W^{1,2}(\Omega) \times W^{1,2}(\Omega) \times [W_0^{1,2}(\Omega) \cap L^n(\Omega)]$$

if Ω is (bounded and) locally Lipschitz]. Consequently, in such a case, in the second part of the lemma, the statement

$$\boldsymbol{\varphi} \in H^1(\Omega)$$

should be replaced by

$$\boldsymbol{\varphi} \in \widetilde{H}^1(\Omega),$$

where $\widetilde{H}^1(\Omega)$ is the completion of $\mathcal{D}(\Omega)$ in the norm

$$\|\boldsymbol{\varphi}\|_{\widetilde{1,2}} \equiv \|\boldsymbol{\varphi}\|_{1,2} + \|\boldsymbol{\varphi}\|_n.$$

Other continuity properties of the trilinear form a will be given in Exercise IX.2.1 and Lemma X.2.1. ■

As in the linear case (see Lemma IV.1.1), the next result shows, in particular, that to every generalized solution $\boldsymbol{v}$, it is possible to associate a suitable pressure field p.

Lemma IX.1.2 *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n = 2, 3$, and let*

$$\boldsymbol{f} \in W_0^{-1,2}(\Omega') \quad \text{for any bounded domain } \Omega', \text{ with } \overline{\Omega}' \subset \Omega.$$

Then a vector field $\boldsymbol{v} \in W^{1,2}_{loc}(\Omega)$ satisfies (IX.1.2) *for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ if and only if there is $p \in L^2_{loc}(\Omega)$ satisfying the identity*

$$\nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) + (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\psi}) = (p, \nabla \cdot \boldsymbol{\psi}) - \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle \tag{IX.1.11}$$

for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. If, moreover, Ω is bounded and locally Lipschitz and

$$\boldsymbol{f} \in D_0^{-1,2}(\Omega), \quad \boldsymbol{v} \in D^{1,2}(\Omega),$$

then

$$p \in L^2(\Omega) \ \text{ with } \ \int_\Omega p = 0,$$

and (IX.1.11) *holds for all $\boldsymbol{\psi} \in W_0^{1,2}(\Omega)$.*

Proof. Clearly, (IX.1.11) implies (IX.1.2). Thus assume (IX.1.2) and Ω locally Lipschitzian. By Lemma IX.1.1 the functional

$$\mathcal{F}(\boldsymbol{\psi}) \equiv \nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) + (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\psi}) + \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle \tag{IX.1.12}$$

is (linear and) bounded in $\boldsymbol{\psi} \in W_0^{1,2}(\Omega)$ and vanishes when $\boldsymbol{\psi} \in \mathcal{D}_0^{1,2}(\Omega)$ $(= H^1(\Omega))$. Therefore, by Corollary III.5.1 there exists $p \in L^2(\Omega)$ such that

$$\mathcal{F}(\boldsymbol{\psi}) = (p, \nabla \cdot \boldsymbol{\psi}), \tag{IX.1.13}$$

for all $\boldsymbol{\psi} \in D_0^{1,2}(\Omega)$ $(= W_0^{1,2}(\Omega))$, thus satisfying (IX.1.11). If Ω is an *arbitrary* domain, we use Corollary III.5.2 to deduce the existence of $p \in L^2_{loc}(\Omega)$ satisfying (IX.1.13) for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. The proof is thus complete. □

Remark IX.1.5 If $n = 4$, Lemma IX.1.2 continues to hold since $(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\psi})$ is still a bounded functional in $\boldsymbol{\psi} \in W_0^{1,2}(\Omega)$, see Remark IX.1.4. If $n \geq 5$ this property no longer holds and Lemma IX.1.2 fails, unless $\boldsymbol{v} \in L^n(\Omega)$. Nevertheless, if

$$\boldsymbol{f} \in D_0^{-1,n/(n-2)}(\Omega),$$

we can again define a pressure field

$$p \in L^{n/(n-2)}(\Omega)$$

satisfying (IX.1.11) if Ω is bounded and locally Lipschitz. If Ω has no regularity or if

$$\boldsymbol{f} \in W_0^{-1,n/(n-2)}(\Omega'), \quad \text{for all domains } \Omega' \text{ with } \overline{\Omega}' \subset \Omega,$$

we then only have

$$p \in L^{n/(n-2)}_{loc}(\Omega).$$

Actually, if Ω is locally Lipschitz, $n \geq 5$, and $\boldsymbol{v}$ is a generalized solution, then

$$\boldsymbol{v} \in W^{1,n/(n-2)}(\Omega).$$

Furthermore, by inequalities (IX.1.6), (IX.1.9), and the Sobolev inequality (II.3.7) we deduce

$$|(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\psi})| \leq c\|\boldsymbol{v}\|_{1,2}|\boldsymbol{\psi}|_{1,n/2}.$$

As a consequence, the functional (IX.1.12) is bounded for $\boldsymbol{\psi} \in D_0^{1,n/2}(\Omega)$ $(=W_0^{1,n/2}(\Omega))$ and vanishes when $\boldsymbol{\psi} \in \mathcal{D}_0^{1,n/2}(\Omega)$ $(=H^1_{n/2}(\Omega))$. By Corollary III.5.1 we then conclude the validity of (IX.1.13) for some $p \in L^{n/(n-2)}(\Omega)$ and all $\boldsymbol{\psi} \in D_0^{1,n}(\Omega)$ $(=W_0^{1,n}(\Omega))$. If Ω has no regularity, by these arguments and Corollary III.5.2 we derive $p \in L^{n/(n-2)}_{loc}(\Omega)$. ■

IX.2 On the Uniqueness of Generalized Solutions

We shall begin to establish a general uniqueness result for weak solutions in a bounded domain. To this end, we need some further information about the trilinear form $a(\boldsymbol{u}, \boldsymbol{v},\boldsymbol{w})$ defined in Lemma IX.1.1.

Lemma IX.2.1 *Let Ω be a bounded locally Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$, and let $\boldsymbol{v} \in W^{1,2}(\Omega)$ with $\nabla \cdot \boldsymbol{v} = 0$. Then*

$$a(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{u}) = 0 \quad \text{for all } \ \boldsymbol{u} \in W_0^{1,2}(\Omega). \tag{IX.2.1}$$

Thus, in particular,

$$a(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w}) = -a(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u}) \quad \text{for all } \ \boldsymbol{u}, \boldsymbol{w} \in W_0^{1,2}(\Omega). \tag{IX.2.2}$$

If $\boldsymbol{v} \in H^1(\Omega)$, the same conclusions hold with no regularity on Ω.

Proof. Property (IX.2.2) is a direct consequence of (IX.2.1) when we replace the function $\boldsymbol{u}$ in it with $\boldsymbol{u}+\boldsymbol{w}$ and use the multilinear properties of a. To prove (IX.2.1), in view of Lemma IX.1.1, it is enough to prove it for $\boldsymbol{u} \in C_0^\infty(\Omega)$. However, for such functions we have

$$a(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{u}) = \int_\Omega \boldsymbol{v} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u} = \frac{1}{2} \int_\Omega \boldsymbol{v} \cdot \nabla u^2 = 0$$

since $\boldsymbol{v}$ is weakly divergence-free. The lemma is thus proved. □

Remark IX.2.1 If $n = 4$ the preceding lemma remains unchanged. If $n \geq 5$ it holds, by replacing $W_0^{1,2}(\Omega)$ with $\widetilde{W}_0^{1,2}(\Omega)$, where this latter space is defined as the completion of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|\widetilde{}_{1,2}$ introduced in Remark IX.1.4. ■

Exercise IX.2.1 Let Ω be a domain of $\mathbb{R}^n$, $n \geq 2$. Show that the trilinear form (IX.1.3) is continuous in $L^q(\Omega) \times W^{1,r}(\Omega) \times L^{qr'/(q-r')}(\Omega)$, $q \in (1,\infty)$, $r \in (q/(q-1),\infty)$. Thus, assuming $\boldsymbol{v} \in \widetilde{H}_s(\Omega)$[1] with $\nabla \cdot \boldsymbol{v} = 0$, show that the conclusions of Lemma IX.2.1 continue to hold under any of the following assumptions

(i) $s = n$, if $n \geq 3$, and $\boldsymbol{u}, \boldsymbol{w} \in W_0^{1,2}(\Omega)$;
(ii) $s > n$, if $n = 2$, and $\boldsymbol{u}, \boldsymbol{w} \in W_0^{1,2}(\Omega)$;
(iii) $s < n$, if $n = 2$, and $\boldsymbol{u}, \boldsymbol{w} \in W_0^{1,\sigma}(\Omega)$, $\sigma > s'$.

Let us next observe that, in view of Lemma IX.1.1 it follows, in particular, that

$$|(\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{u})| \leq k|\boldsymbol{v}|_{1,2}|\boldsymbol{u}|_{1,2}^2 \tag{IX.2.3}$$

for all $\boldsymbol{v} \in W^{1,2}(\Omega), \boldsymbol{u} \in H^1(\Omega)$, and with k defined in (IX.1.5). We are now in a position to prove the following uniqueness theorem.

Theorem IX.2.1 *Let Ω be a bounded locally Lipschitz domain in $\mathbb{R}^n$, $n = 2,3$, and let $\boldsymbol{v}$ be a generalized solution to* (IX.0.1), (IX.0.2) *corresponding to $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$ and $\boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega)$. If we denote by $\boldsymbol{w}$ another generalized solution corresponding to the same data, $\boldsymbol{v} \equiv \boldsymbol{w}$, provided that*

$$|\boldsymbol{v}|_{1,2} < \nu/k \tag{IX.2.4}$$

where k is defined in (IX.1.5).

Proof. Let $\boldsymbol{u} \equiv \boldsymbol{w} - \boldsymbol{v}$. From (IX.1.2) and Lemma IX.1.1 we deduce that $\boldsymbol{u}$ satisfies the following identity

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{\varphi}) + (\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi}) + (\boldsymbol{v} \cdot \nabla \boldsymbol{u}, \boldsymbol{\varphi}) = 0 \tag{IX.2.5}$$

for all $\boldsymbol{\varphi} \in H^1(\Omega)$. Clearly, $\mathbf{u}$ has zero trace at the boundary and, consequently, from Remark IX.1.2 and Theorem II.4.2, we have $\boldsymbol{u} \in W_0^{1,2}(\Omega)$. Since $\boldsymbol{u}$ is weakly divergence-free and Ω locally Lipschitz, from the results of Section III.4.1 it follows that $\boldsymbol{u} \in H^1(\Omega)$. We may thus substitute $\boldsymbol{\varphi}$ with $\boldsymbol{u}$ into (IX.2.5) and employ Lemma IX.2.1 to obtain

$$\nu|\boldsymbol{u}|_{1,2}^2 + (\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{u}) = 0. \tag{IX.2.6}$$

Using estimate (IX.2.3) in (IX.2.6) yields

$$(\nu - k|\boldsymbol{v}|_{1,2})|\boldsymbol{u}|_{1,2}^2 \leq 0$$

which, in turn, by (IX.2.4) implies uniqueness. □

[1] We recall that this space is defined in (III.2.4).

Remark IX.2.2 If Ω has no regularity the procedure adopted earlier a priori *does not furnish uniqueness even in the case of zero boundary data.* This fact is related to the definition of generalized solution given by me in Section IX.1. In fact, if $\boldsymbol{v}_* = 0$, we require that $\boldsymbol{v}$ belongs to the space $\widehat{\mathcal{D}}_0^{1,2}(\Omega)$ which, a priori is *larger* than $\mathcal{D}_0^{1,2}(\Omega)$, if Ω has no regularity (cf. Section III.5). Therefore, the field $\boldsymbol{u}$ introduced in the proof of Theorem IX.2.1 cannot be put in place of $\boldsymbol{\varphi}$ in the identity (IX.1.7) and we can no longer deduce uniqueness. Nevertheless, uniqueness can be still recovered in the (a priori smaller) class of generalized solutions belonging to the space $\mathcal{D}_0^{1,2}(\Omega)$. ∎

Remark IX.2.3 If $n = 4$ Theorem IX.2.1 continues to hold. Its proof, however, does not apply if $n \geq 5$ since, in this case, identity (IX.2.5) is valid for $\boldsymbol{\varphi} \in \widetilde{H}^1(\Omega)$ (cf. Remark IX.1.4) and we cannot take $\boldsymbol{\varphi} = \boldsymbol{u}$, for $\boldsymbol{u}$ does not belong a priori to $\widetilde{H}^1(\Omega)$. Nevertheless, as we shall see in Remark IX.5.5, in dimension $n \geq 5$ one can construct more regular solutions (at the cost, however, of imposing some restriction on the size of the data) and to discuss their uniqueness. For the existence of regular solutions for $n \geq 5$ without restrictions on the size of the data, we refer the reader to Frehse and Růžička (1994a, 1994b, 1995, 1996), and Struwe (1995). ∎

Because of the nonlinearity of the Navier–Stokes equations, we would expect that if a solution does not satisfy condition (IX.2.4), namely, if the coefficient of kinematic viscosity ν is not "sufficiently large," then uniqueness may be lost. Employing the ideas of Youdovich (1967), in the remaining part of this section we shall prove that this is indeed the case. We begin with the following result.

Lemma IX.2.2 *Let Ω be a domain of $\mathbb{R}^n$, $n \geq 2$, and let $\boldsymbol{a}$ be a smooth solenoidal vector in $\overline{\Omega}$. Moreover, assume that there is $\mu > 0$ for which the following problem admits a nontrivial solution $\boldsymbol{\varphi}$, τ*

$$\left.\begin{aligned} &\frac{1}{\mu}\Delta\boldsymbol{\varphi} = \boldsymbol{a}\cdot\nabla\boldsymbol{\varphi} + \boldsymbol{\varphi}\cdot\nabla\boldsymbol{a} + \nabla\tau \\ &\nabla\cdot\boldsymbol{\varphi} = 0 \end{aligned}\right\} \text{ in } \Omega \qquad \boldsymbol{\varphi} = 0 \ \text{ at } \partial\Omega. \tag{IX.2.7}$$

Then, there are vector fields $\boldsymbol{F}$ and $\boldsymbol{v}_*$ such that the Navier–Stokes problem

$$\left.\begin{aligned} &\nu\Delta\boldsymbol{v} = \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p + \boldsymbol{F} \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega \qquad \boldsymbol{v} = \boldsymbol{v}_* \ \text{ at } \partial\Omega \tag{IX.2.8}$$

with $\nu = 1/2\mu$ admits two distinct solutions corresponding to the same data $\boldsymbol{F}$ and $\boldsymbol{v}_*$.

Proof. Set

$$\boldsymbol{v}_1 = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{\varphi}), \quad \boldsymbol{v}_2 = \frac{1}{2}(\boldsymbol{a} - \boldsymbol{\varphi}).$$

Clearly, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ satisfy $(2.8_{2,3})$ with

$$\boldsymbol{v}_* = \boldsymbol{a}_*/2, \tag{IX.2.9}$$

where $\boldsymbol{a}_*$ is the value of $\boldsymbol{a}$ at the boundary $\partial\Omega$. Moreover, by means of a direct calculation, we find the following identities

$$\begin{aligned}\nu\Delta\boldsymbol{v}_1 - \boldsymbol{v}_1\cdot\nabla\boldsymbol{v}_1 &= \frac{\nu}{2}\Delta\boldsymbol{a} - \frac{1}{4}(\boldsymbol{a}\cdot\nabla\boldsymbol{a} + \boldsymbol{\varphi}\cdot\nabla\boldsymbol{\varphi})\\ &\quad + \frac{1}{4}[2\nu\Delta\boldsymbol{\varphi} - (\boldsymbol{a}\cdot\nabla\boldsymbol{\varphi} + \boldsymbol{\varphi}\cdot\nabla\boldsymbol{a})]\end{aligned}$$

$$\begin{aligned}\nu\Delta\boldsymbol{v}_2 - \boldsymbol{v}_2\cdot\nabla\boldsymbol{v}_2 &= \frac{\nu}{2}\Delta\boldsymbol{a} - \frac{1}{4}(\boldsymbol{a}\cdot\nabla\boldsymbol{a} + \boldsymbol{\varphi}\cdot\nabla\boldsymbol{\varphi})\\ &\quad - \frac{1}{4}[2\nu\Delta\boldsymbol{\varphi} - (\boldsymbol{a}\cdot\nabla\boldsymbol{\varphi} + \boldsymbol{\varphi}\cdot\nabla\boldsymbol{a})]\end{aligned}$$

and so, recalling that $\boldsymbol{\varphi}$ satisfies (IX.2.7), if $\nu = 1/2\mu$ we deduce that the pairs $\boldsymbol{v}_1, p_1$ and $\boldsymbol{v}_2, p_2$ with

$$p_1 = \tau/4, \quad p_2 = -\tau/4$$

are two distinct solutions to (2.8) corresponding to

$$\boldsymbol{F} = \frac{\nu}{2}\Delta\boldsymbol{a} - \frac{1}{4}(\boldsymbol{a}\cdot\nabla\boldsymbol{a} + \boldsymbol{\varphi}\cdot\nabla\boldsymbol{\varphi})$$

and to $\boldsymbol{v}_*$ given in (IX.2.9). The lemma is therefore proved. □

Remark IX.2.4 If Ω is a bounded locally Lipschitz domain of $\mathbb{R}^n$, $n \le 4$, by the type of reasoning employed in the proof of Theorem IX.2.1, from (IX.2.7) we show that, for a given $\boldsymbol{a} \in W^{1,2}(\Omega)$, if ν $(=1/2\mu)$ is such that

$$|\boldsymbol{a}|_{1,2} < 2\nu/k,$$

then problem (IX.2.7) admits only the zero solution. Therefore, to obtain nonuniqueness (if any) ν has to be sufficiently small. ■

We also have the following lemma.

Lemma IX.2.3 *Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth body of revolution around an axis* r. *We suppose that Ω does not include points of* r *so that, for instance, Ω can be a torus of arbitrary bounded smooth section. In a system of cylindrical coordinates (r, θ, z) with the z-axis coinciding with* r, *we consider the vector field $\boldsymbol{a}$ $=(a_r, a_\theta, a_z)$ with*

$$a_r = a_z = 0,\ a_\theta = r^{-3}. \tag{IX.2.10}$$

Then there is $\mu > 0$ such that problem (IX.2.7) *admits at least one nontrivial smooth solution $\boldsymbol{\varphi}, \tau$.*

Proof. We look for a solution to problem (IX.2.7) in the class $\Re$ of functions having rotational symmetry, that is, they are independent of the angle θ. Taking into account that, in cylindrical coordinates, for $\boldsymbol{\varphi} \in \Re$,

$$\boldsymbol{\varphi} \cdot \nabla \boldsymbol{a} + \boldsymbol{a} \cdot \nabla \boldsymbol{\varphi} = \left(-2a_\theta \varphi_\theta, \left(\frac{da_\theta}{dr} + \frac{a_\theta}{r} \right) \varphi_r, 0 \right),$$

equations (IX.2.7) with the choice (IX.2.10) and in the class $\Re$ become

$$\left.\begin{aligned} &\frac{1}{\mu}(\Delta_1 \varphi_r - \frac{\varphi_r}{r^2}) = -g(r)\varphi_\theta + \frac{\partial \tau}{\partial r} \\ &\frac{1}{\mu}(\Delta_1 \varphi_\theta - \frac{\varphi_\theta}{r^2}) = -g(r)\varphi_r \\ &\frac{1}{\mu}\Delta_1 \varphi_z = \frac{\partial p}{\partial z} \\ &\frac{1}{r}\frac{\partial}{\partial r}(r\varphi_r) + \frac{\partial \varphi_z}{\partial z} = 0 \end{aligned}\right\} \text{ in } \Omega \tag{IX.2.11}$$

$$\boldsymbol{\varphi} = 0 \ \text{ on } \partial\Omega$$

where

$$\Delta_1 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{\partial^2}{\partial z^2}$$

and

$$g(r) = 2r^{-4}.$$

Set

$$I(\boldsymbol{\varphi}) = -\int_\Omega g\varphi_\theta\varphi_r$$

$$\begin{aligned} D(\boldsymbol{\varphi}) = \int_\Omega \Big\{ &(\frac{\partial \varphi_r}{\partial r})^2 + (\frac{\partial \varphi_\theta}{\partial r})^2 + (\frac{\partial \varphi_z}{\partial r})^2 \\ &+ (\frac{\partial \varphi_r}{\partial z})^2 + (\frac{\partial \varphi_\theta}{\partial z})^2 + (\frac{\partial \varphi_z}{\partial z})^2 + \frac{\varphi_r^2}{r^2} + \frac{\varphi_\theta^2}{r^2} \Big\} \end{aligned}$$

and consider the maximum problem

$$\frac{1}{\mu} = \max_{H_0^1(\Omega), \boldsymbol{\varphi} \neq 0} \left\{ \frac{I(\boldsymbol{\varphi})}{D(\boldsymbol{\varphi})} \right\} \tag{IX.2.12}$$

with $H_0^1(\Omega)$ denoting the completion of the subspace of $\mathcal{D}(\Omega)$ of rotationally symmetric fields in the norm $[D(\boldsymbol{\varphi})]^{1/2}$. In view of inequality (II.5.5), we at once conclude that the functional $I(\boldsymbol{\varphi})/D(\boldsymbol{\varphi})$ is bounded from above. Furthermore, taking among possible competitors $\boldsymbol{\varphi}$ those vectors such that $\varphi_\theta = -\varphi_r$ (which is an allowed choice) we find

$$\sup_{H_0^1(\Omega),\boldsymbol{\varphi}\neq 0} \left\{ \frac{I(\boldsymbol{\varphi})}{D(\boldsymbol{\varphi})} \right\} > 0. \tag{IX.2.13}$$

Also, from Theorem II.3.4, it easily follows that $H_0^1(\Omega)$ is compactly embedded into the subspace of $L^2(\Omega)$ constituted by vectors having rotational symmetry. Therefore, classical results on the maximum of quadratic functionals (see, e.g., Galdi & Straughan 1985, Lemma 3), ensure that the maximum (IX.2.12) exists and that the maximizing function satisfies (IX.2.11). Moreover, by (IX.2.13), $\mu > 0$. Finally, by the regularity theory for the Stokes problem developed in Section IV.4, Section IV.5, we easily show that the solution $\boldsymbol{\varphi} \in H_0^1(\Omega)$ to (IX.2.11) and the corresponding "pressure field" τ are smooth for Ω smooth. The lemma is thus proved. □

From Lemma IX.2.2 and Lemma IX.2.3 we deduce the following nonuniqueness result.

Theorem IX.2.2 *Let Ω be as in* Lemma IX.2.3. *Then there are smooth fields $\boldsymbol{F}$ and $\boldsymbol{v}_*$ and a value of $\nu > 0$ such that the steady Navier–Stokes problem* (2.8) *corresponding to these data admits at least two distinct solutions.*

IX.3 Existence and Uniqueness with Homogeneous Boundary Data

It was first noticed by J. Leray in his celebrated memoir of 1933 on the existence of steady solutions to the Navier–Stokes equations that every smooth solution to (IX.0.1), (IX.0.2) with $\boldsymbol{v}_* \equiv 0$ admits the following a priori estimate (see Leray 1933, Lemme A, p. 23)

$$\int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{v} \leq M \tag{IX.3.1}$$

where M is independent of $\boldsymbol{v}$. Actually, multiplying *formally* $(\text{IX.0.1})_1$ by $\boldsymbol{v}$ and using the incompressibility condition $(\text{IX.0.1})_2$ leads to

$$\nu \nabla \boldsymbol{v} : \nabla \boldsymbol{v} = \nabla \cdot \left(\frac{\nu}{2} \nabla v^2 - p\boldsymbol{v} - \frac{1}{2} v^2 \boldsymbol{v} \right) - \boldsymbol{f} \cdot \boldsymbol{v}$$

which, after integration over Ω, in view of the assumed homogeneous boundary conditions implies

$$\nu \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{v} = -\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}. \tag{IX.3.2}$$

Thus, (IX.3.1) follows with $M \equiv |\boldsymbol{f}|^2_{-1,2}/\nu^2$. We wish to point out the remarkable fact that (IX.3.1) and (IX.3.2) *are exactly the same relations we obtain from the linearized Stokes equations*. Then it is not surprising that Fujita (1961) and independently Vorovich & Youdovich (1961) (cf. also Vorovich & Youdovich 1959) employed a well-known device for constructing solutions to *linear* equations, *i.e.*, the Galerkin method (cf. Galerkin 1915) along with estimate (IX.3.1) to show the existence of generalized solutions to the *nonlinear* Navier–Stokes problem. Here we shall follow the ideas of these authors.

We begin to consider the case of homogeneous boundary conditions ($\boldsymbol{v}_* \equiv 0$), while in the next section we will handle the more general nonhomogeneous case. However, we need some preparatory results. The first concerns the zeros of continuous mappings of $\mathbb{R}^m$ into itself which generalizes to the case $m > 1$, the well-known fact that a real continuous function that attains values of opposite signs at the endpoints of an interval must then vanish at some interior point. Specifically, we have (see, e.g., Lions 1969, Lemme 4.3, p. 53):[1]

Lemma IX.3.1 *Let $\boldsymbol{P}$ be a continuous function of $\mathbb{R}^m$, $m \geq 1$, into itself such that for some $\rho > 0$*

$$\boldsymbol{P}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq 0 \quad \textit{for all } \boldsymbol{\xi} \in \mathbb{R}^m \textit{ with } |\boldsymbol{\xi}| = \rho.$$

Then there exists $\boldsymbol{\xi}_0 \in \mathbb{R}^m$ with $|\boldsymbol{\xi}_0| \leq \rho$ such that $\boldsymbol{P}(\boldsymbol{\xi}_0) = 0$.

Proof. Assume, *per absurdum*, $\boldsymbol{P}(\boldsymbol{\xi}) \neq 0$ for all $\boldsymbol{\xi}$ belonging to the closed ball $\overline{B}_\rho$ of $\mathbb{R}^m$ of radius ρ and centered at the origin. The map

$$\Pi : \boldsymbol{\xi} \to -\boldsymbol{P}(\boldsymbol{\xi}) \frac{\rho}{|\boldsymbol{P}(\boldsymbol{\xi})|}$$

is then continuous and goes from $\overline{B}_\rho$ into $\overline{B}_\rho$. By the Brouwer theorem (see, e.g., Kantorovich & Akilov 1964, Lemma 5, p. 639) we then obtain that the map Π has a fixed point $\overline{\boldsymbol{\xi}}$, i.e.,

$$\overline{\boldsymbol{\xi}} = -\boldsymbol{P}(\overline{\boldsymbol{\xi}}) \frac{\rho}{|\boldsymbol{P}(\overline{\boldsymbol{\xi}})|}, \quad |\overline{\boldsymbol{\xi}}| = \rho.$$

Dotting both sides of this relation by $\boldsymbol{P}(\overline{\boldsymbol{\xi}})$ yields

$$\boldsymbol{P}(\overline{\boldsymbol{\xi}}) \cdot \overline{\boldsymbol{\xi}} = -\rho |\boldsymbol{P}(\overline{\boldsymbol{\xi}})| < 0$$

contradicting the assumption. □

The lemma just shown allows us to prove a general existence result concerning certain algebraic systems.

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[1] In fact, a proof of Lemma IX.3.1 can already be found in Miranda (1940), where it is also shown that the lemma is equivalent to Brouwer's fixed point theorem. (For this latter, see, e.g., Kantorovich & Akilov 1964, Lemma 5, p. 639).) I would like to thank Professor P.N. Srikanth for bringing Miranda's paper to my attention.

Lemma IX.3.2 *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \geq 1$, and let $\{\boldsymbol{\psi}_k\}$, $k = 1, \dots, m$, be a set of functions from $C_0^\infty(\Omega)$. such that*

$$(\boldsymbol{\psi}_k, \boldsymbol{\psi}_{k'}) = \delta_{kk'}, \quad k, k' = 1, \dots, m.$$

Consider the algebraic system in the unknown $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$

$$\nu(\nabla \boldsymbol{w}, \nabla \boldsymbol{\psi}_k) = F_k(\boldsymbol{\xi}), \quad k = 1, \dots, m, \tag{IX.3.3}$$

where $\nu > 0$,

$$\boldsymbol{w} = \sum_{k=1}^m \xi_k \boldsymbol{\psi}_k, \tag{IX.3.4}$$

and $\boldsymbol{F}$ is a continuous function of $\mathbb{R}^m$ into itself. Then if there are positive constants c and $\alpha < \nu$ such that[2]

$$\boldsymbol{F}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \leq c|\boldsymbol{w}|_{1,2} + \alpha|\boldsymbol{w}|_{1,2}^2,$$

problem (IX.3.3)–(IX.3.4) *admits at least one solution.*

Proof. Consider the map $\boldsymbol{P}$, which to every vector $\boldsymbol{\xi} \in \mathbb{R}^m$, associates the vector $\boldsymbol{P}(\boldsymbol{\xi}) \in \mathbb{R}^m$ whose kth component is

$$(\boldsymbol{P}(\boldsymbol{\xi}))_k \equiv \nu(\nabla \boldsymbol{w}, \nabla \boldsymbol{\psi}_k) - F_k(\boldsymbol{\xi}).$$

Evidently, $\boldsymbol{P}$ is continuous. Let us prove that there exists $\rho > 0$ such that $\boldsymbol{P}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq 0$ for all $\boldsymbol{\xi}$ such that $|\boldsymbol{\xi}| = \rho$. Actually, by assumption,

$$\boldsymbol{P}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} = \nu|\boldsymbol{w}|_{1,2}^2 - \boldsymbol{F}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq |\boldsymbol{w}|_{1,2}\left[(\nu - \alpha)|\boldsymbol{w}|_{1,2} - c\right].$$

Denote by $K \subset \Omega$ a compact set containing the supports of all functions $\boldsymbol{\psi}_k$, $k = 1, \dots, m$. By inequality (II.5.5) and the normalization condition on the functions $\boldsymbol{\psi}_k$ we then have, for a suitable $c_1 = c_1(n)$,

$$\begin{aligned}\boldsymbol{P}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} &\geq |\boldsymbol{w}|_{1,2}\left[(\nu - \alpha)c_1|K|^{-1/n}\|\boldsymbol{w}\|_2 - c\right] \\ &= |\boldsymbol{w}|_{1,2}\left[(\nu - \alpha)c_1|K|^{-1/n}|\boldsymbol{\xi}| - c\right].\end{aligned}$$

Therefore,

$$\boldsymbol{P}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq 0 \quad \text{for} \quad |\boldsymbol{\xi}| = \frac{c\,|K|^{1/n}}{(\nu - \alpha)c_1},$$

and the lemma follows from Lemma IX.3.1. □

We are now in a position to prove the following existence result.

[2] The stated assumption on $\boldsymbol{F}$ can be widely generalized. However, such a generalization is not needed in the cases treated in this book.

Theorem IX.3.1 *Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$. Given $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$ there exists at least one generalized solution $\boldsymbol{v}$ to problem* (IX.0.1), (IX.0.2) *with $\boldsymbol{v}_* = 0$. This solution satisfies the estimate*

$$\nu|\boldsymbol{v}|_{1,2} \leq |\boldsymbol{f}|_{-1,2}. \tag{IX.3.5}$$

Furthermore, if Ω is locally Lipschitz, we have

$$\|p\|_2 \leq c\,(|\boldsymbol{f}|_{-1,2} + |\boldsymbol{v}|_{1,2}^2 + \nu|\boldsymbol{v}|_{1,2}) \tag{IX.3.6}$$

where p is the pressure field associated to $\boldsymbol{v}$ by Lemma IX.1.2 *and $c = c(n, \Omega)$.*

Proof. Denote by $\{\boldsymbol{\psi}_k\}$ a denumerable set of functions of $\mathcal{D}(\Omega)$ whose linear hull is dense in $H^1(\Omega)$. We normalize it as

$$(\boldsymbol{\psi}_k, \boldsymbol{\psi}_{k'}) = \delta_{kk'}.$$

For each $m = 1, 2, \ldots$, we then look for an "approximating solution" $\boldsymbol{v}_m$ to (IX.1.2) as follows:

$$\begin{gathered}\boldsymbol{v}_m = \sum_{k=1}^{m} \xi_{km} \boldsymbol{\psi}_k \\ \nu(\nabla \boldsymbol{v}_m, \nabla \boldsymbol{\psi}_k) + (\boldsymbol{v}_m \cdot \nabla \boldsymbol{v}_m, \boldsymbol{\psi}_k) + \langle \boldsymbol{f}, \boldsymbol{\psi}_k \rangle = 0, \quad k = 1, 2, \ldots, m.\end{gathered} \tag{IX.3.7}$$

Relation (IX.3.7) represents a system of nonlinear equations in the m unknowns ξ_{km}, $k = 1, \ldots, m$. Since $\boldsymbol{v}_m \in \mathcal{D}(\Omega)$, by Lemma IX.2.1 it follows that

$$\sum_{k=1}^{m} (\boldsymbol{v}_m \cdot \nabla \boldsymbol{v}_m, \xi_k \boldsymbol{\psi}_k) = (\boldsymbol{v}_m \cdot \nabla \boldsymbol{v}_m, \boldsymbol{v}_m) = 0. \tag{IX.3.8}$$

Moreover,

$$\left| \sum_{k=1}^{m} \langle \boldsymbol{f}, \xi_k \boldsymbol{\psi}_k \rangle \right| \leq |\boldsymbol{f}|_{-1,2} |\boldsymbol{v}_m|_{1,2} \tag{IX.3.9}$$

and so, by Lemma IX.3.2, we deduce that problem (IX.3.7) admits a solution for all $m \in \mathbb{N}$. Multiplying $(\text{IX.3.7})_2$ by ξ_{km}, summing over k from 1 to m, and using (IX.3.8) and (IX.3.9) we deduce also that

$$\nu|\boldsymbol{v}_m|_{1,2}^2 = -\langle \boldsymbol{f}, \boldsymbol{v}_m \rangle, \tag{IX.3.10}$$

and since

$$-\langle \boldsymbol{f}, \boldsymbol{v}_m \rangle \leq |\boldsymbol{f}|_{-1,2} |\boldsymbol{v}_m|_{1,2},$$

we have

$$|\boldsymbol{v}_m|_{1,2} \leq |\boldsymbol{f}|_{-1,2}/\nu. \tag{IX.3.11}$$

Therefore, the sequence $\{\boldsymbol{v}_m\}$ is uniformly bounded in $\mathcal{D}_0^{1,2}(\Omega)$ and, by Theorem II.1.3, there is a subsequence, denoted again by $\{\boldsymbol{v}_m\}$, and a field $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega)$ such that

$$\boldsymbol{v}_m \to \boldsymbol{v} \;\text{ weakly in } \mathcal{D}_0^{1,2}(\Omega). \tag{IX.3.12}$$

Moreover, by inequality (II.5.5), $\{\boldsymbol{v}_m\}$ is also uniformly bounded in $L^2(\Omega)$ and $\boldsymbol{v} \in L^2(\Omega)$. Thus, by compactness Theorem II.5.2 we may take this subsequence such that

$$\boldsymbol{v}_m \to \boldsymbol{v} \;\text{ strongly in } L^2(\Omega). \tag{IX.3.13}$$

With (IX.3.12) and (IX.3.13) in hand we pass next to the limit as $m \to \infty$ in $(\text{IX.3.7})_2$. By (IX.3.12) it follows that

$$(\nabla \boldsymbol{v}_m, \nabla \psi_k) \to (\nabla \boldsymbol{v}, \nabla \psi_k). \tag{IX.3.14}$$

In addition,

$$\begin{aligned} I_m &\equiv |(\boldsymbol{v}_m \cdot \nabla \boldsymbol{v}_m, \boldsymbol{\psi}_k) - (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\psi}_k)| \\ &\le |((\boldsymbol{v}_m - \boldsymbol{v}) \cdot \nabla \boldsymbol{v}_m, \boldsymbol{\psi}_k)| + |(\boldsymbol{v} \cdot \nabla(\boldsymbol{v}_m - \boldsymbol{v}), \boldsymbol{\psi}_k)| \\ &\equiv I_m^{(1)} + I_m^{(2)}. \end{aligned} \tag{IX.3.15}$$

Now, by (IX.3.11) we have

$$I_m^{(1)} \le \sup_{\Omega} |\psi_k| \, \|\boldsymbol{v}_m - \boldsymbol{v}\|_2 |\boldsymbol{v}|_{1,2} \le \sup_{\Omega} |\boldsymbol{\psi}_k| \, \|\boldsymbol{v}_m - \boldsymbol{v}\|_2 |\boldsymbol{f}|_{-1,2}/\nu$$

so that by (IX.3.13)

$$\lim_{m\to\infty} I_m^{(1)} = 0. \tag{IX.3.16}$$

Also, by Lemma IX.2.1,

$$(\boldsymbol{v} \cdot \nabla(\boldsymbol{v}_m - \boldsymbol{v}), \boldsymbol{\psi}_k) = -(\boldsymbol{v} \cdot \nabla \boldsymbol{\psi}_k, (\boldsymbol{v}_m - \boldsymbol{v})),$$

since $\boldsymbol{v} \in H^1(\Omega)$. Therefore

$$I_m^{(2)} \le \sup_{\Omega} |\nabla \psi_k| \|\boldsymbol{v}\|_2 \|\boldsymbol{v}_m - \boldsymbol{v}\|_2$$

and, again by (IX.3.13)

$$\lim_{m\to\infty} I_m^{(2)} = 0. \tag{IX.3.17}$$

From (IX.3.15)–(IX.3.17) we then conclude

$$\lim_{m\to\infty} I_m = 0. \tag{IX.3.18}$$

Replacing (IX.3.14) and (IX.3.18) into $(\text{IX.3.7})_2$ it follows that the field $\boldsymbol{v}$ (belongs to $H^1(\Omega)$ and) satisfies the equation

$$\nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}_k) + (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\psi}_k) + \langle \boldsymbol{f}, \boldsymbol{\psi}_k \rangle = 0, \tag{IX.3.19}$$

for all $k = 1, 2, \ldots$ However, any $\boldsymbol{\varphi} \in H^1(\Omega)$ can be approximated by linear combinations of functions $\boldsymbol{\psi}_k$ through suitable coefficients. Since every term in (IX.3.19) defines a bounded linear functional in $\boldsymbol{\psi}_k \in H^1(\Omega)$ (cf. Lemma IX.1.1) we may conclude from (IX.3.19) that the field $\boldsymbol{v}$ satisfies (IX.1.2) for all $\boldsymbol{\varphi} \in H^1(\Omega)$. Existence is then established. As far as the validity of estimates (IX.3.5)–(IX.3.6) is concerned, we notice that from (IX.3.11) and from Theorem II.2.4 we have

$$|\boldsymbol{v}|_{1,2} \le \liminf_{m \to \infty} |\boldsymbol{v}_m|_{1,2} \le |\boldsymbol{f}|_{-1,2}/\nu,$$

which proves (IX.3.5). Assuming Ω locally Lipschitz, from Lemma IX.1.2 follows the existence of a pressure field $p \in L^2(\Omega)$ satisfying (IX.1.11) and

$$\int_\Omega p = 0. \tag{IX.3.20}$$

Consider the problem

$$\begin{aligned} \nabla \cdot \boldsymbol{\Psi} &= p \\ \boldsymbol{\Psi} &\in W_0^{1,2}(\Omega) \\ \|\boldsymbol{\Psi}\|_{1,2} &\le c\|p\|_2. \end{aligned} \tag{IX.3.21}$$

Since p is in $L^2(\Omega)$ and satisfies (IX.3.20), problem (IX.3.21) is solvable in virtue of Theorem III.3.1. From (IX.1.11), (IX.3.21), and (IX.1.4) we then deduce

$$\|p\|_2^2 \le c_1(|\boldsymbol{f}|_{-1,2} + |\boldsymbol{v}|_{1,2} + \nu|\boldsymbol{v}|_{1,2})\|p\|_2$$

with $c_1 = c_1(n, \Omega)$, which shows (IX.3.6). The theorem is therefore proved. □

Remark IX.3.1 The theorem just shown extends with no changes to cover the case $n = 4$. If $n \ge 5$ the only change we need is in the choice of the set $\{\boldsymbol{\psi}_k\} \subset \mathcal{D}(\Omega)$ whose linear hull, this time, has to be dense in $\widetilde{H}^1(\Omega)$; see Remark IX.1.4. Moreover, estimate (IX.3.5) continues to be valid, while (IX.3.6) must be suitably changed according to the fact that now the pressure field $p \in L^{n/(n-2)}(\Omega)$ (cf. Remark IX.1.5). Specifically, if Ω is locally Lipschitz, by letting

$$C \equiv \frac{1}{|\Omega|}\int_\Omega p|p|^{(4-n)/(n-2)}$$

and introducing a function $\boldsymbol{\Psi}$ that instead of (IX.3.21) solves the problem:

$$\nabla \cdot \boldsymbol{\Psi} = p\,|p|^{(4-n)/(n-2)} - C \equiv g$$

$$\boldsymbol{\Psi} \in W_0^{1,n/2}(\Omega)$$

$$\|\boldsymbol{\Psi}\|_{1,n/2} \le c_1\|g\|_{n/2} \le c_2\|p\|_{n/(n-2)}$$

one may use the arguments adopted in the proof of Theorem IX.3.1 to show the validity of the following inequality.

$$\|p\|_{n/(n-2)} \leq c\,(|\boldsymbol{f}|_{-1,2} + |\boldsymbol{v}|_{1,2}^2 + \nu|\boldsymbol{v}|_{1,2}).$$

■

Remark IX.3.2 As the reader may have noticed, the method employed in the proof of Theorem IX.3.1 –being based on Lemma IX.3.2– is in fact essentially *independent* of the boundedness of Ω. Actually, as we shall see in the next chapters, with an appropriate choice of the basis $\{\boldsymbol{\psi}_k\}$, it can be equally applied to the case when Ω is unbounded. ■

Uniqueness of the solutions just constructed is easily discussed by means of Theorem IX.2.1. Actually, from the estimate (IX.3.5) we deduce that condition (IX.2.4) is certainly satisfied if

$$|\boldsymbol{f}|_{-1,2} < \nu^2/k. \tag{IX.3.22}$$

We thus have the following theorem.

Theorem IX.3.2 *The generalized solution determined in* Theorem IX.3.1 *is the only generalized solution corresponding to* $\boldsymbol{f}$*, provided* $\boldsymbol{f}$ *satisfies* (IX.3.22).

Remark IX.3.3 Theorem IX.3.2 continues to hold if $n = 4$. For $n \geq 5$, we refer the reader to Remark IX.5.5. ■

Exercise IX.3.1 Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 2$. Show that the generalized solution $\boldsymbol{v}$ constructed by the method of Theorem IX.3.1 (see also Remark IX.3.1) satisfies the *energy inequality*:

$$\nu|\boldsymbol{v}|_{1,2}^2 \leq -\langle \boldsymbol{f}, \boldsymbol{v}\rangle. \tag{IX.3.23}$$

Hint: Use (IX.3.10).

Furthermore, show that if $n = 2, 3, 4$, every generalized solution corresponding to $\boldsymbol{v}_* = 0$ and $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$ satisfies the energy *equality*, that is, (IX.3.23) with the equality sign. Finally, show that if $n \geq 5$, the energy equality holds for those generalized solutions that belong to $L^n(\Omega)$.

IX.4 Existence and Uniqueness with Nonhomogeneous Boundary Data

As in the linear case, to prove existence when $\boldsymbol{v}_* \not\equiv 0$, we may look for solutions $\boldsymbol{v}$ of the form

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{V},$$

where $\boldsymbol{V}$ is a (sufficiently smooth) solenoidal extension to Ω of the velocity $\boldsymbol{v}_*$ at the boundary. From (IX.0.1), (IX.0.2) we then conclude that $\boldsymbol{u}$ solves the problem

$$\left.\begin{aligned}&\nu\Delta\boldsymbol{u}=\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{V}+\boldsymbol{V}\cdot\nabla\boldsymbol{u}+\nabla p-\nu\Delta\boldsymbol{V}+\boldsymbol{V}\cdot\nabla\boldsymbol{V}+\boldsymbol{f}\\&\nabla\cdot\boldsymbol{u}=0\end{aligned}\right\}\text{ in }\Omega$$
$$\boldsymbol{u}=0 \ \text{ at } \partial\Omega. \tag{IX.4.1}$$

However, in order to show existence by the technique used in Theorem IX.3.1, we need, as already observed, to find a uniform bound on $|\boldsymbol{u}|_{1,2}^2$ depending only on the data. Now, formally multiplying $(\text{IX.4.1})_1$ by $\boldsymbol{u}$, integrating by parts over Ω and using $(\text{IX.4.1})_2$, we obtain

$$\nu\int_\Omega\nabla\boldsymbol{u}:\nabla\boldsymbol{u}=-\int_\Omega\boldsymbol{u}\cdot\nabla\boldsymbol{V}\cdot\boldsymbol{u}-\int_\Omega\boldsymbol{f}\cdot\boldsymbol{u}-\nu\int_\Omega\nabla\boldsymbol{V}:\nabla\boldsymbol{u}-\int_\Omega\boldsymbol{V}\cdot\nabla\boldsymbol{V}\cdot\boldsymbol{u}. \tag{IX.4.2}$$

Using the Schwarz inequality and Lemma IX.1.1 one easily shows that the last three terms of this relation can be increased by

$$C\left(\int_\Omega\nabla\boldsymbol{u}:\nabla\boldsymbol{u}\right)^{1/2}$$

with $C=C(n,\nu,\Omega,\boldsymbol{f},\boldsymbol{V})$. Therefore, from (IX.4.2) we find the estimate

$$\nu\int_\Omega\nabla\boldsymbol{u}:\nabla\boldsymbol{u}\le-\int_\Omega\boldsymbol{u}\cdot\nabla\boldsymbol{V}\cdot\boldsymbol{u}+C\left(\int_\Omega\nabla\boldsymbol{u}:\nabla\boldsymbol{u}\right)^{1/2},$$

and so a way of recovering a uniform bound on $|\boldsymbol{u}|_{1,2}^2$, is to require that the extension field $\boldsymbol{V}$ satisfy the one-sided inequality

$$-\int_\Omega\boldsymbol{u}\cdot\nabla\boldsymbol{V}\cdot\boldsymbol{u}\le\alpha|\boldsymbol{u}|_{1,2}^2, \tag{IX.4.3}$$

for some $\alpha<\nu$ and for all $\boldsymbol{u}\in H^1(\Omega)$.[1] If we do not want to impose restrictions from below on the kinematic viscosity, then Ω and $\boldsymbol{v}_*$ should satisfy the following *extension condition* (referred to by the abbreviation *EC*): *for any* $\alpha>0$ *there is a solenoidal extension* $\boldsymbol{V}=\boldsymbol{V}(\alpha)$ *of* $\boldsymbol{v}_*$ *satisfying* (IX.4.3). As a consequence, *in contrast with the linear case*, to prove existence it is not enough to pick *any* (sufficiently smooth) extension of the boundary data.

If $\boldsymbol{V}$ were *not* required to be solenoidal, every (sufficiently regular) Ω and $\boldsymbol{v}_*$ would satisfy *EC*. Actually, for a given $\varepsilon>0$ we could choose

$$\boldsymbol{V}=\psi_\varepsilon\boldsymbol{W} \tag{IX.4.4}$$

[1] In fact, according to the Galerkin method, it would suffice to require the validity of (IX.4.3) for all $\boldsymbol{u}\in\mathcal{D}(\Omega)$.

with $\boldsymbol{W}$ an extension of $\boldsymbol{v}_*$ and ψ_ε the "cut-off" function introduced in Lemma III.6.2. Thus, integrating by parts and using the Hölder inequality together with the embedding Theorem II.3.4 it follows that

$$\begin{aligned}\left|\int_\Omega \boldsymbol{u}\cdot\nabla(\psi_\varepsilon\boldsymbol{W})\cdot\boldsymbol{u}\right| &= \left|\int_\Omega \psi_\varepsilon\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{W}\right|\\ &\le \|\boldsymbol{u}\|_4|\boldsymbol{u}|_{1,2}\|\psi_\varepsilon\boldsymbol{W}\|_4\\ &\le c|\boldsymbol{u}|_{1,2}^2\|\psi_\varepsilon\boldsymbol{W}\|_4\end{aligned}$$

and since, by the properties of ψ_ε,

$$\|\psi_\varepsilon\boldsymbol{W}\|_4 \to 0 \quad \text{as } \varepsilon\to 0,$$

we recover (IX.4.3) for any $\alpha > 0$. Nevertheless, following the ideas of Leray (1933, pp. 40-41), completed and clarified by Hopf (1941, §2) (see also Hopf 1957, pp. 12–14), instead of (IX.4.4) one can take[2]

$$\boldsymbol{V} = \nabla\times(\psi_\varepsilon\boldsymbol{W}) \tag{IX.4.5}$$

with a suitable choice of the field $\boldsymbol{W}$. Then $\boldsymbol{V}$ is solenoidal and, by arguments slightly more complicated than those employed before, one can show that (IX.4.3) can be satisfied by any $\alpha > 0$. Recalling that the incompressibility of the fluid requires

$$\int_{\partial\Omega}\boldsymbol{v}_*\cdot\boldsymbol{n} = 0 \tag{IX.4.6}$$

we may conclude that, if $\partial\Omega$ has only one connected component, the choice (IX.4.5) ensures that any (sufficiently smooth) Ω and $\boldsymbol{v}_*$ satisfy *EC*. However, if $\partial\Omega$ has more than one connected component Γ_i, say, *i.e.*, for $m > 0$

$$\partial\Omega = \bigcup_{i=1}^{m+1}\Gamma_i,$$

with the choice (IX.4.5) we have, as a consequence of the Stokes theorem, that *Ω and $\boldsymbol{v}_*$ satisfy EC if*

$$\Phi_i \equiv \int_{\Gamma_i}\boldsymbol{v}_*\cdot\boldsymbol{n} = 0, \quad i = 1,2,\dots,m+1. \tag{IX.4.7}$$

Observe that (IX.4.7) is stronger than the compatibility condition (IX.4.6) and that, in particular, it does not allow for the presence of separated sinks

[2] If $n = 2$, one takes

$$\boldsymbol{V} = (\partial(\psi_\varepsilon w)/\partial x_2, -\partial(\psi_\varepsilon w)/\partial x_1) \equiv \nabla\times(\psi_\varepsilon w).$$

and sources of liquids into the region of flow. At this point we may think of choosing a field $\boldsymbol{V}$ in a form other than (IX.4.5) so that Ω and $\boldsymbol{v}_*$ may satisfy *EC* under the *sole* condition (IX.4.6). However, this is not possible, in general. In fact, it is easy to bring examples of smooth domains Ω for which *EC holds (if and) only if* (IX.4.7) *is satisfied, whatever the choice of* $\boldsymbol{v}_*$ *may be*, cf. Takeshita (1993, Theorem 1). For instance, let us suppose that Ω is the annular region

$$\Omega = \left\{x \in \mathbb{R}^2 : R_1 < |x| < R_2\right\}$$

and set

$$\Phi = \int_{\Gamma_2} \boldsymbol{v}_* \cdot \boldsymbol{n} = -\int_{\Gamma_1} \boldsymbol{v}_* \cdot \boldsymbol{n} \tag{IX.4.8}$$

where

$$\Gamma_1 = \left\{x \in \mathbb{R}^2 : |x| = R_1\right\}, \quad \Gamma_2 = \left\{x \in \mathbb{R}^2 : |x| = R_2\right\}.$$

We take $\Phi < 0$ (inflow condition). Assuming that Ω and $\boldsymbol{v}_*$ satisfy *EC* means that for any $\alpha > 0$ there is an extension $\boldsymbol{V} = \boldsymbol{V}(\alpha) = (V_r, V_\theta)$ of $\boldsymbol{v}_*$ verifying (IX.4.3), where, as usual, (r, θ) denotes a system of polar coordinates in the plane. Then, because $\boldsymbol{V}$ is solenoidal

$$\frac{\partial(rV_r)}{\partial r} + \frac{\partial V_\theta}{\partial \theta} = 0$$

and by (IX.4.8), it follows for all $r \in (R_1, R_2)$

$$r\int_0^{2\pi} V_r(r,\theta)d\theta = \Phi. \tag{IX.4.9}$$

We take, next, $\boldsymbol{u} = u(r)\boldsymbol{e}_\theta$ with $u \in C_0^\infty((R_1, R_2))$, and observe that $\boldsymbol{u} \in \mathcal{D}(\Omega)$. With this choice of $\boldsymbol{u}$ and in view of (IX.4.9), we find

$$\int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{V} \cdot \boldsymbol{u} = \int_\Omega \left(\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r}\right) u^2 = \Phi \int_{R_1}^{R_2} \frac{u^2}{r} dr.$$

Thus admitting *EC* would imply

$$|\Phi| \int_{R_1}^{R_2} \frac{u^2}{r} dr \le \alpha |\boldsymbol{u}|_{1,2}^2,$$

for any $\alpha > 0$, that is, $\Phi = 0$.

Remark IX.4.1 This example can be extended, in an simple way, to more general two-dimensional bounded domains, Ω, satisfying the following properties: (i) $\partial\Omega$ is constituted by two connected components, Γ_1 and Γ_2; (ii) Γ_1 surrounds a circle, C_1, and Γ_2 lies within a circle, C_2, with both C_1 and C_2 contained in Ω; (iii) The centers of C_1, C_2 are in the interior of the bounded connected component of $\mathbb{R}^2 - \Omega$; (iv) $C_1 \cap C_2 = \emptyset$. ■

Remark IX.4.2 As already noticed, the above example only works if $\Phi < 0$. An example when $\Phi > 0$ (outflow condition) has been recently furnished by Heywood (2010). ■

Remark IX.4.3 By similar ideas and a slightly more complicated reasoning, one is able to prove the invalidity of *EC* when Ω is the 3-dimensional spherical shell

$$\left\{x \in \mathbb{R}^3 : R_1 < |x| < R_2\right\} ; \tag{IX.4.10}$$

see Takeshita (1989, Theorem 1) and Farwig, Kozono & Yanagisawa (2010, Theorem 1). ■

Remark IX.4.4 It is interesting to observe that the counterexample given previously to the validity of *EC* implies, indirectly, that the problem

$$\begin{aligned}
&\nabla \cdot \boldsymbol{w} = f \;\text{ in } \Omega, \quad \int_\Omega f = 0, \\
&\boldsymbol{w} \in W_0^{1,q}(\Omega) \\
&|\boldsymbol{w}|_{1,q} \leq c\|f\|_q \\
&\|\boldsymbol{w}\|_q \leq c\|f\|_{-1,q}
\end{aligned} \tag{IX.4.11}$$

is, in general, *not solvable even if f is in divergence form*. Actually let Ω be the annular region

$$\left\{x \in \mathbb{R}^2 : R_1 < |x| < R_2\right\} .$$

If (IX.4.11) were solvable for some $q > 2$, we could add to the extension (IX.4.4) a field $\boldsymbol{w}$ verifying (IX.4.11) with $f = -\nabla \cdot (\psi_\varepsilon \boldsymbol{W})$. The vector field $\boldsymbol{U} \equiv \boldsymbol{V} + \boldsymbol{w}$ is then solenoidal and assumes the value $\boldsymbol{v}_*$ at $\partial\Omega$. Furthermore, by the Hölder inequality, by the embedding Theorem II.3.4, and by $(\text{IX.4.11})_4$ we have

$$\begin{aligned}
\left|\int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{U} \cdot \boldsymbol{u}\right| &= \left|\int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{U}\right| \\
&\leq c_1 |\boldsymbol{u}|_{1,2}^2 \|\boldsymbol{U}\|_q \\
&\leq c_2 |\boldsymbol{u}|_{1,2}^2 \left(\|\psi_\varepsilon \boldsymbol{W}\|_q + \|\nabla \cdot (\psi_\varepsilon \boldsymbol{W})\|_{-1,q}\right),
\end{aligned}$$

and being

$$\|\nabla \cdot (\psi_\varepsilon \boldsymbol{W})\|_{-1,q} \leq \|\psi_\varepsilon \boldsymbol{W}\|_q \to 0 \;\text{ as } \varepsilon \to 0,$$

we obtain that, for all $\alpha > 0$ there is an extension $\boldsymbol{U} = \boldsymbol{U}(\alpha)$ such that,

$$\left|\int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{U} \cdot \boldsymbol{u}\right| \leq \alpha |\boldsymbol{u}|_{1,2}^2$$

for any $\boldsymbol{u} \in H^1(\Omega)$, thus allowing *EC* for Ω. By the same token, one can show that problem $(\text{IX.4.11})_1$ with $f \in C(\overline{\Omega})$ does not admit a solution $\boldsymbol{v} \in C^1(\overline{\Omega})$ such that

$$\|\boldsymbol{v}\|_{C^1} \leq c\|f\|_C$$

with $c = c(n, \Omega)$. ■

The example shown previously rules out the general validity of *EC*, but, on the other side, it also suggests that condition (IX.4.7) can possibly be replaced by the weaker one:

$$\sum_{i=1}^{m+1} |\Phi_i| < c\nu, \tag{IX.4.12}$$

for some positive constant c. In fact, this is indeed the case. Specifically, following the work of Galdi (1991), by suitably modifying the Leray-Hopf extension (IX.4.5), we shall prove existence of solutions under the *sole* condition that the fluxes Φ_i of $\boldsymbol{v}_*$ through each component Γ_i of $\partial\Omega$ satisfy condition (IX.4.12), with a *computable* constant c that depends only on Ω and n.[3] However, *if $\partial\Omega$ has more than one connected component, existence of solutions satisfying merely* (IX.4.6) *with no restriction on the size of the fluxes Φ_i remains open.*[4]

To show our main result we need some preparatory steps.

Lemma IX.4.1 *Let Ω be a bounded locally Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$. Denote by ω_i, $i = 1, \ldots, m$, the* (*bounded*) *connected components of $\mathbb{R}^n - \overline{\Omega}$ and set*

$$\omega \equiv \bigcup_{i=1}^{m} \omega_i.^5$$

Then, given $\boldsymbol{a} \in W^{1/2,2}(\partial\Omega)$ verifying the condition

$$\int_{\Gamma_i} \boldsymbol{a} \cdot \boldsymbol{n} = 0, \quad i = 1, 2, \quad \ldots, m+1, \tag{IX.4.13}$$

where $\boldsymbol{n}$ is the outer normal to $\partial\Omega$ and

$$\Gamma_i \equiv \partial\omega_i, \quad \text{for } i = 1, \ldots, m, \quad \Gamma_{m+1} \equiv \partial(\Omega \cup \overline{\omega}),$$

---

[3] We wish to emphasize that, clearly, the condition on the "smallness" of Φ_i does not imply a priori "smallness" of $\boldsymbol{v}_*$. On the other hand, if we assume that the trace norm of $\boldsymbol{v}_*$ at the boundary is small with respect to ν, then existence with nonzero small fluxes Φ_i is proved in a direct elementary way.

[4] For another approach to existence with nonhomogeneous data, again due to Leray, we refer the reader to the Notes for this Chapter. In that context, we shall also give other existence results due to Amick (1984), Morimoto (1992) and Morimoto and Ukai (1996), where the restriction (IX.4.7) can be removed.

[5] Clearly, the number m is finite since $\partial\Omega$ is compact and, furthermore,

$$\min_{i=1,\ldots,m} \text{dist}\,(\omega_i, \partial(\Omega \cup \overline{\omega})) > 0,$$

(cf. also Griesinger 1990a).

there is $\boldsymbol{w} \in W^{2,2}(\Omega)$ *if* $n = 3$ [*respectively* $w \in W^{2,2}(\Omega)$, *if* $n = 2$] *such that* $\boldsymbol{a} = \nabla \times \boldsymbol{w}$ [*respectively* $\boldsymbol{a} = \nabla \times w$] *in the trace sense at* $\partial\Omega$. *Moreover, the following inequality holds*

$$\|\boldsymbol{w}\|_{2,2} \leq c\|\boldsymbol{a}\|_{1/2,2(\partial\Omega)} \quad [\textit{respectively} \;\; \|w\|_{2,2} \leq c\|\boldsymbol{a}\|_{1/2,2(\partial\Omega)}\,] \tag{IX.4.14}$$

where $c = c(n, \Omega)$.

Proof. Since

$$\partial\Omega = \bigcup_{i=1}^{m+1} \Gamma_i,$$

from (IX.4.13) it follows that

$$\int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} = 0$$

and so, by Exercise III.3.5, we may extend $\boldsymbol{a}$ to a field $\boldsymbol{v}_0 \in W^{1,2}(\Omega)$ with $\nabla \cdot \boldsymbol{v}_0 = 0$, and verifying the inequality

$$\|\boldsymbol{v}_0\|_{1,2} \leq c\|\boldsymbol{a}\|_{1/2,2(\partial\Omega)}. \tag{IX.4.15}$$

If $n = 2$, for a fixed $x_0 \in \Omega$ we define a function w through the line integral

$$w(x) = \int_{x_0}^{x} (v_{01}dx_2 - v_{02}dx_1), \quad x \in \Omega,$$

i.e., w is the stream function associated to $\boldsymbol{v}_0$. Since (IX.4.13) holds, w is singlevalued. Furthermore,

$$\frac{\partial w}{\partial x_2} = v_{01}, \quad \frac{\partial w}{\partial x_1} = -v_{02}$$

and so

$$|w|_{1,2} + |w|_{2,2} \leq c_1\|v_0\|_{1,2}. \tag{IX.4.16}$$

Also, we can modify w by an additive constant in such a way that

$$\int_{\Omega} w = 0$$

so that by inequality (II.5.10), (IX.4.15), and (IX.4.16) we deduce (IX.4.14), proving the lemma if $n = 2$. To prove it for $n = 3$, we notice that, again by Exercise III.3.5, we can extend $\boldsymbol{a}$ at $\partial\omega_i$, $i = 1, \ldots, m$ into each ω_i to a solenoidal vector field $\boldsymbol{v}_i \in W^{1,2}(\omega_i)$ satisfying the estimate

$$\|\boldsymbol{v}_i\|_{1,2,\omega_i} \leq c_2\|\boldsymbol{a}\|_{1/2,2(\partial\Omega)}, \quad i = 1, \ldots, m. \tag{IX.4.17}$$

Moreover, denoting by B an open ball with $B \supset \overline{\Omega}$, since

$$\int_{\Gamma_{m+1}} \boldsymbol{a}\cdot\boldsymbol{n} = 0,$$

by Corollary III.3.1 we can extend $\boldsymbol{a}$ at $\partial(\Omega\cup\overline{\omega})$ to a solenoidal vector field, $\boldsymbol{v}_{m+1}$, in $\omega_{m+1}\equiv B-(\Omega\cup\overline{\omega})$ such that

$$\boldsymbol{v}_{m+1}\in W^{1,2}(\omega_{m+1})\,,\quad \boldsymbol{v}_{m+1}(x)=\boldsymbol{0}\quad x\in\partial B$$

and, moreover,

$$\|\boldsymbol{v}_{m+1}\|_{1,2,\omega_{m+1}}\le c_2\|\boldsymbol{a}\|_{1/2,2(\partial\Omega)}. \tag{IX.4.18}$$

It is then immediately verified that the vector field:

$$\boldsymbol{v}: x\in B\to\begin{cases}\boldsymbol{v}_0 \text{ if } x\in\Omega\\ \boldsymbol{v}_i \text{ if } x\in\omega_i,\ i=1,\dots,m+1\end{cases} \tag{IX.4.19}$$

satisfies the following properties

(i) $\boldsymbol{v}\in W^{1,2}(B)$,
(ii) $\nabla\cdot\boldsymbol{v}=0$ in B,
(iii) $\boldsymbol{v}=0$ at ∂B,

implying $\boldsymbol{v}\in H^1(B)$. However, by means of an explicit representation formula it can be easily proved (see Exercise IX.4.1) that, given $\boldsymbol{v}\in H^1(B)$ there is $\boldsymbol{w}\in W^{2,2}(B)$ such that

$$\boldsymbol{v}=\nabla\times\boldsymbol{w}$$

$$\|\boldsymbol{w}\|_{2,2}\le c_1\|\boldsymbol{v}\|_{1,2}.$$

This last relation, together with (IX.4.17)–(IX.4.19) implies (IX.4.14) and the restriction of $\boldsymbol{w}$ to Ω verifies all requirements stated in the lemma. The proof is therefore complete. □

Exercise IX.4.1 Let Ω be a bounded domain in $\mathbb{R}^3$ and let $\boldsymbol{v}\in H^1(\Omega)$. Show that there exists $\boldsymbol{w}\in W^{2,2}(\Omega)$ such that $\boldsymbol{v}=\nabla\times\boldsymbol{w}$. *Hint:* Take first $\boldsymbol{v}\in\mathcal{D}(\Omega)$ and consider the function $\boldsymbol{U}=\mathcal{E}*\boldsymbol{v}$, where $\mathcal{E}$ is the fundamental solution of Laplace's equation. Then $\boldsymbol{v}=\nabla\times\boldsymbol{w}$, where $\boldsymbol{w}=-\nabla\times\boldsymbol{U}$. By the Calderón-Zygmund Theorem II.7.4 and Young's inequality (II.5.3) it follows that

$$\|\boldsymbol{w}\|_{2,2,\Omega}\le C\|\boldsymbol{v}\|_{1,2,\Omega},$$

where $C=C(\Omega)$. The result is then a consequence of this inequality and of the density of $\mathcal{D}(\Omega)$ into $H^1(\Omega)$.

Remark IX.4.5 Using the same lines of proof, we at once recognize that Lemma IX.4.1 is valid, more generally, with $\boldsymbol{w}\in W^{2,q}(\Omega), 1<q<\infty$, provided $\boldsymbol{a}\in W^{1-1/q}(\partial\Omega)$. In particular, $\boldsymbol{w}$ obeys the following estimate

$$\|\boldsymbol{w}\|_{2,q}\le c\|\boldsymbol{a}\|_{1-1/q,q(\partial\Omega)}.$$

■

Remark IX.4.6 Lemma IX.4.1 admits of a suitable immediate extension to arbitrary dimension $n \geq 4$, which will be appropriate to our purposes. Actually, if we set $\boldsymbol{W} = \nabla \boldsymbol{U}$ (*i.e.*, $W_{ij} = \partial U_j / \partial x_i$) with $\boldsymbol{U}$ defined in Exercise IX.4.1, then it is easily seen that $\boldsymbol{v} = \nabla \cdot \boldsymbol{W}$ (*i.e.* $v_j = \partial W_{ij} / \partial x_i$) and that W_{ij} satisfies an estimate of the type (IX.4.14). More generally, if $\boldsymbol{a} \in W^{1-1/q,q}(\partial\Omega)$, $1 < q < \infty$, then for all $i, j = 1, \ldots, n$ we have

$$\|W_{ij}\|_{2,q} \leq c\|\boldsymbol{a}\|_{1-1/q,q(\partial\Omega)}.$$

■

The result just shown allows us to construct the desired extension of the field $\boldsymbol{v}_*$. Let us introduce some notation first. We denote by $c = c(n, \Omega)$ the constant entering the problem:

$$\begin{aligned} \nabla \cdot \boldsymbol{b} &= h \ \text{ in } \Omega \\ \boldsymbol{b} &\in W_0^{1,2}(\Omega) \\ |\boldsymbol{b}|_{1,2} &\leq c\|h\|_2. \end{aligned} \tag{IX.4.20}$$

Moreover, if $\partial\Omega$ has more than one boundary, $\Gamma_1, .., \Gamma_{m+1}$, with $\Gamma_i, i = 1, \ldots, m$, the "interior" boundaries and Γ_{m+1} the "outer" one, we set

$$\begin{aligned} d &\equiv \min_{i,j} \operatorname{dist}(\Gamma_i, \Gamma_j) \\ \Omega_{i,d} &= \{x \in \Omega : \operatorname{dist}(x, \Gamma_i) < d/2\} \end{aligned} \tag{IX.4.21}$$

and, indicated by ω_i, $i = 1, ..., m$, the (bounded) connected components of $\mathbb{R}^n - \overline{\Omega}$,

$$\begin{aligned} \boldsymbol{\sigma}_i(x) &= -\nabla \mathcal{E}(x - x_i), \quad x_i \in \omega_i, \quad i = 1, \ldots, m \\ \boldsymbol{\sigma}_{m+1}(x) &= -\boldsymbol{\sigma}_1(x), \end{aligned} \tag{IX.4.22}$$

where $\mathcal{E}(\xi)$ is the fundamental solution of Laplace's equation defined in (II.9.1). Clearly, we have

$$\int_{\Gamma_i} \boldsymbol{\sigma}_i \cdot \boldsymbol{n} = 1, \quad i = 1, \ldots, m+1, \tag{IX.4.23}$$

where $\boldsymbol{n}$ denotes the outer normal to $\partial\Omega$ at Γ_i. The following extension lemma holds.

Lemma IX.4.2 *Let Ω be a bounded locally Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$, and let $\boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega)$ satisfy*

$$\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} = 0\,. \tag{IX.4.24}$$

Then, for any $\eta > 0$, there exist $\varepsilon = \varepsilon(\eta, \boldsymbol{v}_*, n, \Omega) > 0$, and a solenoidal vector field $\boldsymbol{V} = \boldsymbol{V}(\varepsilon)$ such that

$$\boldsymbol{V} \in W^{1,2}(\Omega), \quad \textit{with } \boldsymbol{V} = \boldsymbol{v}_* \textit{ at } \partial\Omega$$

and verifying

$$\left|\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{V}\cdot\boldsymbol{u}\right| \le \left\{\eta + \sum_{i=1}^{m+1}\left(\kappa^2\frac{4c\kappa_2}{d}\|\boldsymbol{\sigma}_i\|_{2,\Omega_{i,d}} + \kappa\|\boldsymbol{\sigma}_i\|_{4,\Omega_{i,d}}\right)|\Phi_i|\right\}|\boldsymbol{u}|_{1,2}^2 \tag{IX.4.25}$$

for all $\boldsymbol{u} \in H^1(\Omega)$. *Here* κ, κ_2 *are constants depending on* n *and defined in* (IX.4.34), (IX.4.35), *and* Lemma III.6.1, *respectively,* $c = c(n,\Omega)$ *is defined in* (IX.4.20) *and*

$$\Phi_i = \int_{\Gamma_i} \boldsymbol{v}_*\cdot\boldsymbol{n}, \quad i = 1,\ldots,m+1.$$

Furthermore, $\boldsymbol{\sigma}_i$ *and* $\Omega_{i,d}$ *are given in* (IX.4.21) *and* (IX.4.22). *Finally, if* $\boldsymbol{v}_*$ *lies in a ball of* $W^{1/2,2}(\partial\Omega)$, *namely,* $\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \le M$, *for some* $M > 0$, *then there is* $C = C(n,\Omega,\eta,M) > 0$ *such that*

$$\|\boldsymbol{V}\|_{1,2} \le C\,\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\,. \tag{IX.4.26}$$

Proof. We shall first consider the case $m > 0$. Let

$$\delta_i(x) \equiv \operatorname{dist}(x,\Gamma_i), \quad x \in \Omega, \quad i = 1,\ldots,m+1,$$

and denote by $\rho_i(x)$ the regularized distance of x from Γ_i, in the sense of Stein (cf. Lemma III.6.1). Set

$$\psi(t) = \begin{cases} 1 & t \le 1 \\ 2-t & 1 \le t \le 2 \\ 0 & t \ge 2 \end{cases}$$

and define

$$\psi_i(x) \equiv \psi(4\rho_i(x)/d), \quad i = 1,\ldots,m+1. \tag{IX.4.27}$$

Recalling the properties of $\rho_i(x)$, we have that the functions (IX.4.27) are piecewise differentiable and that, moreover,

$$\begin{aligned} \psi_i(x) &= 1 \quad \text{if } \delta_i(x) < d/4\kappa_1 \\ \psi_i(x) &= 0 \quad \text{if } \delta_i(x) \ge d/2 \\ |\psi_i(x)| &\le 1 \\ |\nabla\psi_i(x)| &\le 4\kappa_2/d \\ \operatorname{supp}(\nabla\psi_i) &\subset \{x \in \Omega : d/4\kappa_1 \le \delta_i(x) \le d/2\} \end{aligned} \tag{IX.4.28}$$

where κ_1 and κ_2 are the constants introduced in Lemma III.6.1. In view of (IX.4.23) and (IX.4.28) we recover that the field

$$\boldsymbol{v}_1(x) := \boldsymbol{v}_*(x) - \sum_{i=1}^{m+1} \Phi_i \psi_i(x) \boldsymbol{\sigma}_i(x), \quad x \in \partial\Omega \tag{IX.4.29}$$

satisfies the $m+1$ conditions

$$\int_{\Gamma_i} \boldsymbol{v}_1 \cdot \boldsymbol{n} = 0, \quad i = 1, \dots, m+1.$$

By Lemma IX.4.1 we then have that, if $n = 3$, there is a $\boldsymbol{w} \in W^{2,2}(\Omega)$ [respectively $w \in W^{2,2}(\Omega)$, if $n = 2$] such that $\boldsymbol{v}_1(x) = \nabla \times \boldsymbol{w}(x)$, $x \in \partial\Omega$ [respectively $\boldsymbol{v}_1(x) = \nabla \times w(x)$]. For given $\varepsilon > 0$ we set

$$\boldsymbol{V}_\varepsilon = \nabla \times (\psi_\varepsilon \boldsymbol{w}) \quad [\text{respectively } \boldsymbol{V}_\varepsilon = \nabla \times (\psi_\varepsilon w)] \tag{IX.4.30}$$

where ψ_ε is the "cut-off" function defined in Lemma III.6.2. From the properties of ψ_ε and $\boldsymbol{w}$ we easily realize that the field

$$\boldsymbol{U}(x) = \boldsymbol{V}_\varepsilon(x) + \sum_{i=1}^{m+1} \Phi_i \psi_i(x) \boldsymbol{\sigma}_i(x), \quad x \in \Omega,$$

is a $W^{1,2}(\Omega)$-extension of $\boldsymbol{v}_*$. However, $\boldsymbol{U}$ is not solenoidal and, therefore, in order to obtain the desired extension of $\boldsymbol{v}_*$, we have to modify $\boldsymbol{U}$ appropriately. To this end, let us consider the field $\boldsymbol{b}$ defined by the following properties:

$$\begin{aligned} \nabla \cdot \boldsymbol{b} &= -\sum_{i=1}^{m+1} \boldsymbol{\sigma}_i(x) \cdot \nabla(\Phi_i \psi_i(x)) \equiv h(x) \\ \boldsymbol{b} &\in W_0^{1,2}(\Omega) \\ |\boldsymbol{b}|_{1,2} &\le c\,\|h\|_2. \end{aligned} \tag{IX.4.31}$$

Since, by (IX.4.24) and (IX.4.28),

$$h \in L^q(\Omega), \quad \text{for all } q \in (1, \infty)$$

$$\int_\Omega h = 0,$$

Theorem III.3.1 ensures the existence of at least one vector $\boldsymbol{b}$ satisfying (IX.4.31). Furthermore, using $(\text{IX.4.28})_{3,4}$, we obtain

$$|\boldsymbol{b}|_{1,2} \le \frac{4c\kappa_2}{d} \sum_{i=1}^{m+1} \|\boldsymbol{\sigma}_i\|_{2,\Omega_{i,d}} |\Phi_i|. \tag{IX.4.32}$$

The desired extension of $\boldsymbol{v}_*$ is then given by the field:

$$\boldsymbol{V}(x) := \boldsymbol{V}_\varepsilon(x) + \sum_{i=1}^{m+1} \Phi_i \psi_i(x) \boldsymbol{\sigma}_i(x) + \boldsymbol{b}(x) \equiv \boldsymbol{V}_\varepsilon(x) + \boldsymbol{V}_{\boldsymbol{\sigma}}(x) + \boldsymbol{b}(x). \tag{IX.4.33}$$

In fact, $\boldsymbol{V}$ is solenoidal, belongs to $W^{1,2}(\Omega)$, and its trace at $\partial\Omega$ is $\boldsymbol{v}_*$. Let us now estimate the trilinear form:

$$a(\boldsymbol{u}, \boldsymbol{V}, \boldsymbol{u}) \equiv \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{V} \cdot \boldsymbol{u}, \quad \boldsymbol{u} \in H^1.$$

In this respect, we recall the inequality:

$$\|\boldsymbol{u}\|_4 \leq \kappa |\boldsymbol{u}|_{1,2} \tag{IX.4.34}$$

where

$$\kappa = \begin{cases} 2^{7/4} 3^{-13/8} |\Omega|^{1/12} & \text{if } n = 3 \\ |\Omega|^{1/4}/\sqrt{2} & \text{if } n = 2. \end{cases} \tag{IX.4.35}$$

Actually, (IX.4.34) and (IX.4.35) follow from (II.3.9), (II.3.10), and (II.5.5). By the Hölder inequality, (IX.4.32), (IX.4.34), and (IX.4.35) we obtain

$$|a(\boldsymbol{u}, \boldsymbol{b}, \boldsymbol{u})| \leq \|\boldsymbol{u}\|_4^2 |\boldsymbol{b}|_{1,2} \leq \sum_{i=1}^{m+1} \left(\kappa^2 \frac{4c\kappa_2}{d} \|\boldsymbol{\sigma}_i\|_{2,\Omega_{i,d}} |\Phi_i| \right) |\boldsymbol{u}|_{1,2}^2. \tag{IX.4.36}$$

Furthermore, by an easily justified integration by parts we find

$$|a(\boldsymbol{u}, \boldsymbol{V}_{\boldsymbol{\sigma}}, \boldsymbol{u})| = |a(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{V}_{\boldsymbol{\sigma}})|$$

and so, again by the Hölder inequality, $(\text{IX.4.28})_3$, (IX.4.33), and (IX.4.34), it follows that

$$|a(\boldsymbol{u}, \boldsymbol{V}_{\boldsymbol{\sigma}}, \boldsymbol{u})| \leq \|\boldsymbol{u}\|_4 |\boldsymbol{u}|_{1,2} \|\boldsymbol{V}_{\boldsymbol{\sigma}}\|_4 \leq \kappa |\boldsymbol{u}|_{1,2}^2 \sum_{i=1}^{m+1} \|\boldsymbol{\sigma}_i\|_{4,\Omega_{i,d}} |\Phi_i|. \tag{IX.4.37}$$

It remains to estimate the term

$$a(\boldsymbol{u}, \boldsymbol{V}_\varepsilon, \boldsymbol{u}).$$

From the properties of the function ψ_ε established in Lemma III.6.2 we find that

$$|\boldsymbol{V}_\varepsilon(x)| \begin{cases} \leq \dfrac{\varepsilon\kappa_2}{\delta(x)} |\boldsymbol{w}(x)| + |\nabla \boldsymbol{w}(x)|, & \text{if } x \in \Omega_\varepsilon \\ = 0, & \text{if } x \notin \Omega_\varepsilon \end{cases} \tag{IX.4.38}$$

where $\delta(x)$ is the distance from $x \in \Omega$ to $\partial\Omega$,

$$\Omega_\varepsilon \equiv \{x \in \Omega : \delta(x) \leq 2\gamma(\varepsilon)\};$$

and $\gamma(\varepsilon) := \exp(-1/\varepsilon)$. Observe that, for any $k > 0$,

$$|\Omega_\varepsilon|^k \leq c_0\, \varepsilon\,, \tag{IX.4.39}$$

with $c_0 = c_0(n, k, \Omega)$. Moreover, from the embedding Theorem II.3.4,

$$\begin{aligned} |\boldsymbol{w}(x)| &\le c_1 \|\nabla \boldsymbol{w}\|_{2,2} \\ \|\nabla \boldsymbol{w}\|_4 &\le c_1 \|\boldsymbol{w}\|_{2,2} \end{aligned} \tag{IX.4.40}$$

which, by Lemma IX.4.1, in turn implies

$$\|\nabla \boldsymbol{w}\|_4 + |\boldsymbol{w}(x)| \le c_2 \|\boldsymbol{v}_1\|_{1/2,2(\partial\Omega)}. \tag{IX.4.41}$$

Thus, (IX.4.38), along with (IX.4.41), (IX.4.29) and (II.3.7), gives for all $\boldsymbol{u} \in H^1(\Omega)$

$$\begin{aligned} \| |\boldsymbol{u}| |\boldsymbol{V}_\varepsilon| \|_2 &\le \left[\varepsilon \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \|\boldsymbol{u}\delta^{-1}\|_2 + \left(\int_{\Omega_\varepsilon} u^2 |\nabla \boldsymbol{w}|^2 \right)^{1/2} \right] \\ &\le c_3 \left(\varepsilon \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \|\boldsymbol{u}\delta^{-1}\|_2 + |\boldsymbol{u}|_{1,2} \|\nabla \boldsymbol{w}\|_{3,\Omega_\varepsilon} \right), \end{aligned} \tag{IX.4.42}$$

By Lemma III.6.3, we have

$$\|\boldsymbol{u}\delta^{-1}\|_2 \le c_4 |\boldsymbol{u}|_{1,2},$$

while, by Hölder inequality and (IX.4.41),

$$\|\nabla \boldsymbol{w}\|_{3,\Omega_\varepsilon} \le |\Omega_\varepsilon|^{\frac{1}{12}} \|\nabla \boldsymbol{w}\|_{4,\Omega} \le c_5 \, |\Omega_\varepsilon|^{\frac{1}{12}} \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \, .$$

Thus, from (IX.4.39) and (IX.4.42), we infer

$$\| |\boldsymbol{u}| |\boldsymbol{V}_\varepsilon| \|_2 \le c_6 \, \varepsilon \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} |\boldsymbol{u}|_{1,2} \, , \tag{IX.4.43}$$

where $c_6 = c_6(n, \Omega)$. Fix arbitrary $\eta > 0$ and choose

$$\varepsilon \le \frac{\eta}{c_6 \, \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}}. \tag{IX.4.44}$$

From (IX.4.43), (IX.4.44), Lemma IX.2.1, and the Schwarz inequality we then conclude that

$$|a(\boldsymbol{u}, \boldsymbol{V}_\varepsilon, \boldsymbol{u})| = |a(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{V}_\varepsilon)| \le \eta \, |\boldsymbol{u}|^2_{1,2}. \tag{IX.4.45}$$

Collecting (IX.4.33), (IX.4.36), (IX.4.37), and (IX.4.45) yields

$$\begin{aligned} |a(\boldsymbol{u}, \boldsymbol{V}, \boldsymbol{u})| \le \Bigg\{ \eta + \sum_{i=1}^{m+1} \Bigg(&\kappa^2 \tfrac{4c\kappa_2}{d} \|\boldsymbol{\sigma}_i\|_{2,\Omega_{i,d}} \\ &+ \kappa \frac{4\kappa_2}{d} \|\boldsymbol{\sigma}_i\|_{4,\Omega_{i,d}} \Bigg) |\Phi_i| \Bigg\} |\boldsymbol{u}|^2_{1,2} \end{aligned} \tag{IX.4.46}$$

which coincides with (IX.4.25) if $m > 0$. If $m = 0$ the proof is simpler, since, then, one can take $\boldsymbol{V} = \boldsymbol{V}_\varepsilon$ and proceed as before to arrive formally at (IX.4.46) with identically vanishing Φ_i. The first part of the lemma is therefore proved. In order to show (IX.4.26), we observe that, from (IX.4.32), (IX.4.33), it obviously follows that

$$\|\boldsymbol{V}_{\boldsymbol{\sigma}}\|_{1,2}+\|\boldsymbol{b}\|_{1,2}\le c_7\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\,. \tag{IX.4.47}$$

It remains to give an analogous estimate for $\boldsymbol{V}_\varepsilon$. To this end, we notice that, under the stated hypothesis on $\boldsymbol{v}_*$, (IX.4.44) is certainly satisfied if we choose $\varepsilon=\eta/(c_6\,M)\equiv\varepsilon_1$. Thus, from (IX.4.30) and the properties of the function ψ_ε, we easily obtain that

$$\|\boldsymbol{V}_\varepsilon\|_{1,2}\le c_8\left(\|\boldsymbol{w}\|_{2,2}+\|\boldsymbol{w}/(\delta^2+\delta)\|_{2,\Omega'_\varepsilon}+\|\nabla\boldsymbol{w}/\delta\|_{2,\Omega'_\varepsilon}\right)$$

where

$$\Omega'_\varepsilon:=\left\{x\in\Omega:\ \gamma^2(\varepsilon)/(2\kappa_1)\le\delta(x)\le 2\gamma(\varepsilon)\right\}\,.$$

and $c_8=c_8(n,\Omega,\varepsilon_1)$. Consequently, from (IX.4.14) and the last two displayed equations we deduce

$$\|\boldsymbol{V}_\varepsilon\|_{1,2}\le c_9\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \tag{IX.4.48}$$

where $c_9=c_9(n,\Omega,\eta,M)$. The proof of the lemma is completed.

□

Remark IX.4.7 It is simple to generalize Lemma IX.4.2 to dimension $n\ge 4$, provided we make appropriate changes in the proof just given. Actually, it suffices to use, instead of (IX.4.34), the Sobolev inequality (II.3.7), to take the field $\boldsymbol{b}$ as solution to the following problem

$$\nabla\cdot\boldsymbol{b}=h$$

$$\boldsymbol{b}\in W_0^{1,n/2}(\Omega)$$

$$|\boldsymbol{b}|_{1,n/2}\le c\|h\|_{n/2}$$

and, finally, to choose

$$\boldsymbol{V}_\varepsilon=\nabla\cdot(\psi_\varepsilon\boldsymbol{W})$$

as an extension of the field $\boldsymbol{v}_1$, with $\boldsymbol{W}$ defined in Remark IX.4.5. However, in order that $\boldsymbol{V}_\varepsilon$ satisfies the estimate needed in the lemma, we should require that $\boldsymbol{v}_*$ has slightly more regularity. Actually, in dimensions higher than three, (IX.4.40) need not hold and we have, instead,

$$\left.\begin{aligned}|\boldsymbol{W}(x)|\ &\le c_1\|\boldsymbol{W}\|_{2,q}\\ \|\nabla\boldsymbol{W}\|_s&\le c_1\|\boldsymbol{W}\|_{2,q}\end{aligned}\right\}\quad q>n/2,\ s>n; \tag{IX.4.49}$$

see Theorem II.3.4. On the other hand, taking into account Remark IX.4.6, the right-hand side of (IX.4.49) is finite provided $\boldsymbol{v}_*\in W^{1-1/q,q}(\partial\Omega)$, $q>n/2$. Therefore, if $n\ge 4$, under this additional condition on $\boldsymbol{v}_*$ the vector field (IX.4.33) belongs to $W^{1,q}(\Omega)$,[6] and satisfies (IX.4.45). One can then show that the trilinear form $a(\boldsymbol{u},\boldsymbol{V},\boldsymbol{u})$ satisfies the estimate

[6] Notice that, since $h\in L^r(\Omega)$, for all $r>1$, we may take $\boldsymbol{b}\in W^{1,q}(\Omega)$; see Remark III.3.4.

$$|a(\boldsymbol{u},\boldsymbol{V},\boldsymbol{u})| \le \left\{\eta + \sum_{i=1}^{m+1} \left(c_3\|\boldsymbol{\sigma}_i\|_{n/2,\Omega_{i,d}} + c_4\|\boldsymbol{\sigma}_i\|_{n,\Omega_{i,d}}\right)|\Phi_i|\right\}|\boldsymbol{u}|_{1,2}^2$$

with c_3 and c_4 suitable constants. Likewise, inequality (IX.4.26) is replaced by the following one

$$\|\boldsymbol{V}\|_{1,q} \le C\,\|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)}\,,$$

with $C = C(n,\Omega,\eta,M)$, provided $\|\boldsymbol{v}_*\|_{1-1/q,q(\partial\Omega)} \le M$. ■

Exercise IX.4.2 (Alekseev & Tereshko, 1998) By adopting (and simplifying) the arguments used in the proof of Lemma IX.4.2, show the following result. Suppose Ω and $\boldsymbol{v}_*$ satisfy the assumption of Lemma IX.4.2 with $\Phi_i = 0$, $i = 1,\dots,m+1$. Then, given $\eta > 0$ there exists a solenoidal extension, $\boldsymbol{V}_\eta \in W^{1,2}(\Omega)$ of $\boldsymbol{v}_*$, such that

$$\left|\int_\omega \boldsymbol{u}\cdot\nabla\boldsymbol{V}_\eta\cdot\boldsymbol{u}\right| \le \eta\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\,|\boldsymbol{u}|_{1,2}^2\,,$$

for all $\boldsymbol{u}\in H^1(\Omega)$. Moreover, there is a constant $c = c(\eta,n,\Omega)$ such that

$$\|\boldsymbol{V}_\eta\|_{1,2} \le c\,\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\,.$$

We are now in a position to prove the main results of this section.

Theorem IX.4.1 *Let Ω be a bounded locally Lipschitz domain of $\mathbb{R}^n$, $n = 2,3$, with $\partial\Omega$ constituted by $m+1$ connected components $\Gamma_1\dots\Gamma_{m+1}$, $m\ge 0$, and let*

$$\boldsymbol{v}_*\in W^{1/2,2}(\partial\Omega),\quad \boldsymbol{f}\in D_0^{-1,2}(\Omega),$$

with $\boldsymbol{v}_*$ satisfying (IX.4.24). *The following properties hold.*

(i) Existence. *If*

$$\Phi \equiv \sum_{i=1}^{m+1}\left(\kappa^2\frac{4c\kappa_2}{d}\|\boldsymbol{\sigma}_i\|_{n/2,\Omega_{i,d}} + \kappa\|\boldsymbol{\sigma}_i\|_{n,\Omega_{i,d}}\right)\left|\int_{\Gamma_i}\boldsymbol{v}_*\cdot\boldsymbol{n}\right| < \nu\,, \tag{IX.4.50}$$

there is at least one generalized solution $\boldsymbol{v}$ to problem (IX.0.1), (IX.0.2), *with corresponding pressure field $p\in L^2(\Omega)$, associated to $\boldsymbol{v}$ by* Lemma IX.1.2, *that satisfies the inequality*

$$\|p\|_2 \le c\left(|\boldsymbol{f}|_{-1,2} + \|\boldsymbol{v}\|_{1,2}^2 + \nu\|\boldsymbol{v}\|_{1,2}\right) \tag{IX.4.51}$$

where $c = c(n,\Omega)$.

(ii) Estimate by the data. (a) *Let*

$$\mathfrak{W}_M^{1/2,2}(\partial\Omega) := \{\boldsymbol{\phi}\in W^{1/2,2}(\partial\Omega):\ \|\boldsymbol{\phi}\|_{1/2,2(\partial\Omega)}\le M\},\quad \textit{some } M>0. \tag{IX.4.52}$$

If $\boldsymbol{v}_* \in \mathfrak{W}_M^{1/2,2}(\partial\Omega)$ *and* $\Phi \le \nu/2$*, any generalized solution* $\boldsymbol{v}$ *corresponding to* $\boldsymbol{v}_*$ *and* $\boldsymbol{f}$ *satisfies the estimate:*

$$\|\boldsymbol{v}\|_{1,2} \le \frac{c_1}{\nu}|\boldsymbol{f}|_{-1,2} + c_2\left(\|\boldsymbol{v}_*\|^2_{1/2,2(\partial\Omega)} + \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\right), \tag{IX.4.53}$$

where $c_1 = c_1(n,\Omega)$*, while* $c_2 = c_2(n,\Omega,\nu,M)$*.*
(b) *There exists* $c_3 = c_3(n,\Omega)$ *such that if*

$$\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \le c_3\nu/2,$$

any generalized solution $\boldsymbol{v}$ *corresponding to* $\boldsymbol{v}_*$ *and* $\boldsymbol{f}$ *verifies the following estimate*

$$\|\boldsymbol{v}\|_{1,2} \le \frac{c_4}{\nu}\left(|\boldsymbol{f}|_{-1,2} + \|\boldsymbol{v}_*\|^2_{1/2,2(\partial\Omega)} + \nu\,\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\right), \tag{IX.4.54}$$

with $c_4 = c_4(n,\Omega)$*. Estimates for* p *follow from* (IX.4.51)–(IX.4.54).

Proof. We look for a solution of the form $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{V}$, where $\boldsymbol{V}$ is the extension of the field $\boldsymbol{v}_*$ constructed in Lemma IX.4.2. We then consider a sequence $\{\boldsymbol{u}_s\}$ of "approximating solutions" as in Theorem IX.3.1, *i.e.*,

$$\begin{gathered}\boldsymbol{u}_s = \sum_{k=1}^{s}\xi_{ks}\boldsymbol{\psi}_k \\ \nu(\nabla\boldsymbol{u}_s,\nabla\boldsymbol{\psi}_k) + (\boldsymbol{u}_s\cdot\nabla\boldsymbol{u}_s,\boldsymbol{\psi}_k) + (\boldsymbol{u}_s\cdot\nabla\boldsymbol{V},\boldsymbol{\psi}_k) + (\boldsymbol{V}\cdot\nabla\boldsymbol{u}_s,\boldsymbol{\psi}_k) \\ = -\langle\boldsymbol{f},\boldsymbol{\psi}_k\rangle - \nu(\nabla\boldsymbol{V},\nabla\boldsymbol{\psi}_k) - (\boldsymbol{V}\cdot\nabla\boldsymbol{V},\boldsymbol{\psi}_k), \quad k = 1,2,\ldots,s.\end{gathered} \tag{IX.4.55}$$

From this point on we can repeat step by step the proof of Theorem IX.3.1, provided we show a uniform bound on $|\boldsymbol{u}_s|_{1,2}$. But, as already seen, this is easily achieved thanks to the particular choice of the field $\boldsymbol{V}$. Actually, multiplying $(\text{IX.4.55})_2$ by ξ_{ks}, summing over k from one to s and recalling (IX.3.8) we have

$$\nu|\boldsymbol{u}_s|^2_{1,2} + (\boldsymbol{u}_s\cdot\nabla\boldsymbol{V},\boldsymbol{u}_s) = -\langle\boldsymbol{f},\boldsymbol{u}_s\rangle - \nu(\nabla\boldsymbol{V},\nabla\boldsymbol{u}_s) - (\boldsymbol{V}\cdot\nabla\boldsymbol{V},\boldsymbol{u}_s). \tag{IX.4.56}$$

Using (IX.4.25) with $\eta = (\nu - \Phi)/2$ (say) and Lemma IX.1.1, from (IX.4.56) it follows that

$$\frac{1}{2}(\nu - \Phi)|\boldsymbol{u}_s|^2_{1,2} \le (|\boldsymbol{f}|_{-1,2} + C(\boldsymbol{V},\nu))\,|\boldsymbol{u}_s|_{1,2},$$

which, in view of (IX.4.50), furnishes the desired bound on $|\boldsymbol{u}_s|_{1,2}$. Moreover, (IX.4.51) is established exactly as in Theorem IX.3.1. The proof of the existence can be then considered complete. We shall now show the second part of the theorem. Let $\boldsymbol{v}$ be a generalized solution corresponding to $\boldsymbol{v}_*$ and $\boldsymbol{f}$. We write $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{U}$, with $\boldsymbol{U} \in W^{1,2}(\Omega)$ solenoidal extension of $\boldsymbol{v}_*$ to be specified later on. From (IX.1.2) we find

$$\begin{aligned}\nu(\nabla \boldsymbol{w}, \nabla \boldsymbol{\varphi}) + (\boldsymbol{w}\cdot\nabla \boldsymbol{w}, \boldsymbol{\varphi}) + (\boldsymbol{w}\cdot\nabla \boldsymbol{U}, \boldsymbol{\varphi}) + (\boldsymbol{U}\cdot\nabla \boldsymbol{w}, \boldsymbol{\varphi})\\ = -\langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle - \nu(\nabla \boldsymbol{U}, \nabla \boldsymbol{\varphi}) - (\boldsymbol{U}\cdot\nabla \boldsymbol{U}, \boldsymbol{\varphi}).\end{aligned}$$

Since $\boldsymbol{w} \in \widehat{H}^1(\Omega)$, from Lemma IX.1.1 and Section III.4.1 we can replace $\boldsymbol{\varphi}$ with $\boldsymbol{w}$ in the previous identity to obtain

$$\nu|\boldsymbol{w}|_{1,2}^2 + (\boldsymbol{w}\cdot\nabla \boldsymbol{U}, \boldsymbol{w}) = -\langle \boldsymbol{f}, \boldsymbol{w}\rangle - \nu(\nabla \boldsymbol{U}, \nabla \boldsymbol{w}) - (\boldsymbol{U}\cdot\nabla \boldsymbol{U}, \boldsymbol{w}). \quad \text{(IX.4.57)}$$

We now choose $\boldsymbol{U} \equiv \boldsymbol{V}$, where $\boldsymbol{V}$ is the extension constructed in Lemma IX.4.2. Thus, on the account that $\Phi \le \nu/2$ and by taking $\eta = \nu/4$, from (IX.4.57), Lemma IX.4.2 and Lemma IX.1.1, we find

$$\frac{\nu}{4}|\boldsymbol{w}|_{1,2} \le C_1\left(|\boldsymbol{f}|_{-1,2} + \|\boldsymbol{V}\|_{1,2}^2 + \nu\|\boldsymbol{V}\|_{1,2}\right)$$

where $C_1 = C_1(n,\Omega)$. Since $\boldsymbol{v}_* \in \mathfrak{W}_M^{1/2,2}(\partial\Omega)$, from Lemma IX.4.2, we find that the extension $\boldsymbol{V}$ satisfies (IX.4.26). Thus, (IX.4.53) follows from this latter displayed inequality and (IX.4.26).[7] It remains to show (IX.4.54). To this end, we choose $\boldsymbol{U} \in W^{1,2}(\Omega)$ to be the solenoidal extension of $\boldsymbol{v}_*$ given in Exercise III.3.5. From (IX.4.57), Lemma IX.1.1 and the condition (Exercise III.3.5)

$$\|\boldsymbol{U}\|_{1,2} \le c\,\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)},$$

we then easily obtain

$$\begin{aligned}\nu|\boldsymbol{w}|_{1,2}^2 \le c\Big\{\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}|\boldsymbol{w}|_{1,2}^2 + \Big(|\boldsymbol{f}|_{-1,2} + \nu\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\\ +\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}^2\Big)|\boldsymbol{w}|_{1,2}\Big\},\end{aligned}$$

and (IX.4.54) follows from this latter inequality and the assumption on $\boldsymbol{v}_*$.

□

Remark IX.4.8 The estimate of generalized solution in terms of the boundary data given in (IX.4.53) deserves some comments. We wish to emphasize that the method we employed, which goes back to J. Leray and E. Hopf (see the Notes for this Chapter for more details), does not seem to furnish the estimate (IX.4.53) unless we require the boundedness of the set of the boundary data (that is, $\boldsymbol{v}_* \in \mathfrak{W}_M^{1/2,2}(\partial\Omega)$. Moreover, the dependence of the constant c_2 in (IX.4.53) on the coefficient of kinematic viscosity ν may be very complicated. However, if $\nu \ge \nu_0$, for some positive ν_0, then c_2 depends only on ν_0. These facts seem to have been overlooked by several authors, including myself; see Finn (1961a, Theorem 2.3), Ladyzhenskaya (1969, Chapter 5, Section 4), Galdi (1994b, Theorem VIII.4.1), Finn & Solonnikov (1997, Theorem 3). ■

[7] Notice that the constant c_2 in (IX.4.53) depends also on ν, because the choice of η depends on ν, and the constant C in (IX.4.26) depends on η.

Sufficient conditions for the uniqueness of generalized solutions are at once derived from Theorem IX.2.1 and (IX.4.54), and we find the following.

Theorem IX.4.2 *The generalized solution $\boldsymbol{v}$ constructed in* Theorem IX.4.1 *is unique in the class of generalized solutions corresponding to the same $\boldsymbol{f}$ and $\boldsymbol{v}_*$ provided*

$$\frac{1}{\nu}\Big(|\boldsymbol{f}|_{-1,2} + \|\boldsymbol{v}_*\|^2_{1/2,2(\partial\Omega)}\Big) + \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} < C\,\nu,$$

where $C = \min\{c_3/2, 1/\sqrt{c_4 k}\}$, while c_3, c_4 and k are defined in Theorem IX.4.1 *and in* Theorem IX.2.1, *respectively.*

Remark IX.4.9 If $m = 0$, that is, if the boundary of Ω is constituted by only one connected surface (line, for plane flow) Γ, say, condition (IX.4.50) is automatically satisfied, since, by the incompressibility condition,

$$\int_\Gamma \boldsymbol{v}_* \cdot \boldsymbol{n} \equiv \int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} = 0.$$

Moreover, if $m > 1$, condition (IX.4.50) furnishes a *computable* bound on the fluxes Φ_i in terms of ν. It may be of a certain interest to evaluate this bound when Ω is an annulus. In fact, as we have noticed at the beginning of this section, such domains cannot admit, in general, an extension field $\boldsymbol{V}(\alpha)$ of $\boldsymbol{v}_*$ obeying (IX.4.3) for arbitrary $\alpha > 0$, and, as a consequence, the Leray-Hopf construction of steady-state solutions would require identically vanishing Φ_i. To fix the ideas, take Ω to be the annulus bounded by R and $2R$, that is,

$$\Omega = \big\{x \in \mathbb{R}^2 : R < |x| < 2R\big\}. \tag{IX.4.58}$$

Thus, in the notation of Lemma IX.4.1 and Theorem IX.4.1, we have[8]

$$\begin{gathered} d = 2R - R = R, \quad \kappa_2 = 1 \\ \boldsymbol{\sigma}_1(x) = -\boldsymbol{\sigma}_2(x) = -\nabla(\log|x|)/2\pi = -1/(2\pi|x|), \\ \Omega_{1,d} = \big\{x \in \mathbb{R}^2 : R < |x| < 3R/2\big\}, \\ \Omega_{2,d} = \big\{x \in \mathbb{R}^2 : 3R/2 < |x| < 2R\big\}. \end{gathered}$$

Moreover, we have to give explicit values to the constants κ and c defined in (IX.4.31) and (IX.4.35), respectively. Concerning κ, from (IX.4.35) and (IX.4.58) we at once obtain

$$\kappa = (3\pi)^{1/4}\sqrt{R/2} \simeq 1.238\sqrt{R}.$$

However, a sharper estimate can be obtained on κ. Actually, from the Ladyzhenskaya inequality (II.3.9)

[8] For the value of κ_2, see Remark III.6.1.

$$\|u\|_4 \le 2^{-1/4}\|u\|_2^{1/2}\|\nabla u\|_2^{1/2}$$

and from (IX.4.34) we find that κ can be estimated by the product of $2^{-1/4}$ times the fourth root of the Poincaré constant μ, defined as

$$\mu = \mu(\Omega) = \max_{u\in W_0^{1,2}(\Omega)} \left(\frac{\|u\|_2^2}{|u|_{1,2}^2}\right);$$

cf. (II.5.3). The value of μ can be calculated from the formula

$$R/\sqrt{\mu} = \pi - 1/(16\pi) + 163/(3072\pi^2) - 93029/(491520\pi^5) + \ldots;$$

cf. McLachlan (1961,§1.62, eq.(4)). We then recover $\mu \simeq 0.102\cdot R^2$ and

$$\kappa \simeq 0.476\sqrt{R}. \tag{IX.4.59}$$

To evaluate the constant c, we observe that, since the products $\boldsymbol{\sigma}_i\cdot\nabla\psi_i$, $i=1,2$, depend only on $r\equiv|x|$, the function h in (IX.4.35) depends only on r. Therefore, a solution $\boldsymbol{b}$ to (IX.4.35) with Ω given in (IX.4.57) can be chosen of the form

$$\boldsymbol{b}(x) = \frac{\boldsymbol{x}}{r^2}\int_R^r \xi h(x)d\xi, \quad x\in\Omega.$$

Since

$$\frac{\partial b_2}{\partial x_1} = \frac{\partial b_1}{\partial x_2},$$

by a direct computation we show that

$$|\boldsymbol{b}|_{1,2} = \|\nabla\cdot\boldsymbol{b}\|_2 = \|h\|_2,$$

and we conclude that $c=1$. Collecting all these data and setting $\Phi\equiv -\Phi_1 = \Phi_2$, condition (IX.4.50) becomes

$$H|\Phi| < \nu, \tag{IX.4.60}$$

where

$$H \equiv \frac{\kappa}{2\pi}\left[\frac{4\kappa}{2\pi}(A+B)+(C+D)\right],$$

$$A = \left(2\pi\int_R^{3R/2}\xi^{-1}d\xi\right)^{1/2}, \quad B = \left(2\pi\int_{3R/2}^{R}\xi^{-1}d\xi\right)^{1/2},$$

$$C = \left(2\pi\int_R^{3R/2}\xi^{-3}d\xi\right)^{1/4}, \quad D = \left(2\pi\int_{3R/2}^{R}\xi^{-3}d\xi\right)^{1/4},$$

and κ is given in (IX.4.59). Evaluation of H furnishes

$$H \simeq 0.58$$

and the flux condition (IX.4.60) becomes

$$|\Phi| < 1.72\nu.$$

■

Remark IX.4.10 The existence result of Theorem IX.4.1 can be extended to any dimension $n \geq 4$, provided

$$\boldsymbol{v}_* \in W^{1-1/q,q}(\partial\Omega), \quad q > n/2,$$

and

$$\Phi_{(n)} \equiv \sum_{i=1}^{m+1} \left(c_1\|\boldsymbol{\sigma}_i\|_{n/2,\Omega_{i,d}} + c_2\|\boldsymbol{\sigma}\|_{n,\Omega_{i,d}}\right) | \int_{\Gamma_i} \boldsymbol{v}_* \cdot \boldsymbol{n}| < \nu.$$

In fact, by Remark IX.4.7 and (IX.4.56) we obtain

$$\frac{1}{2}(\nu - \Phi_{(n)})|\boldsymbol{u}_s|_{1,2}^2 \leq (|\boldsymbol{f}|_{-1,2} + \nu|\boldsymbol{V}|_{1,2})\,|\boldsymbol{u}_s|_{1,2} + |(\boldsymbol{V}\cdot\nabla\boldsymbol{V},\boldsymbol{u}_s)|.$$

Moreover, from the Hölder inequality it follows that

$$|(\boldsymbol{V}\cdot\nabla\boldsymbol{V},\boldsymbol{u}_s)| \leq c\|\boldsymbol{V}\|_4^2|\boldsymbol{u}_s|_{1,2},$$

and since $W^{1,q}(\Omega) \subset L^4(\Omega)$ for $q > n/2$ and $n \geq 4$ (see Theorem II.3.4), we then have

$$|(\boldsymbol{V}\cdot\nabla\boldsymbol{V},\boldsymbol{u}_s)| \leq c'\|\boldsymbol{V}\|_{1,q}^2|\boldsymbol{u}_s|_{1,2}.$$

Thus,

$$\frac{1}{2}(\nu - \Phi_{(n)})|\boldsymbol{u}_s|_{1,2} \leq \left(|\boldsymbol{f}|_{-1,2} + \nu\|\boldsymbol{V}\|_{1,2} + c'\|\boldsymbol{V}\|_{1,q}^2,\right)$$

which furnishes the desired uniform bound on $|\boldsymbol{u}_s|_{1,2}$. Concerning the estimate for the pressure, we refer to Remark IX.3.1. Under the stated assumptions on the trace norm of $\boldsymbol{v}_*$, estimates similar to (IX.4.53), (IX.4.54) continue to hold for generalized solutions also for $n = 4$. Therefore, by Remark IX.2.3, the uniqueness result of Theorem IX.4.2 extends in the same form to $n = 4$. For uniqueness in dimension $n \geq 5$, we refer to Remark IX.5.5. ■

IX.5 Regularity of Generalized Solutions

We shall now show certain L^q-estimates for weak solutions to problem (IX.0.1), (IX.0.2). These estimates will imply, in particular, that if Ω and the data are smooth then the corresponding weak solutions are also smooth.

The key tool is a very general result proved in the next Lemma IX.5.1, regarding a *linearized* version of problem (IX.0.1), (IX.0.2).

Lemma IX.5.1 *Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 2$, of class C^2, $\boldsymbol{u} \in L^n(\Omega)$ with $\nabla \cdot \boldsymbol{u} = 0$ in Ω, in the weak sense, and $q \in (1, n)$. Then, for any*

$$\boldsymbol{F} \in L^q(\Omega)\,, \quad g \in W^{1,q}(\Omega)\,, \quad \boldsymbol{w}_* \in W^{2-1/q,q}(\partial\Omega)\,,$$

with

$$\int_\Omega g = \int_{\partial\Omega} \boldsymbol{w}_* \cdot \boldsymbol{n}\,,$$

the problem

$$\left.\begin{aligned} &\Delta \boldsymbol{w} = \boldsymbol{u} \cdot \nabla \boldsymbol{w} + \nabla \pi + \boldsymbol{F} \\ &\operatorname{div} \boldsymbol{w} = g \end{aligned}\right\} \quad \text{in } \Omega \qquad \text{(IX.5.1)}$$
$$\boldsymbol{w} = \boldsymbol{w}_* \quad \text{at } \partial\Omega$$

has at least one solution $(\boldsymbol{w}, \pi) \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$, that satisfies, in addition, the following estimate

$$\|\boldsymbol{w}\|_{2,q} + \|\pi\|_{1,q} \leq C \left(\|\boldsymbol{F}\|_q + \|g\|_{1,q} + \|\boldsymbol{w}_*\|_{2-1/q,q,\partial\Omega}\right)\,, \qquad \text{(IX.5.2)}$$

with $C = C(n, q, \Omega, \boldsymbol{u})$. Furthermore, let $\overline{\boldsymbol{w}} \in D^{1,s}(\Omega)$, for some $s \in (1, \infty)$, satisfy (IX.5.1)$_{2,3}$ *along with the equation*

$$(\nabla \overline{\boldsymbol{w}}, \nabla \boldsymbol{\varphi}) - (\boldsymbol{u} \cdot \nabla \boldsymbol{\varphi}, \overline{\boldsymbol{w}}) = (\boldsymbol{F}, \boldsymbol{\varphi})\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega)\,. \qquad \text{(IX.5.3)}$$

Then, if $n \geq 3$, necessarily $\overline{\boldsymbol{w}} = \boldsymbol{w}$, a.e. *in Ω, while, if $n = 2$, the same conclusion holds provided $s \in [2, \infty)$ and $\boldsymbol{u} \in L^{q_0}(\Omega)$, for some $q_0 > 2$.*

Proof. We begin to show the existence result. In this respect, we claim that it is enough to show it with $g \equiv 0$ and $\boldsymbol{w}_* \equiv \boldsymbol{0}$. Actually, under the assumption of the lemma, let $(\boldsymbol{w}_1, \pi_1) \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$ (with $\overline{\pi_1}_\Omega = 0$) be a solution to the following Stokes problem

$$\left.\begin{aligned} &\Delta \boldsymbol{w}_1 = \nabla \pi_1 \\ &\operatorname{div} \boldsymbol{w}_1 = g \end{aligned}\right\} \quad \text{in } \Omega$$
$$\boldsymbol{w}|_{\partial\Omega} = \boldsymbol{w}_*\,.$$

In view of Theorem IV.6.1, this solution exists (uniquely) and satisfies the estimate

$$\|\boldsymbol{w}_1\|_{2,q} + \|\pi_1\|_{1,q} \leq C \left(\|g\|_{1,q} + \|\boldsymbol{w}_*\|_{2-1/q,q,\partial\Omega}\right)\,, \qquad \text{(IX.5.4)}$$

with $C = C(n, q, \Omega)$. If we then write the solution to (IX.5.1) in the form $(\boldsymbol{w} = \boldsymbol{w}_1 + \boldsymbol{w}_2, \pi = \pi + \pi_2)$, we immediately recognize that $(\boldsymbol{w}_2, \pi_2)$ solves (IX.5.1) with $g = \boldsymbol{w}_* = 0$ and with $\boldsymbol{F}$ replaced by $\boldsymbol{F}' := \boldsymbol{F} - \boldsymbol{u} \cdot \nabla \boldsymbol{w}_1$. However, by the Hólder and Sobolev inequalities (see Theorem II.3.4), we find

$$\|\boldsymbol{u} \cdot \nabla \boldsymbol{w}_1\|_q \leq \|\boldsymbol{u}\|_n \|\nabla \boldsymbol{w}_1\|_{nq/(n-q)} \leq C\, \|\boldsymbol{u}\|_n \|\boldsymbol{w}_1\|_{2,q}$$

which ensures $\boldsymbol{F}' \in L^q(\Omega)$. Consequently, the above claim follows from this property and from (IX.5.4). We shall thus prove the lemma when $g \equiv 0$ and $\boldsymbol{w}_* \equiv \boldsymbol{0}$. By Theorem III.2.1 and Exercise II.2.6, given $\varepsilon > 0$, there are sequences $\{\boldsymbol{v}^{(j)}\} \subset C^\infty(\overline{\Omega})$, $\{\boldsymbol{F}^{(j)}\} \subset C_0^\infty(\Omega)$, and an integer $\overline{j} = \overline{j}(\varepsilon) > 0$ such that

$$\|\boldsymbol{u} - \boldsymbol{u}^{(j)}\|_n + \|\nabla \cdot \boldsymbol{u}^{(j)}\|_n + \|\boldsymbol{F} - \boldsymbol{F}^{(j)}\|_q < \varepsilon\,, \quad \text{for all } j \geq \overline{j}\,. \tag{IX.5.5}$$

Consider the following sequence of problems

$$\left.\begin{aligned} &\Delta \boldsymbol{w} = \boldsymbol{U}^{(j)} \cdot \nabla \boldsymbol{w} + \boldsymbol{u}^{(\overline{j})} \cdot \nabla w + \nabla \pi + F^{(j)} \\ &\nabla \cdot \boldsymbol{w} = 0 \end{aligned}\right\} \text{ in } \Omega \tag{IX.5.6}$$
$$\boldsymbol{w}|_{\partial\Omega} = 0\,,$$

where $\boldsymbol{U}^{(j)} := \boldsymbol{u}^{(j)} - \boldsymbol{u}^{(\overline{j})}$. If we formally multiply (IX.5.6)$_1$ by $\boldsymbol{w}$ and integrate by parts over Ω, we obtain

$$|\boldsymbol{w}|_{1,2}^2 = \tfrac{1}{2}(\nabla \cdot \boldsymbol{u}^{(j)}, |\boldsymbol{w}|^2) - (\boldsymbol{F}^{(j)}, \boldsymbol{w})\,. \tag{IX.5.7}$$

From the embedding Theorem II.3.4 we get

$$\|\boldsymbol{w}\|_{2n/(n-1)} \leq C_1\,|\boldsymbol{w}|_{1,2}\,,$$

where $C_1 = C_1(\Omega, n)$, and so, with the help of the Hölder inequality, it follows that

$$|(\nabla \cdot \boldsymbol{u}^{(j)}, |\boldsymbol{w}|^2)| \leq C_1^2\,\|\nabla \cdot \boldsymbol{u}^{(j)}\|_n |\boldsymbol{w}\|_{1,2}^2\,. \tag{IX.5.8}$$

Moreover, by the Poincaré inequality (II.5.1), we find

$$|(\boldsymbol{F}^{(j)}, \boldsymbol{w})| \leq C_2 \|\boldsymbol{F}^{(j)}\|_2 |\boldsymbol{w}|_{1,2}\,, \tag{IX.5.9}$$

where $C_2 = C_2(\Omega)$. Thus, choosing $\varepsilon < 1/C_1^2$, from (IX.5.7)–(IX.5.9) it follows that

$$|\boldsymbol{w}|_{1,2} \leq C_2 \|\boldsymbol{F}^{(j)}\|_2\,.$$

Thanks to this estimate, we may then use the method employed in the proof of Theorem IX.3.1 to show, for each fixed $j \geq \overline{j}$, the existence of a generalized solution $(\boldsymbol{w}^{(j)}, \pi^{(j)}) \in W_0^{1,2}(\Omega) \times L^2(\Omega)$ to (IX.5.6). Let us prove that this solution is, in fact, more regular and that it satisfies, in particular

$$(\boldsymbol{w}^{(j)}, \pi^{(j)}) \in W^{2,t}(\Omega) \times W^{1,t}(\Omega)\,, \quad \text{for all } t \in [1, \infty)\,. \tag{IX.5.10}$$

Let us begin to show that

$$(\boldsymbol{w}^{(j)}, \pi^{(j)}) \in W^{1,s}(\Omega) \times L^s(\Omega), \quad \text{for all } s \in [1, \infty). \tag{IX.5.11}$$

Actually, since

$$\boldsymbol{a}^{(j)} \equiv \boldsymbol{u}^{(j)} \cdot \nabla \boldsymbol{w}^{(j)} \in L^2(\Omega),$$

from the results on the Stokes problem established in Theorem IV.6.1 it then follows that

$$(\boldsymbol{w}^{(j)}, \pi^{(j)}) \in W^{2,2}(\Omega) \times W^{1,2}(\Omega)\,.$$

Thus, by the embedding theorem Theorem II.3.4 we infer that $\boldsymbol{a}^{(j)} \in L^r(\Omega)$ with $r = 2n/(n-2)$ if $n > 2$ and all $r > 1$ if $n = 2$. Employing again Theorem IV.6.1, we find that

$$(\boldsymbol{w}^{(j)}, \pi^{(j)}) \in W^{2,r}(\Omega) \times W^{1,r}(\Omega).$$

If $r \geq n$, that is, $n \leq 4$, then (IX.5.11) is proved; otherwise,

$$\boldsymbol{a}^{(j)} \in L^{r_1}(\Omega), \quad r_1 = 2n/(n-4)\ (> r)$$

and, again by Theorem IV.6.1 we deduce

$$(\boldsymbol{w}^{(j)}, \pi^{(j)}) \in W^{2,r_1}(\Omega) \times W^{1,r_1}(\Omega).$$

If $r_1 \geq n$ we arrive at (IX.5.11); if not, we iterate the argument as many times as we please until we derive (IX.5.11). Once (IX.5.11) has been established, we use the classical results of Theorem IV.6.1 one more time to prove, in particular, the validity of (IX.5.10), in any space dimension $n \geq 2$. Next, by means of the Hölder inequality, the embedding Theorem II.3.4, and relations (IX.5.5) we deduce, for any $q \in (1, n)$,

$$\begin{aligned} \|\boldsymbol{u}^{(j)} \cdot \nabla \boldsymbol{w}^{(j)}\|_q &\leq \|\boldsymbol{U}^{(j)}\|_n \|\nabla \boldsymbol{w}^{(j)}\|_{nq/(n-q)} + \max_{\Omega} |\boldsymbol{u}^{(\overline{j})}|\, \|\nabla \boldsymbol{w}^{(j)}\|_q \\ &\leq \varepsilon\, \|\boldsymbol{w}^{(j)}\|_{2,q} + \max_{\Omega} |\boldsymbol{u}^{(\overline{j})}|\, \|\nabla \boldsymbol{w}^{(j)}\|_q. \end{aligned} \qquad \text{(IX.5.12)}$$

In view of (IX.5.10) we can now apply Theorem IV.6.1 to problem (IX.5.6) to recover, with $q \in (1, n)$,

$$\|\boldsymbol{w}^{(j)}\|_{2,q} + \|\pi^{(j)}\|_{1,q} \leq C_3 \big(\varepsilon \|\boldsymbol{w}^{(j)}\|_{2,q} + \max_{\Omega} |\boldsymbol{u}^{(\overline{j})}|\, \|\nabla \boldsymbol{w}^{(j)}\|_q + \|\boldsymbol{F}^{(j)}\|_q\big)\,,$$

where $C_3 = C_3(q, n, \Omega)$. So, taking $\varepsilon < \min\{1/C_1^2, 1/C_3\}$, from the preceding inequality we derive the following one

$$\|\boldsymbol{w}^{(j)}\|_{2,q} + \|\pi^{(j)}\|_{1,q} \leq C_3 \big(M(\boldsymbol{u}) \|\boldsymbol{w}^{(j)}\|_{1,q} + \|\boldsymbol{F}^{(j)}\|_q\big)\,, \quad q \in (1, n)\,, \qquad \text{(IX.5.13)}$$

with $M(\boldsymbol{u}) \equiv \max_\Omega |\boldsymbol{u}^{(\overline{j})}|$. Our next task is to prove the existence of a positive constant $C_4 = C_4(q, n, \Omega, \boldsymbol{u})$, but otherwise independent of j, such that

$$\|\boldsymbol{w}^{(j)}\|_{1,q} \leq C_4 \|\boldsymbol{F}^{(j)}\|_q\,. \qquad \text{(IX.5.14)}$$

This will be proved by the, by now familiar, contradiction argument. Thus, assuming the invalidity of (IX.5.14), for any integer m we could find $\{\boldsymbol{F}^{(m)}\}$

such that denoted by $\{(\boldsymbol{w}^{(m)}, \pi^{(m)})\}$ the corresponding solutions to (IX.5.6) (with $j = m$), the inequality

$$\|\boldsymbol{w}^{(m)}\|_{1,q} > m\,\|\boldsymbol{F}^m\|_q\,, \quad \text{for all } m \in \mathbb{N},$$

would hold. By the linearity of the problem, we can take, without loss,

$$\|\boldsymbol{w}^{(m)}\|_{1,q} = 1, \quad \text{for all } m \in \mathbb{N}, \tag{IX.5.15}$$

so that the preceding inequality furnishes

$$\|\boldsymbol{F}^{(m)}\|_q < 1/m\,, \quad \text{for all } m \in \mathbb{N}\,. \tag{IX.5.16}$$

From the estimate (IX.5.13), we deduce that $\|\boldsymbol{w}^{(m)}\|_{2,q}$ is uniformly bounded in m, and by Remark II.3.1 and Theorem II.5.2 we can infer the existence of a field $\boldsymbol{W} \in W^{2,q}(\Omega)$ and of a subsequence, again denoted by $\{\boldsymbol{w}^{(m)}\}$, such that

$$\begin{aligned} &\boldsymbol{w}^{(m)} \overset{w}{\to} \boldsymbol{W} \ \text{ in } W^{2,q}(\Omega) \\ &\boldsymbol{w}_m \to \boldsymbol{W} \ \text{ in } W_0^{1,t}(\Omega)\,, \quad \text{for all } t \in [1, nq/(n-q)). \end{aligned} \tag{IX.5.17}$$

Furthermore, in view of (IX.5.5),

$$\boldsymbol{u}^{(m)} \to \boldsymbol{u} \ \text{ in } L^n(\Omega). \tag{IX.5.18}$$

Passing to the limit $m \to \infty$ in (IX.5.6) (with $j = m$) and using (IX.5.16)–(IX.5.18), it follows that the limit function $\boldsymbol{W}$ satisfies

$$\begin{aligned} &(\nabla \boldsymbol{W}, \nabla \boldsymbol{\varphi}) + (\boldsymbol{u} \cdot \nabla \boldsymbol{W}, \boldsymbol{\varphi}) = 0, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega), \\ &\nabla \cdot \boldsymbol{W} = 0\,, \quad \boldsymbol{W} \in W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)\,. \end{aligned} \tag{IX.5.19}$$

Let us prove that $\boldsymbol{W} \equiv 0$. If $n = 2$, since $q \in (1,2)$ by the embedding Theorem II.3.4 we infer $\boldsymbol{W} \in H^1_{r_1}(\Omega)$, with $r_1 = 2q/(2-q) > 2$, so that, in particular, $\boldsymbol{W} \in H^1(\Omega)$. Let $\{\boldsymbol{\varphi}_k\} \subset \mathcal{D}(\Omega)$ with $\boldsymbol{\varphi}_k \to \boldsymbol{W}$ in $H^1_{r_1}(\Omega)$, and thus, in particular, in $H^1(\Omega)$. This ensures that

$$\lim_{k\to\infty} (\nabla \boldsymbol{W}, \nabla \boldsymbol{\varphi}_k) = |\boldsymbol{W}|^2_{1,2}\,.$$

Moreover, for any $s \in (r_1', 2)$, we have $\boldsymbol{u} \in L^s(\Omega)$ with $\nabla \cdot \boldsymbol{u} = 0$, and so, using the properties of $\boldsymbol{W}$ and $\boldsymbol{\varphi}_k$ along with the embedding Theorem II.3.4, with the help of Exercise IX.2.1 we find

$$\lim_{k\to\infty} (\boldsymbol{u} \cdot \nabla \boldsymbol{W}, \boldsymbol{\varphi}_k) = (\boldsymbol{u} \cdot \nabla \boldsymbol{W}, \boldsymbol{W}) = 0\,.$$

We next replace $\boldsymbol{\varphi}$ with $\boldsymbol{\varphi}_k$ in (IX.5.19)$_1$, pass to the limit $k \to \infty$ and use the properties stated in the last two displayed equation to obtain $\boldsymbol{W} \equiv 0$. On the other hand, from (IX.5.15) and (IX.5.17) we also have

$$\|\nabla \boldsymbol{W}\|_q = 1, \tag{IX.5.20}$$

leading to a contradiction. We shall next consider the case $n \geq 3$. We begin to suppose $q \in [2n/(n+2), n)$. In this situation, by Theorem II.3.4, we get $\boldsymbol{W} \in W_0^{1,2}(\Omega)$. The trilinear form $(\boldsymbol{v} \cdot \nabla \boldsymbol{z}_1, \boldsymbol{z}_2)$ is continuous in $L^n(\Omega) \times W^{1,2}(\Omega) \times L^{2n/(n-2)}(\Omega)$ (see Exercise IX.2.1), and so, by the embedding Theorem II.3.2, it is continuous in $L^n(\Omega) \times H^1(\Omega) \times H^1(\Omega)$. Since, by definition, $\mathcal{D}(\Omega)$ is dense in $H^1(\Omega)$, for all $\sigma \in (1, \infty)$, by a standard approximating procedure we are then allowed to take $\boldsymbol{W} = \boldsymbol{\varphi}$ in (IX.5.19) to get

$$0 = |\boldsymbol{W}|_{1,2}^2 + (\boldsymbol{u} \cdot \nabla \boldsymbol{W}, \boldsymbol{W})\,.$$

However, $(\boldsymbol{u} \cdot \nabla \boldsymbol{W}, \boldsymbol{W}) = 0$, by Exercise IX.2.1, because $\boldsymbol{W} \in W_0^{1,2}(\Omega)$ and $\boldsymbol{u} \in L^n(\Omega)$ with $\nabla \cdot \boldsymbol{u} = 0$. As a result, we infer $\boldsymbol{W} \equiv 0$, which contradicts (IX.5.20). Inequality (IX.5.14) is therefore established, if $n = 2$, for $q \in (1, 2)$, and, if $n \geq 3$, for $q \in [2n/(n+2), n)$. From (IX.5.13) and (IX.5.14) we obtain

$$\|\boldsymbol{w}^{(j)}\|_{2,q} + \|\pi^{(j)}\|_{1,q} \leq C_5\, \|\boldsymbol{F}^{(j)}\|_q \tag{IX.5.21}$$

with $C_5 = C_5(n, q, \Omega, \boldsymbol{u})$. From (IX.5.21), Remark II.3.1 and Theorem II.5.2 it follows that there are subsequences $\{\boldsymbol{w}^{(j')}\}$ and $\{\pi^{(j')}\}$, and two fields $\boldsymbol{w} \in W^{2,q}(\Omega)$ and $\pi \in W^{1,q}(\Omega)$, such that

$$\begin{aligned} &\boldsymbol{w}^{j'} \overset{w}{\to} \boldsymbol{w} \;\; \text{in } W^{2,q}(\Omega)\\ &\boldsymbol{w}^{j'} \to \boldsymbol{w} \;\; \text{in } W^{1,q}(\Omega) \end{aligned}$$

and

$$\pi^{j'} \overset{w}{\to} \pi \;\; \text{weakly in } W^{1,q}(\Omega).$$

Clearly, $(\boldsymbol{w}, \pi)$ is a solution to (IX.5.1), and the lemma is thus proved for $n = 2$, and, for $n \geq 3$, under the condition that $q \in [2n/(n+2), n)$. Let us next assume $n \geq 3$, and $q \in (1, 2n/(n+2))$. Taking into account the procedure previously used, the result will be proved provided we show that (IX.5.19) has only the solution $\boldsymbol{W} \equiv \boldsymbol{0}$. We begin to observe that, since $2n/(n+2) < n/2$ (for $n \geq 3$) by the embedding Theorem II.3.4, we deduce

$$\begin{aligned} &\boldsymbol{W} \in H^1_{r_1}(\Omega) \cap L^{r_2}(\Omega)\,,\\ &r_1 \in (1,2)\,, \quad r_2 = \frac{n r_1}{n - r_1}\,. \end{aligned} \tag{IX.5.22}$$

Consider now the problem (adjoint to (IX.5.1))

$$\left.\begin{aligned} &\Delta \boldsymbol{\varphi} + \boldsymbol{u} \cdot \nabla \boldsymbol{\varphi} = \nabla \tau + \boldsymbol{G}\\ &\nabla \cdot \boldsymbol{\varphi} = 0 \end{aligned}\right\} \;\; \text{in } \Omega \tag{IX.5.23}$$
$$\boldsymbol{\varphi}\,|_{\partial\Omega} = 0\,.$$

Noticing that

$$\sigma \equiv \frac{nr_1'}{n+r_1'} \in (\frac{2n}{n+2}, n)\,, \quad \frac{n\sigma}{n-\sigma} = r_1'\,,$$

from what we have previously shown, and, again, Theorem II.3.4, we know that, for each $\boldsymbol{G} \in C_0^\infty(\Omega)$, there is at least one corresponding solution $\overline{\boldsymbol{\varphi}}, \overline{\tau}$ to (IX.5.23) such that

$$\overline{\boldsymbol{\varphi}} \in W^{2,\frac{nr_1'}{n+r_1'}}(\Omega) \cap H^1_{r_1'}(\Omega)\,, \quad \overline{\tau} \in W^{1,\frac{nr_1'}{n+r_1'}}(\Omega). \tag{IX.5.24}$$

Now, let $\{\boldsymbol{\varphi}_k\} \subset \mathcal{D}(\Omega)$ converge to $\overline{\boldsymbol{\varphi}}$ in $H^1_\sigma(\Omega)$, and replace $\boldsymbol{\varphi}_k$ for $\boldsymbol{\varphi}$ in (IX.5.19). Clearly, we have

$$\lim_{k\to\infty} (\nabla \boldsymbol{W}, \nabla \boldsymbol{\varphi}_k) = (\nabla \boldsymbol{W},\ \nabla \overline{\boldsymbol{\varphi}})\,. \tag{IX.5.25}$$

Moreover, by an easily justified integration by parts, based on density arguments, we find

$$(\boldsymbol{u} \cdot \nabla \boldsymbol{W}, \boldsymbol{\varphi}) = -(\boldsymbol{u} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{W})\,, \quad \boldsymbol{\varphi} \in \mathcal{D}(\Omega)\,. \tag{IX.5.26}$$

Furthermore, by the Hölder inequality (see also Exercise IX.2.1),

$$|(\boldsymbol{u} \cdot \nabla \boldsymbol{\varphi}_k, \boldsymbol{W})| \leq \|\boldsymbol{u}\|_n \|\nabla \boldsymbol{\varphi}_k\|_{r_1'} \|\boldsymbol{W}\|_{nr_1/(n-r_1)}\,,$$

and so, recalling the summability properties (IX.5.22) of $\boldsymbol{W}$, we deduce

$$\lim_{k\to\infty} (\boldsymbol{u} \cdot \nabla \boldsymbol{W}, \boldsymbol{\varphi}_k) = (\boldsymbol{u} \cdot \nabla \overline{\boldsymbol{\varphi}}, \boldsymbol{W})\,. \tag{IX.5.27}$$

Therefore, from (IX.5.19)$_1$ with $\boldsymbol{\varphi} \equiv \boldsymbol{\varphi}_k$, and (IX.5.25)–(IX.5.27) we conclude, in the limit $k \to \infty$,

$$(\nabla \boldsymbol{W}, \nabla \overline{\boldsymbol{\varphi}}) - (\boldsymbol{u} \cdot \nabla \overline{\boldsymbol{\varphi}}, \boldsymbol{W}) = 0\,. \tag{IX.5.28}$$

Next, let $\{\boldsymbol{W}_k\} \subset C_0^\infty(\Omega)$ converge to $\boldsymbol{W}$ in $W_0^{1,r_1}(\Omega)$, so that, by embedding, it converges to $\boldsymbol{W}$ also in $L^{nr_1/(n-r_1)}(\Omega)$; see Theorem II.3.4. Thus, taking into account

$$(nr_1/(n-r_1))' = nr_1'/(n+r_1')\,, \tag{IX.5.29}$$

along with (IX.5.24), we have

$$(\nabla \boldsymbol{W}, \nabla \overline{\boldsymbol{\varphi}}) = \lim_{k\to\infty} (\nabla \boldsymbol{W}_k, \nabla \overline{\boldsymbol{\varphi}}) = -\lim_{k\to\infty} (\boldsymbol{W}_k, \Delta \overline{\boldsymbol{\varphi}}) = -(\boldsymbol{W}, \Delta \overline{\boldsymbol{\varphi}})\,.$$

As a consequence, with the help of (IX.5.28), we infer

$$(\boldsymbol{W}, \Delta \overline{\boldsymbol{\varphi}} + \boldsymbol{u} \cdot \nabla \overline{\boldsymbol{\varphi}}) = 0\,,$$

which, in turn, recalling that $(\overline{\boldsymbol{\varphi}}, \overline{\tau})$ satisfy (IX.5.23), is equivalent to the following

$$(\boldsymbol{W}, \boldsymbol{G}) = -(\boldsymbol{W}, \nabla \overline{\tau}) . \tag{IX.5.30}$$

However, by (IX.5.22) and Lemma III.2.2, $\boldsymbol{W} \in H_{r_2}(\Omega)$, $r_2 \equiv \frac{nr_1}{n-r_1}$, whereas by (IX.5.24), and (IX.5.29), $\overline{\tau} \in G_{r_2'}(\Omega)$, and so, by Lemma III.2.1, we obtain $(\boldsymbol{W}, \nabla\overline{\tau}) = 0$. Replacing this information back into (IX.5.30), we deduce $(\boldsymbol{W}, \boldsymbol{G}) = 0$, which, by the arbitrariness of $\boldsymbol{G} \in C_0^\infty(\Omega)$, allows us to conclude $\boldsymbol{W} = 0$ a.e. in Ω. The proof of the existence part of the lemma is thus completed. We shall now show the uniqueness part. Setting $\boldsymbol{z} := \boldsymbol{w} - \overline{\boldsymbol{w}}$, we have that $\boldsymbol{z}$ satisfies the following problem

$$\begin{gathered}(\nabla \boldsymbol{z}, \nabla \boldsymbol{\varphi}) - (\boldsymbol{u} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{z}) = 0\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega) \\ \boldsymbol{z} \in H_r^1(\Omega)\,, \quad r = \min\{s, q\}\,.\end{gathered} \tag{IX.5.31}$$

Let us first consider the case $n \geq 3$. The result immediately follows if $s \geq 2$. Actually, we then have $\boldsymbol{z} \in H^1(\Omega)$ and therefore, by employing the standard density argument that we have already used previously in the proof (right after (IX.5.20)), we may replace $\boldsymbol{\varphi}$ with $\boldsymbol{z}$ in (IX.5.31) to obtain, as before, $|\boldsymbol{z}|_{1,2} = 0$, namely $\boldsymbol{z} = 0$ a.e. in Ω. Assume then $s \in (1, 2)$. However, in this case, $\boldsymbol{z}$ satisfies the same assumption and the same equation satisfied by $\boldsymbol{W}$; see (IX.5.22), (IX.5.19), and (IX.5.26)) . Therefore, following exactly the same argument used to show $\boldsymbol{W} = \boldsymbol{0}$, we also prove $\boldsymbol{z} = \boldsymbol{0}$, and uniqueness is completely recovered if $n \geq 3$. If $n = 2$, we observe that, since by the embedding Theorem II.3.4, $\boldsymbol{w} \in H^1(\Omega)$, we have $\boldsymbol{z} \in H^1(\Omega)$. Let $\{\boldsymbol{\varphi}_k\} \subset \mathcal{D}(\Omega)$ with $\boldsymbol{\varphi}_k \to \boldsymbol{z}$ in $H^1(\Omega)$. Furthermore, again by that theorem, we have also $H^1(\Omega) \hookrightarrow L^{2q_0/(q_0-2)}(\Omega)$. Therefore, setting $\boldsymbol{\varphi} = \boldsymbol{\varphi}_k$ into (IX.5.31), then letting $k \to \infty$ and using the results of Exercise IX.2.1, we find

$$|\boldsymbol{z}|_{1,2}^2 = -(\boldsymbol{u} \cdot \nabla \boldsymbol{z}, \boldsymbol{z}) = 0\,,$$

which implies $\boldsymbol{z} = 0$ a.e. in Ω. The proof of the lemma is complete. □

The next result provides, in particular, sufficient conditions for the *interior regularity* of weak solutions *in arbitrary space dimensions* $n \geq 2$.

Theorem IX.5.1 *Let Ω be any domain of $\mathbb{R}^n$, $n \geq 2$, and let $\boldsymbol{v}$ be such that:*

(i) $\boldsymbol{v} \in L^s_{loc}(\Omega)$, *where $s = n$, if $n \geq 3$, while $s = s_0 > 2$, if $n = 2$;*
(ii) $\nabla \cdot \boldsymbol{v} = 0$, *in the weak sense;*
(iii) $\boldsymbol{v}$ *satisfies the following equation*

$$\nu(\boldsymbol{v}, \Delta\boldsymbol{\varphi}) + (\boldsymbol{v} \cdot \nabla\boldsymbol{\varphi}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle\,, \quad \textit{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega)\,. \tag{IX.5.32}$$

The following properties hold.

(a) *If*

$$\boldsymbol{f} \in L^q_{loc}(\Omega)\,, \quad 1 < q < \infty\,,$$

then

$$\boldsymbol{v} \in W^{2,q}_{loc}(\Omega)\,,$$

and there exists $p \in W^{1,q}_{loc}(\Omega)$ such that (IX.0.1) *is satisfied a.e. in Ω.*

(b) *If, moreover,*

$$\boldsymbol{f} \in W^{m,q}_{loc}(\Omega)$$

where $m \geq 1$, *and*

$$q \in (1,\infty), \quad \textit{if } n = 2,$$

while

$$q \in [n/2,\infty), \quad \textit{if } n > 2, [1]$$

then

$$\boldsymbol{v} \in W^{m+2,q}_{loc}(\Omega), \quad p \in W^{m+1,q}_{loc}(\Omega). \tag{IX.5.33}$$

Proof. We begin to show part (a). Let B be a ball of $\mathbb{R}^n$, with $\overline{B} \subset \Omega$. Then, from (IX.5.32) and Lemma IV.4.1 we find that, for all sufficiently small $\varepsilon > 0$, the mollification, $\boldsymbol{v}_\varepsilon$, of $\boldsymbol{v}$ satisfies the following system

$$\left.\begin{aligned} &\nu \Delta \boldsymbol{v}_\varepsilon = \nabla p^{(\varepsilon)} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})_\varepsilon + \boldsymbol{f}_\varepsilon \\ &\nabla \cdot \boldsymbol{v}_\varepsilon = 0 \end{aligned}\right\} \text{ in } B\,, \tag{IX.5.34}$$

for some $p^{(\varepsilon)} \in C^\infty(B)$. We next denote by $\boldsymbol{w} = \boldsymbol{w}(\varepsilon), \tau = \tau(\varepsilon)$, $\overline{\tau}_B = 0$, the solution to the following Stokes problem

$$\begin{aligned} &\left.\begin{aligned} &\nu \Delta \boldsymbol{w} = \nabla \tau + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})_\varepsilon + \boldsymbol{f}_\varepsilon \\ &\nabla \cdot \boldsymbol{w} = 0 \end{aligned}\right\} \text{ in } B \\ &\qquad\qquad \boldsymbol{w} = 0 \text{ at } \partial B\,, \end{aligned} \tag{IX.5.35}$$

with

$$(\boldsymbol{w}, \tau) \in W^{1,r/2}(B) \times L^{r/2}(B)\,, \quad r > 2\,.$$

In view of the properties of the mollification and of Theorem IV.6.1, such a solution exists and satisfies the inequality

$$\|\boldsymbol{w}\|_{1,r/2} + \|\tau\|_{r/2} \leq c_1 \left(\|\boldsymbol{v}_\varepsilon\|^2_{r,B} + \|\boldsymbol{f}_\varepsilon\|_{-1,r/2,B} \right) . \tag{IX.5.36}$$

Our next task is to estimate $\|\boldsymbol{f}_\varepsilon\|_{-1,r/2,B}$ in terms of $\|\boldsymbol{f}\|_{q,B}$. The starting point is the obvious inequality

$$|(\boldsymbol{f}_\varepsilon, \boldsymbol{\Phi})| \leq \|\boldsymbol{f}\|_{q,B} \|\boldsymbol{\Phi}\|_{q'}\,, \quad \boldsymbol{\Phi} \in W_0^{1,(r/2)'}(B)\,, \tag{IX.5.37}$$

that follows from (II.2.9) and the Hölder inequality. We distinguish several cases. Suppose first $n = 2$. Without loss we may assume $s_0 \leq 4$. Thus, if we choose $r = s_0$, from the embedding Theorem II.3.4 it follows $W_0^{1,(r/2)'}(B) \hookrightarrow L^t(B)$, for all $t \in (1,\infty)$, so that from (IX.5.36), we deduce

$$\|\boldsymbol{f}_\varepsilon\|_{-1,r/2,B} \leq c_2 \|\boldsymbol{f}\|_{q,B}\,, \quad q \in (1,\infty)\,,\ r := s_0\,,\ n = 2. \tag{IX.5.38}$$

[1] The lower bound $n/2$ for q is not necessarily the best possible. However, it suffices for our aims.

If $n = 3$, we observe that $(n/2)' = n = 3$. Consequently, if we choose $r = 3$ by the embedding Theorem II.3.4 we have $W_0^{1,(r/2)'}(B) \equiv W_0^{1,3}(B) \hookrightarrow L^t(B)$, for all $t \in (1, \infty)$, and (IX.5.36) furnishes

$$\|\boldsymbol{f}_\varepsilon\|_{-1,r/2,B} \le c_3 \|\boldsymbol{f}\|_{q,B}\,, \quad q \in (1,\infty)\,,\ r := 3\,,\ n = 3. \tag{IX.5.39}$$

We next analyze the case $n \ge 4$. In such a case we find $(n/2)' < n$, and so, if $q \in [n/3, \infty)$, it follows that $q' \le n/(n-3) = n(n/2)'/(n-(n/2)')$. Now, by Theorem II.3.4, $W_0^{1,(n/2)'}(B) \hookrightarrow L^{q'}(B)$ and, therefore, by choosing $r = n$, from (IX.5.36), we obtain

$$\|\boldsymbol{f}_\varepsilon\|_{-1,r/2,B} \le c_3 \|\boldsymbol{f}\|_{q,B}\,, \quad q \in [n/3,\infty)\,, \quad r := n\,,\ n \ge 4. \tag{IX.5.40}$$

Finally, if $n \ge 4$ and $q \in (1, n/3)$, we take $r = 2nq/(n-q)$ and observe that $(r/2)' < n$ and that $q' = n(r/2)'/(n-(r/2)')$. Thus, since by Theorem II.3.4 $W^{1,(r/2)'}(B) \hookrightarrow L^{q'}(B)$, again from (IX.5.36) we conclude

$$\|\boldsymbol{f}_\varepsilon\|_{-1,r/2,B} \le c_3 \|\boldsymbol{f}\|_{q,B}\,, \quad q \in (1,n/3)\,,\ r := \frac{2nq}{n-q}\,,\ n \ge 4\,. \tag{IX.5.41}$$

We next notice that, from (IX.5.34) and (IX.5.35), the fields $\boldsymbol{z} := \boldsymbol{v}_\varepsilon - \boldsymbol{w}$ and $\chi := p^{(\varepsilon)} - \tau$ satisfy the following Stokes system

$$\left.\begin{aligned} \nu \Delta \boldsymbol{z} &= \nabla \chi \\ \nabla \cdot \boldsymbol{z} &= 0 \end{aligned}\right\} \ \text{in } B\,. \tag{IX.5.42}$$

From Theorem IV.4.4 and Remark IV.4.2 we then find, in particular,

$$\|\boldsymbol{z}\|_{1,r/2,B_1} + \|\chi\|_{r/2,B_1} \le c_3\, \|\boldsymbol{z}\|_{r/2,B}\,,$$

where B_1 is an open ball with $\overline{B_1} \subset B$. Using this inequality, and taking into account that, by assumption, $\|\boldsymbol{v}_\varepsilon\|_{r,B} \le \|\boldsymbol{v}\|_{r,B} < \infty$ for all values of r, q and n specified in (IX.5.38)–(IX.5.41), from (IX.5.36), (IX.5.38)–(IX.5.41), and Exercise II.3.5 we deduce

$$\begin{aligned} &\boldsymbol{v} \in W^{1,s_0/2}(B_1)\,, \quad n = 2\,, \quad q \in (1,\infty) \\ &\boldsymbol{v} \in W^{1,n/2}(B_1)\,, \quad n = 3\,, \quad q \in (1,\infty) \\ &\boldsymbol{v} \in W^{1,n/2}(B_1)\,, \quad n \ge 4\,, \quad q \in [n/3,\infty) \\ &\boldsymbol{v} \in W^{1,r/2}(B_1)\,, \quad n \ge 4\,, \quad q \in (1,n/3)\,, \quad r := \frac{2nq}{n-q} \end{aligned} \tag{IX.5.43}$$

Moreover, by a similar argument that employs also Theorem II.2.4(ii), we show the existence of a scalar field p such that

$$\begin{aligned} &p \in L^{s_0/2}(B_1)\,, \quad n = 2\,, \quad q \in (1,\infty) \\ &p \in L^{n/2}(B_1)\,, \quad n = 3\,, \quad q \in (1,\infty) \\ &p \in L^{n/2}(B_1)\,, \quad n \ge 4\,, \quad q \in [n/3,\infty) \\ &p \in L^{r/2}(B_1)\,, \quad n \ge 4\,, \quad q \in (1,n/3)\,, \quad r := \frac{2nq}{n-q} \end{aligned} \tag{IX.5.44}$$

and, further, the pair $(\boldsymbol{v}, p)$ satisfies (IX.1.11) for all $\boldsymbol{\psi} \in C_0^\infty(B_1)$. However, the ball B_1 is arbitrary with $\overline{B_1} \subset \Omega$, so that from (IX.5.43)–(IX.5.43) we find that

$$
\begin{aligned}
&(\boldsymbol{v}, p) \in W^{1,s_0/2}_{loc}(\Omega) \times L^{s_0/2}_{loc}(\Omega)\,, \quad n=2\,, \quad q \in (1,\infty)\\
&(\boldsymbol{v}, p) \in W^{1,n/2}_{loc}(\Omega) \times L^{n/2}_{loc}(\Omega)\,, \quad n=3\,, \quad q \in (1,\infty)\\
&(\boldsymbol{v}, p) \in W^{1,n/2}_{loc}(\Omega) \times L^{n/2}_{loc}(\Omega)\,, \quad n\ge 4\,, \quad q \in [n/3,\infty)\\
&(\boldsymbol{v}, p) \in W^{1,r/2}_{loc}(\Omega) \times L^{r/2}_{loc}(\Omega)\,, \quad n\ge 4\,, \quad q \in (1,n/3)\,, \quad r := \frac{2nq}{n-q}\,,
\end{aligned}
\tag{IX.5.45}
$$

and, in addition, that $(\boldsymbol{v}, p)$ satisfies (IX.1.11) for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. Our next objective is to show that

$$
(\boldsymbol{v}, p) \in W^{2,q}_{loc}(\Omega) \times W^{1,q}_{loc}(\Omega)\,, \quad q \in (1,n)\,,\ n \ge 2\,. \tag{IX.5.46}
$$

It is clear that, in order to show (IX.5.46), it is enough to show the stated summability properties on an arbitrary bounded subdomain of Ω. To this end, let Ω', Ω'' be bounded domains in $\mathbb{R}^n$ with $\overline{\Omega'} \subset \Omega''$, $\overline{\Omega''} \subset \Omega$ and let $\phi \in C^\infty(\mathbb{R}^n)$ be one in Ω' and zero outside Ω''.[2] Writing $\phi\boldsymbol{\psi}$ in place of $\boldsymbol{\psi}$ into (IX.1.11) and extending $\boldsymbol{v}$ to zero outside Ω'', we readily recognize that $\boldsymbol{v}' \equiv \phi\boldsymbol{v}$ is a weak solution to the problem

$$
\left.
\begin{aligned}
&\Delta \boldsymbol{v}' = \boldsymbol{u}\cdot\nabla\boldsymbol{v}' + \nabla p' + \boldsymbol{F}\\
&\nabla\cdot\boldsymbol{v}' = g
\end{aligned}
\right\}\ \text{in } B_0
\qquad
\boldsymbol{v}' = 0 \ \text{ at } \partial B_0
\tag{IX.5.47}
$$

where

$$
\begin{aligned}
p' &\equiv \phi p/\nu\\
\boldsymbol{u} &\equiv \boldsymbol{v}/\nu\\
\boldsymbol{F} &\equiv \phi\boldsymbol{f}/\nu + 2\nabla\phi\cdot\nabla\boldsymbol{v} + \boldsymbol{v}\Delta\phi - \boldsymbol{v}\cdot\nabla\phi\boldsymbol{v}/\nu + p\nabla\phi/\nu\\
g &\equiv \boldsymbol{v}\cdot\nabla\phi
\end{aligned}
\tag{IX.5.48}
$$

and B_0 is an open ball containing $\overline{\Omega''}$. The leading idea in the proof of (IX.5.46), is to use a boot-strap argument that starts from (IX.5.45) and uses several times Lemma IX.5.1 applied to the problem (IX.5.47)–(IX.5.48). Suppose, at first, $n = 2$, the case that, seemingly, requires more effort. We begin to show that

$$
\boldsymbol{v} \in W^{1,2}_{loc}(\Omega)\,. \tag{IX.5.49}
$$

If $s_0 \ge 4$, this is obvious from $(\text{IX.5.45})_1$, and so we shall assume $s_0 \in (2,4)$. Let B an open ball of $\mathbb{R}^2$ with $\overline{B} \subset \Omega$, and consider the following two Stokes problems

---

[2] For the construction of ϕ, see the proof of Theorem IV.4.1.

$$\begin{aligned}&\nu(\nabla \boldsymbol{v}_1, \nabla \boldsymbol{\varphi}) = (\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{v})\,, \quad \text{for all } \boldsymbol{\varphi}\in\mathcal{D}(B)\,,\ \boldsymbol{v}\in H^1_t(B)\\ &\nu(\nabla \boldsymbol{v}_2, \nabla \boldsymbol{\varphi}) = -(\boldsymbol{f}, \boldsymbol{\varphi}) \ \text{ for all } \boldsymbol{\varphi}\in\mathcal{D}(B)\,,\ \boldsymbol{v}\in W^{2,q}(B)\cap H^1_q(B)\,.\end{aligned} \tag{IX.5.50}$$

Clearly, the function $\boldsymbol{\Phi} := \boldsymbol{v}-\boldsymbol{v}_1-\boldsymbol{v}_2$ satisfies $(\nabla\boldsymbol{\Phi}, \nabla\boldsymbol{\varphi}) = 0$, for all $\boldsymbol{\varphi}\in\mathcal{D}(\Omega)$, and, consequently, in view of the properties of $\boldsymbol{v}$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and of Theorem IV.4.3, we find $\boldsymbol{\Phi}\in C^\infty(B)$. Furthermore, by Theorem IV.6.1(a), $\boldsymbol{v}_2$ exists and, by the embedding Theorem II.3.4, $\boldsymbol{v}_2\in W^{1,2}(B)$. Therefore, to show (IX.5.49), it is enough to show that $\boldsymbol{v}_1\in W^{1,2}(B)$. We shall prove this by means of a recurrence argument based on a repeated use of Theorem IV.6.1(b) and of the embedding Theorem II.3.4. Actually, from the former theorem and $(\text{IX.5.45})_1$ we can take $t\equiv t_0 = s_0/2$, which, by Theorem II.3.4 and the fact that $s_0\in(2,4)$, implies $\boldsymbol{v}\in L^{2t_0/(2-t_0)}(B)$. Thus, again by Theorem IV.6.1, we may take $t\equiv t_1 = t_0/(2-t_0)$, which, in turn, by Theorem II.3.4 furnishes $\boldsymbol{v}\in L^{2t_1/(2-t_1)}(B)$, and so on. We thus obtain the following recurrence relation for the exponents t_k:

$$t_{k+1} = \frac{t_k}{2-t_k}\,, \quad k\in\mathbb{N}\,, \quad t_0 = s_0/2\,. \tag{IX.5.51}$$

We notice that, since by assumption $t_0 > 1+\eta$, for some positive η, the sequence $\{t_k\}$ is increasing, and so, in particular, $t_k > 1+\eta$, for all $k\in\mathbb{N}$. We claim that there is $\overline{k}\in\mathbb{N}$ such that $t_{\overline{k}+1} = 2$. By assuming the contrary, we would have $t_k < 2$, for all $k\in\mathbb{N}$, and so, due to the fact that $\{t_k\}$ is increasing, there is $t_* > 0$ such that $t_k\to t_*$ as $k\to\infty$. However, by taking the limit $k\to\infty$ in (IX.5.51), we would find $t_* = 1$, which furnishes a contradiction. As a consequence, (IX.5.49) is proved. By an argument entirely analogous to that used to show (IX.5.38), we show $\boldsymbol{f}\in W^{-1,2}(\omega)$ for all bounded domains ω with $\overline{\omega}\subset\Omega$. Thus, from (IX.5.51) and Lemma IX.2.1, we deduce $p\in L^2_{loc}(\Omega)$. This latter property along with (IX.5.51) and $(\text{IX.5.48})_{3,4}$, allows us to conclude $\boldsymbol{F}\in L^q(B_0)$, $g\in W^{1,q}(B_0)$, $q\in(1,2)$, so that by Lemma IX.5.1 and (IX.5.49) we prove (IX.5.46) for $n=2$. We next consider the case $n=3$. If $q\in(1,3/2]$, from (IX.5.45) and (IX.5.47) we easily find that

$$\boldsymbol{F}\in L^{\overline{q}}(B_0)\,, \quad g\in W^{1,\overline{q}}(B_0)\,, \tag{IX.5.52}$$

with $\overline{q} = q$, and so, by Lemma IX.5.1, the property (IX.5.46) follows for the above values of q. If $q\in(3/2,3)$, again by (IX.5.45), we find that $\boldsymbol{F}, g$ satisfy (IX.5.52) with $\overline{q} = 3/2$, which, in turn, by Lemma IX.5.1 and the arbitrariness of Ω' implies $(\boldsymbol{v}, p)\in W^{2,3/2}_{loc}(\Omega)\times W^{1,3/2}_{loc}(\Omega)$. From this latter property, by the embedding Theorem II.3.4, we infer $(\boldsymbol{v}, p)\in\left(W^{1,3}_{loc}(\Omega)\cap L^\infty_{loc}(\Omega)\right)\times L^3_{loc}(\Omega)$. Thus, we deduce the validity of (IX.5.52), with $\overline{q} = q$, $q\in(3/2,3)$, which, by Lemma IX.5.1, concludes the proof of (IX.5.46), in the case $n=3$. If $n\geq 4$, suppose first $q\in(1,n/3)$. Then, from $(\text{IX.5.45})_4$, we find $r/2 > q$, which implies (IX.5.52), with $\overline{q} = q$, and so, in turn, (IX.5.46) for these values of q. If $q\in[n/3,n/2]$, then, by $(\text{IX.5.45})_3$, we may take $\overline{q} = q$ in (IX.5.52), which, again by Lemma IX.5.1, furnishes (IX.5.46) also for these values of q. Finally, if $q \in [n/2, \infty)$, the argument is identical to that used for the case $n = 3$ by replacing $3/2$ with $n/2$. The property (IX.5.46) is thus established, and so is part (a) of the theorem if $q \in (1, n)$. Assume next $q \geq n$. Then (IX.5.46) is satisfied for *all* $q \in (1, n)$, and so, by the embedding Theorem II.3.4, we get

$$\boldsymbol{v} \in L^\infty_{loc}(\Omega) \cap W^{1,t}_{loc}(\Omega), \quad p \in L^t_{loc}(\Omega), \quad \text{for all } t \in (1, \infty),$$

yielding, in particular,

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} \in L^q_{loc}(\Omega).$$

From the interior estimates for the Stokes problem proved in Theorem IV.4.1 it then follows $\boldsymbol{v} \in W^{2,q}_{loc}(\Omega)$, $p \in W^{1,q}_{loc}(\Omega)$, which completes the proof of part (a) of the theorem. In order to prove part (b), we shall use an inductive argument. Since, by the results just established, (IX.5.33) is true for $l = 0$, let us assume that it holds for $l = k - 1$, $k \geq 1$, and let us show that it continues to hold for $l = k$. This amounts to proving that if, for the values of q specified in the statement of the theorem,

$$\boldsymbol{f} \in W^{k,q}_{loc}(\Omega), \quad \boldsymbol{v} \in W^{k+1,q}_{loc}(\Omega), \quad p \in W^{k,q}_{loc}(\Omega), \tag{IX.5.53}$$

necessarily

$$\boldsymbol{v} \in W^{k+2,q}_{loc}(\Omega), \quad p \in W^{k+1,q}_{loc}(\Omega), \tag{IX.5.54}$$

By Theorem IV.4.1, (IX.5.54) holds whenever

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} \in W^{k,q}_{loc}(\Omega). \tag{IX.5.55}$$

However, by the inductive assumption, we know that

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} \in W^{k-1,q}_{loc}(\Omega)$$

and so to obtain (IX.5.55), and consequently (IX.5.54), we have to show that

$$D^\alpha(\boldsymbol{v} \cdot \nabla \boldsymbol{v}) \in L^q(\Omega'), \quad |\alpha| = k, \tag{IX.5.56}$$

where Ω' is any bounded subdomain of $\mathbb{R}^n$ with $\overline{\Omega}' \subset \Omega$. Without loss, we may assume Ω' to be a ball. Expanding (IX.5.56) according to the Leibniz rule we deduce

$$D^\alpha(\boldsymbol{v} \cdot \nabla \boldsymbol{v}) = \sum_{\beta \leq \alpha} D^\beta \boldsymbol{v} \cdot \nabla D^{\alpha-\beta} \boldsymbol{v} \tag{IX.5.57}$$

where

$$D^{\alpha-\beta} \equiv \frac{\partial^{|\alpha|-|\beta|}}{\partial x_1^{\alpha_1-\beta_1} \ldots \partial x_n^{\alpha_n-\beta_n}}, \qquad \binom{\alpha}{\beta} \equiv \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha}{\beta},$$

and $\beta \leq \alpha$ means $\beta_i \leq \alpha_i$, $i = 1, \ldots, n$. Take first $q > n/2$. Then, by the embedding Theorem II.3.4 we have $\boldsymbol{v} \in C^{k-1}(\overline{\Omega}')$ and so (IX.5.57) yields

$$\sum_{|\alpha|=k} \|D^\alpha(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\|_{q,\Omega'} \le c\Big(\|\boldsymbol{v}\|_{C^{k-1}(\Omega)}\|\boldsymbol{v}\|_{2,q,\Omega'} + \sum_{|\alpha|=k}\|D^\alpha\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{q,\Omega'}\Big). \tag{IX.5.58}$$

Thus, since $k \ge 1$, the first term on the right-hand side of (IX.5.58) is bounded in view of (IX.5.53). Furthermore, by Theorem II.3.4, if $k = 1$ it follows that $\boldsymbol{v} \in W^{1,r}(\Omega')$ for all $r \in (1,\infty)$, while if $k > 1$, $\boldsymbol{v} \in C^1(\overline{\Omega}')$ and so, in any case, the second term on the right-hand side of (IX.5.57) is finite, thus proving (IX.5.56) when $q > n/2$ and, therefore, the theorem for $n = 2$. Assume now that $q = n/2$, $n > 2$. By Theorem II.3.4 it follows that $\boldsymbol{v} \in C^{k-2}(\overline{\Omega}')$ and so (IX.5.57) gives, with $|\alpha'| = k-2$,

$$\begin{aligned}|D^\alpha(\boldsymbol{v}\cdot\nabla\boldsymbol{v})| \le c\Big(&\sum_{\beta\le\alpha'}|D^\beta\boldsymbol{v}\cdot\nabla D^{\alpha-\beta}\boldsymbol{v}| + \sum_{|\beta|=k-1}|D^\beta\boldsymbol{v}\cdot\nabla D^{\alpha-\beta}\boldsymbol{v}|\\ &+\sum_{|\beta|=k}|D^\beta\boldsymbol{v}\cdot\nabla\boldsymbol{v}|\Big).\end{aligned} \tag{IX.5.59}$$

The first term in (IX.5.59) is increased as before, and we can show that it belongs to $L^q(\Omega')$. [3] Again by Theorem II.3.4 and (IX.5.53) we have

$$\begin{gathered}\boldsymbol{v} \in W^{k,n}(\Omega'),\\ \boldsymbol{v} \in W^{k-1,r}(\Omega'), \quad \text{for } all \ \ r \in (1,\infty)\end{gathered} \tag{IX.5.60}$$

and so the third term in (IX.5.59) belongs to $L^q(\Omega') \equiv L^{n/2}(\Omega')$. As far as the second term is concerned, we notice that its norm in $L^{n/2}(\Omega')$ can be increased by

$$\sum_{|\beta|=k-1}\|D^\beta\boldsymbol{v}\|_n\|\boldsymbol{v}\|_{2,n}$$

which, if $k \ge 2$, is finite by (IX.5.60). If $k = 1$, the second term is increased by $N \equiv |\boldsymbol{v}||D^2\boldsymbol{v}|$ and, again by (IX.5.60), it follows that it belongs to $L^{n/2-\varepsilon}(\Omega')$, for $\varepsilon \in (0, n/6)$. Thus,

$$\boldsymbol{v}\cdot\nabla\boldsymbol{v} \in W^{1,n/2-\varepsilon}(\Omega')$$

and by the estimates for the Stokes problem of Theorem IV.4.1 we deduce, in particular, that

$$\boldsymbol{v} \in W^{3,n/2-\varepsilon}(\Omega')$$

which, by Theorem II.3.4, in turn implies $\boldsymbol{v} \in C(\overline{\Omega'})$. From this and (IX.5.53) we then conclude that $N \in L^{n/2}(\Omega')$. The theorem is therefore proved. □

Remark IX.5.1 From the embedding Theorem II.3.4 it follows that the assumptions (i)–(iii) on $\boldsymbol{v}$ in Theorem IX.5.1 are satisfied if $\boldsymbol{v}$ is a generalized solution to (IX.0.1), (IX.0.2) and $n \le 4$. ■

[3] If, of course, $k \ge 2$; otherwise that term does not appear.

An important consequence of Theorem IX.5.1 is the following.

Corollary IX.5.1 *Let $\boldsymbol{v}$ obey conditions* (i)–(iii) *in* Theorem IX.5.1. *Then, if $\boldsymbol{f} \in C^\infty(\Omega)$, $\boldsymbol{v}$ and the associated pressure field p belong to $C^\infty(\Omega)$. The above conditions are satisfied if $\boldsymbol{v}$ is a generalized solution to* (IX.0.1), (IX.0.2) *and $n \le 4$.*

Remark IX.5.2 Assuming less regularity on $\boldsymbol{f}$, we can obtain intermediate regularity results on $\boldsymbol{v}$ and p. ■

Regularity up to the boundary of a generalized solution follows as a particular case of the following one.

Theorem IX.5.2 *Let Ω be a bounded domain of $\mathbb{R}^n$, $n \ge 2$, of class C^2, and let $\boldsymbol{v}$ be such that:*

(i) $\boldsymbol{v} \in W^{1,s}(\Omega) \cap L^n(\Omega)$, $s \in (1,\infty)$, *if $n \ge 3$, while $\boldsymbol{v} \in W^{1,2}(\Omega)$, if $n = 2$;*
(ii) *$\boldsymbol{v}$ is weakly divergence free, and obeys* (IX.0.2) *in the trace sense ;*
(iii) *$\boldsymbol{v}$ satisfies* (IX.1.2) .

Then, the following properties hold.

(a) *If*
$$\boldsymbol{f} \in L^q(\Omega), \quad \boldsymbol{v}_* \in W^{2-1/q,q}(\partial\Omega), \quad q \in (1,\infty),$$
then
$$\boldsymbol{v} \in W^{2,q}(\Omega),$$
and there exists $p \in W^{1,q}(\Omega)$, such that (IX.0.1) *is satisfied a.e. in Ω.*

(b) *If, moreover, Ω is of class C^{m+2} and*
$$\boldsymbol{f} \in W^{m,q}(\Omega), \quad \boldsymbol{v}_* \in W^{m+2-1/q}(\partial\Omega)$$
where $m \ge 1$, and
$$q \in (1,\infty), \quad \text{if } n = 2,$$
while
$$q \in [n/2,\infty), \quad \text{if } n > 2, \; [4]$$
then
$$\boldsymbol{v} \in W^{m+2,q}(\Omega), \quad p \in W^{m+1,q}(\Omega).$$

Proof. The proof of part (a) is an immediate consequence of Lemma IX.5.1, and we leave it to the reader as an exercise. The proof of part (b), is entirely analogous to that of Theorem IX.5.2(b), and will be, therefore, omitted. □

Remark IX.5.3 In view of the embedding Theorem II.3.4, the assumptions (i)–(iii) of Theorem IX.5.2 are satisfied by any generalized solution for $n \le 4$. ■

[4] The lower bound $n/2$ for q is not necessarily the best possible.

Corollary IX.5.2 *Let* $\boldsymbol{v}$ *obey conditions* (i)–(iii) *in* Theorem IX.5.2. *If the bounded domain* Ω *is of class* C^∞, *and* $\boldsymbol{f} \in C^\infty(\overline{\Omega})$, $\boldsymbol{v}_* \in C^\infty(\partial\Omega)$, *then* $\boldsymbol{v}$ *and the associated pressure field* p *belong to* $C^\infty(\overline{\Omega})$. *The above conditions are satisfied if* $\boldsymbol{v}$ *is a generalized solution to* (IX.0.1)–(IX.0.2) *and* $n \le 4$.

Remark IX.5.4 Assuming less regularity on Ω, $\boldsymbol{f}$ and $\boldsymbol{v}_*$ we can obtain intermediate regularity results on $\boldsymbol{v}$ and p. ■

Remark IX.5.5 If $n \ge 5$, the theory developed in Section IX.3 does not guarantee existence of solutions verifying the assumptions of Corollary IX.5.1 and Corollary IX.5.2, since we don't know, in such a case, if a generalized solution belongs to $L^n(\Omega)$. The question of existence of *regular* solutions for $n \ge 5$ and *without restrictions* on the size of the data has been addressed by Frehse and Růžička (1994a, 1994b, 1995, 1996), and Struwe (1995). Here, we would like to show that, if the size of the data is sufficiently "small," existence of regular solutions in arbitrary dimension $n \ge 5$ is easily established by means of the theory developed for the linearized Stokes problem. We assume Ω of class C^2, $\boldsymbol{f} \in W_0^{-1,n/2}(\Omega)$, $\boldsymbol{v}_* \in W^{(n-2)/n,n/2}(\partial\Omega)$, with

$$\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} = 0,$$

and set

$$\mathcal{D} = \|\boldsymbol{f}\|_{-1,n/2} + \|\boldsymbol{v}_*\|_{(n-2)/n,n/2(\partial\Omega)}.$$

We next introduce a sequence of approximating solutions $\{\boldsymbol{v}_m, p_m\}$, $m \in \mathbb{N}$, defined as follows

$$\begin{aligned} \nu(\nabla\boldsymbol{v}_m, \nabla\boldsymbol{\psi}) + (\boldsymbol{v}_{m-1}\cdot\nabla\boldsymbol{v}_{m-1}, \boldsymbol{\psi}) - (p_m, \nabla\cdot\boldsymbol{\psi}) + \langle\boldsymbol{f}, \boldsymbol{\psi}\rangle = 0 \\ \nabla\cdot\boldsymbol{v}_m = 0 \ \text{ in } \Omega \\ \boldsymbol{v}_m = \boldsymbol{v}_* \ \text{ at } \partial\Omega, \end{aligned} \tag{IX.5.61}$$

where $\boldsymbol{v}_0 \equiv 0$ and $\boldsymbol{\psi}$ is arbitrary from $C_0^\infty(\Omega)$. By the existence theory for the Stokes problem of Theorem IV.6.1, we know that (IX.5.61) for $m = 1$ admits a unique solution $\{\boldsymbol{v}_1, p_1\}$ (p_1 up to a constant) with $\boldsymbol{v}_1 \in W^{1,n/2}(\Omega)$, $p_1 \in L^{n/2}(\Omega)$, such that

$$\|\boldsymbol{v}_1\|_{1,n/2} + \frac{1}{\nu}\|p_1\|_{n/2/\mathbb{R}} \le 2\frac{c}{\nu}\mathcal{D} \tag{IX.5.62}$$

where $c = c(n, \Omega)$ is the constant entering estimate (IV.6.10). Let us show, by induction, the existence of $\{\boldsymbol{v}_m, p_m\}$ with $\boldsymbol{v}_m \in W^{1,n/2}(\Omega)$, $p_m \in L^{n/2}(\Omega)$ satisfying (IX.5.61), (IX.5.62) for all $m \in \mathbb{N}$. We assume that $\{\boldsymbol{v}_{m-1}, p_{m-1}\}$ obeys (IX.5.62). For any $\boldsymbol{\psi} \in W_0^{1,n/(n-2)}(\Omega)$ we have

$$|(\boldsymbol{v}_{m-1}\cdot\nabla\boldsymbol{v}_{m-1}, \boldsymbol{\psi})| = |(\boldsymbol{v}_{m-1}\otimes\boldsymbol{v}_{m-1}, \nabla\boldsymbol{\psi})| \le \|\boldsymbol{v}_{m-1}\|_n^2\|\boldsymbol{\psi}\|_{1,n/(n-2)}$$

and so, by the embedding Theorem II.3.4:

$$\|\boldsymbol{v}_{m-1}\|_n \leq \gamma \|\boldsymbol{v}_{m-1}\|_{1,n/2}, \tag{IX.5.63}$$

and by the induction hypothesis (IX.5.62) for the $(m-1)$th solution we deduce that

$$|(\boldsymbol{v}_{m-1} \cdot \nabla \boldsymbol{v}_{m-1}, \boldsymbol{\psi})| \leq 4\frac{\gamma^2 c^2}{\nu^2}\mathcal{D}^2 \|\boldsymbol{\psi}\|_{1,n/(n-2)}. \tag{IX.5.64}$$

So

$$\boldsymbol{v}_{m-1} \cdot \nabla \boldsymbol{v}_{m-1} \in W_0^{-1,n/2}(\Omega)$$

and (IX.5.64) together with Theorem IV.6.1 ensure for all $m \in \mathbb{N}$ the existence of a pair $\{\boldsymbol{v}_m, p_m\}$ satisfying (IX.5.61) along with the inequality

$$\|\boldsymbol{v}_m\|_{1,n/2} + \frac{1}{\nu}\|p_m\|_{n/2/\mathbb{R}} \leq \frac{c}{\nu}\mathcal{D}\left(4\frac{\gamma^2 c^2}{\nu^2}\mathcal{D} + 1\right). \tag{IX.5.65}$$

However, if

$$\mathcal{D} < \frac{\nu^2}{4\gamma^2 c^2}, \tag{IX.5.66}$$

relation (IX.5.65) furnishes

$$\|\boldsymbol{v}_m\|_{1,n/2} + \frac{1}{\nu}\|p_m\|_{n/2/\mathbb{R}} \leq 2\frac{c}{\nu}\mathcal{D} \tag{IX.5.67}$$

which is therefore proved for all $m \in \mathbb{N}$. Let us next establish that $\{\boldsymbol{v}_m, p_m\}$ is a Cauchy sequence in $W^{1,n/2}(\Omega) \times \left\{L^{n/2}(\Omega)/\mathbb{R}\right\}$. To this end, it is enough to show that for all $m \geq 1$

$$\|\boldsymbol{v}_m - \boldsymbol{v}_{m-1}\|_{1,n/2} + \frac{1}{\nu}\|p_m - p_{m-1}\|_{n/2/\mathbb{R}} \leq \alpha^m \tag{IX.5.68}$$

where $\alpha \in (0,1)$.[5] From (5.35_1) we have

$$\begin{aligned}\nu(\nabla(\boldsymbol{v}_m - \boldsymbol{v}_{m-1}), \nabla\boldsymbol{\psi}) &+ ((\boldsymbol{v}_{m-1} - \boldsymbol{v}_{m-2}) \cdot \nabla \boldsymbol{v}_{m-1}, \boldsymbol{\psi}) \\ &+(\boldsymbol{v}_{m-2} \cdot \nabla(\boldsymbol{v}_{m-1} - \boldsymbol{v}_{m-2}), \boldsymbol{\psi}) + ((p_m - p_{m-1}), \boldsymbol{\psi}) = 0\end{aligned}$$

and with the aid of Theorem IV.6.1, it follows for all $m \in \mathbb{N}$, that

[5] In fact, for all $m' = m + k$, $k > 0$,

$$\begin{aligned}&\|\boldsymbol{v}_m - \boldsymbol{v}_{m'}\|_{1,n/2} + \tfrac{1}{\nu}\|p_m - p_{m'}\|_{n/2/\mathbb{R}} \\ &\quad\leq \textstyle\sum_{i=1}^k \left(\|\boldsymbol{v}_{m+i} - \boldsymbol{v}_{m+i-1}\|_{1,n/2} + \tfrac{1}{\nu}\|p_{m+i} - p_{m+i-1}\|_{n/2/\mathbb{R}}\right) \\ &\quad\leq \alpha^m \textstyle\sum_{i=1}^k \alpha^i \leq \dfrac{\alpha^{m+1}}{1-\alpha}.\end{aligned}$$

$$\begin{aligned}&\|\boldsymbol{v}_m-\boldsymbol{v}_{m-1}\|_{1,n/2}+\frac{1}{\nu}\|p_m-p_{m-1}\|_{n/2/\mathbb{R}}\\&\le\frac{c}{\nu}\left(\|\boldsymbol{v}_{m-1}-\boldsymbol{v}_{m-2}\|_n\|\boldsymbol{v}_{m-1}\|_n+\|\boldsymbol{v}_{m-2}\|_n\|\boldsymbol{v}_{m-1}-\boldsymbol{v}_{m-2}\|_n\right).\end{aligned}$$

Using (IX.5.67) and (IX.5.63) in this relation yields

$$\begin{aligned}&\|\boldsymbol{v}_m-\boldsymbol{v}_{m-1}\|_{1,n/2}+\frac{1}{\nu}\|p_m-p_{m-1}\|_{n/2/\mathbb{R}}\\&\le 4\frac{\gamma^2c^2}{\nu^2}\mathcal{D}\left(\|\boldsymbol{v}_{m-1}-\boldsymbol{v}_{m-2}\|_{1,n/2}+\frac{1}{\nu}\|p_{m-1}-p_{m-2}\|_{n/2/\mathbb{R}}\right),\end{aligned}$$

which in turn implies (IX.5.68) with

$$\alpha=\frac{4\gamma^2c^2}{\nu^2}\mathcal{D}.$$

Thus, if $\boldsymbol{f},\boldsymbol{v}_*$ satisfy (IX.5.66), there are $\boldsymbol{v}\in W^{1,n/2}(\Omega)$, $p\in L^{n/2}(\Omega)$ such that

$$\boldsymbol{v}_m\to\boldsymbol{v}\ \text{ strongly in } W^{1,n/2}(\Omega)$$

$$p_m\to p\ \text{ strongly in } L^{n/2}(\Omega)/\mathbb{R}.$$

From (IX.5.61) it follows at once that $\boldsymbol{v}$, p satisfy (IX.1.11), which completes the proof of existence. In addition, by (IX.5.63), (IX.5.65), and (IX.5.66) we have

$$\frac{1}{\gamma}\|\boldsymbol{v}\|_n+\|\boldsymbol{v}\|_{1,n/2}+\frac{1}{\nu}\|p\|_{n/2/\mathbb{R}}\le 2\frac{c}{\nu}\left(\|\boldsymbol{f}\|_{-1,n/2}+\|\boldsymbol{v}_*\|_{(n-2)/n,n/2(\partial\Omega)}\right). \tag{IX.5.69}$$

Taking into account Theorem IX.5.2, we may then conclude with the following result.

Theorem IX.5.3 *Let Ω be a bounded domain in $\mathbb{R}^n$, $n\ge 5$, of class C^2 and let $\boldsymbol{f}\in W_0^{-1,n/2}(\Omega)$, $\boldsymbol{v}_*\in W^{(n-2)/n,n/2}(\partial\Omega)$ with*

$$\int_{\partial\Omega}\boldsymbol{v}_*\cdot\boldsymbol{n}=0.$$

Then, there exists a positive $C=C(n,\Omega)$ such that if

$$\|\boldsymbol{f}\|_{-1,n/2}+\|\boldsymbol{v}_*\|_{(n-2)/n,n/2(\partial\Omega)}<C\nu^2,$$

there is a generalized solution $\boldsymbol{v}$ to (IX.0.1), (IX.0.2) *such that*

$$\boldsymbol{v}\in W^{1,n/2}(\Omega),\quad p\in L^{n/2}(\Omega)$$

where p is the pressure field associated to $\boldsymbol{v}$ by Lemma IX.1.2. *Moreover, $\boldsymbol{v},p$ satisfy* (IX.5.69). *Finally, if Ω is of class C^∞, $\boldsymbol{f}\in C^\infty(\overline{\Omega})$ and $\boldsymbol{v}_*\in C^\infty(\partial\Omega)$, then $\boldsymbol{v}$, $p\in C^\infty(\overline{\Omega})$.*

Concerning the uniqueness of these solutions, we observe that if

$$\|\boldsymbol{f}\|_{-1,n/2} + \|\boldsymbol{v}_*\|_{(n-2)/n,n/2(\partial\Omega)} \tag{IX.5.70}$$

is sufficiently small, by the method used in the proof of Theorem IX.2.1 and Remark IX.2.3, it follows immediately that they are unique in the class of generalized solutions which, in addition, satisfy $\boldsymbol{v} \in L^n(\Omega)$. However, we can prove a more general result which ensures uniqueness in the class of generalized solutions that obey the energy inequality.[6] We shall sketch the proof of this result in the special case when $\boldsymbol{v}_* = 0$. To this end, we notice that, by Exercise IX.3.1, we can construct, in any dimension $n \geq 2$, a generalized solution satisfying the following *energy inequality*

$$\nu|\boldsymbol{v}|_{1,2}^2 \leq -\langle \boldsymbol{f}, \boldsymbol{v}\rangle. \tag{IX.5.71}$$

Let now $\boldsymbol{v}_1$ be a solution corresponding to $\boldsymbol{f}$ as given in Theorem IX.5.3. In view of Remark IX.1.4, we can show that $\boldsymbol{v}_1$ satisfies the energy equality

$$\nu|\boldsymbol{v}_1|_{1,2}^2 = -\langle \boldsymbol{f}, \boldsymbol{v}_1\rangle. \tag{IX.5.72}$$

Again by Remark IX.1.4, we can take $\boldsymbol{\varphi} = \boldsymbol{v}_1$ in (IX.1.2) to obtain

$$-\nu(\nabla\boldsymbol{v}, \nabla\boldsymbol{v}_1) - (\boldsymbol{v}\cdot\nabla\boldsymbol{v}, \boldsymbol{v}_1) = \langle \boldsymbol{f}, \boldsymbol{v}_1\rangle. \tag{IX.5.73}$$

By the same token, we can consider (IX.1.2) with $\boldsymbol{v}$ replaced by $\boldsymbol{v}_1$, and choose $\boldsymbol{\varphi} = \boldsymbol{v}$. We then get

$$-\nu(\nabla\boldsymbol{v}_1, \nabla\boldsymbol{v}) - (\boldsymbol{v}_1\cdot\nabla\boldsymbol{v}_1, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle. \tag{IX.5.74}$$

If we add together (IX.5.71)–(IX.5.74), and use again Remark IX.1.4, we can show that

$$\nu|\boldsymbol{w}|_{1,2}^2 \leq (\boldsymbol{w}\cdot\nabla\boldsymbol{w}, \boldsymbol{v}_1), \tag{IX.5.75}$$

where $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v}_1$. We now use (IX.1.6) and (II.3.7) on the right-hand side of (IX.5.75) to deduce

$$(\nu - c_0\|\boldsymbol{v}_1\|_n)\,|\boldsymbol{w}|_{1,2}^2 \leq 0,$$

with $c_0 = c_0(n)$. This latter inequality along with (IX.5.69) written with $\boldsymbol{v} = \boldsymbol{v}_1$, proves uniqueness if the norm (IX.5.70) of the data is sufficiently small. We finally observe that this uniqueness result implies that for $n \geq 5$, *any* generalized solution $\boldsymbol{v}$ corresponding to smooth small data and satisfying (IX.5.71) is smooth. ■

[6] We recall that if $n = 2, 3, 4$, every generalized solution satisfies the energy *equality* (that is, (5.45) with equality sign), while, for $n \geq 5$, this equality is satisfied by those generalized solutions which are also in $L^n(\Omega)$; see Exercise IX.3.1.

IX.6 Limit of Infinite Viscosity: Transition to the Stokes Problem

As we noticed in the Introduction to Chapter IV, the Stokes system is to be regarded as a *formal* approximation of the Navier–Stokes system, whenever the inertial term $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$ becomes small compared to the viscous term $\nu \Delta \boldsymbol{v}$ or, in dimensionless language, whenever the Reynolds number $\mathcal{R}$ becomes vanishingly small. The question arises quite naturally of whether one can give a *rigorous* mathematical justification of this approximation. The affirmative answer to such a problem, in the case of a bounded region of flow, is essentially due to Odqvist (1930, §6); cf. also Finn (1961a, Section 7a)), and is founded upon the estimate of the Green tensor associated to the Stokes system. Here, we shall follow a different approach based, mainly, on Theorem IV.6.1. Specifically, we shall show that every generalized solution $\boldsymbol{v}$ to (IX.0.1), (IX.0.2), as $\nu \to \infty$ (or $\mathcal{R} \to 0$) tends to the generalized solution $\boldsymbol{w}$ of the Stokes problem (IV.0.1) corresponding to the same data, cf. Theorem IX.6.1. Furthermore, if Ω, $\boldsymbol{f}$, and $\boldsymbol{v}_*$ are sufficiently smooth, we show that the following estimates are valid

$$\left.\begin{aligned} &\|\boldsymbol{v} - \boldsymbol{w}\|_{C^2(\Omega)} \leq C_1/\nu \\ &\|p' - \pi\|_{C^1(\Omega)} \leq C_2/\nu \end{aligned}\right\} \text{for all } \nu \geq \nu_0 \tag{IX.6.1}$$

where $p' \equiv p/\nu, \pi$ are the pressure fields associated to $\boldsymbol{v}$ and to $\boldsymbol{w}$, respectively, and C_1, C_2 are known functions of the data and of the positive number ν_0.

Theorem IX.6.1 *Let Ω , $\boldsymbol{f}$, and $\boldsymbol{v}_*$ be as in* Theorem IX.4.1. *Denote by $\boldsymbol{v} = \boldsymbol{v}(x; \nu), p = p(x; \nu)$ the family of generalized solutions to* (IX.0.1), (IX.0.2) *corresponding to $\boldsymbol{f}$, $\boldsymbol{v}_*$ and parameterized in $\nu > 0$, whose existence has been established in* Theorem IX.4.1. *Moreover, let $\boldsymbol{w}$, π be the generalized solution to the Stokes problem* (IV.0.1) *and the associated pressure field, respectively, corresponding to $\boldsymbol{f}$ and $\boldsymbol{v}_*$, whose existence is guaranteed by* Theorem IV.1.1. *Then*

$$\|\boldsymbol{v} - \boldsymbol{w}\|_{1,2} + \|p' - \pi\|_2 \leq c/\nu, \quad \nu \geq \nu_0$$

where c is a known function of the data and of ν_0 (cf. (IX.6.3), (IX.6.4)*) and ν_0 is any positive fixed number.*

Proof. Setting $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{w}$, by the definition of a generalized solution and Remark IX.1.1 we obtain

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) = -\frac{1}{\nu}(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi}), \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega). \tag{IX.6.2}$$

Since Ω is locally Lipschitz, we have $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ (cf. Section III.5, and we may take $\boldsymbol{u} = \boldsymbol{\varphi}$ in (IX.6.2) to obtain

$$|\boldsymbol{u}|_{1,2}^2 = \frac{1}{\nu}|(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{u})|.$$

Thus, Lemma IX.1.1 and inequality (II.5.5) imply

$$\|\boldsymbol{u}\|_{1,2} \leq \frac{C}{\nu}\|\boldsymbol{v}\|_{1,2}^2,$$

with $C = C(n, \Omega)$. Substituting into this inequality (IX.4.51) with $\Phi = \nu/2$ (say) furnishes

$$\|\boldsymbol{u}\|_{1,2} \leq \frac{CC_1^2}{\nu}\left(\frac{1}{\nu}|\boldsymbol{f}|_{-1,2} + \frac{1}{\nu}\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}^2 + \frac{1+\nu}{\nu}\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\right)^2 \tag{IX.6.3}$$

where $C_1 = 2c_1$. Furthermore, using (IX.1.11) and (IV.1.3) (the latter written with $\boldsymbol{w}$ and π in place of $\boldsymbol{v}$ and p), we deduce that

$$\nu(\nabla\boldsymbol{u}, \nabla\boldsymbol{\psi}) + (\boldsymbol{v}\cdot\nabla\boldsymbol{v}, \boldsymbol{\psi}) = +(p - \nu\pi, \nabla\cdot\boldsymbol{\psi})$$

for all $\boldsymbol{\psi} \in D_0^{1,2}(\Omega)$ and so, reasoning as in the proof of Theorem IX.3.1, we obtain

$$\|p - \nu\pi\|_2 \leq C_3\left(\nu|\boldsymbol{u}|_{1,2} + \|\boldsymbol{v}\|_{1,2}^2\right) \tag{IX.6.4}$$

which, on account of (IX.6.3) and of (IX.4.51), completes the proof of the theorem. □

Remark IX.6.1 Theorem IX.6.1 is also valid in dimension $n = 4$. If $n \geq 5$, as already noticed several times, we cannot take $\boldsymbol{\varphi} = \boldsymbol{u}$ into (IX.6.2), because a generalized solution need not belong a priori to $L^n(\Omega)$. However, using the more regular solutions constructed in Theorem IX.5.3, the reader will prove without difficulty that Theorem IX.6.1 continues to hold with ν_0 depending, this time, on the magnitude of $\boldsymbol{f}$ and $\boldsymbol{v}_*$. ■

We shall next show that, if Ω is of class C^3 and if the data satisfy

$$\boldsymbol{f} \in W^{1,\sigma}(\Omega), \quad \boldsymbol{v}_* \in W^{3-1/\sigma,\sigma}(\partial\Omega), \quad \sigma = \frac{nr}{n+r}, \ r > n, \tag{IX.6.5}$$

then estimates (IX.6.1) hold. To fix the ideas, we suppose $n = 3$, the case where $n = 2$ is treated similarly. If $\boldsymbol{v}$ is a generalized solution to (IX.0.1), (IX.0.2), then $\boldsymbol{v}\cdot\nabla\boldsymbol{v} \in W_0^{-1,3}(\Omega)$ and from the estimates for the Stokes problem derived in Theorem IV.6.1 we obtain $\boldsymbol{v} \in W^{1,3}(\Omega)$; furthermore

$$\begin{aligned}\|\boldsymbol{v}\|_{1,3} &\leq \frac{c}{\nu}\left(\|\boldsymbol{v}\|_6^2 + \|\boldsymbol{f}\|_{-1,3} + \nu\|\boldsymbol{v}_*\|_{2/3,3(\partial\Omega)}\right)\\ &\leq \frac{c_1}{\nu}\left(\|\boldsymbol{v}\|_{1,2}^2 + \|\boldsymbol{f}\|_{-1,3} + \nu\|\boldsymbol{v}_*\|_{2/3,3(\partial\Omega)}\right)\end{aligned} \tag{IX.6.6}$$

where, in the second inequality, we have used the embedding Theorem II.3.4. Again by this latter theorem we deduce

$$\|\boldsymbol{v}\|_s \leq c_2\|\boldsymbol{v}\|_{1,3}, \quad \text{for all } s > 1, \tag{IX.6.7}$$

and, by the Hölder inequality,

$$\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_q \le \|\boldsymbol{v}\|_{3q/(3-q)}\|\boldsymbol{v}\|_{1,3}, \quad \text{for all } q\in(1,3). \tag{IX.6.8}$$

Theorem IV.6.1 then furnishes $\boldsymbol{v}\in W^{2,q}(\Omega)$, for all $q\in(1,3)$, and, by (IX.6.6)–(IX.6.8),

$$\|\boldsymbol{v}\|_{2,q} \le \frac{c_3}{\nu}\left(\|\boldsymbol{v}\|^2_{3q/(3-q)} + \|\boldsymbol{f}\|_q + \nu\|\boldsymbol{v}_*\|_{2-1/q,q(\partial\Omega)}\right). \tag{IX.6.9}$$

On the other hand, by Theorem II.3.4,

$$\left.\begin{aligned}\|\boldsymbol{v}\|_\infty &\le c_4\|\boldsymbol{v}\|_{2,q}\\ \|\boldsymbol{v}\|_{1,3q/(3-q)} &\le c_4\|\boldsymbol{v}\|_{2,q}\end{aligned}\right. \qquad \text{for all } q\in(3/2,3), \tag{IX.6.10}$$

which, for these values of q, together with (IX.6.8), implies

$$\begin{aligned}\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{1,q} &\le 2(\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_q + \|\nabla(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\|_q)\\ &\le 2(\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_q + \|\boldsymbol{v}\|^2_{1,2q} + \|\boldsymbol{v}\|_\infty\|\boldsymbol{v}\|_{2,q})\\ &\le c_5\|\boldsymbol{v}\|^2_{2,q}.\end{aligned}$$

In view of Theorem IV.6.1 we then obtain for all $q\in(3/2,3)$

$$\begin{aligned}\|\boldsymbol{v}\|_{3,q} &\le \frac{c_6}{\nu}\left(\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{1,q} + \|\boldsymbol{f}\|_{1,q} + \nu\|\boldsymbol{v}_*\|_{3-1/q,q(\partial\Omega)}\right)\\ &\le \frac{c_7}{\nu}\left(\|\boldsymbol{v}\|^2_{2,q} + \|\boldsymbol{f}\|_{1,q} + \nu\|\boldsymbol{v}_*\|_{3-1/q,q(\partial\Omega)}\right).\end{aligned} \tag{IX.6.11}$$

Now, setting as before $\boldsymbol{u}=\boldsymbol{v}-\boldsymbol{w}$ and, further, $\tau=(p/\nu-\pi)$, we have

$$\left.\begin{aligned}\Delta\boldsymbol{u} &= \frac{1}{\nu}\boldsymbol{v}\cdot\nabla\boldsymbol{v}+\nabla\tau\\ \nabla\cdot\boldsymbol{u} &= 0\end{aligned}\right\}\ \text{in } \Omega$$
$$\boldsymbol{u}=0 \ \text{ at } \partial\Omega$$

and, by Theorem IV.6.1, we recover for $r>3$

$$\|\boldsymbol{u}\|_{3,r} + \|\tau\|_{2,r} \le \frac{1}{\nu}\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{1,r} \le \frac{c_8}{\nu}\left(\|\boldsymbol{v}\|^2_{1,2r} + \|\boldsymbol{v}\|_\infty\|\boldsymbol{v}\|_{2r}\right). \tag{IX.6.12}$$

By Theorem II.3.4 we have, with $\sigma=3r/(3+r)$

$$\|\boldsymbol{v}\|_{2,r} \le c_9\|\boldsymbol{v}\|_{3,\sigma} \tag{IX.6.13}$$

and

$$\|\boldsymbol{v}\|_{1,2r} \le c_{10}\|\boldsymbol{v}\|_{3,6r/(3+4r)} \le c_{11}\|\boldsymbol{v}\|_{3,\sigma} \tag{IX.6.14}$$

since $r>3$. Therefore, $(\text{IX.6.10})_1$, (IX.6.12)–(IX.6.14) furnish

$$\begin{aligned}\|\boldsymbol{u}\|_{3,r} + \|\tau\|_{2,r} &\le \frac{c_{12}}{\nu}\|\boldsymbol{v}\|^2_{3,\sigma} \\ &\le \frac{c_{13}}{\nu^3}\left(\|\boldsymbol{v}\|^2_{2,\sigma} + \|\boldsymbol{f}\|_{1,\sigma} + \nu\|\boldsymbol{v}_*\|_{3-1/\sigma,\sigma(\partial\Omega)}\right)^2\end{aligned}$$

and estimates (IX.6.1) become a consequence of this latter inequality, of Theorem II.3.4, of (IX.6.6), (IX.6.9), (IX.6.11), and (IX.4.51).

We have then proved the following result:

Theorem IX.6.2 *Let Ω, $\boldsymbol{f}$, $\boldsymbol{v}_*$, $\boldsymbol{v}$, p, $\boldsymbol{w}$, and π be as in* Theorem IX.6.1 *with Ω of class C^3 and $\boldsymbol{f}$, $\boldsymbol{v}_*$ satisfying* (IX.6.5)*. Then $\boldsymbol{v} - \boldsymbol{w} \in C^2(\overline{\Omega})$, $p - \pi \in C^1(\overline{\Omega})$ and they obey estimate* (IX.6.1) *with ν_0 any positive, fixed number.*

Remark IX.6.2 If $n \ge 4$, Theorem IX.6.2 continues to hold provided we use the more regular solutions constructed in Theorem IX.5.3. However, this time, ν_0 will depend on the size of the data; cf. also Remark IX.6.1. ■

IX.7 Notes for the Chapter

Section IX.1. The introduction of the pressure field p for weak solutions as an element of a suitable L^q-space is essentially due to the work of Solonnikov & Ščadilov (1973), cf. also Lemma IV.1.1. Actually, these authors show the existence of p in the case of the linearized Stokes system but their ideas carry over, without conceptual difficulties, to the nonlinear Navier–Stokes problem.

The general results derived in Lemma IX.1.2 and Remark IX.1.5, whose proofs are based on ideas of Solonnikov & Ščadilov, are due to me.

Section IX.2. Uniqueness of generalized solutions in the form presented here can be traced back to the papers of Hopf (1941, 1957); cf. Finn (1961a Section 8).

The condition (IX.2.4), ensuring that a generalized solution is unique, can be improved in several respects. For instance, following the work of Serrin (1959a) one can give a variational formulation of uniqueness that is aimed, among other things, at yielding the "best" upper bound on $\boldsymbol{v}$ for uniqueness to hold. It is interesting to observe that such conditions likewise ensure the nonlinear energy stability of $\boldsymbol{v}$, cf. Joseph (1976), Galdi & Rionero (1985).

Section IX.3. An interesting variant of Leray's existence theorem that shows existence of generalized solutions has been given by Ladyzhenskaya (1959b, Theorem 2); cf. also Vorovich & Youdovich (1961, Theorem 1). The variant is based on the following steps. First, as we already remarked, Ladyzhenskaya gives a variational formulation of the boundary-value problem (IX.0.1), (IX.0.2). Such a formulation leads to an equivalent operator equation for the velocity field $\boldsymbol{v}$ of the form

$$\boldsymbol{v} = \frac{1}{\nu}\left(A\boldsymbol{v} + \boldsymbol{f}\right), \tag{$*$}$$

with A a suitable nonlinear operator in an appropriate Hilbert space $\mathcal{H}$. She then proves that A is completely continuous and that every possible solution $\boldsymbol{v}$ to $(*)$ that is in $\mathcal{H}$ admits a uniform bound independent of $1/\nu \in (0, B)$. This fact, together with a simplified version of the Leray-Schauder theorem, produces existence in the physically relevant cases of two and three dimensions. For the sake of mathematical generality, however, it should be noted that neither Leray's nor Ladyzhenskaya's argument works in four dimensions due to the fact that the operator A is no longer completely continuous. This problem was taken up and solved by Shinbrot (1964) who proves that a property for A weaker than complete continuity is sufficient to recover existence to $(*)$. In light of this consideration, the method of Fujita which we used here assumes even more relevance since, as we have seen, it applies in *all* space dimensions.[1]

Existence of q-generalized solutions (see Remark IX.1.3) has been investigated by Serre (1983). Specifically, he proves that if $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is (bounded) and of class C^2, then given

$$\boldsymbol{f} \in D_0^{-1,q}(\Omega), \quad q \in (n/2, 2),$$

there is at least one corresponding q-generalized solution to (IX.0.1), (IX.0.2) with $\boldsymbol{v}_* = 0$. The result is also extended to the case $\boldsymbol{v}_* \neq 0$. The case $q = 3/2$, for $n = 3$, left out by Serre has been lately covered by Kim (2009, Remark 6).

More recently, there has been an increasing interest in the study of the properties of *very weak solutions*. These latter are characterized by a weakly divergence free velocity field $\boldsymbol{v} \in L^n(\Omega)^2$ satisfying the condition

$$(\boldsymbol{v}, \Delta\boldsymbol{\psi}) = -(\boldsymbol{v} \cdot \nabla\boldsymbol{\psi}, \boldsymbol{v}) + \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle - \langle \boldsymbol{n} \cdot \nabla\boldsymbol{\psi}, \boldsymbol{v}_* \rangle_{\partial\Omega}\,, \qquad (**)$$

for all $\boldsymbol{\psi} \in W^{2,\frac{n}{n-1}}(\Omega) \cap H^1_{\frac{n}{n-1}}(\Omega) \equiv \widetilde{W}_0(\Omega)$, where, we recall, $\langle \cdot, \cdot \rangle_{\partial\Omega}$ is the duality pairing between the spaces $W^{-1/q,q}(\partial\Omega)$ and $W^{1-1/q',q'}(\partial\Omega)$. As mentioned in the Notes for Section IV.6, even though, at the outset, very weak solutions are not weakly differentiable, they still possess a well defined trace at the (sufficiently smooth) boundary, in an appropriate space; see Galdi, Simader & Sohr (2005, Theorem 1). The study of existence and uniqueness of very weak solutions in a smooth (for instance, $C^{1,1}$), bounded and simply connected domain of $\mathbb{R}^n$, $n = 2, 3$, was initiated by Marušić-Paloka (2000). In particular, when $n = 3$, this author proves existence for *any* $\boldsymbol{f} \in W_0^{-1,2}(\Omega)$ and $\boldsymbol{v}_* \in L^2(\partial\Omega)$. Successively, when Ω is a bounded domain of $\mathbb{R}^3$ of class

[1] As a matter of fact, the method, as presented in the paper of Fujita (1961) fails for space dimension $n > 4$. However, the slight modification given by Finn (1965b §2.7)) and adopted by me permits the demonstration of existence of generalized solutions for all $n \geq 2$.

[2] One may replace L^n with L^q, with $q \geq n$, on condition of choosing appropriate test functions $\boldsymbol{\psi}$ in $(**)$; see the literature cited below. However, the case $q = n$, is, of course, the more interesting, because it is the case of less regularity.

$C^{2,1}$, Galdi, Simader & Sohr (2005) showed existence under the more general assumption $\boldsymbol{v}_* \in W^{-1/3,3}(\partial\Omega)$. (See Farwig, Galdi & Sohr (2006), for similar results when $n = 2$.) However, such a result requires the data to be "sufficiently small". This latter restriction was further removed by Kim (2009, Theorem 1), by cleverly combining arguments of Galdi, Simader & Sohr with those of Gehrardt (1979) and Marušić-Paloka. For the sake of precision, it is worth noticing that Kim's definition of "very weak" solution (as well as Marušić-Paloka's) is, in principle, more restrictive than the one adopted by Galdi, Simader & Sohr, where the test function in $(**)$ is chosen from the space $\mathcal{C}_0^2(\Omega)$, defined in equation $(*)$ in the Notes for Section IV.6, instead of the space $\widetilde{W}_0(\Omega)$. Of course, whenever $\mathcal{C}_0^2(\Omega)$ is dense in $\widetilde{W}_0(\Omega)$, the two definitions coincide. This happens, for example, if Ω is of class $C^{2,\alpha}$, $\alpha \in (0,1)$. Whether or not the same is true for domains with less regularity remains to be seen. It must be finally noticed that, in the above papers with the exception of that by Marušić-Paloka, the more general condition $\nabla \cdot \boldsymbol{v} = g$, with g suitably prescribed, is considered.

Section IX.4. The case of nonhomogeneous boundary conditions was first treated by Leray (1933, pp. 28-30, pp. 40-41) by means of two different approaches. The first is essentially that described at the beginning of the section, and it was successively completed and clarified by Hopf (1941, 1957); cf. also Ladyzhenskaya (1959b, Chapter I, §2), Finn (1961a, Lemma 2.1 and Section 2a)). The second is based on a clever contradiction argument, which we would like to sketch here. As we have mentioned several times, the clue for existence is to show a uniform bound on the Dirichlet integral of all possible solutions $\boldsymbol{v}$ to (IX.0.1), (IX.0.2) (in a suitable regularity class):

$$\int_\Omega \nabla\boldsymbol{v} : \nabla\boldsymbol{v} \le M, \tag{D}$$

with M independent of ν ranging in some bounded interval $[\nu_0, \nu_1]$, $\nu_0 > 0$. Contradicting (D) means that there is a sequence of solutions $\{\boldsymbol{v}_k, \pi_k\}$ to (IX.0.1), (IX.0.2) and of positive numbers $\{\nu_k\}$ such that as $k \to \infty$

$$J_k \equiv \int_\Omega \nabla\boldsymbol{v}_k : \nabla\boldsymbol{v}_k \to \infty, \quad \nu_k \to \nu$$

for a suitable $\nu \in [\nu_0, \nu_1]$. On the other hand, it is easy to show (cf. Leray 1933, p. 28), that, denoting by $\boldsymbol{V}$ a (sufficiently smooth) solenoidal extension of $\boldsymbol{v}_*$ the following identity holds for all $k \in \mathbb{N}$

$$\nu_k \int_\Omega \nabla\boldsymbol{v}_k : \nabla\boldsymbol{v}_k = -\frac{1}{2}\int_{\partial\Omega} v_*^2 \boldsymbol{v}_* \cdot \boldsymbol{n} + \nu_k \int_\Omega \nabla\boldsymbol{v}_k : \nabla\boldsymbol{V} + \int_\Omega \boldsymbol{v}_k \cdot \nabla\boldsymbol{v}_k \cdot \boldsymbol{V}. \tag{$***$}$$

Setting $\boldsymbol{w}_k = \boldsymbol{v}_k/\sqrt{\nu J_k}$ and using standard compactness arguments one shows that $\boldsymbol{w}_k$ tends to a suitable solenoidal field $\boldsymbol{w}$ that vanishes at the boundary, and that, after dividing both sides of $(***)$ by J_k and letting $k \to \infty$, satisfies

$$\int_\Omega \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{V} = 1. \tag{I}$$

Note that if $\boldsymbol{V}$ is an extension of $\boldsymbol{v}_*$, $\boldsymbol{V} + \boldsymbol{\varphi}$ is such for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$, and from (I) it follows that

$$\int_\Omega \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{\varphi} = 0 \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega).$$

In view of Lemma III.1.1, this latter condition, together with the fact that $\boldsymbol{w}$ is solenoidal and vanishes at the boundary, tells us that $\boldsymbol{w}$ solves the Euler-type problem:

$$\left.\begin{aligned} \boldsymbol{w} \cdot \nabla \boldsymbol{w} &= \nabla \pi \\ \nabla \cdot \boldsymbol{w} &= 0 \end{aligned}\right\} \text{ in } \Omega \qquad \boldsymbol{w} = 0 \text{ at } \partial\Omega \tag{II}$$

for some "pressure field" π. Therefore, the uniform bound (∗∗), and hence existence of solutions to (IX.0.1), (IX.0.2), will be proved *whenever we show that conditions* (I) *and* (II) *are incompatible*. It is readily seen that (I) and (II) are certainly incompatible if the *stronger* condition (IX.4.7) is satisfied. Actually, from $(\mathrm{II})_{1,3}$ it follows that π is a constant π_i (say) on each connected component Γ_i of $\partial\Omega$, $i = 1, \ldots, m+1$.[3] Multiplying $(\mathrm{II})_1$ by $\boldsymbol{V}$ and integrating by parts over Ω we find

$$\int_\Omega \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{V} = \sum_{i=1}^{m+1} \pi_i \int_{\Gamma_i} \boldsymbol{v}_* \cdot \boldsymbol{n}$$

and so, if we assume (IX.4.7), we obtain

$$\int_\Omega \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{V} = 0,$$

which contradicts (I). Incompatibility under the *sole* condition (IX.4.6) was left open by Leray. However, it is readily seen that Leray's approach *as it stands* does not lead in general to a contradiction. In fact, there are examples where conditions (I) and (II) are not incompatible. Consider, for instance, the annular domain

$$\Omega = \left\{x \in \mathbb{R}^2 : R_1 < |x| < R_2\right\} \tag{III}$$

and set

$$\Phi = \int_{\Gamma_2} \boldsymbol{v}_* \cdot \boldsymbol{n} = -\int_{\Gamma_1} \boldsymbol{v}_* \cdot \boldsymbol{n},$$

where

$$\Gamma_1 = \left\{x \in \mathbb{R}^2 : |x| = R_1\right\}, \quad \Gamma_2 = \left\{x \in \mathbb{R}^2 : |x| = R_2\right\}.$$

[3] A detailed proof of this property is given by Kapitanskiĭ & Pileckas (1983).

It is readily seen that the field $\boldsymbol{u} = u(r)\boldsymbol{e}_\theta$ solves (II) with $\pi = -\int (u^2/r)$. Moreover, taking into account (IX.4.9), we show

$$\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{V} = -\Phi\int_{R_1}^{R_2}\frac{u^2}{r}.$$

As a consequence, if $\Phi < 0$ the field

$$\boldsymbol{w} = \frac{\boldsymbol{u}}{\left(-\Phi\int_{R_1}^{R_2}\frac{u^2}{r}\right)^{1/2}}$$

satisfies *both* conditions (I) and (II) that are, therefore, compatible.

The contradiction method of Leray just described has been more recently used by Kapitanskiĭ & Pileckas (1983, §4), Borchers & Pileckas (1994) and, independently, by Amick (1984). In particular, the latter author pushes Leray's argument a bit further to show that conditions (I) and (II) are indeed incompatible in a class of plane domains Ω and fields $\boldsymbol{v}_*$, $\boldsymbol{w}$ possessing certain suitable symmetry. As a result, under these assumptions on the data, Amick proves existence requiring only the compatibility condition (IX.4.6) on the flux. The result of Amick has been successively rediscovered by Sazonov (1993). Another, constructive proof of Amick's result can be found in Fujita (1998). If Ω is the annulus (III), Morimoto (1992) and Morimoto & Ukai (1996) have furnished existence under the general condition (IX.4.6), provided, however, the boundary data satisfy some extra restrictions. In this respect, see also Morimoto (1995), Fujita & Morimoto (1997), and Russo & Starita (2008).

All mentioned papers need a sufficiently high degree of smoothness on both $\boldsymbol{v}_*$ and Ω, *e.g.*, of class C^3. The first general result with $\boldsymbol{v}_*$ in the "natural" trace space $W^{1/2,2}(\partial\Omega)$ is due to Foiaş & Temam (1978), under the assumption that Ω is C^2-smooth.

Existence results in locally Lipschitz domains, and with boundary data of lesser regularity than the $W^{1/2,2}(\partial\Omega)$ one, have been proved in the interesting paper of Russo (2003). The approach followed by this author is based on potential-theoretic methods which, in turn, rely upon previous work of Fabes, Kenig & Verchota (1988).

A maximum modulus result for (bounded) locally Lipschitz three-dimensional domains has been shown by Russo (2011).

Uniqueness under conditions of the type given in Theorem IX.4.2 but with a computable constant c_1 can be found in Payne (1965).

Section IX.5. Regularity of generalized solutions was first proved by Ladyzhenskaya (1959b, Chapter II, §2). An independent proof is given by Fujita (1961, §§4,5); see also Shapiro (1976a). Fujita also observes that analyticity of a generalized solution $(\boldsymbol{v}, p)$, corresponding to an analytic $\boldsymbol{f}$ in Ω, follows from the fact that $\boldsymbol{v}, p \in C^\infty(\Omega)$ along with classical results of Morrey (1958) and Friedman (1958) on nonlinear analytic elliptic systems. A completely different (and formally simpler) proof of regularity, based on the estimates of the Stokes problem, is due to Temam (1977, Chapter II, Proposition 1.1).

All the above results hold in dimension $n = 2, 3$ but fail if $n \geq 4$. The problem of regularity in higher dimensions has been considered by von Wahl (1978, Satz II.1) who gave a first, partial answer for $n = 4$. Successively, Gerhardt (1979) proved regularity of generalized solutions in dimension $n = 4$. An analogous theorem has been later obtained, by different tools, by Giaquinta & Modica (1982). Results found by all the above authors, however, do not admit a direct generalization to higher dimensions. Regularity of generalized solution $\boldsymbol{v}$ in arbitrary dimension $n \geq 2$ was first proved by von Wahl (1986) as a by-product of his study on the unsteady Navier–Stokes equations; cf. also Sohr & von Wahl (1984). von Wahl's assumptions on $\boldsymbol{v}$ are a particular case of those of Theorem IX.5.2, obtained by setting there $s = 2$. He also requires extra regularity on Ω.

As already emphasized, our proof of smoothness of generalized (and q-generalized) solutions is based on Lemma IX.5.1. This lemma improves an analogous result shown in Galdi (1994b, Chapter VIII, Lemma 5.1)

An interior regularity result similar to that of Theorem IX.5.1(a) is given by Kim & Kozono (2006, Corollary 5). However, these authors require $n \geq 3$ and $p \in L^1_{loc}(\Omega)$, which are not needed in Theorem IX.5.1(a).

In the paper of Kim & Kozono, the interesting problem of removable singularities is also addressed. More precisely, assuming that a solution $(\boldsymbol{v}, p)$ to the steady-state Navier–Stokes equation is regular in the punctured ball $B - \{x_0\}$, the question is to find conditions on the behavior of $\boldsymbol{v}(x)$ as $x \to x_0$, that ensure that the solution is regular in the whole of B. Similar questions were previously addressed by Dyer & Edmunds (1970), who first studied the problem, Shapiro (1974, 1976b, 1976c), and Choe & Kim (2000).

The important question of *regularity of very weak solutions* has been investigated by several authors. For *interior regularity*, we recall the contribution of Kim & Kozono previously mentioned and its improved version furnished by me in Theorem IX.5.1(a). Results on *regularity up to the boundary* can be found in Marušíc-Paloka (2000, Remark 3), Farwig, Galdi & Sohr (2006, Corollary 1.5), Farwig & Sohr (2009, Theorem 1.5), and Kim (2009, Theorem 3). In particular, in the latter two papers, the authors independently prove a result analogous to Theorem IX.5.2(a), under the following assumptions (i) $\boldsymbol{v} \in L^{q_0}(\Omega)$, $q_0 = n$, if $n \geq 3$, $q_0 > 2$, if $n = 2$; (ii) $\nabla \cdot \boldsymbol{v} = g$, $g \in L^{\frac{nq_0}{n+q_0}}(\Omega) \cap W^{1,q}(\Omega)$ with a sufficiently "small" norm; (iii) $\boldsymbol{v}$ satisfies $(**)$ for all $\boldsymbol{\psi} \in \mathcal{C}^2_0(\Omega)$, and Ω of class $C^{2,1}$ (Farwig & Sohr), all $\boldsymbol{\psi} \in \widetilde{W}_0(\Omega)$, and Ω of class C^2 (Kim). All the above results leave *open* the intriguing question of weather, for $n = 2$, a very weak solution, with $\boldsymbol{v} \in L^2(\Omega)$ and corresponding to regular data, is regular.

X

Steady Navier–Stokes Flow in Three-Dimensional Exterior Domains. Irrotational Case

F.F. CHOPIN, Ballade op.23, bars 7-8.

Introduction

Objective of this and the next two chapters is to investigate the mathematical properties of steady flow of a viscous incompressible fluid that fills the entire space outside a finite number of "bodies", $\Omega_1, \ldots, \Omega_s$, and whose motion is governed by the fully nonlinear Navier–Stokes equations. More specifically, we shall be concerned with the following boundary value problem

$$\left.\begin{aligned} &\nu\Delta \boldsymbol{v} = \boldsymbol{v}\cdot\nabla\boldsymbol{v} + 2\boldsymbol{\omega}\times\boldsymbol{v} + \nabla p + \boldsymbol{f} \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega \qquad \boldsymbol{v} = \boldsymbol{v}_* \text{ at } \partial\Omega \tag{X.0.1}$$

to which we append the condition at infinity

$$\lim_{|x|\to\infty} (\boldsymbol{v}(x) + \boldsymbol{v}_\infty(x)) = \boldsymbol{0}\,. \tag{X.0.2}$$

Here, as usual, Ω is a domain of $\mathbb{R}^n$, $n = 2, 3$, exterior to $\Omega_0 := \cup_{i=1}^s \Omega_i$, with Ω_i compact, $i = 1, \ldots, s$, and $\Omega_i \cap \Omega_j = \emptyset$, for $i \neq j$, representing the

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region of flow, $\boldsymbol{f}$ and $\boldsymbol{v}_*$ are prescribed fields in Ω and $\partial\Omega$, respectively, while $\boldsymbol{v}_\infty(x) := \boldsymbol{v}_0 + \boldsymbol{\omega}\times\boldsymbol{x}$, with $\boldsymbol{v}_0$ and $\boldsymbol{\omega}$ given constant vectors. As explained in Section I.2, this problem is of particularly great relevance in the study of the steady flow past a rigid body $\mathcal{B}$ that translates and rotates, when described from a frame, $\mathcal{S}$, attached to $\mathcal{B}$. In such a case, $s = 1$, $\mathcal{B} \equiv \Omega_1$, and $\boldsymbol{\omega}$ represents the (constant) angular velocity of the body, while $\boldsymbol{v}_0$ is the velocity of its center of mass referred to $\mathcal{S}$, supposed to be constant.[1]

Since the type of difficulties encountered, the methods used and the results found in solving problem (X.0.1)–(X.0.2) are different according to whether $\boldsymbol{\omega} = \boldsymbol{0}$ or $\boldsymbol{\omega} \neq \boldsymbol{0}$, and $n = 2$ or $n = 3$, we wish to treat the various cases in as many chapters. In the present one we shall consider the irrotational case $\boldsymbol{\omega} = \boldsymbol{0}$, so that problem (X.0.1) reduces to

$$\left.\begin{aligned} &\nu\Delta\boldsymbol{v} = \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p + \boldsymbol{f} \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega \qquad \boldsymbol{v} = \boldsymbol{v}_* \text{ at } \partial\Omega \tag{X.0.3}$$

along with the condition at infinity

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = -\boldsymbol{v}_\infty \tag{X.0.4}$$

where $\boldsymbol{v}_\infty \equiv \boldsymbol{v}_0 \in \mathbb{R}^3$.

With few exceptions, which will be either remarked on or explicitly noticed in the statements of our results, *we shall be concerned with three–dimensional flows*, while we shall treat the analogous, more involved two-dimensional problem in Chapter XII. The three-dimensional case with $\boldsymbol{\omega} \neq \boldsymbol{0}$ will be studied in Chapter XI. To date, no significant results are available in two dimensions when $\boldsymbol{\omega} \neq \boldsymbol{0}$.

We shall next describe the problems and the main results presented in this chapter. Following the work of Leray (1933), Ladyzhenskaya (1959b), and Fujita (1961), one gives, as we already did in previous chapters, a *variational* formulation of the problem and proves the existence of a *weak* solution $\boldsymbol{v}$ to (X.0.3), (X.0.4) without restrictions on the "size" of the data and even with a nonzero (but *small*) total flux of $\boldsymbol{v}_*$ through $\partial\Omega$. Such solutions, characterized by having a *finite Dirichlet integral*:

$$\int_\Omega \nabla\boldsymbol{v} : \nabla\boldsymbol{v} \le M, \tag{X.0.5}$$

where M depends only on the data, are sometimes referred to as *D-solutions*, or also *Leray solutions.* In the investigation of the features of D-solutions one has to face *two* different kinds of problems. On the one hand, similarly to generalized solutions in bounded domains, one has to analyze their *differentiability* properties. However, this is simply achieved by using the regularity

[1] See footnote 14 in Chapter I.

theory developed in the case Ω bounded and we can prove that D-solutions are, in fact, smooth provided the data are smooth. On the other hand, one has to show that D-solutions have those properties expected from the physical point of view and tightly related to their *behavior at large distances*. For example, they have to verify the *energy equation*:

$$2\nu\int_{\Omega}\boldsymbol{D}(\boldsymbol{v}):\boldsymbol{D}(\boldsymbol{v})-\int_{\partial\Omega}\Big[\,(\boldsymbol{v}_*+\boldsymbol{v}_\infty)\cdot\boldsymbol{T}(\boldsymbol{v},p)\\ -\tfrac{1}{2}(\boldsymbol{v}_*+\boldsymbol{v}_\infty)^2\boldsymbol{v}_*\Big]\cdot\boldsymbol{n}+\int_{\Omega}\boldsymbol{f}\cdot(\boldsymbol{v}+\boldsymbol{v}_\infty)=0 \tag{X.0.6}$$

with $\boldsymbol{D}$ and $\boldsymbol{T}$ stretching and stress tensors (IV.8.6), which describes the balance between the power of the work of external force, the work done on the "body" Ω_1, and the energy dissipated by the viscosity. Also, if $\boldsymbol{f}, \boldsymbol{v}_*$, and $\boldsymbol{v}_\infty$ are "sufficiently small" with respect to the viscosity ν, the corresponding D-solutions must be unique. In addition, in the case $s=1$, $\boldsymbol{v}_*=\boldsymbol{f}=\boldsymbol{0}$ (rigid, impermeable body translating in the liquid with constant velocity $\boldsymbol{v}_0$), the flow must exhibit an infinite wake extending in the direction opposite to $\boldsymbol{v}_\infty$: inside the wake the flow is essentially vortical and the order of convergence of $\boldsymbol{v}$ to $\boldsymbol{v}_\infty$ is different depending on whether it is calculated inside or outside the wake. Finally, according to the boundary-layer concept, the flow must be potential outside the close vicinity of the body Ω_1 and of the wake, which means (at least) that the vorticity should decay exponentially fast at large distances and outside the wake.

Despite of the efforts of many mathematicians, for quite a long time these questions had no answer to the point that, in 1959, R. Finn was led to introduce another class of solutions characterized by the requirement that the velocity field $\boldsymbol{v}$ obeys, as $|x|\to\infty$,

$$\begin{aligned}\boldsymbol{v}(x)+\boldsymbol{v}_\infty&=O(|x|^{-1/2-\varepsilon}),\quad \text{some }\ \varepsilon>0,\text{ if }\boldsymbol{v}_\infty\neq 0\\ \boldsymbol{v}(x)&=O(|x|^{-1}),\quad \text{if }\ \boldsymbol{v}_\infty=0.\end{aligned} \tag{X.0.7}$$

In a series of fundamental papers, Finn and his coworkers were then able to show that any such solution has all the basic properties previously mentioned. For this reason, he called solutions satisfying (X.0.7) *physically reasonable* (PR). Moreover, in 1965, by extremely careful and painstaking estimates of the Green's tensor function, Finn showed that if the data are "small enough" there exists a unique corresponding PR-solution.

It then appeared quite natural to study the relation between a D-solution and a PR-solution and, therefore, to ascertain if D-solutions can effectively describe the real world or are just mathematical inventions. However, although it is a relatively simple task to prove that a PR-solution is a D-solution, the question of whether the converse implication holds true has remained open for years and, even today, it presents some aspects that are yet to be clarified. Specifically, if $\boldsymbol{v}_\infty\neq 0$, K.I. Babenko (1973) has shown that every

D-solution $\boldsymbol{v}$ corresponding to a body force of bounded support is a PR-solution (cf. also Galdi 1992b, Farwig & Sohr 1998) while Galdi (1992c) has proved the same property for $\boldsymbol{v}_\infty = 0$ *provided, however, that* $\boldsymbol{v}$ *obeys a the "energy inequality"* (that is, (X.0.6) with "=" replaced by "$\le$") *and that the viscosity is sufficiently large.* In this respect, it is worth noticing that the class of D-solutions obeying the energy inequality is certainly nonempty. The nonhomogeneity of these results for the cases $\boldsymbol{v}_\infty \neq 0$ and $\boldsymbol{v}_\infty = 0$ is essentially due to the following reason. They both rely on the asymptotic properties of solutions to the linearized approximations of (X.0.3), (X.0.4). Now, if $\boldsymbol{v}_\infty \neq 0$, such an approximation leads to the Oseen system, while if $\boldsymbol{v}_\infty = 0$ it leads to the Stokes system and we know from Chapters V and VII that the asymptotic properties of solutions to the Stokes system are, in an appropriate sense, weaker than those of the Oseen one.

In the present chapter, we shall follow the basic ideas of Galdi (1992a, 1992b, 1992c), with several changes in the details. Thus, after proving existence of D-solutions, we shall investigate, among other things, their asymptotic behavior at large distances and shall show that it coincides with that of the Stokes or the Oseen fundamental tensor according to whether $\boldsymbol{v}_\infty$ is or is not zero. However, if $\boldsymbol{v}_\infty = 0$, we are able to prove this only for *large* viscosity and for D-solutions obeying the energy inequality.

It should also be emphasized that if $\boldsymbol{v}_\infty = 0$, several fundamental questions *remain open.* For example, it is not known if, for given $\boldsymbol{f}$ and $\boldsymbol{v}_*$ there are solutions obeying the energy equation (X.0.6) *without restriction on the size of the data* or, equivalently, *for all values of the viscosity.* Moreover, it is not known if, when $\Omega = \mathbb{R}^3$, the only D-solution corresponding to $\boldsymbol{f} = 0$ is the zero solution; see Remark X.9.4.

In what follows, we shall find it convenient to put (X.0.3), (X.0.4) into a dimensionless form, and so we need comparison length d and velocity V. If $\boldsymbol{v}_\infty \neq 0$ and $|\Omega^c| \neq \emptyset$, we can take $d = \delta(\Omega^c)$, $V = |\boldsymbol{v}_\infty|$ and so, introducing the *Reynolds number*

$$\mathcal{R} = \frac{Vd}{\nu},$$

the system (X.0.3) becomes

$$\left.\begin{aligned} &\Delta \boldsymbol{v} = \mathcal{R}\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p + \mathcal{R}\boldsymbol{f} \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega \qquad \text{(X.0.8)}$$
$$\boldsymbol{v} = \boldsymbol{v}_* \text{ at } \partial\Omega$$

where $\boldsymbol{v}$, $\boldsymbol{v}_*$, p, and $\boldsymbol{f}$ are now nondimensional quantities. In such a case the nondimensional velocity $\boldsymbol{v}_\infty$ *becomes a unit vector that, without loss, we will take coincident with the unit vector* $\boldsymbol{e}_1$. If either $\boldsymbol{v}_\infty = 0$ or $\Omega \equiv \mathbb{R}^n$ this choice of d and V is no longer possible, even though we can still give meaning to (X.0.8), which is what we shall do throughout the chapter.

We wish to observe that, whenever possible, we shall explicitly remark if and in what form a certain result we find can be generalized to dimension

$n \geq 4$. If no comment is made, it will be tacitly understood that either the result does not admit of a straightforward generalization to higher dimension or, even, that such a generalization constitutes an open question.

Finally, we remark that all material presented here concerns the "classical" formulation of the exterior problem. However, there is a wide variety of exterior problems which can not be included within this formulation and are of great physical relevance like, for example, the steady fall of a body or the steady motion of a self-propelled body in a viscous liquid. We shall not treat these questions and refer the interested reader to the works of Weinberger (1972, 1973, 1974), Serre (1887), Galdi (1999a), and to the comprehensive article of Galdi (2002) and to the references cited therein.

X.1 Generalized Solutions. Preliminary Considerations and Regularity Properties

We begin to give a generalized formulation of the Navier–Stokes problem (X.0.8) , (X.0.4) and to investigate some basic properties of the corresponding solutions, including their regularity. To this end, assuming at first $\boldsymbol{v}$, p, and $\boldsymbol{f}$ are sufficiently smooth, we multiply $(\text{X.0.8})_1$ by $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and integrate by parts over Ω to obtain

$$\int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{\varphi} + \mathcal{R} \int_\Omega \boldsymbol{v} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{\varphi} = -\mathcal{R} \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}. \tag{X.1.1}$$

Thus, every regular solution to $(\text{X.0.8})_1$ satisfies identity (X.1.1) for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. Conversely, as in the case where Ω is bounded (cf. Section IX.1), it is easy to show by means of Lemma III.1.1 that if $\boldsymbol{v}$ is of class $C^2(\Omega)$, say, and satisfies (X.1.1) for some $\boldsymbol{f} \in C(\Omega)$ and all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$, then $\boldsymbol{v}$ obeys $(\text{X.0.8})_1$ for some $p \in C^1(\Omega)$. However, if $\boldsymbol{v}$ merely satisfies (X.1.1) and is not a priori sufficiently regular, we cannot go from (X.1.1) to $(\text{X.0.8})_1$ and therefore (X.1.1) is the *weak* version of (X.0.8).

As in the linear case, we shall consider the more general situation when the right-hand side of (X.1.1) is defined by a linear functional $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$.

Thus, in analogy with Definition V.1.1, we give the following.

Definition X.1.1. Let Ω be an exterior domain of $\mathbb{R}^n$, $n \geq 2$. A vector field $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ is called a *weak* (or *generalized*) *solution to the Navier–Stokes problem* (X.0.8), (X.0.4) if and only if[1]

[1] Often, in the literature, generalized solutions are called *D-solutions* (or *Leray solutions*), where D means that they have a finite Dirichlet integral. We shall avoid this nomenclature in the present chapter, while in Chapter XII, which treats the two-dimensional case, it will be used to mean a field satisfying the requirements (i)-(iii) and (v) of Definition X.1.1 but not necessarily condition (iv). It should be emphasized that in three dimensions any such field obeys (iv) for *some* $\boldsymbol{v}_\infty \in \mathbb{R}^3$, as a consequence of Theorem II.6.1.

(i) $\boldsymbol{v} \in D^{1,2}(\Omega)$;
(ii) $\boldsymbol{v}$ is (weakly) divergence-free in Ω;
(iii) $\boldsymbol{v}$ satisfies the boundary condition $(\mathrm{X}.0.8)_3$ (in the trace sense) or, if $\boldsymbol{v}_* \equiv 0$, then $\vartheta \boldsymbol{v} \in D_0^{1,2}(\Omega)$ where $\vartheta \in C_0^1(\overline{\Omega})$ and $\vartheta(x) = 1$ if $x \in \Omega_{R/2}$ and $\vartheta(x) = 0$ if $x \in \Omega^R$, $R > 2\delta(\Omega^c)$;
(iv) $\lim\limits_{|x|\to\infty} \int_{S^{n-1}} |\boldsymbol{v}(x) + \boldsymbol{v}_\infty| = 0$;
(v) $\boldsymbol{v}$ satisfies the identity

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) + \mathcal{R}(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi}) = -\mathcal{R}\,[\boldsymbol{f}, \boldsymbol{\varphi}] \tag{X.1.2}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$.

Remark X.1.1 Remark V.1.1 with $q = 2$, and Remark IX.1.1, Remark IX.1.3 equally apply to generalized solutions of Definition X.1.1. ■

Our next objective is to ascertain the existence of a pressure field associated to a generalized solution and to investigate the corresponding properties. Existence is, in fact, readily established. Actually, if

$$\boldsymbol{f} \in W_0^{-1,2}(\Omega'), \quad \text{for every bounded domain } \Omega' \text{ with } \overline{\Omega'} \subset \Omega,$$

from Lemma II.6.1 and Lemma IX.1.2 it follows that there is a field

$$p \in L^2_{loc}(\Omega)$$

verifying the identity

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) + \mathcal{R}(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\psi}) = (p, \nabla \cdot \boldsymbol{\psi}) - \mathcal{R}[\boldsymbol{f}, \boldsymbol{\psi}] \tag{X.1.3}$$

for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. In particular, if

$$\boldsymbol{f} \in W_0^{-1,2}(\Omega_R),$$

and Ω is locally Lipschitz, again from Lemma II.6.1 and Lemma IX.1.2, we deduce

$$p \in L^2(\Omega_R).$$

However, unlike the case where Ω is bounded, we are not able a priori to draw any conclusion concerning the L^2-summability of p on the *whole* of Ω. This is due to the circumstance that if we *only* have

$$\boldsymbol{v} \in D^{1,2}(\Omega)$$

we cannot guarantee the existence of $c = c(\boldsymbol{v})$ such that

$$|(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\psi})| \le c|\boldsymbol{\psi}|_{1,2}. \tag{X.1.4}$$

Nevertheless, if we restrict ourselves to three-dimensional flows, we can show summability for p with a suitable exponent in a neighborhood of infinity. In this regard, we can show the following.

Lemma X.1.1 *Let Ω be an exterior domain of $\mathbb{R}^n$, $n = 2, 3$, and let $\boldsymbol{v}$ be a generalized solution to* (X.0.8), (X.0.4) *in Ω. Then, if*

$$\boldsymbol{f} \in W_0^{-1,2}(\Omega') \tag{X.1.5}$$

for every bounded domain Ω' with $\overline{\Omega'} \subset \Omega$, there exists

$$p \in L^2_{loc}(\Omega)$$

satisfying (X.1.3) *for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. If Ω is locally Lipschitz and for some $R > \delta(\Omega^c)$*

$$\boldsymbol{f} \in W_0^{-1,2}(\Omega_R),$$

we have

$$p \in L^2(\Omega_R).$$

Moreover, if $\Omega \subseteq \mathbb{R}^3$ and, in addition to (X.1.5),

$$\boldsymbol{f} \in D_0^{-1,q}(\Omega^R), \quad \text{some } R > \delta(\Omega^c), \text{ and } q \in (3/2, \infty),$$

then p (up to an additive constant) admits the following decomposition

$$p = p_1 + p_2\,, \quad p_1 \in L^3(\Omega^\rho)\,, \quad p_2 \in L^q(\Omega^\rho)\,, \quad \rho > R. \tag{X.1.6}$$

For this latter results to hold, no regularity on Ω is needed.

Proof. By what we already said, we have to prove only (X.1.6). We shall do this under the assumption $\boldsymbol{v}_\infty = \boldsymbol{0}$, the case $\boldsymbol{v}_\infty \neq \boldsymbol{0}$ being treated in an entirely analogous way. For $i = 1, 2$, let

$$(\boldsymbol{v}_i, p_i) \in \mathcal{D}_0^{1,q_i}(\Omega^R) \times L^{q_i}(\Omega^R), \quad q_1 = 3\,, \quad q_2 = q\,,$$

be q_i-generalized solution and associated pressure to the following Stokes problems

$$\begin{aligned} &(\nabla \boldsymbol{v}_i, \nabla \boldsymbol{\psi}) = (p_i, \nabla \cdot \boldsymbol{\psi}) + [\boldsymbol{f}_i, \boldsymbol{\psi}]\ ,\ \boldsymbol{\psi} \in C_0^\infty(\Omega^R)\,, \\ &\boldsymbol{f}_1 := -\mathcal{R}\boldsymbol{v} \cdot \nabla \boldsymbol{v}\,, \quad \boldsymbol{f}_2 := -\mathcal{R}\boldsymbol{f}\,. \end{aligned} \tag{X.1.7}$$

Since by (iv) of Definition X.1.1 and Theorem II.6.1 we have

$$\boldsymbol{v} \in L^6(\Omega^R)$$

and therefore, for all $\boldsymbol{\psi} \in C_0^\infty(\Omega^R)$,

$$|(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\psi})| = |(\boldsymbol{v} \otimes \boldsymbol{v}, \nabla \boldsymbol{\psi})| \leq \|\boldsymbol{v}\|^2_{6,\Omega^R} |\boldsymbol{\psi}|_{1,3/2}\,,$$

we conclude that

$$\boldsymbol{f}_1 \in D_0^{-1,3}(\Omega^R).$$

By this and by the assumption on $\boldsymbol{f}$, with the help of Theorem V.5.1[2] we show the existence of the above solutions. Next, from (X.1.7) and (X.1.3), it follows that

[2] And of Theorem VII.7.2 and Remark VII.7.3 when $\boldsymbol{v}_\infty \neq \boldsymbol{0}$.

$$(\nabla(\boldsymbol{v}_1 + \boldsymbol{v}_2 - \boldsymbol{v}), \nabla\boldsymbol{\psi}) = (p_1 + p_2 - p, \nabla \cdot \boldsymbol{\psi}), \quad \boldsymbol{\psi} \in C_0^\infty(\Omega^R),$$

and so, from Theorem IV.4.3 we find that $\boldsymbol{V} := \boldsymbol{v}_1 + \boldsymbol{v}_2 - \boldsymbol{v}$ and $P := p_1 + p_2 - p$ satisfy the following properties

$$\Delta \boldsymbol{V} = \nabla P \ \text{ in } \Omega^R, \quad \boldsymbol{V}, P \in C^\infty(\Omega^R). \tag{X.1.8}$$

Suppose, at first, $q \geq 2$. By the Hölder inequality, Lemma II.6.3, and Exercise II.6.3 we find (with $|x| = r$, $x \in \Omega^R$)

$$\begin{aligned} I_2(r) := \int_{S^2} |\boldsymbol{V}(r,\omega)|^2 d\omega &\leq c_1 \left(\|\boldsymbol{v}_1\|^2_{3,S^2} + \|\boldsymbol{v}_2\|^2_{q,S^2} + \|\boldsymbol{v}\|^2_{2,S^2} \right) \\ &\leq c_2 \left[(\ln r)^{4/3} + g(r) + r^{-1} + 1 \right], \end{aligned}$$

where $g(r) = (\ln r)^{4/3}$ if $q = 3$, while $g(r) = r^{2-6/q}$, if $q \neq 3$. If $q \in (3/2, 2)$, by the same token, we estimate as follows

$$\begin{aligned} I_q(r) := \int_{S^2} |\boldsymbol{V}(r,\omega)|^q d\omega &\leq c_3 \left(\|\boldsymbol{v}_1\|^q_{3,S^2} + \|\boldsymbol{v}_2\|^q_{q,S^2} + \|\boldsymbol{v}\|^q_{2,S^2} \right) \\ &\leq c_4 \left[(\ln r)^{2q/3} + r^{q-3} + r^{-q/2} + 1 \right]. \end{aligned}$$

As a consequence, we have, for all $\rho > R$,

$$\begin{aligned} &\int_\rho^\infty \frac{I_2(\tau)}{(1+\tau)^5} \tau^2 d\tau < \infty, \quad \text{if } q \in [2, \infty) \\ &\int_\rho^\infty \frac{I_q(\tau)}{(1+\tau)^{3+q}} \tau^2 d\tau < \infty, \quad \text{if } q \in (3/2, 2), \end{aligned}$$

so that $\boldsymbol{V}$ satisfies the assumption of Theorem V.3.2, from which we derive, in particular, $P \in L^s(\Omega^\rho)$, for all $\rho > R$, and all $s > 3/2$. The proof of the theorem is then completed. □

Remark X.1.2 If $n = 2$, the proof just given fails, because Theorem II.6.1 does not ensure any summability for $\boldsymbol{v} + \boldsymbol{v}_\infty$ on Ω^R. Consequently, for plane motions, a generalized solution has a priori no summability at all in the neighborhood of infinity. However, by a more detailed study performed in Chapter XII, where we will show that *if* $\boldsymbol{v}_\infty \neq 0$ the pressure field of any weak solution satisfying some mild extra property and corresponding to suitable body force is in $L^q(\Omega^R)$ for *all* $q > 2$. *If* $\boldsymbol{v}_\infty = 0$, *the question is completely open.* ■

Remark X.1.3 Lemma X.1.1 is readily extended to dimension $n \geq 4$. In fact, under the assumption

$$\boldsymbol{f} \in D_0^{-1,q}(\Omega^R), \quad q \in (n/(n-1), \infty)$$

one can show that

$$p = p_1 + p_2\,, \quad p_1 \in L^{n/(n-2)}(\Omega^R)\,, \quad p_2 \in L^q(\Omega^R).$$

This result is obtained by the same technique used in Lemma X.1.1, observing that, by Theorem II.6.1, one has

$$(\boldsymbol{v} + \boldsymbol{v}_\infty) \cdot \nabla \boldsymbol{v} \in D_0^{-1,n/(n-2)}(\Omega^R).$$

In particular, if Ω is locally Lipschitz and $q = n/(n-2)$, we have

$$p \in L^{n/(n-2)}(\Omega),$$

thus recovering for p the *same* property proved in the case Ω bounded; cf. Remark IX.1.5. It is interesting to observe that if the asymptotic properties of $\boldsymbol{v}$ merely reduce to $\boldsymbol{v} \in D^{1,2}(\Omega)$, the exponent $n/(n-2)$ is sharp, even if $\boldsymbol{f} \in C_0^\infty(\Omega)$, as the following argument shows. Consider (X.0.8), (X.0.4) in $\Omega \equiv \mathbb{R}^n$, $n \geq 3$, with $\boldsymbol{v}_\infty = 0$ and assume $\boldsymbol{v}$, p to be a smooth solution corresponding to $\boldsymbol{f} = \nabla \cdot \boldsymbol{F}$, where $\boldsymbol{F} \equiv \{F_{ij}\} \in C_0^\infty(\mathbb{R}^n)$. (As will be proved in Theorem X.4.1, such a solution does exist.) By taking the divergence operator of both sides of (X.0.8) we then recover that the pressure field satisfies the following equation

$$\frac{1}{\mathcal{R}} \Delta p = \sum_{i,j=1}^{n} D_i D_j (v_i v_j + F_{ij}).$$

Starting with this relation, we can then give an *explicit* representation for p. Actually, if p tends to a limit p_0 at infinity, employing classical arguments we can easily show (with $c_i = c_i(n)$, $i = 1, 2$) that

$$\begin{aligned} p(x) = p_0 + \mathcal{R}\{c_1 &\sum_{i=1}^{n} [v_i^2(x) + F_{ii}(x)] \\ &+ c_2 \sum_{i,j=1}^{n} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3 - B_\varepsilon(x)} (v_i v_j(y) + F_{ij}(y))[D_i D_j |x-y|^{2-n}] dy\} \\ \equiv p_0 + p_1(x) &+ p_2(x) + p_3(x) + p_4(x). \end{aligned}$$

If

$$\boldsymbol{v} \in D^{1,2}(\mathbb{R}^n), \tag{X.1.9}$$

by the Sobolev inequality it follows that

$$v_i v_j \in L^{n/(n-2)}(\mathbb{R}^n) \tag{X.1.10}$$

and so

$$p_1 \in L^{n/(n-2)}(\mathbb{R}^n).$$

Clearly,

$$p_2 \in L^q(\mathbb{R}^n) \quad \text{for all } q > 1.$$

Moreover, the quantity

$$D_i D_j |x-y|^{2-n}$$

is a singular kernel, and so, by the Calderón–Zygmund theorem, by (X.1.9) and the properties of $\boldsymbol{F}$ we have

$$p_3 \in L^{n/(n-2)}(\mathbb{R}^n)$$
$$p_4 \in L^q(\mathbb{R}^n) \ \text{ for all } q > 1.$$

We then conclude that

$$p \in L^{n/(n-2)}(\mathbb{R}^n). \tag{X.1.11}$$

Since (X.1.10) cannot be improved under the *sole* assumption (X.1.9), the estimates on p_1 and p_3 are sharp and therefore (X.1.11) is sharp too. ■

Exercise X.1.1 Let Ω be a exterior domain of $\mathbb{R}^n$, $n \geq 3$, of class C^2. By the same argument used in the proof of Lemma X.1.1, show that if $\boldsymbol{v}$ is a weak solution to (X.0.8), (X.0.4) with

$$\boldsymbol{v}_* \in W^{1-1/q,q}(\partial\Omega), \quad \boldsymbol{f} \in D_0^{-1,q}(\Omega)\,, \quad q \in (n/(n-1), \infty)$$

then $p = p_1 + p_2$ with $p_1 \in L^{n/(n-2)}(\Omega)$, $p_2 \in L^q(\Omega)$, where p is the pressure field associated to $\boldsymbol{v}$ by Lemma X.1.1.

In the final part of this section we shall discuss the differentiability properties of a generalized solution. This will be done with the help of Theorem IX.5.1 and Theorem IX.5.2. Specifically, we have the following

Theorem X.1.1 *Let $\boldsymbol{v}$ be a generalized solution to* (X.0.8), (X.0.4) *in an exterior domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$ with $\boldsymbol{v} \in L^n_{loc}(\Omega)$.*[3] *Then, if*

$$\boldsymbol{f} \in W^{m,q}_{loc}(\Omega), \quad m \geq 0\,, \tag{X.1.12}$$

where $q \in (1, \infty)$ if either $m = 0$ or $n = 2$, while $q \in [n/2, \infty)$ if $m > 0$ and $n \geq 3$, it follows that

$$\boldsymbol{v} \in W^{m+2,q}_{loc}(\Omega), \quad p \in W^{m+1,q}_{loc}(\Omega),$$

where p is the pressure associated to $\boldsymbol{v}$ by Lemma X.1.1.[4] *Thus, in particular, if*

[3] These assumptions on on $\boldsymbol{v}$ can be replaced by the weaker requirements (i)–(iii) of Theorem IX.5.1; see also Remark IX.5.1.

[4] Observe that if $\boldsymbol{f}$ satisfies (XI.1.23) for the specified values of m and q, then for all bounded Ω' with $\overline{\Omega'} \subset \Omega$ it holds that

$$\boldsymbol{f} \in W_0^{-1,2}(\Omega'), \quad \text{if } n = 2, 3$$

while

$$\boldsymbol{f} \in W_0^{-1,n/(n-2)}(\Omega'), \quad \text{if } n \geq 4.$$

$$\boldsymbol{f} \in C^{\infty}(\Omega),$$

then

$$\boldsymbol{v}, p \in C^{\infty}(\Omega).$$

Assume, further, that $\boldsymbol{v} \in L^n(\Omega_R)$, *for some* $R > \delta(\Omega^c)$. *Then if* Ω *is of class* C^{m+2} *and*

$$\boldsymbol{v}_* \in W^{m+2-1/q,q}(\partial\Omega), \quad \boldsymbol{f} \in W^{m,q}(\Omega_R) \tag{X.1.13}$$

with the values of m *and* q *specified earlier, we have*

$$\boldsymbol{v} \in W^{m+2,q}(\Omega_R), \quad p \in W^{m+1,q}(\Omega_R). \tag{X.1.14}$$

Thus, in particular, if Ω *is of class* C^{∞} *and*

$$\boldsymbol{v}_* \in C^{\infty}(\partial\Omega), \quad \boldsymbol{f} \in C^{\infty}(\overline{\Omega}_R),$$

it follows that

$$\boldsymbol{v},\ p \in C^{\infty}(\overline{\Omega}_R).$$

Proof. The first part of the theorem (local regularity) has already been shown in Theorem IX.5.1. To show the second, from the hypothesis on $\boldsymbol{f}$ made in (X.1.13) we obtain, again by Theorem IX.5.1,

$$\boldsymbol{v} \in W^{m+2,q}_{loc}(\Omega),$$

and, as a consequence,

$$\boldsymbol{v} \in W^{m+2-1/q,q}(\partial B_R).$$

Therefore, $\boldsymbol{v}$ is a generalized solution to the Navier–Stokes problem in the bounded domain Ω_R with $\boldsymbol{v} \in L^n(\Omega_R)$ and corresponding to data satisfying the assumption of Theorem IX.5.2. Therefore, (X.1.14) follows and the result is completely proved. $\square$

X.2 On the Validity of the Energy Equation for Generalized Solutions

When we give a mathematical definition of solution to a system of equations relating to physics, one of the first questions that arises is that of establishing if these solutions meet all basic physical principles such as conservation laws or energy balance. Thus, in the present situation, a first issue to investigate is to ascertain if a generalized solution satisfies the *energy equation.* In the case when the velocity at the boundary vanishes identically[1] this equation in its classical formulation reads:

$$2\int_{\Omega} \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\boldsymbol{v}) - \boldsymbol{v}_\infty \cdot \int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{v},p)\cdot\boldsymbol{n} + \mathcal{R}\int_{\Omega} \boldsymbol{f}\cdot(\boldsymbol{v}+\boldsymbol{v}_\infty) = 0 \quad \text{(X.2.1)}$$

where $\boldsymbol{D}$ and $\boldsymbol{T}$ are stretching and stress tensors defined in (IV.8.6). The meaning of (X.2.1) is quite clear: it represents the balance between the dissipation due to the viscosity, the power of external body force and, for $\boldsymbol{v}_\infty \neq 0$, the total force exerted by the liquid on the bodies. If Ω is *bounded*, the proof of the energy equation (i.e., the relation formally obtained from (X.2.1) with $\boldsymbol{v}_\infty = 0$) depends *solely* on the *regularity in* $\overline{\Omega}$ of the solution and one readily shows that, if $\boldsymbol{f}$ has a mild degree of smoothness, every corresponding generalized solution satisfies the energy equation; see Exercise IX.3.1. However, if Ω is an *exterior domain*, the proof of (X.2.1) demands not only certain regularity in $\overline{\Omega}_R$, for all $R > \delta(\Omega^c)$, but it also depends in an essential way on the *asymptotic properties* of the solution. To see this, assume $\boldsymbol{v}, p$ is a classical solution to (X.0.8), (X.0.4) with $\boldsymbol{v}_* \equiv 0$. Multiplying (X.0.8) by $\boldsymbol{u} := \boldsymbol{v}+\boldsymbol{v}_\infty$, integrating by parts over Ω_R and taking into account $(\text{X.0.8})_2$, we deduce

$$\begin{aligned} 2\int_{\Omega_R} \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\boldsymbol{v}) &- \boldsymbol{v}_\infty \cdot \int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{v},p)\cdot\boldsymbol{n} + \mathcal{R}\int_{\Omega_R} \boldsymbol{f}\cdot(\boldsymbol{v}+\boldsymbol{v}_\infty) \\ &= \mathcal{R}\int_{\partial B_R} \boldsymbol{u}\cdot\boldsymbol{T}(\boldsymbol{v},p)\cdot\boldsymbol{n} - \frac{\mathcal{R}}{2}\int_{\partial B_R} u^2\boldsymbol{v}\cdot\boldsymbol{n}. \end{aligned} \quad \text{(X.2.2)}$$

If $\boldsymbol{v}$ is a generalized solution,

$$\boldsymbol{v} \in D^{1,2}(\Omega)\,. \quad \text{(X.2.3)}$$

So, if $\boldsymbol{f}$ is “well-behaved” at infinity, letting $R \to \infty$ into (X.2.2), the left-hand side of (X.2.2) tends to the left-hand side of (X.2.1); nevertheless the sole condition (X.2.3), together with condition (X.1.6) on the pressure,[2] is not enough to ensure that the surface integral on the right-hand side of (X.2.2) tends to zero as $R \to \infty$, even along a sequence. Thus, we are not able to conclude the validity of (X.2.1), unless we have *further* information on the behavior of $\boldsymbol{v}$ at large distances.

Another (equivalent) way of looking at the problem is the following one. Assume, for simplicity, $\boldsymbol{v}_* \equiv \boldsymbol{v}_\infty \equiv 0$. Then, taking into account that $|\boldsymbol{v}|_{1,2}^2 = 2\|\boldsymbol{D}(\boldsymbol{v})\|_2^2$, in our standard notation, the energy equation (X.2.1) becomes

$$|\boldsymbol{v}|_{1,2}^2 = -\mathcal{R}\,[\boldsymbol{f},\boldsymbol{v}]\,. \quad \text{(X.2.4)}$$

[1] This assumption is made here for the sake of simplicity. The case $\boldsymbol{v}_* \neq 0$ will be considered later in this section.

[2] Recall that, by Remark X.1.3, (X.1.6) is sharp for a generalized solution. Moreover, (X.1.6) holds if $\Omega \subset \mathbb{R}^3$, otherwise, if $\Omega \subset \mathbb{R}^2$, we do not even know if $p \in L^q(\Omega^R)$ for some $q \in [1,\infty)$.

From the definition of generalized solution we know that

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) + \mathcal{R}(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi}) = -\mathcal{R}\,[\boldsymbol{f}, \boldsymbol{\varphi}] \tag{X.2.5}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. The natural step would then be to put $\boldsymbol{v}$ into (X.2.5) in place of $\boldsymbol{\varphi}$. However, unlike the case where Ω is bounded (see Exercise IX.3.1), we are not allowed a priori to do this, since $\boldsymbol{v}$ does not have the appropriate summability properties at large distances. This fact is intimately related to the *continuity properties* of the trilinear form

$$a(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \equiv (\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{w}) \tag{X.2.6}$$

which, for an arbitrary domain Ω are specified in the following.

Lemma X.2.1 *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \geq 2$. Then the trilinear form* (X.2.6) *is continuous in the space*

$$L^q(\Omega) \times \dot{D}^{1,r}(\Omega) \times L^s(\Omega), \quad q^{-1} + r^{-1} + s^{-1} = 1.$$

If $n \geq 3$, it is also continuous in the space

$$D_0^{1,2}(\Omega) \times \dot{D}^{1,2}(\Omega) \times L^n(\Omega).$$

Proof. The first assertion is a trivial consequence of the Hölder inequality, while the second is proved by choosing $q = 2n/(n-2)$, $r = n$ and using the Sobolev inequality (II.3.7):

$$\|\boldsymbol{u}\|_{2n/(n-2)} \leq \frac{q(n-1)}{2(n-q)\sqrt{n}} |\boldsymbol{u}|_{1,2}.$$

$\square$

In view of this lemma we can thus *formally*[3] replace $\boldsymbol{\varphi}$ in (X.2.5) with the generalized solution $\boldsymbol{v}$ if, for instance, $\boldsymbol{v} \in L^4(\Omega)$ (by choosing $q = s = 4, r = 2$) or, if $n \geq 3$, $\boldsymbol{v} \in L^n(\Omega)$.

The reasonings just developed explain why, in the case of an exterior domain Ω, a solution should still be considered a "generalized" solution even though it is smooth in $\overline{\Omega_R}$ for any $R > \delta(\Omega^c)$. Actually, unlike the case where Ω is bounded, where the word "generalized" expresses only a possible lack of differentiability, for Ω exterior it is also to mean the possibility that the solution need not be as "regular" in the neighborhood of infinity as expected from the physical (and intuitive) point of view.

Our objective in this section is to investigate what regularity must be imposed on the velocity field at large distances in order that it satisfies equation (X.2.1). Successively (cf. Section X.7 and Section X.9), we shall analyze

[3] We have, in fact, to ascertain that $\boldsymbol{v}$ can be suitably approximated by functions from $\mathcal{D}(\Omega)$.

whether such a regularity can be deduced by making suitable assumptions on the data (on $\boldsymbol{f}$ in particular). Specifically, when the region of flow is three-dimensional, we shall prove that if $\boldsymbol{f}$ enjoys certain summability conditions, every generalized solution corresponding to suitable $\boldsymbol{f}$ and $\boldsymbol{v}_\infty \neq 0$ obeys the energy equation. If $\boldsymbol{v}_\infty = 0$, however, we are able to show the same result only for small Reynolds number $\mathcal{R}$. Thus, *if* $\boldsymbol{v}_\infty = 0$ *the validity of* (X.2.1) *for arbitrary* $\mathcal{R}$ *remains open.* If the region of flow is two-dimensional the picture is even less clear and will be treated in detail in Chapter XI. Specifically, for plane flows one shows that, if $\boldsymbol{f}$ is suitable and $\boldsymbol{v}_\infty \neq 0$, every generalized solution satisfying a mild extra property obeys (X.2.1) (even though existence is only known for small $\mathcal{R}$) while, if $\boldsymbol{v}_\infty = 0$, the question remains entirely open.

Before performing the above investigation, however, there is a formal aspect that must be fixed. If $\boldsymbol{v}_\infty \neq 0$ and $\boldsymbol{v}$ merely belongs to $W^{1,2}_{loc}(\Omega_R)$ for all $R > \delta(\Omega^c)$, as prescribed by Definition X.1.1, the second integral in (X.2.1) need not be meaningful. Thus, in such a case, we have to introduce a suitable generalization of (X.2.1). Let us denote by $\boldsymbol{a}$ any vector field in Ω verifying

$$
\begin{aligned}
&\text{(i)} \ \ \boldsymbol{a} \in C^\infty(\Omega);\\
&\text{(ii)} \ \ \nabla \cdot \boldsymbol{a} = 0;\\
&\text{(iii)} \ \ \boldsymbol{a} = 0 \ \text{ in } \Omega_d \text{ and } \boldsymbol{a} + \boldsymbol{v}_\infty \equiv 0 \text{ in } \Omega^{2d} \text{ for some } d > \delta(\Omega^c).
\end{aligned}
\tag{X.2.7}
$$

For instance, we may take $\boldsymbol{a}$ as in (V.2.5). If $\boldsymbol{v}_\infty = 0$ we take $\boldsymbol{a} \equiv 0$. The field $\boldsymbol{a}$ will be called *an extension of* $\boldsymbol{v}_\infty$. We then give the following.

Definition X.2.1. Let $\boldsymbol{v}$ be a generalized solution to (X.0.8), (X.0.4) corresponding to $\boldsymbol{v}_* \equiv 0$ and let $\boldsymbol{a}$ be an extension of $\boldsymbol{v}_\infty$. The relation

$$
|\boldsymbol{v}|^2_{1,2} + \mathcal{R}[\boldsymbol{f}, \boldsymbol{v} + \boldsymbol{a}] = (\nabla \boldsymbol{v}, \nabla \boldsymbol{a}) - \mathcal{R}(\boldsymbol{v} \cdot \nabla \boldsymbol{a}, \boldsymbol{v} + \boldsymbol{a}) \tag{X.2.8}
$$

is called *generalized energy equation.*

To justify the preceding definition, we notice that, if $\boldsymbol{v}_\infty = 0$, (X.2.8) coincides with (X.2.1), while it reduces to (X.2.1) for $\boldsymbol{v}_\infty \neq 0$, provided Ω, $\boldsymbol{f}$, $\boldsymbol{v}$, and p are sufficiently smooth. For example, if Ω is of class C^1 and

$$
\boldsymbol{v} \in C^1(\overline{\Omega_R}) \cap C^2(\Omega_R), \quad p \in C(\overline{\Omega_R}) \cap C^1(\Omega_R), \quad \boldsymbol{f} \in C(\Omega),
$$

multiplying (X.0.8) by $\boldsymbol{a} + \boldsymbol{v}_\infty$, integrating by parts over Ω_R and observing that supp $(\nabla \boldsymbol{a}) \subset \overline{\Omega_{d,2d}}$ it follows that

$$
\int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{T}(\boldsymbol{v}, p) \cdot \boldsymbol{v}_\infty = \mathcal{R}[\boldsymbol{f}, \boldsymbol{a} + \boldsymbol{v}_\infty] + 2(\boldsymbol{D}(\boldsymbol{v}), \boldsymbol{D}(\boldsymbol{a})) - \mathcal{R}(\boldsymbol{v} \cdot \nabla \boldsymbol{a}, \boldsymbol{v} - \boldsymbol{a}). \tag{X.2.9}
$$

Taking into account that[4]

[4] Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in D^{1,2}(\Omega)$, with $\boldsymbol{u}_1 - \boldsymbol{u}_2 \in \mathcal{D}^{1,2}_0(\Omega)$. Then, the following identity holds

$$|\boldsymbol{v}|_{1,2}^2 - (\nabla \boldsymbol{v}, \nabla \boldsymbol{a}) = 2[\|\boldsymbol{D}(\boldsymbol{v})\|_2^2 - (\boldsymbol{D}(\boldsymbol{v}), \boldsymbol{D}(\boldsymbol{a}))],$$

(X.2.9) along with (X.2.8), implies (X.2.1).

Exercise X.2.1 The validity of (X.2.9) can be shown under very mild regularity assumptions on $\boldsymbol{f}$. For example, suppose that, for some $R > \delta(\Omega^c)$, $\boldsymbol{f} \in L^r(\Omega_R)$, where r is arbitrary in $(1, \infty)$. Assume, further, $\boldsymbol{v}_* \equiv 0$ and Ω is of class C^2. Show that every generalized solution $\boldsymbol{v}$ corresponding to these data satisfies (X.2.9) with p pressure field associated to $\boldsymbol{v}$ by Lemma X.1.1. *Hint:* Use Theorem X.1.1 and Exercise II.4.3.

Remark X.2.1 In Section X.4 we shall prove that, regardless of the dimension $n \geq 2$, for any $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$ there exists a corresponding generalized solution that (for $\boldsymbol{v}_* \equiv 0$) satisfies the *generalized energy inequality*, that is, relation (X.2.8) with "=" replaced by " $\leq$ ". However, the case $\Omega = \mathbb{R}^2$ appears to be open; see Remark X.4.4 ■

The main results of this section are contained in Theorem X.2.1 and Theorem X.2.2, which we are now going to prove.

Theorem X.2.1 *Let $\boldsymbol{v}$ be a generalized solution to the Navier–Stokes problem* (X.0.8), (X.0.4) *in an exterior locally Lipschitz domain of $\mathbb{R}^n$, $n = 2, 3$, corresponding to $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$, $\boldsymbol{v}_* = 0$ and to some $\boldsymbol{v}_\infty \in \mathbb{R}^n$. Then if $\boldsymbol{v}_\infty = 0$, a sufficient condition in order that $\boldsymbol{v}$ satisfies the generalized energy equation* (X.2.8) *with $\boldsymbol{a} \equiv 0$ is*

$$\boldsymbol{v} \in L^4(\Omega) \tag{X.2.10}$$

while, if $\boldsymbol{v}_\infty \neq 0$, a sufficient condition in order that $\boldsymbol{v}$ satisfies (X.2.8) *for any extension $\boldsymbol{a}$ of $\boldsymbol{v}_\infty$ is*

$$\begin{gathered} \boldsymbol{v} + \boldsymbol{v}_\infty \in L^4(\Omega) \cap L^q(\Omega) \\ \boldsymbol{v}_\infty \cdot \nabla \boldsymbol{v} \in L^{q'}(\Omega) \end{gathered} \tag{X.2.11}$$

for some $q \in (1, \infty)$, $q' = q/(q-1)$.

Proof. Let us first consider the case where $\boldsymbol{v}_\infty = 0$. By Definition X.1.1 and Remark XI.1.1 we have

$$\boldsymbol{v} \in \widehat{\mathcal{D}}_0^{1,2}(\Omega)$$

and, since Ω is locally Lipschitz, by the results of Section III.5,

$$\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega).$$

$$|\boldsymbol{u}_1|_{1,2}^2 - (\nabla \boldsymbol{u}_1, \nabla \boldsymbol{u}_2) = 2[\|\boldsymbol{D}(\boldsymbol{u}_1)\|_2^2 - (\boldsymbol{D}(\boldsymbol{u}_1), \boldsymbol{D}(\boldsymbol{u}_2))]\,.$$

In fact, let $\{\boldsymbol{w}_k\} \subset \mathcal{D}(\Omega)$ be such that $|\boldsymbol{w}_k - (\boldsymbol{u}_1 - \boldsymbol{u}_2)|_{1,2} \to 0$ as $k \to \infty$. Multiplying both sides of the identity $2\nabla \cdot \boldsymbol{D}(\boldsymbol{w}_k) = \Delta \boldsymbol{w}_k$ by $\boldsymbol{u}_1$, integrating by parts over Ω and letting $k \to \infty$ proves the result.

Denote by $\{\boldsymbol{v}_k\} \subset \mathcal{D}(\Omega)$ a sequence of functions converging to $\boldsymbol{v}$ in the space $\mathcal{D}_0^{1,2}(\Omega) \cap L^4(\Omega)$. In view of (X.2.10) and Theorem III.6.2 this sequence certainly exists. Replacing $\boldsymbol{\varphi}$ with $\boldsymbol{v}_k$ into (X.1.2) furnishes

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{v}_k) + \mathcal{R}(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{v}_k) = -\mathcal{R}[\boldsymbol{f}, \boldsymbol{v}_k]. \tag{X.2.12}$$

Clearly, as $k \to \infty$,

$$\begin{aligned} (\nabla \boldsymbol{v}, \nabla \boldsymbol{v}_k) &\to |\boldsymbol{v}|_{1,2}^2 \\ [\boldsymbol{f}, \boldsymbol{v}_k] &\to [\boldsymbol{f}, \boldsymbol{v}]\,. \end{aligned} \tag{X.2.13}$$

Furthermore, by Lemma X.2.1, we have

$$|(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{v}_k) - (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{v})| \le \|\boldsymbol{v}\|_4 |\boldsymbol{v}|_{1,2} \|\boldsymbol{v} - \boldsymbol{v}_k\|_4$$

and so, in virtue of (X.2.10),

$$(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{v}_k) \to (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{v}). \tag{X.2.14}$$

However, it is immediately seen that

$$(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{v}) = 0. \tag{X.2.15}$$

Actually, (X.2.15) is a consequence of Lemma X.2.1 and the fact that, by Lemma IX.2.1,

$$(\boldsymbol{v}_k \cdot \nabla \boldsymbol{v}_k, \boldsymbol{v}_k) = 0, \quad \text{for all } k \in \mathbb{N}.$$

The proof of the theorem when $\boldsymbol{v}_\infty = 0$ then follows from (X.2.12)–(X.2.15). Consider next the case where $\boldsymbol{v}_\infty \neq 0$. To this end, let $\boldsymbol{a}$ be any extension of $\boldsymbol{v}_\infty$ and set

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{a}.$$

Since Ω is locally Lipschitz, by (i)-(iv) of Definition X.1.1, Theorem II.6.3, and the results of Section III.5 we obtain, as before,

$$\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega). \tag{X.2.16}$$

Furthermore, from $(\text{X.2.11})_1$ we also have

$$\boldsymbol{u} \in L^4(\Omega) \cap L^q(\Omega). \tag{X.2.17}$$

In view of (X.2.16) and (X.2.17), Theorem III.6.2 implies the existence of a sequence $\{\boldsymbol{u}_k\} \subset \mathcal{D}(\Omega)$ converging to $\boldsymbol{u}$ in the space

$$\mathcal{D}_0^{1,2}(\Omega) \cap L^4(\Omega) \cap L^q(\Omega).$$

Setting $\boldsymbol{\varphi} = \boldsymbol{u}_k$ into (X.1.2) we thus find

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{u}_k) + \mathcal{R}((\boldsymbol{v} + \boldsymbol{v}_\infty) \cdot \nabla \boldsymbol{v}, \boldsymbol{u}_k) - \mathcal{R}(\boldsymbol{v}_\infty \cdot \nabla \boldsymbol{v}, \boldsymbol{u}_k) = -\mathcal{R}\,[\boldsymbol{f}, \boldsymbol{u}_k]\,. \tag{X.2.18}$$

Taking into account Lemma X.2.1, (X.2.11), and (X.2.17), we recover, in the limit $k \to \infty$

$$\begin{aligned} (\nabla \boldsymbol{v}, \nabla \boldsymbol{u}_k) &\to (\nabla \boldsymbol{v}, \nabla \boldsymbol{u}) \\ ((\boldsymbol{v}+\boldsymbol{v}_\infty)\cdot \nabla \boldsymbol{v}, \boldsymbol{u}_k) &\to ((\boldsymbol{v}+\boldsymbol{v}_\infty)\cdot \nabla \boldsymbol{v}, \boldsymbol{u}) \\ (\boldsymbol{v}_\infty \cdot \nabla \boldsymbol{v}, \boldsymbol{u}_k) &\to (\boldsymbol{v}_\infty \cdot \nabla \boldsymbol{v}, \boldsymbol{u}). \end{aligned} \tag{X.2.19}$$

Moreover, since for all $k \in \mathbb{N}$

$$((\boldsymbol{v}+\boldsymbol{v}_\infty)\cdot \nabla \boldsymbol{u}_k, \boldsymbol{u}_k) = 0,$$

we have, again by Lemma X.2.1,

$$((\boldsymbol{v}+\boldsymbol{v}_\infty)\cdot \nabla \boldsymbol{u}, \boldsymbol{u}) = 0$$

and (X.2.18), (X.2.19) furnish

$$|\boldsymbol{v}|_{1,2}^2 + \mathcal{R}\,[\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{a}] = (\nabla \boldsymbol{v}, \nabla \boldsymbol{a}) - \mathcal{R}\,(\boldsymbol{v}\cdot\nabla\boldsymbol{a}, \boldsymbol{v}-\boldsymbol{a}) - \mathcal{R}\,(\boldsymbol{v}_\infty\cdot\nabla\boldsymbol{u}, \boldsymbol{u})\,. \tag{X.2.20}$$

We have now to show that the last term on the right-hand side of (X.2.20) vanishes identically. First of all, we observe that it is clearly well-defined since, by (X.2.11)_2 and (X.2.17) it follows that

$$\boldsymbol{v}_\infty \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u} \in L^1(\Omega). \tag{X.2.21}$$

Next, taking $\boldsymbol{v}_\infty = \boldsymbol{e}_1$, for $\rho > \delta(\Omega^c)$ we let $\{\mathcal{C}_{k,\rho}\}$, $k \in \mathbb{N}$, denote a family of cylinder-like domains of the form

$$\mathcal{C}_{k,\rho} = \{x \in \Omega : |x'| < k, |x_1| < \rho\}\,,$$

where $x' = (x_2, \ldots, x_n)$. For sufficiently large k and ρ it is $\mathcal{C}_{k,\rho} \supset \Omega^c$ and, correspondingly, we find

$$\int_{\mathcal{C}_{k,\rho}} \boldsymbol{v}_\infty \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u} = \int_{\mathcal{C}_{k,\rho}} \frac{\partial \boldsymbol{u}}{\partial x_1} \cdot \boldsymbol{u} = \int_{\Sigma_1(k,\rho)} u^2 - \int_{\Sigma_2(k,\rho)} u^2, \tag{X.2.22}$$

where

$$\begin{aligned} \Sigma_1(k,\rho) &= \{x \in \mathbb{R} : x_1 = \rho,\ |x'| \le k\}\,, \\ \Sigma_2(k,\rho) &= \{x \in \mathbb{R} : x_1 = -\rho,\ |x'| \le k\}\,. \end{aligned}$$

By (X.2.11)_1, it is easy to show that, for each fixed k,

$$\lim_{\rho\to\infty} \int_{\Sigma_i(k,\rho)} u^2 = 0, \quad i = 1, 2. \tag{X.2.23}$$

In fact, setting $x_\rho = (\rho, 0, \ldots, 0)$, by the boundary inequality (II.4.1), it follows that

$$\left(\int_{\Sigma_i(k,\rho)} u^2\right)^{1/2} \leq c_1 \left(\|\boldsymbol{u}\|_{2,B_k(x_\rho)} + \|\nabla \boldsymbol{u}\|_{2,B_k(x_\rho)}\right)$$
$$\leq c_2 \left(\|\boldsymbol{u}\|_{4,B_k(x_\rho)} + \|\nabla \boldsymbol{u}\|_{2,B_k(x_\rho)}\right)$$

with $c_i = c_i(n,k)$, and so (X.2.23) with $i = 1$ is recovered. By an analogous argument one shows (X.2.23) with $i = 2$. Putting

$$\mathcal{C}_k = \{x \in \Omega : |x'| < k,\ x_1 \in \mathbb{R}\},$$

from (X.2.22), and (X.2.23) we find

$$\int_{\mathcal{C}_k} \boldsymbol{v}_\infty \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u} = 0 \tag{X.2.24}$$

for all sufficiently large k. On the other hand, by (X.2.21),

$$\lim_{k\to\infty} \int_{\mathcal{C}_k} \boldsymbol{v}_\infty \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u} = \int_\Omega \boldsymbol{v}_\infty \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u}$$

and so, by (X.2.24),

$$(\boldsymbol{v}_\infty \cdot \nabla \boldsymbol{u}, \boldsymbol{u}) = 0,$$

which completes the proof of the theorem. □

Remark X.2.2 The theorem continues to hold under different assumptions on $\boldsymbol{f}$. In fact, $\boldsymbol{f}$ is required to satisfy

$$[\boldsymbol{f}, \boldsymbol{v}_k - \boldsymbol{a}] \to [\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{a}].$$

Therefore, one can take

$$\boldsymbol{f} \in L^{4/3}(\Omega), \tag{X.2.25}$$

or else, if $\boldsymbol{v}_\infty \neq 0$,

$$\boldsymbol{f} \in L^{q'}(\Omega).$$

■

Remark X.2.3 If $n = 3$, Ω is of class C^2, and $\boldsymbol{f}$ satisfies (X.2.25), then condition (X.2.10) alone is enough to ensure the validity of the energy equality also in the case $\boldsymbol{v}_\infty \neq \boldsymbol{0}$. In fact, under the stated assumptions on $\boldsymbol{v}$ and $\boldsymbol{f}$, by the Hölder inequality we find $[(\boldsymbol{v} + \boldsymbol{v}_\infty) \cdot \nabla \boldsymbol{v} + \boldsymbol{f}] \in L^{4/3}(\Omega)$. Therefore, from Theorem VII.7.1 and Theorem VII.6.2, it easily follows, in particular, that $\boldsymbol{v}_\infty \cdot \nabla \boldsymbol{v} \in L^{4/3}(\Omega)$, and condition (X.2.11) remains satisfied with $q = 4$. ■

Remark X.2.4 Theorem X.2.1 continues to hold, as it stands, in dimension $n \geq 4$, the proof remaining completely unchanged. In this respect, it is interesting to observe that if $n = 4$, from Theorem II.6.1 it follows that every generalized solution corresponding to $\boldsymbol{v}_* \equiv \boldsymbol{v}_\infty \equiv 0$ satisfies (X.2.10). We may then conclude that *in dimension 4 any generalized solution corresponding to* $\boldsymbol{v}_* \equiv \boldsymbol{v}_\infty \equiv 0$ *satisfies the energy equality.* Such a result is not known in dimension $n = 3$. ■

Theorem X.2.1 can be extended, with no essential technical changes, to the case when the velocity $\boldsymbol{v}_*$ at the boundary is not identically zero. To this end, we need a suitable extension of *both* $\boldsymbol{v}_*$ and $\boldsymbol{v}_\infty$. Following what we already did in the proof of Theorem V.2.1, for Ω locally Lipschitz and

$$\boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega),$$

we may take

$$\boldsymbol{A} = \boldsymbol{V} + \Phi\nabla\mathcal{E} - \boldsymbol{a}. \tag{X.2.26}$$

Here $\boldsymbol{V} \in W^{1,2}(\Omega)$ is an extension of $\boldsymbol{v}_* - \Phi\nabla\mathcal{E}$ vanishing in Ω^ρ for some $\rho > \delta(\Omega^c)$, Φ is the flux of $\boldsymbol{v}_*$ through $\partial\Omega$, and $\boldsymbol{a}$ is an extension of $\boldsymbol{v}_\infty$ in the sense of (X.2.7). Taking into account the elementary properties of the fundamental solution of Laplace's equation, from (X.2.26) it is apparent that

$$\begin{aligned}
&\text{(i)}\quad |\boldsymbol{A}(x) + \boldsymbol{v}_\infty| \le c|\Phi|\,|x|^{-n+1} \text{ in } \Omega^\rho, \quad c = c(n);\\
&\text{(ii)}\;\; \boldsymbol{A} \in D^{1,s}(\Omega^\rho) \cap W^{1,2}(\Omega_\rho) \cap C^\infty(\Omega^\rho), \quad \text{for all } s > 1;\\
&\text{(iii)}\; \boldsymbol{A} = \boldsymbol{v}_* \;\text{ at } \partial\Omega;\\
&\text{(iv)}\; \nabla\cdot\boldsymbol{A} = 0 \;\text{ in } \Omega.
\end{aligned} \tag{X.2.27}$$

In particular, if $\Phi = 0$, $\boldsymbol{A} + \boldsymbol{v}_\infty$ is of bounded support in Ω. Any field $\boldsymbol{A}$ satisfying conditions (i)-(iv) in (X.2.27) will be called *an extension of* $\boldsymbol{v}_*$ *and* $\boldsymbol{v}_\infty$. Thus, generalizing Definition X.2.1, we give the following definition.

Definition X.2.2. Let $\boldsymbol{v}$ be a generalized solution to (X.0.8), (X.0.4) and let $\boldsymbol{A}$ be an extension of $\boldsymbol{v}_*$ and $\boldsymbol{v}_\infty$. The relation

$$|\boldsymbol{v}|_{1,2}^2 + \mathcal{R}[\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{A}] = (\nabla\boldsymbol{v}, \nabla\boldsymbol{A}) - \mathcal{R}(\boldsymbol{v}\cdot\nabla\boldsymbol{A}, \boldsymbol{v} - \boldsymbol{A}) \tag{X.2.28}$$

is called *the generalized energy equation.*

As expected, under suitable regularity assumptions on the data and $\boldsymbol{v}$, identity (X.2.28) reduces to the energy equation in its classical formulation, cf. Exercise X.2.2.

By a procedure completely analogous to that used in the proof of Theorem X.2.1, one can show the following result.

Theorem X.2.2 *Let* $\boldsymbol{v}$ *be a generalized solution to the Navier–Stokes problem* (X.0.8), (X.0.4) *in an exterior locally Lipschitz domain of* $\mathbb{R}^n$, $n = 2, 3$, *corresponding to*

$$\boldsymbol{f} \in D_0^{-1,2}(\Omega), \quad \boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega), \quad \boldsymbol{v}_\infty \in \mathbb{R}^n.$$

Then, if $\boldsymbol{v}_\infty = 0$, *a sufficient condition in order that* $\boldsymbol{v}$ *satisfies the generalized energy equation* (X.2.28) *for any extension* $\boldsymbol{A}$ *of* $\boldsymbol{v}_*$ *and* $\boldsymbol{v}_\infty$ *is that* (X.2.10) *holds. Likewise, if* $\boldsymbol{v}_\infty \neq 0$, *a sufficient condition in order that* $\boldsymbol{v}$ *satisfies* (X.2.28) *for any extension* $\boldsymbol{A}$ *of* $\boldsymbol{v}_*$ *and* $\boldsymbol{v}_\infty$ *is that* (X.2.11) *holds for some* $q > n/(n-1)$.

Remark X.2.5 The restriction on q (not needed in Theorem X.2.1) is due to the fact that, if $\Phi \neq 0$, the field $\boldsymbol{A}$ belongs to $L^q(\Omega^\rho)$ for $q > n/(n-1)$ only. However, if $\Phi = 0$, $\boldsymbol{A}$ can be taken of bounded support and we may assume $q \in (1, \infty)$. ■

Remark X.2.6 Remark X.2.2 equally applies to Theorem X.2.2. The same holds for Remark X.2.3, provided we assume $\boldsymbol{v}_* \in W^{5/4,4/3}(\partial\Omega)$. ■

Remark X.2.7 Theorem X.2.2 continues to hold in dimension $n \geq 4$. ■

Exercise X.2.2 Let $\boldsymbol{f}$, $\boldsymbol{v}$, and Ω satisfy the assumptions of Exercise X.2.1 and Theorem X.2.2. Suppose, further, that

$$\boldsymbol{v}_* \in W^{2-1/r,r}(\partial\Omega),$$

with r as in Exercise X.2.1. Then, if $\Phi = 0$, show the identity

$$\begin{aligned}\int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{T}(\boldsymbol{v}, p) \cdot (\boldsymbol{v}_* + \boldsymbol{v}_\infty) - \frac{\mathcal{R}}{2}\int_{\partial\Omega} (\boldsymbol{v}_* + \boldsymbol{v}_\infty)^2 \boldsymbol{v}_* \cdot \boldsymbol{n} \\ = \mathcal{R}[\boldsymbol{f}, \boldsymbol{A} + \boldsymbol{v}_\infty] + 2(\boldsymbol{D}(\boldsymbol{v}), \boldsymbol{D}(\boldsymbol{A})) - \mathcal{R}(\boldsymbol{v} \cdot \nabla \boldsymbol{A}, \boldsymbol{v} - \boldsymbol{A})\end{aligned}$$

with $\boldsymbol{A}$ an extension of $\boldsymbol{v}_*$ and $\boldsymbol{v}_\infty$. If $\Phi \neq 0$, show the validity of the same identity under the additional assumption $\boldsymbol{f} \in D_0^{-1,3}(\Omega)$. Thus, since[5]

$$|\boldsymbol{v}|_{1,2}^2 - (\nabla \boldsymbol{v}, \nabla \boldsymbol{A}) = 2[\|\boldsymbol{D}(\boldsymbol{v})\|_2^2 - (\boldsymbol{D}(\boldsymbol{v}), \boldsymbol{D}(\boldsymbol{A}))]$$

under the stated assumptions, in view of Theorem X.2.2, $\boldsymbol{v}$ obeys the energy equation

$$2\int_\Omega \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\boldsymbol{v}) - \int_{\partial\Omega} [(\boldsymbol{v}_\infty + \boldsymbol{v}_*) \cdot \boldsymbol{T} - \frac{\mathcal{R}}{2}(\boldsymbol{v}_* + \boldsymbol{v}_\infty)^2 \boldsymbol{v}_*] \cdot \boldsymbol{n} + \mathcal{R}\int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} + \boldsymbol{v}_\infty) = 0. \tag{X.2.29}$$

X.3 Some Uniqueness Results

The aim of this section is to determine conditions under which a weak solution $\boldsymbol{v}$ is unique. The problem offers more or less the same type of technical difficulties encountered in the preceding section, as we are about to explain. Let us *denote by $\mathcal{C}_v$ the class of weak solutions achieving the same data as* $\boldsymbol{v}$. Then, as we shall prove, in order for a weak solution to be unique in $\mathcal{C}_v$, in addition to the (expected) restriction on the "size" of $\boldsymbol{v}$ in suitable norms of the type made in Theorem IX.2.1 for Ω bounded, we have to require on $\boldsymbol{v}$ some extra conditions at large spatial distances that a priori do not follow directly from Definition X.1.1. Thus, it becomes necessary to ascertain if such conditions are met by prescribing the data appropriately. This will be done in Section X.7 and Section X.9. It should be also said that, in fact, we are

[5] The proof of this identity is similar to that given in footnote 4 in this section.

able to prove uniqueness only in the a priori smaller class $\mathcal{C}'_v$ constituted by those elements of $\mathcal{C}_v$ that satisfy the energy inequality (cf. (X.3.1)) and, if $\boldsymbol{v}_\infty \neq 0$, verify further summability conditions at large distances (cf. (X.3.2)). Nevertheless, in Section X.4 we will prove that $\mathcal{C}'_v$ is certainly nonempty.

To accomplish our objective, however, we need to employ a method that is a bit different from that adopted for flows in bounded domains. This is because the use of such a method would lead to a uniqueness result that does not impose extra conditions directly on $\boldsymbol{v}$ but, rather, on $\boldsymbol{v} - \boldsymbol{w}$, $\boldsymbol{w} \in \mathcal{C}_v$, in contrast to what stated previously. To see why this happens, we recall that the starting point of the method is the identity (IX.2.5) which, according to the nondimensionalization used in the present chapter, now reads

$$\mathcal{R}^{-1}(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{\varphi}) + (\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi}) + (\boldsymbol{v} \cdot \nabla \boldsymbol{u}, \boldsymbol{\varphi}) = 0.$$

The next step is to substitute $\boldsymbol{u}$ for $\boldsymbol{\varphi}$ into this relation and this can be done via the usual approximating procedure that employs the continuity of the trilinear form (X.2.6). According to Lemma X.2.1, we must then require some extra conditions on $\boldsymbol{u}$. However, $\boldsymbol{u}$ is the *difference* of two generalized solutions and the method would lead to a uniqueness result different from that stated at the beginning of the current section.

The method we shall adopt here is due to Galdi (1992a, 1992c) and relies upon an idea introduced by Leray (1934, §32) in a completely different context, namely, that of local regularity of weak solutions to the *initial* value problem for the Navier–Stokes equations, and successively generalized by Serrin and Sather; cf. Serrin (1963, Theorem 6), Sather (1963, Theorem 5.1). In the case of steady flows in exterior domains with $\boldsymbol{v}_* \equiv \boldsymbol{v}_\infty \equiv 0$, the method was first considered by Kozono & Sohr (1993).

Before proving the main results, we need to define the class $\mathcal{C}'_v$ properly.

Definition X.3.1 $\mathcal{C}'_v$ denotes the subclass of $\mathcal{C}_v$ constituted by those generalized solutions $\boldsymbol{w}$ satisfying the *generalized energy inequality*

$$|\boldsymbol{w}|^2_{1,2} + \mathcal{R}[\boldsymbol{f}, \boldsymbol{w} - \boldsymbol{A}] \leq (\nabla \boldsymbol{w}, \nabla \boldsymbol{A}) - \mathcal{R}(\boldsymbol{w} \cdot \nabla \boldsymbol{A}, \boldsymbol{w} - \boldsymbol{A}) \tag{X.3.1}$$

with $\boldsymbol{A}$ an extension of $\boldsymbol{v}_*$ and $\boldsymbol{v}_\infty$ in the sense of Definition X.2.2. Moreover, if $\boldsymbol{v}_\infty \neq 0$, we denote by $\mathcal{C}'_{v,q}$ the subclass of $\mathcal{C}'_v$ of functions $\boldsymbol{w}$ such that

$$\begin{aligned} \boldsymbol{w} + \boldsymbol{v}_\infty &\in L^q(\Omega) \\ \boldsymbol{v}_\infty \cdot \nabla \boldsymbol{w} &\in L^{q'}(\Omega) \end{aligned} \tag{X.3.2}$$

for some $q > n/(n-1)$, $q' = q/(q-1)$.

Remark X.3.1 The condition $q > n/(n-1)$ is required only because, if the flux Φ of $\boldsymbol{v}_*$ through $\partial\Omega$ is nonzero, $\boldsymbol{A}+\boldsymbol{v}_\infty$ is not summable at large distances for $q \leq n/(n-1)$ so that the term

$$(\boldsymbol{w} \cdot \nabla \boldsymbol{A}, \boldsymbol{w} - \boldsymbol{A})$$

can be meaningless. On the other hand, if $\Phi = 0$ we can take $\boldsymbol{A} - \boldsymbol{v}_\infty$ of bounded support in Ω (see Definition X.2.2) so that the restriction $q > n/(n-1)$ can be dropped. ■

We are now in a position to prove the following.

Theorem X.3.1 *Let Ω be a locally Lipschitz exterior domain of $\mathbb{R}^3$. Assume $\boldsymbol{v}$ is a generalized solution to the Navier–Stokes problem* (X.0.8), (X.0.4) *corresponding to data*

$$\boldsymbol{f} \in D_0^{-1,2}(\Omega), \quad \boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega), \quad \boldsymbol{v}_\infty \in \mathbb{R}^3,$$

and such that

$$\boldsymbol{v} + \boldsymbol{v}_\infty \in L^3(\Omega). \tag{X.3.3}$$

If $\boldsymbol{v}_\infty \neq 0$, suppose, further, that

$$\boldsymbol{v} + \boldsymbol{v}_\infty \in L^q(\Omega), \quad \boldsymbol{v}_\infty \cdot \nabla \boldsymbol{v} \in L^{q'}(\Omega) \tag{X.3.4}$$

for some $q > 3/2$, $q' = q/(q-1)$. Then, if

$$\|\boldsymbol{v} + \boldsymbol{v}_\infty\|_3 < \frac{\sqrt{3}}{2\mathcal{R}}, \tag{X.3.5}$$

$\boldsymbol{v}$ is the unique solution in the class $\mathcal{C}'_v$ if $\boldsymbol{v}_\infty = 0$ and in the class $\mathcal{C}'_{v,q}$ if $\boldsymbol{v}_\infty \neq 0$.

Proof. Let $\boldsymbol{w}$ be any element from $\mathcal{C}'_v$ or $\mathcal{C}'_{v,q}$, according to whether $\boldsymbol{v}_\infty$ is zero or not. The field $\boldsymbol{w} - \boldsymbol{A}$ has zero trace at the boundary, is divergence-free and vanishes at infinity in the sense of Definition X.1.1(iv). Therefore, from the results of Theorem III.5.1, it follows that

$$\boldsymbol{w} - \boldsymbol{A} \in \mathcal{D}_0^{1,2}(\Omega). \tag{X.3.6}$$

Furthermore, if $\boldsymbol{v}_\infty \neq 0$, from (X.3.2) and (X.2.27)

$$\boldsymbol{w} - \boldsymbol{A} \in L^q(\Omega). \tag{X.3.7}$$

By (X.3.6), (X.3.7), and Theorem III.6.2 we may find a sequence $\{\boldsymbol{\varphi}_k\} \subset \mathcal{D}(\Omega)$ converging to $\boldsymbol{w} - \boldsymbol{A}$ in $\mathcal{D}_0^{1,2}(\Omega) \cap L^q(\Omega)$. Replacing $\boldsymbol{\varphi}_k$ for $\boldsymbol{\varphi}$ into (X.1.2) and reasoning as in the proof of Theorem X.2.1 (with slight difference in details) we deduce that

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{w}) + \mathcal{R}[\boldsymbol{f}, \boldsymbol{w} - \boldsymbol{A}] - (\nabla \boldsymbol{v}, \nabla \boldsymbol{A}) + \mathcal{R}(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{w} - \boldsymbol{A}) = 0. \tag{X.3.8}$$

Likewise, observing that $\boldsymbol{v} - \boldsymbol{A}$ has the same summability properties of $\boldsymbol{w} - \boldsymbol{A}$ and that, in addition, by (X.3.3)

$$\boldsymbol{v} - \boldsymbol{A} \in L^3(\Omega),$$

one shows with no difficulty that

$$(\nabla \boldsymbol{w}, \nabla \boldsymbol{v}) + \mathcal{R}[\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{A}] - (\nabla \boldsymbol{w}, \nabla \boldsymbol{A}) + \mathcal{R}(\boldsymbol{w} \cdot \nabla \boldsymbol{w}, \boldsymbol{v} - \boldsymbol{A}) = 0. \tag{X.3.9}$$

Finally, noticing that, by Theorem II.6.1,

$$\boldsymbol{v} + \boldsymbol{v}_\infty \in L^6(\Omega),$$

from (X.3.3) we have by interpolation (cf. (II.2.10))

$$\boldsymbol{v} + \boldsymbol{v}_\infty \in L^4(\Omega),$$

and by Theorem X.2.2 we conclude

$$|\boldsymbol{v}|_{1,2}^2 + \mathcal{R}[\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{A}] = (\nabla \boldsymbol{v}, \nabla \boldsymbol{A}) - \mathcal{R}(\boldsymbol{v} \cdot \nabla \boldsymbol{A}, \boldsymbol{v} - \boldsymbol{A}). \tag{X.3.10}$$

Addition of (X.3.8) and (X.3.9) yields

$$\begin{aligned} -2(\nabla \boldsymbol{w}, \nabla \boldsymbol{v}) &- \mathcal{R}(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{w} - \boldsymbol{A}) - \mathcal{R}(\boldsymbol{w} \cdot \nabla \boldsymbol{w}, \boldsymbol{v} - \boldsymbol{A}) + (\nabla \boldsymbol{v}, \nabla \boldsymbol{A}) \\ &+ (\nabla \boldsymbol{w}, \nabla \boldsymbol{A}) - \mathcal{R}[\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{A}] - \mathcal{R}[\boldsymbol{f}, \boldsymbol{w} - \boldsymbol{A}] = 0. \end{aligned} \tag{X.3.11}$$

Summing side by side (X.3.1), (X.3.10), and (X.3.11) we deduce

$$\frac{1}{\mathcal{R}}|\boldsymbol{u}|_{1,2}^2 \le (\boldsymbol{w}\cdot\nabla \boldsymbol{w}, \boldsymbol{v}-\boldsymbol{A}) + (\boldsymbol{v}\cdot\nabla \boldsymbol{v}, \boldsymbol{w}-\boldsymbol{A}) - (\boldsymbol{w}\cdot\nabla \boldsymbol{A}, \boldsymbol{w}-\boldsymbol{A}) - (\boldsymbol{v}\cdot\nabla \boldsymbol{A}, \boldsymbol{v}-\boldsymbol{A})$$

with $\boldsymbol{u} = \boldsymbol{w} - \boldsymbol{v}$. Adding and subtracting to the right-hand side of this relation the quantity

$$(\boldsymbol{v} \cdot \nabla \boldsymbol{w}, \boldsymbol{v} - \boldsymbol{A}) + (\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{v} - \boldsymbol{A})$$

(notice that each term in the sum is well-defined), we obtain

$$\begin{aligned} \frac{1}{\mathcal{R}}|\boldsymbol{u}|_{1,2}^2 \le\ & (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{v}_\infty) - (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}_\infty + \boldsymbol{A}) \\ & + (\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{v} - \boldsymbol{A}) + (\boldsymbol{v} \cdot \nabla \boldsymbol{w}, \boldsymbol{v} - \boldsymbol{A}) - (\boldsymbol{v} \cdot \nabla \boldsymbol{A}, \boldsymbol{v} - \boldsymbol{A}) \\ & - (\boldsymbol{w} \cdot \nabla \boldsymbol{A}, \boldsymbol{w} - \boldsymbol{A}) + (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{w} - \boldsymbol{A}). \end{aligned}$$

Since

$$-(\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}_\infty + \boldsymbol{A}) + (\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{v} - \boldsymbol{A}) = (\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{v} + \boldsymbol{v}_\infty) - (\boldsymbol{u} \cdot \nabla \boldsymbol{w}, \boldsymbol{v}_\infty + \boldsymbol{A}),$$

it follows that

$$\begin{aligned} \frac{1}{\mathcal{R}}|\boldsymbol{u}|_{1,2}^2 \le\ & (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{v}_\infty) + (\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{v} + \boldsymbol{v}_\infty) \\ & - (\boldsymbol{u} \cdot \nabla \boldsymbol{w}, \boldsymbol{v}_\infty + \boldsymbol{A}) + (\boldsymbol{v} \cdot \nabla (\boldsymbol{w} - \boldsymbol{A}), \boldsymbol{v} - \boldsymbol{A}) \\ & - (\boldsymbol{w} \cdot \nabla \boldsymbol{A}, \boldsymbol{w} - \boldsymbol{A}) + (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{w} - \boldsymbol{A}). \end{aligned} \tag{X.3.12}$$

The following identities hold

(i) $(\boldsymbol{u}\cdot\nabla\boldsymbol{v},\boldsymbol{v}+\boldsymbol{v}_\infty)=0$;
(ii) $(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{w}-\boldsymbol{A})=-(\boldsymbol{v}\cdot\nabla(\boldsymbol{w}-\boldsymbol{A}),\boldsymbol{v}+\boldsymbol{v}_\infty)$;
(iii) $(\boldsymbol{w}\cdot\nabla\boldsymbol{A},\boldsymbol{w}-\boldsymbol{A})=-(\boldsymbol{w}\cdot\nabla(\boldsymbol{w}-\boldsymbol{A}),\boldsymbol{A}+\boldsymbol{v}_\infty)$;
(iv) $(\boldsymbol{u}\cdot\nabla\boldsymbol{w},\boldsymbol{v}_\infty+\boldsymbol{A})=(\boldsymbol{u}\cdot\nabla(\boldsymbol{w}-\boldsymbol{A}),\boldsymbol{v}_\infty+\boldsymbol{A})$.

To show (i), we let $\{\boldsymbol{u}_k\}\subset\mathcal{D}(\Omega)$ be a sequence approximating $\boldsymbol{u}$ in $\mathcal{D}_0^{1,2}(\Omega)$. Set

$$\Omega_k=\operatorname{supp}(\boldsymbol{u}_k)$$

and denote by $\{\boldsymbol{v}_m\}\subset C_0^\infty(\overline{\Omega}_k)$ a sequence converging, for each fixed k, to $\boldsymbol{v}-\boldsymbol{v}_\infty$ in $W^{1,2}(\overline{\Omega}_k)$. Clearly, for all $m\in\mathbb{N}$ we have

$$(\boldsymbol{u}_k\cdot\nabla\boldsymbol{v}_m,\boldsymbol{v}_m)=\tfrac{1}{2}(\boldsymbol{u}_k,\nabla\boldsymbol{v}_m^2)=0$$

and so, passing to the limit $m\to\infty$, by Lemma X.2.1,

$$(\boldsymbol{u}_k\cdot\nabla\boldsymbol{v},\boldsymbol{v}+\boldsymbol{v}_\infty)=(\boldsymbol{u}_k\cdot\nabla(\boldsymbol{v}+\boldsymbol{v}_\infty),\boldsymbol{v}+\boldsymbol{v}_\infty)=0,\quad\text{for all } k\in\mathbb{N}.$$

This relation, along with Lemma X.2.1, implies (i). In a similar way, one proves (ii) and (iii). Finally, consider the identity

$$(\boldsymbol{u}\cdot\nabla\boldsymbol{w},\boldsymbol{v}_\infty+\boldsymbol{A})=(\boldsymbol{u}\cdot\nabla(\boldsymbol{w}-\boldsymbol{A}),\boldsymbol{v}_\infty+\boldsymbol{A})+(\boldsymbol{u}\cdot\nabla(\boldsymbol{A}+\boldsymbol{v}_\infty),\boldsymbol{A}+\boldsymbol{v}_\infty).$$

By a reasoning completely analogous to that used before, one shows

$$(\boldsymbol{u}\cdot\nabla(\boldsymbol{A}+\boldsymbol{v}_\infty),\boldsymbol{A}+\boldsymbol{v}_\infty)=0,$$

so that also (iv) follows. Replacing (i)-(iv) in (X.3.12) we obtain

$$\begin{aligned}\frac{1}{\mathcal{R}}|\boldsymbol{u}|_{1,2}^2&\le(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v}+\boldsymbol{v}_\infty)-(\boldsymbol{u}\cdot\nabla(\boldsymbol{w}-\boldsymbol{A}),\boldsymbol{v}_\infty+\boldsymbol{A})\\&\quad-(\boldsymbol{v}\cdot\nabla(\boldsymbol{w}-\boldsymbol{A}),\boldsymbol{v}_\infty+\boldsymbol{A})+(\boldsymbol{w}\cdot\nabla(\boldsymbol{w}-\boldsymbol{A}),\boldsymbol{v}_\infty+\boldsymbol{A})\\&=(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v}+\boldsymbol{v}_\infty).\end{aligned}\tag{X.3.13}$$

From Lemma X.2.1 it follows that

$$|(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v}+\boldsymbol{v}_\infty)|\le\frac{2}{\sqrt{3}}|\boldsymbol{u}|_{1,2}^2\|\boldsymbol{v}+\boldsymbol{v}_\infty\|_3,$$

which, once substituted into (X.3.13), furnishes

$$|\boldsymbol{u}|_{1,2}^2(\mathcal{R}^{-1}-2(3)^{-1/2}\|\boldsymbol{v}+\boldsymbol{v}_\infty\|_3)\le 0.\tag{X.3.14}$$

Thus, if (X.3.5) holds, (X.3.14) implies $\boldsymbol{u}(x)=0$ for a.a. $x\in\Omega$, and the theorem is proved. □

Remark X.3.2 Theorem X.2.1 is easily extended to any space dimension $n\ge 4$. Actually, it is enough to replace (X.3.3) with

$$\boldsymbol{v} + \boldsymbol{v}_\infty \in L^n(\Omega), \tag{X.3.15}$$

(X.3.5) with

$$\|\boldsymbol{v} + \boldsymbol{v}_\infty\|_n < \frac{(n-2)\sqrt{n}}{\mathcal{R}(n-1)}, \tag{X.3.16}$$

and the condition $q > 3/2$ in (X.3.4) by $q > n/(n-1)$. It is interesting to observe that if $n = 4$, by Theorem II.6.1, every generalized solution satisfies (X.3.15) and so, if $\boldsymbol{v}_\infty = 0$, every generalized solution $\boldsymbol{v}$ satisfying (X.3.16) is unique in the class $\mathcal{C}'_v$. For the case of plane motions, we refer the reader to Section XII.2. ■

Sometimes, it is convenient to formulate uniqueness theorems with summability assumptions replaced by pointwise bounds. Actually, such a type of result is of great relevance when investigating the asymptotic structure of generalized solutions when $\boldsymbol{v}_\infty = 0$. Our goal is then to show a uniqueness result in such a direction. For the applications we have in mind, it will be enough to prove this when $\boldsymbol{v}_* \equiv \boldsymbol{v}_\infty \equiv 0$. Nevertheless, the result admits a straightforward extension to the more general case of nonzero $\boldsymbol{v}_*$ and $\boldsymbol{v}_\infty$, which we leave to the reader as an interesting exercise.

Theorem X.3.2 *Let Ω be as in* Theorem X.3.1 *and let $\boldsymbol{v}$ be a generalized solution to the Navier–Stokes problem* (X.0.8), (X.0.4) *corresponding to the data*

$$\boldsymbol{f} \in D_0^{-1,2}(\Omega)\,, \quad \boldsymbol{v}_* \equiv \boldsymbol{v}_\infty \equiv 0,$$

and such that

$$|x||\boldsymbol{v}(x)| \le M, \tag{X.3.17}$$

for a.a. $x \in \Omega$ and some $M > 0$. Then, if

$$M < (2\mathcal{R})^{-1} \tag{X.3.18}$$

$\boldsymbol{v}$ is the unique solution in the class $\mathcal{C}'_v$.

Proof. Let $\boldsymbol{w}$ be any element from $\mathcal{C}'_v$. By Definition X.1.1 and the regularity of Ω,

$$\boldsymbol{w} \in \mathcal{D}_0^{1,2}(\Omega).$$

It is easy to show the validity of (X.3.8) with $\boldsymbol{A} \equiv 0$. In fact, let $\{\boldsymbol{\varphi}_k\} \subset \mathcal{D}(\Omega)$ be a sequence converging to $\boldsymbol{w}$ in $\mathcal{D}_0^{1,2}(\Omega)$ and set $\boldsymbol{\varphi} = \boldsymbol{\varphi}_k$ into (X.1.2). Evidently, as $k \to \infty$,

$$\begin{aligned} (\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}_k) &\to (\nabla \boldsymbol{v}, \nabla \boldsymbol{w}) \\ [\boldsymbol{f}, \boldsymbol{\varphi}_k] &\to [\boldsymbol{f}, \boldsymbol{w}]\,. \end{aligned} \tag{X.3.19}$$

Furthermore, from (X.3.17) and Theorem II.6.1 we recover

$$|(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi}_k - \boldsymbol{w})| \le M|\boldsymbol{v}|_{1,2}\|(\boldsymbol{\varphi}_k - \boldsymbol{w})/|x|\|_2 \le c|\boldsymbol{v}|_{1,2}|\boldsymbol{\varphi}_k - \boldsymbol{w}|_{1,2}$$

and so

$$(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{\varphi}_k)\to(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{w}).$$

Thus, from this latter relation, (X.1.2) with $\boldsymbol{\varphi}\equiv\boldsymbol{\varphi}_k$, and (X.3.19), we then arrive at the identity

$$(\nabla\boldsymbol{v},\nabla\boldsymbol{w})+\mathcal{R}\,[\boldsymbol{f},\boldsymbol{w}]+\mathcal{R}(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{w})=0. \tag{X.3.20}$$

We shall next show the following relation:

$$(\nabla\boldsymbol{w},\nabla\boldsymbol{v})+\mathcal{R}\,[\boldsymbol{f},\boldsymbol{v}]+\mathcal{R}(\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{v})=0. \tag{X.3.21}$$

To this end we notice that, by Lemma X.1.1, for all $\boldsymbol{\psi}\in C_0^\infty(\Omega)$,

$$(\nabla\boldsymbol{w},\nabla\boldsymbol{\psi})+\mathcal{R}\,[\boldsymbol{f},\boldsymbol{\psi}]+\mathcal{R}(\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{\psi})=(p,\nabla\cdot\boldsymbol{\psi}) \tag{X.3.22}$$

where $p\in L^2(\Omega_R)$, any $R>\delta(\Omega^c)$, is the pressure field associated to $\boldsymbol{w}$. By this and Lemma IX.1.1, (X.3.22) continues to hold for every $\boldsymbol{\psi}\in W_0^{1,2}(\Omega_R)$. Let $\psi_R(x)$ be a smooth "cut-off" function such that $\psi_R(x)=1$ if $|x|\le R$, $\psi_R(x)=0$ if $|x|\ge 2R$ and, furthermore,

$$|\nabla\psi_R(x)|\le CR^{-1}$$

for some positive C independent of R and x. Taking

$$\boldsymbol{\psi}=\psi_R\boldsymbol{v},$$

we find

$$(\psi_R\nabla\boldsymbol{w},\nabla\boldsymbol{v})+(\nabla\boldsymbol{w},\boldsymbol{v}\otimes\nabla\psi_R)+\mathcal{R}(\boldsymbol{w}\cdot\nabla\boldsymbol{w},\psi_R\boldsymbol{v})+\mathcal{R}\,[\boldsymbol{f},\psi_R\boldsymbol{v}]=(p,\nabla\psi_R\cdot\boldsymbol{v}) \tag{X.3.23}$$

where we have used the condition $\nabla\cdot\boldsymbol{v}=0$ in Ω. Recalling the properties of ψ_R we have

$$\begin{aligned}
|(\nabla\boldsymbol{w},\boldsymbol{v}\otimes\nabla\psi_R)|&\le c\,|\boldsymbol{w}|_{1,2,\Omega_{R,2R}}\|\boldsymbol{v}/|x|\|_{2,\Omega_{R,2R}}\\
|\,[\boldsymbol{f},(\psi_R-1)\boldsymbol{v}]\,|&\le|\boldsymbol{f}|_{-1,2}|(\psi_R-1)\boldsymbol{v}|_{1,2}\\
&\le c\,|\boldsymbol{f}|_{-1,2}\left(\|(\psi_R-1)\nabla\boldsymbol{v}\|_2+\|\boldsymbol{v}/|x|\|_{2,\Omega_{R,2R}}\right)\\
|((\boldsymbol{w}\cdot\nabla\boldsymbol{w},(\psi_R-1)\boldsymbol{v})|&\le M\|(\psi_R-1)\nabla\boldsymbol{w}\|_2\|\boldsymbol{w}/|x|\|_2.
\end{aligned}$$

Employing these inequalities along with Theorem II.6.1 into (X.3.23) and then letting $R\to\infty$ furnishes

$$(\nabla\boldsymbol{w},\nabla\boldsymbol{v})+\mathcal{R}\,[\boldsymbol{f},\boldsymbol{v}]+\mathcal{R}(\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{v})=\lim_{R\to\infty}(p,\nabla\psi_R\cdot\boldsymbol{v}). \tag{X.3.24}$$

To show (X.3.21), it remains to show that the limit on the right-hand side of (X.3.24) is zero. To this end, we recall that, from Lemma X.1.1 and the assumption on $\boldsymbol{f}$, we have

$$p = p_1 + p_2\,, \quad p_1 \in L^3(\Omega^\rho)\,, \quad p_2 \in L^2(\Omega^\rho)\,, \quad \rho > \delta(\Omega^c)\,. \tag{X.3.25}$$

Thus, from the Hölder inequality and (II.6.48), (II.6.49) we have

$$\begin{aligned}\jmath(R) \equiv |(p, \nabla\psi_R \cdot \boldsymbol{v}) &\le \|p_1\|_{3,\Omega_{R,2R}} \|\nabla\psi_R \cdot \boldsymbol{v}\|_{3/2,\Omega_{R,2R}} \\ &\quad + \|p_2\|_{2,\Omega_{R,2R}} \|\nabla\psi_R \cdot \boldsymbol{v}\|_{2,\Omega_{R,2R}} \\ &\le c \left[\|p_1\|_{3,\Omega_{R,2R}} \left(\int_R^{2R} r^{-1} dr \right)^{2/3} + \|p_2\|_{2,\Omega_{R,2R}} \right] |\boldsymbol{v}|_{1,2}.\end{aligned}$$

However, from (X.3.25) it follows that

$$\|p_1\|_{3,\Omega_{R,2R}} + \|p_2\|_{2,\Omega_{R,2R}} \;\to 0 \;\text{ as } R \to \infty$$

and so

$$\lim_{R\to\infty} \jmath(R) = 0.$$

The validity of (X.3.21) is thus established. We next observe that by (X.3.17) (and (i) of Definition X.1.1, Lemma II.6.1, and Theorem II.3.4 if $\Omega = \mathbb{R}^3$) it follows that

$$\boldsymbol{v} \in L^4(\Omega) \tag{X.3.26}$$

so that, in view of Theorem X.2.1, $\boldsymbol{v}$ obeys the energy equality

$$|\boldsymbol{v}|_{1,2}^2 = -\mathcal{R}\,[\boldsymbol{f}, \boldsymbol{v}]\,. \tag{X.3.27}$$

On the other hand, by assumption, $\boldsymbol{w}$ obeys the energy inequality

$$|\boldsymbol{w}|_{1,2}^2 \le -\mathcal{R}\,[\boldsymbol{f}, \boldsymbol{w}]\,. \tag{X.3.28}$$

Adding the four displayed equations (X.3.20), (X.3.21), (X.3.27), and (X.3.28) yields

$$\mathcal{R}^{-1} |\boldsymbol{u}|_{1,2}^2 \le (\boldsymbol{v} \cdot \nabla\boldsymbol{v}, \boldsymbol{w}) + (\boldsymbol{w} \cdot \nabla\boldsymbol{w}, \boldsymbol{v}) \tag{X.3.29}$$

with $\boldsymbol{u} = \boldsymbol{w} - \boldsymbol{v}$. The following identities hold:

(i) $(\boldsymbol{v} \cdot \nabla\boldsymbol{v}, \boldsymbol{w}) = -(\boldsymbol{v} \cdot \nabla\boldsymbol{w}, \boldsymbol{v})$;
(ii) $(\boldsymbol{B} \cdot \nabla\boldsymbol{v}, \boldsymbol{v}) = 0, \quad \boldsymbol{B} = \boldsymbol{v}, \boldsymbol{w}$.

To show (i), let $\{\boldsymbol{w}_k\} \subset \mathcal{D}(\Omega)$ be a sequence converging to $\boldsymbol{w}$ in $\mathcal{D}_0^{1,2}(\Omega)$. For fixed $k \in \mathbb{N}$, we choose $\rho > \delta(\Omega^c)$ with $\Omega_\rho \supset \operatorname{supp}(\boldsymbol{w}_k)$ and denote by $\{\boldsymbol{v}_m\} \subset C_0^\infty(\overline{\Omega}_\rho)$ a sequence converging to $\boldsymbol{v}$ in $W^{1,2}(\Omega_\rho)$. Clearly,

$$(\boldsymbol{v}_m \cdot \nabla\boldsymbol{v}_m, \boldsymbol{w}_k) = -(\boldsymbol{v}_m \cdot \nabla\boldsymbol{w}_k, \boldsymbol{v}_m) + (\nabla \cdot \boldsymbol{v}_m, \boldsymbol{w}_k \cdot \boldsymbol{v}_m).$$

Letting $m \to \infty$ into this identity, with the aid of Lemma IX.1.1 and the fact that, as $m \to \infty$,

$$\|\nabla \cdot \boldsymbol{v}_m\|_{2,\Omega_\rho} \to 0,$$

we recover

$$(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{w}_k) = -(\boldsymbol{v}\cdot\nabla\boldsymbol{w}_k,\boldsymbol{v}) \tag{X.3.30}$$

for all $k\in\mathbb{N}$. Using (X.3.17) and Theorem II.6.1 it follows that

$$|(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{w}_k-\boldsymbol{w})| \le M|\boldsymbol{v}|_{1,2}\|(\boldsymbol{w}_k-\boldsymbol{w})/|x|\|_2 \le c|\boldsymbol{v}|_{1,2}|\boldsymbol{w}_k-\boldsymbol{w}|_{1,2} \tag{X.3.31}$$

while Lemma X.2.1 furnishes

$$|(\boldsymbol{v}\cdot\nabla(\boldsymbol{w}_k-\boldsymbol{w}),\boldsymbol{v})| \le \|\boldsymbol{v}\|_4^2|\boldsymbol{w}_k-\boldsymbol{w}|_{1,2}. \tag{X.3.32}$$

Recalling (X.3.26), we pass to the limit $k\to\infty$ in (X.3.30) and use (X.3.31), (X.3.32) to show the validity of (i). The proof of (ii) is analogous, and therefore it will be omitted. In view of (i) and (ii) from (X.3.29) we have

$$\begin{aligned}\mathcal{R}^{-1}|\boldsymbol{u}|_{1,2}^2 &\le -(\boldsymbol{v}\cdot\nabla\boldsymbol{w},\boldsymbol{v})+(\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{v})+(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{v})-(\boldsymbol{w}\cdot\nabla\boldsymbol{v},\boldsymbol{v})\\ &=(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v}).\end{aligned} \tag{X.3.33}$$

However, by assumption and (II.6.10),

$$|(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v})| \le M\|\boldsymbol{u}/|x|\|_2|\boldsymbol{u}|_{1,2} \le 2M|\boldsymbol{u}|_{1,2}^2$$

which, once replaced into (X.3.33), furnishes

$$|\boldsymbol{u}|_{1,2}(\mathcal{R}^{-1}-2M)\le 0.$$

By (X.3.18) this inequality implies $\boldsymbol{u}(x)=0$ for a.a. $x\in\Omega$, and the theorem is proved. □

Remark X.3.3 Theorem X.3.2 continues to hold in any dimension $n\ge 4$, provided we replace the bound on M given in (X.3.18) with a suitable one depending on n. This generalization has been considered by Miyakawa (1995). ■

X.4 Existence of Generalized Solutions

As in the case of a bounded domain, the fundamental contribution to the existence of steady-state solutions to the Navier–Stokes problem in exterior domains is due to J.Leray (1933, Chapitre III). Again, the leading idea is to employ the following a priori estimate for solutions to (X.0.3), (X.0.4) with $\boldsymbol{v}_*=0$:

$$\int_\Omega \nabla\boldsymbol{v}:\nabla\boldsymbol{v}\le M \tag{X.4.1}$$

where M depends only on $\boldsymbol{f}$, Ω and ν. Actually, for any $R>\delta(\Omega^c)$, Leray looked for a solution to (X.0.3), (X.0.4) in Ω_R with $\boldsymbol{v}_*=0$ at $\partial\Omega$ and $\boldsymbol{v}=\boldsymbol{v}_\infty$

at ∂B_R and he was able to prove the existence of a solution $\boldsymbol{v}_R, p_R$ such that the Dirichlet integral admits a uniform bound

$$\int_{\Omega_R} \nabla \boldsymbol{v}_R : \nabla \boldsymbol{v}_R \leq M \tag{X.4.2}$$

with M independent of R. Upon taking a suitable sequence, as $R \to \infty$, a solution $\boldsymbol{v}$, p to (X.0.3), (X.0.4) was found. Because of (X.4.2), this solution satisfies (X.4.1). In the case of three–dimensional flow, the estimate (X.4.1) is enough to ensure all requirements of a generalized solution, including the behavior at infinity. This latter property is in fact a consequence of Lemma II.6.2. However, for plane flow the bound (X.4.1) is not enough to control the behavior at infinity of the solution. It is this circumstance that renders the problem of existence of solutions in a two-dimensional exterior problem a very difficult one and, in several respects, it has to be considered still an open question; cf. Chapter XII.

The method we shall employ to show existence is, in fact, different from Leray's and it is the same used for the case of a bounded domain, namely, the Galerkin method. Such an approach is due to Fujita (1961). The proofs given in Theorem IX.3.1 and Theorem IX.4.1 remain essentially unchanged to cover the present case (cf. Remark IX.3.2). The only point that deserves some attention is the construction of a suitable extension $\boldsymbol{V}$ of the velocity field at the boundary and at infinity. Actually, as in the case where Ω is bounded, we have to require that $\boldsymbol{V}$ satisfies an inequality of the type (IX.4.3) for all $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ and some $\alpha < \nu$. The first part of this section is devoted to the preceding question. In this regard, we observe that, for such a field $\boldsymbol{V}$ to exist, the vanishing of the overall flux Φ of the velocity field through the bounding walls is *not* needed; rather, it is enough that Φ be "sufficiently small" in a way that will be made precise later. However, it is not known whether an upper bound for Φ is in fact necessary and consequently, unlike the corresponding linearized Stokes and Oseen approximations, *the problem of existence of steady Navier–Stokes flow in exterior domains with arbitrary flux at the boundary remains open.*

With a view to the rotational case, that will be treated in the following chapter, we shall construct the extension field $\boldsymbol{V}$ under more general assumptions on $\boldsymbol{v}_\infty$ than needed here.

To begin with, we need to introduce certain quantities. We recall that $\Omega = \mathbb{R}^3 - \cup_{i=1}^s \Omega_i$, $s \geq 1$, where each Ω_i is compact and, we assume, with a non-empty interior. Furthermore, $\Omega_i \cap \Omega_j = \emptyset$, for $i \neq j$. We thus set

$$\boldsymbol{\sigma}_i(x) = \frac{1}{4\pi} \nabla \left(\frac{1}{|x - x_i|} \right), \quad x_i \in \mathring{\Omega}_i, \quad i = 1, \ldots, s, \tag{X.4.3}$$

and observe that

$$\int_{\partial\Omega_i} \boldsymbol{\sigma}_j \cdot \boldsymbol{n} = \delta_{ij}, \quad i, j = 1, \ldots, s\,, \tag{X.4.4}$$

where $\boldsymbol{n}$ denotes the outer normal to $\partial\Omega$ at $\partial\Omega_i$.

Lemma X.4.1 *Let Ω be a locally Lipschitz domain of $\mathbb{R}^3$ exterior to $s \geq 1$ compact and disjoint sets $\Omega_1, \dots, \Omega_s$, and let*

$$\boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega), \quad \mathfrak{v}_\infty := \boldsymbol{a} + \boldsymbol{A} \cdot \boldsymbol{x}\,,$$

where $\boldsymbol{a} \in \mathbb{R}^3$ and $\boldsymbol{A}$ is a second order tensor with $\operatorname{trace}(\boldsymbol{A}) = 0$. *Then, for any $\eta > 0$, there exists $\varepsilon = \varepsilon(\eta, \boldsymbol{v}_*, \Omega) > 0$ and $\boldsymbol{V} = \boldsymbol{V}(\varepsilon) : \Omega \to \mathbb{R}^3$ such that, for some $R > \delta(\Omega^c)$ and all $q \in (1,\infty)$, $r \in (3/2,\infty)$:*

(i) $\boldsymbol{V} + \mathfrak{v}_\infty \in W^{1,2}(\Omega_R) \cap D^{1,q}(\Omega^R)$;
(ii) $\boldsymbol{V} + \mathfrak{v}_\infty \in L^r(\Omega^R)$;
(iii) $\boldsymbol{V} = \boldsymbol{v}_*$ *at* $\partial\Omega$;
(iv) $\nabla \cdot \boldsymbol{V} = 0$ *in* Ω.

Furthermore, for all $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$, it holds that

$$|(\boldsymbol{u} \cdot \nabla(\boldsymbol{V} + \mathfrak{v}_\infty), \boldsymbol{u})| \leq \left\{\eta + \frac{1}{4\pi}\sum_{i=1}^{s}|\Phi_i| \max_{x\in\Omega} \frac{1}{|x - x_i|}\right\}|\boldsymbol{u}|_{1,2}^2. \tag{X.4.5}$$

where

$$\Phi_i = \int_{\partial\Omega_i} \boldsymbol{v}_* \cdot \boldsymbol{n}, \quad i = 1, \dots, s,$$

is the flux of $\boldsymbol{v}_*$ through the boundary of Ω_i. Finally, if $\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \leq M$, for some $M > 0$, then

$$\begin{aligned} &\|\boldsymbol{V} + \mathfrak{v}_\infty\|_{1,2} \leq C_1\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \\ &\|\boldsymbol{V} + \mathfrak{v}_\infty\|_{r,\Omega^R} + |\boldsymbol{V} + \mathfrak{v}_\infty|_{1,q,\Omega^R} \leq C_2\,\|\boldsymbol{v}_*\|_{1/2,2,(\partial\Omega)}\,, \end{aligned} \tag{X.4.6}$$

where $C_1 = C_1(\eta, M, \Omega)$, and $C_2 = C_2(r, q, R, \Omega)$.

Proof. The procedure is similar to, but simpler than, that used in the proof of Lemma IX.4.2. We set

$$\boldsymbol{v}_1(x) = \boldsymbol{v}_*(x) - \sum_{i=1}^{s} \Phi_i \boldsymbol{\sigma}_i(x) + \mathfrak{v}_\infty, \quad x \in \partial\Omega\,.$$

Clearly, because of (X.4.4),

$$\int_{\partial\Omega_i} \boldsymbol{v}_1 \cdot \boldsymbol{n} = 0, \quad i = 1, \dots, s,$$

and so, according to Lemma IX.4.1, there is $\boldsymbol{w} \in W^{2,2}(\Omega)$ such that, setting

$$\boldsymbol{U} := \nabla \times \boldsymbol{w} \in W^{1,2}(\Omega)$$

we have

$$\boldsymbol{U} = \boldsymbol{v}_1 \quad \text{at } \partial\Omega$$

$$\boldsymbol{U}(x) = 0, \quad \text{for all } x \in \Omega^R, \ \ R > \delta(\Omega^c).$$

Given $\varepsilon > 0$, we set

$$\boldsymbol{V}_\varepsilon = \nabla \times (\psi_\varepsilon \boldsymbol{w})$$

with ψ_ε the "cut-off" function defined in Lemma III.6.2. The desired extension of $\boldsymbol{v}_*$ is then given by

$$\boldsymbol{V} = \boldsymbol{V}_\varepsilon + \sum_{i=1}^{s} \Phi_i \sigma_i - \mathfrak{v}_\infty \equiv \boldsymbol{V}_\varepsilon + \boldsymbol{V}_\sigma - \mathfrak{v}_\infty \,. \tag{X.4.7}$$

Taking into account the properties of the function $\boldsymbol{w}$ given in Lemma IX.4.1, with the help of (X.4.3) it is immediate to show that $\boldsymbol{V}$ satisfies (i)-(iv). Let us now estimate the trilinear form

$$a(\boldsymbol{u}, \boldsymbol{V} + \mathfrak{v}_\infty, \boldsymbol{u}) \equiv (\boldsymbol{u} \cdot \nabla(\boldsymbol{V} + \mathfrak{v}_\infty), \boldsymbol{u}) = a(\boldsymbol{u}, \boldsymbol{V}_\varepsilon + \boldsymbol{V}_\sigma, \boldsymbol{u}), \quad \boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega).$$

Fix $\eta > 0$. By reasoning entirely analogous to that used in the proof of Lemma IX.4.2, and which we shall leave to the reader, we have

$$|a(\boldsymbol{u}, \boldsymbol{V}_\varepsilon, \boldsymbol{u})| \le \eta \, |\boldsymbol{u}|_{1,2}^2 \,, \tag{X.4.8}$$

provided $\varepsilon \le \eta / \left(c_4 \, \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\right)$, for suitable $c_4 = c_4(\Omega)$. Furthermore, integrating by parts and recalling (X.4.3), we find

$$|a(\boldsymbol{u}, \boldsymbol{V}_\sigma, \boldsymbol{u})| \le \sum_{i=1}^{s} |\Phi_i| \int_\Omega |\nabla \boldsymbol{u}|^2 |\mathcal{E}(x - x_i)| \le \frac{1}{4\pi} \sum_{i=1}^{s} |\Phi_i| \max_{x \in \Omega} \frac{1}{|x - x_i|} |\boldsymbol{u}|_{1,2}^2 \tag{X.4.9}$$

Inequality (X.4.5) then follows from (X.4.8) and (X.4.9). It remains to show (X.4.6), under the further stated assumptions on $\boldsymbol{v}_*$. By the same argument used in the proof of Lemma IX.4.1 (specifically, the argument leading to (IX.4.48)), we easily infer

$$\|\boldsymbol{V}_\varepsilon\|_{1,2} \le c_1 \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \tag{X.4.10}$$

with $c_1 = c_1(\eta, M, \Omega)$. Moreover, from (X.4.3) we deduce

$$|\boldsymbol{V}_\sigma|_{1,q} + \|\boldsymbol{V}_\sigma\|_r \le c_1 \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \,, \quad \text{for all } q \in (1, \infty), \, r \in (3/2, \infty) \,, \tag{X.4.11}$$

where $c_2 = c_2(\Omega)$. Then, noticing that $\boldsymbol{V}(x) + \mathfrak{v}_\infty = \boldsymbol{V}_\sigma$ for all $|x| \ge R$, Ω_R containing the support of ψ_ε,[1] we conclude that (X.4.6) follows from (X.4.10) and (X.4.11). □

[1] Notice that $\cup_{\varepsilon > 0} \text{supp}\,(\psi_\varepsilon)$ is bounded.

Remark X.4.1 If $s = 1$, choosing $x_1 = 0$, we have

$$\max_{x\in\Omega} \frac{1}{|x|} \le \frac{1}{r_0}, \quad r_0 = \text{dist}\,(0, \partial\Omega),$$

condition (X.4.5) becomes

$$|(\boldsymbol{u}\cdot\nabla(\boldsymbol{V}+\mathfrak{v}_\infty), \boldsymbol{u})| \le \left\{\eta + \frac{|\Phi_1|}{4\pi r_0}\right\} |\boldsymbol{u}|^2_{1,2}.$$

■

Remark X.4.2 We would like to discuss the generalization of Lemma X.4.1 to arbitrary dimension $n \ge 2$, $n \ne 3$, in the case $\mathfrak{v}_\infty \equiv \boldsymbol{v}_\infty \in \mathbb{R}^n$. If $n \ge 4$, the generalization is straightforward, provided we require a little more regularity on $\boldsymbol{v}_*$ of the type

$$\boldsymbol{v}_* \in W^{1-1/s,s}(\partial\Omega), \quad s > n/2.$$

The extra regularity is needed to construct the extension field $\boldsymbol{V}_\varepsilon$, cf. Remark IX.4.7. If we assume this, we can prove the existence of a field $\boldsymbol{V}$ satisfying (i)-(iv) for all $q \in (1,\infty)$ and all $r \in (n/(n-1), \infty)$. Furthermore, the following inequality holds

$$|(\boldsymbol{u}\cdot\nabla\boldsymbol{V}, \boldsymbol{u})| \le (\eta + \Phi_{(n)})|\boldsymbol{u}|^2_{1,2}$$

where

$$\Phi_{(n)} \equiv c\sum_{i=1}^{s} |\Phi_i| \max_{x\in\Omega} \frac{1}{|x-x_i|^{n-2}}$$

and $c = c(n)$. In the two-dimensional case, the starting point is, again, (X.4.7), with

$$\boldsymbol{\sigma}_i(x) = \frac{1}{2\pi}\nabla\left(\log|x-x_i|\right). \tag{X.4.12}$$

By following exactly the same arguments used for the proof of Lemma X.4.1, we thus show that $\boldsymbol{V}$ satisfies (i)-(iv) with $q \in (1,\infty)$ and $r \in (2,\infty)$, as well as the inequalities in (X.4.6), under the stated further assumption on $\boldsymbol{v}_*$. Moreover, we have

$$|(\boldsymbol{u}\cdot\nabla\boldsymbol{V}, \boldsymbol{u})| \le (\eta + \frac{1}{2\pi}\sum_{i=1}^{s}|\Phi_i|)|\boldsymbol{u}|^2_{1,2}.$$

Given the properties of $\boldsymbol{V}_\varepsilon$, the proof of this latter is, as in Lemma X.4.1, reduced to show the estimate

$$|(\boldsymbol{u}\cdot\nabla\boldsymbol{V}_\sigma, \boldsymbol{u})| \le \frac{1}{2\pi}\sum_{i=1}^{s}|\Phi_i||\boldsymbol{u}|^2_{1,2}. \tag{X.4.13}$$

However, (X.4.13) can not be obtained by a procedure similar to that used to obtain (X.4.9), and we have to argue differently. The following argument is due to A. Russo (2010b, Lemma 3). By integration by parts, the proof of (X.4.13) reduces to show ($i = 1, \dots, s$)

$$\mathscr{I}_i := 2\pi \left| \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{\sigma}_i \right| \leq |\boldsymbol{u}|_{1,2}^2 \,. \tag{X.4.14}$$

We introduce a system of polar coordinates (r, θ) with the origin at x_i, and set

$$\overline{f} := \frac{1}{2\pi} \int_0^{2\pi} f(r,\theta) d\theta \,.$$

Extending $\boldsymbol{u}$ to $\boldsymbol{0}$ in Ω^c, we find

$$\mathscr{I}_i = 2\pi \left| \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \frac{\boldsymbol{x}}{|x|^2} \right| = \left| \int_0^\infty \frac{dr}{r} \int_0^{2\pi} \left[(u_1 - \overline{u_1}) \frac{\partial u_2}{\partial \theta} - (u_2 - \overline{u_2}) \frac{\partial u_1}{\partial \theta} \right] d\theta \right| , \tag{X.4.15}$$

where we used

$$\int_0^{2\pi} \overline{u_1} \frac{\partial u_2}{\partial \theta} d\theta = \int_0^{2\pi} \overline{u_2} \frac{\partial u_1}{\partial \theta} d\theta = 0 \,.$$

Employing the Schwarz inequality, along with the Wirtinger inequality (II.5.17) and Exercise II.5.12, we deduce

$$\begin{aligned} \left| \int_0^{2\pi} (u_1 - \overline{u_1}) \frac{\partial u_2}{\partial \theta} \right| &\leq \left(\int_0^{2\pi} |u_1 - \overline{u_1}|^2 \int_0^{2\pi} \left| \frac{\partial u_2}{\partial \theta} \right|^2 \right)^{1/2} \\ &\leq \left(\int_0^{2\pi} \left| \frac{\partial u_1}{\partial \theta} \right|^2 \int_0^{2\pi} \left| \frac{\partial u_2}{\partial \theta} \right|^2 \right)^{1/2} , \end{aligned} \tag{X.4.16}$$

and, likewise,

$$\left| \int_0^{2\pi} (u_2 - \overline{u_2}) \frac{\partial u_1}{\partial \theta} \right| \leq \left(\int_0^{2\pi} \left| \frac{\partial u_2}{\partial \theta} \right|^2 \int_0^{2\pi} \left| \frac{\partial u_1}{\partial \theta} \right|^2 \right)^{1/2} \tag{X.4.17}$$

Therefore, observing that $|\dfrac{\partial \boldsymbol{u}}{\partial \theta}| \leq r |\nabla \boldsymbol{u}|$, we find that (X.4.14) follows from (X.4.15)–(X.4.17). ■

Theorem X.4.1 *Let Ω be a locally Lipschitz domain of $\mathbb{R}^3$, exterior to compact and disjoint domains $\Omega_1, \dots, \Omega_s$, $s \geq 1$.[2] Moreover, let*

$$\boldsymbol{f} \in D_0^{-1,2}(\Omega), \quad \boldsymbol{v}_* \in W^{1/2}(\partial\Omega), \quad \boldsymbol{v}_\infty \in \mathbb{R}^3 \,,$$

and set

$$\Phi := \frac{1}{4\pi} \sum_{i=1}^s \max_{x \in \Omega} \frac{1}{|x - x_i|} \left| \int_{\partial\Omega_i} \boldsymbol{v}_* \cdot \boldsymbol{n} \right| \,.$$

The following properties hold.

[2] The theorem continues to hold in the case where $\Omega = \mathbb{R}^3$.

(i) Existence. *If $\Phi < 1/\mathcal{R}$, there exists at least one generalized solution $\boldsymbol{v}$ to the Navier–Stokes problem* (X.0.8), (X.0.4). *Such a solution verifies the conditions:*

$$\begin{gathered}\int_{S^2} |\boldsymbol{v}(x)+\boldsymbol{v}_\infty| = O(1/\sqrt{|x|}) \ \text{ as } |x|\to\infty \\ \|p\|_{2,\Omega_R/\mathbb{R}} \le c\,(|\boldsymbol{v}|_{1,2} + \mathcal{R}\|\boldsymbol{v}\|^2_{1,2\Omega_R} + \mathcal{R}|\boldsymbol{f}|_{1,2})\end{gathered} \tag{X.4.18}$$

for all $R > \delta(\Omega^c)$. In (X.4.18) *p is the pressure field associated to $\boldsymbol{v}$ by* Lemma X.1.1, *while $c = c(\Omega, R)$ with $c \to \infty$ as $R \to \infty$. Furthermore, $\boldsymbol{v}$ obeys the* generalized energy inequality

$$|\boldsymbol{v}|^2_{1,2} + \mathcal{R}\,[\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{V}] \le (\nabla\boldsymbol{v}, \nabla\boldsymbol{V}) - \mathcal{R}(\boldsymbol{v}\cdot\nabla\boldsymbol{V}, \boldsymbol{v}-\boldsymbol{V}) \tag{X.4.19}$$

where $\boldsymbol{V}$ is the extension of $\boldsymbol{v}_*$ and $\boldsymbol{v}_\infty$ constructed in Lemma X.4.1.

(ii) Estimate by the data. *If $\boldsymbol{v}_* \in \mathfrak{M}^{1/2,2}_M(\partial\Omega)$ (defined in* (IX.4.52)*) and $\Phi \le 1/(2\mathcal{R})$, then the generalized solution determined in* (i) *satisfies the following estimate:*

$$|\boldsymbol{v}|_{1,2} \le 4\mathcal{R}|\boldsymbol{f}|_{-1,2} + C\,\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\left[1 + \mathcal{R}\left(1 + \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} + |\boldsymbol{v}_\infty|\right)\right], \tag{X.4.20}$$

where $C = C(\Omega, \mathcal{R}, M)$.

Proof. We shall employ the Galerkin method. We look for a solution of the form

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{V},$$

where $\boldsymbol{V}$ is the extension of $\boldsymbol{v}_*$ and $\boldsymbol{v}_\infty$ constructed in Lemma X.4.1 and corresponding to some $\eta > 0$ that will be fixed successively. Let $\{\boldsymbol{\psi}_k\} \subset \mathcal{D}(\Omega)$ be the basis of $\mathcal{D}^{1,2}_0(\Omega)$, introduced in Lemma VII.2.1. A sequence of approximating solutions $\{\boldsymbol{u}_m\}$ is then sought of the form

$$\begin{gathered}\boldsymbol{u}_m = \sum_{k=1}^{m} \xi_{km}\boldsymbol{\psi}_k \\ \frac{1}{\mathcal{R}}(\nabla\boldsymbol{u}_m, \nabla\boldsymbol{\psi}_k) + (\boldsymbol{u}_m\cdot\nabla\boldsymbol{u}_m, \boldsymbol{\psi}_k) + (\boldsymbol{u}_m\cdot\nabla\boldsymbol{V}, \boldsymbol{\psi}_k) + (\boldsymbol{V}\cdot\nabla\boldsymbol{u}_m, \boldsymbol{\psi}_k) \\ = -\,[\boldsymbol{f}, \boldsymbol{\psi}_k] - \frac{1}{\mathcal{R}}(\nabla\boldsymbol{V}, \nabla\boldsymbol{\psi}_k) - (\boldsymbol{V}\cdot\nabla\boldsymbol{V}, \boldsymbol{\psi}_k), \quad k = 1, 2, \ldots, m.\end{gathered} \tag{X.4.21}$$

For each $m \in \mathbb{N}$ we may establish existence to (X.4.21) in the same way as in Theorem IX.3.1, provided we show a suitable *uniform* bound for $|\boldsymbol{u}_m|_{1,2}$ in terms of the data. Thus, multiplying $(\text{X.4.21})_2$ by ξ_{km}, summing over k from 1 to m and recalling that, for all $m \in \mathbb{N}$,

$$(\boldsymbol{V}\cdot\nabla\boldsymbol{u}_m, \boldsymbol{u}_m) = (\boldsymbol{u}_m\cdot\nabla\boldsymbol{u}_m, \boldsymbol{u}_m) = 0, \tag{X.4.22}$$

(cf. Lemma IX.2.1), we obtain

$$\frac{1}{\mathcal{R}}|\boldsymbol{u}_m|_{1,2}^2 + (\boldsymbol{u}_m \cdot \nabla \boldsymbol{V}, \boldsymbol{u}_m) = -[\boldsymbol{f}, \boldsymbol{u}_m] - \frac{1}{\mathcal{R}}(\nabla \boldsymbol{V}, \nabla \boldsymbol{u}_m) - (\boldsymbol{V} \cdot \nabla \boldsymbol{V}, \boldsymbol{u}_m). \tag{X.4.23}$$

Recalling the properties of $\boldsymbol{V}$ given in Lemma X.4.1, we have, by the Schwarz inequality,

$$|(\nabla \boldsymbol{V}, \nabla \boldsymbol{u}_m)| \le |\boldsymbol{u}_m|_{1,2} |\boldsymbol{V}|_{1,2}, \tag{X.4.24}$$

and, by the Hölder and Sobolev inequalities

$$\begin{aligned} |(\boldsymbol{V} \cdot \nabla \boldsymbol{V}, \boldsymbol{u}_m)| &\le \left| \int_{\Omega_R} \boldsymbol{V} \cdot \nabla \boldsymbol{V} \cdot \boldsymbol{u}_m \right| + \left| \int_{\Omega^R} (\boldsymbol{V} + \boldsymbol{v}_\infty) \cdot \nabla \boldsymbol{V} \cdot \boldsymbol{u}_m \right| \\ &\qquad + \left| \int_{\Omega^R} \boldsymbol{v}_\infty \cdot \nabla \boldsymbol{V} \cdot \boldsymbol{u}_m \right| \\ &\le [c_1 \|\boldsymbol{V}\|_{1,2,\Omega_R} + \frac{2}{\sqrt{3}} (\|\boldsymbol{V} + \boldsymbol{v}_\infty\|_{3,\Omega^R} |\boldsymbol{V}|_{1,2} \\ &\qquad + |\boldsymbol{v}_\infty| |\boldsymbol{V}|_{1,6/5})] |\boldsymbol{u}_m|_{1,2} \end{aligned} \tag{X.4.25}$$

Furthermore, again from Lemma X.4.1,

$$|(\boldsymbol{u}_m \cdot \nabla \boldsymbol{V}, \boldsymbol{u}_m)| \le (\eta + \Phi) |\boldsymbol{u}_m|_{1,2}^2. \tag{X.4.26}$$

Collecting (X.4.23)– (X.4.26), we find

$$\begin{aligned} (1/\mathcal{R} - \eta - \Phi) \, |\boldsymbol{u}_m|_{1,2} &\le |\boldsymbol{f}|_{-1,2} + \frac{1}{\mathcal{R}} |\boldsymbol{V}|_{1,2} + c_1 \|\boldsymbol{V}\|_{1,2,\Omega_R} \\ &\qquad + \frac{2}{\sqrt{3}} \left(\|\boldsymbol{V} + \boldsymbol{v}_\infty\|_{3,\Omega^R} |\boldsymbol{V}|_{1,2} + |\boldsymbol{v}_\infty| \, |\boldsymbol{V}|_{1,6/5} \right) . \end{aligned} \tag{X.4.27}$$

Thus, if $\Phi < 1/\mathcal{R}$, we may choose $\eta \in (0, 1/\mathcal{R} - \Phi)$ to obtain the following uniform estimate

$$|\boldsymbol{u}_m|_{1,2} \le C, \tag{X.4.28}$$

with $C = C(\boldsymbol{f}, \boldsymbol{v}_*, \boldsymbol{v}_\infty, \mathcal{R})$. By (X.4.28) and Lemma IX.3.2 we may show existence of solutions to (X.4.21) for all m. Moreover, we can select a subsequence, denoted again by $\{\boldsymbol{u}_m\}$, such that as $m \to \infty$

$$\boldsymbol{u}_m \overset{w}{\to} \boldsymbol{u} \ \text{ in } \mathcal{D}_0^{1,2}(\Omega). \tag{X.4.29}$$

Also, by Exercise II.5.8 and the Cantor diagonalization argument, we may select another subsequence, that we still call $\{\boldsymbol{u}_m\}$, such that, as $m \to \infty$,

$$\boldsymbol{u}_m \to \boldsymbol{u} \ \text{ in } L^q(\Omega_R), \quad q \in [1, 6)\,, \tag{X.4.30}$$

for any $R > \delta(\Omega^c)$. By (X.4.28) and Theorem II.1.3 it follows that the limiting field $\boldsymbol{u}$ also obeys inequality (X.4.28). Let us now pass to the limit $m \to \infty$ into $(\text{X.4.21})_2$. Observe that, setting

$$\Omega_k \equiv \operatorname{supp}(\boldsymbol{\psi}_k),$$

we have

$$(D_j V_i)\boldsymbol{\psi}_k, \quad V_i\boldsymbol{\psi}_k \in L^2(\Omega_k),$$

and so, from (X.4.29) and (X.4.30) we obtain

$$\begin{aligned}
(\nabla \boldsymbol{u}_m, \nabla \boldsymbol{\psi}_k) &\to (\nabla \boldsymbol{u}, \nabla \boldsymbol{\psi}_k) \\
(\boldsymbol{u}_m \cdot \nabla \boldsymbol{V}, \boldsymbol{\psi}_k) &\to (\boldsymbol{u} \cdot \nabla \boldsymbol{V}, \boldsymbol{\psi}_k) \\
(\boldsymbol{V} \cdot \nabla \boldsymbol{u}_m, \boldsymbol{\psi}_k) &\to (\boldsymbol{V} \cdot \nabla \boldsymbol{u}, \boldsymbol{\psi}_k).
\end{aligned} \tag{X.4.31}$$

Furthermore, reasoning as in the proof of Theorem IX.3.1[3] we have

$$(\boldsymbol{u}_m \cdot \nabla \boldsymbol{u}_m, \boldsymbol{\psi}_k) \to (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{\psi}_k). \tag{X.4.32}$$

Employing (X.4.31) and (X.4.32) into $(\text{X.4.21})_2$ yields

$$\begin{aligned}
\frac{1}{\mathcal{R}}(\nabla \boldsymbol{u}, \nabla \boldsymbol{\psi}_k) &+ (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{\psi}_k) + (\boldsymbol{u} \cdot \nabla \boldsymbol{V}, \boldsymbol{\psi}_k) + (\boldsymbol{V} \cdot \nabla \boldsymbol{u}, \psi_k) \\
&= -[\boldsymbol{f}, \boldsymbol{\psi}_k] - \frac{1}{\mathcal{R}}(\nabla \boldsymbol{V}, \nabla \boldsymbol{\psi}_k) - (\boldsymbol{V} \cdot \nabla \boldsymbol{V}, \boldsymbol{\psi}_k).
\end{aligned} \tag{X.4.33}$$

It is now immediately verified that the field

$$\boldsymbol{v} \equiv \boldsymbol{u} + \boldsymbol{V}$$

is a generalized solution to the problem, in the sense of Definition X.1.1. Actually, issues (i)-(iii) of that definition are at once established. Also, from Lemma X.4.1 and Lemma II.6.2 it follows, as $|x| \to \infty$, that

$$\int_{S^2} |\boldsymbol{v}(x) + \boldsymbol{v}_\infty| \le \int_{S^2} |\boldsymbol{u}(x)| + \int_{S^2} |\boldsymbol{V}(x) + \boldsymbol{v}_\infty| = O(1/\sqrt{|x|}) \tag{X.4.34}$$

and so $(\text{X.4.18})_1$ is proved. Finally, from (X.4.33) we obtain

$$\frac{1}{\mathcal{R}}(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}_k) + (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\psi}_k) = -[\boldsymbol{f}, \boldsymbol{\psi}_k]. \tag{X.4.35}$$

We have

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} = \boldsymbol{u} \cdot \nabla \boldsymbol{v} + (\boldsymbol{V} + \boldsymbol{v}_\infty) \cdot \nabla \boldsymbol{v} - \boldsymbol{v}_\infty \cdot \nabla \boldsymbol{v}.$$

Thus, recalling that $\boldsymbol{u}, (\boldsymbol{V} + \boldsymbol{v}_\infty) \in L^6(\Omega)$, by the Hölder inequality it follows that

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} \in L^{3/2}(\Omega) + L^2(\Omega).$$

[3] Specifically, one has to follow the part of the proof that goes from (IX.3.15) to (IX.3.20). In these formulas, for the case at hand, the integration has to be performed over the *bounded* domain Ω_k.

Since any function $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ can be approximated in the $W^{1,s}$-norm by linear combinations of $\boldsymbol{\psi}_k$ for all $s \geq 2$, the issue (v) follows from this latter property and (X.4.35). Let us next show the estimate for the pressure field p, whose existence is guaranteed by Lemma X.1.1. In particular, by this lemma,

$$p \in L^2_{loc}(\overline{\Omega}). \tag{X.4.36}$$

Fix $R > \delta(\Omega^c)$ and add to p the constant

$$C = C(R) = -\frac{1}{|\Omega_R|}\int_{\Omega_R} p,$$

so that

$$\int_{\Omega_R} (p + C) = 0. \tag{X.4.37}$$

Successively, choose $\boldsymbol{\psi}$ such that

$$\begin{aligned}
&\nabla \cdot \boldsymbol{\psi} = p + C \quad \text{in } \Omega_R \\
&\boldsymbol{\psi} \in W_0^{1,2}(\Omega_R) \\
&\|\boldsymbol{\psi}\|_{1,2} \leq c_5 \|p + C\|_{2,\Omega_R}.
\end{aligned} \tag{X.4.38}$$

Because of (X.4.36) and (X.4.37), Theorem III.3.1 guarantees the existence of $\boldsymbol{\psi}$ and it is clear that we can replace this function into identity (X.1.3). Thus, from (X.1.3), and the Schwarz and Hölder inequalities, we find

$$\|p + C\|^2_{2,\Omega_R} \leq c_6 [(|\boldsymbol{v}|_{1,2} + \mathcal{R}|\boldsymbol{f}|_{-1,2})|\boldsymbol{\psi}|_{1,2} + \mathcal{R}\|\boldsymbol{v}\|_{4,\Omega_R}|\boldsymbol{v}|_{1,2,\Omega_R}\|\boldsymbol{\psi}\|_{4,\Omega_R}]. \tag{X.4.39}$$

Recalling that $W^{1,2}(\Omega_R) \hookrightarrow L^4(\Omega_R)$, and using $(\text{X.4.38})_3$, from the above relation we prove $(\text{X.4.18})_2$. To show the theorem completely, we have to prove the generalized energy inequality (X.4.19). To this end, we observe that, by (i) and (ii) of Lemma X.4.1, we have

$$\boldsymbol{V} \cdot \nabla \boldsymbol{V} \in L^{6/5}(\Omega), \tag{X.4.40}$$

and since, by the Sobolev inequality (II.3.7),

$$\|\boldsymbol{u}_m\|_6 \leq \gamma\, |\boldsymbol{u}_m|_{1,2} \leq M \tag{X.4.41}$$

we deduce (along a subsequence, at least)

$$\lim_{m\to\infty} (\boldsymbol{V} \cdot \nabla \boldsymbol{V}, \boldsymbol{u}_m) = (\boldsymbol{V} \cdot \nabla \boldsymbol{V}, \boldsymbol{u}). \tag{X.4.42}$$

Moreover, for fixed $R > \delta(\Omega^c)$, by the Hölder inequality,

$$\begin{aligned}
|(\boldsymbol{u}_m \cdot \nabla \boldsymbol{V}, \boldsymbol{u}_m) - &(\boldsymbol{u} \cdot \nabla \boldsymbol{V}, \boldsymbol{u})| \\
&\leq |(\boldsymbol{u}_m - \boldsymbol{u}) \cdot \nabla \boldsymbol{V}, \boldsymbol{u}_m)| + |(\boldsymbol{u} \cdot \nabla \boldsymbol{V}, \boldsymbol{u}_m - \boldsymbol{u})| \\
&\leq (\|\boldsymbol{u}_m\|_{4,\Omega_R} + \|\boldsymbol{u}\|_{4,\Omega_R})\|\boldsymbol{u}_m - \boldsymbol{u}\|_{4,\Omega_R}|\boldsymbol{V}|_{1,2} \\
&\qquad + (\|\boldsymbol{u}_m\|_6 + \|\boldsymbol{u}\|_6)|\boldsymbol{V}|_{1,3/2,\Omega^R}.
\end{aligned} \tag{X.4.43}$$

Again by (i) of Lemma X.4.1, for a given $\varepsilon > 0$ we may choose R so that

$$|\boldsymbol{V}|_{1,3/2,\Omega^R} < \varepsilon,$$

while, by (X.4.30) (possibly along a new subsequence),

$$\lim_{m\to\infty} \|\boldsymbol{u}_m - \boldsymbol{u}\|_{4,\Omega_R} = 0$$

and so from (X.4.43) and (X.4.41), it follows that

$$\limsup_{m\to\infty} |(\boldsymbol{u}_m \cdot \nabla \boldsymbol{V}, \boldsymbol{u}_m) - (\boldsymbol{u} \cdot \nabla \boldsymbol{V}, \boldsymbol{u})| \le 4M^2\varepsilon$$

which, in turn, by the arbitrariness of ε furnishes

$$\lim_{m\to\infty} (\boldsymbol{u}_m \cdot \nabla \boldsymbol{V}, \boldsymbol{u}_m) = (\boldsymbol{u} \cdot \nabla \boldsymbol{V}, \boldsymbol{u}). \tag{X.4.44}$$

Since, clearly,

$$\lim_{m\to\infty} (\nabla \boldsymbol{V}, \nabla \boldsymbol{u}_m) = (\nabla \boldsymbol{V}, \nabla \boldsymbol{u}) \tag{X.4.45}$$

from (X.4.22), (X.4.23), (X.4.42), (X.4.44), (X.4.45), and Theorem II.1.3 we conclude

$$\begin{aligned}
\frac{1}{\mathcal{R}}|\boldsymbol{u}|_{1,2}^2 &\le \frac{1}{\mathcal{R}} \lim_{m\to\infty} |\boldsymbol{u}_m|_{1,2}^2 \\
&= -(\boldsymbol{u} \cdot \nabla \boldsymbol{V}, \boldsymbol{u}) - \frac{1}{\mathcal{R}}(\nabla \boldsymbol{V}, \nabla \boldsymbol{u}) - (\boldsymbol{V} \cdot \nabla \boldsymbol{V}, \boldsymbol{u}) - [\boldsymbol{f}, \boldsymbol{u}]\,.
\end{aligned}$$

Recalling that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{V}$, we then recover (X.4.19) and the proof of part (i) is complete. In order to show part (ii), we notice that from (X.4.27) for $\Phi \le 1/(2\mathcal{R})$ and $\eta = 1/(4\mathcal{R})$, we deduce

$$\begin{aligned}
|\boldsymbol{u}_m|_{1,2} &\le 4\mathcal{R}|\boldsymbol{f}|_{-1,2} + 4|\boldsymbol{V}|_{1,2} + 4\mathcal{R}c_1\|\boldsymbol{V}\|_{1,2,\Omega_R} \\
&\quad + \frac{8\mathcal{R}}{\sqrt{3}} \left(\|\boldsymbol{V} + \boldsymbol{v}_\infty\|_{3,\Omega^R} |\boldsymbol{V}|_{1,2} + |\boldsymbol{v}_\infty|\, |\boldsymbol{V}|_{1,6/5} \right) .
\end{aligned} \tag{X.4.46}$$

Passing to the limit $m \to \infty$ in (X.4.46) and employing Theorem II.1.3 and the estimates (X.4.6) for $\boldsymbol{V}$, we infer the validity of (X.4.20), which concludes the proof of the theorem. □

Remark X.4.3 If the number s of compact regions Ω_i is one, existence is proved under the following condition on Φ:

$$\Phi \equiv \frac{|\Phi_1|}{4\pi r_0} < \frac{1}{\mathcal{R}};$$

cf. Remark X.4.1. ■

Remark X.4.4 We would like to make some comments about the extension of Theorem X.4.1 to the two-dimensional case. Taking into account Remark X.4.2, we verify at once that, for $\Omega \subset \mathbb{R}^2$, with $\Omega \neq \mathbb{R}^2$, under the assumption that the flux of $\boldsymbol{v}_*$ through $\partial\Omega$ satisfies the following restriction

$$\sum_{i=1}^{s} |\Phi_i| < \frac{2\pi}{\mathcal{R}}\,, \tag{X.4.47}$$

there is at least one field $\boldsymbol{v}$ satisfying conditions (i)–(iii) and (v) of Definition X.1.1.[4] Nevertheless, with the information derived so far, we are not able to draw any conclusion about the behavior at infinity of $\boldsymbol{v}$. This is because, unlike the three-dimensional case, we have no Sobolev-like (or weighted) inequality which ensures some type of decay for $\boldsymbol{v} + \boldsymbol{v}_\infty$ as $|x| \to \infty$. A fundamental problem is then to analyze what is the behavior at infinity of vector fields satisfying merely (i)-(iii) and (v) of Definition X.1.1, when Ω is a planar domain. This will be the object of Section XII.3. Let us now consider the case $\Omega = \mathbb{R}^2$. In such a circumstance the procedure adopted in Theorem X.4.1 does not produce any kind of existence. Actually, we can still construct an approximating sequence $\{\boldsymbol{u}_m\}$ and show that it satisfies estimate (X.4.28). Therefore, we can find a field $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\mathbb{R}^2)$ for which (X.4.29) holds. However, we can not establish (X.4.30) and, as a consequence, (X.4.32). In fact, in view of the example given in Exercise II.7.3, we know that condition (X.4.28) *alone* is not sufficient to ensure any kind of convergence of $\{\boldsymbol{u}_m\}$ to $\boldsymbol{u}$ in any space $L^q(B_R)$, $q \geq 1$, $R > 0$. Because of this, we are not able to show (X.4.32) and, as a consequence, we can not prove that the field $\boldsymbol{u}$ ($\equiv \boldsymbol{v}$) satisfies the identity (X.1.2) ■

Remark X.4.5 In dimension $n \geq 4$, existence of generalized solutions to (X.0.8), (X.0.4) can be proved along the same lines of Theorem X.4.1 provided

$$\boldsymbol{v}_* \in W^{1-1/s,s}(\partial\Omega), \quad \Phi_{(n)} < 1/\mathcal{R},$$

where $\Phi_{(n)}$ is defined in Remark X.4.2. In such a case, the asymptotic estimate $(\text{X.4.18})_2$ becomes

$$\int_{S^{n-1}} |\boldsymbol{v}(x) + \boldsymbol{v}_\infty| = O(1/|x|^{n/2-1}) \ \text{ as } |x| \to \infty.$$

■

Remark X.4.6 If $\boldsymbol{v}_* \equiv \boldsymbol{v}_\infty \equiv 0$, Theorem X.4.1, with the exception of $(\text{X.4.18})_3$, holds without requiring any regularity of the domain. This is because, the only point in the proof where regularity is needed is in the construction of the extension $\boldsymbol{V}$ which, in this case, can be taken as identically zero. ■

[4] The first proof of this result is due to Russo (2009).

X.5 On the Asymptotic Behavior of Generalized Solutions: Preliminary Results and Representation Formulas

In the present section we begin to study the asymptotic behavior of a generalized solution $\boldsymbol{v}$ to the Navier–Stokes equation in a three-dimensional exterior domain (we postpone the analogous two-dimensional case until Chapter XII). Specifically, we shall prove that if the body force has a certain degree of summability at infinity, then $\boldsymbol{v}$ behaves essentially as the corresponding solution of the linearized approximations (cf. Theorem V.3.1 and Theorem VII.6.1), that is, we have

$$\lim_{|x|\to\infty} |\boldsymbol{v}(x) + \boldsymbol{v}_\infty| = 0$$

$$\lim_{|x|\to\infty} |D^\alpha \boldsymbol{v}(x)| = 0, \quad 1 \le |\alpha| \le s,$$

uniformly, where s is related to the degree of summability of the derivatives of $\boldsymbol{f}$. An analogous property is shown for the pressure field p associated to $\boldsymbol{v}$ by Lemma X.1.1. Successively, we prove that, as in the linear problem, $\boldsymbol{v}$ and p admit an integral representation valid for almost all points in Ω. We have the following theorem.

Theorem X.5.1 *Let $\boldsymbol{v}$ be a generalized solution to the Navier–Stokes problem in an exterior three-dimensional domain Ω. Assume that for some $R > \delta(\Omega^c)$ and some $q \in (3/2, \infty)$,*

$$\boldsymbol{f} \in L^q(\Omega^R). \tag{X.5.1}$$

Then

$$\lim_{|x|\to\infty} |\boldsymbol{v}(x) + \boldsymbol{v}_\infty| = 0 \tag{X.5.2}$$

uniformly. Furthermore, if for $m \ge 0$ and some $r \in (3, \infty)$, $q \in (3/2, 2]$

$$\boldsymbol{f} \in W^{m,r}(\Omega^R) \cap L^q(\Omega^R), \tag{X.5.3}$$

then

$$\lim_{|x|\to\infty} |D^\alpha \boldsymbol{v}(x)| = 0, \quad 1 \le |\alpha| \le m+1, \tag{X.5.4}$$

uniformly. Finally, under assumption (X.5.3) *there is $p_1 \in \mathbb{R}$ such that*

$$\lim_{|x|\to\infty} |D^\alpha (p(x) - p_1)| = 0, \quad 0 \le |\alpha| \le m, \tag{X.5.5}$$

uniformly, where p is the pressure field associated to $\boldsymbol{v}$ by Lemma X.1.1.

Proof. We shall consider throughout the case $\boldsymbol{v}_\infty = 0$, since the case $\boldsymbol{v}_\infty \neq 0$ is treated in a completely analogous way. Moreover, to simplify notation, we shall put $\mathcal{R} = 1$ throughout the proof. From Lemma V.3.1 we have for *a.a.* $x \in \Omega^R$ with $\operatorname{dist}(x, \partial\Omega_R) > d$

$$\begin{aligned} v_j(x) = & \int_{B_d(x)} U_{ij}^{(d)}(x-y)[f_i(y) + v_l D_l v_i(y)]dy \\ & - \int_{\beta(x)} H_{ij}^{(d)}(x-y) v_i(y) dy \\ \equiv & \, I_1(x) + I_2(x) + I_3(x). \end{aligned} \tag{X.5.6}$$

As in the proof of Theorem V.3.1 we show, for $q > 3/2$,

$$|I_1(x)| \le c_1 \|\boldsymbol{f}\|_{q,B_d(x)}. \tag{X.5.7}$$

Furthermore, recalling that

$$|U_{ij}^{(d)}(x-y)| \le c_2 |x-y|^{-1}, \quad y \in B_d(x),$$

we find

$$|I_2(x)| \le c_3 \|\boldsymbol{v}/|x-y|\|_{2,B_d(x)} |\boldsymbol{v}|_{1,2,B_d(x)}.$$

On the other hand, by Theorem II.6.1,

$$\|\boldsymbol{v}/|x-y|\|_{2,B_d(x)} \le c_4 |\boldsymbol{v}|_{1,2}$$

and therefore

$$|I_2(x)| \le c_5 |\boldsymbol{v}|_{1,2,B_d(x)}. \tag{X.5.8}$$

Finally, as in the proof of Theorem V.3.1,

$$|I_3(x)| \le c_6 \|\boldsymbol{v}\|_{6,B_d(x)} \tag{X.5.9}$$

and so, recalling that, by Theorem II.6.1,

$$\boldsymbol{v} \in L^6(\Omega^R) \tag{X.5.10}$$

the property (X.5.2) follows from (X.5.1) and (X.5.6)–(X.5.10). Let us now show (X.5.4). We begin to notice that, by the regularity Theorem X.1.1, we have

$$\boldsymbol{v} \in L^\infty(\Omega_{R,R_1}) \ \text{ for all } R_1 > R,$$

which, because of (X.5.1), implies

$$\boldsymbol{v} \in L^\infty(\Omega^R). \tag{X.5.11}$$

As a consequence, from the inequality

$$\|\boldsymbol{v} \cdot \nabla \boldsymbol{v}\|_t \le \|\boldsymbol{v}\|_{2t/(2-t)} |\boldsymbol{v}|_{1,2}, \quad 1 \le t \le 2,$$

with a view at (X.5.10), (X.5.11) we obtain

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} \in L^s(\Omega^R), \quad \text{for all } s \in [3/2, 2]. \tag{X.5.12}$$

The assumption on $\boldsymbol{f}$ and (X.5.12) lead to

$$(\boldsymbol{f} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) \in L^q(\Omega^R), \quad \text{for some } q \in [3/2, 2],$$

and thus, by Theorem V.5.2, it follows that

$$\boldsymbol{v} \in D^{2,q}(\Omega^{R_1}), \quad p \in D^{1,q}(\Omega^{R_1}), \quad R_1 > R. \tag{X.5.13}$$

Using (X.5.13) and Theorem II.6.1 yields

$$\boldsymbol{v} \in D^{1,3q/(3-q)}(\Omega^{R_1}) \tag{X.5.14}$$

and from (X.5.11), (X.5.14) we conclude that

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} \in L^{3q/(3-q)}(\Omega^{R_1}). \tag{X.5.15}$$

Next, recalling that $I_3 \in C^\infty(\mathbb{R}^3)$, from (X.5.6) with $\boldsymbol{F} = \boldsymbol{f} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}$ we find

$$D_k v_j(x) = D_k \int_{B_d(x)} U_{ij}^{(d)}(x-y) F_i(y) dy - \int_{\beta(x)} D_k H_{ij}^{(d)}(x-y) v_i(y) dy. \tag{X.5.16}$$

Using the definition of generalized differentiation we easily show that

$$D_k \int_{B_d(x)} U_{ij}^{(d)}(x-y) F_i(y) dy = \int_{B_d(x)} (D_k U_{ij}^{(d)}(x-y)) F_i(y) dy \tag{X.5.17}$$

and so, bearing in mind that

$$|D_k U_{ij}^{(d)}(x-y)| \le c_7 |x-y|^{-2}, \quad y \in B_d(x),$$

we recover, by (X.5.17) and the Hölder inequality,

$$\begin{aligned} I(x) \equiv & \left| D_k \int_{B_d(x)} U_{ij}^{(d)}(x-y) F_i(y) dy \right| \\ & \le c_8 (\| |x-y| \|_{r', B_d(x)} \|\boldsymbol{f}\|_{r, B_d(x)} \\ & \quad \| |x-y|^{-2} \|_{3q/(4q-3), B_d(x)} \|\boldsymbol{v} \cdot \nabla \boldsymbol{v}\|_{3q/(3-q), B_d(x)}). \end{aligned}$$

Since both r' and $3q/(4q-3)$ are strictly less than $3/2$, from (X.5.15) and (X.5.2) we obtain

$$I(x) \to 0 \;\; \text{as } |x| \to \infty. \tag{X.5.18}$$

Since

$$\left| \int_{\beta(x)} D_k H_{ij}^{(d)}(x-y) v_i(y) dy \right| \le c_9 \|\boldsymbol{v}\|_{6, B_d},$$

from this inequality, and from (X.5.16) and (X.5.18) we infer (X.5.4) for $|\alpha| = 1$. Let us now pass to higher-order derivatives. To this end, we begin to show

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} \in W^{1,r}(\Omega^R) \tag{X.5.19}$$

for all sufficiently large R.[1] By (X.5.11) and the fact that $\boldsymbol{v}$ is a generalized solution, we find

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} \in L^s(\Omega^R) \ \text{ for all } s \geq 2 \tag{X.5.20}$$

and therefore

$$\boldsymbol{f} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} \in L^r(\Omega^R).$$

From Theorem V.5.3 we then deduce

$$\boldsymbol{v} \in D^{2,r}(\Omega^R) \ \ p \in D^{1,r}(\Omega^R) \tag{X.5.21}$$

and so, since

$$D_k(\boldsymbol{v} \cdot \nabla \boldsymbol{v}) = (D_k \boldsymbol{v}) \cdot \nabla \boldsymbol{v} + \boldsymbol{v} \cdot D_k \nabla \boldsymbol{v} \tag{X.5.22}$$

from (X.5.20), (X.5.11), and (X.5.21) we prove, in particular, (X.5.19). With the help of Lemma V.3.1 we then have

$$D_k D_t v_j(x) = D_k \int_{B_d(x)} U_{ij}^{(d)}(x-y) D_t[f_i(y) + v_l D_l v_i(y)] dy$$

$$- \int_{\beta(x)} D_k D_t H_{ij}^{(d)}(x-y) D^\alpha v_i(y) dy,$$

and since

$$D_t[\boldsymbol{f} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}] \in L^r(\Omega^R) \ \ r > 3,$$

we reason as in the case where $|\alpha| = 1$ and conclude that

$$\lim_{|x| \to \infty} |D^\alpha \boldsymbol{v}(x)| = 0 \ \ |\alpha| = 2 \tag{X.5.23}$$

uniformly. Relation (X.5.23) implies

$$D^2 \boldsymbol{v} \in L^\infty(\Omega^R). \tag{X.5.24}$$

On the other hand, from (X.5.19), (X.5.21), and Lemma V.4.3, it follows that in particular

$$\boldsymbol{v} \in D^{3,r}(\Omega^R) \tag{X.5.25}$$

and so, differentiating one more time (X.5.22) and employing (X.5.19), (X.5.21), (X.5.24), and (X.5.25) we obtain

[1] In the remaining part of the proof the number R need not be the same for all formulas. However, it is understood that the indicated property holds for "sufficiently large" R.

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} \in W^{2,r}(\Omega^R), \quad r > 3.$$

Consequently, by the same reasoning previously used, we prove (X.5.2) with $|\alpha| = 3$. Iterating this procedure as many times as needed, we then show (X.5.4) in the general case. Let us now turn to the pressure p. Collecting (X.5.13) and (X.5.21) under assumption (X.5.3) we obtain

$$p \in D^{1,r}(\Omega^R) \cap D^{1,q}(\Omega^R) \tag{X.5.26}$$

where, we recall, $r > 3$ and $2 > q$. Theorem II.9.1 then implies (X.5.5) with $\alpha = 0$. The proof of the general case (for $m \geq 1$) is a direct consequence of the momentum equation, that is,

$$\nabla p = -\boldsymbol{f} + \Delta \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}. \tag{X.5.27}$$

In fact, if (X.5.3) holds with some $m \geq 1$, then from the embedding Theorem II.3.4, it follows that

$$D^\alpha \boldsymbol{f}(x) \to 0 \ \text{ as } |x| \to \infty, \quad \text{for all } |\alpha| \in [0, m-1]. \tag{X.5.28}$$

By means of (X.5.2), (X.5.4), (X.5.27), and (X.5.28) we thus conclude that as $|x| \to \infty$

$$D^\alpha[-\boldsymbol{f}(x) + \Delta \boldsymbol{v}(x) + \boldsymbol{v}(x) \cdot \nabla \boldsymbol{v}(x)] \to 0$$

uniformly for all multi-index α with $|\alpha| = m - 1$. The theorem is therefore proved. □

Remark X.5.1 The arguments used in the proof of the preceding theorem fail in dimension $n \geq 4$. In fact, they do not ensure pointwise decay even for $\boldsymbol{v}$ itself. The reason is that, in such a case, we cannot increase the term $I_2(x)$ by a function vanishing at large distances. However, it is probably true that results of the type presented in Theorem X.5.1 continue to hold, at least, for $n = 4$. ■

We shall now derive representation formulas for $\boldsymbol{v}$ and p analogous to those derived in the linear case. Specifically, we have the following theorem.

Theorem X.5.2 *Let $\boldsymbol{v}$ be a generalized solution to the Navier–Stokes problem* (X.0.8), (X.0.4) *in an exterior three-dimensional domain Ω of class C^2 with*

$$\boldsymbol{v} \in W^{2,r}_{loc}(\overline{\Omega}), \quad r \in (1, \infty).$$

Then if for some $R > \delta(\Omega^c)$

$$\boldsymbol{f} \in L^q(\Omega^R) \cap L^s_{loc}(\overline{\Omega}), \tag{X.5.29}$$

with $q \in (1, 3/2)$ and $s \in (1, \infty)$, the following representations hold for all $x \in \Omega$: if $\boldsymbol{v}_\infty = 0$

$$
\begin{aligned}
v_j(x) = \mathcal{R}\int_\Omega U_{ij}(x-y)f_i(y)dy + \mathcal{R}\int_\Omega U_{ij}(x-y)v_l(y)D_l v_i(y)dy \\
+ \int_{\partial\Omega}[v_i(y)T_{il}(\boldsymbol{u}_j, q_j)(x-y) \\
- U_{ij}(x-y)T_{il}(\boldsymbol{v}, p)(y)]n_l d\sigma_y,
\end{aligned}
\tag{X.5.30}
$$

and

$$
\begin{aligned}
v_j(x) = \mathcal{R}\int_\Omega U_{ij}(x-y)f_i(y)dy - \mathcal{R}\int_\Omega v_l(y)v_i(y)D_l U_{ij}(x-y)dy \\
+ \int_{\partial\Omega}[v_i(y)T_{il}(\boldsymbol{u}_j, q_j)(x-y) - U_{ij}(x-y)(T_{il}(\boldsymbol{v}, p)(y) \\
- \mathcal{R}v_l(y)v_i(y))]n_l d\sigma_y;
\end{aligned}
\tag{X.5.31}
$$

if $\boldsymbol{v}_\infty \neq 0 \quad (\boldsymbol{v}_\infty = (1,0,0))$, *setting* $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{v}_\infty$

$$
\begin{aligned}
u_j(x) = \mathcal{R}\int_\Omega E_{ij}(x-y)f_i(y)dy + \mathcal{R}\int_\Omega E_{ij}(x-y)u_l(y)D_l u_i(y)dy \\
+ \int_{\partial\Omega}[u_i(y)T_{il}(\boldsymbol{w}_j, e_j)(x-y) - E_{ij}(x-y)T_{il}(\boldsymbol{u}, p)(y) \\
- \mathcal{R}u_i(y)E_{ij}(x-y)\delta_{1l}]n_l d\sigma_y
\end{aligned}
\tag{X.5.32}
$$

and

$$
\begin{aligned}
u_j(x) = \mathcal{R}\int_\Omega E_{ij}(x-y)f_i(y)dy - \mathcal{R}\int_\Omega u_l(y)u_i(y)D_l E_{ij}(x-y)dy \\
+ \int_{\partial\Omega}[u_i(y)T_{il}(\boldsymbol{w}_j, e_j)(x-y) \\
- E_{ij}(x-y)(T_{il}(\boldsymbol{u}, p)(y) - \mathcal{R}u_l(y)u_i(y)) \\
- \mathcal{R}u_i(y)E_{ij}(x-y)\delta_{1l}]n_l d\sigma_y;
\end{aligned}
\tag{X.5.33}
$$

where $\boldsymbol{U}, \boldsymbol{q}$ *and* $\boldsymbol{E}, \boldsymbol{e}$ *are Stokes and Oseen fundamental solutions, respectively, while* $\boldsymbol{u}_j$ *and* $\boldsymbol{w}_j$ *are defined in* (IV.8.11)$_1$ *and* (VII.6.3). *All volume integrals in* (X.5.30)–(X.5.33) *are absolutely convergent.*

Furthermore, if

$$
\boldsymbol{f} \in L^r(\Omega^R) \cap L^t_{loc}(\Omega),
$$

for some $r \in (1, 3/2)$ *and* $t \in (3, \infty)$, *then, denoting by* p *the pressure field associated to* $\boldsymbol{v}$ *by* Lemma X.1.1, *we have for a.a.* $x \in \Omega$: *if* $\boldsymbol{v}_\infty = 0$

$$
\begin{aligned}
p(x) = p_0 - \mathcal{R}\int_\Omega q_i(x-y)f_i(y)dy - \mathcal{R}\int_\Omega q_i(x-y)v_l(y)D_l v_i(y)dy \\
+ \int_{\partial\Omega}[q_i(x-y)T_{il}(\boldsymbol{v}, p)(y) - 2v_i(y)\frac{\partial}{\partial x_l}q_i(x-y)]n_l d\sigma_y\,;
\end{aligned}
\tag{X.5.34}
$$

if $\boldsymbol{v}_\infty \neq 0$

$$\begin{aligned}p(x) = p_0' - \mathcal{R}\int_\Omega e_i(x-y)f_i(y)dy - \mathcal{R}\int_\Omega e_i(x-y)u_l(y)D_l u_i(y)dy \\ + \int_{\partial\Omega}\{e_i(x-y)T_{il}(\boldsymbol{u},p)(y) - 2u_i(y)\frac{\partial}{\partial x_l}e_i(x-y) \\ - \mathcal{R}[e_1(x-y)u_l(y) - u_i(y)e_i(x-y)\delta_{1l}]n_l d\sigma_y,\end{aligned} \tag{X.5.35}$$

where p_0 *and* p_0' *are constants. All volume integrals in* (X.5.34), (X.5.35) *are absolutely convergent.*

Proof. We show (X.5.30), (X.5.31) and (X.5.34), since the proof of (X.5.32), (X.5.33) and (X.5.35) is somehow similar and, therefore, left to the reader as an exercise. From the assumptions made and (V.3.6) with $\alpha = 0$, we obtain for $R > \delta(\Omega^c)$

$$v_j(x) = \mathcal{R}\int_\Omega U_{ij}^{(R)}(x-y)F_i(y)dy - \int_\Omega H_{ij}^{(R)}(x-y)v_i(y)dy + s_i(x), \tag{X.5.36}$$

where

$$\boldsymbol{F} = \boldsymbol{f} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}$$

and $\boldsymbol{s}(x)$ is the surface integral in (X.5.30). We recall that

$$|\boldsymbol{U}(x-y)|,\quad |\boldsymbol{U}^{(R)}(x-y)| \le c|x-y|^{-1},\quad x,y\in\mathbb{R}^3,\quad x\neq y$$

and so, taking into account (V.3.2), (V.3.3) we have

$$\int\left|[U_{ij}^{(R)}(x-y) - U_{ij}(x-y)]F_i(y)dy\right| \le c\int_{\Omega_{R/2,R}(x)}\frac{|\boldsymbol{F}(y)|}{|x-y|}dy$$

with

$$\Omega_{R/2,R}(x) = \{y\in\Omega : R/2 < |x-y| < R\}.$$

Since

$$\begin{aligned}\int_{\Omega_{R/2,R}(x)}\frac{|\boldsymbol{F}(y)|}{|x-y|}dy \le \||x-y|^{-1}\|_{q',\Omega_{R/2,R}(x)}\|\boldsymbol{f}\|_{q,\Omega_{R/2,R}(x)} \\ + \|\boldsymbol{v}/|x-y|\|_{2,\Omega_{R/2,R}(x)}|\boldsymbol{v}|_{1,2,\Omega_{R/2,R}(x)},\end{aligned}$$

from Theorem II.6.1 and the assumption made on $\boldsymbol{f}$ we derive for all $x\in\Omega$

$$\lim_{R\to\infty}\int_\Omega U_{ij}^{(R)}(x-y)F_i(y)dy = \int_\Omega U_{ij}(x-y)F_i(y)dy \tag{X.5.37}$$

in the sense of absolute convergence. Furthermore, bearing in mind that $H_{ij}^{(R)}(x-y)$ is identically vanishing outside $\Omega_{R/2,R}(x)$, from (V.3.5) and Theorem II.6.1 we recover in the limit $R\to\infty$

$$\left|\int_\Omega H_{ij}^{(R)}(x-y)v_i(y)dy\right| \le c\|\boldsymbol{v}\|_{6,\Omega_{R/2,R}} \to 0. \tag{X.5.38}$$

Formula (X.5.30) then follows from (X.5.36)–(X.5.38). To prove (X.5.31) we notice that

$$\begin{aligned}\int_\Omega U_{ij}^{(R)}(x-y)v_l(y)D_l v_j(y)dy = &\int_\Omega v_l(y)v_j(y)D_l U_{ij}^{(R)}(x-y)dy \\ &+\int_{\partial\Omega} U_{ij}(x-y)v_l(y)v_i(y)n_l d\sigma_y.\end{aligned} \tag{X.5.39}$$

Since

$$|\nabla \boldsymbol{U}(x-y)|, \quad |\nabla \boldsymbol{U}^{(R)}(x-y)| \le c|x-y|^{-2}, \quad x,y\in\mathbb{R},$$

we find

$$\left|\int_\Omega v_l(y)v_j(y)(D_l U_{ij}^{(R)}(x-y) - D_l U_{ij}(x-y))dy\right| \le c\|\boldsymbol{v}/|x-y|\|_{2,\Omega_{R,2R}}^2$$

and so

$$\lim_{R\to\infty}\int_\Omega v_l(y)v_j(y)D_l U_{ij}^{(R)}(x-y)dy = \int_\Omega v_l(y)v_j(y)D_l U_{ij}(x-y)dy$$

in the sense of absolute convergence. This latter relation, together with (X.5.39) and (X.5.30), proves (X.5.31). Let us now consider representation (X.5.34) for the pressure field. Setting $\widetilde{p} = p - p_1$ with p_1 given in Theorem X.5.1, from (IV.8.19) it follows, for all sufficiently large R, that

$$\begin{aligned}\widetilde{p}(x) = p_{0R} - \mathcal{R}\int_{\Omega_R(x)} q_i(x-y)F_i(y)dy + \int_{\partial B_R(x)}[q_i(x-y)T_{il}(\boldsymbol{v},p)(y) \\ -2v_i(y)\frac{\partial q_l(x-y)}{\partial x_i}]n_l(y)d\sigma_y + \sigma(x),\end{aligned} \tag{X.5.40}$$

where $\Omega_R(x) = \Omega\cap B_R(x)$, $\sigma(x)$ is the surface integral in (X.5.34) and p_{0R} is a constant, possibly depending on R. Recalling that

$$|q_i(x-y)| \le |x-y|^{-2}$$
$$|D_l q_i(x-y)| \le c|x-y|^{-3}$$

from Theorem X.5.1 and the assumption on $\boldsymbol{f}$ we immediately deduce for all $x\in\Omega$ that

$$\lim_{R\to\infty}\int_{\partial B_R(x)}[q_i(x-y)T_{il}(\boldsymbol{v},p)(y) - 2v_i(y)D_l q_i(x-y)]n_l(y)d\sigma_y = 0. \tag{X.5.41}$$

Furthermore, for $d < \operatorname{dist}(x,\partial\Omega)$, setting $\Omega^d(x) = \Omega - B_d(x)$ we have

$$\int_\Omega |q_i(x-y)f_i(y)|dy \le c\,(\||x-y|^{-2}\|_{t',B_d(x)}\|\boldsymbol{f}\|_{t,B_d(x)} + \||x-y|^{-2}\|_{r',\Omega^d(x)}\|\boldsymbol{f}\|_{r,\Omega^d(x)}) \tag{X.5.42}$$

and, by Theorem II.6.1,

$$\begin{aligned}\int_\Omega |q_i(x-y)v_l(y)D_lv_i(y)(y)|dy &\le c\,(\||x-y|^{-2}\|_{t',B_d(x)}\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{t,B_d(x)} \\ &\quad +\|\boldsymbol{v}/|x-y|\|_{2,\Omega^d(x)}|\boldsymbol{v}|_{1,2,\Omega^d(x)}) \\ &\le c_1(\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{t,B_d(x)} + |\boldsymbol{v}|_{1,2,\Omega^d(x)}).\end{aligned} \tag{X.5.43}$$

By the hypothesis on $\boldsymbol{f}$ and Theorem X.1.1,

$$\boldsymbol{v} \in W^{2,t}(B_d(x)), \quad t > 3$$

and so, the embedding Theorem II.3.4 yields

$$\boldsymbol{v},\ \nabla\boldsymbol{v} \in L^\infty(B_d(x)). \tag{X.5.44}$$

From (X.5.42)–(X.5.44) we conclude that the integral

$$\int_\Omega q_i(x-y)F_i(y)dy$$

is absolutely convergent and therefore

$$\lim_{R\to\infty}\int_{\Omega_R(x)} q_i(x-y)F_i(y)dy = \int_\Omega q_i(x-y)F_i(y)dy. \tag{X.5.45}$$

Combining (X.5.40), (X.5.41), and (X.5.45) furnishes for *a.a.* $x \in \Omega$

$$\widetilde{p}(x) = p_0 - \mathcal{R}\int_\Omega q_i(x-y)F_i(y)dy + \sigma(x) \tag{X.5.46}$$

with

$$\lim_{R\to\infty} p_{0R} = p_0,$$

and (X.5.34) is proved. □

Employing the same type of arguments used to show the asymptotic formulas (V.3.19), (V.3.20) and (VII.6.18), (VII.6.19) (see also Exercise VII.6.3), from Theorem X.5.2 we obtain the following result whose proof is left to the reader as an exercise.

Theorem X.5.3 *Let $\boldsymbol{v}$ be a generalized solution to the Navier–Stokes equations in a domain Ω of class C^2, with*

$$\boldsymbol{v} \in W^{2,r}(\Omega_R), \quad \textit{for some } R > \delta(\Omega^c) \ \textit{ and } r \in (1,\infty),$$

and let p be the associated pressure field. Then if $\boldsymbol{f}$ is of bounded support and

$$\boldsymbol{f} \in L^s(\Omega) \quad \textit{for some } s \in (1, \infty),$$

the following asymptotic representation formulas hold as $|x| \to \infty$. If $\boldsymbol{v}_\infty = 0$:

$$\begin{aligned} v_j(x) &= \mathcal{T}_i U_{ij}(x) + \mathcal{R}\int_\Omega U_{ij}(x-y) v_l(y) D_l v_i(y) dy + \sigma_j^{(1)}(x) \\ v_j(x) &= \mathcal{T}_i' U_{ij}(x) - \mathcal{R}\int_\Omega v_i(y) v_l(y) D_l U_{ij}(x-y) dy + \sigma_j^{(2)}(x) \\ p(x) &= p_0 - \mathcal{T}_i q_i(x) - \mathcal{R}\int_\Omega q_i(x-y) v_l(y) D_l v_i(y) dy + \eta(x) \end{aligned} \tag{X.5.47}$$

where $p_0 \in \mathbb{R}$,

$$\begin{aligned} \mathcal{T}_i &= -\int_{\partial\Omega} T_{il}(\boldsymbol{v}, p) n_l + \mathcal{R}\int_\Omega f_i \\ \mathcal{T}_i' &= -\int_{\partial\Omega} (T_{il}(\boldsymbol{v}, p) - \mathcal{R} v_i v_l) n_l + \mathcal{R}\int_\Omega f_i \end{aligned} \tag{X.5.48}$$

and, for all $|\alpha| \geq 0$,

$$\begin{aligned} D^\alpha \sigma_j^{(k)}(x) &= O(|x|^{-2-|\alpha|}), \quad k = 1, 2 \\ D^\alpha \eta(x) &= O(|x|^{-3-|\alpha|}). \end{aligned} \tag{X.5.49}$$

If $\boldsymbol{v}_\infty \neq 0$ $(\boldsymbol{v}_\infty = (1,0,0))$, setting $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_\infty$:

$$\begin{aligned} u_j(x) &= \mathcal{M}_i E_{ij}(x) + \mathcal{R}\int_\Omega E_{ij}(x-y) u_l(y) D_l u_i(y) dy + s_j^{(1)}(x) \\ u_j(x) &= m_i E_{ij}(x) - \mathcal{R}\int_\Omega u_i(y) u_l(y) D_l E_{ij}(x-y) dy + s_j^{(2)}(x) \\ p(x) &= p_0' - \mathcal{M}_i^* e_i(x) - \mathcal{R}\int_\Omega e_i(x-y) u_l(y) D_l u_i(y) dy + h(x) \end{aligned} \tag{X.5.50}$$

where $p_0' \in \mathbb{R}$

$$\begin{aligned} \mathcal{M}_i &= -\int_{\partial\Omega} T_{il}(\boldsymbol{u}, p) n_l + \mathcal{R}\delta_{1l} u_i] n_l + \mathcal{R}\int_\Omega f_i \\ m_i &= -\int_{\partial\Omega} (T_{il}(\boldsymbol{u}, p) + \mathcal{R}(\delta_{1l} u_i - u_i u_l) n_l + \mathcal{R}\int_\Omega f_i \\ \mathcal{M}_i^* &= -\int_{\partial\Omega} \{T_{il}(\boldsymbol{u}, p) n_l + \mathcal{R}[\delta_{1l} u_i - \delta_{1i} u_l]\} n_l + \mathcal{R}\int_\Omega f_i \end{aligned} \tag{X.5.51}$$

and, for all $|\alpha| \geq 0$, all $q \in (3/2, \infty]$ and $j = 1, 2$,

$$
\begin{aligned}
D^\alpha s_j^{(k)}(x) &= O(|x|^{-(3+|\alpha|)/2}),\\
s_j^{(k)} &\in L^q(\Omega) \qquad\qquad (\mathrm{X}.5.52)\\
D^\alpha h(x) &= O(|x|^{-3-|\alpha|}).
\end{aligned}
$$

Global Summability of Generalized Solutions when $\boldsymbol{v}_\infty \neq \mathbf{0}$

A fundamental step in deriving the asymptotic structure of generalized solutions is to establish "good" summability properties at large distances, that is, in a domain Ω^R, for sufficiently large R. In this direction, in the three-dimensional case, the only information that we have at the outset is that the velocity field $\boldsymbol{v}$ satisfies $(\boldsymbol{v}+\boldsymbol{v}_\infty) \in D^{1,2}(\Omega) \cap L^6(\Omega)$.

The objective of the present section is to show that if $\boldsymbol{v}_\infty \neq \mathbf{0}$ and if, for some $q_0 > 3$, it is assumed that

$$
\begin{aligned}
&\boldsymbol{f} \in L^q(\Omega), \quad \text{for all } q \in (1, q_0]\\
&\boldsymbol{v}_* \in W^{2-1/q_0, q_0}(\partial\Omega)
\end{aligned}
\qquad (\mathrm{X}.6.1)
$$

then

$$
\boldsymbol{v}+\boldsymbol{v}_\infty \in L^r(\Omega) \;\; \text{for all } r > 2 \qquad (\mathrm{X}.6.2)
$$

and, likewise,[1]

$$
\begin{aligned}
&\frac{\partial \boldsymbol{v}}{\partial x_1} \in L^s(\Omega), \quad \text{for all } s > 1\\
&\boldsymbol{v} \in D^{1,t}(\Omega), \quad \text{for all } t > 4/3\\
&p \in L^\sigma(\Omega), \quad \text{for all } \sigma > 3/2.
\end{aligned}
\qquad (\mathrm{X}.6.3)
$$

Conditions (X.6.2) and (X.6.3) tell us, in particular, that under the assumption (X.6.1) on the data, any corresponding generalized solution and associated pressure field have at large distances the same summability properties of the Oseen fundamental solution $\boldsymbol{E}$, $\boldsymbol{e}$, or, what amounts to the same thing, of the solution of the Oseen problem with the same data $\boldsymbol{f}$ and $\boldsymbol{v}_*$. Moreover, as in the linearized theory, if $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv 0$ and $\boldsymbol{v}_\infty \neq 0$, we show that

$$
\boldsymbol{v}+\boldsymbol{v}_\infty \notin L^r(\Omega), \quad \text{for all } r \in (1, 2]. \qquad (\mathrm{X}.6.4)
$$

Taking into account that for our model of liquid the density is a constant, relation (X.6.4) with $r = 2$ shows that *the kinetic energy in a steady motion of a liquid in which a body moves with a constant velocity is, in general, infinite*, a fact first pointed out by Finn (1960).

[1] p is possibly modified by addition of a constant.

In order to show all the above, we introduce the following function space

$$X_q(\Omega) = \Big\{(\boldsymbol{u},\phi) \in L^1_{loc}(\Omega) : \boldsymbol{u} \in D^{2,q}(\Omega)\cap D^{4q/(4-q)}(\Omega)\cap L^{2q/(2-q)}(\Omega)\,;\\ \frac{\partial \boldsymbol{u}}{\partial x_1} \in L^q(\Omega); \phi \in D^{1,q}(\Omega)\cap L^{3q/(3-q)}(\Omega)\Big\},\ q\in(1,2)\,. \tag{X.6.5}$$

It is readily checked that X_q becomes a Banach space when endowed with the "natural" norm

$$\|(\boldsymbol{u},\phi)\|_{X_q} := \|\boldsymbol{u}\|_{2q/(2-q)} + |\boldsymbol{u}|_{1,4q/(4-q)} + \left\|\frac{\partial \boldsymbol{u}}{\partial x_1}\right\|_q + |\boldsymbol{u}|_q + \|\phi\|_{3q/(3-q)} + |\phi|_{1,q}\,. \tag{X.6.6}$$

The following result holds

Lemma X.6.1 *Let Ω be a C^2-smooth exterior three-dimensional domain, and assume, for some $q\in(1,2)$, that*

$$\boldsymbol{f} \in L^q(\Omega)\cap L^{3/2}(\Omega)\,,\quad \boldsymbol{v}_* \in W^{2-1/q,q}(\partial\Omega)\cap W^{4/3,3/2}(\partial\Omega)\,. \tag{X.6.7}$$

Then, every generalized solution $\boldsymbol{v}$ to the Navier–Stokes problem corresponding to $\boldsymbol{f}$, $\boldsymbol{v}_*$ and to $\boldsymbol{v}_\infty \neq \boldsymbol{0}$, and the associated pressure field p[2] *satisfy $(\boldsymbol{v}+\boldsymbol{v}_\infty, p) \in X_q(\Omega)$.*

Proof. Without loss, we assume $\boldsymbol{v}_\infty = \boldsymbol{e}_1$. Also, since the actual value of $\mathcal{R}$ is irrelevant in the proof, we set $\mathcal{R}=1$, for simplicity. Recalling that $\boldsymbol{v} \in D^{1,2}(\Omega)$, we may find a sequence of second-order tensors $\{\boldsymbol{G}_k\}$ with components in $C_0^\infty(\Omega)$ such that $\boldsymbol{G}_k \to \nabla \boldsymbol{v}$ in $L^2(\Omega)$. Consider now the problem

$$\left.\begin{aligned} &\Delta \boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial x_1} = \boldsymbol{u}\cdot \boldsymbol{A}_k + \nabla\phi + \boldsymbol{F}_k \\ &\nabla\cdot \boldsymbol{u} = 0 \end{aligned}\right\}\ \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{u}_* := \boldsymbol{v}_* + \boldsymbol{e}_1 \ \text{ at } \partial\Omega\,, \tag{X.6.8}$$

where $\boldsymbol{A}_k := \nabla \boldsymbol{v} - \boldsymbol{G}_k$, $\boldsymbol{F}_k := \boldsymbol{v}\cdot\boldsymbol{G}_k + \boldsymbol{f}$, while the integer k will be specified successively. Clearly, $(\boldsymbol{v}+\boldsymbol{e}_1, p)$ is a solution to (X.6.8), for all $k\in\mathbb{N}$. Our plan is to show the existence of a solution to (X.6.8) in the class $X_q(\Omega)$, and then prove that such a solution coincides with $(\boldsymbol{v}+\boldsymbol{e}_1,p)$. To this end, we begin to observe that, in view of the assumption on $\boldsymbol{f}$ and the fact that $(\boldsymbol{v}+\boldsymbol{e}_1) \in L^6(\Omega)$, it follows that $\boldsymbol{F}_k \in L^q(\Omega)\cap L^{3/2}(\Omega)$, for all $k\in\mathbb{N}$. We now set $X_{q,3/2}(\Omega) = X_q(\Omega)\cap X_{3/2}(\Omega)$, endowed with the norm $\|\cdot\|_{X_{q,3/2}} := \|\cdot\|_{X_q} + \|\cdot\|_{X_{3/2}}$, and consider the map

$$M : (\boldsymbol{w},\tau) \in X_{q,3/2}(\Omega) \to (\boldsymbol{u},\phi) := M(\boldsymbol{w},\tau)$$

where $(\boldsymbol{u},\phi)$ satisfies:

[2] Possibly modified by the addition of a constant.

$$\left.\begin{aligned}&\Delta \boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial x_1} = \boldsymbol{w}\cdot \boldsymbol{A}_k + \nabla\phi + \boldsymbol{F}_k\\ &\nabla\cdot\boldsymbol{u} = 0\end{aligned}\right\}\ \text{in } \Omega$$
$$\boldsymbol{u} = \boldsymbol{u}_* \ \text{ at } \partial\Omega\,. \tag{X.6.9}$$

Observing that, by the Hölder inequality and the fact that $\boldsymbol{v}\in D^{1,2}(\Omega)$,

$$\|\boldsymbol{w}\cdot\boldsymbol{A}_k\|_s \le \|\boldsymbol{w}\|_{2s/(2-s)}\|\boldsymbol{A}_k\|_2 < \infty\,, \quad s\in(1,2)\,, \tag{X.6.10}$$

with the help of Theorem VII.7.1 we deduce, on the one hand, that the map M is well-defined and, on the other hand, that there exists a (unique) solution $(\boldsymbol{u},\phi)\in X_{q,3/2}(\Omega)$ to (X.6.9) obeying the estimate (with $q_1=q$, $q_2=3/2$)

$$\begin{aligned}\|(\boldsymbol{u},\phi)\|_{X_{q,3/2}} &\le c\sum_{i=1}^{2}\left(\|\boldsymbol{A}_k\|_2\|\boldsymbol{w}\|_{2q_i/(2-q_i)} + \|\boldsymbol{F}_k\|_{q_i} + \|\boldsymbol{u}_*\|_{2-2/q_i(\partial\Omega)}\right)\\ &\le c\left(\|\boldsymbol{A}_k\|_2\|(\boldsymbol{w},\tau)\|_{X_{q,3/2}} + \sum_{i=1}^{2}\left(\|\boldsymbol{F}_k\|_{q_i} + \|\boldsymbol{u}_*\|_{2-2/q_i(\partial\Omega)}\right)\right)\,.\end{aligned} \tag{X.6.11}$$

Thus, if we choose k such that

$$\|\boldsymbol{A}_k\|_2 \le 1/(2c) \tag{X.6.12}$$

and define

$$\delta := 2c\sum_{i=1}^{2}\left(\|\boldsymbol{F}_k\|_{q_i} + \|\boldsymbol{u}_*\|_{2-2/q_i(\partial\Omega)}\right)\,,$$

from (X.6.11) it follows at once that M maps the closed ball $\{(\boldsymbol{u},\phi)\in X_{q,3/2}(\Omega) : \|(\boldsymbol{u},\phi)\|_{X_{q,3/2}} \le \delta\}$ into itself. Moreover, in view of the linearity of the map M and of (X.6.12), from (X.6.11) with $\|\boldsymbol{F}_k\|_{q_i}=0$, $i=1,2$, we deduce that M is a contraction and, therefore, there exists one and only one $(\boldsymbol{u},\phi)\in X_{q,3/2}$ solution to (X.6.8). Next, set $(\boldsymbol{w},\tau) := (\boldsymbol{u}-\boldsymbol{v}-\boldsymbol{e}_1, \phi-p)$. We thus find

$$\left.\begin{aligned}&\Delta \boldsymbol{w} + \frac{\partial \boldsymbol{w}}{\partial x_1} = \boldsymbol{w}\cdot \boldsymbol{A}_k + \nabla\tau\\ &\nabla\cdot\boldsymbol{w} = 0\end{aligned}\right\}\ \text{in } \Omega$$
$$\boldsymbol{w} = \boldsymbol{0} \ \text{ at } \partial\Omega\,. \tag{X.6.13}$$

Recalling that, by the assumption on $\boldsymbol{v}$ and (X.6.10), it is $\boldsymbol{g} := \boldsymbol{w}\cdot\boldsymbol{A}_k \in L^{3/2}(\Omega)$, with the help of Theorem VII.7.1 we infer that the problem

$$\left.\begin{aligned}&\Delta \widetilde{\boldsymbol{w}} + \frac{\partial \widetilde{\boldsymbol{w}}}{\partial x_1} = \boldsymbol{g} + \nabla\widetilde{\tau}\\ &\nabla\cdot\widetilde{\boldsymbol{w}} = 0\end{aligned}\right\}\ \text{in } \Omega$$
$$\widetilde{\boldsymbol{w}} = \boldsymbol{0} \ \text{ at } \partial\Omega\,. \tag{X.6.14}$$

has a (unique) solution $(\widetilde{\boldsymbol{w}}, \widetilde{\tau})$ in the class $X_{3/2}(\Omega)$. We claim that $(\widetilde{\boldsymbol{w}}, \widetilde{\tau}) = (\boldsymbol{w}, \tau)$. In fact, the fields $\boldsymbol{z} := \widetilde{\boldsymbol{w}} - \boldsymbol{w}$ and $\chi := \widetilde{\tau} - \tau$ solve the homogeneous Oseen problem (X.6.14) with $\boldsymbol{g} \equiv \boldsymbol{0}$. Furthermore, $\boldsymbol{z} \in L^6(\Omega)$, because $\widetilde{\boldsymbol{w}}, \boldsymbol{u} \in X_{3/2}(\Omega)$, while $\boldsymbol{v} + \boldsymbol{e}_1 \in L^6(\Omega)$ by assumption. Therefore, from Theorem VII.6.2 and Exercise VII.6.2 we obtain $\boldsymbol{z} \equiv \nabla\chi \equiv \boldsymbol{0}$. Consequently, $\boldsymbol{w} \in X_{3/2}(\Omega)$, and so, again by Theorem VII.7.1 applied to (X.6.13), we obtain

$$\|(\boldsymbol{w}, \phi)\|_{X_{3/2}} \leq c\,\|\boldsymbol{w} \cdot \boldsymbol{A}_k\|_{3/2} \leq c\,\|\boldsymbol{A}_k\|_2 \|(\boldsymbol{w}, \phi)\|_{X_{3/2}} .$$

Thus, by using (X.6.12) into this latter inequality, we deduce $\boldsymbol{w} \equiv \nabla\tau \equiv \boldsymbol{0}$, that is $(\boldsymbol{u}, \phi) = (\boldsymbol{v} + \boldsymbol{e}_1, p + C)$, for some $C \in \mathbb{R}$, and the proof of the lemma is complete.

□

Combining the result of the previous lemma with those of Theorem X.5.1 we prove the following.

Theorem X.6.4 *Let Ω be a C^2-smooth, exterior three-dimensional domain and assume that*

$$\boldsymbol{f} \in L^q(\Omega), \quad \boldsymbol{v}_* \in W^{2-1/q_0,1/q_0}(\partial\Omega), \quad \boldsymbol{v}_\infty \neq \boldsymbol{0},$$

for some $q_0 > 3$, and all $q \in (1, q_0]$. Then every corresponding generalized solution $\boldsymbol{v}$ to the Navier–Stokes problem (X.0.8), (X.0.4) *satisfies the following summability properties*

$$(\boldsymbol{v} + \boldsymbol{v}_\infty) \in L^r(\Omega)\,, \ \frac{\partial \boldsymbol{v}}{\partial x_1} \in L^s(\Omega)\,, \ \boldsymbol{v} \in D^{1,t}(\Omega)\,, \ p \in L^\sigma(\Omega),$$

for all $r \in (2, \infty]$, $s \in (1, \infty]$, $t \in (4/3, \infty]$ and $\sigma \in (3/2, \infty]$ where p is (up to a constant) the pressure field associated to $\boldsymbol{v}$ by Lemma X.1.1. *If in addition, Ω is of class C^3, and $\boldsymbol{f} \in W^{1,q_0}(\Omega)$, $\boldsymbol{v}_* \in W^{3-1/q_0,1/q_0}(\partial\Omega)$, then we have also*

$$\boldsymbol{v} \in D^{2,\tau}(\Omega)\,, \ p \in D^{1,\tau}(\Omega)\,, \tag{X.6.15}$$

for all $\tau \in (1, \infty]$.

Proof. The stated summability properties, in the domain Ω^R, follow at once from Lemma X.6.1 and Theorem X.5.1. On the other hand, in the domain Ω_R, they are a consequence of Theorem IV.5.1. □

Remark X.6.2 It is worth emphasizing that Theorem X.6.4 does not require the vanishing of the flux of $\boldsymbol{v}_*$ through the boundary $\partial\Omega$. ■

Remark X.6.3 Summability properties at large distances for higher order derivatives can be likewise obtained by using Theorem VII.7.1, with $\boldsymbol{f}$ replaced by $\boldsymbol{f} + (\boldsymbol{v} + \boldsymbol{v}_\infty) \cdot \nabla \boldsymbol{v}$. To show this, let us assume, for simplicity, $\boldsymbol{f}$ of bounded support in Ω. Observing that, by Theorem X.6.4 and by (X.6.15) for sufficiently large $R > \delta(\Omega^c)$, we have

$$\boldsymbol{N} \equiv (\boldsymbol{v} + \boldsymbol{v}_\infty) \cdot \nabla \boldsymbol{v} \in W^{1,\tau}(\Omega^R),$$

from Theorem VII.7.1, it follows that

$$\boldsymbol{v} \in D^{3,\tau}(\Omega^R), \quad p \in D^{2,\tau}(\Omega^R),$$

for all $\tau > 1$. Then

$$\boldsymbol{N} \in W^{2,\tau}(\Omega)$$

for all $\tau > 1$, and so on. Therefore, we can conclude, by iteration,

$$\boldsymbol{v} \in D^{m+2,\tau}(\Omega^R), \quad p \in D^{m+1,\tau}(\Omega^R),$$

for all $m \geq 0$ and all $\tau > 1$. ■

The remaining part of this section is devoted to investigate the finiteness of the kinetic energy of the liquid. To this end, we begin to observe that, from Theorem X.5.3, it follows that for $\boldsymbol{f} \in L^s(\Omega)$, $s > 1$, of bounded support in Ω, the field $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{v}_\infty \ \ (\boldsymbol{v}_\infty = \boldsymbol{e}_1)$ admits the following representation:

$$u_j(x) = E_{ij}(x) m_i - \mathcal{R} \int_\Omega u_l(y) u_i(y) D_l E_{ij}(x-y) dy + s_j(x) \tag{X.6.16}$$

where

$$m_i = -\int_{\partial\Omega} \left[T_{il}(\boldsymbol{u}, p) + \mathcal{R}\left(\delta_{1l} u_i - u_l u_i\right)\right] n_l + \mathcal{R} \int_\Omega f_i \tag{X.6.17}$$

and

$$\boldsymbol{s} \in L^\tau(\Omega^R), \quad \text{for all } \tau > 3/2, \tag{X.6.18}$$

see Exercise VII.6.3. Observing that, by (VII.3.21) and (VII.3.33),

$$\nabla \boldsymbol{E} \in L^r(\mathbb{R}^3) \ \text{ for all } r \in (4/3, 3/2),$$

and that, by Theorem X.6.4,

$$u_i u_l \in L^s(\Omega) \ \text{ for all } s > 1,$$

from Young's theorem on convolutions it follows that

$$\int_\Omega u_l(y) u_i(y) D_l E_{ij}(x-y) dy \in L^t(\Omega), \quad \text{for all } t > 4/3. \tag{X.6.19}$$

In view of (X.6.16)–(X.6.19), we then conclude that, for sufficiently large R,

$$\boldsymbol{u} \in L^q(\Omega^R), \quad q \in (3/2, 2] \tag{X.6.20}$$

if and only if

$$E_{ij} m_j \in L^q(\Omega^R), \quad q \in (3/2, 2]. \tag{X.6.21}$$

Consider now the quadratic form

$$\mathcal{Q} \equiv E_{ij}(x)E_{ij}(x)m_i m_k .$$

Starting from (VII.3.20) and using the symmetry properties of the tensor field $E_{ij}(x)$, it is not hard to show that, for any $\rho > 0$,

$$\int_{\partial B_\rho} E_{ij}(x)E_{ik}(x) = 0, j \neq k,$$

and so, the integrability of $\mathcal{Q}$ is reduced to that of

$$\begin{aligned}\mathcal{Q}' = m_1^2 E_{11}^2 + m_2^2 E_{22}^2 + m_3^2 E_{33}^2 + (m_1^2 + m_2^2)E_{12}^2 \\ +(m_1^2 + m_3^2)E_{13}^2 + (m_2^2 + m_3^2)E_{23}^2 .\end{aligned} \tag{X.6.22}$$

However, as we know from (VII.3.30), for $m_i \not\equiv 0$, no term in the sum (X.6.22) is integrable over Ω^R and so (X.6.21) with $q = 2$ can not hold unless $m_i \equiv 0$. In fact, we can say more. Actually, since $\boldsymbol{E}(x)$ tends to zero as $|x| \to \infty$, (X.6.21) cannot hold for *any* of the specified values of q, unless $m_i \equiv 0$. We thus conclude that property (X.6.20) can hold if and only if some restrictions are imposed on the motion itself and which are described by the conditions

$$m_i \equiv - \int_{\partial \Omega} \left[T_{il}(\boldsymbol{u}, p) + \mathcal{R}(\delta_{1l} u_i - u_l u_i) \right] n_l + \mathcal{R} \int_\Omega f_i = 0, \quad i = 1, 2, 3. \tag{X.6.23}$$

From the physical point of view, (X.6.23) means that there is no net external force applied to the "body" Ω^c. This circumstance occurs in the case of steady flow around a body which, for instance, propels itself either by maintaining a momentum flux across portion of its boundary or by moving tangentially portions of its boundary (as by belts). However, the existence theory related to problems of this kind may be completely different than that developed so far for the "classical" problem (X.0.3)–(X.0.4), in that the solution must obey the extra conditions expressed by (X.6.23). As a consequence, one has to introduce another unknown into the problem which, as suggested by physics, can be either the velocity at the boundary, or the (nonzero) velocity at infinity. This type of questions has been considered by several authors. Among others, we refer to Sennitskii (1978, 1984) for flow around symmetric self-propelled bodies, and to Galdi (1999a, 2002) for a general existence and uniqueness theory. The asymptotic behavior of velocity and pressure fields has been investigated in full detail by Pukhnacev (1989).

Another worth of mentioning circumstance where (X.6.23) occurs is the case when $\Omega = \mathbb{R}^3$ and f has zero average on Ω. In such a situation we thus obtain, in particular, that the kinetic energy of the liquid is finite. For this type of problems we refer to the papers of Bjorland & Schonbek (2009), Bjorland, Brandolese, Iftimie & Schonbek (2011), and Silvestre (2009), this latter also considering a more general choice of $\boldsymbol{v}_\infty$.

However, there is also a very significant case where (X.6.23) can *not* hold and, as a consequence, the total kinetic energy of the liquid is infinite. Specifically, consider the situation when Ω is exterior to just one compact body $\mathcal{B}$ (say)[3] and that $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv 0$. Physically, this means that $\mathcal{B}$ is steadily moving into the liquid with velocity $\boldsymbol{v}_\infty$. In such a case (X.6.23) is equivalent to

$$\int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{u}, p) \cdot \boldsymbol{n} = 0. \tag{X.6.24}$$

On the other hand, in Theorem X.7.1 of the following section it will be proved that $\boldsymbol{v}$ and p obey the energy equation

$$\int_{\Omega} \nabla \boldsymbol{v} : \nabla \boldsymbol{v} = \boldsymbol{v}_\infty \cdot \int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{u}, p) \cdot \boldsymbol{n}\,, \tag{X.6.25}$$

and, being $\boldsymbol{v}_\infty \neq 0$, from (X.6.24) and (X.6.25) it follows $\boldsymbol{v} \equiv 0$ in Ω, which gives an absurd conclusion. Therefore, (X.6.24) cannot hold and, consequently, $\boldsymbol{u}$ ($\equiv \boldsymbol{v} + \boldsymbol{v}_\infty$) cannot satisfy (X.6.20) for *any* of the specified values of q, which proves, in particular, that the kinetic energy of the liquid is infinite.

The above considerations are collected in the following.

Theorem X.6.5 *Let $\boldsymbol{v}$ be a generalized solution to the Navier–Stokes problem* (X.0.8), (X.0.4) *in a three-dimensional exterior domain of class C^2 with $\boldsymbol{v}_\infty \neq \boldsymbol{0}$ ($\boldsymbol{v}_\infty = \boldsymbol{e}_1$). Assume that, for some r, $s > 1$*

$$\boldsymbol{v} \in W^{2,r}_{loc}(\overline{\Omega}) \quad \boldsymbol{f} \in L^s(\Omega),$$

with $\boldsymbol{f}$ of bounded support and let $\Omega_R \supset \operatorname{supp}(\boldsymbol{f})$. Then,

$$\boldsymbol{u} \equiv \boldsymbol{v} + \boldsymbol{v}_\infty \in L^q(\Omega^R), \quad \textit{for some } q \in (3/2, 2] \tag{X.6.26}$$

if and only if

$$m_i \equiv -\int_{\partial\Omega} \left[T_{il}(\boldsymbol{u}, p) + \mathcal{R}\left(\delta_{1l} u_i - u_l u_i\right)\right] n_l + \mathcal{R} \int_{\Omega} f_i = 0, \quad i = 1, 2, 3.$$

Moreover, if $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv 0$, it follows that $m_i \not\equiv 0$, and therefore (X.6.26) *can not hold. Thus, in particular, under these latter conditions on the data, the total kinetic energy of the liquid is infinite.*

Exercise X.6.1 Under the assumptions of Theorem X.6.5, prove that if

$$\boldsymbol{v} + \boldsymbol{v}_\infty \in L^q(\Omega), \quad \text{for some } q \in (1, 2], \tag{X.6.27}$$

then $m_i \equiv 0$. Thus, if $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv 0$, (X.6.27) can not hold.

Exercise X.6.2 Let the assumptions of Theorem X.6.5 be satisfied. Suppose, also, that $\boldsymbol{f} \equiv 0$, $\boldsymbol{v}_* \equiv \boldsymbol{v}_0 = \text{const}$. Show that (X.6.27) holds if and only if $\boldsymbol{v} \equiv \boldsymbol{v}_0 \equiv \boldsymbol{e}_1$.

[3] This restriction is, of course, unnecessary. As assumed so far, Ω can be exterior to $s \geq 1$ compact bodies.

X.7 The Energy Equation and Uniqueness for Generalized Solutions when $\boldsymbol{v}_\infty \neq \boldsymbol{0}$

Our objective here is two-fold. On one hand, we show that, under suitable regularity assumption on the data, weak solutions corresponding to $\boldsymbol{v}_\infty \neq \boldsymbol{0}$ satisfy the energy equation, and, on the other hand, that if the data are "sufficiently small", they are unique in their own class.

We begin to give a sufficient condition for the validity of the energy equality.

Theorem X.7.1 *Let Ω be a C^2-smooth exterior three-dimensional domain, and let $\boldsymbol{v}$ be a generalized solution to the Navier–Stokes problem* (X.0.8), (X.0.4) *corresponding to the data*

$$\boldsymbol{f} \in L^{4/3}(\Omega) \cap L^{3/2}(\Omega)\,, \quad \boldsymbol{v}_* \in W^{4/3,3/2}(\partial\Omega)\,, \quad \boldsymbol{v}_\infty \neq \boldsymbol{0}\,. \tag{X.7.1}$$

Then $\boldsymbol{v}$ verifies the energy equation (X.2.29), *where p is the pressure field associated to $\boldsymbol{v}$ by* Lemma X.1.1.

Proof. Under the assumption (X.7.1), from Lemma X.6.1 we have, in particular,

$$\boldsymbol{v} + \boldsymbol{v}_\infty \in L^4(\Omega), \quad \boldsymbol{v}_\infty \cdot \nabla \boldsymbol{v} \in L^{4/3}(\Omega),$$

so that (X.2.11) holds with $q = 4$. Thus, the result follows from Remark X.2.6 and Exercise X.2.2. □

An important corollary to Theorem X.7.1 is the following result of Liouville-type.

Theorem X.7.2 *Let $\boldsymbol{v}$ be a generalized solution to the Navier–Stokes problem* (X.0.8), (X.0.4) *in $\mathbb{R}^3$ corresponding to $\boldsymbol{f} \equiv 0$ and $\boldsymbol{v}_\infty \neq 0$. Then $\boldsymbol{v}(x) = -\boldsymbol{v}_\infty$ for all $x \in \mathbb{R}^3$.*

Remark X.7.1 Theorem X.7.2 necessitates the condition $\boldsymbol{v}_\infty \neq 0$. It is not known if the result continues to be valid when $\boldsymbol{v}_\infty = 0$. In this respect, cf. Remark X.9.4. ■

We shall now investigate the uniqueness problem. To this end, we begin to show that if $\boldsymbol{f}$ (at large distances) and $\boldsymbol{v}_*$ have some property in addition to those required in Theorem X.7.1, and if the data are suitably "small," the corresponding generalized solution satisfies conditions (X.3.3) and (X.3.5).

Lemma X.7.1 *Let the assumptions of* Theorem X.7.1 *be satisfied and assume, in addition, that*

$$\boldsymbol{f} \in L^{6/5}(\Omega)$$

$$\mathcal{R}\,\|\boldsymbol{f}\|_{6/5} + \|\boldsymbol{v}_* + \boldsymbol{v}_\infty\|_{7/6,6/5(\partial\Omega)} < \frac{a_1}{\mathcal{R}} \min\left\{\frac{1}{4c^2}, \frac{\sqrt{3}}{4c}\right\} \tag{X.7.2}$$

where

$$c = c\,(\Omega, B) \quad \textit{for all } \mathcal{R} \in [0, B]$$

and

$$a_1 = \min\{1, \mathcal{R}^{1/2}\}.$$

Then any generalized solution corresponding to $\boldsymbol{f}, \boldsymbol{v}_*$, *and* $\boldsymbol{v}_\infty$ *satisfies* $(\boldsymbol{v} + \boldsymbol{v}_\infty) \in L^3(\Omega)$[1] *along with the inequality*

$$\|\boldsymbol{v} + \boldsymbol{v}_\infty\|_3 < \frac{\sqrt{3}}{2\mathcal{R}}. \tag{X.7.3}$$

Proof. We begin to show that under hypothesis (X.7.2) there exists a generalized solution $\boldsymbol{w}'$, say, verifying the condition

$$\|\boldsymbol{w}' + \boldsymbol{v}_\infty\|_3 < \frac{\sqrt{3}}{2\mathcal{R}} \tag{X.7.4}$$

To this end, we may employ, for example, the method of successive approximations. We introduce a sequence of approximating solutions $\{\boldsymbol{w}_k, \pi_k\}$, defined by $\boldsymbol{w}_0 \equiv \pi_0 \equiv 0$ and, for $k \geq 1$,

$$\left.\begin{aligned} &\Delta \boldsymbol{w}_k + \mathcal{R}\frac{\partial \boldsymbol{w}_k}{\partial x_1} = \mathcal{R}\boldsymbol{w}_{k-1} \cdot \nabla \boldsymbol{w}_{k-1} + \nabla \pi_k + \mathcal{R}\boldsymbol{f} \\ &\nabla \cdot \boldsymbol{w}_k = 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\lim_{|x|\to\infty} \boldsymbol{w}_k(x) = 0,$$
$$\boldsymbol{w}_k = \boldsymbol{v}_* + \boldsymbol{v}_\infty \ \text{ at } \partial\Omega. \tag{X.7.5}$$

By Theorem VII.7.1 with $m = 0$ and $q = 6/5$, we know that, for $k = 1$, there is a solution $\boldsymbol{w}_1, \pi_1$ such that

$$\boldsymbol{w}_1 \in L^3(\Omega) \cap D^{1,12/7}(\Omega) \cap D^{2,6/5}(\Omega),$$
$$\pi_1 \in D^{1,6/5}(\Omega),$$

and obeying the estimate

$$\begin{aligned} a_1\|\boldsymbol{w}_1\|_3 + a_2|\boldsymbol{w}_1|_{1,12/7} + |\boldsymbol{w}_1|_{2,6/5} + |\pi_1|_{1,6/5} \\ \leq c_1\left(\mathcal{R}\|\boldsymbol{f}\|_{6/5} + \|\boldsymbol{v}_* - \boldsymbol{v}_\infty\|_{7/6,6/5(\partial\Omega)}\right) \\ \equiv c_1\,\mathcal{D}, \end{aligned} \tag{X.7.6}$$

where $c_1 = c_1(q, \Omega)$ is independent of $\mathcal{R}$ for $\mathcal{R} \in (0, B]$ and

[1] Observe that this property follows from the assumptions on the data and Lemma X.6.1.

$$a_1 = \min\{1, \mathcal{R}^{1/2}\}, \quad a_2 = \min\{1, \mathcal{R}^{1/4}\}.$$

Since, by Theorem II.6.1,

$$|\boldsymbol{w}_1|_{1,2} \leq c_2 |\boldsymbol{w}_1|_{2,6/5}, \tag{X.7.7}$$

(X.7.6) furnishes, in particular,

$$a_1 \|\boldsymbol{w}_1\|_3 + |\boldsymbol{w}_1|_{1,2} + |\boldsymbol{w}_1|_{2,6/5} + |\pi_1|_{1,6/5} \leq c\mathcal{D}, \tag{X.7.8}$$

with $c = c(\Omega, B)$ independent of $\mathcal{R} \in (0, B]$. We next show, by induction, that the following inequality is verified for all $k \in \mathbb{N}$

$$a_1 \|\boldsymbol{w}_k\|_3 + |\boldsymbol{w}_k|_{1,2} + |\boldsymbol{w}_k|_{2,6/5} + |\pi_k|_{1,6/5} \leq 2c\mathcal{D}, \tag{X.7.9}$$

provided $\mathcal{D}$ is "small enough." Thus, assuming $\boldsymbol{w}_k, \pi_k$ obey (X.7.9), by Theorem VII.7.1 we recover

$$\begin{aligned} a_1 \|\boldsymbol{w}_{k+1}\|_3 + |\boldsymbol{w}_{k+1}|_{1,2} + |\boldsymbol{w}_{k+1}|_{2,6/5} + |\pi_{k+1}|_{1,6/5} & \\ \leq c \left(\mathcal{D} + \mathcal{R}\|\boldsymbol{w}_k \cdot \nabla \boldsymbol{w}_k\|_{6/5}\right). & \end{aligned} \tag{X.7.10}$$

Now we have

$$\|\boldsymbol{w}_k \cdot \nabla \boldsymbol{w}_k\|_{6/5} \leq \|\boldsymbol{w}_k\|_3 |\boldsymbol{w}_k|_{1,2}$$

and, by induction hypothesis,

$$\|\boldsymbol{w}_k \cdot \nabla \boldsymbol{w}_k\|_{6/5} \leq 4c^2\mathcal{D}^2/a_1,$$

so that (X.7.10) shows that if

$$\mathcal{D} < a_1/4c^2\mathcal{R}, \tag{X.7.11}$$

then inequality (X.7.9) is satisfied for all $k \in \mathbb{N}$. It is now easy to prove that $\{\boldsymbol{w}_k, \pi_k\}$ is a Cauchy sequence in the space

$$\mathcal{S} \equiv \left(L^3(\Omega) \cap \dot{D}^{1,2}(\Omega) \cap \dot{D}^{2,6/5}(\Omega)\right) \times \dot{D}^{1,6/5}(\Omega).$$

In fact, from (X.7.5) we deduce that

$$\begin{aligned} a_1 \|\boldsymbol{w}_{k+1} - \boldsymbol{w}_k\|_3 + |\boldsymbol{w}_{k+1} - \boldsymbol{w}_k|_{1,2} + |\boldsymbol{w}_{k+1} - \boldsymbol{w}_k|_{2,6/5} + |\pi_{k+1} - \pi_k|_{1,6/5} & \\ \leq c\mathcal{R}\, \|\boldsymbol{w}_k \cdot \nabla \boldsymbol{w}_k - \boldsymbol{w}_{k-1} \cdot \nabla \boldsymbol{w}_{k-1}\|_{6/5}, & \end{aligned}$$

and since

$$\begin{aligned} \|\boldsymbol{w}_k \cdot \nabla \boldsymbol{w}_k - \boldsymbol{w}_{k-1} \cdot \nabla \boldsymbol{w}_{k-1}\|_{6/5} \leq \|\boldsymbol{w}_k - \boldsymbol{w}_{k-1}\|_3 |\boldsymbol{w}_k|_{1,2} & \\ + \|\boldsymbol{w}_k\|_3 |\boldsymbol{w}_k - \boldsymbol{w}_{k-1}|_{1,2}\,, & \end{aligned}$$

in view of (X.7.9) we conclude, for all $k \geq 1$,

$$
\begin{aligned}
a_1\|\boldsymbol{w}_{k+1}-\boldsymbol{w}_k\|_3 &+ |\boldsymbol{w}_{k+1}-\boldsymbol{w}_k|_{1,2}\\
&+ |\boldsymbol{w}_{k+1}-\boldsymbol{w}_k|_{2,6/5} + |\pi_{k+1}-\pi_k|_{1,6/5}\\
&\le (4c^2\mathcal{R}\mathcal{D}/a_1)(a_1\|\boldsymbol{w}_k-\boldsymbol{w}_{k-1}\|_3 + |\boldsymbol{w}_k-\boldsymbol{w}_{k-1}|_{1,2}).
\end{aligned}
$$

From this inequality we receive, for all $k \ge 1$,

$$
\begin{gathered}
a_1\|\boldsymbol{w}_{k+1}-\boldsymbol{w}_k\|_3 + |\boldsymbol{w}_{k+1}-\boldsymbol{w}_k|_{1,2}\\
+|\boldsymbol{w}_{k+1}-\boldsymbol{w}_k|_{2,6/5} + |\pi_{k+1}-\pi_k|_{1,6/5} \le (4c^2\mathcal{R}\mathcal{D}/a_1)^{k+1},
\end{gathered}
$$

which, by (X.7.11) and virtue of a standard argument, implies that the sequence $\{\boldsymbol{w}_k, \pi_k\}$ is a Cauchy sequence in the space $\mathcal{S}$. Denoting by $\boldsymbol{w}, \pi$ the limiting field, we then have

$$
\begin{gathered}
\boldsymbol{w} \in L^3(\Omega) \cap D^{1,2}(\Omega) \cap D^{2,6/5}(\Omega)\\
\pi \in D^{1,6/5}(\Omega)
\end{gathered}
$$

and, moreover, $\boldsymbol{w}, \pi$ obey the estimate (X.7.9). In addition, setting

$$
\boldsymbol{w}' = \boldsymbol{w} + \boldsymbol{v}_\infty, \tag{X.7.12}
$$

and recalling that, in particular, $\boldsymbol{w}$ is a generalized solution, from Lemma X.6.1 we find

$$
\boldsymbol{w}' + \boldsymbol{v}_\infty \in L^4(\Omega)\,, \quad \boldsymbol{v}_\infty \cdot \nabla \boldsymbol{w}' \in L^{4/3}(\Omega) \tag{X.7.13}
$$

and, if the data satisfy $(\text{X.7.2})_2$, we also have that $\boldsymbol{w}'$ obeys (X.7.4). Now let $\boldsymbol{v}$ denote *any* generalized solution corresponding to $\boldsymbol{f}$, $\boldsymbol{v}_*$, and $\boldsymbol{v}_\infty$. From Lemma X.6.1 we find

$$
\begin{gathered}
\boldsymbol{v} + \boldsymbol{v}_\infty \in L^4(\Omega)\\
\boldsymbol{v}_\infty \cdot \nabla \boldsymbol{v} \in L^{4/3}(\Omega)
\end{gathered} \tag{X.7.14}
$$

and, by Theorem X.7.1, we obtain that

$$
\boldsymbol{v} \text{ verifies the generalized energy equality (X.2.28)} \tag{X.7.15}
$$

for any extension $\boldsymbol{A}$ of $\boldsymbol{v}_*$ and $\boldsymbol{v}_\infty$.[2] Observing that $\boldsymbol{f} \in L^{6/5}(\Omega)$ implies $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$ (as a consequence of Sobolev inequality (II.3.11)), the uniqueness Theorem X.3.1, together with (X.7.12)–(X.7.15)[3] and (X.7.4), furnishes $\boldsymbol{w} = \boldsymbol{v}$ *a.e.* in Ω. The lemma is proved. □

From Lemma X.7.1 and Theorem X.7.1 we immediately obtain the following *uniqueness theorem for generalized solutions corresponding to* $\boldsymbol{v}_\infty \neq \boldsymbol{0}$.

---

[2] Actually, $\boldsymbol{v}$ satisfies the energy equation (X.2.29) in its classical form.
[3] See footnote 1 in this section.

Theorem X.7.3 *Let Ω be a three-dimensional exterior domain of class C^2. Further, let*

$$\boldsymbol{f} \in L^{6/5}(\Omega) \cap L^{4/3}(\Omega), \quad \boldsymbol{v}_* \in W^{5/4,4/3}(\partial\Omega), \quad \boldsymbol{v}_\infty \neq \boldsymbol{0}.$$

Then if $(\text{X.7.2})_2$ *is satisfied,* $\boldsymbol{v}$ *is the only generalized solution achieving these data.*

X.8 The Asymptotic Structure of Generalized Solutions when $\boldsymbol{v}_\infty \neq \boldsymbol{0}$

In this section we conclude the study of the asymptotic behavior of generalized solutions with $\boldsymbol{v}_\infty \neq \boldsymbol{0}$, by showing that they have the same structure as the corresponding solutions to the Oseen problem. In particular, they exhibit a "downstream" paraboloidal wake region and the rate of convergence of the velocity field $\boldsymbol{v}$ to $-\boldsymbol{v}_\infty$ is more rapid outside the wake than inside.

For the reader's convenience, we collect the estimates on the tensor field $\boldsymbol{E}$ derived in (VII.3.24), (VII.3.32), and Exercise VII.3.1, which will be frequently used in the sequel:

$$\begin{aligned} |E_{ij}(x-y)| &\leq c\,|x-y|^{-1} \\ |\nabla E_{ij}(x-y)| &\leq c\,|x-y|^{-3/2} \\ \int_{\partial B_R(x)} |\nabla E_{ij}(x-y)| &\leq c\,R^{-1/2}. \end{aligned} \tag{X.8.1}$$

Our first objective is to recover an appropriate uniform estimate for $\boldsymbol{u}(x) = \boldsymbol{v}(x)+\boldsymbol{v}_\infty$, for large values of $|x|$, where, as usual and without loss of generality, we take $\boldsymbol{v}_\infty = \boldsymbol{e}_1$. For simplicity, we shall assume that the body force $\boldsymbol{f}$ is of bounded support. Let Ω denote, temporarily, the exterior of B_ρ, where ρ is taken large enough to satisfy $B_\rho \supset \operatorname{supp}(\boldsymbol{f})$. By Theorem X.1.1 we then have that $\boldsymbol{v}$, $p \in C^\infty(\Omega)$ and that $\boldsymbol{v}$, p enjoy the summability properties proved in Theorem X.6.4. From the asymptotic formulas (X.5.50)–(X.5.52) we have

$$\boldsymbol{u}(x) = \boldsymbol{N}[\boldsymbol{u}(x)] + O(1/|x|) \;\text{ as } |x| \to \infty \tag{X.8.2}$$

where

$$N_j = (\boldsymbol{N}[\boldsymbol{u}(x)])_j = \int_\Omega E_{ij}(x-y)u_l(y)D_l u_i(y)dy.$$

We wish to give a uniform estimate for $\boldsymbol{N}$. To this end, setting $|x| = 2R$, we may write

$$\begin{aligned} N_j &= \int_{\Omega_R} E_{ij}(x-y)u_l(y)D_l u_i(y)dy + \int_{\Omega^R} E_{ij}(x-y)u_l(y)D_l u_i(y)dy \\ &\equiv \mathcal{N}_1 + \mathcal{N}_2. \end{aligned} \tag{X.8.3}$$

Recalling $(\mathrm{X.8.1})_1$, it follows that

$$|\mathcal{N}_1| \le \frac{c}{R}\|\boldsymbol{u}\|_{3,\Omega_R}|\boldsymbol{u}|_{1,3/2,\Omega_R} \le \frac{2c}{|x|}\|\boldsymbol{u}\|_3|\boldsymbol{u}|_{1,3/2}.$$

This inequality and Theorem X.6.4 yield

$$|\mathcal{N}_1| \le c_1|x|^{-1}. \tag{X.8.4}$$

Furthermore, from Theorem II.6.1 (cf. (II.6.20)) and $(\mathrm{X.8.1})_1$ we also have

$$|\mathcal{N}_2| \le \left(\int_{\Omega^R} \frac{u^2}{|x-y|^2}\right)^{1/2} \left(\int_{\Omega^R} \nabla\boldsymbol{u} : \nabla\boldsymbol{u}\right)^{1/2} \le c\int_{\Omega^R} \nabla\boldsymbol{u} : \nabla\boldsymbol{u} \tag{X.8.5}$$

with c_2 independent of $\boldsymbol{u}$ and R. Setting

$$G(R) \equiv \int_{\Omega^R} \nabla\boldsymbol{u} : \nabla\boldsymbol{u} \tag{X.8.6}$$

from (X.8.2)–(X.8.5) we deduce that estimating the nonlinear term $\boldsymbol{N}[\boldsymbol{u}(x)]$ is reduced to estimating the functions $G(R)$, $R = |x|/2$. This latter question will be analyzed in the next two lemmas. We begin with a simple but very useful result.

Lemma X.8.1 *Suppose that for all* $t > a \ge 0$,

(i) $y \in C^1([a,t])$,
(ii) $y(t) \ge 0$,
(iii) $y'(t) \le 0$,

and that, for some $\beta \in [0,1)$,

$$\int_a^\infty y(s)s^{-\beta}ds < \infty.$$

Then, for all $t \ge a$,

$$y(t)t^{1-\beta} \le (1-\beta)\int_a^\infty y(s)s^{-\beta}ds + y(a)a^{1-\beta}.$$

Proof. The assertion is an immediate consequence of the identity

$$y(t)t^{1-\beta} = \int_a^t \frac{d}{ds}[y(s)s^{1-\beta}]ds + y(a)a^{1-\beta}$$

and of the assumptions made on y. □

Lemma X.8.1 allows us to prove the following estimate for $G(R)$.

Lemma X.8.2 *Let $\boldsymbol{v}$ be a generalized solution to the Navier–Stokes problem in $\Omega = \mathbb{R}^3 - B_\rho$ corresponding to $\boldsymbol{f} = 0$ and $\boldsymbol{v}_\infty \neq \boldsymbol{0}$. Then, setting $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{v}_\infty$, for all $R > \rho$, it holds that*

$$G(R) \leq cR^{-1+\varepsilon} \tag{X.8.7}$$

where G is defined in (X.8.6), *ε is an arbitrary positive number and c is independent of R.*

Proof. As usual, we assume $\boldsymbol{v}_\infty = \boldsymbol{e}_1$. We recall that $\boldsymbol{v}$ and the corresponding pressure p are in $C^\infty(\Omega)$. Multiplying (X.0.8) by $\boldsymbol{u}$ and integrating over Ω_{R,R^*}, ($R^* > R$), it follows that

$$\int_{\Omega_{R,R^*}} \nabla \boldsymbol{u} : \nabla \boldsymbol{u} = \int_{\partial B_R \cup \partial B_{R^*}} \left\{ \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial n} - \frac{\mathcal{R}}{2} u^2 \boldsymbol{v} \cdot \boldsymbol{n} - p(\boldsymbol{u} \cdot \boldsymbol{n}) \right\}. \tag{X.8.8}$$

From Theorem X.6.4 we derive, in particular,

$$\Psi \equiv |\boldsymbol{u}|^3 + |\nabla \boldsymbol{u}|^{3/2} + |\boldsymbol{u}|^{5/2} + |p|^{5/3} \in L^1(\Omega). \tag{X.8.9}$$

Therefore, there exists a sequence $\{R_k\}$ tending to infinity as k tends to infinity, such that

$$\int_{\partial B_{R_k}} \Psi(R_k, \omega) d\omega = o(R_k^{-1}). \tag{X.8.10}$$

Using Young's inequality several times and the Hölder inequality, we deduce (with r denoting either R or R^*)

$$\begin{aligned} F(r) &\equiv \int_{\partial B_r} \left[\boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial n} - \frac{\mathcal{R}}{2} u^2 \boldsymbol{v} \cdot \boldsymbol{n} - p(\boldsymbol{u} \cdot \boldsymbol{n}) \right] \\ &\leq c \left\{ \int_{\partial B_r} \left[u^3 + |\nabla \boldsymbol{u}|^{3/2} + u^{5/2} + p^{5/3} \right] + r^{2/q'} \left(\int_{\partial B_r} u^{2q} \right)^{1/q} \right\} \end{aligned} \tag{X.8.11}$$

where q is an arbitrary number greater than one. Taking in (X.8.11) $r \equiv R^* = R_k$, $q = 3/2$ and using (X.8.9), (X.8.10), it follows that

$$\lim_{k \to \infty} F(R_k) = 0,$$

and(X.8.8) furnishes

$$G(R) = F(R). \tag{X.8.12}$$

For any $\varepsilon \in (0, 1)$, by Young's inequality,

$$H(R) \equiv R^{-\varepsilon} R^{2/q'} \left(\int_{\partial B_R} u^{2q} \right)^{1/q} \leq c \left(R^{-\varepsilon q' + 2} + \int_{\partial B_R} u^{2q} \right)$$

and so, taking $q < 3/(3-\varepsilon)$, and recalling that, by Theorem X.6.4, $\boldsymbol{u} \in L^s(\Omega)$ for all $s > 2$, we have

$$H \in L^1(\rho, \infty). \tag{X.8.13}$$

Collecting (X.8.9), (X.8.11), and (X.8.13) yields

$$R^{-\varepsilon} F(R) \in L^1(\rho, \infty) \tag{X.8.14}$$

and, since

$$G'(R) = -\int_{\partial B_R} \nabla \boldsymbol{u} : \nabla \boldsymbol{u} < 0, \tag{X.8.15}$$

from (X.8.12), (X.8.14), and (X.8.15), with the help of Lemma X.8.1, we obtain (X.8.7), and the proof is complete. □

Using (X.8.3)–(X.8.5), together with the results of Lemma X.8.2, we then have the following uniform estimate for the nonlinear term:

$$|\boldsymbol{N}[\boldsymbol{u}(x)]| \leq c|x|^{-1+\varepsilon}, \quad \text{for any } \varepsilon \in (0,1], \tag{X.8.16}$$

for some $c = c(\varepsilon)$ ($c \to \infty$ as $\varepsilon \to 0$). From (X.8.2) and (X.8.16) we thus obtain the following.

Lemma X.8.3 *Let $\boldsymbol{v}$ be a generalized solution to the Navier–Stokes problem in a three-dimensional exterior domain Ω with $\boldsymbol{f}$ of bounded support and $\boldsymbol{v}_\infty \neq 0$. Then, for all sufficiently large $|x|$, the following uniform estimate holds:*

$$\boldsymbol{v}(x) + \boldsymbol{v}_\infty = O(1/|x|^{1-\varepsilon}), \quad \textit{for any } \varepsilon \in (0,1].$$

With this result in hand, we can now show the following theorem which furnishes the asymptotic structure of any generalized solution corresponding to a body force of bounded support.

Theorem X.8.1 *Let $\boldsymbol{v}$ be a generalized solution to the Navier–Stokes problem* (X.0.8)–(X.0.4) *in a three-dimensional exterior domain of class C^2, corresponding to $\boldsymbol{v}_\infty \neq \boldsymbol{0}$, $\boldsymbol{f}$ of bounded support in Ω, and, moreover, with*

$$\boldsymbol{f} \in L^q(\Omega), \quad \boldsymbol{v}_* \in W^{2-1/q_0,1/q_0}(\partial\Omega),$$

for some $q_0 > 3$ and all $q \in (1, q_0]$. Then for all sufficiently large $|x|$, $\boldsymbol{v}$ admits the following representation:

$$\boldsymbol{v}(x) + \boldsymbol{v}_\infty = \boldsymbol{m} \cdot \boldsymbol{E}(x) + \boldsymbol{\mathcal{V}}(x) \tag{X.8.17}$$

where $\boldsymbol{E}$ is the Oseen fundamental tensor,

$$m_i = -\int_{\partial\Omega} [T_{il}(\boldsymbol{u}, p) + \mathcal{R}\,(\delta_{1l} u_i - u_i u_\ell)] n_l + \mathcal{R} \int_\Omega f_i, \tag{X.8.18}$$

with $\boldsymbol{u} := \boldsymbol{v} + \boldsymbol{v}_\infty$, and $\boldsymbol{\mathcal{V}}(x)$ satisfies the estimate

$$\boldsymbol{\mathcal{V}}(x) = O(|x|^{-3/2+\delta}) \quad \textit{for any } \delta > 0. \tag{X.8.19}$$

Proof. In view of Theorem X.5.3, formula (X.5.50)$_2$, it suffices to show the uniform estimate

$$\left| \int_\Omega u_l(y) u_i(y) D_l E_{ij}(x-y)dy \right| = O(|x|^{-3/2+\delta}). \tag{X.8.20}$$

Setting $|x| = R$ sufficiently large, we divide the region Ω into three parts:

$$\Omega_{R/2}, \quad \Omega_{R/2,3R}, \quad \Omega^{3R}$$

and denote the corresponding integrals over these regions by I_1, I_2, and I_3, respectively. Using (X.8.1)$_2$ and the Hölder inequality, we find

$$|I_1| \le \frac{c}{R^{3/2}} \int_{\Omega_{R/2}} u^2 \le c_1 R^{3(1/q'-1/2)} \|\boldsymbol{u}\|_{2q}^2$$

and so, choosing $q = 3/(3-\delta)$, from Theorem X.6.4 it follows that

$$|I_1| \le c_2 |x|^{-3/2+\delta}. \tag{X.8.21}$$

Furthermore, recalling Lemma X.8.3, we have

$$|I_2| \le c_3 R^{-2+2\varepsilon} \int_{\Omega_{R/2,3R}} |\boldsymbol{E}(x-y)|\, dy \le c_3 R^{-2+2\varepsilon} |\boldsymbol{E}|_{1,1,B_{4R}(x)}$$

and so, in view of (X.8.1)$_3$, choosing $\varepsilon = \delta/2$, we recover

$$|I_2| \le c_4 |x|^{-3/2+\delta}. \tag{X.8.22}$$

Finally, from Lemma X.8.3 and an obvious majoration it follows that

$$|I_3| \le c_5 \int_{\Omega^{3R}} |y|^{-2+2\varepsilon} |\nabla \boldsymbol{E}(x-y)| dy \le c_5 \int_{B^{2R}(x)} |y|^{-2+2\varepsilon} |\nabla \boldsymbol{E}(x-y)| dy\,,$$

where we have used the fact that $|x| = R$. Now, for any $y \in B^{2R}(x)$ we find $|x-y| \le |y| + R \le 2|y|$, so that the above inequality leads to the following one

$$|I_3| \le c_6 \int_{B^{2R}(x)} |x-y|^{-2+2\varepsilon} |\nabla \boldsymbol{E}(x-y)| dy\,,$$

which, in turn, by (X.8.1)$_3$ furnishes

$$|I_3| \le c_7 \int_{2R}^\infty r^{-2+2\varepsilon} r^{-1/2} dr \le c_8 R^{-3/2+2\varepsilon}\,.$$

This latter, with the choice $\varepsilon = \delta/2$, implies

$$|I_3| \le c_9 |x|^{-3/2+\delta}. \tag{X.8.23}$$

The validity of (X.8.20) is a consequence of (X.8.21)–(X.8.23) and the theorem is proved. $\square$

Remark X.8.1 From (X.8.17) and the properties of the tensor field $\boldsymbol{E}$ (cf. (VII.3.24)–(VII.3.26)), we deduce, in particular, that any generalized solution of Theorem X.8.1 exhibits a paraboloidal wake region in the direction of $\boldsymbol{v}_\infty (= \boldsymbol{e}_1)$. Specifically, for all sufficiently large $|x|$ we have the uniform estimate

$$\boldsymbol{v}(x) + \boldsymbol{v}_\infty = O(1/|x|). \tag{X.8.24}$$

On the other hand, denoting by φ the angle made by a ray starting from the origin (in Ω^c, say) with the negatively directed x_1-axis, for all x satisfying[1]

$$|x|(1+\cos\varphi) \geq |x|^{2\sigma}, \quad \sigma \in (0, 1/2], \tag{X.8.25}$$

we have

$$\boldsymbol{v}(x) + \boldsymbol{v}_\infty = O(1/|x|^{1+\alpha}) \tag{X.8.26}$$

where

$$\alpha = \min(2\sigma, 1/2 - \delta). \tag{X.8.27}$$

Relations (X.8.24)–(X.8.27) then show the mentioned behavior of $\boldsymbol{v}$. Comparing (X.8.26) with the analogous estimate for $\boldsymbol{E}$ given in (VII.3.26) or, what amounts to the same thing, with the estimate for the velocity field of the corresponding Oseen linearized problem given by (VII.6.18), we recognize that (X.8.26) is apparently weaker, because we cannot take $\alpha = 2\sigma$, for $\sigma \geq 1/4$. This latter circumstance is due to the fact that, in the proof of Theorem X.8.1, we have given a *uniform* bound for the nonlinear term

$$\boldsymbol{N}(x) \equiv \int_\Omega u_l(y) u_i(y) D_l E_{ij}(x-y) dy.$$

However, also for $\boldsymbol{N}$, we can prove estimates whose decay order is faster than (X.8.21) if x belongs to the region $\mathcal{R}_\sigma$ described by (X.8.25). This will, in turn, furnish more accurate estimates for the remainder $\boldsymbol{\mathcal{V}}(x)$, defined in (X.8.17), for $x \in \mathcal{R}_\sigma$. For example, Finn (1959b, Theorem 8) has shown the following asymptotic bound for $\boldsymbol{\mathcal{V}}(x)$:

$$\boldsymbol{\mathcal{V}}(x) = O\left(|x|^{-3/2+\delta-2\sigma}\right), \quad \text{for all } x \in \mathcal{R}_\sigma.$$

Sharper estimates for $\boldsymbol{\mathcal{V}}(x)$ can be found in the work of Finn (1965a, Theorem 5.1) and Vasil'ev (1973) and, under suitable restriction on $\boldsymbol{m}$, in that of Babenko & Vasil'ev (1973). ■

An immediate consequence of Theorem X.8.1 is that the uniform estimate (X.8.24) is sharp in the sense specified by the following result, whose first formulation traces back to the work of Udeschini (1941), Berker (1952), and Finn (1959b).[2]

[1] See Remark IX.3.1.
[2] See also Exercise X.8.1.

Corollary X.8.1 *Let the assumptions of* Theorem X.8.1 *be satisfied for some* $\boldsymbol{v}_\infty \neq 0$. *Then*

$$\boldsymbol{v} + \boldsymbol{v}_\infty = o\,(1/|x|)$$

if and only if the vector $\boldsymbol{m}$ *defined in* (X.8.18) *is zero.*[3]

Proof. We can take, as usual, $\boldsymbol{v}_\infty = \boldsymbol{e}_1$. By Theorem X.8.1 we have at once that, if $\boldsymbol{m}$ is zero, then $\boldsymbol{v}+\boldsymbol{v}_\infty$ satisfies the stated property. Conversely, assume that such a property holds. On the ray $x_2 = x_3 = 0$, $x_1 > 0$, from (VII.3.20),

$$E_{21}(x) = E_{31}(x) = 0 \quad |E_{11}(x)| > c|x|^{-1}$$

and so, by Theorem X.8.1 it follows that $m_1 = 0$. A similar argument proves, in turn, $m_2 = m_3 = 0$, and the corollary follows. □

Exercise X.8.1 Let the assumptions of Corollary X.8.1 hold. Suppose, further, that $\boldsymbol{f} \equiv 0$ and $\boldsymbol{v}_* \equiv \boldsymbol{v}_0 =$ const. Show that $\boldsymbol{v} + \boldsymbol{v}_\infty = o(1/|x|)$ if and only if $\boldsymbol{v} \equiv \boldsymbol{v}_0 \equiv -\boldsymbol{v}_\infty$.

Let us now turn our attention to the behavior at infinity of the first derivatives of the velocity field. Our starting point is again the representation $(\text{X.5.50})_2$. However, we cannot differentiate this formula under the sign of integration because of the singularity of the term $D_l E_{ij}(x-y)$. Nevertheless, observing that

$$\begin{aligned}
&\int_\Omega u_l(y) D_l u_i(y)\; D_k E_{ij}(x-y)dy \\
&\qquad = -\int_{\Omega - B_1(x)} u_l(y) u_i(y) D_l D_k E_{ij}(x-y) dy \\
&\qquad\quad + \int_{B_1(x)} D_k E_{ij}(x-y) u_l(y) D_l u_i(y) dy \\
&\qquad\quad + \int_{\partial B_1(x)} D_k E_{ij}(x-y) u_l(y) u_i(y) n_l(y) d\sigma_y \\
&\qquad\quad + \int_{\partial\Omega} D_k E_{ij}(x-y) u_l(y) u_i(y) n_l(y) d\sigma_y
\end{aligned}$$

we may use this identity in $(\text{X.5.50})_1$ to find

$$D_k u_j(x) = m_i D_k E_{ij}(x) + D_k s_j^{(2)}(x) - \mathcal{R}\left(N_j(x) + \mathcal{I}_j(x) - \imath_j(x)\right) \quad \text{(X.8.28)}$$

where

[3] Compare this result with that of Theorem X.6.5.

$$N_j(x) = \int_{\Omega - B_1(x)} u_l(y)u_i(y)D_kD_lE_{ij}(x-y)dy$$

$$\mathcal{I}_j(x) = \int_{B_1(x)} D_kE_{ij}(x-y)u_l(y)D_lu_i(y)dy$$

$$\imath_j(x) = \int_{\partial B_1(x)} D_kE_{ij}(x-y)u_l(y)u_i(y)n_l(y)dy$$

and $\boldsymbol{s}^{(2)}$ satisfies $(\text{X.5.52})_{1,2}$. Clearly, in view of (VII.3.32) and $(\text{X.5.52})_1$ we need to estimate only the term in brackets in (X.8.28). To this end, we recall the following bounds (cf. (VII.3.35) and Exercise VII.3.1),

$$\begin{gathered}|D_kD_l\boldsymbol{E}(x-y)| \le c\,|x-y|^{-2}\\ \int_{\partial B_R(x)} |D_kD_l\boldsymbol{E}(x-y)| \le c\,R^{-1}.\end{gathered} \tag{X.8.29}$$

Setting $|x| = R$ and splitting N_j as the sum of three integrals, n_1, n_2, and n_3, over $\Omega_{R/2}$, $\Omega_{R/2,2R} - B_1(x)$, and Ω^{2R}, respectively, from $(\text{X.8.29})_1$ and Theorem X.6.4 we deduce

$$|n_1| \le \frac{1}{R^2}\int_{\Omega_{R/2}} u^2 \le cR^{3/q'-2}\|u\|_{2q}^2 \le c_1|x|^{-2+\varepsilon} \tag{X.8.30}$$

where ε is a positive number that can be taken arbitrarily close to zero (at the cost, of course, of increasing the value of c_1). The other two terms, n_2 and n_3, are estimated exactly as the integrals I_2 and I_3 introduced in the proof of Theorem X.8.1 (cf. (X.8.22), (X.8.23)) using, this time, $(\text{X.8.29})_2$ in place of $(\text{X.8.1})_3$. We then obtain

$$|n_2| + |n_3| \le c_2|x|^{-2+\varepsilon}. \tag{X.8.31}$$

Moreover, bearing in mind that $\boldsymbol{v}$ satisfies the uniform estimate (X.8.24), from $(\text{X.8.1})_2$ we recover

$$|\imath_j| \le c_3|x|^{-2}. \tag{X.8.32}$$

It remains to give an upper bound to $\mathcal{I}_j(x)$. To this purpose, we notice that from Lemma VII.6.3 it follows for all sufficiently large $|x|$

$$D_ku_j(x) = -\int_{B_d(x)}\Big(u_l(y)D_lu_i(y)D_kE_{ij}^{(d)}(x-y) - u_i(y)D_k\mathcal{H}_{ij}^{(d)}(x-y)\Big)dy \tag{X.8.33}$$

and since, by Theorem X.5.1,

$$|\nabla \boldsymbol{u}(x)| \le M$$

for sufficiently large $|x|$ and with M independent of x, we recover

$$|\nabla \boldsymbol{u}(x)| \leq c_4 |x|^{-1}. \tag{X.8.34}$$

Thus, from $(\text{X.8.1})_2$, (X.8.24), and (X.8.34) we conclude that

$$|\mathcal{I}_j(x)| \leq c_5 |x|^{-2}. \tag{X.8.35}$$

Therefore, identity (X.8.28) along with (VII.3.32), $(\text{X.5.52})_1$, (X.8.30)–(X.8.32), and (X.8.35), allows us to deduce the following theorem.

Theorem X.8.2 *Let the assumptions of* Theorem X.8.1 *hold. Then for all sufficiently large* $|x|$, $D_k\boldsymbol{v}(x)$, $k = 1, 2, 3$, *admits the following representation:*

$$D_k\boldsymbol{v}(x) = \boldsymbol{m} \cdot D_k\boldsymbol{E}(x) + \boldsymbol{\mathcal{D}}_k(x) \tag{X.8.36}$$

where $\boldsymbol{E}$ *is the Oseen fundamental tensor,* $\boldsymbol{m}$ *is given in* (X.8.18) *and* $\boldsymbol{\mathcal{D}}_k(x)$ *satisfies the estimate*

$$\boldsymbol{\mathcal{D}}_k(x) = O(|x|^{-2+\varepsilon}), \tag{X.8.37}$$

where the positive number ε *can be taken arbitrarily close to zero.*

Remark X.8.2 Taking into account the asymptotic properties of $D_k\boldsymbol{E}(x)$, cf. (VII.3.31), (VII.3.32), from (X.8.28) it is possible to derive sharper estimates for the gradient of the velocity, according to whether we are or are not in the paraboloidal wake region. In this regard, it should be observed that the uniform estimate (X.8.37) can also be slightly improved. For example, Finn (1959b, Theorem 9) has proved the following bound

$$|\boldsymbol{\mathcal{D}}_k(x)| \leq \begin{cases} c\,|x|^{-2}, & \text{uniformly in } x \in \Omega \\ c\,|x|^{-2-4\overline{\sigma}}, & x \text{ in the region (X.8.25)}, \end{cases} \tag{X.8.38}$$

for all $\overline{\sigma} < \sigma$. For more accurate bounds, we refer the reader to Finn (1965a, Theorem 5.3) and to Babenko & Vasil'ev (1973, Section 3.3). ■

Remark X.8.3 It should be emphasized that the method adopted in the proof of Theorem X.8.2 does not apply to higher-order derivatives in the sense that it does not furnish for such derivatives improved bounds at large distances. For example, consider $D^2\boldsymbol{v}(x)$. We would expect that as $|x| \to \infty$

$$D^2\boldsymbol{v}(x) = \boldsymbol{m} \cdot D^2\boldsymbol{E}(x) + O(|x|^{-2-\alpha}),$$

for some $\alpha > 0$. Now, differentiating twice $(\text{X.5.50})_1$ and suitably manipulating the volume integral as we did in the proof of Theorem X.8.2, we obtain the following relation

$$
\begin{aligned}
D_k D_s u_j(x) = & m_i D_k D_s E_{ij}(x) + D_k D_s s_j^{(2)} \\
& -\mathcal{R} \int_{\Omega - B_1(x)} u_l(y) u_i(y) D_l D_k D_s E_{ij}(x-y) dy \\
& +\mathcal{R} \int_{\partial B_1(x)} u_l(y) D_l u_i(y) D_s E_{ij}(x-y) n_k(y) d\sigma_y \\
& +\mathcal{R} \int_{B_1(x)} D_k (u_l(y) D_l u_i(y)) D_s E_{ij}(x-y) dy \\
& -\mathcal{R} \int_{\partial B_1(x)} u_l(y) u_i(y) D_k D_s E_{ij}(x-y) n_l(y) d\sigma_y .
\end{aligned}
\tag{X.8.39}
$$

Using Lemma VII.6.3 and the estimate

$$
|\nabla \boldsymbol{u}(x)| = O(|x|^{-3/2}), \tag{X.8.40}
$$

we can show

$$
|D^2 \boldsymbol{u}(x)| = O(|x|^{-3/2}). \tag{X.8.41}
$$

Employing (X.8.40), (X.8.41) into (X.8.39) together with the asymptotic bounds for $\boldsymbol{E}$ and $\boldsymbol{s}^{(2)}$, and recalling that

$$
\boldsymbol{v} + \boldsymbol{v}_\infty = O(|x|^{-1}), \tag{X.8.42}
$$

we can show

$$
\begin{aligned}
D_k D_s u_j(x) = & m_i D_k D_s E_{ij}(x) + O(|x|^{-5/2+\varepsilon}) \\
& -\mathcal{R} \int_{\partial B_1(x)} u_l(y) u_i(y) D_k D_s E_{ij}(x-y) n_l(y) d\sigma_y ,
\end{aligned}
$$

where ε is a positive number arbitrarily close to zero. However, for the last term on the right-hand side of this relation we can only say that it behaves as $|x|^{-2}$ for large $|x|$, and so no improved bound can be deduced on the second derivatives of $\boldsymbol{v}$.

The problem of the asymptotic structure of the second derivatives of a generalized solution has been taken up and sharply solved by Deuring (2005). In particular, this author proves that, similarly to $\boldsymbol{v}$ and $\nabla \boldsymbol{v}$, also $D_k D_s \boldsymbol{v}$ behaves “almost” like the corresponding quantities calculated for the Oseen tensor $\boldsymbol{E}$. More precisely, he shows the following estimate as $|x| \to \infty$

$$
D_k D_s \boldsymbol{v}(x) = \boldsymbol{\alpha} \cdot (D_k D_s \boldsymbol{E}(x)) + O\left(|x|^{-5/2} \ln^2 |x| \right) ,
$$

where $\boldsymbol{\alpha} \in \mathbb{R}^3$ is suitable. In the same paper, an estimate of the asymptotic behavior of ∇p is also provided. ■

Our next objective is to obtain an asymptotic formula for the pressure field p associated to the generalized solution $\boldsymbol{v}$. The starting point is the representation (X.5.50)$_3$, which can be written as

$$p(x) = p_0' - \boldsymbol{\mathcal{M}}^* \cdot \boldsymbol{e}(x) + \mathcal{R} \sum_{i=1}^{3} P_i(x) + h(x) \tag{X.8.43}$$

with $\boldsymbol{\mathcal{M}}^*$ and $h(x)$ given by (X.5.51)$_3$ and (X.5.52)$_3$, respectively, and

$$P_1(x) = -\mathcal{R} \int_{\Omega_{R/2}} e_i(x-y) u_l(y) D_l u_i(y) dy$$

$$P_2(x) = -\mathcal{R} \int_{\Omega_{R/2,2R}} e_i(x-y) u_l(y) D_l u_i(y) dy$$

$$P_3(x) = -\mathcal{R} \int_{\Omega^{2R}} e_i(x-y) u_l(y) D_l u_i(y) dy$$

where $R = |x|$. From the inequality

$$|e_i(x-y)| \leq c|x-y|^{-2}$$

(cf. (VII.3.17)), from (X.8.42) and the uniform bound on the gradient of $\boldsymbol{v}$ given by (X.8.36), (X.8.38), we easily obtain

$$\begin{aligned}
|P_1(x)| &\leq \frac{c}{|x|^2} \|\boldsymbol{u}\|_3 |\boldsymbol{u}|_{1,3/2} \leq c_1 |x|^{-2} \\
|P_2(x)| &\leq \frac{c_2}{|x|^3} \int_{\Omega_{R/2,2R}} |\boldsymbol{e}(x-y)| dy \leq c_3 |x|^{-2} \\
|P_3(x)| &\leq c_4 \int_{2R}^{\infty} \rho^{-3} d\rho \leq c_5 |x|^{-2} \, ,
\end{aligned} \tag{X.8.44}$$

where we have employed the summability properties of $\boldsymbol{u}$, $\nabla \boldsymbol{u}$ as established in Theorem X.6.4. Collecting (X.8.43), (X.8.44) and recalling the asymptotic properties of $\boldsymbol{e}$ and h we then have the following.

Theorem X.8.3 *Let the assumptions of* Theorem X.8.1 *hold and let p be the pressure field associated to $\boldsymbol{v}$ by* Lemma X.1.1. *Then there exists a constant p_0' such that, for all sufficiently large $|x|$, $p(x)$ admits the following representation:*

$$p(x) = p_0' - \boldsymbol{\mathcal{M}}^* \cdot \boldsymbol{e}(x) + \mathcal{P}(x) \tag{X.8.45}$$

where $\boldsymbol{e}$ is the Oseen fundamental pressure field (VII.3.14), *$\boldsymbol{\mathcal{M}}^*$ is defined in* (X.5.51)$_3$ *and $\mathcal{P}(x)$ satisfies the estimate*

$$\mathcal{P}(x) = O(|x|^{-2}).$$

Remark X.8.4 Also for the pressure field one can give an asymptotic estimate that is sharper than (X.8.45). For example, Finn (1959b, Theorem 10) has shown that, provided $\boldsymbol{\mathcal{M}}^*$ is modified by adding to it a suitable (constant) vector, one has

$$|\mathcal{P}(x)| \le \begin{cases} c|x|^{-2} & \text{uniformly in } x \in \Omega \\ c|x|^{-2-2\overline{\sigma}} & x \text{ in the region (X.8.25)} \end{cases}$$

for all $\overline{\sigma} < \sigma$. Further asymptotic estimates on the pressure can be found in Finn (1965a, Theorem 5.4). ■

We end this section by describing the asymptotic structure of the vorticity field $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ associated to a solution $\boldsymbol{v}$ to (X.0.8), (X.0.4) with $\boldsymbol{v}_\infty \neq 0$. This problem has been studied in full detail by several authors; cf. Clark (1971), and Babenko & Vasil'ev (1973, §4). The main result states, essentially, that if $\boldsymbol{f}$ is of bounded support and $\boldsymbol{v}$ satisfies an estimate of the form

$$\boldsymbol{v}(x) + \boldsymbol{v}_\infty = O(|x|^{-1/2-\varepsilon}) \tag{X.8.46}$$

for some $\varepsilon > 0$, then the principal term in the representation of $\boldsymbol{\omega}$ at large distances is the curl of the principal term of the representation (X.8.17) for $\boldsymbol{v}$. However, as we know from Theorem X.8.1, every generalized solution obeys (X.8.46) and so the vorticity field of every generalized solution satisfies the preceding property. Thus, in particular, combining Theorem X.8.1 and a theorem of Clark (1971, Theorem 3.5) one can show the following result, whose rather long proof will be omitted.

Theorem X.8.4 *Let the assumptions of* Theorem X.8.1 *be satisfied. Then the vorticity field* $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ *obeys the following representation for all sufficiently large* $|x|$

$$\boldsymbol{\omega}(x) = \nabla \boldsymbol{\mathcal{G}}(x) \times \boldsymbol{m} + \boldsymbol{\mathcal{H}}(x),$$

where

$$\boldsymbol{\mathcal{G}}(x) \equiv \frac{e^{-\rho}}{4\pi\mathcal{R}|x|}, \quad \rho = \frac{\mathcal{R}}{2}(|x| + x_1),$$

$\boldsymbol{m}$ *is defined in* (X.5.51), *and*

$$\boldsymbol{\mathcal{H}}(x) = O(e^{-\rho}|x|^{-2}\log|x|).$$

From this theorem it follows, in particular, that the vorticity field of any generalized solution decays exponentially fast in the region (X.8.25) with $\sigma = 1/2$, i.e., outside any semi-infinite straight cone of finite aperture having the axis coincident with the negative x_1-axis.

X.9 On the Asymptotic Structure of Generalized Solutions when $\boldsymbol{v}_\infty = 0$

The methods we used to investigate the asymptotic structure of a generalized solution corresponding to $\boldsymbol{v}_\infty \neq 0$, no longer apply when $\boldsymbol{v}_\infty = 0$. The reason is basically due to the different properties possessed at infinity by solutions to the Oseen and Stokes problems, respectively. More specifically, what we cannot do when $\boldsymbol{v}_\infty = 0$ is to show (under suitable assumptions on the body force) an analog of Lemma X.6.1 that would ensure that the velocity field $\boldsymbol{v}$ belongs to $L^q(\Omega^R)$ for some $q < 6$. As a matter of fact, existence of solutions corresponding to $\boldsymbol{v}_\infty = \mathbf{0}$ and to data of arbitrary "size", in the class L^q in a neighborhood of infinity, with $q \in (1, 6)$, is, to date, an *open question.*

Nevertheless, by using a completely different approach due to Galdi (1992c), we can still draw some interesting conclusion on the asymptotic structure of generalized solutions corresponding to $\boldsymbol{v}_\infty = 0$ which, further, satisfy the energy inequality (X.4.19). (As we know from the existence Theorem X.4.1, this class of solutions is certainly not empty.) Specifically, we shall show that, provided a certain norm of the data is sufficiently small compared to $\mathcal{R}^{-2}$ (namely, to the square of the kinematic viscosity), every corresponding generalized solution $\boldsymbol{v}$ satisfying the energy inequality behaves for large $|x|$ as $|x|^{-1}$. Moreover, employing a simple scaling argument due to Šverák & Tsai (2000), we can prove that, if $\boldsymbol{f}$ is of bounded support,[1] the derivatives $D^\alpha \boldsymbol{v}$, behave like $|x|^{-|\alpha|-1}$, while the derivatives, $D^\alpha p$, of the corresponding pressure field p, decay like $|x|^{-|\alpha|-2}$. In other words, $\boldsymbol{v}, p$ possess the asymptotic properties of the fundamental Stokes tensor $\boldsymbol{U}, \boldsymbol{e}$. *The question of whether such a result continues to hold for large data also remains open.*

It must be also emphasized that, as shown by Deuring & Galdi (2000), even though $\boldsymbol{v}$ behaves, for large $|x|$, like $\boldsymbol{U}$, it does *not* admit an asymptotic expansion where the leading term is of the form $\boldsymbol{m} \cdot \boldsymbol{U}$, for some (constant) non-zero vector $\boldsymbol{m}$. This issue has been taken up by Nazarov & Pileckas (2000) and, successively, by Korolev & Šverák (2007, 2011). In particular, the latter authors have demonstrated that the leading term coincides with a suitable exact (and singular) solution of the full nonlinear problem obtained by Landau (1944); see Remark X.9.3.

The proof of the main result is based on a certain number of steps. To render the presentation simpler, we shall restrict ourselves to the case where $\boldsymbol{v}_* \equiv 0$ and shall suppose that the body force $\boldsymbol{f}$ can be written in divergence form, namely, $\boldsymbol{f} = \nabla \cdot \boldsymbol{F}$, with $\boldsymbol{F}$ a second-order tensor field.[2] For $\boldsymbol{G}$ a vector or second order tensor field, we recall the notation:

$$[\![\boldsymbol{G}]\!]_\beta \equiv \sup_{x \in \Omega} \left[(1 + |x|^\beta) |\boldsymbol{G}(x)| \right].$$

[1] Concerning this assumption, see footnote 4.

[2] This latter condition is not, in fact, a restriction, provided we give some regularity on $\boldsymbol{f}$; cf. Exercise III.3.1. See also Lemma VIII.5.1.

We begin with the following.

Lemma X.9.1 *Let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain of class C^2. Suppose that the second order tensor field $\boldsymbol{F}$ in Ω satisfies*

$$(1 + |x|^2)\boldsymbol{F} \in L^\infty(\Omega).$$

Then there exists a positive constant $A = A(\Omega, q)$ such that if

$$[\![\boldsymbol{F}]\!]_2 < A/\mathcal{R}^2 \tag{X.9.1}$$

there is at least one generalized solution $\boldsymbol{v}$ to the Navier–Stokes problem (X.0.8), (X.0.4) *with $\boldsymbol{v}_* \equiv \boldsymbol{v}_\infty \equiv 0$ and $\boldsymbol{f} = \nabla \cdot \boldsymbol{F}$* [3] *such that*

$$\boldsymbol{v} \in \mathcal{D}_0^{1,q}(\Omega), \quad p \in L^q(\Omega) \quad \textit{for each } q > 3/2,$$
$$[\![\boldsymbol{v}]\!]_1 < \infty.$$

Moreover, for any $q > 3/2$, this solution satisfies the following estimate

$$[\![\boldsymbol{v}]\!]_1 + |\boldsymbol{v}|_{1,q} + \|p\|_q \le B\mathcal{R}[\![\boldsymbol{F}]\!]_2, \tag{X.9.2}$$

with $B = B(\Omega, q)$.

Proof. The solution can be determined (for instance) by the successive approximation method. Thus, we look for a sequence of solutions to the sequence of problems, $m \in \mathbb{N}$,

$$\left.\begin{aligned} &\frac{1}{\mathcal{R}}(\nabla \boldsymbol{v}_m, \nabla \boldsymbol{\psi}) + (\boldsymbol{v}_{m-1} \cdot \nabla \boldsymbol{v}_{m-1}, \boldsymbol{\psi}) - (p_m, \nabla \cdot \boldsymbol{\psi}) - (\boldsymbol{F}, \nabla \boldsymbol{\psi}) = 0 \\ &\qquad\qquad \boldsymbol{v}_m \in \mathcal{D}_0^{1,q}(\Omega), \quad p_m \in L^q(\Omega) \end{aligned}\right. \tag{X.9.3}$$

where $\boldsymbol{v}_0 \equiv 0$ and $\boldsymbol{\psi}$ is arbitrary from $C_0^\infty(\Omega)$. By the existence theory for the Stokes problem of Theorem V.8.1, we know that (X.9.3) with $m = 1$ admits a unique solution $\{\boldsymbol{v}_1, p_1\}$ such that for all $q > 3/2$

$$[\![\boldsymbol{v}_1]\!]_1 + |\boldsymbol{v}_1|_{1,q} + \|p\|_q \le 2c\mathcal{R}[\![\boldsymbol{F}]\!]_2 \tag{X.9.4}$$

where $c = c(q, \Omega)$ is the constant entering the estimate (V.8.25). Let us show, by induction, the existence of $\{\boldsymbol{v}_m, p_m\}$ satisfying (X.9.4) for all $n \in \mathbb{N}$. Thus, assume $\{\boldsymbol{v}_{m-1}, p_{m-1}\}$ obeys (X.9.4). Since

$$\boldsymbol{v}_{m-1} \cdot \nabla \boldsymbol{v}_{m-1} = \nabla \cdot (\boldsymbol{v}_{m-1} \otimes \boldsymbol{v}_{m-1})$$
$$[\![\boldsymbol{v}_{m-1}^2]\!]_2 \le [\![\boldsymbol{v}_{m-1}]\!]_1^2 < \infty,$$

from Theorem V.8.1 we establish the existence of $\boldsymbol{v}_m, p_m$ obeying (X.9.3) along with the estimate

[3] In the weak sense.

$$\|\boldsymbol{v}_m\|_1 + |\boldsymbol{v}_m|_{1,q} + \|p_m\|_q \le c\mathcal{R}\left(\|\boldsymbol{F}\|_2 + \|\boldsymbol{v}_{m-1}\|_1^2\right). \tag{X.9.5}$$

However, by the induction hypothesis,

$$\|\boldsymbol{v}_{m-1}\|_1^2 \le 4c^2\mathcal{R}^2\|\boldsymbol{F}\|_2^2,$$

and from (X.9.5) we find

$$\|\boldsymbol{v}_m\|_1 + |\boldsymbol{v}_m|_{1,q} + \|p_m\|_q \le c\mathcal{R}\|\boldsymbol{F}\|_2\left(1 + 4c^2\mathcal{R}^2\|\boldsymbol{F}\|_2\right). \tag{X.9.6}$$

Therefore, if (X.9.1) holds with $A = 1/4c^2$, the approximating solution satisfies, for all $m \in \mathbb{N}$, the following inequality

$$\|\boldsymbol{v}_m\|_1 + |\boldsymbol{v}_m|_{1,q} + \|p_m\|_q \le 2c\mathcal{R}\|\boldsymbol{F}\|_2. \tag{X.9.7}$$

Notice that this inequality coincides with (X.9.2) with $\boldsymbol{v}_m$ in place of $\boldsymbol{v}$ and $B = 2c$. It is now a standard procedure, which we already employed several times, and which we therefore omit here (see, for instance, the proof of Theorem IX.5.3), to show that if (X.9.7) is verified then $\boldsymbol{v}_m, p_m$ converges strongly in $\mathcal{D}_0^{1,q}(\Omega) \times L^q(\Omega)$ to functions $\boldsymbol{v}, p$ such that

$$\boldsymbol{v} \in \mathcal{D}_0^{1,q}(\Omega), \quad p \in L^q(\Omega).$$

Moreover,

$$\|\boldsymbol{v}\|_1 < \infty$$

and (X.9.2) is satisfied. The theorem is thus proved. □

Remark X.9.1 Lemma X.9.1, under the same assumptions, remains valid in any number of dimensions $n \ge 4$, the only change in its statement being that the restriction from below on q becomes $q > n/2$. ■

Exercise X.9.1 (Finn (1965a)) Show that, for sufficiently small $\mathcal{R}$, the solution constructed in Lemma X.9.1 can be expanded as a power in $\mathcal{R}$:

$$\boldsymbol{v}(x) = \boldsymbol{v}_0(x) + \sum_{k=1}^{\infty} \mathcal{R}^k \boldsymbol{v}_k(x), \quad x \in \Omega, \tag{X.9.8}$$

where $\boldsymbol{v}_0(x)$ is a solution to the following Stokes problem

$$\left.\begin{aligned} \Delta \boldsymbol{v}_0 &= \nabla p_0 + \mathcal{R}\boldsymbol{F} \\ \nabla \cdot \boldsymbol{v}_0 &= 0 \end{aligned}\right\} \text{ in } \Omega$$

$$\boldsymbol{v}_0 = 0 \text{ at } \partial\Omega$$

$$\lim_{|x|\to\infty} \boldsymbol{v}_0(x) = 0.$$

Hint: A solution of the form (X.9.8) can be constructed by recurrence, that is,

$$\left.\begin{aligned}&\Delta \boldsymbol{v}_{m+1} = \textstyle\sum_{j=0}^{m} \boldsymbol{v}_{m-j}\cdot\nabla\boldsymbol{v}_j + \nabla p_{m+1}\\ &\nabla\cdot\boldsymbol{v}_{m+1} = 0\end{aligned}\right\}\ \text{in } \Omega$$

$$\boldsymbol{v}_{m+1} = 0 \ \text{ at } \partial\Omega$$

$$\lim_{|x|\to\infty} \boldsymbol{v}_{m+1}(x) = 0$$

$m = 0, 1, 2, \ldots$. This sequence of problems can then be solved with the help of Theorem V.8.1. Moreover,

$$V_{m+1} \le c\sum_{j=0}^{m} V_{m-j}V_j \tag{X.9.9}$$

where $c = c(\Omega)$ and $V_m = [\![\boldsymbol{v}_m]\!]_1 + |\boldsymbol{v}_m|_{1,q}$. By (X.9.9) and the Cauchy product formula for the series, the series

$$\sum_{k=0}^{\infty} \mathcal{R}^k V_k$$

is converging whenever $\mathcal{R} < 1/4cV_0$, that is, since $V_0 \le c\mathcal{R}[\![\boldsymbol{F}]\!]_2$, whenever $\mathcal{R}$ satisfies a restriction of the type (X.9.1). Finally, the uniqueness Theorem X.3.2 implies that (X.9.8) coincides with the solution constructed in Lemma X.9.1.

Lemma X.9.2 *Let $\boldsymbol{v}$ be a generalized solution to the Navier–Stokes problem* (X.0.8), (X.0.4) *corresponding to $\boldsymbol{v}_* \equiv \boldsymbol{v}_\infty \equiv 0$ and $\boldsymbol{f}$ of bounded support. If $\boldsymbol{v}(x) = O(|x|^{-1})$, as $|x| \to \infty$, then*

$$\begin{aligned}&D^\alpha \boldsymbol{v}(x) = O(|x|^{-|\alpha|-1}),\\ &D^\alpha(p(x) - p_0) = O(|x|^{-|\alpha|-2})\end{aligned}$$

for some $p_0 \in \mathbb{R}$ and for all $|\alpha| \in \mathbb{N}$.

Proof. To show the decay of the velocity field, we follow the scaling argument of Šverák & Tsai (2000).[4] Fix $x \in \Omega$, set $R = |x|/3$ sufficiently large so that $\operatorname{supp}(\boldsymbol{f}) \subset \Omega_R$, and define

$$\boldsymbol{v}_R(y) := R\,\boldsymbol{v}(Ry + x), \quad p_R(y) := R^2\,p(Ry + x), \tag{X.9.10}$$

where p is the pressure field associated to $\boldsymbol{v}$ by Lemma IX.1.2. Since, by Theorem X.1.1, $\boldsymbol{v}, p \in C^\infty(\Omega^R)$, and satisfy in Ω^R $(\text{X.0.8})_{1,2}$ with $\boldsymbol{f} \equiv \boldsymbol{0}$, it easily follows that $\boldsymbol{v}_R, p_R$ is a smooth solution to the following system

$$\left.\begin{aligned}&\Delta \boldsymbol{v}_R = \mathcal{R}\,\boldsymbol{v}_R\cdot\nabla\boldsymbol{v}_R + \nabla p_R\\ &\nabla\cdot\boldsymbol{v}_R = 0\end{aligned}\right\}\ \text{in } B_2\,. \tag{X.9.11}$$

We now apply to (X.9.11) the interior estimate for the Stokes problem given in (IV.4.20) to obtain, in particular,

[4] Actually, the result of Šverák & Tsai only requires that $\boldsymbol{f}$ decays sufficiently rapidly (depending on $|\alpha|$) as $|x| \to \infty$.

$$\|\boldsymbol{v}_R\|_{1,s,B_1} \le c\left(\|\boldsymbol{v}_R\|_{s,B_2} + \|\boldsymbol{v}_R\|^2_{2s,B_2}\right). \tag{X.9.12}$$

In view of the assumption on $\boldsymbol{v}$, we may take s is arbitrary in $(1,\infty)$, and thus obtain

$$\|\boldsymbol{v}_R\|_{1,s,B_1} \le M\,, \tag{X.9.13}$$

for all $s \in (1,\infty)$ for a constant M independent of R. We then use (IV.4.4), Remark IV.4.1 along with the embedding Theorem II.3.4 to find

$$\|\boldsymbol{v}_R\|_{2,s,B_a} \le c\left(\|\boldsymbol{v}_R\|_{1,s,B_1} + \|\boldsymbol{v}_R\|^2_{1,s,B_1}\right),$$

where $a < 1$. This inequality combined with (X.9.13) yields

$$\|\boldsymbol{v}_R\|_{2,s,B_a} \le M_1 \tag{X.9.14}$$

with M_1 independent of R, and then by the embedding Theorem II.3.4,

$$\max_{y\in B_a} |\nabla \boldsymbol{v}_R(y)| \le M_2$$

with M_2 independent of R. Therefore, from (X.9.10) it follows that

$$R^2|\nabla \boldsymbol{v}(x)| \le M_2$$

which shows the result for $k = 1$. The estimate of the higher order derivatives is obtained by a simple boot-strap argument. Thus, for example, from (IV.4.4), Remark IV.4.1, the embedding Theorem II.3.4, and (X.9.14) we deduce $\|\boldsymbol{v}_R\|_{3,s,B_b} \le M_3$, $b < a$, which, by (X.9.10), and Theorem II.3.4 in turn implies $R^3|D^2\boldsymbol{v}(x)| \le M_4$, and so forth. We next come to the estimate for the pressure. For $|\alpha| \ge 1$ they immediately follow from the estimate just proved for $\boldsymbol{v}$ and from $(\mathrm{X}.0.8)_1$ (with $\boldsymbol{f} \equiv \boldsymbol{0}$). In order to show the stated decay for $|\alpha| = 0$, we notice that from $(\mathrm{X}.5.47)_3$, (VII.3.14) and $(\mathrm{X}.5.47)_2$ we find, for some $p_0 \in \mathbb{R}$, and all sufficiently large $|x| := 3R$

$$\begin{aligned} p(x) &= p_0 - \mathcal{R}\int_{\Omega^R} q_i(x-y)v_l(y)D_lv_i(y)dy + O(1/|x|^2) \\ &= -\mathcal{R}\int_{\Omega_{R,2R}} q_i(x-y)v_l(y)D_lv_i(y)dy - \mathcal{R}\int_{\Omega^{2R}} q_i(x-y)v_l(y)D_lv_i(y)dy \\ &\qquad + O(1/|x|^2) \\ &:= \mathcal{R}\,I_1(R) + \mathcal{R}\,I_2(R) + O(1/|x|^2)\,. \end{aligned} \tag{X.9.15}$$

Recalling that $|q(\xi)| \le c\,|\xi|^{-2}$, the asymptotic estimates for $\boldsymbol{v}$, and that $|x| = 3R$ we at once obtain

$$\begin{aligned} |I_1(R)| &\le \frac{c_1}{R^2}\int_{\Omega_{R,2R}} |\boldsymbol{v}\cdot\nabla\boldsymbol{v}| \le \frac{c_2}{R^5}|\Omega_{R,2R}| \le \frac{c_3}{|x|^2} \\ |I_2(R)| &\le \frac{c_4}{|x|}\int_{\mathbb{R}^3} \frac{dy}{|x-y|^2|y|^2} \le \frac{c_5}{|x|^2}\,, \end{aligned} \tag{X.9.16}$$

where, in the last inequality, we have used Lemma II.9.2. The desired estimate for the pressure then follows from (X.9.15), (X.9.16), which completes the proof of the lemma. □

Collecting the results of Lemma X.9.1 and Lemma X.9.2, we prove the following result that provides, at least for "small" data, the asymptotic behavior of a weak solution corresponding to $\boldsymbol{v}_\infty = 0$ and verifying the energy inequality.

Theorem X.9.1 *Let Ω be as in* Lemma X.9.1 *and let $\boldsymbol{v}$ be a generalized solution to the Navier–Stokes problem* (X.0.8), (X.0.4) *corresponding to* $\boldsymbol{v}_* \equiv \boldsymbol{v}_\infty \equiv 0$ *and* $\boldsymbol{f} = \nabla \cdot \boldsymbol{F}$[5] *with*

$$(1+|x|^2)\boldsymbol{F} \in L^\infty(\Omega).$$

Assume, further, that $\boldsymbol{v}$ obeys the energy inequality

$$|\boldsymbol{v}|_{1,2}^2 \le \mathcal{R}(\boldsymbol{F}, \nabla \boldsymbol{v}).$$

Then:

(i) *If*

$$[\![\boldsymbol{F}]\!]_2 < \frac{1}{\mathcal{R}^2}\min\{A, 1/2B\} \tag{X.9.17}$$

with A and B given in (X.9.1) *and* (X.9.2), *respectively, we have*

$$\boldsymbol{v} \in \mathcal{D}_0^{1,q}(\Omega), \quad p \in L^q(\Omega) \ \ \textit{for each } q > 3/2,$$
$$(1+|x|)\boldsymbol{v} \in L^\infty(\Omega).$$

(ii) *If, in addition to* (X.9.17), $\boldsymbol{f}$ *is of bounded support then*

$$\begin{aligned} D^\alpha \boldsymbol{v}(x) &= O(|x|^{-|\alpha|-1}), \\ D^\alpha p(x) &= O(|x|^{-|\alpha|-2}) \end{aligned} \tag{X.9.18}$$

for all $|\alpha| \in \mathbb{N}$.

Proof. In view of (X.9.17), by Lemma X.9.1 we can construct a generalized solution $\boldsymbol{w}$ (say) that, together with the associate pressure field π, satisfies

$$\boldsymbol{w} \in \mathcal{D}_0^{1,q}(\Omega), \quad \pi \in L^q(\Omega) \ \ \text{for each } q > 3/2,$$
$$[\![\boldsymbol{w}]\!]_1 < \infty,$$

and

$$[\![\boldsymbol{w}]\!]_1 + |\boldsymbol{w}|_{1,q} + \|\pi\|_q \le B\mathcal{R}[\![\boldsymbol{F}]\!]_2.$$

However, again by (X.9.17), with the help of the uniqueness Theorem X.3.2 and Lemma X.1.1, we conclude that $\boldsymbol{v} \equiv \boldsymbol{w}$, $p \equiv \pi$ and the first part of the theorem follows. The second part is an obvious consequence of Lemma X.9.2. □

[5] In the weak sense.

Remark X.9.2 By the same argument employed in the proof of Theorem X.9.1, in view of Remark X.9.1 we can show that in dimension $n \geq 4$ every generalized solution satisfying the energy inequality and corresponding to "small" $\boldsymbol{f}$, behaves at large distances as $1/|x|$. ■

Let us point out two important consequences of Theorem X.9.1. Observing that the generalized solution $\boldsymbol{v}$ of Theorem X.9.1 satisfies

$$\boldsymbol{v} \in L^4(\Omega),$$

from Theorem X.2.1 we at once obtain the following theorem.[6]

Theorem X.9.2 *Let the assumptions of* Theorem X.9.1(i) *be satisfied. Then $\boldsymbol{v}$ obeys the energy equation*

$$|\boldsymbol{v}|_{1,2}^2 = \mathcal{R}(\boldsymbol{F}, \nabla \boldsymbol{v}).$$

Likewise, from Theorem X.3.2, it follows.

Theorem X.9.3 *Let the assumptions of* Theorem X.9.1(i) *be verified. Then $\boldsymbol{v}$ is the only generalized solution corresponding to $\boldsymbol{f}$ and satisfying the energy inequality.*

The results of Theorem X.9.1(i) show that every generalized solution obeying the energy inequality and corresponding to "small" data of bounded support has the *same* asymptotic behavior as the Stokes fundamental solution $(\boldsymbol{U}, \boldsymbol{e})$. It is then natural to wonder whether, in analogy with the case $\boldsymbol{v}_\infty \neq \boldsymbol{0}$ (see (X.8.17)), the velocity field $\boldsymbol{v}$ admits an asymptotic expansion of the type

$$\boldsymbol{v}(x) = \boldsymbol{\alpha} \cdot \boldsymbol{U}(x) + O(1/|x|^{1+\beta}) \quad \text{as } |x| \to \infty\,, \tag{X.9.19}$$

for some $\boldsymbol{\alpha} \in \mathbb{R}^3$ and $\beta > 0$. This problem has been considered by Deuring & Galdi (2000), who showed that, in fact, an expansion like (X.9.19) is possible only if $\boldsymbol{\alpha} = \boldsymbol{0}$. More precisely, we have the following result for whose proof we refer to Theorem 3.1 of the paper by Deuring & Galdi.

Theorem X.9.4 *Let $\boldsymbol{v}$ be a generalized solution to the Navier–Stokes problem* (X.0.8), (X.0.4) *corresponding to $\boldsymbol{v}_\infty \equiv 0$ and $\boldsymbol{f} \in L^2_{loc}(\Omega)$. Assume there is $R > \delta(\Omega^c)$ such that for all $x \in \Omega^R$ the following conditions hold:*

(i) $|\boldsymbol{f}(x)| \leq C_1/|x|^\gamma$ *for some* $\gamma > 3$, *and* $C_1 > 0$;
(ii) $|\boldsymbol{v}(x)| \leq C_2/|x|$ *for some* $C_2 > 0$.

Then, if there is a constant $M > 0$ such that

$$|x|^{1+\beta}\, |\boldsymbol{v}(x) - \boldsymbol{\alpha} \cdot \boldsymbol{U}(x)| \leq M\,,$$

for some $\boldsymbol{\alpha} \in \mathbb{R}^3$, some $\beta > 0$ and all $x \in \Omega^R$, necessarily $\boldsymbol{\alpha} = \boldsymbol{0}$.

[6] See also Exercise X.9.2.

Remark X.9.3 Theorem X.9.4 leaves open the intriguing question of whether a generalized solution satisfying the assumption (ii) of Theorem X.9.4 can still admit an asymptotic expansion of the type (X.9.19), *but with* $\boldsymbol{\alpha}\cdot\boldsymbol{U}(x)$ *replaced by some other vector field that behaves like* $1/|x|$ *as* $|x| \to \infty$.

Such a question finds an answer in the work of Nazarov & Pileckas (2000, Theorem 3.2 and Remark 3.3) who proved the following result.[7] Under the assumptions that Ω, $\boldsymbol{v}_*$, $\boldsymbol{f}$ are sufficiently smooth with $\boldsymbol{v}_*$ and $\boldsymbol{f}$ sufficiently small in suitable norms, and $\boldsymbol{f}$ of bounded support, there exists a (unique) corresponding (smooth) solution $(\boldsymbol{v}, p)$ to the Navier–Stokes problem (X.0.8), (X.0.4) with $\boldsymbol{v}_\infty = \boldsymbol{0}$, such that the following asymptotic representation holds:

$$\begin{aligned} \boldsymbol{v}(x) &= \frac{\boldsymbol{V}}{|x|} + O(1/|x|^{1+\beta}), \\ p(x) &= \frac{P}{|x|^2} + O(1/|x|^{2+\beta}), \end{aligned} \tag{X.9.20}$$

where $\boldsymbol{V}$ and P are functions defined on the unit sphere S^2, and $\beta \in (0, 1)$.

This interesting result has been further clarified by Šverák (2006) and, successively, its proof simplified by Korolev & Šverák (2007, 2011). Specifically, these authors show that the terms of order $|x|^{-1}$ in the expansion (X.9.20) must coincide with a specific velocity and pressure field of the family of solutions to the Navier–Stokes equations obtained by Landau (1944) and, independently, by Squire (1951); see (X.9.21). More precisely, let $\boldsymbol{b} \in \mathbb{R}^3 - \{\boldsymbol{0}\}$, and let (r, θ, ϕ) be a system of polar coordinates, with polar axis oriented in the direction $\boldsymbol{b}/|\boldsymbol{b}|$ which, without loss, we may take coinciding with the positive x_1–direction. The *Landau solution* $(\boldsymbol{U}^b, P^b)$ corresponding to $\boldsymbol{b}$ is then defined as follows

$$\begin{aligned} U_r^b &= \frac{2}{r}\left[\frac{A^2-1}{(A-\cos\theta)^2} - 1\right], \\ U_\theta^b &= -\frac{2\sin\theta}{r(A-\cos\theta)}, \\ U_\phi^b &= 0, \\ P^b &= \frac{4(A\cos\theta - 1)}{r^2(A-\cos\theta)^2}, \end{aligned} \tag{X.9.21}$$

where $A \in (1, \infty)$ is a parameter chosen in such a way that

[7] As a matter of fact, the time line of the events goes as follows. I learned the main ideas of the proof of Nazarov & Pileckas result in April 1996, after a conversation I had with Professor Serguei Nazarov, while he was visiting Professor Padula and me at the University of Ferrara. I then got the feeling of the invalidity of (X.9.19) with $\boldsymbol{\alpha} \neq \boldsymbol{0}$ and discussed the matter with Professor Paul Deuring, at an Oberwolfach meeting in August 1997. However, both papers of Nazarov & Pileckas and Deuring & Galdi were published only later on, in the year 2000.

$$16\pi\left(A + \frac{1}{2}A^2 \log\frac{A-1}{A+1} + \frac{4A}{3(A^2-1)}\right) = b \tag{X.9.22}$$

By direct inspection, we find that the function on the left-hand side is monotonically decreasing in $A \in (1,\infty)$ and that its range covers the entire positive line $(0,\infty)$. Therefore, for any given b (>0) we find one and only one A satisfying (X.9.22), namely, one and only one Landau solution $(\boldsymbol{U}^b, P^b)$.

Suppose now $(\boldsymbol{v}, p)$ is a regular solution to $(\text{X.0.8})_{1,2}$ with Ω sufficiently smooth, and set

$$\boldsymbol{b} := -\int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{v}, p)\cdot\boldsymbol{n}\,,$$

Then in Theorem 1 of Korolev & Šverák (2007, 2011) it is proved that for each $\beta \in (0,1)$ there exists $\varepsilon > 0$ such that, if $\boldsymbol{v}$ satisfies the assumption of Theorem X.9.4(ii) with a (sufficiently small) $C_2 = C_2(\varepsilon)$, necessarily $\boldsymbol{v}$ and p have the following asymptotic behavior as $|x| \to \infty$

$$\begin{aligned}\boldsymbol{v}(x) &= \boldsymbol{U}^b(x) + O(1/|x|^{1+\beta}),\\ p(x) &= P^b(x) + O(1/|x|^{2+\beta})\,.\end{aligned}$$

Whether or not in these formulas (or in (X.9.20)) we may take $\beta = 0$ *is open.* ■

Remark X.9.4 The following Liouville problem is *open*: *Are there nonidentically vanishing smooth solutions* $\boldsymbol{v}, p$ *to the Navier–Stokes problem*[8]

$$\left.\begin{aligned}\Delta\boldsymbol{v} &= \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p\\ \nabla\boldsymbol{v} &= 0\end{aligned}\right\}\quad \text{in } \mathbb{R}^3$$
$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = 0 \tag{X.9.23}$$

such that

$$\int_{\mathbb{R}^3} \nabla\boldsymbol{v} : \nabla\boldsymbol{v} < \infty.^{9} \tag{X.9.24}$$

Even though an answer to this question is not yet available, we can look for conditions on $\boldsymbol{v}$ under which (X.9.23) admits the null solution only. In this respect, we have the following theorem.

Theorem X.9.5 *Suppose* $\boldsymbol{v}$ *is a smooth solution*[10] *to* (X.9.23) *such that* [11]

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[8] Temporarily, we set $\mathcal{R} = 1$.

[9] We recall that the analogous question when the limiting velocity at infinity is nonzero is solved in Theorem X.7.2.

[10] We may equivalently require $\boldsymbol{v} \in L^3_{loc}(\mathbb{R}^3)$. Actually, by Theorem X.1.1 this would imply $\boldsymbol{v}$, $p \in C^\infty(\mathbb{R}^3)$ as required. Notice that *condition* (X.9.24) *is not needed.*

[11] In view of $(\text{X.9.23})_3$ and the regularity of $\boldsymbol{v}$, we may equally assume

$$\boldsymbol{v} \in L^q(\mathbb{R}^3) \ \text{ for some } q \in [1, 9/2].$$

$$\boldsymbol{v} \in L^{9/2}(\mathbb{R}^3) \tag{X.9.25}$$

then $\boldsymbol{v} \equiv 0$.

Proof. For $R > 0$, let ψ_R be a real nonincreasing smooth function defined in $\mathbb{R}^3$ such that $\psi_R(x) = 0$ for $|x| \geq 2R$, $\psi_R(x) = 1$ for $|x| \leq R$ and satisfying

$$|\nabla \psi_R(x)| \leq M/R,$$

for some (positive) constant M independent of $x \in \mathbb{R}^3$ and R. Setting $B(R) = B_{2R} \backslash B_R$, from this latter inequality it follows that

$$\Psi \equiv \int_{B(R)} |\nabla \psi_R|^3 \leq C$$

for some C independent of R. Multiplying $(\text{X.9.23})_1$ by $\psi_R \boldsymbol{v}$, integrating by parts over $\mathbb{R}^3$ and taking into account $(\text{X.9.23})_2$ yields

$$\begin{aligned}\int_{\mathbb{R}^3} \psi_R \nabla \boldsymbol{v} : \nabla \boldsymbol{v} &= \int_{\mathbb{R}^3} \left\{ -\nabla \psi_R \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{v} + \frac{1}{2} v^2 \boldsymbol{v} \cdot \nabla \psi_R + p \boldsymbol{v} \cdot \nabla \psi_R \right\} \\ &\equiv I_1 + I_2 + I_3.\end{aligned} \tag{X.9.26}$$

We have

$$\begin{aligned}|I_1| &\leq \Psi^{1/3} |\boldsymbol{v}|_{1,9/4,B(R)} \|\boldsymbol{v}\|_{9/2,B(R)} \\ |I_2| &\leq \Psi^{1/3} \|\boldsymbol{v}\|^3_{9/2,B(R)} \\ |I_3| &\leq \Psi^{1/3} \|p\|_{9/4,B(R)} \|\boldsymbol{v}\|_{9/2,B(R)}.\end{aligned} \tag{X.9.27}$$

Observing that

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} \in D_0^{-1,9/4}(\mathbb{R}^3),$$

from Theorem IV.2.2, Theorem V.3.2, and Theorem V.3.5, it follows that

$$\nabla \boldsymbol{v} \in L^{9/4}(\mathbb{R}^3), \quad p \in L^{9/4}(\mathbb{R}^3),$$

and so from inequality (X.9.27) we find

$$\lim_{R \to \infty} I_i = 0, \quad i = 1, 2, 3. \tag{X.9.28}$$

Relations (X.9.26) and (X.9.28) imply, by the monotone convergence theorem, $\nabla \boldsymbol{v} \equiv 0$. Since $\boldsymbol{v}$ satisfies (X.9.25), this latter condition delivers $\boldsymbol{v} \equiv 0$ and the proof of the theorem is complete. □

It is worth noticing that, in view of the Sobolev inequality (II.3.7), a generalized solution $\boldsymbol{v}$ to (X.9.23) (in dimension 3) satisfies only

$$\boldsymbol{v} \in L^6(\mathbb{R}^3)$$

which, in the sense of the behavior at large distances, is weaker than (X.9.25).

Theorem X.9.5 can be extended, with formal changes in the proof, to the n-dimensional case, $n \geq 4$, provided we replace (X.9.25) with the assumption that

$$\boldsymbol{v} \in L^{3n/(n-1)}(\mathbb{R}^n). \tag{X.9.29}$$

Now if $\boldsymbol{v}$ satisfies (X.9.24), from the Sobolev inequality (II.3.7) we deduce

$$\boldsymbol{v} \in L^{2n/(n-2)}(\mathbb{R}^n), \tag{X.9.30}$$

and since $\boldsymbol{v}$ is smooth and obeys the vanishing condition $(\text{X.9.23})_3$, it follows that (X.9.30) implies (X.9.29) and we may conclude that every smooth solution to (X.9.23) in $\mathbb{R}^n$ with $n \geq 4$, satisfying (X.9.24) is identically zero.[12] As we shall see in Chapter XII (cf. Theorem XII.3.1), this property continues to hold also for $n = 2$ (plane flows), so that *the three-dimensional case is the only one that remains open.* ■

Exercise X.9.2 (Kozono, Sohr & Yamazaki 1997) Let $\boldsymbol{v}$ be a generalized solution to (X.0.8), (X.0.4) corresponding to $\boldsymbol{v}_* \equiv \boldsymbol{v}_\infty \equiv \boldsymbol{0}$ and $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$. Show that if, in addition, $\boldsymbol{v} \in L^{9/2}(\Omega)$, then $\boldsymbol{v}$ satisfies the energy equality (X.2.4). *Hint*: Use Theorem V.5.1, along with the argument adopted in the proof of Theorem X.9.5.

X.10 Limit of Vanishing Reynolds Number: Transition to the Stokes Problem

The aim of this section is to show that every generalized solution $\boldsymbol{v}$ to the Navier–Stokes problem (X.0.8)–(X.0.4), corresponding to sufficiently smooth data, tends in an appropriate sense, as the Reynolds number $\mathcal{R} \to 0$, to the solution $\boldsymbol{v}_0$ of the linearized Stokes system corresponding to the same data.

If the limiting velocity $\boldsymbol{v}_\infty$ is zero, such a problem has been already considered in Exercise X.9.1. Actually, from the results of this exercise, it follows that any generalized solution can be expressed (for small $\mathcal{R}$) as a perturbation series in $\mathcal{R}$ around $\boldsymbol{v}_0$, that is,

$$\boldsymbol{v}(x) = \boldsymbol{v}_0(x) + \sum_{k=1}^{\infty} \mathcal{R}^k \boldsymbol{v}_k(x). \tag{X.10.1}$$

We shall, therefore, turn our attention to the more involved case where $\boldsymbol{v}_\infty \neq 0$. As a matter of fact, in such a situation an expansion of the type (X.10.1) is no longer expected, as we are going to show. Actually, assume that we try to express $\boldsymbol{v}$ in the form (X.10.1), with $\boldsymbol{v}_0$ solving the following Stokes problem[1]

[12] In dimension $n = 4$, this also follows from Remark X.2.4.

[1] For the sake of simplicity, we assume that there are no body forces acting on the liquid.

$$\left.\begin{aligned}\Delta \boldsymbol{v}_0 &= \nabla p_0\\ \nabla\cdot\boldsymbol{v}_0 &= 0\end{aligned}\right\}\ \text{in } \Omega$$
$$\boldsymbol{v}_0 = \boldsymbol{v}_* \ \text{ at } \partial\Omega\,, \qquad \lim_{|x|\to\infty}\boldsymbol{v}_0(x) = \boldsymbol{e}_1. \tag{X.10.2}$$

Then, at first order in $\mathcal{R}$, we should have

$$\left.\begin{aligned}\Delta \boldsymbol{v}_1 &= \boldsymbol{v}_0\cdot\nabla\boldsymbol{v}_0 + \nabla p_1\\ \nabla\cdot\boldsymbol{v}_1 &= 0\end{aligned}\right\}\ \text{in } \Omega$$
$$\boldsymbol{v}_1 = 0 \ \text{ at } \partial\Omega \tag{X.10.3}$$

and

$$\lim_{|x|\to\infty}\boldsymbol{v}_1(x) = 0. \tag{X.10.4}$$

In view of the asymptotic properties established in Theorem V.3.2 (cf. also Exercise V.3.4), we have

$$\boldsymbol{v}_0\cdot\nabla\boldsymbol{v}_0 \in L^q(\Omega), \quad \text{for } q > 3/2,$$

whereas

$$\boldsymbol{v}_0\cdot\nabla\boldsymbol{v}_0 \notin L^q(\Omega), \quad \text{for } q \le 3/2.$$

Thus, from Theorem V.4.7 we infer the existence of a solution $\boldsymbol{v}_1, p_1$ to the system (X.10.3) such that

$$\boldsymbol{v}_1 \in D^{2,q}(\Omega), \quad \text{for } q > 3/2.$$

However, this property is not enough to control the behavior of $\boldsymbol{v}_1$ at large distances and, consequently, we cannot prove the validity of (X.10.4). The situation just described is a form of the so-called *Whitehead paradox*; cf. Whitehead (1888) and Oseen (1927, p. 163).

In view of all this, we expect that $\boldsymbol{v}_0, p_0$ should approximate $\boldsymbol{v}, p$ as $\mathcal{R}\to 0$ in a form weaker than that required by the expansion (X.10.1). We shall, in fact, prove that this approximation holds in the sense of uniform convergence; cf. Theorem X.10.1.

In order to reach our objective, we need several preliminary results. For the sake of formal simplicity, we assume that there are no body forces acting on the liquid. For the same reason, the assumption we shall make on the regularity of Ω and $\boldsymbol{v}_*$ will be somewhat stronger than that actually needed. Weakening of these hypotheses will be left to the interested reader as an exercise.

Lemma X.10.1 *Let Ω be an exterior domain of $\mathbb{R}^3$ of class C^4 and let $\boldsymbol{v}$ be a generalized solution to* (X.0.8), (X.0.4) *corresponding to $\boldsymbol{v}_\infty = \boldsymbol{e}_1$, $\boldsymbol{f}\equiv 0$ and $\mathcal{R}\in(0,B]$, for some $B>0$. Then, if*

$$\boldsymbol{v}_* \in W^{11/2,2}(\partial\Omega),$$

there exists $C = C(\Omega, \boldsymbol{v}_*, B, R) > 0$ *such that*

$$\|p\|_{2,\Omega_R} + |\boldsymbol{v}|_{1,2} + \|\boldsymbol{v}\|_{2,\infty} \le C \tag{X.10.5}$$

for all $R > \delta(\Omega^c)$.

Proof. Throughout the proof, by the letter C we mean any positive constant depending on $\Omega, \boldsymbol{v}_*, B$, and R, but otherwise independent of $\mathcal{R}$. The value of C may, however, change within the same context. By Theorem X.4.1 and Theorem X.7.3, we know that there is a $B > 0$ such that

$$\|\boldsymbol{v}\|_{1,2,\Omega_R} + \|p\|_{2,\Omega_R} + |\boldsymbol{v}|_{1,2} \le C \tag{X.10.6}$$

where p has been modified by the addition of a suitable constant. From Theorem IV.5.1 we easily derive, for all $R > r > \delta(\Omega^c)$,

$$\|\boldsymbol{v}\|_{m+2,q,\Omega_r} \le C(\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{m+2,q,\Omega_R} + \|\boldsymbol{v}\|_{1,q,\Omega_R} + \|p\|_{q,\Omega_R} + 1)$$

with $m = 0, 1, 2$, $q > 1$ and so, by (X.10.6), it follows that

$$\|\boldsymbol{v}\|_{m+2,q,\Omega_r} \le C(\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{m+2,q,\Omega_R} + 1). \tag{X.10.7}$$

Since

$$\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{m+2,q,\Omega_R} \le \|(\boldsymbol{v}-\boldsymbol{v}_\infty)\cdot\nabla\boldsymbol{v}\|_{m+2,q,\Omega_R} + \|\boldsymbol{v}_\infty\cdot\nabla\boldsymbol{v}\|_{m+2,q,\Omega_R},$$

with the aid of the Hölder inequality and the inequality

$$\|\boldsymbol{v}-\boldsymbol{v}_\infty\|_6 \le \gamma_1|\boldsymbol{v}|_{1,2} \tag{X.10.8}$$

(cf. Theorem II.6.1), we obtain

$$\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{3/2,\Omega_R} \le \|\boldsymbol{v}-\boldsymbol{v}_\infty\|_6|\boldsymbol{v}|_{1,2} \le C,$$

where use has been made of (X.10.6). Inserting this information into (X.10.7) with $m = 0$, $q = 3/2$, and $r = R_1$ furnishes

$$\|\boldsymbol{v}\|_{2,3/2,\Omega_{R_1}} \le C.$$

With the help of the embedding Theorem II.5.2, we then easily prove

$$\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{2,\Omega_{R_1}} \le C$$

and (X.10.7) implies

$$\|\boldsymbol{v}\|_{2,2,\Omega_{R_2}} \le C,$$

for $R_2 < R_1$. Again using Theorem II.5.2, together with this latter relation, we show

$$\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{1,2,\Omega_{R_2}} \le C$$

and (X.10.7) then yields

$$\|\boldsymbol{v}\|_{3,2,\Omega_{R_3}} \le C,$$

for $R_3 < R_2$. Iterating this procedure one more time, we finally arrive at

$$\|\boldsymbol{v}\|_{4,2,\Omega_R} \le C,$$

for all $R > \delta(\Omega^c)$. This condition with the aid of Theorem II.3.4 then furnishes

$$\|\boldsymbol{v}\|_{2,\infty,\Omega_R} \le C. \tag{X.10.9}$$

Now, from Theorem X.5.1 we have, for sufficiently large R,

$$\|\boldsymbol{v}\|_{2,\infty,\Omega^R} \le C, \tag{X.10.10}$$

and the lemma is thus a consequence of (X.10.6), (X.10.9), and (X.10.10). □

Lemma X.10.2 *Let the assumptions of* Lemma X.10.1 *be satisfied. Then for all* $\mathcal{R} \in (0, B]$ *and all* $q \in [3/2, 2]$ *we have*

$$\|(\boldsymbol{v} - \boldsymbol{v}_\infty) \cdot \nabla \boldsymbol{v}\|_{2,q} \le C,$$

where $C = C(q, \Omega, \boldsymbol{v}_*, B)$.

Proof. Throughout the proof, by the letter C we mean any positive constant depending, at most, on q, Ω, $\boldsymbol{v}_*$, and B. Set

$$\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_\infty.$$

From Lemma V.4.3 it follows, in particular, with $m = 0, 1, 2$, that

$$|\boldsymbol{v}|_{m+2,2} \le C(\|\boldsymbol{u} \cdot \nabla \boldsymbol{v}\|_{m+2,2} + \|\boldsymbol{v}\|_{2,\Omega_R} + \|p\|_{2,\Omega_R} + 1).$$

Using Lemma X.10.1 in this inequality we find

$$|\boldsymbol{v}|_{m+2,2} \le C(\|\boldsymbol{u} \cdot \nabla \boldsymbol{v}\|_{m+2,2} + 1). \tag{X.10.11}$$

Again employing Lemma X.10.1 we find

$$\|\boldsymbol{u} \cdot \nabla \boldsymbol{v}\|_2 \le C$$

so that (X.10.11) furnishes

$$|\boldsymbol{v}|_{2,2} \le C. \tag{X.10.12}$$

Using this inequality, together with Lemma X.10.1, we infer that

$$\|\boldsymbol{u} \cdot \nabla \boldsymbol{v}\|_{1,2} \le C$$

and, therefore, (X.10.11) implies

$$|\boldsymbol{v}|_{3,2} \le C. \tag{X.10.13}$$

With the help of this estimate and Lemma X.10.1, we derive

$$\|\boldsymbol{u}\cdot\nabla\boldsymbol{v}\|_{2,2} \le C. \tag{X.10.14}$$

From (X.10.8) and Lemma X.10.1, we find

$$\|\boldsymbol{u}\cdot\nabla\boldsymbol{v}\|_{3/2} \le \|\boldsymbol{u}\|_6|\boldsymbol{v}|_{1,2} \le C. \tag{X.10.15}$$

Moreover, by Theorem II.6.1 and (X.10.11), it follows that

$$|\boldsymbol{v}|_{1,6} \le \gamma_1|\boldsymbol{v}|_{2,2} \le C \tag{X.10.16}$$

and so, by Lemma X.10.1 and interpolation, we derive

$$|\boldsymbol{v}|_{1,3} \le C.$$

Consequently,

$$|\boldsymbol{u}\cdot\nabla\boldsymbol{v}|_{1,3/2} \le |\boldsymbol{v}|_{1,3}^2 + \|\boldsymbol{u}\|_6|\boldsymbol{v}|_{2,2} \le C. \tag{X.10.17}$$

In addition, from (X.10.8), (X.10.12), (X.10.13), and (X.10.16) we obtain

$$|\boldsymbol{u}\cdot\nabla\boldsymbol{v}|_{2,3/2} \le 3|\boldsymbol{v}|_{1,6}|\boldsymbol{v}|_{2,2} + \|\boldsymbol{u}\|_6|\boldsymbol{v}|_{3,2} \le C. \tag{X.10.18}$$

Collecting (X.10.15)–(X.10.17) we find

$$\|\boldsymbol{u}\cdot\nabla\boldsymbol{v}\|_{2,3/2} \le C,$$

and the lemma becomes a consequence of this inequality, (X.10.14), and the elementary L^q convexity inequality (II.2.7). □

Lemma X.10.3 *Let the assumptions of* Lemma X.10.1 *be satisfied. Denote by $\boldsymbol{u}_0$, π_0 the solution to the following Oseen problem*

$$\left.\begin{aligned} \Delta\boldsymbol{u}_0 - \mathcal{R}\frac{\partial\boldsymbol{u}_0}{\partial x_1} &= \nabla\pi_0 \\ \nabla\cdot\boldsymbol{u}_0 &= 0 \end{aligned}\right\} \ \text{in } \Omega$$
$$\lim_{|x|\to\infty}\boldsymbol{u}_0(x) = 0 \tag{X.10.19}$$
$$\boldsymbol{u}_0 = \boldsymbol{v}_* - \boldsymbol{e}_1 \equiv \boldsymbol{u}_* \ \text{ at } \partial\Omega.$$

Then, there exists $B > 0$ such that for any $\mathcal{R} \in (0, B]$, all $q \in [3/2, 2]$, all $t \in [12/5, 3)$, and all $s \in (2, 3)$ we have

$$\begin{aligned} |\boldsymbol{v}-\boldsymbol{u}_0|_{1,3q/(3-q)} + \|D^2(\boldsymbol{v}-\boldsymbol{u}_0)\|_{2,q} + \|p-\pi_0\|_{3q/(3-q)} & \\ + \|\nabla(p-\pi_0)\|_{2,q} &\le C\,\mathcal{R} \\ \|\boldsymbol{v}-\boldsymbol{u}_0-\boldsymbol{e}_1\|_{3t/(3-t)} &\le C\,\mathcal{R}^{2-3/t} \\ |\boldsymbol{u}_0|_{1,3q/(3-q)} + \|D^2\boldsymbol{u}_0\|_{2,q} + \|\nabla\pi_0\|_{2,q} &\le C \\ \|\boldsymbol{u}_0\|_s &\le C\,\mathcal{R}^{1-3/s}, \end{aligned} \tag{X.10.20}$$

where $C = C(\Omega, \boldsymbol{v}_*, B, q, t, s)$.

Proof. Setting $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{e}_1$, $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u}_0$, and $\pi = p - \pi_0$ we deduce

$$\left.\begin{aligned} &\Delta \boldsymbol{w} - \mathcal{R}\frac{\partial \boldsymbol{w}}{\partial x_1} = \mathcal{R}(\boldsymbol{u}\cdot\nabla\boldsymbol{v}) + \nabla\pi \\ &\nabla\cdot\boldsymbol{w} = 0 \end{aligned}\right\} \text{ in } \Omega$$

$$\lim_{|x|\to\infty} \boldsymbol{w}(x) = 0$$

$$\boldsymbol{w} = 0 \text{ at } \partial\Omega.$$

Thus, $(\text{X.10.20})_1$ follows directly from Theorem II.6.1, Theorem VII.7.1, and Lemma X.10.2. Since from Lemma X.10.2 we have

$$\boldsymbol{u}\cdot\nabla\boldsymbol{v} \in L^{3/2}(\Omega),$$

Theorem VII.7.1 in turn furnishes

$$|\boldsymbol{w}|_{1,12/5} \le C\mathcal{R}^{3/4}.$$

Thus, by $(\text{X.10.20})_1$ and the interpolation inequality (II.2.7), we deduce with $\theta = 4(3-t)/t$

$$|\boldsymbol{w}|_{1,t} \le |\boldsymbol{w}|^{\theta}_{12/5}|\boldsymbol{w}|_3^{(1-\theta)} \le C\mathcal{R}^{2-3/t}.$$

As a consequence, the estimate $(\text{X.10.20})_2$ follows from this inequality and Theorem II.6.1. Moreover, $(\text{X.10.20})_3$ is an immediate consequence of Theorem II.6.1 and Theorem VII.7.1. In the rest of the proof the letter C will have the same meaning as in Lemma X.10.2. Let us denote by $\boldsymbol{E}(x;\mathcal{R})$ the Oseen tensor corresponding to the Reynolds number $\mathcal{R}$. From Exercise VII.3.5 we know that $\boldsymbol{E}$ obeys the homogeneity condition

$$\boldsymbol{E}(x;\mathcal{R}) = \mathcal{R}\,\boldsymbol{E}(\mathcal{R}x;1). \tag{X.10.21}$$

We next notice that, in view of Theorem VII.6.2, we have

$$u_{0j}(x) = \int_{\partial\Omega} [u_{*i}T_{il}(\boldsymbol{w}_j, e_j)(x-y) - E_{ij}(x-y)T_{il}(\boldsymbol{u}_0,\pi_0)(y)$$
$$+ \mathcal{R}u_{*j}(y)E_{ij}(x-y)\delta_{1l}]n_l(y)d\sigma_y.$$

Therefore, assuming, without loss, that $\Omega^c \subset B_{1/2}$, it follows that

$$|\boldsymbol{u}_0(x)| \le CD\{\sup_{y\in\Omega_{1/2}} |\boldsymbol{E}(x-y;\mathcal{R})| + \sup_{y\in\Omega_{1/2}} [|\boldsymbol{e}(x-y)| + |\nabla_x\boldsymbol{E}(x-y;\mathcal{R})|]\}. \tag{X.10.22}$$

where

$$D = \|\boldsymbol{u}_0\|_{1,1,\partial\Omega} + \|\pi_0\|_{1,\partial\Omega}.$$

From (X.10.21) and (X.10.22) we deduce

$$\|\boldsymbol{u}_0\|^s_{s,\Omega^1} \le CD^s\mathcal{R}^{s-3}\int_{|y|\ge\mathcal{R}} \sup_{|z|\le\mathcal{R}/2} \Big\{|\boldsymbol{E}(y-z;1)|^s + \mathcal{R}^s[|\boldsymbol{e}(y-z)|^s + |\nabla_y\boldsymbol{E}(y-z;1)|^s]\Big\}dy. \tag{X.10.23}$$

To estimate the first integral on the right-hand side of this inequality, we observe that

$$\int_{|y|\ge\mathcal{R}} \sup_{|z|\le\mathcal{R}/2} |\boldsymbol{E}(y-z;1)|^s dy \le \int_{2\ge|y|\ge\mathcal{R}} \sup_{|z|\le B/2} |\boldsymbol{E}(y-z;1)|^s dy + \int_{|y|\ge 2} \sup_{|z|\le B/2} |\boldsymbol{E}(y-z;1)|^s dy\,. \tag{X.10.24}$$

From (VII.3.21) we have

$$|\boldsymbol{E}(y-z;1)| \le c(|y-z|^{-1}+1), \quad |y|,|z|\le 2$$

and since

$$|z|\le\mathcal{R}/2, \quad |y|\ge\mathcal{R} \ \text{ implies } |y-z|\ge \tfrac{1}{2}|y|, \tag{X.10.25}$$

recalling that $s<3$, we find

$$\int_{2\ge|y|\ge\mathcal{R}} [\sup_{|z|\le\mathcal{R}/2} |\boldsymbol{E}(y-z;1)|]^s dy \le C. \tag{X.10.26}$$

Furthermore, by the mean value theorem,

$$|(E_{ij}(y-z)-E_{ij}(y))| = |z_l\frac{\partial}{\partial x_l}E_{ij}(y-\beta z)|, \quad \beta\in(0,1),$$

and so, from (VII.3.32) it follows for all $|y|$ sufficiently large ($\ge |y_0|$, say) that

$$\int_{|y|\ge|y_0|} \sup_{|z|\le B/2} |\boldsymbol{E}(y-z;1)|^s dy \le C\int_{|y|\ge|y_0|} \Big(|\boldsymbol{E}(y;1)|^s + |y|^{-3s/2}\Big)\, dy. \tag{X.10.27}$$

Taking into account that $s>2$, from (X.10.27), the local regularity of $\boldsymbol{E}$, and (VII.3.28), we have

$$\int_{|y|\ge 2} \sup_{|z|\le B/2} |\boldsymbol{E}(y-z;1)|^s dy \le C. \tag{X.10.28}$$

In addition, from (VII.3.14) and (X.10.25) we infer that

$$\begin{aligned}\mathcal{R}^s\int_{|y|\ge\mathcal{R}} \sup_{|z|\le\mathcal{R}/2} |\boldsymbol{e}(y-z)|^s dy &\le C\,\mathcal{R}^s\{\int_{2\ge|y|\ge\mathcal{R}} |y|^{-2s}dy + \int_{|y|\ge 2} \sup_{|z|\le B/2} |y-z|^{-2s}\} \\ &\le C\,(1+\mathcal{R}^{3-s}).\end{aligned} \tag{X.10.29}$$

Likewise,

$$\int_{|y|\geq\mathcal{R}}\sup_{|z|\leq\mathcal{R}/2}|\nabla_y\boldsymbol{E}(y-z;1)|^s dy \leq \int_{2\geq|y|\geq\mathcal{R}}\sup_{|z|\leq B/2}|\nabla_y\boldsymbol{E}(y-z;1)|^s dy$$
$$+\int_{|y|\geq 2}\sup_{|z|\leq B/2}|\nabla_y\boldsymbol{E}(y-z;1)|^s dy\,. \tag{X.10.30}$$

Since, from (VII.3.21),

$$|\nabla\boldsymbol{E}(y-z;1)|\leq C|y-z|^{-2},\quad |y|,\ |z|\leq 2,$$

by virtue of (X.10.25), it follows that

$$\int_{2\geq|y|\geq\mathcal{R}}\sup_{|z|\leq\mathcal{R}/2}|\nabla_y\boldsymbol{E}(y-z;1)|^s dy \leq C\int_{2\geq|y|\geq\mathcal{R}}|y|^{-2s}dy\leq C(1+\mathcal{R}^{3-2s}). \tag{X.10.31}$$

Also, we use the asymptotic properties of $\nabla\boldsymbol{E}(y;1)$, cf. (VII.3.31), together with the following ones on the second derivatives

$$|D^2\boldsymbol{E}(y)|\leq c|y|^{-2},$$

cf. (VII.3.35), to establish, as we did for (X.10.28), the following estimate:

$$\int_{|y|\geq 2}\sup_{|z|\leq B/2}|\nabla_y\boldsymbol{E}(y-z;1)|^s dy\leq C. \tag{X.10.32}$$

Thus, from (X.10.30)–(X.10.32) we recover

$$\mathcal{R}^s\int_{|y|\geq\mathcal{R}}\sup_{|z|\leq\mathcal{R}/2}|\nabla_y\boldsymbol{E}(y-z;1)|^s dy\leq C(1+\mathcal{R}^{3-s}). \tag{X.10.33}$$

Collecting (X.10.23), (X.10.26), (X.10.28), (X.10.29), and (X.10.33) we find

$$\|\boldsymbol{u}_0\|_{s,\Omega^1}\leq D\mathcal{R}^{1-3/s}. \tag{X.10.34}$$

On the other hand, by Theorem VII.2.1 and Theorem VII.6.2, and with the help of the embedding Theorem II.3.4, we find

$$\|\pi_0\|_{1,2,\Omega_2}+\|\boldsymbol{u}_0\|_{s,\Omega_2}+|\boldsymbol{u}_0|_{1,2}\leq C \tag{X.10.35}$$

where π_0 has been modified by the addition of a suitable constant. Moreover, from Theorem IV.4.1 and Theorem IV.5.1 we obtain, in particular,

$$\|\boldsymbol{u}_0\|_{2,2,\Omega_1}\leq C\left(\|\boldsymbol{u}_0\|_{1,2,\Omega_2}+\|\pi_0\|_{1,2,\Omega_2}\right). \tag{X.10.36}$$

Use of the trace Theorem II.4.4 yields

$$D \equiv \|\boldsymbol{u}_0\|_{1,1,\partial\Omega} + \|\pi_0\|_{1,\partial\Omega} \le C\left(\|\boldsymbol{u}_0\|_{2,2,\Omega_1} + \|\pi_0\|_{1,2,\Omega_1}\right)$$

and so this inequality, together with (X.10.35), (X.10.36), implies

$$D \le C.$$

Estimate $(\text{X.10.20})_4$ then follows from this relation and (X.10.34), (X.10.35), and the lemma is proved. $\square$

Lemma X.10.4 *Let the assumptions of* Lemma X.10.1 *be satisfied. Denote by* $\boldsymbol{v}_0$, p_0 *the solution to the following Stokes problem*

$$\left.\begin{aligned} \Delta \boldsymbol{v}_0 &= \nabla p_0 \\ \nabla \cdot \boldsymbol{v}_0 &= 0 \end{aligned}\right\} \ \text{in } \Omega$$
$$\boldsymbol{v}_0 = \boldsymbol{v}_* \ \text{ at } \partial\Omega \qquad \text{(X.10.37)}$$
$$\lim_{|x|\to\infty} \boldsymbol{v}_0(x) = \boldsymbol{e}_1$$

and by $\boldsymbol{u}_0$, π_0 *the solution to* (X.10.19). *Then there exists* $B > 0$ *such that for any* $\mathcal{R} \in (0, B]$, *all* $q \in [3/2, 2]$ *and all* $s \in (2,3)$ *we have*

$$\|\boldsymbol{u}_0 - \boldsymbol{v}_0 + \boldsymbol{e}_1\|_{3s/(3-s)} + \|\pi_0 - p_0\|_s + |\boldsymbol{u}_0 - \boldsymbol{v}_0|_{1,3q/(3-q)}$$
$$+\|D^2(\boldsymbol{u}_0 - \boldsymbol{v}_0)\|_{2.q} + \|\nabla(\pi_0 - p_0)\|_{2,q} \le C\mathcal{R}^{2-3/s}$$

where $C = C(\Omega, \boldsymbol{v}_*, B, q, s)$.

Proof. Setting

$$\boldsymbol{z} = \boldsymbol{u}_0 - \boldsymbol{v}_0 + \boldsymbol{e}_1, \quad \tau = \pi_0 - p_0,$$

we find

$$\left.\begin{aligned} \Delta \boldsymbol{z} &= \mathcal{R}\frac{\partial \boldsymbol{u}_0}{\partial x_1} + \nabla\tau \\ \nabla \cdot \boldsymbol{z} &= 0 \end{aligned}\right\} \ \text{in } \Omega$$
$$\lim_{|x|\to\infty} \boldsymbol{z}(x) = 0$$
$$\boldsymbol{z} = 0 \ \text{ at } \partial\Omega.$$

By Lemma X.10.3 we have, in particular, for all $s \in (2,3)$

$$\left\|\frac{\partial \boldsymbol{u}_0}{\partial x_1}\right\|_{-1,s} \le C\mathcal{R}^{1-3/s}.$$

Therefore, with the help of Theorem V.5.1, we obtain

$$\|\boldsymbol{z}\|_{3s/(3-s)} + |\boldsymbol{z}|_{1,s} + \|\tau\|_s \le C\mathcal{R}^{2-3/s}. \qquad \text{(X.10.38)}$$

In addition, from Lemma V.4.3 and Lemma X.10.3, we have

$$|\boldsymbol{z}|_{1,3q/(3-q)} + \|D^2\boldsymbol{z}\|_{2,q} + \|\nabla\tau\|_{2,q} \le C\left(\mathcal{R} + \|\boldsymbol{z}\|_{2,\Omega_R} + \|\tau\|_{2,\Omega_R}\right),$$

and the proof becomes a consequence of this latter inequality and (X.10.38). □

We are now in a position to prove the following main result.

Theorem X.10.1 *Let Ω be an exterior domain of $\mathbb{R}^3$ of class C^4 and let $\boldsymbol{v}$ be a generalized solution to* (X.0.8), (X.0.4) *corresponding to $\boldsymbol{v}_\infty = \boldsymbol{e}_1$ and to $\boldsymbol{f} \equiv 0$. Moreover, let $\boldsymbol{v}_0$, p_0 be the solution to the Stokes problem* (X.10.1). *Then, if*

$$\boldsymbol{v}_* \in W^{11/2,2}(\partial\Omega),$$

there exist $B > 0$ and $C = C(\Omega, \boldsymbol{v}_*, B, \varepsilon) > 0$ such that

$$\|\boldsymbol{v} - \boldsymbol{v}_0\|_{C^2(\Omega)} + \|p - p_0\|_{C^1(\Omega)} \le C\mathcal{R}^{1-\varepsilon}$$

where p is the pressure field associated to $\boldsymbol{v}$ and ε is a positive number that can be taken arbitrarily close to zero.[2]*.*

Proof. Set

$$\boldsymbol{z} = \boldsymbol{u}_0 - \boldsymbol{v}_0 + \boldsymbol{e}_1, \quad \tau = \pi_0 - p_0$$

with $\boldsymbol{u}_0$, π_0 and $\boldsymbol{v}_0$, p_0 solving (X.10.19) and (X.10.37). Then, by Lemma X.10.4 and a repeated use of Theorem II.6.1 we have

$$\|\nabla\boldsymbol{z}\|_{1,3q/(3-q)} \le C\mathcal{R}^{2-3/s}$$

and so, being $3q/(3-q) > 3$, by the embedding Theorem II.3.4 and taking s arbitrarily close to 3, we deduce

$$\|\nabla\boldsymbol{z}\|_{C^1(\Omega)} \le C\mathcal{R}^{1-\varepsilon}, \tag{X.10.39}$$

where ε satisfies the property stated in the lemma. Likewise, again by Lemma X.10.4 and Theorem II.3.4, we have

$$\|\nabla\tau\|_{C(\Omega)} \le C\mathcal{R}^{1-\varepsilon}. \tag{X.10.40}$$

Next, in view of (X.10.39) and Lemma X.10.4, it follows that

$$\|\boldsymbol{z}\|_{1,3s/(3-s)} \le C\mathcal{R}^{1-\varepsilon}$$

for all s arbitrarily close to 3. Therefore, with the help of Lemma X.10.4 and Theorem II.3.4, we find

$$\|\boldsymbol{z}\|_{C(\Omega)} \le C\mathcal{R}^{1-\varepsilon}. \tag{X.10.41}$$

Likewise, we prove

$$\|\tau\|_{C(\Omega)} \le C\mathcal{R}^{1-\varepsilon}. \tag{X.10.42}$$

[2] $C \to \infty$ as $\varepsilon \to 0$.

In a similar way, making use of the results of Lemma X.10.3, we have

$$\|\boldsymbol{v} - \boldsymbol{u}_0 - \boldsymbol{e}_1\|_{C^2(\Omega)} + \|p - \pi_0\|_{C^1(\Omega)} \leq C\mathcal{R}^{1-\varepsilon}. \tag{X.10.43}$$

Since

$$\|\boldsymbol{v} - \boldsymbol{v}_0\|_{C^2(\Omega)} \leq \|\boldsymbol{v} - \boldsymbol{u}_0 - \boldsymbol{e}_1\|_{C^2(\Omega)} + \|\boldsymbol{u}_0 - \boldsymbol{v}_0 + \boldsymbol{e}_1\|_{C^2(\Omega)}$$

$$\|p - p_0\|_{C^1(\Omega)} \leq \|p - \pi_0\|_{C^1(\Omega)} + \|\pi_0 - p_0\|_{C^1(\Omega)}$$

the result follows from (X.10.39)–(X.10.43). □

Remark X.10.1 If $\overline{\Omega}$ is the complement of a bounded domain, Fischer, Hsiao, & Wendland (1985, Theorem 1) have performed a more detailed analysis than that of Theorem X.10.1 of the way in which $\boldsymbol{v}$ approaches $\boldsymbol{v}_0$. Such an analysis is based on boundary integrals and pseudo-differential operators and leads to the following main result.

Theorem X.10.1′. *Let Ω be a smooth exterior domain in $\mathbb{R}^3$ bounded by a simple closed surface and let $\boldsymbol{v}$, p and $\boldsymbol{v}_0$, p_0 be solutions to* (X.0.8), (X.0.4) *and* (X.10.2), *respectively, corresponding to $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv 0$. Then,*

$$\boldsymbol{v}(x) = \boldsymbol{v}_0(x) + \boldsymbol{q}(x; \mathcal{R}),$$

where

$$\boldsymbol{q}(x; \mathcal{R}) = O(\mathcal{R}), \quad \text{as } \mathcal{R} \to 0, \text{ uniformly in } \Omega.$$

Moreover, there exists $\boldsymbol{w} = \boldsymbol{w}(x)$ defined in $\overline{\Omega}$ such that, for any given compact subset $\mathcal{K}$ of $\overline{\Omega}$,

$$\boldsymbol{v}(x) = \boldsymbol{v}_0(x) + \mathcal{R}\boldsymbol{w}(x) + O(\mathcal{R}^2 \ln \mathcal{R}^{-1}), \quad \text{as } \mathcal{R} \to 0,$$

holds uniformly on $\mathcal{K}$. ■

Theorem X.10.1 allows us to draw some interesting conclusions of the way in which the force $\boldsymbol{\mathcal{D}}$ exerted by the liquid on the boundary $\partial\Omega$ of the region of flow in the nonlinear motion described by (X.0.8), (X.0.4) approaches the same quantity $\boldsymbol{\mathcal{D}}_0$ calculated in the Stokes approximation (X.10.1). To show this, we observe that since

$$\boldsymbol{\mathcal{D}} = -\int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{v}, p) \cdot \boldsymbol{n}, \quad \boldsymbol{\mathcal{D}}_0 = -\int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{v}_0, p_0) \cdot \boldsymbol{n},$$

with $\boldsymbol{T}$ the Cauchy stress tensor (IV.8.6) and $\boldsymbol{n}$ outer unit normal at $\partial\Omega$, under the assumptions of Theorem X.10.1, it follows at once that

$$|\boldsymbol{\mathcal{D}} - \boldsymbol{\mathcal{D}}_0| \leq C\mathcal{R}^{1-\varepsilon}, \tag{X.10.44}$$

where $C = C(\Omega, \boldsymbol{v}_*, B)$ and $\mathcal{R} \in (0, B]$.

Remark X.10.2 Under the same assumptions of Theorem X.10.1′, one can show more detailed versions of (X.10.44), cf. Babenko (1976, *eq.* (6.7)), Fischer, Hsiao, & Wendland (1985, Theorem 2). ■

X.11 Notes for the Chapter

Section X.1. The variational formulation of the exterior problem in the way used here has been introduced by Ladyzhenskaya (1959b).

The introduction of the pressure field p associated to a generalized solution in the first part of Lemma X.1.1, as a member of L^2_{loc}, can be deduced from the paper of Solonnikov & Ščadilov (1973). Nevertheless, from this paper no summability properties for p in a neighborhood of infinity are directly obtainable. In fact, the second part of Lemma X.1.1 is due to me.

The special case $q = 2$ and $\boldsymbol{v}_\infty = \boldsymbol{0}$ of the results stated in Exercise X.1.1 can be found in Kozono & Sohr (1992a, Theorem 3(i)). Their proof, different than that indicated in Exercise X.1.1, employs an interpolation theorem of Aronszajn-Gagliardo.

Differentiability properties of generalized solutions were first determined by Ladyzhenskaya (1959b, Chapter II).

Section X.2. The validity of the energy equation in exterior domains was established by Finn (1959b, §III). However, the assumptions made there on the body force and on the asymptotic behavior of solutions are a priori stronger than those stated in Theorem X.2.1 and Theorem X.2.2. In particular, the latter refer to the so-called physically reasonable solutions we mentioned in the Introduction to this chapter. For general properties of these solutions, we refer the reader to the review articles of Finn (1965b, 1973); cf. also Finn (1961b, 1963, 1970).

Section X.3. The connection between generalized solutions satisfying the energy inequality and their uniqueness was first pointed out by H. Kozono and H. Sohr in a preprint in 1991, published later as their paper (1993). However, Theorem X.3.1 and Theorem X.3.2 are placed in a different context and have been independently obtained by Galdi (1992a, 1992c).

Section X.4. The first existence result for the Navier–Stokes problem in exterior domains is due to Leray (1933).

Lemma X.4.1 is based on an idea of Finn (1961a, §2c) and it is due to me. The proof of existence of generalized solutions satisfying the energy inequality (X.4.19) is also due to me; cf. Theorem X.4.1.

Existence in weighted Sobolev spaces has been investigated by Farwig (1990, 1992a, 1992b) and, successively, by Farwig and Sohr (1995, 1998), in the case where $\boldsymbol{v}_\infty \neq 0$. A similar approach when $\boldsymbol{v}_\infty = 0$ is not known.

An interesting question, that has been addresses by several authors, is the existence of q-weak solutions for $q \neq 2$ and $\boldsymbol{v}_\infty = \boldsymbol{0}$;[1] see Galdi & Padula (1991), Kozono & Sohr (1993), Borchers & Miyakawa (1995), Miyakawa (1995, 1999), Kozono, Sohr, & Yamazaki (1997), Kozono & Yamazaki (1998). Now, if $q > 2$, the answer is positive and quite trivial, in the light of the asymptotic results of Theorem X.5.1, and provided $\boldsymbol{f}$ satisfies suitable summability

[1] If $\boldsymbol{v}_\infty = \boldsymbol{0}$, we know that generalized solutions are, in fact, q-solutions as well, under suitable assumptions on the data; see Section X.5.

properties at large distances. On the other hand, if $q \in (1,2)$ the situation is more involved, as we are going to explain. Take $\boldsymbol{v}_* \equiv \mathbf{0}$ (for simplicity) and assume $\boldsymbol{f} \in D_0^{-1,2}(\Omega) \cap D_0^{-1,q}(\Omega)$. By a simple argument one shows that, due to the structure of the nonlinear term, in order to prove existence by a fixed point approach we have to choose $q = 3/2$, in the three-dimensional case;[2] see Kozono & Yamazaki (1998). However, existence of such a (3/2)-weak solution means $\boldsymbol{v} \in \mathcal{D}_0^{1,3/2}(\Omega) \cap \mathcal{D}_0^{1,2}(\Omega)$, and, therefore, $p \in L^{3/2}(\Omega) \cap L^2(\Omega)$.[3] In turn, this implies, at once, that the solution $(\boldsymbol{v}, p)$, if exists, must obey the following *nonlocal compatibility condition*

$$\int_{\partial\Omega} (\boldsymbol{T}(\boldsymbol{v},p) + \boldsymbol{F}) \cdot \boldsymbol{n} = \mathbf{0}\,, \tag{$*$}$$

where $\boldsymbol{T}$ is the Cauchy stress tensor, and, without loss of generality, we have written $\boldsymbol{f} = \nabla \cdot \boldsymbol{F}$ in the weak sense, where $\boldsymbol{F} \in L^{3/2}(\Omega) \cap L^2(\Omega)$; see Theorem II.8.2.[4] Condition $(*)$ is immediately *formally* obtained by integrating $(\text{X.0.8})_1$ (with $\boldsymbol{f} \equiv \nabla \cdot \boldsymbol{F}$) over Ω_R, then letting $R \to \infty$, and showing that, due to the summability properties of $\boldsymbol{v}$ and p the surface integrals converge to zero, along a sequence of surfaces, at least. From the physical viewpoint, $(*)$ means that the total net force exerted on the "obstacle", Ω^c, must vanish. Clearly, if $\Omega \equiv \mathbb{R}^3$, $(*)$ is automatically satisfied and, in fact, one shows existence (and uniqueness) for small data; Kozono & Nakao (1996) and also Maremonti (1991). On the other hand, if $\Omega^c \neq \emptyset$ and sufficiently smooth, one can show that such solutions can exist only for $\boldsymbol{f}$ in a subset of $D_0^{-1,2}(\Omega) \cap D_0^{-1,3/2}(\Omega)$ with empty interior; see Galdi (2009). In other words, for a *generic* $\boldsymbol{F}$ with the specified summability properties, the above (3/2)-weak solution does not exist. Finally, we wish to observe that, as shown by Kozono & Yamazaki (1998), if the space $\mathcal{D}_0^{1,3/2}(\Omega)$ is restricted to a suitable *larger* homogeneous Sobolev space of Lorenz type, the compatibility condition $(*)$ is no longer necessary, and existence of corresponding solutions, with velocity field in this latter space, can be still recovered.

Section X.5. In his paper of 1933, Leray left out the question of whether a generalized solution tends, uniformly and pointwise, to the prescribed vector $\boldsymbol{v}_\infty$, in the case when $\boldsymbol{v}_\infty \neq 0$; cf. Leray (1933, pp. 57-58). The general case was successively proved by Finn (1959a, Theorem 2) and, independently, by M. D. Faddeyev in his thesis published at Leningrad University in 1959; cf. Ladyzhenskaya (1969, Chapter 5, Theorem 8). Convergence of higher-order derivatives for the velocity field and for the pressure field was first investigated by Finn (1959b). In the same paper, Finn shows representation formulas of

[2] In general, $q = n/2$ for $n \geq 3$.

[3] This property for p can be easily established by the same argument used in the proof of Lemma X.1.1.

[4] Of course, $(*)$ has to be understood in the trace sense, according to Theorem III.2.2.

the type derived in Theorem X.5.2, for solutions having suitable behavior at large distances.

Section X.6. The first study of the global summability properties of a generalized solution, $\boldsymbol{v}$, when $\boldsymbol{v}_\infty \neq \mathbf{0}$, goes back to Babenko (1973). In this paper, the crucial step is to show that

$$\boldsymbol{v} + \boldsymbol{v}_\infty \in L^{4-\varepsilon}(\Omega) \tag{$**$}$$

for some small positive ε. Actually, once this condition is established, it is relatively simple to prove that $\boldsymbol{v}$ enjoys the same summability properties (and hence has the same asymptotic structure) of the Oseen fundamental solution; see Babenko (1973, pp. 11-21). I regret that some steps of Babenko's proof remain obscure to me.

The proof of global summability properties of generalized solutions developed in Lemma X.6.1 and Theorem X.6.4 is due to me, and is inspired by the paper of Galdi & Sohr (1995).

Theorem X.6.5 with $q = 2$ is due to Finn (1960).

Section X.8. Employing the key property $(**)$, Babenko (1973) showed that every generalized solution corresponding to $\boldsymbol{v}_\infty \neq 0$ is "physically reasonable" in the sense of Finn, that is, it behaves asymptotically as the Oseen fundamental solution; cf. also the Introduction to this chapter. As remarked previously, however, some steps of Babenko's proof of $(*)$ remain unclear to me. The proof given here, cf. Theorem X.8.1, is completely independent of Babenko's and is taken from the work of Galdi (1992b). Another proof has been successively provided by Farwig & Sohr (1998)

The proofs of Theorem X.8.2 and Theorem X.8.3 rely on ideas of Finn (1959b).

Section X.9. Lemma X.9.1 is due to Galdi & Simader (1994). Theorem X.9.1, Theorem X.9.2(i), and Theorem X.9.3 are due to Galdi (1992a, 1992c). Theorem X.9.5 is due to me. In this connection, see also Novotný & Padula (1995)

Section X.10. The relation between Navier–Stokes and Stokes problems in exterior domains, in the limit of vanishing Reynolds number, was first investigated by Finn (1961a). The approach followed in this section is due to me. For related questions, see also Arai (1995).

XI

Steady Navier–Stokes Flow in Three-Dimensional Exterior Domains. Rotational Case

F.F. CHOPIN, Polonaise op. 53, bars 1–2.

Introduction

This chapter is devoted to the study of the mathematical properties of solutions to the exterior boundary-value problem (X.0.1)–(X.0.2), in the case $\boldsymbol{\omega} \neq \mathbf{0}$, so that in general, $\boldsymbol{v}_\infty = \boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{x}$. As we observed in the Introduction to Chapter VIII, this study has an important bearing on those applied fields involving the free motion of rigid bodies in viscous liquids, such as sedimentation and self-propulsion phenomena.

Without loss of generality, we take $\boldsymbol{\omega} = \omega\, \boldsymbol{e}_1$ ($\omega > 0$), whereas $\boldsymbol{v}_0 = v_0\, \boldsymbol{e}$, with $\boldsymbol{e}$ a unit vector (unrelated, in principle, to $\boldsymbol{e}_1$) and $v_0 \geq 0$. Then, we may use the Mozzi–Chasles transformation (VIII.0.5)–(VIII.0.6) in (X.0.1)–(X.0.2), so that our task reduces to the study of the following boundary-value problem:

$$\left.\begin{aligned} &\nu \Delta \boldsymbol{v} = \boldsymbol{v} \cdot \nabla \boldsymbol{v} + 2\omega \boldsymbol{e}_1 \times \boldsymbol{v} + \nabla p + \boldsymbol{f} \\ &\nabla \cdot \boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega \qquad \text{(XI.0.1)}$$

$$\boldsymbol{v} = \boldsymbol{v}_* \quad \text{at } \partial\Omega,$$

G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems*, Springer Monographs in Mathematics,
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along with the side condition

$$\lim_{|x|\to\infty}(\boldsymbol{v}(x)+\boldsymbol{v}_\infty)=\boldsymbol{0}\,,\quad \boldsymbol{v}_\infty:=v_0(\boldsymbol{e}\cdot\boldsymbol{e}_1)\boldsymbol{e}_1+\omega\,\boldsymbol{e}_1\times\boldsymbol{x}. \tag{XI.0.2}$$

Our investigation of problem (XI.0.1)–(XI.0.2) will be *limited to the three-dimensional case only*, in that no general results are available, to date, for a two-dimensional domain, with the exception of very special cases.[1] Furthermore, for the sake of simplicity, *we assume that* $\mathring{\Omega}^c$ *is connected* .

The fundamental challenge with (XI.0.1)–(XI.0.2) consists in the fact that the velocity field becomes *unbounded at large distances from* $\partial\Omega$. An immediate consequence of this circumstance is that problem (XI.0.1)–(XI.0.2) can*not* by any means be viewed as a perturbation to the analogous problem with $\boldsymbol{\omega}=0$, which we analyzed in the previous chapter.

Notwithstanding this difficulty, thanks to the remarkable fact that the total power of the "rotational term" $\int_\Omega(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{u}-\boldsymbol{e}_1\times\boldsymbol{u})\cdot\boldsymbol{u}$ vanishes identically along differentiable vector functions $\boldsymbol{u}$ compactly supported in Ω, one can prove that all solutions (in a suitable class) to problem (XI.0.1)–(XI.0.2) satisfy an a priori estimate analogous to (X.0.5), that is,

$$\int_\Omega \nabla(\boldsymbol{v}+\boldsymbol{v}_\infty):\nabla(\boldsymbol{v}+\boldsymbol{v}_\infty)\le M\,, \tag{XI.0.3}$$

with M depending only on the data. As a consequence, by appropriately modifying the procedure used for the proof of Theorem X.4.1, we can prove the existence of a generalized solution, that is, D-solutions, for data of *arbitrary "size,"* also in the case $\boldsymbol{\omega}\neq\boldsymbol{0}$. This fact was first pointed out by Leray (1933, Chapter III); see also Borchers (1992, Korollar 4.1).

Now, though it is a simple job to show that to every generalized solution $\boldsymbol{v}$ one can associate a locally integrable pressure field p and that the pair $(\boldsymbol{v},p)$ is smooth (for as long as the data and the domain may allow), the question whether a generalized solution is also *physically reasonable* appears to be a much more challenging task. In accordance to what we discussed in the Introduction to the previous chapter, by "physically reasonable" (PR) we mean a solution that satisfies the energy equation (X.0.6), that for "small" data is unique, and that moreover, under suitable circumstances, shows a "wake-like" behavior, namely, a region outside which the velocity field converges to its asymptotic value much more rapidly than inside. As we know from the study of the case $\boldsymbol{\omega}=\boldsymbol{0}$, the proof of all these properties can be achieved if a solution has "good" behavior at large distances. However, for a D-solution, in addition to (XI.0.3), the only other information we have *at the outset* on its asymptotic behavior is

$$\int_\Omega |\boldsymbol{v}+\boldsymbol{v}_\infty|^6\le M_1\,, \tag{XI.0.4}$$

[1] For example, if Ω is the exterior of a rotating circle, the solution assumes the very simple form given in $(\text{V.0.8})_1$, (V.0.9).

as a consequence of (XI.0.3) and Theorem II.6.1.

The question of existence of *PR* solutions was initiated by Galdi (2003) for the case $\mathcal{R}' = 0$ (rotation without translation, that is), and continued and, to some extent, completed in the general case, by Galdi & Silvestre (2007a, 2007b). Specifically, by an entirely different approach from that one adopted by Finn (1965a) for the case $\boldsymbol{\omega} = \mathbf{0}$, in those papers it is proved that *if the data* $\boldsymbol{f}$ *and* $\boldsymbol{v}_*$ *are "small"* in a suitable sense, then problem (XI.0.1)–(XI.0.2) possesses one (and, in fact, only one) *PR solution.* As in the case $\boldsymbol{\omega} = 0$, while it is immediate to show that these solutions are also D-solutions, the converse property is in no way obvious, even for "small" data.

The problem of whether a D-solution is also physically reasonable has been recently investigated by Galdi & Kyed (2010, 2011a). Their main results heavily rely on the analysis of the generalized Oseen problem performed in Chapter VIII, and can be summarized as follows.

Suppose, at first, $v_0 \neq 0$ and $\boldsymbol{e} \cdot \boldsymbol{e}_1 \neq 0$, namely,

$$\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq 0\,, \tag{XI.0.5}$$

and let $(\boldsymbol{v}, p)$ be a solution to (XI.0.1) with $\boldsymbol{f}$ mildly regular and of bounded support,[2] and $\boldsymbol{v}$ satisfying (XI.0.3), (XI.0.4). Then, there exists a semi-infinite cone, $\mathcal{C}$, whose axis is directed along the $\pm\boldsymbol{e}_1$-direction according to whether $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \lesseqgtr 0$, and such that for all sufficiently large $|x|$ and $\delta > 0$,

$$\begin{aligned} \boldsymbol{v}(x) + \boldsymbol{v}_\infty(x) &= \begin{cases} O(|x|^{-1}) & \text{uniformly}\,, \\ O(|x|^{-3/2+\delta}) & \text{if } x \notin \mathcal{C}\,, \end{cases} \\ \nabla\left(\boldsymbol{v}(x) + \boldsymbol{v}_\infty(x)\right) &= \begin{cases} O(|x|^{-3/2}) & \text{uniformly}\,, \\ O(|x|^{-2+\delta}) & \text{if } x \notin \mathcal{C}\,. \end{cases} \end{aligned} \tag{XI.0.6}$$

Furthermore, there is $p_0 \in \mathbb{R}$, such that

$$\widetilde{p}(x) - p_0 = O(|x|^{-2}\ln|x|)\,, \tag{XI.0.7}$$

where $\widetilde{p}(x) = p(x) + \frac{\omega^2}{2}(x_2^2 + x_3^2)$. [3] Notice that the asymptotic properties given in (XI.0.6) *coincide* with those of the same quantity in the case $\boldsymbol{\omega} = \mathbf{0}$ and $\boldsymbol{v}_0 \neq \mathbf{0}$, given in Theorem X.8.1 and Theorem X.8.2. However, the main, and in principle substantial, difference between the cases $\boldsymbol{\omega} = \mathbf{0}$ and $\boldsymbol{\omega} \neq \mathbf{0}$ relies on the fact that in the rotational case it is not known, to date, whether there existence a leading term in the asymptotic behavior of the velocity field, while in absence of rotation we proved that such a term exists and coincides with the Oseen fundamental tensor $\boldsymbol{E}$, in the sense specified in Theorem X.8.1 and Theorem X.8.2. As a consequence, it is not known whether

[2] This latter assumption can be fairly weakened, by imposing only that f decays "sufficiently fast" at large distances.

[3] Observe that the field $\widetilde{p}$ is, in fact, the real pressure of the liquid; see Section I.2.

the asymptotic estimates given in (XI.0.6) are *optimal.* In particular, *it is not known whether the kinetic energy of the liquid when $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv \boldsymbol{0}$ is infinite,*[4] *as in the irrotational case, or else is finite.*

It is interesting to give an interpretation of the above results and of condition (XI.0.5), in the case in which $\boldsymbol{v}_0$ is the translational velocity of the center of mass G and $\boldsymbol{\omega}$ is the angular velocity of the "body" $\mathcal{B} \equiv \Omega^c$, moving in a viscous liquid that is quiescent at spatial infinity. In such a case, we recall[5] that condition (XI.0.5) ensures that with respect to an inertial frame $\mathcal{I}$, the velocity $\boldsymbol{\eta}$ of G has a nonzero component η_1 in the direction $\boldsymbol{e}_1$ of $\boldsymbol{\omega}$. On physical grounds, we thus expect the formation of a wake region behind the body in the direction opposite to η_1. The cone $\mathcal{C}$ is then exactly representative of this wake region. We also would like to emphasize that, again on physical grounds, condition (XI.0.5) is necessary for the formation of the wake. In fact, if $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0$, then the motion of $\mathcal{B}$ in $\mathcal{I}$ reduces to a pure rotation[6] where, of course, no wake region is expected.

In the process of proving the estimates (XI.0.6) and (XI.0.7) we also find,[7] on the one hand, that *every D-solution is unique in its own class if the data are "sufficiently small,"* and on the other hand, that *every D-solution satisfies the energy equation.*

If $v_0 = 0$ or $\boldsymbol{e} \cdot \boldsymbol{e}_1 = 0$, namely,

$$\boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0\,, \tag{XI.0.8}$$

the picture is less clear, and the corresponding results resemble those found in the previous chapter when $\boldsymbol{v}_\infty = \boldsymbol{0}$. More precisely, following Galdi & Kyed (2010), we show that every D-solution satisfying the *energy inequality*, that is, (X.0.6) with "=" replaced by "$\le$",[8] and corresponding to "sufficiently small" data, has asymptotic behavior similar to that of the fundamental Stokes solution. More precisely,

$$D^\alpha(\boldsymbol{v}(x)+\boldsymbol{\omega}\times\boldsymbol{x}) = O(|x|^{-|\alpha|-1})\,,\quad D^\alpha(\widetilde{p}(x)-p_0) = O(|x|^{-|\alpha|-2})\,,\ \ |\alpha| = 0,1\,, \tag{XI.0.9}$$

for some $p_0 \in \mathbb{R}$. The proof of (XI.0.9) is similar, in principle, to that of Theorem X.9.1 in the case $\boldsymbol{\omega} = \boldsymbol{0}$. However, unlike this latter, the proof of the appropriate summability properties of the pressure field associated to a D-solution is quite elaborate.

The question whether the above results continue to hold for data of "arbitrary" size, as in the case $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0$, remains open. However, on the bright

[4] This happens, for example, in the very important situation of a rigid body $\mathcal{B}$ translating *and* rotating in a viscous liquid when the walls of $\mathcal{B}$ are fixed and impermeable; see also the comments in the next paragraph.

[5] See footnote 14 of Chapter I.

[6] See footnote 14 of Chapter I.

[7] Under more general assumptions on the body force $\boldsymbol{f}$; see Theorem XI.5.1 and Theorem XI.5.3.

[8] This class is proved to be not empty.

side, unlike the case $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq 0$, for the case at hand, Farwig, Galdi & Kyed (2010), relying on the previous work of Galdi (2003) and Farwig and Hishida (2009), were able to show, for small data at least, the existence of a leading term in the pointwise asymptotic behavior of a generalized solution. In particular, regarding the velocity, one shows that it coincides with the velocity field of a suitable Landau solution.[9]

We conclude this introductory section by rewriting (XI.0.1), (XI.0.2) in a suitable nondimensional form. If $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq 0$, we scale the velocity by $v_0 \boldsymbol{e} \cdot \boldsymbol{e}_1$ and the length by $d := \delta(\Omega^c)$, so that by a simple computation, (XI.0.1) becomes

$$\left.\begin{aligned} &\Delta \boldsymbol{v} = \mathcal{R}\, \boldsymbol{v} \cdot \nabla \boldsymbol{v} + 2\mathcal{T}\, \boldsymbol{e}_1 \times \boldsymbol{v} + \nabla p + \boldsymbol{f} \\ &\nabla \cdot \boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\boldsymbol{v} = \boldsymbol{v}_* \ \text{ at } \partial\Omega \,, \tag{XI.0.10}$$

where $\mathcal{R}$ and $\mathcal{T}$ are defined in $(\text{VIII.0.6})_4$ and $(\text{VIII.0.3})_2$, respectively. If, on the other hand, $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0$, then we scale the velocity with $\omega\, d$, which amounts to setting (formally) in (XI.0.10) $\mathcal{R} = \mathcal{T}$. As far as (XI.0.2) is concerned, we have

$$\lim_{|x|\to\infty} (\boldsymbol{v}(x) + \boldsymbol{v}_\infty(x)) = \boldsymbol{0}\,, \quad \boldsymbol{v}_\infty(x) := \begin{cases} \boldsymbol{e}_1 + \frac{\mathcal{T}}{\mathcal{R}}\, \boldsymbol{e}_1 \times \boldsymbol{x}\,, & \text{if } \boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq 0\,, \\ \boldsymbol{e}_1 \times \boldsymbol{x}\,, & \text{if } \boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0\,. \end{cases} \tag{XI.0.11}$$

Throughout this chapter, we will focus on the study of the boundary-value problem (XI.0.10)–(XI.0.11).

XI.1 Generalized Solutions. Existence of the Pressure and Regularity Properties

The weak formulation of the boundary-value problem (XI.0.10), (XI.0.11) is given with a by now standard procedure that is the natural generalization of its irrotational counterpart. Specifically, if we formally multiply $(\text{XI.0.10})_1$ by $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and integrate by parts over Ω, we obtain

$$\int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{\varphi} + \mathcal{R} \int_\Omega \boldsymbol{v} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{\varphi} + 2\mathcal{T}\, \boldsymbol{e}_1 \times \int_\Omega \boldsymbol{v} \cdot \boldsymbol{\varphi} = - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}. \tag{XI.1.1}$$

As usual, we shall consider the more general situation in which the right-hand side of (XI.1.1) is defined by a linear functional $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$.

Thus, in complete analogy with Definition X.1.1 (and Definition VIII.1.1), we give the following.

Definition XI.1.1. Let Ω be an exterior domain of $\mathbb{R}^3$.[1] A vector field $\boldsymbol{v} : \Omega \to \mathbb{R}^3$ is called a *weak* (or *generalized*) *solution to the Navier–Stokes problem* (XI.0.10), (XI.0.11) if

[9] See Remark X.9.3 for the definition of a Landau solution.

[1] We recall that we will assume throughout that Ω^c is connected.

(i) $\boldsymbol{v} \in D^{1,2}(\Omega)$;
(ii) $\boldsymbol{v}$ is (weakly) divergence-free in Ω;
(iii) $\boldsymbol{v}$ satisfies the boundary condition $(\text{XI.0.10})_3$ (in the trace sense) or, if $\boldsymbol{v}_* \equiv 0$, then $\vartheta \boldsymbol{v} \in D_0^{1,2}(\Omega)$, where $\vartheta \in C_0^1(\overline{\Omega})$, and $\vartheta(x) = 1$ if $x \in \Omega_{R/2}$ and $\vartheta(x) = 0$ if $x \in \Omega^R$, $R > 2\delta(\Omega^c)$;
(iv) $\displaystyle\lim_{|x|\to\infty} \int_{S^2} |\boldsymbol{v}(x) + \boldsymbol{v}_\infty| = 0\,;$
(v) $\boldsymbol{v}$ satisfies the identity

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) + \mathcal{R}(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi}) + 2\mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{v}, \boldsymbol{\varphi}) = -\,[\boldsymbol{f}, \boldsymbol{\varphi}] \tag{XI.1.2}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$.

Remark XI.1.1 Remark V.1.1 with $q = 2$, and Remark IX.1.1 equally apply to generalized solutions of Definition XI.1.1. ■

We shall next investigate the existence and the properties of the pressure field associated to a generalized solution. While existence is a simple task, the proof of *global* summability properties requires more effort, and heavily relies on the properties of weak solutions to the generalized Oseen problem established in Section VIII.2. We begin with the following.

Lemma XI.1.1 *Let $\boldsymbol{v}$ be a generalized solution to* (XI.0.10), (XI.0.11). *Then, if*

$$\boldsymbol{f} \in W_0^{-1,2}(\Omega') \tag{XI.1.3}$$

for every bounded domain Ω' with $\overline{\Omega'} \subset \Omega$, there exists

$$p \in L^2_{loc}(\Omega)$$

such that

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) + \mathcal{R}(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\psi}) + 2\mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{v}, \boldsymbol{\psi}) = (p, \nabla \cdot \boldsymbol{\psi}) - [\boldsymbol{f}, \boldsymbol{\psi}] \tag{XI.1.4}$$

for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. Furthermore, if Ω is locally Lipschitz and for some $R > \delta(\Omega^c)$,

$$\boldsymbol{f} \in W_0^{-1,2}(\Omega_R),$$

then we have

$$p \in L^2(\Omega_R).$$

Proof. The proof of the lemma is completely analogous to that given in Lemma X.1.1 and will be therefore left to the reader. □

Lemma XI.1.2 *Let $\boldsymbol{v}$ and $\boldsymbol{f}$ be as in* Lemma XI.1.1. *Suppose, in addition,*

$$\boldsymbol{f} \in L^2(\Omega^\rho)\,. \ \text{ for some } \rho > \delta(\Omega^c),$$

Then $\boldsymbol{v} \in D^{2,2}(\Omega^r)$, for all $r > \rho$.

Proof. Under the stated assumptions on $\boldsymbol{f}$, in the following Theorem XI.1.2 it is shown that $(\boldsymbol{v}, p) \in W^{2,2}_{loc}(\Omega^\rho) \times W^{1,2}_{loc}(\Omega^\rho)$. Therefore, setting $\boldsymbol{u} := \boldsymbol{v} + \boldsymbol{v}_\infty$, from (XI.1.4), we obtain

$$\left.\begin{array}{c}\Delta \boldsymbol{u} + \mathcal{R}\dfrac{\partial \boldsymbol{u}}{\partial x_1} + \mathcal{T}\left(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{u}\right) = \mathcal{R}\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \widetilde{p} + \boldsymbol{f} \\ \nabla \cdot \boldsymbol{u} = 0 \end{array}\right\} \text{a.e. in } \Omega^\rho , \tag{XI.1.5}$$

where[2]

$$\widetilde{p} := \begin{cases} p + \frac{\mathcal{T}^2}{2\mathcal{R}}(x_2^2 + x_3^2), & \text{if } \boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq 0 \\ p + \frac{\mathcal{T}}{2}(x_2^2 + x_3^2), & \text{if } \boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0 . \end{cases} \tag{XI.1.6}$$

We have to show that

$$D^2\boldsymbol{u} \,(\equiv D^2\boldsymbol{v}) \in L^2(\Omega^\rho) . \tag{XI.1.7}$$

Let ψ_R be the "cut-off" function used in the proof of Lemma VIII.2.1. By dot-multiplying both sides of (XI.1.5)$_1$ by $-\nabla \times (\psi_R \nabla \times \boldsymbol{u})$, integrating over Ω^ρ, and taking into account (VIII.2.4), we derive[3]

$$\begin{aligned} \|\sqrt{\psi_R}\Delta \boldsymbol{u}\|_2^2 = & -(\mathcal{R}\frac{\partial \boldsymbol{u}}{\partial x_1} + \boldsymbol{f}, \psi_R \Delta \boldsymbol{u} + (\nabla \times \boldsymbol{u}) \times \nabla \psi_R) \\ & -\tfrac{1}{2}((\nabla \times \boldsymbol{u}) \times \nabla(\sqrt{\psi_R}), \sqrt{\psi_R}\Delta \boldsymbol{u}) \\ & +\mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{u}, \nabla \times (\psi_R \nabla \times \boldsymbol{u}) - (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{u}, \nabla \times (\psi_R \nabla \times \boldsymbol{u})) \\ & +\mathcal{R}(\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \psi_R \Delta \boldsymbol{u} + (\nabla \times \boldsymbol{u}) \times \nabla \psi_R) . \end{aligned}$$

With the help of (VIII.2.6), (VIII.2.7), (VIII.2.9) and (VIII.2.11), from this relation we deduce

$$\|\sqrt{\psi_R}\Delta \boldsymbol{u}\|_2^2 \leq c\,(\|\boldsymbol{f}\|_2^2 + |\boldsymbol{u}|_{1,2}^2) + \mathcal{R}(\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \psi_R \Delta \boldsymbol{u} + (\nabla \times \boldsymbol{u}) \times \nabla \psi_R) , \tag{XI.1.8}$$

with c independent of R. We now estimate the last two terms on the right-hand side of (XI.1.8). To this end, we begin by observing that since $\boldsymbol{u}$ belongs to $D^{1,2}(\Omega)$ and satisfies condition (iv) in Definition XI.1.1, by Theorem II.6.1(i) it follows that $\boldsymbol{u} \in L^6(\Omega)$ and

$$\|\boldsymbol{u}\|_6 \leq c_0 |\boldsymbol{u}|_{1,2} , \tag{XI.1.9}$$

where $c_0 = c_0(\Omega)$. By the Hölder inequality and (XI.1.9), we have

[2] See footnote 3 in the Introduction to this chapter.

[3] This formal calculation can be easily made rigorous by a simple approximation procedure and by the use of the results of Exercise II.4.3. For simplicity, throughout the proof, we set $\|\cdot\|_{2,\Omega^\rho} \equiv \|\cdot\|_2$, and $(\cdot,\cdot)_{\Omega^\rho} \equiv (\cdot,\cdot)$.

$$
\begin{aligned}
(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\psi_R\Delta\boldsymbol{u}) &\le \|\sqrt{\psi_R}\nabla\boldsymbol{u}\|_3\|\boldsymbol{u}\|_6\|\sqrt{\psi_R}\Delta\boldsymbol{u}\|_2\\
&= \|\nabla(\sqrt{\psi_R}\boldsymbol{u})-\boldsymbol{u}\otimes\nabla\sqrt{\Psi_R}\|_3\|\boldsymbol{u}\|_6\|\sqrt{\psi_R}\Delta\boldsymbol{u}\|_2\\
&\le c\left(\|\nabla(\sqrt{\psi_R}\boldsymbol{u})\|_3+\|\boldsymbol{u}\otimes\nabla\sqrt{\Psi_R}\|_3\right)|\boldsymbol{u}|_{1,2}\|\sqrt{\psi_R}\Delta\boldsymbol{u}\|_2\,.
\end{aligned}
\tag{XI.1.10}
$$

From Nirenberg's Lemma II.3.3 and Exercise II.7.4, we also obtain

$$
\begin{aligned}
\|\nabla(\sqrt{\psi_R}\boldsymbol{u})\|_3 &\le c_1\|\nabla(\sqrt{\psi_R}\boldsymbol{u})\|_2^{1/2}\|D^2(\sqrt{\psi_R}\boldsymbol{u})\|_2^{1/2}\\
&= c_1\|\nabla(\sqrt{\psi_R}\boldsymbol{u})\|_2^{1/2}\|\Delta(\sqrt{\psi_R}\boldsymbol{u})\|_2^{1/2}\,.
\end{aligned}
\tag{XI.1.11}
$$

We next observe that by the properties of the function ψ_R, (XI.1.9), and again the Hölder inequality, we easily obtain

$$
\begin{aligned}
\|\Delta(\sqrt{\psi_R}\boldsymbol{u})\|_2 &\le \|\sqrt{\Psi_R}\Delta\boldsymbol{u}\|_2+2\|\nabla\boldsymbol{u}\cdot\nabla\sqrt{\psi_R}\|_2+\|\boldsymbol{u}\Delta\sqrt{\psi_R}\|_2\\
&\le \|\sqrt{\Psi_R}\Delta\boldsymbol{u}\|_2+c_2|\boldsymbol{u}|_{1,2}+c_3\left(\int_{B_{\rho,r}}|\boldsymbol{u}|^2+\int_{B_{R,2R}}\frac{|\boldsymbol{u}|^2}{R^2}\right)^{1/2}\\
&\le \|\sqrt{\Psi_R}\Delta\boldsymbol{u}\|_2+c_4(|\boldsymbol{u}|_{1,2}+\|\boldsymbol{u}\|_6)\\
&\le \|\sqrt{\Psi_R}\Delta\boldsymbol{u}\|_2+c_5|\boldsymbol{u}|_{1,2}\,,
\end{aligned}
\tag{XI.1.12}
$$

with c_5 independent of R. By a similar argument, we also obtain

$$
\begin{aligned}
\|\nabla(\sqrt{\psi_R}\boldsymbol{u})\|_2 &\le c_6\left(\int_{B_{\rho,r}}|\boldsymbol{u}|^2+\int_{B_{R,2R}}\frac{|\boldsymbol{u}|^2}{R^2}\right)^{1/2}+|\boldsymbol{u}|_{1,2}\\
&\le c_7\left(\|\boldsymbol{u}\|_6+|\boldsymbol{u}|_{1,2}\right)\le c_8|\boldsymbol{u}|_{1,2}\,,
\end{aligned}
\tag{XI.1.13}
$$

with c_8 independent of R. Thus, inserting the above inequalities into (XI.1.11), we deduce

$$
\|\nabla(\sqrt{\psi_R}\boldsymbol{u})\|_3 \le c_9\left(|\boldsymbol{u}|_{1,2}^{1/2}\|\sqrt{\psi_R}\Delta\boldsymbol{u}\|_2^{1/2}+|\boldsymbol{u}|_{1,2}\right).
$$

Since by an argument similar to that leading to (XI.1.13) we get

$$
\|\boldsymbol{u}\otimes\nabla\sqrt{\psi_R}\|_3\le c_{10}\left(\int_{B_{\rho,r}}|\boldsymbol{u}|^3+\int_{B_{R,2R}}\frac{|\boldsymbol{u}|^3}{R^3}\right)^{1/2}\le c_{11}\|\boldsymbol{u}\|_6\le c_{12}|\boldsymbol{u}|_{1,2}\,,
$$

from (XI.1.10) we deduce

$$
(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\psi_R\Delta\boldsymbol{u})\le c_{13}\left(|\boldsymbol{u}|_{1,2}^{1/2}\|\sqrt{\psi_R}\Delta\boldsymbol{u}\|_2^{1/2}+|\boldsymbol{u}|_{1,2}\right)|\boldsymbol{u}|_{1,2}\|\sqrt{\psi_R}\Delta\boldsymbol{u}\|_2\,.
$$

If we use Young's inequality (II.2.5) in this relation, we finally conclude that

$$
(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\psi_R\Delta\boldsymbol{u})\le c_{14}\left(|\boldsymbol{u}|_{1,2}^6+|\boldsymbol{u}|_{1,2}^4\right)+\varepsilon\|\sqrt{\psi_R}\Delta\boldsymbol{u}\|_2\,,
\tag{XI.1.14}
$$

where $\varepsilon > 0$ is arbitrary and c_{14} depends on ε, but is otherwise independent of R. It remains to estimate the last term on the right-hand side of (XI.1.8). Using one more time the Hölder inequality, (XI.1.9), and Nirenberg's Lemma II.3.3, we deduce

$$\begin{aligned}(\boldsymbol{u}\cdot\nabla\boldsymbol{u},(\nabla\times\boldsymbol{u})\times\nabla\psi_R) &\le c_{12}\|\boldsymbol{u}\|_6|\boldsymbol{u}|_{1,2}\|\sqrt{\psi_R}\nabla\boldsymbol{u}\|_3\\ &\le c_{13}|\boldsymbol{u}|_{1,2}^2\|\sqrt{\psi_R}\nabla\boldsymbol{u}\|_2^{1/2}\|\nabla(\sqrt{\psi_R}\nabla\boldsymbol{u})\|_2^{1/2}\\ &\le c_{14}|\boldsymbol{u}|_{1,2}^{5/2}\left(\|D^2(\sqrt{\psi_R}\boldsymbol{u})\|_2+\|\boldsymbol{u}D^2\sqrt{\psi_R}\|_2\right)^{1/2}.\end{aligned}$$

Thus, by (XI.1.12), (XI.1.9), the properties of ψ_R, and Young's inequality, from the previous inequality we obtain

$$\begin{aligned}(\boldsymbol{u}\cdot\nabla\boldsymbol{u},(\nabla\times\boldsymbol{u})\times\nabla\psi_R) &\le c_{15}\left(|\boldsymbol{u}|_{1,2}^{5/2}\|\sqrt{\psi_R}\Delta\boldsymbol{u}\|_2^{1/2}+|\boldsymbol{u}|_{1,2}^3\right)\\ &\le c_{16}\left(|\boldsymbol{u}|_{1,2}^{10/3}+|\boldsymbol{u}|_{1,2}^3\right)+\varepsilon\|\sqrt{\psi_R}\Delta\boldsymbol{u}\|_2^2,\end{aligned}\tag{XI.1.15}$$

where $\varepsilon > 0$ is arbitrary and c_{16} depends on ε, but is otherwise independent of R. Collecting (XI.1.8), (XI.1.14), and (XI.1.15), and choosing ε sufficiently small, we conclude that $\|\sqrt{\psi_R}\Delta\boldsymbol{u}\|_2 \le M$, with a constant M independent of R. This inequality formally coincides with (VIII.2.12), so that, arguing as in the proof of Lemma VIII.2.1, we prove (XI.1.7). □

The previous lemma has the following interesting consequence.

Corollary XI.1.1 *Let $\boldsymbol{v}$ be a generalized solution to* (XI.0.10), (XI.0.11) *corresponding to $\boldsymbol{f}$ that satisfies the assumption of* Lemma XI.1.2. *Then, $\boldsymbol{v}\in L^\infty(\Omega^r)$, $r>\rho$, and moreover,*

$$\lim_{|x|\to\infty}(\boldsymbol{v}(x)+\boldsymbol{v}_\infty(x))=\boldsymbol{0}\,,\quad \textit{uniformly}.\tag{XI.1.16}$$

Proof. Under the given assumptions on $\boldsymbol{f}$, in Theorem XI.1.2 below it will be proved that $\boldsymbol{v}\in W^{2,2}(\Omega_{r,R})$, for all $R>r$, and so by the embedding Theorem II.3.4, $\boldsymbol{v}+\boldsymbol{v}_\infty\in L^\infty(\Omega_{r,R})$, for all $R>r$. Consequently, in order to prove the corollary, it is enough to prove (XI.1.16). Since $\boldsymbol{v}$ is a generalized solution (see (XI.1.9)), we know that

$$\boldsymbol{u}\in L^6(\Omega^r)\cap D^{1,2}(\Omega^r)\,,\tag{XI.1.17}$$

where $\boldsymbol{u}:=\boldsymbol{v}+\boldsymbol{v}_\infty$. Moreover, from Lemma XI.1.2 we also have $\boldsymbol{u}\in D^{2,2}(\Omega^r)$, which, in turn, by (XI.1.17) and Theorem II.6.1, implies $\boldsymbol{u}\in D^{1,6}(\Omega^r)$. The property (XI.1.16) then follows from this latter, (XI.1.17), and Theorem II.9.1.

□

We are now in a position to prove the global summability properties of the pressure.

Theorem XI.1.1 *Let* $\boldsymbol{v}$ *be a generalized solution to* (XI.0.10), (XI.0.11), *and let* $\boldsymbol{f}$ *satisfy* (XI.1.4) *and, in addition,*

$$\boldsymbol{f} \in L^2(\Omega^\rho) \cap L^q(\Omega^\rho)\,, \quad \text{some } q \in (1,\infty) \text{ and } \rho > \delta(\Omega^c)\,.$$

Let p *be the pressure field associated to* $\boldsymbol{v}$ *by* Lemma XI.1.1, *with* $\widetilde{p}$ *defined in* (XI.1.6). *Then* $\widetilde{p}$, *possibly modified by the addition of a constant, admits the following decomposition:*

$$\widetilde{p} = p_1 + p_2\,, \tag{XI.1.18}$$

where

$$p_1 \in L^6(\Omega^r) \cap D^{1,2}(\Omega^r) \cap D^{1,q}(\Omega^r)\,, \quad p_2 \in L^3(\Omega^r)\,, \quad r > \rho. \tag{XI.1.19}$$

Moreover, if $q \in (1,3)$, *we also have*

$$p_1 \in L^{3q/(3-q)}(\Omega^r)\,. \tag{XI.1.20}$$

Thus, in particular, if $q \in (1,3/2)$, *we conclude that*

$$\widetilde{p} \in L^3(\Omega^r)\,. \tag{XI.1.21}$$

Proof. We set $\boldsymbol{u} := \boldsymbol{v} + \boldsymbol{v}_\infty$, and recall that, by Theorem XI.1.2, stated below, and the assumptions on $\boldsymbol{f}$, we can take $(\boldsymbol{u}, \widetilde{p}) \in W^{2,2}_{loc}(\Omega^\rho) \times W^{1,2}_{loc}(\Omega^\rho)$, so that in particular, $(\boldsymbol{u}, \widetilde{p})$ satisfies (XI.1.5). Therefore, we obtain

$$\left.\begin{array}{c} \Delta \boldsymbol{u} + \mathcal{R}\dfrac{\partial \boldsymbol{u}}{\partial x_1} + \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{u}) = \nabla \widetilde{p} + \boldsymbol{f} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) \\ \nabla \cdot \boldsymbol{u} = 0 \end{array}\right\} \text{a.e. in } \Omega^r. \tag{XI.1.22}$$

We next notice that by (XI.1.9),

$$\boldsymbol{u} \otimes \boldsymbol{u} \in L^3(\Omega^r)\,.$$

This latter property, along with the assumption on $\boldsymbol{f}$, allows us to conclude, by Lemma VIII.2.2, the validity of (XI.1.18)–(XI.1.21). The proof of the theorem is complete. □

Remark XI.1.2 We notice that in view of Remark VIII.2.1, Theorem XI.1.1 continues to hold if $\mathcal{T} = 0$. ■

We conclude this section by summarizing, in the next theorem, the differentiability properties of generalized solutions. Their proof is entirely analogous (and simpler, since we are in three dimensions) to that of Theorem X.1.1, and we leave it to the reader.

Theorem XI.1.2 *Let* $\boldsymbol{v}$ *be a generalized solution to* (XI.0.10), (XI.0.11). *Then, if*

$$\boldsymbol{f} \in W^{m,q}_{loc}(\Omega), \quad m \geq 0\,, \tag{XI.1.23}$$

where $q \in (1, \infty)$ *if* $m = 0$*, while* $q \in [3/2, \infty)$ *if* $m > 0$*, it follows that*

$$\boldsymbol{v} \in W^{m+2,q}_{loc}(\Omega), \quad p \in W^{m+1,q}_{loc}(\Omega),$$

where p *is the pressure associated to* $\boldsymbol{v}$ *by* Lemma XI.1.1. *Thus, in particular, if*

$$\boldsymbol{f} \in C^\infty(\Omega),$$

then

$$\boldsymbol{v}, p \in C^\infty(\Omega).$$

Assume further that Ω *is of class* C^{m+2} *and*

$$\boldsymbol{v}_* \in W^{m+2-1/q,q}(\partial\Omega), \quad \boldsymbol{f} \in W^{m,q}(\Omega_R)\,,$$

for some $R > \delta(\Omega^c)$ *and with the values of* m *and* q *specified earlier. Then, we have*

$$\boldsymbol{v} \in W^{m+2,q}(\Omega_R), \quad p \in W^{m+1,q}(\Omega_R).$$

Therefore, in particular, if Ω *is of class* C^∞ *and*

$$\boldsymbol{v}_* \in C^\infty(\partial\Omega), \quad \boldsymbol{f} \in C^\infty(\overline{\Omega}_R),$$

it follows that

$$\boldsymbol{v},\ p \in C^\infty(\overline{\Omega}_R).$$

XI.2 On the Energy Equation and the Uniqueness of Generalized Solutions

In this section we shall investigate two important properties of generalized solutions that, as shown in Sections X.2 and X.3 in the irrotational case, may be somewhat related. We will begin by giving sufficient conditions for a generalized solution to satisfy the *energy equation*:

$$\begin{aligned} 2\int_\Omega \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\boldsymbol{v}) - \int_{\partial\Omega} \big[(\boldsymbol{v}_\infty + \boldsymbol{v}_*) \cdot \boldsymbol{T}(\boldsymbol{v}, \widetilde{p}) - \tfrac{\mathcal{R}}{2}(\boldsymbol{v}_* + \boldsymbol{v}_\infty)^2 \boldsymbol{v}_*\big] \cdot \boldsymbol{n} \\ + \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} + \boldsymbol{v}_\infty) = 0\,, \end{aligned} \tag{XI.2.1}$$

where $\boldsymbol{v}_\infty$ is defined in (XI.0.11), $\widetilde{p}$ in (XI.1.6), and $\boldsymbol{T}$ is the Cauchy stress tensor (IV.8.6). This relation is formally obtained by dot-multiplying both sides of (XI.0.10) by $\boldsymbol{v} + \boldsymbol{v}_\infty$, integrating over Ω_R, and assuming that all the surface integrals on ∂B_R converge to zero as $R \to \infty$. We also notice that, up to the pressure term, (XI.2.1) coincides with the energy equality (X.2.29) proved for the irrotational case. The reason is that the total power of the rotational contribution vanishes identically.

We shall deal directly with the validity of (XI.2.1), leaving the analogous proof of the "generalized energy equality" (in the sense of Definition X.2.2) as an exercise to the interested reader. We recall that this latter coincides with (XI.2.1) for sufficiently smooth domains (for example, of class C^2) and data.

We thus have the following.

Theorem XI.2.1 *Let Ω be of class C^2, and assume*

$$\boldsymbol{f} \in L^{4/3}(\Omega)\,, \quad \boldsymbol{v}_* \in W^{5/4,4/3}(\partial\Omega)\,. \tag{XI.2.2}$$

Then, any generalized solution corresponding to the above data that in addition satisfies

$$(\boldsymbol{v} + \boldsymbol{v}_\infty) \in L^4(\Omega) \tag{XI.2.3}$$

satisfies the energy equality (XI.2.1).

Proof. Set $\boldsymbol{u} := \boldsymbol{v} + \boldsymbol{v}_\infty$. From (IV.8.9), (XI.2.2), and Theorem XI.1.2 we obtain

$$\left.\begin{aligned} \nabla\cdot\boldsymbol{T}(\boldsymbol{u},\widetilde{p}) + \lambda_1\frac{\partial\boldsymbol{u}}{\partial x_1} + \mathcal{T}\left(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{u} - \boldsymbol{e}_1\times\boldsymbol{u}\right) &= \lambda_2\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{f} \\ \nabla\cdot\boldsymbol{u} &= 0 \end{aligned}\right\}\text{a.e. in } \Omega\,, \tag{XI.2.4}$$

with $\widetilde{p}$ defined in (XI.1.6), and where $\lambda_1 = \lambda_2 = \mathcal{R}$ if $\boldsymbol{v}_0\cdot\boldsymbol{\omega} \neq 0$ (with $\boldsymbol{v}_\infty$ as in (XI.0.11)$_1$), while $\lambda_1 = 0$ and $\lambda_2 = \mathcal{T}$ if $\boldsymbol{v}_0\cdot\boldsymbol{\omega} = 0$ (with $\boldsymbol{v}_\infty$ as in (XI.0.11)$_2$). Since $\boldsymbol{v}$ is a weak solution, we have (see (XI.1.9))

$$\boldsymbol{u} \in D^{1,2}(\Omega) \cap L^6(\Omega)\,. \tag{XI.2.5}$$

We analyze first the case $v_0\cdot\boldsymbol{\omega} \neq 0$ (see (XI.0.11)$_1$), and notice that in particular, by (XI.2.2), (XI.2.3), and the Hölder inequality,

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{f} \in L^{4/3}(\Omega)\,. \tag{XI.2.6}$$

Consequently, from this property, (XI.2.2), (XI.2.5), and Theorem VIII.8.1, it follows that

$$\frac{\partial\boldsymbol{u}}{\partial x_1} \in L^{4/3}(\Omega)\,, \quad \widetilde{p} \in L^{12/5}(\Omega)\,. \tag{XI.2.7}$$

Let ψ_R be the "cut-off" function used in the proof of Theorem X.3.2, and let us dot-multiply both sides of (XI.2.4) by $\psi_R\boldsymbol{u}$. By an easily justified integration by parts on the resulting equation, and taking into account that $\boldsymbol{u} = \boldsymbol{v}_* + \boldsymbol{v}_\infty$ at $\partial\Omega$ (in the trace sense), we obtain

$$2(\boldsymbol{D}(\boldsymbol{u}),\, \boldsymbol{D}(\psi_R\boldsymbol{u})) - \int_{\partial\Omega} \big[(\boldsymbol{v}_\infty + \boldsymbol{v}_*)\cdot \boldsymbol{T}(\boldsymbol{v},\widetilde{p}) - \tfrac{\mathcal{R}}{2}(\boldsymbol{v}_* + \boldsymbol{v}_\infty)^2\boldsymbol{v}_*\big]\cdot\boldsymbol{n}$$
$$+\mathcal{R}\left(\frac{\partial \boldsymbol{u}}{\partial x_1},\psi_R\boldsymbol{u}\right) + \tfrac{1}{2}\mathcal{R}(|\boldsymbol{u}|^2\nabla\psi_R,\boldsymbol{u}) - (\widetilde{p}\boldsymbol{u},\nabla\psi_R) + (\boldsymbol{f},\psi_R\boldsymbol{u}) = 0\,, \tag{XI.2.8}$$

where we have used $\nabla\psi_R\cdot(\boldsymbol{e}_1\times\boldsymbol{x}) = 0$. We now recall the following properties of the function ψ_R:

$$\begin{aligned} &\lim_{R\to\infty}\psi_R(x) = 1\,, \quad \text{for all } x\in\Omega\,,\\ &\operatorname{supp}(\psi_R)\subset\Omega_{R,2R}\,, \quad |\nabla\psi_R|\le C/R \end{aligned} \tag{XI.2.9}$$

for a constant C independent of x and R. Thus, using this latter along with (XI.2.3), (XI.2.5), and (XI.2.7), the reader will have no difficulty in proving the following:

$$\begin{aligned} &\lim_{R\to\infty}(\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\psi_R\boldsymbol{u})) = (\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{u}))\,, \quad \lim_{R\to\infty}(\boldsymbol{f},\psi_R\boldsymbol{u}) = (\boldsymbol{f},\boldsymbol{u})\,,\\ &\lim_{R\to\infty}(|\boldsymbol{u}|^2\nabla\psi_R,\boldsymbol{u}) = 0\,, \quad \lim_{R\to\infty}\left(\frac{\partial\boldsymbol{u}}{\partial x_1},\psi_R\boldsymbol{u}\right) = \left(\frac{\partial\boldsymbol{u}}{\partial x_1},\boldsymbol{u}\right). \end{aligned} \tag{XI.2.10}$$

However, as shown in the proof of Theorem X.2.1, we obtain

$$\left(\frac{\partial\boldsymbol{u}}{\partial x_1},\boldsymbol{u}\right) = 0\,. \tag{XI.2.11}$$

Finally, by the Hölder inequality and by (XI.2.7), we have

$$\begin{aligned} |(\widetilde{p}\boldsymbol{u},\nabla\psi_R)| &\le \|\widetilde{p}\|_{12/5,\Omega_{R,2R}}\|\boldsymbol{u}\|_{4,\Omega_{R,2R}}\|\nabla\psi_R\|_{3,\Omega_{R,2R}}\\ &\le C\,\|\widetilde{p}\|_{12/5,\Omega_{R,2R}}\|\boldsymbol{u}\|_{4,\Omega_{R,2R}}\,, \end{aligned}$$

where in the last inequality, we have used $(\text{XI.2.9})_3$. Thus,

$$\lim_{R\to\infty}(\widetilde{p}\boldsymbol{u},\nabla\psi_R) = 0\,. \tag{XI.2.12}$$

Consequently, letting $R\to\infty$ in (XI.2.8) and employing (XI.2.10)–(XI.2.12), we recover (XI.2.1), which is therefore proved in the case $(\text{XI.0.11})_1$. The proof in the case $(\text{XI.0.11})_2$ is entirely analogous. (We recall that in this situation, in (XI.2.4) we must take $\lambda_1 = 0$ and $\lambda_2 = \mathcal{T}$.) It suffices to observe that from (XI.2.3), (XI.2.6), and Theorem VIII.7.2, it follows that $\widetilde{p}$ (up to an inessential constant) satisfies (XI.2.7). Consequently, following step by step the argument previously used for the case $(\text{XI.0.11})_1$, we prove the desired property also in case $(\text{XI.0.11})_2$. The theorem is completely proved. □

Our next objective is to furnish sufficient conditions for uniqueness of generalized solutions. We have the following.

Theorem XI.2.2 *Let Ω, $\boldsymbol{f}$, and $\boldsymbol{v}_*$ satisfy the assumptions of* Theorem XI.2.1, *and let $\boldsymbol{v}$, $\boldsymbol{v}_1$ be two corresponding generalized solutions satisfying, in addition,* (XI.2.3).[1] *Then, if $(\boldsymbol{v}+\boldsymbol{v}_\infty) \in L^3(\Omega)$ with*

$$\|\boldsymbol{v}+\boldsymbol{v}_\infty\|_3 < \frac{\sqrt{3}}{2\mathcal{R}}\,, \tag{XI.2.13}$$

necessarily $\boldsymbol{v}(x)=\boldsymbol{v}_1(x)$, a.e. in Ω.

Proof. We shall give the proof when $\boldsymbol{v}_0\cdot\boldsymbol{\omega}\neq 0$, so that $\boldsymbol{v}_\infty$ is given in $(\text{XI.0.11})_1$. In fact, as the reader will immediately realize, it remains basically unchanged in the case $\boldsymbol{v}_0\cdot\boldsymbol{\omega}=0$. Set $\boldsymbol{u}:=\boldsymbol{v}_1-\boldsymbol{v}$, $\phi:=\widetilde{p}_1-\widetilde{p}$, $\boldsymbol{w}:=\boldsymbol{v}+\boldsymbol{v}_\infty$, $\boldsymbol{w}_1:=\boldsymbol{v}_1+\boldsymbol{v}_\infty$, where $\widetilde{p}$ and $\widetilde{p}_1$ are the (modified, according to (XI.1.6)) pressure fields associated to $\boldsymbol{v}$ and $\boldsymbol{v}_1$ by Lemma XI.1.1. From (IV.8.9), (XI.2.2), and Theorem XI.1.2 we thus obtain

$$\left.\begin{aligned}\nabla\cdot\boldsymbol{T}(\boldsymbol{u},\phi)+\mathcal{R}\frac{\partial\boldsymbol{u}}{\partial x_1}&+\mathcal{T}\,(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{u}-\boldsymbol{e}_1\times\boldsymbol{u})\\ &=\mathcal{R}\,(\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\boldsymbol{w}\cdot\nabla\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{w})\\ \nabla\cdot\boldsymbol{u}=0&\end{aligned}\right\}\ \text{a.e. in } \Omega\,,$$
$$\boldsymbol{u}=\boldsymbol{0}\ \text{ at } \partial\Omega\,. \tag{XI.2.14}$$

We now follow exactly the same procedure used in the proof of Theorem XI.2.1, that is, after dot-multiplying both sides of $(\text{XI.2.14})_1$ by $\psi_R\boldsymbol{u}$, integrating by parts over Ω, and taking into account the boundary conditions, we let $R\to\infty$. Using the fact that both $\boldsymbol{v}$ and $\boldsymbol{v}_1$ satisfy (XI.2.3) we then show that $\boldsymbol{u}$ obeys the following "perturbed" energy equality:

$$|\boldsymbol{u}|_{1,2}^2=\mathcal{R}(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{w})\,, \tag{XI.2.15}$$

where we have used the relation $|\boldsymbol{u}|_{1,2}=\sqrt{2}\|\boldsymbol{D}(\boldsymbol{u})\|_2$.[2] By (XI.2.15), the Hölder inequality, and the Sobolev inequality (II.3.11), we thus obtain

$$|\boldsymbol{u}|_{1,2}^2\left(1-\mathcal{R}\frac{2}{\sqrt{3}}\|\boldsymbol{w}\|_3\right)\le 0\,,$$

which, in turn, proves the result under the assumption (XI.2.13). □

For future use, we need another uniqueness result that generalizes to the rotational case the one proved in Theorem X.3.2. We shall limit ourselves to the case $\boldsymbol{v}_0\cdot\boldsymbol{\omega}=0$, namely, when $\boldsymbol{v}_\infty$ satisfies $(\text{XI.0.11})_2$, in that it will suffice for our purposes.

Thus, let $\mathfrak{C}$ *be the class of generalized solutions $\boldsymbol{v}$ to* (XI.0.10), (XI.0.11) *with $\boldsymbol{v}_0\cdot\boldsymbol{\omega}=0$ satisfying the energy inequality*

[1] The assumption on $\boldsymbol{v}_1$ can be somehow weakened. See Exercise XI.2.1.

[2] See Footnote 4 in Chapter X.

$$2\int_{\Omega} \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\boldsymbol{v}) - \int_{\partial\Omega} \big[(\boldsymbol{v}_\infty + \boldsymbol{v}_*) \cdot \boldsymbol{T}(\boldsymbol{v}, \widetilde{p}) - \tfrac{\mathcal{T}}{2}(\boldsymbol{v}_* + \boldsymbol{v}_\infty)^2 \boldsymbol{v}_*\big] \cdot \boldsymbol{n}$$
$$+ \int_{\Omega} \boldsymbol{f} \cdot (\boldsymbol{v} + \boldsymbol{v}_\infty) \le 0\,. \tag{XI.2.16}$$

As we shall prove in the following section, under suitable regularity assumptions on Ω, $\boldsymbol{f}$, and $\boldsymbol{v}_*$, the class $\mathfrak{C}$ is not empty.

We have the following.

Theorem XI.2.3 *Let Ω be of class C^2, and let*

$$\boldsymbol{f} \in L^2(\Omega) \cap L^{6/5}(\Omega)\,, \quad \boldsymbol{v}_* \in W^{3/2,2}(\partial\Omega)\,, \quad \boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0.$$

Suppose $\boldsymbol{v}$ is a corresponding generalized solution such that for all $x \in \Omega$,

$$(1 + |x|)|\boldsymbol{v}(x) + \boldsymbol{v}_\infty| \le M\,, \quad M < \frac{1}{2\mathcal{T}}\,. \tag{XI.2.17}$$

Then $\boldsymbol{v}$ is unique in the class $\mathfrak{C}$.

Proof. Let $\boldsymbol{v}_1$ be any element in $\mathfrak{C}$ corresponding to the given data, let p_1 be the associated pressure, and set $\boldsymbol{w} := \boldsymbol{v}_1 + \boldsymbol{v}_\infty$. From (IV.8.9), (XI.2.2), and Theorem XI.1.2 we deduce that $\boldsymbol{w}$ satisfies

$$\left.\begin{aligned} \nabla \cdot \boldsymbol{T}(\boldsymbol{w}, \widetilde{p}_1) + \mathcal{T}\,(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{w} - \boldsymbol{e}_1 \times \boldsymbol{w}) &= \mathcal{T} \boldsymbol{w} \cdot \nabla \boldsymbol{w} + \boldsymbol{f} \\ \nabla \cdot \boldsymbol{w} &= 0 \end{aligned}\right\} \text{ a.e. in } \Omega\,,$$
$$\boldsymbol{w} = \boldsymbol{v}_* + \boldsymbol{v}_\infty \equiv \boldsymbol{d}_* \ \text{ at } \partial\Omega, \tag{XI.2.18}$$

with $\widetilde{p}_1$ the modified pressure as in (XI.1.6). Dot-multiplying both sides of (XI.2.18)$_1$ by $\boldsymbol{u} := \boldsymbol{v} + \boldsymbol{v}_\infty$ and integrating by parts over Ω_R, we obtain

$$\begin{aligned} &-2\int_{\Omega_R} \boldsymbol{D}(\boldsymbol{w}) : \boldsymbol{D}(\boldsymbol{u}) + 2\int_{\partial B_R} \boldsymbol{n} \cdot \boldsymbol{D}(\boldsymbol{w}) \cdot \boldsymbol{u} - \int_{\partial B_R} \widetilde{p}_1 (\boldsymbol{u} \cdot \boldsymbol{n}) \\ &\quad -\mathcal{T}\int_{\Omega_R} \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{u} + \mathcal{T}\int_{\Omega_R} (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{w} - \boldsymbol{e}_1 \times \boldsymbol{w}) \cdot \boldsymbol{u} \\ &\qquad = -\int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{T}(\boldsymbol{w}, \widetilde{p}_1) \cdot \boldsymbol{d}_* + \int_{\Omega_R} \boldsymbol{f} \cdot \boldsymbol{u}\,. \end{aligned} \tag{XI.2.19}$$

Likewise, interchanging the roles of $\boldsymbol{u}$ and $\boldsymbol{w}$, we get

$$\begin{aligned} &-2\int_{\Omega_R} \boldsymbol{D}(\boldsymbol{w}) : \boldsymbol{D}(\boldsymbol{u}) + 2\int_{\partial B_R} \boldsymbol{n} \cdot \boldsymbol{D}(\boldsymbol{u}) \cdot \boldsymbol{w} - \int_{\partial B_R} \widetilde{\phi} (\boldsymbol{w} \cdot \boldsymbol{n}) \\ &\quad -\mathcal{T}\int_{\Omega_R} \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{w} + \mathcal{T}\int_{\Omega_R} (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{u}) \cdot \boldsymbol{w} \\ &\qquad = -\int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{T}(\boldsymbol{u}, \widetilde{\phi}) \cdot \boldsymbol{d}_* + \int_{\Omega_R} \boldsymbol{f} \cdot \boldsymbol{w}\,, \end{aligned} \tag{XI.2.20}$$

where $\widetilde{\phi}$ is the modified pressure, according to (XI.1.6), associated to $\boldsymbol{v}$. Our next task is to show that all integrals on ∂B_R in (XI.2.19) and (XI.2.20) converge to zero as $R \to \infty$, along a sequence at least. To this end, we notice that in (XI.2.18), written with $\boldsymbol{w} \equiv \boldsymbol{u}$ and $\widetilde{p}_1 \equiv \widetilde{\phi}$, its right-hand side can be put in the form $\mathcal{T}\,\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u})+\boldsymbol{f}$. Since $\boldsymbol{u}$ decays pointwise in the way specified in (XI.2.17), and $\boldsymbol{u} \in D^{1,2}(\Omega)$, it follows that $\boldsymbol{u}\otimes\boldsymbol{u} \in L^2(\Omega)$ and $\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}) \in L^2(\Omega)$. Moreover, by assumption, $\boldsymbol{f} \in L^{6/5}(\Omega)$. As a consequence, from Lemma VIII.2.2 we deduce $\widetilde{\phi} \in L^2(\Omega^R)$, for sufficiently large R. Moreover, by Theorem XI.1.2, $\widetilde{p}_1 \in L^3(\Omega^R)$. Thus, recalling also that $\boldsymbol{w} \in L^6(\Omega)$, we can find an unbounded sequence $\{R_k\}$ such that

$$\lim_{k\to\infty} R_k \int_{\partial B_{R_k}} \left(|\widetilde{p}_1|^3 + |\nabla\boldsymbol{w}|^2 + |\boldsymbol{w}|^6 + |\widetilde{\phi}|^2 + |\nabla\boldsymbol{u}|^2 + |\boldsymbol{u}|^2\right) = 0\,. \quad \text{(XI.2.21)}$$

Using this property, we shall now show that all surface integrals over ∂B_R in (XI.2.19), (XI.2.20) tend to zero as $R \to \infty$, at least along the sequence $\{R_k\}$. In fact, setting

$$[\![\boldsymbol{u}]\!]_1 := \sup_{x\in\Omega}(1+|x|)|\boldsymbol{u}(x)|\,,$$

we have, as $k \to \infty$,

$$\begin{aligned}
&\left|\int_{\partial B_{R_k}} \boldsymbol{n}\cdot\boldsymbol{D}(\boldsymbol{w})\cdot\boldsymbol{u}\right| \le c_1\,[\![\boldsymbol{u}]\!]_1 \int_{\partial B_{R_k}} \frac{|\nabla\boldsymbol{w}|}{R_k} \le c_2[\![\boldsymbol{u}]\!]_1\,\|\nabla\boldsymbol{w}\|_{2,\partial B_{R_k}} \to 0\,,\\
&\left|\int_{\partial B_{R_k}} \widetilde{p}_1(\boldsymbol{u}\cdot\boldsymbol{n})\right| \le c_3\,[\![\boldsymbol{u}]\!]_1 \int_{\partial B_{R_k}} \frac{|\widetilde{p}_1|}{R_k} \le c_4[\![\boldsymbol{u}]\!]_1 R_k^{\frac{1}{3}}\,\|\widetilde{p}_1\|_{3,\partial B_{R_k}} \to 0\,,\\
&\left|\int_{\partial B_{R_k}} \boldsymbol{n}\cdot\boldsymbol{D}(\boldsymbol{u})\cdot\boldsymbol{w}\right| \le c_5 R_k^{\frac{2}{3}}\|\nabla\boldsymbol{u}\|_{2,\partial B_{R_k}}\|\boldsymbol{w}\|_{6,\partial B_{R_k}} \to 0\,,\\
&\left|\int_{\partial B_{R_k}} \widetilde{\phi}(\boldsymbol{w}\cdot\boldsymbol{n})\right| \le c_6\,R_k^{\frac{2}{3}}\,\|\widetilde{\phi}\|_{2,\partial B_{R_k}}\|\boldsymbol{w}\|_{6,\partial B_{R_k}} \to 0\,.
\end{aligned}$$

(XI.2.22)

We next investigate the convergence of the volume integrals in (XI.2.19), (XI.2.20). To this end, we begin by observing that in view of the assumptions made on $\boldsymbol{u}$ and Theorem II.6.1(i), we obtain

$$\left|\int_\Omega \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{u}\right| \le c_7[\![\boldsymbol{u}]\!]_1\,|\boldsymbol{w}|_{1,2}^2\,,$$

so that

$$\lim_{k\to\infty}\int_{\Omega_{R_k}} \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{u} = \int_\Omega \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{u}\,. \quad \text{(XI.2.23)}$$

By the same token,

$$\lim_{k\to\infty}\int_{\Omega_{R_k}} \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{w} = \int_{\Omega} \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{w}\,. \tag{XI.2.24}$$

Likewise, since $\boldsymbol{f}\in L^{6/5}(\Omega)$ and $\boldsymbol{u},\boldsymbol{w}\in L^6(\Omega)$, we deduce

$$\lim_{k\to\infty}\int_{\Omega_{R_k}} \boldsymbol{f}\cdot\boldsymbol{w} = \int_{\Omega} \boldsymbol{f}\cdot\boldsymbol{w}\,,\quad \lim_{k\to\infty}\int_{\Omega_{R_k}} \boldsymbol{f}\cdot\boldsymbol{u} = \int_{\Omega} \boldsymbol{f}\cdot\boldsymbol{u}\,. \tag{XI.2.25}$$

We now observe that by an integration by parts, we can show for all $R>\delta(\Omega^c)$ that

$$\begin{aligned}\int_{\Omega_R}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{w}-\boldsymbol{e}_1\times\boldsymbol{w})\cdot\boldsymbol{u}+\int_{\Omega_R}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{u}-\boldsymbol{e}_1\times\boldsymbol{u})\cdot\boldsymbol{w}&\\ =\int_{\partial B_R}(\boldsymbol{e}_1\times\boldsymbol{x})\cdot\boldsymbol{n}\,\boldsymbol{w}\cdot\boldsymbol{u}+\int_{\partial\Omega}(\boldsymbol{e}_1\times\boldsymbol{x})\cdot\boldsymbol{n}\,\boldsymbol{d}^2&\\ =\int_{\partial\Omega}(\boldsymbol{e}_1\times\boldsymbol{x})\cdot\boldsymbol{n}\,\boldsymbol{d}_*^2\,,&\end{aligned} \tag{XI.2.26}$$

where in the last step, we have taken into account that on ∂B_R, $\boldsymbol{n}$ and $\boldsymbol{x}$ are parallel. Adding side by side (XI.2.18) and (XI.2.19), using (XI.2.26), and then passing to the limit $k\to\infty$, with the help of (XI.2.23)–(XI.2.25) we recover

$$\begin{aligned}-4\int_{\Omega}\boldsymbol{D}(\boldsymbol{w}):\boldsymbol{D}(\boldsymbol{u}) &= \mathcal{T}\int_{\Omega}(\boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{w})+\int_{\Omega}(\boldsymbol{f}\cdot\boldsymbol{w}+\boldsymbol{f}\cdot\boldsymbol{u})\\ &\quad+\int_{\partial\Omega}\Big(\boldsymbol{n}\cdot\boldsymbol{T}(\boldsymbol{w},\widetilde{p}_1)+\boldsymbol{n}\cdot\boldsymbol{T}(\boldsymbol{u},\widetilde{\phi})\Big)\cdot\boldsymbol{d}_*\\ &\quad-\mathcal{T}\int_{\partial\Omega}(\boldsymbol{e}_1\times\boldsymbol{x})\cdot\boldsymbol{n}\,\boldsymbol{d}_*^2\,.\end{aligned} \tag{XI.2.27}$$

However, by (XI.2.17) and Theorem XI.2.1, $(\boldsymbol{u},\widetilde{\phi})$ satisfies the energy equality (XI.2.1), while by assumption, $(\boldsymbol{w},\widetilde{p}_1)$ satisfies the energy inequality (XI.2.16). Thus adding, side by side, these two relations and (XI.2.27), and observing that

$$\|\boldsymbol{D}(\boldsymbol{w}-\boldsymbol{u})\|_2^2=\|\boldsymbol{D}(\boldsymbol{w})\|_2^2+\|\boldsymbol{D}(\boldsymbol{u})\|_2^2-2(\boldsymbol{D}(\boldsymbol{w}),\boldsymbol{D}(\boldsymbol{u}))\,,$$

we easily deduce

$$\|\boldsymbol{D}(\boldsymbol{z})\|_2^2\le\mathcal{T}\left((\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{u})+(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{w})\right)-\mathcal{T}\int_{\partial\Omega}\boldsymbol{d}_*^2\boldsymbol{d}_*\cdot\boldsymbol{n}\,, \tag{XI.2.28}$$

with $\boldsymbol{z}:=\boldsymbol{w}-\boldsymbol{u}$. However, by a slight modification of the reasoning used in the proof of Theorem X.3.2[3] we show that

[3] Specifically, see the proof of properties (i) and (ii) after (X.3.29) and the argument that follows.

$$(\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{u})+(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{w})=(\boldsymbol{z}\cdot\nabla\boldsymbol{z},\boldsymbol{u})+\int_{\partial\Omega}\boldsymbol{d}_*^2\boldsymbol{d}_*\cdot\boldsymbol{n}\,,$$

and so, placing this latter into (XI.2.27), we conclude that

$$2\|\boldsymbol{D}(\boldsymbol{z})\|_2^2\le\mathcal{T}(\boldsymbol{z}\cdot\nabla\boldsymbol{z},\boldsymbol{u})\,.$$

Using in this relation the assumption (XI.2.17) along with the Schwarz inequality, we obtain

$$2\|\boldsymbol{D}(\boldsymbol{z})\|_2^2\le\mathcal{T}\,M\left(\int_\Omega\frac{|\boldsymbol{z}|^2}{|x|^2}\right)^{1/2}|\boldsymbol{z}|_{1,2}\,. \tag{XI.2.29}$$

We now observe that for $\boldsymbol{\varphi}\in\mathcal{D}(\Omega)$, by dot-multiplying both sides of the identity $\Delta\boldsymbol{\varphi}=2\nabla\cdot\boldsymbol{D}(\boldsymbol{\varphi})$ by $\boldsymbol{\varphi}$ and integrating by parts, we have

$$2\|\boldsymbol{D}(\boldsymbol{\varphi})\|_2^2=|\boldsymbol{\varphi}|_{1,2}^2. \tag{XI.2.30}$$

Moreover, by (II.6.10), we also have that

$$\int_\Omega\frac{|\boldsymbol{\varphi}|^2}{|x|^2}\le 4|\boldsymbol{\varphi}|_{1,2}^2\,. \tag{XI.2.31}$$

Now, $\boldsymbol{z}\in D^{1,2}(\Omega)$, with $\nabla\cdot\boldsymbol{z}=0$. Furthermore, $\boldsymbol{z}$ has zero trace at the boundary, and so by Theorem III.5.1, $\boldsymbol{z}\in\mathcal{D}_0^{1,2}(\Omega)$. Then, by a simple density argument we prove that (XI.2.30) and (XI.2.31) continue to hold for $\boldsymbol{z}$ too. Thus, placing (XI.2.30) and (XI.2.31) (with $\boldsymbol{\varphi}\equiv\boldsymbol{z}$) into (XI.2.29), we obtain

$$|\boldsymbol{z}|_{1,2}^2\,(1-2\mathcal{T}M)\le 0\,,$$

from which, if $M<1/(2\mathcal{T})$, uniqueness follows. □

Exercise XI.2.1 Show that in the uniqueness Theorem XI.2.2 the hypothesis $\boldsymbol{v}_1\in L^4(\Omega)$ can be replaced by the assumption that $\boldsymbol{v}_1$ and the associated pressure p_1 satisfy the energy inequality (XI.2.16). As shown in the next section, the class of such solutions is not empty.

XI.3 Existence of Generalized Solutions

The objective of this section is to prove the existence of a generalized solution to problem (XI.0.10), (XI.0.11). This will be achieved by a suitable generalization of the arguments used in the proof of Theorem X.4.1.

We begin with the following result.

Lemma XI.3.1 *Let Ω be locally Lipschitz and let*

$$\boldsymbol{v}_*\in W^{1/2,2}(\partial\Omega)\,,\quad \boldsymbol{v}_\infty \textit{ be given in } (\text{XI.0.11})\,.$$

Then, for any $\eta > 0$ *there exist* $\varepsilon = \varepsilon(\eta, \boldsymbol{v}_*, \Omega) > 0$ *and* $\boldsymbol{V} = \boldsymbol{V}(\varepsilon) : \Omega \to \mathbb{R}^3$ *satisfying properties* (i)–(iv) *of* Lemma X.4.1 *with* $\mathfrak{v}_\infty \equiv \boldsymbol{v}_\infty$. *The field* $\boldsymbol{V}$ *can be written as follows:*

$$\boldsymbol{V}(x) = \boldsymbol{V}_\varepsilon(x) + \Phi\boldsymbol{\sigma}(x) - \boldsymbol{v}_\infty(x)\,, \tag{XI.3.1}$$

where $\boldsymbol{V}_\varepsilon(x)$ *is of bounded support in* Ω, *and*

$$\Phi := \int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n}\,, \qquad \boldsymbol{\sigma}(x) = \frac{1}{4\pi}\nabla\left(\frac{1}{|x - x_0|}\right)\,, \tag{XI.3.2}$$

with $x_0 \in \overset{\circ}{\Omega}{}^c$. *Moreover, for all* $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$, *we have*

$$|(\boldsymbol{u}\cdot\nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \boldsymbol{u})| \le \left(\eta + \frac{|\Phi|}{4\pi r_0}\right)|\boldsymbol{u}|_{1,2}^2\,, \tag{XI.3.3}$$

where $r_0 = \operatorname{dist}(x_0, \partial\Omega)$. *Finally, if* $\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \le M$, *for some* $M > 0$, *then* $\boldsymbol{V}$ *satisfies the inequalities given in* (X.4.6), *with* $\mathfrak{v}_\infty \equiv \boldsymbol{v}_\infty$.

Proof. The proof is a direct consequence of that of Lemma X.4.1 (with $\mathfrak{v}_\infty \equiv \boldsymbol{v}_\infty$), and Remark X.4.1.[1] □

We are now in a position to prove the main result of this section.

Theorem XI.3.1 *Let* Ω *be a locally Lipschitz domain of* $\mathbb{R}^3$, *with a connected boundary. Moreover, let*

$$\boldsymbol{f} \in D_0^{-1,2}(\Omega), \quad \boldsymbol{v}_* \in W^{1/2}(\partial\Omega), \quad \boldsymbol{v}_\infty \text{ be given in (XI.0.11)}.$$

The following properties hold, with Φ *defined in* (XI.3.2).

(i) Existence. *If* $|\Phi| < 4\pi r_0/\mathcal{R}$, *there is at least one generalized solution* $\boldsymbol{v}$ *to the Navier–Stokes problem* (XI.0.10), (XI.0.11). *Such a solution satisfies the conditions:*

$$\int_{S^2} |\boldsymbol{v}(x) + \boldsymbol{v}_\infty| = O(1/\sqrt{|x|}) \ \text{ as } |x| \to \infty\,,$$
$$\|p\|_{2,\Omega_R/\mathbb{R}} \le c\,(|\boldsymbol{v}|_{1,2,\Omega_R} + \mathcal{R}\|\boldsymbol{v}\|_{1,2\Omega_R}^2 + \mathcal{T}\|\boldsymbol{v}\|_{2,\Omega_R} + |\boldsymbol{f}|_{1,2})\,, \tag{XI.3.4}$$

for all $R > \delta(\Omega^c)$, *where* p *is the pressure field associated to* $\boldsymbol{v}$ *by* Lemma XI.1.1, *while* $c = c(\Omega, R)$ *with* $c \to \infty$ *as* $R \to \infty$.

Furthermore, if Ω *is of class* C^2, $\boldsymbol{f} \in L^2(\Omega)$, *and* $\boldsymbol{v}_* \in W^{3/2,2}(\partial\Omega)$, *then* $\boldsymbol{v}$ *and the corresponding pressure field* p *(see* Theorem XI.1.2*) satisfy the* energy inequality

[1] We recall that throughout this chapter, we are assuming for simplicity that $\overset{\circ}{\Omega}{}^c$ is connected.

$$2\int_{\Omega} \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\boldsymbol{v}) - \int_{\partial\Omega} [(\boldsymbol{v}_\infty + \boldsymbol{v}_*) \cdot \boldsymbol{T}(\boldsymbol{v}, \widetilde{p}) - \tfrac{\mathcal{R}}{2}(\boldsymbol{v}_* + \boldsymbol{v}_\infty)^2 \boldsymbol{v}_*] \cdot \boldsymbol{n}$$
$$+[\boldsymbol{f}, \boldsymbol{v} + \boldsymbol{v}_\infty] \le 0\,, \tag{XI.3.5}$$

where $\widetilde{p}$ *is defined in* (XI.1.6), *and* $\boldsymbol{T}$ *the Cauchy stress tensor* (IV.8.6).

(ii) Estimate by the data. *If* $\boldsymbol{v}_* \in \mathfrak{M}_M^{1/2,2}(\partial\Omega)$ *(defined in* (IX.4.52)*) and* $|\Phi| \le 2\pi r_0/\mathcal{R}$, *then the generalized solution determined in* (i) *satisfies the following estimate:*

$$|\boldsymbol{v} + \boldsymbol{v}_\infty|_{1,2} \le 4|\boldsymbol{f}|_{-1,2} + C\,\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\big[1 + \mathcal{R}\big(1 + \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\big) + \mathcal{T}\big]\,, \tag{XI.3.6}$$

where $C = C(\Omega, \mathcal{R}, M)$.

Proof. We look for a solution of the form $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{V}$, where $\boldsymbol{V}$ is the extension constructed in Lemma XI.3.1. Placing this latter into (XI.1.2), after a simple manipulation we obtain, for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$,

$$\begin{aligned}(\nabla\boldsymbol{u}, \nabla\boldsymbol{\varphi}) &+ (\nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \nabla\boldsymbol{\varphi}) + \mathcal{R}\,[(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{\varphi}) + (\boldsymbol{u}\cdot\nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \boldsymbol{\varphi})\\ &-(\boldsymbol{u}\cdot\nabla\boldsymbol{v}_\infty, \boldsymbol{\varphi}) + ((\boldsymbol{V} + \boldsymbol{v}_\infty)\cdot\nabla\boldsymbol{u}, \boldsymbol{\varphi}) - (\boldsymbol{v}_\infty\cdot\nabla\boldsymbol{u}, \boldsymbol{\varphi})\\ &+((\boldsymbol{V} + \boldsymbol{v}_\infty)\cdot\nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \boldsymbol{\varphi}) - ((\boldsymbol{V} + \boldsymbol{v}_\infty)\cdot\nabla\boldsymbol{v}_\infty, \boldsymbol{\varphi})\\ &-(\boldsymbol{v}_\infty\cdot\nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \boldsymbol{\varphi})] + 2\mathcal{T}\,[(\boldsymbol{e}_1\times\boldsymbol{u}, \boldsymbol{\varphi}) + (\boldsymbol{e}_1\times(\boldsymbol{V} + \boldsymbol{v}_\infty), \boldsymbol{\varphi})]\\ &\quad = -[\boldsymbol{f}, \boldsymbol{\varphi}]\,,\end{aligned} \tag{XI.3.7}$$

where we have used $(\nabla\boldsymbol{v}_\infty, \nabla\boldsymbol{\varphi}) = (\boldsymbol{e}_1\times\boldsymbol{v}_\infty, \boldsymbol{\varphi}) = 0$. Taking into account the identity $\mathcal{R}\boldsymbol{a}\cdot\nabla\boldsymbol{v}_\infty = \mathcal{T}\boldsymbol{e}_1\times\boldsymbol{a}$, and that by a calculation entirely analogous to that leading to (VIII.1.8),

$$\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{\sigma} - \boldsymbol{e}_1\times\boldsymbol{\sigma} = \boldsymbol{0}, \tag{XI.3.8}$$

with the help of (XI.3.1) we deduce that (XI.3.7) reduces to

$$\begin{aligned}&(\nabla\boldsymbol{u}, \nabla\boldsymbol{\varphi}) + (\nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \nabla\boldsymbol{\varphi}) + \mathcal{R}\,[(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{\varphi}) + (\boldsymbol{u}\cdot\nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \boldsymbol{\varphi})\\ &\quad + ((\boldsymbol{V} + \boldsymbol{v}_\infty)\cdot\nabla\boldsymbol{u}, \boldsymbol{\varphi}) - (\boldsymbol{v}_\infty\cdot\nabla\boldsymbol{u}, \boldsymbol{\varphi}) + ((\boldsymbol{V} + \boldsymbol{v}_\infty)\cdot\nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \boldsymbol{\varphi})\\ &\quad - (\boldsymbol{e}_1\cdot\nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \boldsymbol{\varphi})] + \mathcal{T}\,[(\boldsymbol{e}_1\times\boldsymbol{u}, \boldsymbol{\varphi}) + (\boldsymbol{e}_1\times\boldsymbol{V}_\varepsilon - \boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{V}_\varepsilon, \boldsymbol{\varphi})]\\ &\quad\quad = -[\boldsymbol{f}, \boldsymbol{\varphi}]\,.\end{aligned} \tag{XI.3.9}$$

It is clear that if we find $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ satisfying (XI.3.9) for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$, then $\boldsymbol{v} := \boldsymbol{u} + \boldsymbol{V}$ is a generalized solution to (XI.0.10), (XI.0.11). In order to construct such a $\boldsymbol{u}$ we use the Galerkin method. Let $\{\boldsymbol{\varphi}_k\} \subset \mathcal{D}(\Omega)$ be the basis of $\mathcal{D}_0^{1,2}(\Omega)$, introduced in Lemma VII.2.1. A sequence of approximating solutions $\{\boldsymbol{u}_m\}$ is then sought of the form

$$
\begin{aligned}
&\boldsymbol{u}_m := \sum_{k=1}^{m} \xi_{km} \boldsymbol{\psi}_k \\
&(\nabla \boldsymbol{u}_m, \nabla \boldsymbol{\varphi}_k) = -(\nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \nabla \boldsymbol{\varphi}_k) - \mathcal{R}\left[(\boldsymbol{u}_m \cdot \nabla \boldsymbol{u}_m, \boldsymbol{\varphi}_k)\right. \\
&\quad + (\boldsymbol{u}_m \cdot \nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \boldsymbol{\varphi}_k) + ((\boldsymbol{V} + \boldsymbol{v}_\infty) \cdot \nabla \boldsymbol{u}_m, \boldsymbol{\varphi}_k) - (\boldsymbol{v}_\infty \cdot \nabla \boldsymbol{u}_m, \boldsymbol{\varphi}_k) \\
&\quad \left. + ((\boldsymbol{V} + \boldsymbol{v}_\infty) \cdot \nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \boldsymbol{\varphi}_k) - (\boldsymbol{e}_1 \cdot \nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \boldsymbol{\varphi})\right] \\
&\quad - \mathcal{T}\left[(\boldsymbol{e}_1 \times \boldsymbol{u}, \boldsymbol{\varphi}) + (\boldsymbol{e}_1 \times \boldsymbol{V}_\varepsilon - \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{V}_\varepsilon, \boldsymbol{\varphi})\right] - [\boldsymbol{f}, \boldsymbol{\varphi}_k] := F_k(\boldsymbol{\xi}),
\end{aligned}
\tag{XI.3.10}
$$

$k = 1, \ldots, m$. For each $m \in \mathbb{N}$, we may establish existence of a solution $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)$ to (XI.3.10) by means of Lemma IX.3.2. In fact, since $\nabla \cdot \boldsymbol{v}_\infty = \nabla \cdot (\boldsymbol{V} + \boldsymbol{v}_\infty) = 0$ and $\boldsymbol{u}_m \in \mathcal{D}(\Omega)$, by Lemma IX.2.1 we obtain

$$
(\boldsymbol{u}_m \cdot \nabla \boldsymbol{u}_m, \boldsymbol{u}_m) = (\boldsymbol{v}_\infty \cdot \nabla \boldsymbol{u}_m, \boldsymbol{u}_m) = ((\boldsymbol{V} + \boldsymbol{v}_\infty) \cdot \nabla \boldsymbol{u}_m, \boldsymbol{u}_m) = 0.
$$

We thus deduce

$$
\begin{aligned}
\boldsymbol{F} \cdot \boldsymbol{\xi} = &-(\nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \nabla \boldsymbol{u}_m) - \mathcal{R}[(\boldsymbol{u}_m \cdot \nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \boldsymbol{u}_m) \\
&+ ((\boldsymbol{V} + \boldsymbol{v}_\infty) \cdot \nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \boldsymbol{u}_m) - (\boldsymbol{e}_1 \cdot \nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \boldsymbol{u}_m)] \\
&- \mathcal{T}\left[(\boldsymbol{e}_1 \times \boldsymbol{V}_\varepsilon - \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{V}_\varepsilon, \boldsymbol{u}_m)\right] - [\boldsymbol{f}, \boldsymbol{u}_m].
\end{aligned}
\tag{XI.3.11}
$$

Using the inequalities of Schwarz, Hölder, and Sobolev (see (II.3.11)), and recalling that $\boldsymbol{V}_\varepsilon$ is of bounded support, we deduce

$$
\begin{aligned}
-(\nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \nabla \boldsymbol{u}_m) &\leq |\boldsymbol{V} + \boldsymbol{v}_\infty|_{1,2} |\boldsymbol{u}_m|_{1,2} \\
-((\boldsymbol{V} + \boldsymbol{v}_\infty) \cdot \nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \boldsymbol{u}_m) &\leq \|\boldsymbol{V} + \boldsymbol{v}_\infty\|_3 |\boldsymbol{V} + \boldsymbol{v}_\infty|_{1,2} \|\boldsymbol{u}_m\|_6 \\
&\leq c_1 \|\boldsymbol{V} + \boldsymbol{v}_\infty\|_3 |\boldsymbol{V} + \boldsymbol{v}_\infty|_{1,2} |\boldsymbol{u}_m|_{1,2} \\
(\boldsymbol{e}_1 \cdot \nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \boldsymbol{u}_m) &\leq c_2\, |\boldsymbol{V} + \boldsymbol{v}_\infty|_{1,6/5} \|\boldsymbol{u}_m\|_6 \\
&\leq c_3\, |\boldsymbol{V} + \boldsymbol{v}_\infty|_{1,6/5} |\boldsymbol{u}_m|_{1,2} \\
-(\boldsymbol{e}_1 \times \boldsymbol{V}_\varepsilon, \boldsymbol{u}_m) &\leq \|\boldsymbol{V}_\varepsilon\|_{6/5} \|\boldsymbol{u}_m\|_6 \leq c_4 \|\boldsymbol{V}_\varepsilon\|_{6/5} |\boldsymbol{u}_m|_{1,2} \\
(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{V}_\varepsilon, \boldsymbol{u}_m) &\leq c_5 |\boldsymbol{V}_\varepsilon|_{1,6/5} \|\boldsymbol{u}_m\|_6 \leq c_6 |\boldsymbol{V}_\varepsilon|_{1,6/5} |\boldsymbol{u}_m|_{1,2} \\
-[\boldsymbol{f}, \boldsymbol{u}_m] &\leq |\boldsymbol{f}|_{-1,2} |\boldsymbol{u}_m|_{1,2}.
\end{aligned}
\tag{XI.3.12}
$$

Furthermore, by Lemma XI.3.1, for any given $\eta > 0$ we can choose $\boldsymbol{V}$ such that

$$
-(\boldsymbol{u}_m \cdot \nabla(\boldsymbol{V} + \boldsymbol{v}_\infty), \boldsymbol{u}_m)| \leq \left(\eta + \frac{|\Phi|}{4\pi r_0}\right) |\boldsymbol{u}_m|_{1,2}^2. \tag{XI.3.13}
$$

Therefore, if

$$
|\Phi| < \frac{4\pi r_0}{\mathcal{R}}, \tag{XI.3.14}
$$

we may choose η such that $\eta+\frac{|\Phi|}{4\pi r_0}<1/\mathcal{R}$, and from (XI.3.11)–(XI.3.13) and Lemma IX.3.2 we may conclude that the algebraic system (XI.3.10) has at least one solution for every $m\in\mathbb{N}$. Moreover, from (XI.3.10)–(XI.3.13) it also follows that

$$\left[1-\mathcal{R}\left(\eta+\frac{|\Phi|}{4\pi r_0}\right)\right]|\boldsymbol{u}_m|_{1,2}\leq C\,[|\boldsymbol{V}+\boldsymbol{v}_\infty|_{1,2}(1+\mathcal{R}\|\boldsymbol{V}+\boldsymbol{v}_\infty\|_3)$$
$$+\mathcal{R}|\boldsymbol{V}+\boldsymbol{v}_\infty|_{1,6/5}+\mathcal{T}\|\boldsymbol{V}_\varepsilon\|_{1,6/5}+|\boldsymbol{f}|_{-1,2}]\,, \tag{XI.3.15}$$

which shows that under the condition (XI.3.14), the sequence $\{\boldsymbol{u}_m\}$ is (uniformly) bounded in $\mathcal{D}_0^{1,2}(\Omega)$. Thus, we can select a subsequence, denoted again by $\{\boldsymbol{u}_m\}$, such that as $m\to\infty$,

$$\boldsymbol{u}_m\overset{w}{\to}\boldsymbol{u}\ \text{ in } \mathcal{D}_0^{1,2}(\Omega). \tag{XI.3.16}$$

This property along with Theorem II.1.3(i) and (XI.3.15) implies

$$\left[1-\mathcal{R}\left(\eta+\frac{|\Phi|}{4\pi r_0}\right)\right]|\boldsymbol{u}|_{1,2}\leq C\,[|\boldsymbol{V}+\boldsymbol{v}_\infty|_{1,2}(1+\mathcal{R}\|\boldsymbol{V}+\boldsymbol{v}_\infty\|_3)$$
$$+\mathcal{R}|\boldsymbol{V}+\boldsymbol{v}_\infty|_{1,6/5}+\mathcal{T}\|\boldsymbol{V}_\varepsilon\|_{1,6/5}+|\boldsymbol{f}|_{-1,2}]\,, \tag{XI.3.17}$$

where $C=C(\Omega)$. Furthermore, since $\mathcal{D}_0^{1,2}(\Omega)\hookrightarrow W^{1,2}(\Omega_R)$ for all $R>\delta(\Omega^c)$ (see Lemma II.6.1), from Exercise II.5.8 and the Cantor diagonalization argument we secure the existence of another subsequence, still called $\{\boldsymbol{u}_m\}$, such that as $m\to\infty$,

$$\boldsymbol{u}_m\to\boldsymbol{u}\ \text{ in } L^q(\Omega_R),\quad q\in[1,6)\,, \tag{XI.3.18}$$

for any $R>\delta(\Omega^c)$. At this point, we may employ arguments analogous to those used in the proof of Theorem X.4.1 (in particular, the argument following (X.4.29)) to show that the field $\boldsymbol{u}$ satisfies (XI.3.10) with $\boldsymbol{u}_m\equiv\boldsymbol{u}$, for all $k\in\mathbb{N}$. The final step is to replace, in this resulting equation, $\boldsymbol{\varphi}_k$ with an arbitrary $\boldsymbol{\varphi}\in\mathcal{D}(\Omega)$. This goal is again achieved by the same procedure employed in the proof of Theorem X.4.1, thanks to the properties of the linear hull of the basis $\{\boldsymbol{\varphi}_k\}$; see Lemma VII.2.1. The only term that deserves a little more attention is the "rotational term" $(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{u},\boldsymbol{\varphi}_k)-(\boldsymbol{e}_1\times\boldsymbol{u},\boldsymbol{\varphi}_k)$. However, the replacement of $\boldsymbol{\varphi}_k$ with $\boldsymbol{\varphi}\in\mathcal{D}(\Omega)$ in this latter is carried out exactly as described in the proof of Theorem VIII.1.2, after equation (VIII.1.19). We may then safely conclude that $\boldsymbol{u}$ satisfies (XI.3.9), for all $\boldsymbol{\varphi}\in\mathcal{D}(\Omega)$, that is, the field $\boldsymbol{v}:=\boldsymbol{u}+\boldsymbol{V}$ is a generalized solution to (XI.0.10), (XI.0.11). Moreover, we prove (see (X.4.34)) that $\boldsymbol{v}+\boldsymbol{v}_\infty$ satisfies (XI.3.4). Finally, choosing in (XI.1.4) the function $\boldsymbol{\psi}$ introduced in (X.4.38) and proceeding as in (X.4.39), we easily establish the validity of (XI.3.4). We shall next establish the validity of the energy inequality (XI.3.5). We begin by observing that from (XI.3.10) and (XI.3.11) it follows that

$$
\begin{aligned}
|\boldsymbol{u}_m|_{1,2}^2 = &-(\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\nabla\boldsymbol{u}_m)-\mathcal{R}[(\boldsymbol{u}_m\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{u}_m)\\
&+((\boldsymbol{V}+\boldsymbol{v}_\infty)\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{u}_m)-(\boldsymbol{e}_1\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{u}_m)]\\
&-\mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{V}_\varepsilon-\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{V}_\varepsilon,\boldsymbol{u}_m)-[\boldsymbol{f},\boldsymbol{u}_m]\,.
\end{aligned}
\tag{XI.3.19}
$$

We would like to let $m\to\infty$ in this relation (along a subsequence if necessary). Obviously, in view of (XI.3.16) and (XI.3.18), and the assumption on $\boldsymbol{f}$, and recalling that $\boldsymbol{V}_\varepsilon$ is of bounded support, we obtain

$$
\begin{aligned}
&\lim_{m\to\infty}\{\mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{V}_\varepsilon-\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{V}_\varepsilon,\boldsymbol{u}_m)-[\boldsymbol{f},\boldsymbol{u}_m]\}\\
&\qquad\qquad=\mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{V}_\varepsilon-\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{V}_\varepsilon,\boldsymbol{u})-[\boldsymbol{f},\boldsymbol{u}]\\
&\lim_{m\to\infty}(\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\nabla\boldsymbol{u}_m)=(\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\nabla\boldsymbol{u})\,.
\end{aligned}
\tag{XI.3.20}
$$

Furthermore, by exactly the same procedure used in the proof of Theorem X.4.1 to prove the validity of the generalized energy inequality in the irrotational case (see the argument following (X.4.40)), we obtain

$$
\begin{aligned}
&\lim_{m\to\infty}\{(\boldsymbol{u}_m\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{u}_m)+((\boldsymbol{V}+\boldsymbol{v}_\infty)\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{u}_m)\}\\
&\qquad\qquad=(\boldsymbol{u}\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{u})+((\boldsymbol{V}+\boldsymbol{v}_\infty)\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{u})\\
&\lim_{m\to\infty}(\boldsymbol{e}_1\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{u}_m)=(\boldsymbol{e}_1\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{u})\,.
\end{aligned}
\tag{XI.3.21}
$$

Thus, passing to the limit $m\to\infty$ in (XI.3.19), from (XI.3.20)–(XI.3.21) and Theorem II.1.3(i), we conclude that

$$
\begin{aligned}
|\boldsymbol{u}|_{1,2}^2 &+(\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\nabla\boldsymbol{u})+\mathcal{R}[(\boldsymbol{u}\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{u})\\
&+((\boldsymbol{V}+\boldsymbol{v}_\infty)\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{u})-(\boldsymbol{e}_1\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{u})]\\
&+\mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{V}_\varepsilon-\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{V}_\varepsilon,\boldsymbol{u})+[\boldsymbol{f},\boldsymbol{u}]\le 0\,.
\end{aligned}
\tag{XI.3.22}
$$

We next observe that since $\boldsymbol{u}=\boldsymbol{v}-\boldsymbol{V}$,

$$
\begin{aligned}
|\boldsymbol{u}|_{1,2}^2+(\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\nabla\boldsymbol{u}) &= |\boldsymbol{v}+\boldsymbol{v}_\infty|_{1,2}^2-(\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\nabla(\boldsymbol{v}+\boldsymbol{v}_\infty))\\
&=2\left[\|\boldsymbol{D}(\boldsymbol{v})\|_2^2-(\boldsymbol{D}(\boldsymbol{v}),\boldsymbol{D}(\boldsymbol{V}))\right]\,,
\end{aligned}
\tag{XI.3.23}
$$

where in the last step, we have used the identity given in footnote 4 of Section X.2. Moreover,

$$
(\boldsymbol{u}\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{u})+((\boldsymbol{V}+\boldsymbol{v}_\infty)\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{u})=((\boldsymbol{v}+\boldsymbol{v}_\infty)\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{v}-\boldsymbol{V})\,.
\tag{XI.3.24}
$$

Thus, collecting (XI.3.22)–(XI.3.24), we deduce that

$$
\begin{aligned}
2\|\boldsymbol{D}(\boldsymbol{v})\|_2^2 &-2(\boldsymbol{D}(\boldsymbol{v}),\boldsymbol{D}(\boldsymbol{V}))+\mathcal{R}[((\boldsymbol{v}+\boldsymbol{v}_\infty)\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{v}-\boldsymbol{V})\\
&-(\boldsymbol{e}_1\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{v}-\boldsymbol{V})]+\mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{V}_\varepsilon-\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{V}_\varepsilon,\boldsymbol{v}-\boldsymbol{V})\\
&+[\boldsymbol{f},\boldsymbol{v}-\boldsymbol{V}]\le 0\,.
\end{aligned}
\tag{XI.3.25}
$$

We now notice that in view of the assumptions made on $\boldsymbol{f}$ and $\boldsymbol{v}_*$, by Theorem XI.1.2 we have, on the one hand, that $(\boldsymbol{w} := \boldsymbol{v} + \boldsymbol{v}_\infty, p) \in W^{2,2}(\Omega_\rho) \times W^{1,2}(\Omega_\rho)$, for all $\rho > \delta(\Omega^c)$, and, on the other hand, that $(\boldsymbol{w}, \widetilde{p})$, with $\widetilde{p}$ given in (XI.1.6), satisfies (XI.2.4), that is,

$$\left.\begin{aligned} \nabla\cdot\boldsymbol{T}(\boldsymbol{w},\widetilde{p}) + \mathcal{R}(\boldsymbol{v}_\infty - \boldsymbol{w})\cdot\nabla\boldsymbol{w} - \mathcal{T}\boldsymbol{e}_1\times\boldsymbol{w} = \boldsymbol{f} \\ \nabla\cdot\boldsymbol{w} = 0 \end{aligned}\right\}\text{a.e. in } \Omega \tag{XI.3.26}$$
$$\boldsymbol{w}|_{\partial\Omega} = \boldsymbol{v}_* + \boldsymbol{v}_\infty\,.$$

Let $\psi_R = \psi_R(|x|)$ be a smooth, nonincreasing "cut-off" function that is 0 for $|x| \le R$ and is 1 for $|x| \ge 2R$, $R > \delta(\Omega^c)$, with $|\nabla\psi_R(x)| \le M/R$, where M is a constant independent of R and x. If we then dot-multiply both sides of $(\text{XI.3.26})_1$ by $\psi_R(\boldsymbol{V} + \boldsymbol{v}_\infty)$ and integrate by parts over Ω, we obtain

$$\begin{aligned} &\int_{\partial\Omega} \boldsymbol{n}\cdot\boldsymbol{T}(\boldsymbol{w},\widetilde{p})\cdot(\boldsymbol{v}_* + \boldsymbol{v}_\infty) - 2(\boldsymbol{D}(\boldsymbol{v}),\boldsymbol{D}(\psi_R(\boldsymbol{V}+\boldsymbol{v}_\infty))) - (\widetilde{p}\nabla\psi_R, \boldsymbol{V}+\boldsymbol{v}_\infty) \\ &-\mathcal{R}\left[(\psi_R\boldsymbol{w}\cdot\nabla\boldsymbol{w}, \boldsymbol{V}+\boldsymbol{v}_\infty) + \left(\frac{\partial\psi_R}{\partial x_1}\boldsymbol{w}, \boldsymbol{V}+\boldsymbol{v}_\infty\right) + (\psi_R\boldsymbol{v}_\infty\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{w})\right] \\ &\qquad +\mathcal{T}(\psi_R\boldsymbol{e}_1\times(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{w}) + \mathcal{R}\int_{\partial\Omega}(\boldsymbol{v}_*+\boldsymbol{v}_\infty)^2\boldsymbol{v}_\infty\cdot\boldsymbol{n} = [\boldsymbol{f}, \boldsymbol{V}+\boldsymbol{v}_\infty]\,, \end{aligned} \tag{XI.3.27}$$

where we used $\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\psi_R(x) = 0$ for all $x \in \Omega$. In (XI.3.27) we now employ the form (XI.3.1) of the extension $\boldsymbol{V}$ along with its summability properties and the identity (XI.3.8) for $\boldsymbol{\sigma}$, and recall that, by Theorem XI.1.1 and the assumption on $\boldsymbol{f}$, $\widetilde{p} \in L^6(\Omega) + L^3(\Omega)$. Thus, letting $R \to \infty$, it is not hard to show that (XI.3.27) leads to

$$\begin{aligned} -&\int_{\partial\Omega} \boldsymbol{n}\cdot\boldsymbol{T}(\boldsymbol{w},\widetilde{p})\cdot(\boldsymbol{v}_*+\boldsymbol{v}_\infty) + 2(\boldsymbol{D}(\boldsymbol{v}),\boldsymbol{D}(\boldsymbol{V})) \\ &\qquad + \mathcal{R}[(\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{V}+\boldsymbol{v}_\infty) + (\boldsymbol{e}_1\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{w})] \\ &\qquad - \mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{V}_\varepsilon - \boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{V}_\varepsilon,\boldsymbol{w}) - \mathcal{R}\int_{\partial\Omega}(\boldsymbol{v}_*+\boldsymbol{v}_\infty)^2\boldsymbol{v}_\infty\cdot\boldsymbol{n} \\ &\qquad + [\boldsymbol{f},\boldsymbol{V}+\boldsymbol{v}_\infty] = 0\,. \end{aligned} \tag{XI.3.28}$$

Summing side by side (XI.3.25) and (XI.3.28), and recalling that $\boldsymbol{w} = \boldsymbol{v}+\boldsymbol{v}_\infty$, we obtain

$$\begin{aligned} 2\|\boldsymbol{D}(\boldsymbol{v})\|_2^2 &- \int_{\partial\Omega}\boldsymbol{n}\cdot\boldsymbol{T}(\boldsymbol{w},\widetilde{p})\cdot(\boldsymbol{v}_*+\boldsymbol{v}_\infty) - \mathcal{R}\int_{\partial\Omega}(\boldsymbol{v}_*+\boldsymbol{v}_\infty)^2\boldsymbol{v}_\infty\cdot\boldsymbol{n} \\ &+[\boldsymbol{f},\boldsymbol{v}+\boldsymbol{v}_\infty] + \mathcal{R}[(\boldsymbol{w}\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{w}) + (\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{V}+\boldsymbol{v}_\infty) \\ &+(\boldsymbol{e}_1\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{V}+\boldsymbol{v}_\infty)] \\ &-\mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{V}_\varepsilon - \boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{V}_\varepsilon, \boldsymbol{V}+\boldsymbol{v}_\infty) \le 0\,. \end{aligned} \tag{XI.3.29}$$

By an easily justified integration by parts that uses the summability properties of $\boldsymbol{w}$ and $\boldsymbol{V}+\boldsymbol{v}_\infty$, we obtain

$$(\boldsymbol{w}\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{w})+(\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{V}+\boldsymbol{v}_\infty)=\frac{1}{2}\int_{\partial\Omega}(\boldsymbol{v}_*+\boldsymbol{v}_\infty)^2(\boldsymbol{v}_*+\boldsymbol{v}_\infty)\cdot\boldsymbol{n}\,. \tag{XI.3.30}$$

Likewise,

$$(\boldsymbol{e}_1\cdot\nabla(\boldsymbol{V}+\boldsymbol{v}_\infty),\boldsymbol{V}+\boldsymbol{v}_\infty)=\frac{1}{2}\int_{\partial\Omega}(\boldsymbol{v}_*+\boldsymbol{v}_\infty)^2\boldsymbol{e}_1\cdot\boldsymbol{n}\,. \tag{XI.3.31}$$

Finally, recalling that $\boldsymbol{V}+\boldsymbol{v}_\infty=\boldsymbol{V}_\varepsilon+\Phi\boldsymbol{\sigma}$ (see (XI.3.1)), we obtain

$$\begin{aligned}-(\boldsymbol{e}_1\times\boldsymbol{V}_\varepsilon&-\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{V}_\varepsilon,\boldsymbol{V}+\boldsymbol{v}_\infty)\\ &=(\boldsymbol{e}_1\times\cdot\nabla\boldsymbol{V}_\varepsilon,\boldsymbol{V}_\varepsilon)+\Phi(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{\sigma}-\boldsymbol{e}_1\times\boldsymbol{\sigma},\boldsymbol{V}_\varepsilon)\\ &\quad+\Phi\int_{\partial\Omega}(\boldsymbol{V}_\varepsilon\cdot\boldsymbol{\sigma})\boldsymbol{e}_1\times\boldsymbol{x}\cdot\boldsymbol{n}\\ &=\Phi\int_{\partial\Omega}(\boldsymbol{V}_\varepsilon\cdot\boldsymbol{\sigma})\boldsymbol{e}_1\times\boldsymbol{x}\cdot\boldsymbol{n}+\frac{1}{2}\int_{\partial\Omega}|\boldsymbol{V}_\varepsilon|^2\boldsymbol{e}_1\times\boldsymbol{x}\cdot\boldsymbol{n}\,.\end{aligned} \tag{XI.3.32}$$

We next observe that for any fixed $r>\delta(\Omega^c)$, we have

$$\int_{\partial\Omega}|\boldsymbol{\sigma}|^2\boldsymbol{e}_1\times\boldsymbol{x}\cdot\boldsymbol{n}=\int_{\Omega_r}\nabla\cdot(|\boldsymbol{\sigma}|^2\boldsymbol{e}_1\times\boldsymbol{x})=2\int_{\Omega_r}\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}=0\,,$$

because $\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{\sigma}(x)\cdot\boldsymbol{\sigma}(x)=0$, for all $x\in\Omega$. Therefore, (XI.3.32) delivers

$$-(\boldsymbol{e}_1\times\boldsymbol{V}_\varepsilon-\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{V}_\varepsilon,\boldsymbol{V}+\boldsymbol{v}_\infty)=\frac{1}{2}\int_{\partial\Omega}(\boldsymbol{v}_*+\boldsymbol{v}_\infty)^2\boldsymbol{e}_1\times\boldsymbol{x}\cdot\boldsymbol{n}\,. \tag{XI.3.33}$$

Placing (XI.3.30), (XI.3.31), and (XI.3.33) into (XI.3.29), we then conclude that

$$2\|\boldsymbol{D}(\boldsymbol{v})\|_2^2-\int_{\partial\Omega}\big[\boldsymbol{n}\cdot\boldsymbol{T}(\boldsymbol{w},\widetilde{p})\cdot(\boldsymbol{v}_*+\boldsymbol{v}_\infty)-\tfrac{\mathcal{R}}{2}(\boldsymbol{v}_*+\boldsymbol{v}_\infty)^2\boldsymbol{v}_*\cdot\boldsymbol{n}\big]+[\boldsymbol{f},\boldsymbol{v}+\boldsymbol{v}_\infty]\le 0\,,$$

namely, the validity of the energy inequality. It remains to show the estimate of the solution in terms of the data. Thus, assuming $|\Phi|\le 2\pi r_0/\mathcal{R}$, and choosing (for example) $\eta=1/(4\mathcal{R})$, (XI.3.17) furnishes

$$\begin{aligned}|\boldsymbol{u}|_{1,2}\le 4[C\,&|\boldsymbol{V}+\boldsymbol{v}_\infty|_{1,2}(1+\mathcal{R}\|\boldsymbol{V}+\boldsymbol{v}_\infty\|_3)\\ &+\mathcal{R}|\boldsymbol{V}+\boldsymbol{v}_\infty|_{1,6/5}+\mathcal{T}\|\boldsymbol{V}_\varepsilon\|_{1,6/5}+|\boldsymbol{f}|_{-1,2}]\,.\end{aligned} \tag{XI.3.34}$$

If we now use in (XI.3.34) the inequalities (X.4.6) (with $\mathfrak{v}_\infty\equiv\boldsymbol{v}_\infty$), and recall that $\boldsymbol{u}=(\boldsymbol{v}+\boldsymbol{v}_\infty)-(\boldsymbol{V}+\boldsymbol{v}_\infty)$, we obtain (XI.3.6), which completes the proof of the theorem. □

Remark XI.3.1 For future reference, we observe that in the energy inequality (XI.3.5), if, in addition, $\boldsymbol{f}\in L^{6/5}(\Omega)$,we can then replace $[\boldsymbol{f},\boldsymbol{v}+\boldsymbol{v}_\infty]$ with $(\boldsymbol{f},\boldsymbol{v}+\boldsymbol{v}_\infty)$. ■

XI.4 Global Summability of Generalized Solutions when $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq \mathbf{0}$

As in the irrotational case, the fundamental step in deriving the asymptotic structure of generalized solutions is to prove suitable summability properties at large distances for the velocity field $\boldsymbol{v}$ and the pressure p, much more detailed than those available at the outset, namely, $(\boldsymbol{v}+\boldsymbol{v}_\infty) \in D^{1,2}(\Omega) \cap L^6(\Omega)$, with p satisfying the conditions given in Theorem XI.1.2.

The proof of the desired properties will be achieved by following the same arguments used in the irrotational case, with only the foresight to replace Theorem VII.7.1 with Theorem VIII.8.1. As a matter of fact, once this basic replacement is made, the proof is essentially the same.

Specifically, we have the following result

Theorem XI.4.1 *Let Ω be a C^2-smooth exterior three-dimensional domain, and assume, for some $q \in (1,2)$, that*

$$\boldsymbol{f} \in L^q(\Omega) \cap L^{3/2}(\Omega)\,, \quad \boldsymbol{v}_* \in W^{2-1/q,q}(\partial\Omega) \cap W^{4/3,3/2}(\partial\Omega)\,. \tag{XI.4.1}$$

Then, every generalized solution $\boldsymbol{v}$ to the Navier–Stokes problem (XI.0.10)–(XI.0.11) *corresponding to $\boldsymbol{f}$, $\boldsymbol{v}_*$ and to $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq \mathbf{0}$, and the associated pressure field*[1] *p, satisfy $(\boldsymbol{v}+\boldsymbol{v}_\infty, p) \in X_q(\Omega)$, with $X_q(\Omega)$ defined in* (X.6.5).

Proof. Since their actual value is completely irrelevant, throughout the proof we set, for simplicity, $\mathcal{R} = \mathcal{T} = 1$. Moreover, we set $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{v}_\infty$. Since $\boldsymbol{u} \in D^{1,2}(\Omega)$, we may find a sequence of second-order tensors $\{\boldsymbol{G}_k\}$ with components in $C_0^\infty(\Omega)$ such that $\boldsymbol{G}_k \to \nabla\boldsymbol{u}$ in $L^2(\Omega)$. Then, from (XI.0.10) and Theorem XI.1.2, it follows that $\boldsymbol{u}$ satisfies the problem

$$\left.\begin{aligned} &\Delta\boldsymbol{u} + \frac{\partial\boldsymbol{u}}{\partial x_1} + \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla\boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{u} = \boldsymbol{u} \cdot \boldsymbol{A}_k + \nabla\widetilde{p} + \boldsymbol{F}_k \\ &\nabla \cdot \boldsymbol{u} = 0 \end{aligned}\right\} \quad \text{a.e. in } \Omega$$

$$\boldsymbol{u} = \boldsymbol{u}_* := \boldsymbol{v}_* + \boldsymbol{v}_\infty \quad \text{at } \partial\Omega\,, \tag{XI.4.2}$$

where $\boldsymbol{A}_k := \nabla\boldsymbol{u} - \boldsymbol{G}_k$, $\boldsymbol{F}_k := \boldsymbol{v} \cdot \boldsymbol{G}_k + \boldsymbol{f}$, and $\widetilde{p}$ is defined in (XI.1.6). We next observe that by the assumption on $\boldsymbol{f}$ and the fact that $\boldsymbol{u} \in L^6(\Omega)$, we have $\boldsymbol{F}_k \in L^q(\Omega) \cap L^{3/2}(\Omega)$, for all $k \in \mathbb{N}$. Set $X_{q,3/2}(\Omega) = X_q(\Omega) \cap X_{3/2}(\Omega)$, endowed with the norm $\|\cdot\|_{X_{q,3/2}} := \|\cdot\|_{X_q} + \|\cdot\|_{X_{3/2}}$,[2] and let

$$M : (\boldsymbol{w}, \tau) \in X_{q,3/2}(\Omega) \to (\boldsymbol{z}, \phi) := M(\boldsymbol{w}, \tau)\,,$$

where $(\boldsymbol{z}, \phi)$ satisfies

[1] Possibly modified by the addition of a constant.
[2] See (X.6.6).

$$\left.\begin{array}{l}\Delta \boldsymbol{z} + \dfrac{\partial \boldsymbol{z}}{\partial x_1} + \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{z} - \boldsymbol{e}_1 \times \boldsymbol{z} = \boldsymbol{w} \cdot \boldsymbol{A}_k + \nabla \phi + \boldsymbol{F}_k \\ \nabla \cdot \boldsymbol{u} = 0 \end{array}\right\} \text{ in } \Omega\,,$$

$$\boldsymbol{u} = \boldsymbol{u}_* \text{ at } \partial\Omega\,. \tag{XI.4.3}$$

In view of the Hölder inequality and the fact that $\boldsymbol{u} \in D^{1,2}(\Omega)$,

$$\|\boldsymbol{w} \cdot \boldsymbol{A}_k\|_s \leq \|\boldsymbol{w}\|_{2s/(2-s)} \|\boldsymbol{A}_k\|_2 < \infty\,, \quad s \in (1,2)\,, \tag{XI.4.4}$$

with the help of Theorem VII.8.1 we deduce, on the one hand, that the map M is well-defined and, on the other hand, that there exists a (unique) solution $(\boldsymbol{z}, \phi) \in X_{q,3/2}(\Omega)$ to (XI.4.3) satisfying the estimate (with $q_1 = q$, $q_2 = 3/2$)

$$\begin{aligned}\|(\boldsymbol{z}, \phi)\|_{X_{q,3/2}} &\leq c \sum_{i=1}^{2} \left(\|\boldsymbol{A}_k\|_2 \|\boldsymbol{w}\|_{2q_i/(2-q_i)} + \|\boldsymbol{F}_k\|_{q_i} + \|\boldsymbol{u}_*\|_{2-2/q_i(\partial\Omega)}\right) \\ &\leq c \left(\|\boldsymbol{A}_k\|_2 \|(\boldsymbol{w}, \tau)\|_{X_{q,3/2}} + \sum_{i=1}^{2} \left(\|\boldsymbol{F}_k\|_{q_i} + \|\boldsymbol{u}_*\|_{2-2/q_i(\partial\Omega)}\right)\right).\end{aligned} \tag{XI.4.5}$$

Thus, choosing k such that

$$\|\boldsymbol{A}_k\|_2 \leq 1/(2c) \tag{XI.4.6}$$

and putting

$$\delta := 2c \sum_{i=1}^{2} \left(\|\boldsymbol{F}_k\|_{q_i} + \|\boldsymbol{u}_*\|_{2-2/q_i(\partial\Omega)}\right)\,,$$

from (XI.4.5) it follows at once that M maps the closed ball $\{(\boldsymbol{u}, \phi) \in X_{q,3/2}(\Omega) : \|(\boldsymbol{u}, \phi)\|_{X_{q,3/2}} \leq \delta\}$ into itself. Moreover, in view of the linearity of the map M and of (XI.4.6), from (XI.4.5) with $\|\boldsymbol{F}_k\|_{q_i} = \|\boldsymbol{u}_*\|_{2-2/q_i(\partial\Omega)} 0$, $i = 1, 2$, we infer that M is a contraction, and therefore there exists one and only one solution $(\boldsymbol{z}, \phi) \in X_{q,3/2}$ to (XI.4.3). Next, setting $(\boldsymbol{w}, \tau) := (\boldsymbol{z} - \boldsymbol{u}, \phi - \widetilde{p})$, we obtain

$$\left.\begin{array}{l}\Delta \boldsymbol{w} + \dfrac{\partial \boldsymbol{w}}{\partial x_1} + \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{w} - \boldsymbol{e}_1 \times \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{A}_k + \nabla \tau \\ \nabla \cdot \boldsymbol{w} = 0 \end{array}\right\} \text{ in } \Omega \tag{XI.4.7}$$

$$\boldsymbol{w} = \boldsymbol{0} \text{ at } \partial\Omega\,.$$

Recalling that by the assumption on $\boldsymbol{v}$ and (XI.4.6), $\boldsymbol{g} := \boldsymbol{w} \cdot \boldsymbol{A}_k \in L^{3/2}(\Omega)$, with the help of Theorem VIII.8.1 we infer that the problem

$$\left.\begin{array}{l}\Delta \widetilde{\boldsymbol{w}} + \dfrac{\partial \widetilde{\boldsymbol{w}}}{\partial x_1} + \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \widetilde{\boldsymbol{w}} - \boldsymbol{e}_1 \times \widetilde{\boldsymbol{w}} = \boldsymbol{g} + \nabla \widetilde{\tau} \\ \nabla \cdot \widetilde{\boldsymbol{w}} = 0 \end{array}\right\} \text{ in } \Omega\,, \tag{XI.4.8}$$

$$\widetilde{\boldsymbol{w}} = \boldsymbol{0} \text{ at } \partial\Omega\,,$$

has a (unique) solution $(\widetilde{\boldsymbol{w}}, \widetilde{\tau})$ in the class $X_{3/2}(\Omega)$. It is easy to see that $(\widetilde{\boldsymbol{w}}, \widetilde{\tau}) = (\boldsymbol{w}, \tau)$. Actually, the fields $\boldsymbol{Z} := \widetilde{\boldsymbol{w}} - \boldsymbol{w}$ and $\chi := \widetilde{\tau} - \tau$ solve the homogeneous problem (XI.4.8) with $\boldsymbol{g} \equiv \boldsymbol{0}$. Furthermore, $\boldsymbol{Z} \in L^6(\Omega)$, because $\widetilde{\boldsymbol{w}}, \boldsymbol{u} \in X_{3/2}(\Omega)$, while $\boldsymbol{u} \in L^6(\Omega)$ by assumption. Therefore, from Theorem VIII.8.1 we obtain $\boldsymbol{Z} \equiv \nabla\chi \equiv \boldsymbol{0}$. Consequently, $\boldsymbol{w} \in X_{3/2}(\Omega)$, and so, again by Theorem VIII.8.1 applied to (XI.4.7), we obtain

$$\|(\boldsymbol{w}, \phi)\|_{X_{3/2}} \leq c\,\|\boldsymbol{w} \cdot \boldsymbol{A}_k\|_{3/2} \leq c\,\|\boldsymbol{A}_k\|_2 \|(\boldsymbol{w}, \phi)\|_{X_{3/2}} .$$

Thus, using (XI.4.6) in this latter inequality, we deduce $\boldsymbol{w} \equiv \nabla\tau \equiv \boldsymbol{0}$, that is, $(\boldsymbol{u}, \phi) = (\boldsymbol{v} + \boldsymbol{v}_\infty, \widetilde{p} + C)$, for some $C \in \mathbb{R}$, and the proof of the theorem is complete. □

Remark XI.4.1 Note that Theorem XI.4.1 does not require the vanishing of the flux of $\boldsymbol{v}_*$ through the boundary $\partial\Omega$. ■

Remark XI.4.2 From Theorem XI.4.1 and (XI.0.10), it follows that the "rotational term" $\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla(\boldsymbol{v} + \boldsymbol{v}_\infty) - \boldsymbol{e}_1 \times (\boldsymbol{v} + \boldsymbol{v}_\infty)$ belongs to $L^q(\Omega)$. Therefore, in particular, the results of Exercise VIII.7.1 apply to the component $(\boldsymbol{v} + \boldsymbol{v}_\infty) \cdot \boldsymbol{e}_1 = v_1 + 1$. ■

Remark XI.4.3 If $\mathcal{T} = 0$ (irrotational case), we know that the summability properties established in Theorem X.6.4 in the significant circumstance when

$$\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv \boldsymbol{0} \tag{XI.4.9}$$

(rigid body translating with constant velocity in a viscous liquid) are sharp. More precisely, under the assumption (XI.4.9), we have $(\boldsymbol{v} + \boldsymbol{v}_\infty) \notin L^s(\Omega)$ for all $s \in (1, 2]$ (see Exercise X.6.1), implying, in particular, that the total kinetic energy of the liquid is infinite. It is probable that if (XI.4.9) holds, the same conclusion can be drawn also when $\mathcal{T} \neq 0$ (rigid body translating *and* rotating with constant velocity in a viscous liquid), that is, the summability properties established in Theorem XI.4.1 cannot be improved, and consequently, the total kinetic energy of the liquid is still infinite. *However, no proof (or disproof) of this statement is available to date.* ■

XI.5 The Energy Equation and Uniqueness for Generalized Solutions when $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq 0$

An immediate consequence of Theorem XI.4.1 is stated in the following theorem.

Theorem XI.5.1 *Let Ω be a C^2-smooth exterior three-dimensional domain, and let $\boldsymbol{v}$ be a generalized solution to the Navier–Stokes problem* (XI.0.10), (XI.0.11) *corresponding to the data*

$$\boldsymbol{f} \in L^{4/3}(\Omega) \cap L^{3/2}(\Omega)\,, \quad \boldsymbol{v}_* \in W^{4/3,3/2}(\partial\Omega)\,, \quad \boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq \boldsymbol{0}\,. \tag{XI.5.1}$$

Then $\boldsymbol{v}$ and the corresponding pressure field associated to $\boldsymbol{v}$ by Lemma XI.1.1 *satisfy the energy equation* (XI.2.1).

Proof. Under the assumption (XI.5.1), from Theorem XI.4.1 we have, in particular,

$$\boldsymbol{v} + \boldsymbol{v}_\infty \in L^4(\Omega)\,,$$

so that the result follows at once from Theorem XI.2.1. □

A significant consequence to Theorem XI.5.1 is the following result of Liouville type.

Theorem XI.5.2 *Let $\boldsymbol{v}$ be a generalized solution to the Navier–Stokes problem* (XI.0.10), (XI.0.11) *in $\mathbb{R}^3$ corresponding to $\boldsymbol{f} \equiv 0$ and $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq 0$. Then $\boldsymbol{v}(x) = -\boldsymbol{v}_\infty$ for all $x \in \mathbb{R}^3$.*

Remark XI.5.1 It is not known whether the result of Theorem XI.5.3 continues to hold if $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0$. ■

We shall next furnish sufficient conditions on the data for a corresponding solution to be unique in the class of generalized solutions. The procedure will be similar to that adopted in Theorem X.7.3 for the irrotational case. To this end, we begin by proving the following preliminary result.

Lemma XI.5.1 *Let the assumption of* Theorem XI.5.1 *be satisfied and suppose, in addition, that*

$$\boldsymbol{f} \in L^{6/5}(\Omega)\,,$$
$$\|\boldsymbol{f}\|_{6/5} + \|\boldsymbol{v}_* + \boldsymbol{v}_\infty\|_{7/6,6/5(\partial\Omega)} < \frac{1}{\sqrt{\mathcal{R}}}\min\left\{\frac{1}{4c^2}, \frac{\sqrt{3}}{4c}\right\}, \tag{XI.5.2}$$

with $c = c(\Omega, \mathcal{T}, B)$ for $\mathcal{R} \in [0, B]$. Then, any generalized solution $\boldsymbol{v}$ to (XI.0.10), (XI.0.11) *corresponding to $\boldsymbol{f}$, $\boldsymbol{v}_*$, and $\boldsymbol{v}_\infty$ satisfies $(\boldsymbol{v} + \boldsymbol{v}_\infty) \in L^3(\Omega)$, along with the inequality*

$$\|\boldsymbol{v} + \boldsymbol{v}_\infty\|_3 < \frac{\sqrt{3}}{2\mathcal{R}}\,. \tag{XI.5.3}$$

Proof. In view of Theorem XI.4.1 and the assumption on $\boldsymbol{f}$, we have only to prove (XI.5.3). To this end, let

$$M : (\boldsymbol{w}, \phi) \in X_{6/5}(\Omega) \to (\boldsymbol{u}, \tau) \in X_{6/5}(\Omega)\,,$$

where $X_{6/5}(\Omega)$ is defined in (X.6.5) and $(\boldsymbol{u}, \tau)$ solves

$$\left.\begin{array}{l}\Delta \boldsymbol{u}+\mathcal{R}\dfrac{\partial \boldsymbol{u}}{\partial x_1}+\mathcal{T}\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{u}-\boldsymbol{e}_1\times\boldsymbol{u}=\mathcal{R}\boldsymbol{w}\cdot\nabla\boldsymbol{w}+\nabla\tau+\boldsymbol{f}\\ \nabla\cdot\boldsymbol{u}=0\end{array}\right\}\ \text{a.e. in } \Omega\,,$$

$$\boldsymbol{u}=\boldsymbol{v}_*+\boldsymbol{v}_\infty\ \text{ at } \partial\Omega\,. \tag{XI.5.4}$$

It is convenient to endow the Banach space $X_{6/5}(\Omega)$ with the "scaled" norm

$$\|(\boldsymbol{w},\phi)\|_{X_{6/5}(\Omega)}:=\mathcal{R}^{1/2}\|\boldsymbol{w}\|_3+\mathcal{R}^{1/4}|\boldsymbol{w}|_{1,12/7}+|\boldsymbol{w}|_{2,6/5}+\|\phi\|_2+|\phi|_{1,6/5}\,.$$

Since by Theorem II.6.1, $|\boldsymbol{w}|_{1,2}\le c_1|\boldsymbol{w}|_{2,6/5}$, and by the Hölder inequality,

$$\|\boldsymbol{w}\cdot\nabla\boldsymbol{w}\|_{6/5}\le\|\boldsymbol{w}\|_3|\boldsymbol{w}|_{1,2}$$

the assumption $\boldsymbol{w}\in X_{6/5}(\Omega)$ implies $\boldsymbol{w}\cdot\nabla\boldsymbol{w}\in L^{6/5}(\Omega)$, and also

$$\|\boldsymbol{w}\cdot\nabla\boldsymbol{w}\|_{6/5}\le c_1\mathcal{R}^{-1/2}\|(\boldsymbol{w},\phi)\|^2_{X_{6/5}(\Omega)}\,. \tag{XI.5.5}$$

Consequently, from Theorem VIII.8.1, one deduces, on the one hand, that the map M is well defined, and on the other hand, that the following inequality holds:

$$\|(\boldsymbol{u},\tau)\|_{X_{6/5}(\Omega)}\le c_2\left(\mathcal{R}^{1/2}\|(\boldsymbol{w},\phi)\|^2_{X_{6/5}(\Omega)}+\|\boldsymbol{f}\|_{6/5}+\|\boldsymbol{v}_*+\boldsymbol{v}_\infty\|_{7/6,6/5(\partial\Omega)}\right), \tag{XI.5.6}$$

where $c_2=c_2(\Omega,B,\mathcal{T})$. By the same token, for $(\boldsymbol{w}_i,\phi_i)\in X_{6/5}(\Omega)$, $i=1,2$, we find that the corresponding $(\boldsymbol{u}_i,\tau_i)$ satisfy

$$\begin{aligned}&\|(\boldsymbol{u}_1-\boldsymbol{u}_2,\,\tau_1-\tau_2)\|_{X_{6/5}(\Omega)}\\ &\qquad\le c_2\mathcal{R}^{1/2}\|(\boldsymbol{w}_1-\boldsymbol{w}_2,\phi_1-\phi_2)\|_{X_{6/5}(\Omega)}\left(\sum_{i=1}^{2}\|(\boldsymbol{w}_i,\phi_i)\|_{X_{6/5}(\Omega)}\right).\end{aligned} \tag{XI.5.7}$$

From (XI.5.6) it follows that M maps $\mathcal{B}_\delta$, the ball of $X_{6/5}(\Omega)$ centered at the origin and of radius $\delta=1/(2c_2\sqrt{\mathcal{R}})$, into itself, provided

$$\|\boldsymbol{f}\|_{6/5}+\|\boldsymbol{v}_*+\boldsymbol{v}_\infty\|_{7/6,6/5(\partial\Omega)}\le\frac{1}{4c_2^2\sqrt{\mathcal{R}}}\,,$$

and therefore a fortiori, if

$$\|\boldsymbol{f}\|_{6/5}+\|\boldsymbol{v}_*+\boldsymbol{v}_\infty\|_{7/6,6/5(\partial\Omega)}<\frac{1}{\sqrt{\mathcal{R}}}\min\left\{\frac{1}{4c_2^2},\frac{\sqrt{3}}{4c_2}\right\}. \tag{XI.5.8}$$

Furthermore, with the help of (XI.5.7), we conclude that M is a contraction in $\mathcal{B}_\delta$. Thus, under the assumption (XI.5.8), we may take $\boldsymbol{w}\equiv\boldsymbol{u}$ in (XI.5.4), and deduce that $\boldsymbol{v}':=\boldsymbol{u}-\boldsymbol{v}_\infty$ is a generalized solution to (XI.0.10)–(XI.0.11) corresponding to the same data as $\boldsymbol{v}$. Also, from (XI.5.6) with $\boldsymbol{u} \equiv \boldsymbol{w}$, (XI.5.8) and the fact that $\boldsymbol{u} \in \mathcal{B}_\delta$, we obtain, in particular,

$$\mathcal{R}^{1/2}\|\boldsymbol{v}' + \boldsymbol{v}_\infty\|_3 \leq 2c_2\left(\|\boldsymbol{f}\|_{6/5} + \|\boldsymbol{v}_* + \boldsymbol{v}_\infty\|_{7/6,6/5(\partial\Omega)}\right) < \frac{\sqrt{3}}{2\sqrt{\mathcal{R}}}\,.$$

Since both $\boldsymbol{v}' + \boldsymbol{v}_\infty$ and $\boldsymbol{v} + \boldsymbol{v}_\infty$ are in $L^4(\Omega)$, as a consequence of Theorem XI.4.1 and the assumption on $\boldsymbol{f}$, we infer $\boldsymbol{v} = \boldsymbol{v}'$ a.e. in Ω, and the proof of the lemma is complete. □

From Lemma XI.5.1 and Theorem XI.2.2 we immediately obtain the following *uniqueness theorem for generalized solutions when* $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq \boldsymbol{0}$.

Theorem XI.5.3 *Let Ω be a three-dimensional exterior domain of class C^2. Further, let*

$$\boldsymbol{f} \in L^{6/5}(\Omega) \cap L^{4/3}(\Omega), \quad \boldsymbol{v}_* \in W^{5/4,4/3}(\partial\Omega), \quad \boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq \boldsymbol{0}.$$

Then if the data satisfy $(\text{XI.5.2})_2$, $\boldsymbol{v}$ *is the only generalized solution satisfying these data.*

XI.6 On the Asymptotic Structure of Generalized Solutions When $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq \boldsymbol{0}$

This section is devoted to the study of the pointwise behavior of generalized solutions to (XI.0.10)–(XI.0.11) when $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq 0$. Our main achievement is to show that under the assumption of body force possessing mild regularity and of bounded support,[1] every such solution $\boldsymbol{v}$ is pointwise bounded above by a function that, roughly speaking, behaves like the Oseen fundamental solution and consequently, exhibits a wake-like behavior. Analogous estimates are given for $\nabla\boldsymbol{v}$ and for the "modified" pressure $\widetilde{p}$. However, it is worth emphasizing that *our analysis is not able to establish the existence of a leading term in the asymptotic behavior, which therefore is left as an open question.*

We need some preparatory results, which begin with the following.

Lemma XI.6.1 *Let $\boldsymbol{v}$ be a generalized solution to* (XI.0.10)–(XI.0.11) *in $\Omega := \mathbb{R}^3 - B_{\rho/2}$, corresponding to $\boldsymbol{f} \equiv \boldsymbol{0}$ and $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq 0$. Then, setting $\boldsymbol{u} := \boldsymbol{v} + \boldsymbol{v}_\infty$, we have, for all $\varepsilon > 0$,*

$$|\boldsymbol{u}|^2_{1,2,B^R} \leq c\,R^{-1+\varepsilon}, \quad \textit{all } R > \rho/2\,,$$

with $c = (\varepsilon, \mathcal{R}, \mathcal{T}, \boldsymbol{v}, p)$.

[1] This latter assumption can be weakened to that $\boldsymbol{f}$ decays "sufficiently fast" at large distances. However, we will not state these more general conditions here.

Proof. By the results of Theorem XI.1.2, we know that $\boldsymbol{v}$ and the associated pressure p are in $C^\infty(\Omega)$. Thus, $\boldsymbol{u}$ and p satisfy (XI.1.5)–(XI.1.6). Dot-multiplying both sides of $(\text{XI.1.5})_1$ by $\boldsymbol{u}$, and integrating over B_{R,R^*}, $R^* > R$, we obtain

$$\int_{B_{R,R^*}} \nabla \boldsymbol{u} : \nabla \boldsymbol{u} = \int_{B_{R,R^*}} \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{u}) \cdot \boldsymbol{u}$$
$$+ \int_{\partial B_R \cup \partial B_{R^*}} \left[\boldsymbol{n} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u} + \tfrac{1}{2}\mathcal{R}|\boldsymbol{u}|^2 n_1 - \tfrac{1}{2}\mathcal{R}|\boldsymbol{u}|^2 \boldsymbol{u} \cdot \boldsymbol{n} - p(\boldsymbol{u} \cdot \boldsymbol{n})\right] .$$

We observe that on $\partial B_R \cup \partial B_{R^*}$ we have $\boldsymbol{n} := \frac{\boldsymbol{x}}{|x|}$ and thus

$$\int_{B_{R,R^*}} \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u} = \tfrac{1}{2} \int_{\partial B_R \cup \partial B_{R^*}} |\boldsymbol{u}|^2 (\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \boldsymbol{n} = 0.$$

We thus conclude that

$$\int_{B_{R,R^*}} \nabla \boldsymbol{u} : \nabla \boldsymbol{u}$$
$$= \int_{\partial B_R \cup \partial B_{R^*}} \left[\boldsymbol{n} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u} + \tfrac{1}{2}\mathcal{R}|\boldsymbol{u}|^2 n_1 - \tfrac{1}{2}\mathcal{R}|\boldsymbol{u}|^2 \boldsymbol{u} \cdot \boldsymbol{n} - p(\boldsymbol{u} \cdot \boldsymbol{n})\right] .$$

The rest of the proof follows precisely that of Lemma X.8.2, and will be therefore omitted. □

Lemma XI.6.2 *Let the assumptions of* Lemma XI.6.1 *be satisfied. Then, setting* $\boldsymbol{u} := \boldsymbol{v} + \boldsymbol{v}_\infty$, *the following properties hold:*

(a) $\nabla \boldsymbol{u} \in L^\infty(B^\rho)$ *;*
(b) $\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{u} \in L^q(B^\rho)$, *for all* $q \in (2, \infty)$ *.*

Proof. By Theorem XI.1.2 we know that $\boldsymbol{v}, p$ are in $C^\infty(B^\rho)$. Thus, by Theorem XI.4.1 (after possibly adding a constant to p), we obtain

$$(\boldsymbol{u}, \widetilde{p}) \in X_q(B^\rho) \,, \quad \text{for all } q \in (1, 2) \,, \tag{XI.6.1}$$

with $\widetilde{p}$ defined in (XI.1.6). Let ψ_ρ be a smooth, nonincreasing function that is 0 in B_ρ and is 1 in $\Omega^{2\rho}$. We set

$$\boldsymbol{w} := \psi_\rho \,(\boldsymbol{u} + \Phi\, \boldsymbol{\sigma}) - \boldsymbol{Z} \,, \quad \phi := \psi_\rho \, \widetilde{p} \,, \tag{XI.6.2}$$

where

$$\Phi := \int_{\partial B_{2\rho}} \boldsymbol{v} \cdot \boldsymbol{n} \,,$$

and $\boldsymbol{\sigma}$, we recall, is given by (see (XI.3.2))

$$\boldsymbol{\sigma}(x) = -\nabla \mathcal{E}(x - x_0) \,, \tag{XI.6.3}$$

with $\mathcal{E}$ the fundamental solution to Laplace's equation. Moreover, $\boldsymbol{Z} \in C_0^\infty(B_{2\rho})$ satisfies $\nabla \cdot \boldsymbol{Z} = \nabla \psi_\rho \cdot (\boldsymbol{v} + \Phi\, \boldsymbol{\sigma})$. Since

$$\int_{B_{2\rho}} \nabla \psi_\rho \cdot (\boldsymbol{v} + \Phi\, \boldsymbol{\sigma}) = \int_{\partial B_{2\rho}} (\boldsymbol{v} + \Phi\, \boldsymbol{\sigma}) \cdot \boldsymbol{n} = 0\,,$$

the existence of the function $\boldsymbol{Z}$ is secured by Theorem III.3.3. Using (XI.3.8) and (XI.6.3), we see that $(\boldsymbol{w}, \phi)$ satisfies the following problem:

$$\left.\begin{aligned} &\Delta \boldsymbol{w} + \mathcal{R}\frac{\partial \boldsymbol{w}}{\partial x_1} + \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{w} - \boldsymbol{e}_1 \times \boldsymbol{w}) - \nabla d \\ &\qquad\qquad = \mathcal{R}\nabla \cdot [(\psi_\rho \boldsymbol{u}) \otimes (\psi_\rho \boldsymbol{u})] + \boldsymbol{F} \\ &\nabla \cdot \boldsymbol{w} = 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^3\,, \tag{XI.6.4}$$

with $d = d(x) := \phi(x) - \psi_\rho(x) D_1 \mathcal{E}(x - x_0)$, and $\boldsymbol{F} \in C_0^\infty(\mathbb{R}^3)$. In view of (XI.6.1) and the mentioned regularity properties, it is easy to check that $\nabla \cdot [(\psi_\rho \boldsymbol{u}) \otimes (\psi_\rho \boldsymbol{u})] \in L^q(\mathbb{R}^3)$, for all $q \in (1, \infty)$. Thus, by Lemma VIII.8.2, the uniqueness Lemma VIII.7.1, and the fact that $\psi_\rho = 1$ in $B^{2\rho}$, we conclude that

$$\left.\begin{aligned} &D^2 \boldsymbol{u} \in L^q(B^\rho) \\ &(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{u}) \in L^q(B^\rho) \end{aligned}\right\} \quad \text{for all } q \in (1, \infty)\,. \tag{XI.6.5}$$

However, by (XI.6.1), we have also $\nabla \boldsymbol{u} \in L^s(B^\rho)$ for all $s \in (4/3, \infty)$, so that property (a) follows from $(\text{XI.6.5})_1$ and Theorem II.9.1. Furthermore, again by (XI.6.1), we have $\boldsymbol{e}_1 \times \boldsymbol{u} \in L^r(B^\rho)$, for all $r \in (2, \infty)$, which, by virtue of $(\text{XI.6.5})_2$, implies $\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{u} \in L^q(B^\rho)$, for all $q \in (2, \infty)$. This proves property (b) and concludes the proof of the lemma.

□

With the help of the previous lemma, we prove the following.

Lemma XI.6.3 *Let the assumptions of* Lemma XI.6.1 *be satisfied, and let* $\boldsymbol{Q} = \boldsymbol{Q}(t)$ *be the one-parameter family of proper orthogonal matrices defined in* (VIII.5.10)–(VIII.5.11). *Then, setting* $\boldsymbol{u} := \boldsymbol{v} + \boldsymbol{v}_\infty$, *the following properties hold:*

(a) *For all* $q \in (2, \infty)$,

$$\int_\Omega \big(\sup_{s \geq 0} |\boldsymbol{u}(\boldsymbol{Q}^\top(s) \cdot \boldsymbol{x})|\big)^q dx < \infty\,.$$

(b) *For all* $\varepsilon > 0$,

$$\int_{\Omega_R} \big(\sup_{s \geq 0} |\boldsymbol{u}(\boldsymbol{Q}^\top(s) \cdot \boldsymbol{x})|\big)^2 dx \leq c\, R^\varepsilon\,,$$

where c *is independent of* R *(*$c \to \infty$ *as* $\varepsilon \to 0$*).*

Proof. Property (b) is an immediate consequence of property (a). In fact, by the Hölder inequality, for *any* $q \in (2,\infty)$, we obtain

$$\begin{aligned}\int_{\Omega_R} \big(\sup_{s\ge 0}|\boldsymbol{u}(\boldsymbol{Q}^\top(s)\cdot\boldsymbol{x})|\big)^2 dx &\le |\Omega_R|^{\frac{(q-2)}{q}}\left(\int_{\Omega_R}\big(\sup_{s\ge 0}|\boldsymbol{u}(\boldsymbol{Q}^\top(s)\cdot\boldsymbol{x})|\big)^q dx\right)^{\frac{2}{q}}\\ &\le c\,R^{\frac{3(q-2)}{q}}\left(\int_{\Omega}\big(\sup_{s\ge 0}|\boldsymbol{u}(\boldsymbol{Q}^\top(s)\cdot\boldsymbol{x})|\big)^q dx\right)^{\frac{2}{q}},\end{aligned}$$

and property (b) follows from (a). In order to establish (a), we observe that since $\boldsymbol{Q}$ is periodic of period $T := 2\pi/\mathcal{T}$ (see (VIII.5.12)), it is enough to prove the result by restricting s to the interval $[0,T]$. Set $\boldsymbol{w}(x,t) := \boldsymbol{u}(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{x})$. Taking into account that (see also (VIII.5.15))

$$\left|\frac{\partial \boldsymbol{w}}{\partial t}\right| = \mathcal{T}|\boldsymbol{e}_1\times(\boldsymbol{Q}^\top\cdot\boldsymbol{x})\cdot\nabla\boldsymbol{u}|\,,$$

and that $\boldsymbol{Q}(0) = \boldsymbol{I}$, we obtain, for all $s\in[0,T]$,

$$\begin{aligned}|\boldsymbol{w}(\boldsymbol{x},s)| &\le |\boldsymbol{u}(\boldsymbol{x})| + \int_0^T\left|\frac{\partial \boldsymbol{w}}{\partial t}\right| dt\\ &= |\boldsymbol{u}(\boldsymbol{x})| + \int_0^T |\boldsymbol{e}_1\times(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{x})\cdot\nabla\boldsymbol{u}(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{x})|dt\,,\end{aligned}$$

which implies, by (II.3.3) and Hölder inequality,

$$\begin{aligned}&\left(\sup_{s\in[0,T]}|\boldsymbol{u}(\boldsymbol{Q}^\top(s)\cdot\boldsymbol{x})|\right)^q\\ &\qquad\le 2^{q-1}\left(|\boldsymbol{u}(\boldsymbol{x})|^q + T^{q-1}\int_0^T|\boldsymbol{e}_1\times(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{x})\cdot\nabla\boldsymbol{u}(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{x})|^q dt\right).\end{aligned}\tag{XI.6.6}$$

Recalling that $\boldsymbol{Q}(t)$ is proper orthogonal for all $t\ge 0$, we have

$$\int_\Omega |\boldsymbol{e}_1\times(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{x})\cdot\nabla\boldsymbol{u}(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{x})|^q dx = \int_\Omega|\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{u}(\boldsymbol{x})|^q dx\,,$$

so that from (XI.6.6) we get

$$\int_\Omega\left(\sup_{s\in[0,T]}|\boldsymbol{u}(\boldsymbol{Q}^\top(s)\cdot\boldsymbol{x})|\right)^q dx \le 2^{q-1}\left(\|\boldsymbol{u}\|^q_{q,\Omega} + T^q\|\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{u}\|^q_{q,\Omega}\right).\tag{XI.6.7}$$

From the stated assumptions, Theorem XI.4.1, and Lemma XI.6.2, we know that $\boldsymbol{u}, \boldsymbol{e}_1, \times\boldsymbol{x}\cdot\nabla\boldsymbol{u}\in L^q(\Omega)$ for all $q\in(2,\infty)$, so that claimed property (a) follows from (XI.6.7). □

We are now ready to give pointwise estimates for the velocity field of a generalized solution.

Theorem XI.6.1 *Let* $\boldsymbol{v}$ *be a generalized solution to* (XI.0.10)–(XI.0.11), *corresponding to* $\boldsymbol{f}$ *of bounded support and to* $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq 0$. *Then, for any* $\delta > 0$ *and all sufficiently large* $|x|$,

$$\boldsymbol{v}(x) + \boldsymbol{v}_\infty(x) = O\Big(|x|^{-1}(1+\mathcal{R}\, s(x))^{-1} + |x|^{-3/2+\delta}\Big)\,,$$

and where, we recall, $s(x) := |x| + x_1$.

Proof. Choose ρ so large that $\operatorname{supp}(\boldsymbol{f}) \subset B_\rho$. By Theorem XI.1.2, we know that $\boldsymbol{v}, p$ are in $C^\infty(\Omega^\rho) \cap C^\infty(\Omega_\rho, r)$, $r > \rho$. Moreover, setting $\boldsymbol{u} := \boldsymbol{v} + \boldsymbol{v}_\infty$, by Theorem XI.4.1 (after possibly adding a constant to p), we find that $(\boldsymbol{u}, \widetilde{p})$ satisfies (XI.6.1). Proceeding exactly as in the proof of Lemma XI.6.2, we then establish that the functions ψ_ρ, $\boldsymbol{w}$, and d there defined satisfy problem (XI.6.4). Let $\boldsymbol{Q} = \boldsymbol{Q}(t)$ be the one-parameter family of proper orthogonal matrices defined in (VIII.5.10)–(VIII.5.11), and set

$$\begin{aligned}
&\boldsymbol{y} := \boldsymbol{Q}(t)\cdot \boldsymbol{x}\,,\\
&\boldsymbol{S}(y,t) := \boldsymbol{Q}(t)\cdot \boldsymbol{w}(\boldsymbol{Q}^\top(t)\cdot \boldsymbol{y})\,, \quad \pi(y,t) := d(\boldsymbol{Q}^\top(t)\cdot \boldsymbol{y})\,,\\
&\boldsymbol{V}(y,t) := \boldsymbol{Q}(t)\cdot [\psi_\rho \boldsymbol{u}](\boldsymbol{Q}^\top(t)\cdot \boldsymbol{y})\,, \quad \boldsymbol{H}(y,t) := \boldsymbol{Q}(t)\cdot \boldsymbol{F}(\boldsymbol{Q}^\top(t)\cdot \boldsymbol{y})\,.
\end{aligned} \tag{XI.6.8}$$

From (XI.6.1) and (XI.6.3) we obtain

$$\boldsymbol{w} \in L^r(\mathbb{R}^3)\,, \quad \text{for all } r > 2, \tag{XI.6.9}$$

and hence we have

$$\left.\begin{aligned}
\frac{\partial \boldsymbol{S}}{\partial t} = \Delta \boldsymbol{S} + \mathcal{R}\frac{\partial \boldsymbol{S}}{\partial y_1} - \nabla \pi - \mathcal{R}\nabla\cdot[\boldsymbol{V}\otimes\boldsymbol{V}] - \boldsymbol{H}\\
\nabla\cdot \boldsymbol{S} = 0
\end{aligned}\right\}\ \text{in } \mathbb{R}^3_\infty\,, \tag{XI.6.10}$$

$$\lim_{t\to 0^+} \|\boldsymbol{S}(\cdot,t) - \boldsymbol{w}\|_r = 0\,, \quad \text{all } r \in (2,\infty)\,.$$

Utilizing (XI.6.1), we deduce, in particular,

$$\nabla\cdot[\boldsymbol{V}\otimes\boldsymbol{V}] \in L^{\infty,r}(\mathbb{R}^3_\infty)\,, \quad \text{for all } r \in (1,4)\,.$$

Moreover,

$$\operatorname{supp}(\boldsymbol{H}(\cdot,t)) \subset B_{2\rho}\,, \quad \boldsymbol{H} \in L^{\infty,r}(\mathbb{R}^3_\infty)\,, \quad \text{for all } r \in (1,\infty)\,.$$

In view of all the above, from Theorem VIII.4.1, Theorem VIII.4.2, and Theorem VIII.4.3 we can find a solution $(\widehat{\boldsymbol{S}}, \widehat{\pi})$ to (XI.6.10) such that

$$\begin{aligned}
&(\widehat{\boldsymbol{S}}, 0) \in \mathcal{L}^r(\mathbb{R}^3 \times (\varepsilon, T)), \quad \widehat{\boldsymbol{S}} \in L^r(\mathbb{R}^3_\infty)\\
&(\boldsymbol{0}, \widehat{\pi}) \in \mathcal{L}^r(\mathbb{R}^3_T)\,, \text{ for all } r \in (2,4), \text{ all } \varepsilon > 0, \text{ and all } T > \varepsilon,
\end{aligned}$$

having the following representation:[2]

$$\widehat{\boldsymbol{S}}(y,t) = (4\pi t)^{-3/2}\int_{\mathbb{R}^3} e^{-|y-z+\mathcal{R}t\boldsymbol{e}_1|^2/4t}\boldsymbol{w}(z)\,dz$$
$$-\int_0^t\int_{\mathbb{R}^3}\boldsymbol{\Gamma}(y-z,t-\tau)\cdot\big(\mathcal{R}\nabla\cdot[\boldsymbol{V}\otimes\boldsymbol{V}](z,\tau)+\boldsymbol{H}(z,\tau)\big)\,dz\,d\tau\,. \tag{XI.6.11}$$

Clearly, we have (for example)

$$\frac{\partial \boldsymbol{S}}{\partial t}\,,\;D^2\boldsymbol{S}\,,\;\phi,\;\nabla\phi\in L^2_{loc}((0,T]\times\mathbb{R}^3)$$

and, recalling (XI.6.9), also $\boldsymbol{S}\in L^r(\mathbb{R}^3_T)$, for all $r\in(2,\infty)$. Thus, by Lemma VIII.4.2, we conclude, in particular, that $\boldsymbol{S}=\widehat{\boldsymbol{S}}$. From (XI.6.11) we can therefore, derive the following representation for $\boldsymbol{S}$:

$$\boldsymbol{S}=\boldsymbol{S}_1+\boldsymbol{S}_2+\boldsymbol{S}_3\,, \tag{XI.6.12}$$

where

$$S_{1i}(y,t)=\mathcal{R}\int_0^t\int_{\mathbb{R}^3}D_l\Gamma_{ij}(y-z,t-\tau)V_j(z,\tau)V_l(z,\tau)\,dz\,d\tau\,,\quad i=1,2,3\,, \tag{XI.6.13}$$

$$\boldsymbol{S}_2(y,t)=-\int_0^t\int_{\mathbb{R}^3}\boldsymbol{\Gamma}(y-z,t-\tau)\cdot\boldsymbol{H}(z,\tau)\,dz\,d\tau\,, \tag{XI.6.14}$$

$$\boldsymbol{S}_3(y,t)=(4\pi t)^{-3/2}\int_{\mathbb{R}^3}e^{-|y-z+\mathcal{R}t\boldsymbol{e}_1|^2/4t}\boldsymbol{w}(z)\,dz\,. \tag{XI.6.15}$$

We shall now give pointwise estimates of the functions $\boldsymbol{S}_i$, $i=1,2,3$. Since the numerical values of $\mathcal{R}$ and $\mathcal{T}$ are irrelevant in the proof (provided they are both positive, of course), in what follows we shall put, for simplicity, $\mathcal{R}=\mathcal{T}=1$. By Lemma VIII.3.2, we have

$$\begin{aligned}|\boldsymbol{S}_1(y,t)| &\le c_2\int_0^t\int_{\mathbb{R}^3}(\tau+|y-z+\tau\boldsymbol{e}_1|^2)^{-2}|\boldsymbol{V}(z,t-\tau)|^2\,dz\,d\tau\\ &= c_2\int_0^t\int_{B_R}(\tau+|y-z+\tau\boldsymbol{e}_1|^2)^{-2}|\boldsymbol{V}(z,t-\tau)|^2\,dz\,d\tau\\ &\quad+c_2\int_0^t\int_{B^R}(\tau+|y-z+\tau\boldsymbol{e}_1|^2)^{-2}|\boldsymbol{V}(z,t-\tau)|^2\,dz\,d\tau\\ &:=I_1+I_2\,,\end{aligned} \tag{XI.6.16}$$

where $R=|y|/3>2\rho$. Using the Hölder inequality, for any $r\in(2,\infty)$ and with $r_0:=r/(r-2)$ we obtain

[2] For simplicity, throughout the proof we shall omit, the label $\mathcal{R}$ in the notation of the fundamental tensor $\boldsymbol{\Gamma}$.

$$I_1 \le c_3 \int_0^\infty \left(\int_{B_R} (\tau + |y - z + \tau \boldsymbol{e}_1|^2)^{-2r_0}\, dz \right)^{1/r_0} \|\boldsymbol{V}(t-\tau)\|_r^2\, d\tau\,.$$

From the definition of $\boldsymbol{V}$ given in (XI.6.8), we obtain $\|\boldsymbol{V}(t-\tau)\|_r^2 \le \|\boldsymbol{u}\|_{r,\Omega^\rho}^2$, and so

$$I_1 \le c_3 \|\boldsymbol{u}\|_{r,\Omega^\rho}^2 \int_0^\infty \left(\int_{B_R} (\tau + |y - z + \tau \boldsymbol{e}_1|^2)^{-2r_0}\, dz \right)^{1/r_0} d\tau\,.$$

Furthermore, putting $z' = z - \tau \boldsymbol{e}_1$, we have $|z'| \le 2R$ for $\tau \in [0, R]$ and $z \in B_R$. Thus

$$\begin{aligned}&\int_0^R \left(\int_{B_R} (\tau + |y - z + \tau \boldsymbol{e}_1|^2)^{-2r_0}\, dz \right)^{1/r_0} d\tau \\ &\qquad \le \int_0^R \left(\int_{B_R} (\tau + |y - z'|^2)^{-2r_0}\, dz \right)^{1/r_0} d\tau\,.\end{aligned}$$

Also, for $|z'| \le 2R$ we have $|y - z'| \ge 3R - 2R = R$, and hence

$$\begin{aligned}&\int_0^R \left(\int_{B_R} (\tau + |y - z + \tau \boldsymbol{e}_1|^2)^{-2r_0}\, dz \right)^{1/r_0} d\tau \\ &\qquad \le \int_0^R \left(\int_{B_{2R}} (\tau + R^2)^{-2r_0}\, dz \right)^{1/r_0} d\tau \le c_4 \int_0^R \frac{R^{3/r_0}}{(\tau + R^2)^2} d\tau \le c_5 R^{-2+\frac{3}{r_0}}\,.\end{aligned}$$

Since

$$\begin{aligned}\int_R^\infty \left(\int_{B_R} (\tau + |y - z + \tau \boldsymbol{e}_1|^2)^{-2r_0}\, dz \right)^{1/r_0} d\tau &\le \int_R^\infty \left(\int_{B_R} \tau^{-2r_0}\, dz \right)^{1/r_0} d\tau \\ &\le c_6\, R^{-1+\frac{3}{r_0}}\,,\end{aligned}$$

we infer, for sufficiently large $|y|$,

$$I_1 \le c_7 R^{-1+\frac{3}{r_0}} = c_7 R^{-1+\frac{3(r-2)}{r}}\,,$$

where $c_7 = c_7(r)$. Thus, recalling that r is arbitrary in $(2, \infty)$, we conclude that

$$I_1 \le c_8 |y|^{-1+\varepsilon}\,, \quad \text{for all } \varepsilon > 0\,, \tag{XI.6.17}$$

with $c_8 \to \infty$ as $\varepsilon \to 0$. We shall next estimate I_2. To this end, we set $\delta(\xi) = \sqrt{\xi_2^2 + \xi_3^2}$, $\boldsymbol{\xi} \in \mathbb{R}^3$, and write

$$\begin{aligned}\frac{I_2}{c_2} &= \int_0^t \int_{B^R \cap \{\delta(y-z)<1\}} (\tau + |y - z + \tau \boldsymbol{e}_1|^2)^{-2} |\boldsymbol{V}(z, t-\tau)|^2\, dz\, d\tau \\ &\quad + \int_0^t \int_{B^R \cap \{\delta(y-z)\ge 1\}} (\tau + |y - z + \tau \boldsymbol{e}_1|^2)^{-2} |\boldsymbol{V}(z, t-\tau)|^2\, dz\, d\tau \\ &:= I_{21} + I_{22}\,.\end{aligned}$$

By Hölder's inequality, we obtain for an arbitrary $q_0 > 6$ and with $q_1 := q_0/(q_0-2) \in (1, 3/2)$:

$$I_{21} \le \int_0^t \left(\int_{\{\delta(y-z)<1\}} \frac{dz}{(\tau + |y-z+\tau \boldsymbol{e}_1|^2)^{2q_1}} \right)^{1/q_1} \|\boldsymbol{V}(t-\tau)\|^2_{q_0,B^R} d\tau \,,$$

from which, using the result of Exercise VIII.3.2 and the definition of $\boldsymbol{V}$, we deduce

$$I_{21} \le c_9 \|\boldsymbol{u}\|^2_{q_0,B^R} \le c_9 \|\boldsymbol{u}\|^{\frac{2(q_0-6)}{q_0}}_{\infty,\Omega^\rho} \|\boldsymbol{u}\|^{\frac{2(6-q_0)}{q_0}}_{6,B^R} \,. \tag{XI.6.18}$$

We shall now use in the above relation the inequality

$$\|\boldsymbol{u}\|_{6,B^R} \le c_{10} |\boldsymbol{u}|_{1,2,B^R} \,, \tag{XI.6.19}$$

where, as we know from Exercise II.6.5, the constant c_{10} is independent of R. Combining (XI.6.19) with Lemma XI.6.1 and placing this information back into (XI.6.18), it follows that

$$I_{21} \le c_{11} R^{-1+\varepsilon} = c_{12} |y|^{-1+\varepsilon} \,, \quad \text{for all } \varepsilon > 0 \,, \tag{XI.6.20}$$

where $c_{12} \to \infty$ as $\varepsilon \to 0$. In order to estimate I_{22} we choose arbitrary $q_0 \in (2, 6)$ and set $q_2 := q_0/(q_0-2) \in (3/2, \infty)$. We thus obtain

$$I_{22} \le \int_0^t \left(\int_{\{\delta(y-z)\ge 1\}} \frac{dz}{(\tau + |y-z+\tau \boldsymbol{e}_1|^2)^{2q_2}} \right)^{1/q_2} \|\boldsymbol{V}(t-\tau)\|^2_{q_0,B^R} d\tau \,,$$

from which, using again the result of Exercise VIII.3.2, the definition of $\boldsymbol{V}$, and the interpolation inequality (II.2.7) we deduce

$$I_{22} \le c_{13} \|\boldsymbol{u}\|^2_{q_0,B^R} \le c_{13} \left(\|\boldsymbol{u}\|^{1-\theta}_{6,B^R} \|\boldsymbol{u}\|^{\theta}_{3} \right)^2 \,, \tag{XI.6.21}$$

where $\theta \to 0$ and $c_{13} \to \infty$ as $q_0 \to 6$. Combining (XI.6.21), (XI.6.19) and Lemma XI.6.1, we deduce that

$$I_{22} \le c_{14} R^{-1+\varepsilon} = c_{15} |y|^{-1+\varepsilon} \,, \quad \text{for all } \varepsilon > 0 \,, \tag{XI.6.22}$$

where $c_{15} \to \infty$ as $\varepsilon \to 0$. Thus, (XI.6.16), (XI.6.17), (XI.6.20), and (XI.6.22) allow us to conclude that

$$|\boldsymbol{S}_1(y,t)| \le c_{16} |y|^{-1+\varepsilon} \,, \quad \text{for all } \varepsilon > 0 \,, \tag{XI.6.23}$$

with $c_{16} \to \infty$ as $\varepsilon \to 0$. Concerning $\boldsymbol{w}_2$, from Theorem VIII.4.4 (see (VIII.4.48)) and the definition of $\boldsymbol{H}$, we immediately deduce for all $s > 3$

$$|\boldsymbol{S}_2(y,t)| \le C \frac{\|\boldsymbol{F}\|_s}{(1+|y|)(1+s(y))} \,, \tag{XI.6.24}$$

so that in particular,

$$|\boldsymbol{S}_2(y,t)| \leq c_{17}|y|^{-1}\,. \tag{XI.6.25}$$

We also notice that from Theorem VIII.4.3 (see (VIII.4.42)), it follows that

$$|\boldsymbol{S}_3(y,t)| \leq c_{18} t^{-\frac{1}{4}} \|\boldsymbol{w}\|_6\,. \tag{XI.6.26}$$

Combining (XI.6.2), (XI.6.8), and (XI.6.12) together with (XI.6.24)–(XI.6.26), we thus conclude, for sufficiently large $|x|$, that

$$\begin{aligned}|\boldsymbol{u}(x)| &\leq |\boldsymbol{w}(x)| + |\Phi||\boldsymbol{\sigma}(x)| \leq |\boldsymbol{S}(\boldsymbol{Q}(t)\cdot\boldsymbol{x})| + c_{21}|x|^{-2} \\ &\leq c_{22}\left(|x|^{-1+\varepsilon} + t^{-\frac{1}{4}}\|\boldsymbol{w}\|_6\right), \quad \text{for all } \varepsilon > 0 \text{ and all } t > 0\,,\end{aligned} \tag{XI.6.27}$$

where c_{22} depends on ε, but not on t. Thus, letting $t \to \infty$, from this last inequality we obtain

$$|\boldsymbol{u}(x)| \leq c_{22}|x|^{-1+\varepsilon} \tag{XI.6.28}$$

With the estimate (XI.6.28) in hand, we will give an improved bound on the term $\boldsymbol{S}_1$ given in (XI.6.13). We begin by observing that, clearly, from the definition of $\boldsymbol{V}$ and (XI.6.28), it follows that

$$\begin{aligned}|\boldsymbol{V}(z,t)| &\leq c_{23}(1+|z|)^{-1+\varepsilon}\,, \\ \sup_{s\geq 0}|\boldsymbol{V}(z,s)| &\leq \sup_{s\geq 0}|\boldsymbol{u}(\boldsymbol{Q}^\top(s)\cdot\boldsymbol{z})|\,.\end{aligned} \tag{XI.6.29}$$

We next partition $\mathbb{R}^3$ into three regions:

$$B_R\,, \quad \overline{B_{R,4R}}\,, \quad B^{4R}\,,$$

and denote the corresponding contributions of $\boldsymbol{S}_1$ over these spatial regions by $\mathscr{I}_1$, $\mathscr{I}_2$, and $\mathscr{I}_3$, respectively. Thus, employing Lemma VIII.3.3 and (XI.6.29)$_2$, we may increase $\mathscr{I}_1$ as follows

$$\begin{aligned}|\mathscr{I}_1| &\leq c_{24}\int_{B_R}\left(\sup_{s\geq 0}|\boldsymbol{V}(z,s)|\right)^2\left(\int_0^\infty|\nabla\boldsymbol{\Gamma}(y-z,t)|dt\right)dz \\ &\leq c_{25}\int_{\Omega_R}\frac{\left(\sup_{s\geq 0}|\boldsymbol{u}(\boldsymbol{Q}^\top(s)\cdot\boldsymbol{z})|\right)^2}{|y-z|^{3/2}}\,dz \leq \frac{c_{26}}{|y|^{\frac{3}{2}}}\int_{\Omega_R}\left(\sup_{s\geq 0}|\boldsymbol{u}(\boldsymbol{Q}^\top(s)\cdot\boldsymbol{z})|\right)^2.\end{aligned}$$

Using the result of Lemma XI.6.3 in this latter, we then conclude that

$$|\mathscr{I}_1| \leq c_{27}|y|^{-3/2}R^\delta = c_{28}\,|y|^{-3/2+\delta} \tag{XI.6.30}$$

for arbitrary $\delta > 0$. In a similar fashion, we establish

$$|\mathscr{I}_3| \le c_{29}\int_{B^{4R}}\left(\sup_{s\ge 0}|\boldsymbol{V}(z,s)|\right)^2\left(\int_0^\infty|\nabla\boldsymbol{\Gamma}(y-z,t)|dt\right)dz$$
$$\le c_{30}\int_{\Omega^{4R}}\frac{\left(\sup_{s\ge 0}|\boldsymbol{u}(\boldsymbol{Q}^\top(s)\cdot\boldsymbol{z})|\right)^2}{|y-z|^{3/2}}\,dz\,.$$

Therefore, since for $z\in\Omega^{4R}$ we have $|y-z|\ge\frac14|z|+3R-|y|=\frac14|z|$, with the help of the Hölder inequality we obtain for arbitrary $r\in(1,2)$

$$|\mathscr{I}_3|\le c_{31}\left(\int_{\Omega^{4R}}\left(\sup_{s\ge 0}|\boldsymbol{u}(\boldsymbol{Q}^\top(s)\cdot\boldsymbol{z})|\right)^{2r}dz\right)^{\frac1r}\left(\int_{2R}^\infty\rho^{-\frac{3r}{2(r-1)}+2}d\rho\right)^{\frac{r-1}{r}}\,,$$

which, in turn, by Lemma XI.6.3, furnishes

$$|\mathscr{I}_3|\le c_{32}\,|R|^{-3/2+\delta}=c_{33}\,|y|^{-3/2+\delta}\,, \tag{XI.6.31}$$

where $\delta:=3(r-1)/r$ is positive and arbitrarily close to zero, since r can be chosen arbitrarily close to 1. In order to increase $\mathscr{I}_2$, we notice that

$$\Gamma_1(\xi):=\int_0^\infty|\nabla\boldsymbol{\Gamma}(\xi,t)|dt$$

satisfies the following property:

$$\int_{\partial B_\rho(x_0)}\Gamma_1(x-x_0)\le M\,\rho^{-1/2}\,, \tag{XI.6.32}$$

for all $x_0\in\mathbb{R}^3$ and $\rho>0$. The proof of (XI.6.32) is entirely analogous to the analogous property proved in Exercise VII.3.1 for the fundamental tensor $\boldsymbol{E}$, once we take into account the estimate for Γ_1 furnished in Lemma VIII.3.3. Taking into account $(\text{XI.6.29})_1$, we derive

$$|\mathscr{I}_2|\le c_{34}\int_{B_{R,4R}}(1+|z|)^{-2+2\varepsilon}\left(\int_0^\infty|\nabla\boldsymbol{\Gamma}(y-z,t)|dt\right)dz$$
$$\le c_{35}R^{-2+2\varepsilon}\int_{B_{R,4R}}\Gamma(y-z)\,dz\le c_{35}R^{-2+2\varepsilon}|\Gamma|_{1,B_{7R}(y)}\,.$$

Consequently, by (XI.6.32),

$$|\mathscr{I}_2|\le c_{36}|R|^{-2+2\varepsilon}R^{1/2}=c_{37}\,|y|^{-3/2+\delta} \tag{XI.6.33}$$

for arbitrary $\delta>0$. Recalling that $\boldsymbol{S}_1=\sum_{i=1}^3\mathscr{I}_i$, from (XI.6.30), (XI.6.31), and (XI.6.33) we deduce

$$|\boldsymbol{S}_1(y,t)|\le c_{38}|y|^{-3/2+\delta}\,,\quad\text{for all }\delta>0. \tag{XI.6.34}$$

Putting together (XI.6.9), (XI.6.12), (XI.6.25), (XI.6.26), and (XI.6.34), arguing as in (XI.6.27), and letting $t \to \infty$, we conclude, for sufficiently large $|x|$, that

$$|\boldsymbol{u}(x)| \le c_{39}\big[(1+|x|)^{-1}(1+s(x))^{-1} + |x|^{-3/2+\delta}\big]\,, \quad \text{for all } \delta > 0\,,$$

which completes the proof of the theorem. □

Our next result concerns the asymptotic behavior of $\nabla \boldsymbol{v}$.

Theorem XI.6.2 *Let the assumptions of* Theorem XI.6.1 *be satisfied. Then, for any* $\eta > 0$ *and all sufficiently large* $|x|$,

$$\nabla(\boldsymbol{v}(x) + \boldsymbol{v}_\infty(x)) = O\Bigg(|x|^{-3/2}(1+\mathcal{R}\, s(x))^{-3/2} + |x|^{-2+\eta}\Bigg)\,.$$

Proof. As in the proof of the previous theorem, we set $\boldsymbol{u} := \boldsymbol{v} + \boldsymbol{v}_\infty$. We begin by recalling that, by Lemma XI.6.2, $\nabla \boldsymbol{u}$ is bounded for all large $|x|$. From the definition (XI.6.8) of $\boldsymbol{V}$, we thus obtain

$$|\nabla \boldsymbol{V}(y,t)| \le c_1 \tag{XI.6.35}$$

with c_1 independent of y and t. We also recall that by the same token, from Theorem XI.6.1 we have, for all large $|y|$,

$$|\boldsymbol{V}(y,t)| \le c_2\,|y|^{-1} \tag{XI.6.36}$$

with c_2 independent of y and t. Our next step is to prove that $|\nabla \boldsymbol{u}(x)|$ decays at least like $|x|^{-1}$. Our starting point is again (XI.6.12)–(XI.6.15) for large values of $|y|$, which, as immediately proved, yields for $k, i = 1, 2, 3$,

$$\begin{aligned}
D_k S_{1i}(y,t) &= \mathcal{R}\int_0^t\int_{\mathbb{R}^3} D_k \varGamma_{ij}(y-z,t-\tau) D_l[V_j(z,\tau)V_l(z,\tau)]\,dz\,d\tau\,,\\
D_k \boldsymbol{S}_2(y,t) &= -\int_0^t\int_{\mathbb{R}^3} D_k \boldsymbol{\varGamma}(y-z,t-\tau)\cdot \boldsymbol{H}(z,\tau)\,dz\,d\tau\,,\\
D_k \boldsymbol{S}_3(y,t) &= (4\pi t)^{-3/2} D_k\left[\int_{\mathbb{R}^3} e^{-|y-z+\mathcal{R}t\boldsymbol{e}_1|^2/4t}\boldsymbol{w}(z)\,dz\right].
\end{aligned} \tag{XI.6.37}$$

As done previously, we set, for simplicity, $\mathcal{R} = \mathcal{T} = 1$. We notice that from Theorem VIII.4.4 (see, in particular, the argument after (VIII.4.66)), we get

$$|D_k \boldsymbol{S}_2(y,t)| \le c_3\,|y|^{-3/2}(1+s(y))^{-3/2}\,, \tag{XI.6.38}$$

while from Theorem VIII.4.3 it follows that

$$|D_k \boldsymbol{S}_3(y,t)| \le c_4\, t^{-3/4}\|\boldsymbol{w}\|_6\,. \tag{XI.6.39}$$

It remains to estimate $D_k \boldsymbol{w}_1$. To this end, we split the integral in (XI.6.37)$_1$ into two contributions as follows:

$$\begin{aligned} D_k S_{1i}(y,t) = & \int_0^t \int_{\mathbb{R}^3 \backslash B_1(y)} D_k \Gamma_{ij}(y-z,t-\tau) D_l[V_j(z,\tau)V_l(z,\tau)]\, dz\, d\tau\,, \\ & + \int_0^t \int_{B_1(y)} D_k \Gamma_{ij}(y-z,t-\tau)[V_l(z,\tau) D_l V_j(z,\tau)]\, dz\, d\tau\,, \\ := & \, I_1 + I_2\,, \end{aligned} \tag{XI.6.40}$$

where we have used the condition $\nabla \cdot \boldsymbol{V}(z,t) = 0$ for sufficiently large $|z|$. Thanks to (XI.6.35) and (XI.6.36), we obtain

$$|I_2| \le c_5 |y|^{-1} \int_{B_1(y)} \int_0^\infty |\nabla \boldsymbol{\Gamma}(y-z,t)| dt\, dz\,,$$

so that by Lemma VIII.3.3, we deduce

$$|I_2| \le c_6 |y|^{-1}\,. \tag{XI.6.41}$$

We next notice that by an integration by parts,

$$\begin{aligned} I_1 = & -\int_0^t \int_{\mathbb{R}^3 \backslash B_1(y)} D_k D_l \Gamma_{ij}(y-z,\tau)[V_j(z,t-\tau)V_l(z,t-\tau)]\, dz\, d\tau\,, \\ & + \int_0^t \int_{\partial B_1(y)} D_k \Gamma_{ij}(y-z,\tau)\, V_j(z,t-\tau) V_l(z,t-\tau) n_l\, dz\, d\tau \\ := & \, I_1^{(1)} + I_1^{(2)}\,. \end{aligned} \tag{XI.6.42}$$

Employing (XI.6.36) and (XI.6.32), we obtain

$$|I_1^{(2)}| \le c_7\, |y|^{-2} \int_{\partial B_1(y)} \int_0^\infty |\nabla \boldsymbol{\Gamma}(y-z,t)|\, dt \le c_8\, |y|^{-2}\,. \tag{XI.6.43}$$

In order to estimate $I_1^{(1)}$, we split it as sum of three integrals, $\mathscr{I}_1$, $\mathscr{I}_2$, and $\mathscr{I}_3$, over $B_{R/2}$, $B_{2R,R/2} - B_1(y)$, and B^{2R}, respectively, where $R := |y|$, is sufficiently large. We have

$$|\mathscr{I}_1| \le c_9 \int_{B_{R/2}} \Big(\sup_{s \ge 0} |\boldsymbol{V}(z,s)| \Big)^2 \Big(\int_0^\infty |D^2 \boldsymbol{\Gamma}(y-z,t)| dt \Big) dz\,,$$

and consequently, using Lemma VIII.3.3, (XI.6.29), and Lemma XI.6.3, we get

$$\begin{aligned} |\mathscr{I}_1| & \le c_{10} \int_{B_{R/2}} \Big(\sup_{s \ge 0} |\boldsymbol{V}(z,s)| \Big)^2 |y-z|^{-2} (1 + s(y-z))^{-2}\, dz \\ & \le c_{11} |y|^{-2} \int_{\Omega_{R/2}} \Big(\sup_{s \ge 0} |\boldsymbol{u}(\boldsymbol{Q}^\top(s) \cdot \boldsymbol{z})| \Big)^2 \le c_{12}\, |y|^{-2} R^\eta\,, \end{aligned}$$

for arbitrary positive η. Thus we conclude that

$$|\mathscr{I}_1| \le c_{13}\, |y|^{-2+\eta}\,. \tag{XI.6.44}$$

Likewise, we show

$$\begin{aligned}|\mathscr{I}_3| &\le c_{14} \int_{B_{2R}} \big(\sup_{s\ge 0} |\boldsymbol{V}(z,s)|\big)^2 |y-z|^{-2}(1+\ s(y-z))^{-2}\, dz \\ &\le c_{15} \int_{\Omega_{2R}} \big(\sup_{s\ge 0} |\boldsymbol{u}(\boldsymbol{Q}^\top(s)\cdot \boldsymbol{z})|\big)^2 |y-z|^{-2}\, dz\,.\end{aligned}$$

Therefore, observing that for $z \in \Omega^{2R}$ we have $|y-z| \ge \frac{1}{2}|z| + R - |y| = \frac{1}{2}|z|$, by the Hölder inequality we deduce, for arbitrary $r \in (1,3)$,

$$|\mathscr{I}_3| \le c_{16} \left(\int_{\Omega^{2R}} \left(\sup_{s\ge 0} |\boldsymbol{u}(\boldsymbol{Q}^\top(s)\cdot \boldsymbol{z})|\right)^{2r} dz\right)^{\frac{1}{r}} \left(\int_{2R}^{\infty} \rho^{-\frac{2r}{r-1}+2} d\rho\right)^{\frac{r-1}{r}}\,,$$

which, in turn, by Lemma XI.6.3, delivers

$$|\mathscr{I}_3| \le c_{17}\, |R|^{-2+\eta} = c_{17}\, |y|^{-2+\eta} \tag{XI.6.45}$$

with $\eta := 3(r-1)/r$ positive and arbitrarily close to zero, since r may be taken arbitrarily close to 1. Finally, from (XI.6.36) and Lemma VIII.3.3, we obtain

$$\begin{aligned}|\mathscr{I}_2| &\le c_{18}\, |y|^{-2} \int_{B_{R/2,2R}\setminus B_1(y)} \frac{dz}{|y-z|^2(1+s(y-z))^2} \\ &\le c_{18}\, |y|^{-2} \int_{B_{1,3R}(y)} \frac{dz}{|y-z|^2(1+s(y-z))^2}\,.\end{aligned} \tag{XI.6.46}$$

Since by a simple and direct calculation one shows that

$$\int_{B_{1,3R}(y)} \frac{dz}{|y-z|^2(1+s(y-z))^2} \le c_{19} \ln R$$

with c_{19} independent of R, from (XI.6.46) we get

$$|\mathscr{I}_2| \le c_{20}\, |y|^{-2} \ln R \le c_{20}\, |y|^{-2+\eta} \tag{XI.6.47}$$

for arbitrarily small positive η. Thus, recalling that $I_1^{(1)} = \sum_{i=1}^{3} \mathscr{I}_i$ and collecting (XI.6.40)–(XI.6.47), we conclude that

$$|\nabla \boldsymbol{S}_1(y)| \le c_{21} \big(|y|^{-1} + |y|^{-2+\eta}\big)\,, \tag{XI.6.48}$$

where we would like to emphasize that the term $|y|^{-1}$ comes from estimating the integral $I_1^{(2)}$ occurring in (XI.6.40). If we combine (XI.6.2), (XI.6.8), and

(XI.6.12) together with (XI.6.38), (XI.6.39), and (XI.6.48), we obtain, for sufficiently large $|x|$,

$$
\begin{aligned}
|\nabla \boldsymbol{u}(x)| &\le |\nabla \boldsymbol{w}(x)| + |\Phi||\nabla \boldsymbol{\sigma}(x)| \le |\nabla \boldsymbol{S}(\boldsymbol{Q}(t)\cdot \boldsymbol{x})| + c_{22}|x|^{-3} \\
&\le c_{23}\big(|x|^{-1} + |x|^{-3/2}(1+s(x))^{-3/2} + |x|^{-2+\eta} + t^{-3/4}|\boldsymbol{w}\|_6\big)\,,
\end{aligned}
$$

for all $\eta > 0$ and all $t > 0$. Thus, letting $t \to \infty$ in this relation furnishes

$$
|\nabla \boldsymbol{u}(x)| \le c_{23}\big(|x|^{-1} + |x|^{-3/2}(1+s(x))^{-3/2} + |x|^{-2+\eta}\big)\,. \tag{XI.6.49}
$$

This inequality provides, in particular, $\nabla \boldsymbol{u}(x) = O(|x|^{-1})$, which translates into $|\nabla \boldsymbol{V}(y,t)| \le c\,|y|^{-1}$ with c independent of y and t. Plugging this information back into the estimate for $I_1^{(2)}$ allows us to deduce the improved bound $I_1^{(2)} = O(|y|^{-2})$, and so, recalling that the term $|x|^{-1}$ in (XI.6.49) is due only to the contribution from $I_1^{(2)}$, we may now replace that term with $|x|^{-2}$, and the proof of the theorem is complete.

□

We conclude this section with the following result regarding the asymptotic behavior of the pressure.

Theorem XI.6.3 *Let the assumptions of* Theorem XI.6.1 *be satisfied. Then, for all sufficiently large* $|x|$,

$$
p(x) + \frac{\mathcal{T}}{2\mathcal{R}}(x_2^2 + x_3^2) + p_0 = O(|x|^{-2}\ln|x|)\,,
$$

for some $p_0 \in \mathbb{R}$.

Proof. As usual, we set $\boldsymbol{u} := \boldsymbol{v} + \boldsymbol{v}_\infty$, and take the divergence of both sides of (XI.1.5) (with $\boldsymbol{f} \equiv \boldsymbol{0}$). Recalling that $\nabla\cdot(\boldsymbol{e}_1 \times \boldsymbol{x}\cdot\nabla\boldsymbol{u} - \boldsymbol{e}_1\times\boldsymbol{u}) = 0$, we obtain

$$
\begin{aligned}
&\Delta\widetilde{p} = \nabla\cdot\boldsymbol{G} \quad \text{in } \Omega^\rho\,,\\
&\frac{\partial\widetilde{p}}{\partial n} = g \quad \text{on } \partial\Omega^\rho\,,
\end{aligned} \tag{XI.6.50}
$$

where $\widetilde{p}$ is defined in (XI.1.6) and

$$
\begin{aligned}
&\boldsymbol{G} := \mathcal{R}\,\boldsymbol{u}\cdot\nabla\boldsymbol{u}\,,\\
&g := [\Delta\boldsymbol{u} + \mathcal{R}(D_1\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{u} - \boldsymbol{e}_1\times\boldsymbol{u})]\cdot\boldsymbol{n}\,|_{\partial\Omega^\rho}\ .
\end{aligned} \tag{XI.6.51}
$$

From Lemma II.9.1 we establish the following representation of $\widetilde{p}(x)$, for all $x \in \Omega_{\rho,r}$ and all $r > \rho$:

$$
\begin{aligned}
\widetilde{p}(x) = &\int_{\Omega_{\rho,r}} \boldsymbol{G}(y)\cdot\nabla\mathcal{E}(x-y)dy + \int_{\partial\Omega_{\rho,r}}\mathcal{E}(x-y)\boldsymbol{G}(y)\cdot\boldsymbol{n}\,d\sigma_y \\
&-\int_{\partial B_\rho}\mathcal{E}(x-y)g(y)\,d\sigma_y - \int_{\partial B_r}\mathcal{E}(x-y)\frac{\partial\widetilde{p}}{\partial n}(y)\,d\sigma_y \\
&+\int_{\partial B_\rho}\frac{\partial\mathcal{E}}{\partial n}(x-y)\widetilde{p}(y)\,d\sigma_y + \int_{\partial B_r}\frac{\partial\mathcal{E}}{\partial n}(x-y)\widetilde{p}(y)\,d\sigma_y\,,
\end{aligned} \tag{XI.6.52}
$$

where $\mathcal{E}$ is the Laplace fundamental solution (II.9.1). Using the property

$$|D^\alpha \mathcal{E}(\xi)| \leq c_1 \, |\xi|^{-1-|\alpha|}\,, \quad |\alpha| \geq 0\,, \quad \xi \in \mathbb{R}^3 - \{\mathbf{0}\}\,, \tag{XI.6.53}$$

and (XI.6.1), we readily prove the existence of an unbounded sequence $\{r_k\} \in \mathbb{R}_+$ such that (after a possible modification of $\widetilde{p}$ by the addition of a constant)

$$\lim_{k\to\infty} \int_{\partial B_{r_k}} \left[\mathcal{E}(x-y) \left(\boldsymbol{G}(y) \cdot \boldsymbol{n} - \frac{\partial \widetilde{p}}{\partial n}(y) \right) + \frac{\partial \mathcal{E}}{\partial n}(x-y)\widetilde{p}(y) \right] d\sigma_y = 0\,.$$

Consequently, (XI.6.52) furnishes the following representation, for all $x \in \Omega^\rho$:

$$\begin{aligned} \widetilde{p}(x) = & \int_{\Omega_\rho} \boldsymbol{G}(y) \cdot \nabla \mathcal{E}(x-y) dy + \int_{\partial B_\rho} \mathcal{E}(x-y) \boldsymbol{G}(y) \cdot \boldsymbol{n} \, d\sigma_y \\ & - \int_{\partial B_\rho} \mathcal{E}(x-y) g(y) \, d\sigma_y + \int_{\partial B_\rho} \frac{\partial \mathcal{E}}{\partial n}(x-y) \widetilde{p}(y) \, d\sigma_y \,, \end{aligned} \tag{XI.6.54}$$

where we have used the fact that $\partial \Omega_\rho = \partial B_\rho$. Observing that

$$\begin{aligned} & \int_{\partial B_\rho} \mathcal{E}(x-y) \; [\boldsymbol{G}(y) \cdot \boldsymbol{n} - g(y)] \; d\sigma_y \\ & \qquad = m \, \mathcal{E}(x) + \int_{\partial B_\rho} \big(\mathcal{E}(x-y) - \mathcal{E}(x) \big) \, [\boldsymbol{G}(y) \cdot \boldsymbol{n} - g(y)] \; d\sigma_y \end{aligned}$$

with

$$m := \int_{\partial B_\rho} [\boldsymbol{G}(y) \cdot \boldsymbol{n} - g(y)] \;,$$

with the help of (XI.6.53) and the mean-value theorem, from (XI.6.54) we deduce that

$$\widetilde{p}(x) = P(x) + m \, \mathcal{E}(x) + O(|x|^{-2})\,, \tag{XI.6.55}$$

where

$$P(x) := \int_{\Omega_\rho} \boldsymbol{G}(y) \cdot \nabla \mathcal{E}(x-y) dy\,.$$

We write

$$\begin{aligned} P(x) = & \int_{\Omega_{\rho,R/2}} \boldsymbol{G}(y) \cdot \nabla \mathcal{E}(x-y) dy + \int_{\Omega_{R/2,2R}} \boldsymbol{G}(y) \cdot \nabla \mathcal{E}(x-y) dy \\ & + \int_{\Omega^{2R}} \boldsymbol{G}(y) \cdot \nabla \mathcal{E}(x-y) dy \\ := & \; P_1(x) + P_2(x) + P_3(x)\,, \end{aligned} \tag{XI.6.56}$$

where $|x| = R$, sufficiently large. From (XI.6.51) and (XI.6.1), we readily establish, with the help of Hölder's inequality, $\boldsymbol{G} \in L^1(\Omega^\rho)$, and consequently, employing the properties (XI.6.53) for $\mathcal{E}$, we at once show that

$$P_1(x) + P_3(x) = O(|x|^{-2}) . \tag{XI.6.57}$$

Furthermore, setting $\mathscr{B}_{R,x} := B_{R/2,2R} - B_1(x)$, we have

$$\begin{aligned} P_2(x)/\mathcal{R} = & \int_{\mathscr{B}_{R,x}} D_l D_k \mathcal{E}(x-y) u_l(y) u_k(y)\, dy \\ & + \int_{B_1(x)} D_k \mathcal{E}(x-y) u_l(y) D_l u_k(y)\, dy \\ & + \int_{\partial \mathscr{B}_{R,x}} D_k \mathcal{E}(x-y) u_l(y) u_k(y) n_l \, d\sigma_y \\ := & P_{21}(x) + P_{22}(x) + P_{23}(x) . \end{aligned} \tag{XI.6.58}$$

By Theorem XI.6.1, Theorem XI.6.2, and, again, (XI.6.53), we easily show that

$$\begin{aligned} |P_{21}(x)| &\le c_1 \, |x|^{-2} \int_{B_{1,3R}(x)} |\nabla\nabla \mathcal{E}(y)| dy \le c_2 \, |x|^{-2} \int_1^{3R} r^{-1} dr \le c_3 \, |x|^{-2} \ln |x| \, , \\ |P_{22}(x)| &\le c_5 \, |x|^{-2} \int_{B_1(x)} |\nabla \mathcal{E}(x-y)| \, d\sigma_y \le c_3 \, |x|^{-2} \, , \\ |P_{23}(x)| &\le c_4 \, |x|^{-2} \int_{\partial B_1(x) \cup \partial B_{R/2} \cup \partial B_{2R}} |\nabla \mathcal{E}(x-y)| \, d\sigma_y \le c_5 |x|^{-2} . \end{aligned} \tag{XI.6.59}$$

From (XI.6.55)–(XI.6.59) we then deduce

$$\widetilde{p}(x) = m \, \mathcal{E}(x) + O(|x|^{-2} \ln |x|) . \tag{XI.6.60}$$

However, from (XI.6.1), we obtain $\widetilde{p} \in L^{3/2+\varepsilon}(\Omega^\rho)$, for all ε positive and close to zero. Thus, in (XI.6.60) we must have $m = 0$, and the proof of the theorem is complete. □

Remark XI.6.1 The establishment of the asymptotic behavior of second-order derivatives of $\boldsymbol{v}$, as well as the (related) behavior of ∇p, requires, apparently, a more complicated effort, and we leave it unsettled. ■

XI.7 On the Asymptotic Structure of Generalized Solutions When $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0$

Similarly to the irrotational case, the methods we used to investigate the asymptotic structure of a generalized solution corresponding to $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq 0$, no longer work if $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0$. The fundamental reason resides in the fact that we are not able to prove (under suitable assumptions on the body force) an analogue of Theorem XI.4.1, that would ensure that the velocity field $\boldsymbol{v} + \boldsymbol{v}_\infty$

is in a space $L^q(\Omega^R)$ for some $q < 6$. Actually, the existence of solutions that correspond to $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0$ and to data of arbitrary "size" and that are in L^q in a neighborhood of infinity for some $q \in (1, 6)$ remains an *open question*.

Notwithstanding this difficulty, we can still draw some interesting conclusions on the asymptotic structure of generalized solutions corresponding to $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0$ that, in addition, satisfies the energy inequality (XI.2.16). (As we know from the existence Theorem XI.3.1, this class of solutions is certainly not empty.) Specifically, following the work of Galdi & Kyed (2010), we shall show that provided a certain norm of the data is sufficiently small compared to $\mathcal{T}^{-1}$, every corresponding generalized solution $\boldsymbol{v}$ satisfying the energy inequality behaves for large $|x|$ like $|x|^{-1}$. Combining this result with those of Galdi (2003), one can also deduce $\nabla \boldsymbol{v}(x)$, $p(x) = O(|x|^{-2})$, and $\nabla p(x) = O(|x|^{-3})$.

However, an important feature that one is able to clarify when $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0$ and that, as we previously commented, is still obscure in the case $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq 0$ is the question of the leading terms in the asymptotic expansions of velocity and pressure fields, at least when $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv \boldsymbol{0}$, and $\mathcal{T}$ is sufficiently small.[1] In particular, as shown by Farwig, Galdi, & Kyed (2011), in such a case the leading term of the velocity field is the velocity field of a suitable representative of *Landau solutions*, whose class we have recalled in Section X.9; see Theorem XI.7.4.

In order to present all the above, we begin by proving the following preparatory result.

Lemma XI.7.1 *Let Ω be an exterior domain of class C^2, and suppose that the second-order tensor field $\boldsymbol{F}$, and the boundary data $\boldsymbol{v}_*$ satisfy the conditions*

$$\nabla \cdot \boldsymbol{F} \in L^2(\Omega)\,, \quad [\![\boldsymbol{F}]\!]_2 < \infty\,,^2 \qquad \boldsymbol{v}_* \in W^{3/2,2}(\partial\Omega)\,.$$

Then, there exists a constant $C = C(\Omega, q, B)$ if $\mathcal{T} \in (0, B]$ such that if

$$\|\nabla \cdot \boldsymbol{F}\|_2 + [\![\boldsymbol{F}]\!]_2 + \|\boldsymbol{v}_*\|_{3/2,2,\partial\Omega} \le \frac{1}{8C^2\mathcal{T}}\,,$$

there is at least one generalized solution $(\boldsymbol{v}, p)$ to the Navier–Stokes problem (XI.0.10)–(XI.0.11) *with*[3] $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0$ *and $\boldsymbol{f} \equiv \nabla \cdot \boldsymbol{F}$ such that*

$$\boldsymbol{u} \in W^{2,2}_{loc}(\overline{\Omega}) \cap D^{1,2}(\Omega) \cap D^{2,2}(\Omega)\,, \quad [\![\boldsymbol{u}]\!]_1 < \infty\,,$$

$$\widetilde{p} \in D^{1,2}(\Omega) \cap L^{q_1}(\Omega) \cap L^{q_2}(\Omega^\rho) \ \ \textit{for all } q_1 \in (3/2, 6] \textit{ and all } q_2 \in (6, \infty)\,,$$

where $\boldsymbol{u} := \boldsymbol{v} + \boldsymbol{v}_\infty$, $\widetilde{p}$ is defined in (XI.1.6)*, and ρ is an arbitrary number greater than $\delta(\Omega^c)$. Moreover, this solution satisfies the estimate*

$$|\boldsymbol{u}|_{2,2} + |\boldsymbol{u}|_{1,2} + [\![\boldsymbol{u}]\!]_{1,\mathcal{R}} + |\widetilde{p}|_{1,2} + \|\widetilde{p}\|_{q_1} + \|\widetilde{p}\|_{q_2,\Omega^\rho} \le C_1(\|\nabla \cdot \boldsymbol{F}\|_2 + [\![\boldsymbol{F}]\!]_2 + \|\boldsymbol{v}_* + \boldsymbol{v}_\infty\|_{3/2,2,\partial\Omega})\,, \tag{XI.7.1}$$

[1] This means that the "body" Ω^c, is rotating with "small" angular velocity.
[2] See the notation in (VIII.4.46).
[3] In this connection, see Remark VIII.5.1.

where $C_1 = C_1(\Omega, q_1, q_2, B)$.

Proof. The proof of this lemma will be obtained by a simple contraction argument based on the results of Theorem VIII.6.1. Set

$$X = X(\Omega) := \{\boldsymbol{w} \in D^{1,2}(\Omega) : [\![\boldsymbol{w}]\!]_1 < \infty\}.$$

Clearly, X becomes a Banach space endowed with the norm $\|\boldsymbol{w}\|_X := |\boldsymbol{w}|_{1,2} + [\![\boldsymbol{w}]\!]_1$. Let X_δ be the ball in X of radius δ centered at the origin, and consider the map

$$M : X_\delta \to \boldsymbol{u} = M(\boldsymbol{w}) \in X,$$

where $\boldsymbol{u}$ solves, for suitable π, the following generalized Oseen problem:

$$\left.\begin{aligned} \Delta \boldsymbol{u} + \mathcal{T}\,(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{u}) &= \mathcal{T}\boldsymbol{w} \cdot \nabla \boldsymbol{w} + \nabla \pi + \boldsymbol{f} \\ \nabla \cdot \boldsymbol{u} &= 0 \end{aligned}\right\} \text{ in } \Omega, \tag{XI.7.2}$$
$$\boldsymbol{u} = \boldsymbol{v}_* + \boldsymbol{v}_\infty \quad \text{at } \partial\Omega.$$

It is clear that if we show that the map M has a fixed point in X_δ, for some $\delta > 0$, then the existence of the desired solution $(\boldsymbol{v}, p)$ will be acquired by setting $\boldsymbol{v} := \boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{x}$ and $p := \pi - \frac{\mathcal{T}}{2}(x_2^2 + x_3^2)$. If we let $\boldsymbol{\mathcal{F}} := \mathcal{T}\boldsymbol{w} \otimes \boldsymbol{w} + \boldsymbol{F}$, by assumption and the fact that $\boldsymbol{w} \in X$, we obtain $\nabla \cdot \boldsymbol{\mathcal{F}} \in L^2(\Omega)$ with $[\![\boldsymbol{\mathcal{F}}]\!]_2 < \infty$. Thus, from Theorem VIII.6.1 there exists a solution $(\boldsymbol{u}, \pi)$ to (XI.7.2) such that

$$\begin{aligned} &\boldsymbol{u} \in W^{2,2}_{loc}(\overline{\Omega}) \cap D^{1,2}(\Omega) \cap D^{2,2}(\Omega), \quad [\![\boldsymbol{u}]\!]_1 < \infty \\ &\pi \in D^{1,2}(\Omega) \cap L^{q_1}(\Omega) \cap L^{q_2}(\Omega^\rho) \ \text{ for all } q_1 \in (3/2, 6], \text{ and all } q_2 \in (6, \infty), \end{aligned} \tag{XI.7.3}$$

where ρ is an arbitrary number greater than $\delta(\Omega^c)$. This shows in particular, that $\boldsymbol{u} \in X$, so that M is well defined. Furthermore, from (VIII.6.4) in the same theorem we find that $\boldsymbol{u}$ satisfies the inequality

$$\begin{aligned} |\boldsymbol{u}|_{2,2} + |\boldsymbol{u}|_{1,2} + [\![\boldsymbol{u}]\!]_1 + |\pi|_{1,2} + \|\pi\|_{q_1} + \|\pi\|_{q_2,\Omega^\rho} \\ \le C_1(\|\nabla \cdot \boldsymbol{\mathcal{F}}\|_2 + [\![\boldsymbol{\mathcal{F}}]\!]_2 + \|\boldsymbol{v}_* + \boldsymbol{v}_\infty\|_{3/2,2,\partial\Omega}), \end{aligned}$$

where $C_1 = C_1(\Omega, q_1, q_2, B)$ whenever $\mathcal{T} \in (0, B]$. Therefore, observing that

$$\begin{aligned} \|\nabla \cdot \boldsymbol{\mathcal{F}}\|_2 + [\![\boldsymbol{\mathcal{F}}]\!]_2 &\le \mathcal{T}\big(\|\boldsymbol{w}\|_\infty |\boldsymbol{w}|_{1,2} + [\![\boldsymbol{w}]\!]_1^2\big) + \|\nabla \cdot \boldsymbol{F}\|_2 + [\![\boldsymbol{F}]\!]_2 \\ &\le 2\mathcal{T}\|\boldsymbol{w}\|_X^2 + \|\nabla \cdot \boldsymbol{F}\|_2 + [\![\boldsymbol{F}]\!]_2, \end{aligned}$$

we infer that in particular, $\boldsymbol{u}$ satisfies the following estimate:

$$\|\boldsymbol{u}\|_X \le C\,(\mathcal{T}\,\|\boldsymbol{w}\|_X^2 + \|\nabla \cdot \boldsymbol{F}\|_2 + [\![\boldsymbol{F}]\!]_2 + \|\boldsymbol{v}_* + \boldsymbol{v}_\infty\|_{3/2,2,\partial\Omega}). \tag{XI.7.4}$$

Thus, if we choose

$$\delta = 2C\,(\|\nabla \cdot \boldsymbol{F}\|_2 + [\![\boldsymbol{F}]\!]_2 + \|\boldsymbol{v}_* + \boldsymbol{v}_\infty\|_{3/2,2,\partial\Omega})$$

and require

$$\|\nabla \cdot \boldsymbol{F}\|_2 + [\![\boldsymbol{F}]\!]_2 + \|\boldsymbol{v}_* + \boldsymbol{v}_\infty\|_{3/2,2,\partial\Omega} \le \frac{1}{8C^2\mathcal{T}}\,,$$

from (XI.7.4) it follows that M maps X_δ into itself. Furthermore, setting $\boldsymbol{u}_i = M(\boldsymbol{w}_i)$, $\boldsymbol{w}_i \in X_\delta$, $i = 1, 2$, after a simple calculation that uses again (VIII.6.4), we deduce

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_X \le C\,\mathcal{T}\,(\|\boldsymbol{w}_1\|_X + \|\boldsymbol{w}_2\|_X)\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_X\,. \tag{XI.7.5}$$

Consequently, since $\boldsymbol{w}_i \in X_\delta$, $i = 1, 2$, and $\delta \le 1/(4C\,\mathcal{T})$, from (XI.7.5) it follows that

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_X \le \frac{1}{2}\,\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_X\,,$$

which completes the proof that M is a contraction and, as a consequence, with the help of (XI.7.3), the proof of the lemma.

□

Remark XI.7.1 Sharper and more detailed asymptotic estimates than those stated in the previous lemma can be obtained if, in the contraction-mapping argument used in its proof, we use Theorem VIII.6.2 instead of Theorem VIII.6.1. More precisely, one can show that under the additional assumption

$$[\![D_iF_{ij}\boldsymbol{e}_j]\!]_3 + [\![D_jD_iF_{ij}]\!]_4 < \infty \tag{XI.7.6}$$

on $\boldsymbol{F} = \{F_{ij}\}$, the corresponding generalized solution constructed in Lemma XI.7.1 satisfies the further asymptotic properties

$$\nabla\boldsymbol{u} = O(|x|^{-2})\,,\quad \widetilde{p} = O(|x|^{-2})\,,\quad \nabla\widetilde{p} = O(|x|^{-2})\,,\quad \text{as } |x| \to \infty\,.$$

We leave the details of the proof to the interested reader. ■

From the previous result in combination with Theorem XI.2.3 we immediately obtain the following theorem, which furnishes the pointwise asymptotic behavior of a generalized solution satisfying the energy inequality, at least for small data.

Theorem XI.7.1 *Let Ω be as in* Lemma XI.7.1 *and let $\boldsymbol{v}$ be a generalized solution corresponding to $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0$ and to data*

$$\boldsymbol{f} \in L^2(\Omega) \cap L^{6/5}(\Omega), \quad \boldsymbol{v}_* \in W^{3/2,2}(\Omega)\,.$$

Suppose also that $\boldsymbol{f} = \nabla \cdot \boldsymbol{F}$, where $\boldsymbol{F}$ is a second-order tensor field in Ω with $[\![\boldsymbol{F}]\!]_2 < \infty$.[4] *Finally, assume that $\boldsymbol{v}$ and the associated pressure p satisfy the*

[4] In this connection, see Remark VIII.5.1.

energy inequality (XI.2.16), *and that* $\mathcal{T} \in (0, B]$ *for some* $B > 0$. *Then, there exists a positive constant* $c = c(\Omega, B)$ *such that if*

$$\|\nabla \cdot \boldsymbol{F}\|_2 + [\![\boldsymbol{F}]\!]_2 + \|\boldsymbol{v}_*\|_{3/2,2,\partial\Omega} \leq \frac{c}{\mathcal{T}}\,, \tag{XI.7.7}$$

the fields $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{v}_\infty$ *and* $\widetilde{p} = p + \frac{\mathcal{T}}{2}(x_2^2 + x_3^2)$ *satisfy all the properties stated in* Lemma XI.7.1. *In particular,*

$$\boldsymbol{v} + \boldsymbol{v}_\infty = O(|x|^{-1}) \ \text{ as } |x| \to \infty\,.$$

Proof. Let $c = \min\{1/(8C^2), 2M\}$, where C and M are defined in Lemma XI.7.1 and Theorem XI.2.3, respectively. Then, in view of (XI.7.7), by Lemma XI.7.1 there exists a corresponding generalized solution $\boldsymbol{v}_1$ that by (XI.7.1), satisfies in particular the estimate

$$(|x| + 1)|\boldsymbol{v}_1(x) + \boldsymbol{v}_\infty(x)| \leq \frac{c}{\mathcal{T}}\,.$$

Thus, from Theorem XI.2.3 we infer $\boldsymbol{v} \equiv \boldsymbol{v}_1$, and the desired result follows. □

Remark XI.7.2 If the tensor field $\boldsymbol{F}$ satisfies also condition (XI.7.6), then the generalized solution $\boldsymbol{v}$ of Theorem XI.7.1 and the associated pressure field p satisfy, in addition the asymptotic properties:

$$\nabla(\boldsymbol{v}(x) + \boldsymbol{v}_\infty(x)) = O(|x|^{-2})\,,$$
$$p + \tfrac{\mathcal{T}}{2}(x_2^2 + x_3^2) = O(|x|^{-2})\,, \quad \nabla\big(p + \tfrac{\mathcal{T}}{2}(x_2^2 + x_3^2)\big) = O(|x|^{-3})\,;$$

see Remark XI.7.1. ■

Theorem XI.7.1 along with Theorem XI.2.1 and Theorem XI.2.3, furnishes the following two results whose simple proof is left to the reader.

Theorem XI.7.2 *Let the assumptions of* Theorem XI.7.1 *be satisfied. Then* $\boldsymbol{v}$ *and the corresponding pressure* p *satisfy the energy equality* (XI.2.1)

Theorem XI.7.3 *Let the assumptions of* Theorem XI.7.1 *be satisfied. Then* $\boldsymbol{v}$ *is the only generalized solution corresponding to the given data.*

As a matter of fact, the pointwise estimates of Theorem XI.7.1, as well as those of Remark XI.7.1, can be further refined in the case that $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv \boldsymbol{0}$ and $\mathcal{T}$ is below a certain constant depending on Ω. Actually, as shown by Farwig, Galdi, & Kyed (2011), in such a case one is able to produce the leading terms of the asymptotic expansions of $\boldsymbol{v}$ and p. In order to describe this result, let Ω be of class C^2, and denote by $\boldsymbol{v}$ be the generalized solution constructed in Theorem XI.3.1 corresponding to the above data, and by p the associated pressure field (see Theorem XI.1.2). Further, let $(\boldsymbol{U}^b, P^b)$ be the Landau solution (X.9.21) corresponding to the parameter $\boldsymbol{b} = b\,\boldsymbol{e}_1$, where

$$b := \left(\int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{v}, \widetilde{p}) \cdot \boldsymbol{n} \right) \cdot \boldsymbol{e}_1 \,, \tag{XI.7.8}$$

and, we recall, $\widetilde{p}$ is defined in (XI.1.6), while $\boldsymbol{T}$ is the Cauchy stress tensor (IV.8.6).

The following theorem holds.

Theorem XI.7.4 *Let Ω be an exterior domain of class C^2 and let $\alpha \in (0,1)$. Then there exists $C = C(\alpha, \Omega) > 0$ such that if $\mathcal{T} \in (0, C]$, then any generalized solution $(\boldsymbol{v}, \widetilde{p})$ to* (XI.0.10)–(XI.0.11) *corresponding to $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} = 0$, $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv \boldsymbol{0}$ and satisfying the energy inequality* (XI.2.16) *satisfies the asymptotic expansion*

$$\begin{aligned} &\boldsymbol{v}(x) + \boldsymbol{v}_\infty(x) = \boldsymbol{U}^b(x) + O\left(\frac{1}{|x|^{1+\alpha}}\right) \quad \text{as } |x| \to \infty \,, \\ &\nabla\big(\boldsymbol{v}(x) + \boldsymbol{v}_\infty(x)\big) = \nabla \boldsymbol{U}^b(x) + O\left(\frac{1}{|x|^{2+\alpha}}\right) \quad \text{as } |x| \to \infty \,, \end{aligned} \tag{XI.7.9}$$

and

$$p(x) + \frac{\mathcal{T}}{2}(x_2^2 + x_3^2) = P^b(x) + \boldsymbol{m} \cdot \frac{\boldsymbol{x}}{4\pi |x|^3} + O\left(\frac{1}{|x|^{2+\alpha}}\right) \quad \text{as } |x| \to \infty \,,$$

where $(\boldsymbol{U}^b, P^b)$ is the Landau solution (X.9.21), *with $\boldsymbol{b} = b\,\boldsymbol{e}_1$ and b given in* (XI.7.8), *while*

$$\boldsymbol{m} := (\boldsymbol{I} - \boldsymbol{e}_1 \otimes \boldsymbol{e}_1) \cdot \int_{\partial\Omega} \big[\boldsymbol{T}(\boldsymbol{v}, \widetilde{p}) - (\boldsymbol{e}_1 \times \boldsymbol{x}) \otimes (\boldsymbol{e}_1 \times \boldsymbol{x}) \big] \cdot \boldsymbol{n} \,,$$

with $\boldsymbol{I}$ the identity matrix.

We shall not give a proof of this result, referring the interested reader to the quoted paper of Farwig, Galdi, & Kyed. We shall limit ourselves to observing that one of the key points in the proof is the fact that due to its symmetry properties, the Landau solution of Theorem XI.7.4 solves $(\text{XI.0.10})_{1,2}$ with $\mathcal{R} = 0$ and $\boldsymbol{f} \equiv \boldsymbol{0}$ at each $x \in \mathbb{R}^3 - \{\boldsymbol{0}\}$. This property was first recognized by Farwig & Hishida (2009).

XI.8 Notes for the Chapter

Section XI.1. A proof of Corollary XI.1.1 was first given by Galdi (2002, Theorem 4.6). However, his proof is more complicated and less direct than the one provided here. Furthermore, the assumptions on $\boldsymbol{f}$ are more stringent. In this connection, we would like to recall also the contribution of Silvestre (2004, Theorem 3.1). Unlike Corollary XI.1.1, valid for *any* generalized solution, this latter author proves the existence of *at least one* generalized solution satisfying the pointwise property (XI.1.16).

Section XI.2. All results in this section are due to me.

Section XI.3. A proof of existence of generalized solutions under the assumption that the boundary datum $\boldsymbol{v}_*$ (is sufficiently smooth and) has zero total flux through $\partial\Omega$ is due to Leray (1933, Chapter III). If $\boldsymbol{v}_*$ reduces to a rigid motion, namely, $\boldsymbol{v}_* = \boldsymbol{v}_1 + \boldsymbol{v}_2 \times \boldsymbol{x}$, $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^3$, a simpler proof is given in Borchers (1992, Korollar 4.1). However, a similar result and under the same boundary conditions can be easily deduced also from earlier work of Weinberger (1973) and Serre (1987).

The existence theory in its full generality, as presented in Theorem XI.3.1, is due to me.

Section XI.4. The main result, Theorem XI.4.1, is taken from Galdi & Kyed (2011a, Theorem 4.4). As already pointed out in Remark XI.4.3, we would like to emphasize one more time the interesting open question whether $\boldsymbol{v} + \boldsymbol{v}_\infty$ is square-summable in a neighborhood of infinity (that is, whether the liquid possesses a finite kinetic energy), in the case that $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv \boldsymbol{0}$ and $\boldsymbol{v}_0 \cdot \boldsymbol{\omega} \neq 0$.

Section XI.5. All results in this section are due to me. Similar results, under more stringent assumptions on the data, can be found in Galdi & Kyed (2010, 2011a).

Section XI.6. The main results presented in Theorem XI.6.1–Theorem XI.6.3 are due to Galdi & Kyed (2011a). However, the proof of Theorem XI.6.1 given here differs in some significant details from the analogous one furnished by the above authors.

The significant problem that is left *open* is the determination of leading terms (if any) in the asymptotic expansion of the velocity and pressure fields.

Another interesting problem that deserves attention is the asymptotic behavior of the vorticity field. It is very likely that outside the “wake region” the vorticity decays exponentially fast. Nevertheless, a proof of this property is far from being obvious.

Section XI.7. Theorem XI.7.1 is basically due to Galdi (2003) and Galdi & Kyed (2010).

The fact that a suitable Landau solution is the leading term in the asymptotic expansion of the velocity field was first discovered by Farwig & Hishida (2009). In fact, the main result of these authors, under assumptions slightly different from those of Theorem XI.7.4, consists in the proof of a representation for $\boldsymbol{v}$ similar to (XI.7.9), where, however, the quantity $\boldsymbol{v} + \boldsymbol{v}_\infty - \boldsymbol{U}_b$ is estimated in Lebesgue spaces rather than pointwise.

XII

Steady Navier–Stokes Flow in Two-Dimensional Exterior Domains

F.F. CHOPIN, Scherzo op.31, bars 1-3.

Introduction

In this chapter we shall study plane steady flow occurring in the complement of a compact region. Specifically, we shall investigate existence, uniqueness and asymptotic behavior of solutions $\boldsymbol{v}$, p to the Navier–Stokes system:[1]

$$\left.\begin{aligned}&\Delta \boldsymbol{v} = \mathcal{R}\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p + \mathcal{R}\boldsymbol{f}\\ &\nabla\cdot\boldsymbol{v} = 0\end{aligned}\right\}\ \text{in}\ \ \Omega \qquad\qquad \boldsymbol{v} = \boldsymbol{v}_* \ \ \text{at}\ \ \partial\Omega \tag{XII.0.1}$$

where Ω is an exterior domain of $\mathbb{R}^2$. As in the previous chapter, (XII.0.1) is in a nondimensional form, with $\mathcal{R}$ representing the Reynolds number. To system (XII.0.1) we must append the condition at infinity on the velocity field, that we choose to be of the form

[1] As we mentioned previously in several occasions, the steady-state, two-dimensional exterior problem in a rotating frame still lacks of significant results. For this reason, it will not be treated in this monograph.

G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems*, Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-09620-9_12,

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = -\boldsymbol{v}_\infty, \tag{XII.0.2}$$

with $\boldsymbol{v}_\infty$ a prescribed (constant) vector of $\mathbb{R}^2$. As already pointed out, a significant application of problem (XII.0.1)–(XII.0.2) occurs when $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv \boldsymbol{0}$ and $\boldsymbol{v}_\infty \neq \boldsymbol{0}$, and regards the steady-state motion of a viscous liquid around a long, straight cylinder $\mathcal{C}$ with axis a, assuming that the liquid is at rest at large distances from $\mathcal{C}$, and that $\mathcal{C}$ moves with (constant) translational velocity $\boldsymbol{v}_\infty$ orthogonal to a. Actually, in a region of flow sufficiently far from the two ends of $\mathcal{C}$ and including $\mathcal{C}$, one may expect that the velocity field of the liquid is independent of the coordinate parallel to a and, moreover, that there is no motion in the direction of a. Therefore, the relevant region of flow can be reasonably approximated by a two-dimensional domain placed in a suitable plane orthogonal to a and exterior to the cross-section of $\mathcal{C}$.

Using the methods of Section X.4 (cf. Remark X.4.4), we show that if Ω is locally Lipschitz with $\partial\Omega \neq \emptyset$, and if the flux of $\boldsymbol{v}_*$ through $\partial\Omega$ is sufficiently "small" in the sense of (X.4.47), then (XII.0.1) admits at least one solution $\boldsymbol{v}$, p, *without further restrictions on the size of the data*. Such a solution is also smooth if Ω and the data are likewise smooth. Concerning the behavior at large distances, the only information one can obtain is that $\boldsymbol{v}$ has a finite Dirichlet integral, that is,

$$\int_\Omega \nabla\boldsymbol{v} : \nabla\boldsymbol{v} \leq M, \tag{XII.0.3}$$

with $M = M(\Omega, \boldsymbol{f}, \boldsymbol{v}_*, \mathcal{R})$. However, (XII.0.3) alone is not enough to ensure that $\boldsymbol{v}$ tends (even in a generalized sense) to a constant vector to infinity, as is shown by simple counterexamples; see Section XII.1.[2] Therefore, we don't know if these solutions verify condition (XII.0.2). Furthermore, if $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv \boldsymbol{0}$, we do not know, in general, if this solution is *nontrivial.* [3] A separate consideration deserves the case $\Omega = \mathbb{R}^2$. Actually, in such a case, even the existence of solutions to the system (XII.0.1) is not known, in general.[4] Technically, this is due to the fact that a sequence of functions obeying the bound (XII.0.3) with $\Omega = \mathbb{R}^2$, need not be convergent in any of the spaces $L^q(B_R)$, $q > 1$, $R > 0$ (*cf.* Exercise II.7.3), and this in turn implies that one can not show the convergence of the nonlinear term along the sequence of approximating solutions; see Remark X.4.4.

The investigation of whether solutions to (XII.0.1) satisfying (XII.0.3) may also obey (XII.0.2) has attracted the attention of many writers; see Section XII.3. Specifically, Gilbarg & Weinberger (1974, 1978) have shown that if $\boldsymbol{f}$ is

[2] A sharp estimate at large distances of functions satisfying condition (XII.0.3) has been given in Theorem II.9.1.

[3] If Ω^c is symmetric around some direction, then one can construct nontrivial, suitably symmetric solutions; see the Notes for this Chapter.

[4] Existence can be established if the data satisfy suitable symmetry requirements; see the Notes for this Chapter.

of bounded support, every $\boldsymbol{v}$ satisfying (XII.0.1) for a suitable pressure p, and (XII.0.3), either converges at large distances, in a well-defined sense, to some constant vector $\boldsymbol{v}_0$, or that the L^2-norm of $\boldsymbol{v}$ over the unit circle approaches infinity at large distances. However, they can not show that $\boldsymbol{v}_0 = -\boldsymbol{v}_\infty$. If $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv 0$, Amick (1988) has proved that $\boldsymbol{v} \in L^\infty(\Omega)$. Nevertheless, also in such a case, one cannot infer that $\boldsymbol{v}_0 = -\boldsymbol{v}_\infty$ if $\boldsymbol{v}_\infty \neq 0$ and, what perhaps is more surprising, if $\boldsymbol{v}_0 = 0$, one is able to conclude that $\boldsymbol{v} \equiv 0$ only in the special situation where $\Omega = \mathbb{R}^2$.

At this point we may wonder if we could prove existence by means of different techniques, such as a fixed point argument. This problem, which was first considered by Finn & Smith (1967b), is in fact solvable when the velocity field $\boldsymbol{v}_\infty$ is not zero and *the data are sufficiently small*; see Section XII.5. It is interesting to observe that the corresponding solutions obey, in particular, condition (XII.0.3). It should also be emphasized that if $\boldsymbol{v}_\infty = 0$, the above techniques do not work and so it is not known whether (XII.0.1), (XII.0.2) with $\boldsymbol{v}_\infty = 0$ is resoluble, even for small data.[5] The reason for this unequal result is essentially due to the fact that the approach followed is based on fixed point arguments that rely on linearized versions of (XII.0.1), (XII.0.2). As we know from Chapters V and VII, the linearization when $\boldsymbol{v}_\infty \neq 0$ (Oseen system) produces solutions that, in the neighborhood of infinity, are more regular than those corresponding to the linearization when $\boldsymbol{v}_\infty = 0$ (Stokes system). In this respect, we recall that a similar circumstance occurs also for three-dimensional flows.

In view of all the above, it is natural to ask whether (XII.0.1)–(XII.0.2) may indeed admit a solution for data of arbitrary size. This question has been investigated by Galdi (1999b) for the physically relevant problem where $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv \boldsymbol{0}$, which, as we noted previously, describes the motion of the liquid past a sufficiently long cylinder. In such a case the data are represented by the translational velocity of the cylinder that, in dimensionless form, is given by $\mathcal{R}\boldsymbol{e}_1$ (assuming $\boldsymbol{v}_\infty \| \, \boldsymbol{e}_1$). Let us call $\mathscr{P}$ this particular problem. "Arbitrary data" for $\mathscr{P}$ means then "arbitrary speed" of the cylinder, namely, any $\mathcal{R} > 0$. Galdi has shown that if Ω possesses an axis of symmetry (e.g., Ω is the exterior of a circle) then, if there exists $\overline{\mathcal{R}} > 0$ such that for all $\mathcal{R} \geq \overline{\mathcal{R}}$ problem $\mathscr{P}$ has no solution in a very general regularity class, the *homogeneous problem* obtained by setting in (XII.0.1)–(XII.0.2) $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv \boldsymbol{v}_\infty \equiv \boldsymbol{0}$, and $\mathcal{R} = 1$ must admit a *non-zero* solution, a fact that is very questionable on physical grounds; see Section XII.6.

Another question that arises is the asymptotic structure of a field $\boldsymbol{v}$ satisfying (XII.0.1)–(XII.0.3), for suitable p. One may expect that $\boldsymbol{v}$ can be represented asymptotically by an expansion in "reasonable" functions of $r \equiv |x|$ with coefficients independent of r. However, if $\boldsymbol{v}_\infty = 0$, not every such solution can be represented in this way, for one can exhibit examples, due to Hamel, of solutions to (XII.0.1), (XII.0.2) with $\boldsymbol{v}_\infty = 0$ that obey (XII.0.3) and decay

[5] See, however, the Notes for this Chapter for some partial results.

more slowly than any negative power of r; see Section XII.6. If $\boldsymbol{v}_\infty \neq 0$, however, we have a result completely analogous to the three-dimensional case, namely, if $\boldsymbol{v}$ satisfies (XII.0.1)–(XII.0.3), for a suitable pressure p, and, in addition, $\boldsymbol{f}$ is (sufficiently smooth and) of bounded support, then $\boldsymbol{v}$ has the same asymptotic structure of the Oseen fundamental solution. The proof of this result, based on the papers of Galdi & Sohr (1995) and Sazonov (1999), is due to Sazonov; see Section XII.6 and Section XII.8.

In the last section of the chapter we shall study the behavior of solutions to (XII.0.1), (XII.0.2) in the limit of vanishing Reynolds number for the case $\boldsymbol{f} = 0$.[6] This problem is considered in the class $\mathfrak{C}$ of solutions whose existence has been determined in Section XII.5. We show that the solutions from $\mathfrak{C}$ tend (in an appropriate sense) to solutions of the Stokes problem obtained by formally taking $\mathcal{R} = 0$ into (XII.0.1), (XII.0.2). The interesting feature of this study is that, as expected from the linear theory (cf. Section VII.8), the limit process is singular in that it does not preserve the condition at infinity (XII.0.2). In fact, following the work of Galdi (1993), we show that the limit solution satisfies (XII.0.2) if and only if $\boldsymbol{v}_*$ verifies a suitable condition. In the case when Ω is the exterior of a unit circle, this condition reads:

$$\int_{\partial\Omega} (\boldsymbol{v}_* + \boldsymbol{v}_\infty) = 0.$$

Notation. Throughout this chapter Cartesian coordinates of a point x, are denoted either (as usual) by x_1, x_2, or (more simply) by x,y. When using the latter notation, the components of a vector $\boldsymbol{w}$ are written as w_x, w_y. We shall also frequently employ polar coordinates, r,θ. In such a case, the components of $\boldsymbol{w}$, are denoted, as customary, by w_r, w_θ.

XII.1 Generalized Solutions and D-Solutions

As already observed in Remark X.4.4, if Ω is a locally Lipschitz exterior domain in the plane with $\partial\Omega \neq \emptyset$, and if the flux of $\boldsymbol{v}_*$ through $\partial\Omega$ satisfies (X.4.47), we may use the same method employed in Section X.4 to construct, for all $\mathcal{R} > 0$, a vector field satisfying conditions (i)-(iii) and (v) of Definition IX.1.1, together with the energy inequality (IX.4.16). Moreover, we can associate to $\boldsymbol{v}$ a pressure field p (cf. Lemma X.1.1) and both $\boldsymbol{v}$ and p are smooth provided Ω and the data are smooth; cf. Theorem X.1.1. However, $\boldsymbol{v}$ and p solve in principle only problem (XII.0.1) because, unlike the three-dimensional case, the property (i), namely,

$$\int_\Omega \nabla\boldsymbol{v} : \nabla\boldsymbol{v} < \infty, \tag{XII.1.1}$$

is not enough a priori to control the behavior at infinity of $\boldsymbol{v}$. Thus, the limiting condition (XII.0.2) or even the weaker condition (iv) of Definition X.1.1 need

[6] This assumption is made for the sake of simplicity.

not be satisfied. That the problem of proving the convergence at infinity of a solution $\boldsymbol{v}$ under the sole information (XII.1.1) is far from being obvious, is shown by the fact that there are fields w in Ω having a finite Dirichlet integral and that become unbounded at large distances like a power of $\log|x|$. Take, for instance, $\Omega = \mathbb{R}^2 - \overline{B}(0)$ and $w(x) = (\log|x|)^\alpha$, $0 < \alpha < 1/2$. Therefore, in order to prove the convergence of a solution $\boldsymbol{v}$ at infinity, the equations of motion must play a fundamental role.

The situation becomes completely obscure in the case $\Omega = \mathbb{R}^2$, where the methods of Section X.4 do not even furnish existence of a vector field satisfying (XII.0.1) for a suitable pressure p; see Remark X.4.4.

In what follows, we will make a definite distinction between a generalized solution $\boldsymbol{v}$ to (XII.0.1), (XII.0.2) in the sense of Definition X.1.1, and a field $\boldsymbol{v}$ satisfying only (i)-(iii) and (v) of the same definition. In this latter case, the vector field $\boldsymbol{v}$ will be referred to as a *D-solution*, in spite of the fact that the word "solution" is not the most appropriate, since $\boldsymbol{v}$ need not verify the condition at infinity. A study on the asymptotic behavior of D-solutions is performed in Section XII.3, where it will be shown, among other things, that, as $|x| \to \infty$, $\boldsymbol{v}(x)$ tends in a suitable sense to some vector $\boldsymbol{v}_0$. Whether this vector coincides with the vector $-\boldsymbol{v}_\infty$ prescribed in (iv) of Definition X.1.1 remains, however, a fundamental open question.

Nevertheless, we may wonder if, by using a different technique, one can show existence, at least in the range of small data. As shown in Section XII.5, this is indeed possible, provided $\boldsymbol{v}_\infty$ is not zero. *If $\boldsymbol{v}_\infty$ is zero, the question of existence of generalized solutions, even with small data, is open* and, probably, for "generic" data, it does not admit a positive answer; see also the Notes for this Chapter.

XII.2 On the Uniqueness of Generalized Solutions

Maybe uniqueness of generalized solutions is a more complicated question than existence itself. Indeed it represents a formidable problem that, for its resolution, requires in my opinion the contribution of completely new ideas and methods. To test the difficulty, let $\boldsymbol{v}_1$, p_1 and $\boldsymbol{v}_2$, p_2 be two generalized solutions to problem (XII.0.1), (XII.0.2), which will be assumed smooth for simplicity. Letting

$$\boldsymbol{u} = \boldsymbol{v}_1 - \boldsymbol{v}_2 \quad \pi = (p_1 - p_2)/\mathcal{R}$$

we have

$$\left.\begin{aligned} &\frac{1}{\mathcal{R}}\Delta \boldsymbol{u} = \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{v}_1\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{v}_1 + \nabla\pi \\ &\nabla\cdot\boldsymbol{u} = 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\boldsymbol{u} = 0 \text{ at } \partial\Omega \qquad \text{(XII.2.1)}$$
$$\lim_{|x|\to\infty} \boldsymbol{u}(x) = 0.$$

Multiply $(\mathrm{XII.2.1})_1$ by $\boldsymbol{u}$, integrate by parts over Ω_R and use $(\mathrm{XII.2.1})_2$ to obtain

$$\mathcal{R}^{-1}\int_{\Omega_R}\nabla\boldsymbol{u}:\nabla\boldsymbol{u}=-\int_{\Omega_R}\boldsymbol{u}\cdot\nabla\boldsymbol{v}_1\cdot\boldsymbol{u}+\int_{\partial B_R}[\boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial n}-\frac{1}{2}u^2(\boldsymbol{u}+\boldsymbol{v}_1)\cdot\boldsymbol{n}-\pi\boldsymbol{u}\cdot\boldsymbol{n}]. \tag{XII.2.2}$$

Now assume that $\boldsymbol{u}$, $\boldsymbol{v}_1$, and π behave in such a way that the surface integral tends to zero as $R\to\infty$. (This is not known if $\boldsymbol{v}_\infty=0$, while, if $\boldsymbol{v}_\infty\neq 0$, it is true for any generalized solution which satisfies some further condition; cf. Section XII.8.) From (XII.2.2) in the limit $R\to\infty$ we find

$$\mathcal{R}^{-1}\int_{\Omega}\nabla\boldsymbol{u}:\nabla\boldsymbol{u}=-\int_{\Omega}\boldsymbol{u}\cdot\nabla\boldsymbol{v}_1\cdot\boldsymbol{u}. \tag{XII.2.3}$$

This is the classical relation, which we have used several times to show uniqueness of generalized solutions, sometimes with the equality sign replaced by the inequality one. However, for the case at hand, relation (XII.2.3) is not going to produce uniqueness for small $\mathcal{R}$. Actually, we would need an estimate of the type

$$-\int_{\Omega}\boldsymbol{u}\cdot\nabla\boldsymbol{v}_1\cdot\boldsymbol{u}\le c\int_{\Omega}\nabla\boldsymbol{u}:\nabla\boldsymbol{u} \tag{XII.2.4}$$

for some $c=c(\boldsymbol{v}_1,\Omega)\in(0,\infty)$, to conclude

$$\int_{\Omega}\nabla\boldsymbol{u}:\nabla\boldsymbol{u}=0,\quad\text{for }\mathcal{R}<c^{-1},$$

and hence uniqueness. However, the validity of (XII.2.4) is very unlikely,[1] no matter what summability assumptions on $\boldsymbol{u}$ and $\boldsymbol{v}_1$ are made and, consequently, the "traditional" method gives no information whatsoever about uniqueness, even at small Reynolds numbers.

Nevertheless, what is certainly true is that, if $\mathcal{R}$ is "sufficiently" large and the flux through the wall $\partial\Omega$ is not zero, uniqueness of generalized solutions is lost, as can be seen by means of simple examples. To show this, let us consider an indefinite, circular cylinder $\mathcal{C}$ of radius r_0 immersed in a viscous liquid and assume that the radial component of the velocity of the liquid, when evaluated at the lateral surface of $\mathcal{C}$, is a constant V. The appropriate Reynolds number is then defined by

$$\mathcal{R}=Vr_0/\nu.$$

Using V and r_0 as reference velocity and length, respectively, from (XII.0.1), (XII.0.2) we obtain that the two-dimensional motions of the liquid in every plane orthogonal to $\mathcal{C}$, when referred to polar coordinates (r,θ), must obey the following problem

[1] From (II.5.7) it follows that (XII.2.4) holds if $\nabla\boldsymbol{v}_1(x)=O(|x|^{-2}\log^{-2}|x|)$. However, even admitting that $\boldsymbol{v}_1$ at large distances has the same behavior as the solution of the corresponding linearized problem, we would have only $\nabla\boldsymbol{v}_1(x)=O(|x|^{-1})$; cf. Section V.3 and Section VII.6.

$$\left.\begin{aligned}&\mathcal{R}\left(v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta^2}{r}\right)=-\frac{\partial p}{\partial r}+\Delta v_r-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{v_r}{r^2}\\&\mathcal{R}\left(v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_\theta v_r}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial\theta}+\Delta v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2}\\&\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}=0\end{aligned}\right\}\text{ in }\Omega$$

$$\boldsymbol{v}(1,\theta)=\boldsymbol{v}_*$$

$$\lim_{r\to\infty}\boldsymbol{v}(r,\theta)=-\boldsymbol{v}_\infty \tag{XII.2.5}$$

where $\boldsymbol{v}=(v_r,v_\theta)$, Ω is the exterior of the unit circle, $\boldsymbol{v}_*(\theta)$ is prescribed and

$$\Delta=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Now, by a simple computation, we show that for any choice of γ, $\omega\in\mathbb{R}$ and $\mathcal{R}>1$, $\mathcal{R}\neq 2$, problem (XII.2.5) with

$$\boldsymbol{v}_\infty=0,\quad \boldsymbol{v}_*=(-\mathcal{R},\gamma-\omega/(\mathcal{R}-2)) \tag{XII.2.6}$$

admits the elementary solution [2]

$$\begin{aligned}v_r&=-\frac{\mathcal{R}}{r}\\v_\theta&=\frac{1}{r}\left[\gamma-\frac{\omega}{\mathcal{R}-2}r^{-\mathcal{R}+2}\right]\\p&=\mathcal{R}\int\left(\frac{1}{2}\frac{dv_r^2}{dr}-\frac{v_\theta^2}{r}\right).\end{aligned} \tag{XII.2.7}$$

The flow (XII.2.6), (XII.2.7) which was discovered by Hamel (1916, pp. 51-52) has a simple physical explanation; cf. Preston (1950). Specifically, it represents the motion of a liquid subject to a unit suction in the direction orthogonal to the wall of the circular cylinder, while the cylinder rotates with the angular velocity

$$\gamma-\frac{\omega}{\mathcal{R}-2}.$$

By the arbitrarity of $\mathcal{R}$, γ, and ω, we may choose

$$\omega=\gamma(\mathcal{R}-2) \tag{XII.2.8}$$

and the solution (XII.2.7) specializes to

[2] Notice that (XII.2.7) is not a solution of the corresponding linearized Stokes problem, that is, (XII.2.5) with $\mathcal{R}=0$, since $(\text{XII.2.5})_5$ is violated.

$$\begin{aligned} v_r &= -\frac{\mathcal{R}}{r} \\ v_\theta &= \frac{\omega}{\mathcal{R}-2}\frac{1}{r}\left[1 - r^{-\mathcal{R}+2}\right] \\ p &= \mathcal{R}\int \left(\frac{1}{2}\frac{dv_r^2}{dr} - \frac{v_\theta^2}{r}\right). \end{aligned} \tag{XII.2.9}$$

Moreover, the data are independent of γ, since with the choice (XII.2.8) and for $\mathcal{R} > 1$ and $\mathcal{R} \neq 2$, conditions (XII.2.6) become

$$\boldsymbol{v}_\infty = 0, \quad \boldsymbol{v}_* = (-\mathcal{R}, 0). \tag{XII.2.10}$$

As a consequence, we conclude that if $\mathcal{R} > 1$ and $\mathcal{R} \neq 2$ the fields (XII.2.9) constitute a one-parameter family of solutions to (XII.2.5), parameterized in ω, and corresponding to the same data (XII.2.10). Furthermore, since the gradients of the velocity fields square summable in Ω, they are generalized solutions.

The example just given suggests that, in the exterior two-dimensional problem, the prescription of the velocity field alone at the boundary and at infinity is not in general enough to secure the uniqueness of the solution, and other extra conditions may be needed. For instance, in the class (XII.2.9)–(XII.2.10) uniqueness is recovered if we prescribe at the boundary $r = 1$ the vorticity $\omega = r^{-1}\partial(rv_\theta)/\partial r$.

XII.3 On the Asymptotic Behavior of *D*-Solutions

The aim of this section is to investigate the behavior at large distances of D-solutions, namely, of vector fields $\boldsymbol{v}$ satisfying (XII.0.1) and condition (XII.1.1).

Since we are interested only in the regularity at infinity, we shall suppose that the D-solutions with which we are dealing are as smooth as required by the formal manipulation we shall perform. Of course, to substantiate this procedure, it is enough to assume that the data and the domain Ω have a sufficient degree of regularity, what degree will be tacitly understood. Moreover, since the results we shall find are essentially independent of the Reynolds number $\mathcal{R}$, we shall put, for simplicity, $\mathcal{R} = 1$.

We wish now to collect some notation and formulas that will be frequently used. The scalar field

$$\omega = \frac{\partial w_x}{\partial y} - \frac{\partial w_y}{\partial x}$$

is the *vorticity* of a vector field $\boldsymbol{w}$. From (XII.0.1), it follows that the vorticity of a D-solution $\boldsymbol{v}$ verifies the equation

$$\Delta\omega - \boldsymbol{v}\cdot\nabla\omega = \frac{\partial f_x}{\partial y} - \frac{\partial f_y}{\partial x}. \tag{XII.3.1}$$

Owing to the solenoidality of $\boldsymbol{v}$, we also have

$$\Delta v_x = \frac{\partial \omega}{\partial y}, \quad \Delta v_y = -\frac{\partial \omega}{\partial x}. \tag{XII.3.2}$$

The following result is based on ideas of Gilbarg & Weinberger (1978, Lemma 2.3).

Lemma XII.3.1 *Let* $\boldsymbol{v}$ *be a D-solution to* (XII.0.1) *with* $\boldsymbol{f} \in L^2(\Omega^\rho)$, *some* $\rho > \delta(\Omega^c)$. *Then,* $\nabla\omega \in L^2(\Omega^\rho)$ *and the following identity holds*

$$\int_\Omega |\nabla \omega|^2 = \frac{1}{2}\int_{\partial B_\rho}\left(\frac{\partial \omega^2}{\partial n} - \boldsymbol{v}\cdot\boldsymbol{n}\omega^2\right) + \int_{\Omega^\rho}\left(f_y\frac{\partial\omega}{\partial x} - f_x\frac{\partial\omega}{\partial y}\right).$$

Proof. Let $h \in C^1(\mathbb{R})$ be a positive function with piecewise differentiable first derivatives and let ψ_R be the Sobolev "cut-off" function (II.6.1) with $\exp\sqrt{\ln R} > \rho$. Using (XII.3.1), we show the following identity

$$\begin{aligned}\nabla\cdot[\psi_R\nabla h - h\nabla\psi_R - \psi_R h\boldsymbol{v}] &= \psi_R h''|\nabla\omega|^2 - h\left(\Delta\psi_R + \boldsymbol{v}\cdot\nabla\psi_R\right)\\ &\quad + h'\psi_R\left(\frac{\partial f_x}{\partial y} - \frac{\partial f_y}{\partial x}\right),\end{aligned} \tag{XII.3.3}$$

where prime means differentiation. Setting

$$\omega^* = \max_{x\in\partial B_\rho}|\omega(x)|,$$

we choose

$$h = \begin{cases}\omega^2 & \text{if } |\omega| \le \omega_0\\ \omega_0(2|\omega| - \omega_0) & \text{if } |\omega| \ge \omega_0\end{cases} \tag{XII.3.4}$$

for $\omega_0 \ge \omega^*$. From (XII.3.3) and (XII.3.4) we easily recover

$$\begin{aligned}\int_{\Omega^\rho} h''\psi_R|\nabla\omega|^2 &= \int_{\Omega^\rho} h(\Delta\psi_R + \boldsymbol{v}\cdot\nabla\psi_R) + \tfrac{1}{2}\int_{\partial B_\rho}\left(\frac{\partial\omega^2}{\partial n} - \boldsymbol{v}\cdot\boldsymbol{n}\omega^2\right)\\ &\quad + \textstyle\int_{\Omega^\rho}\left[f_y\left(\dfrac{\partial h'\psi_R}{\partial x}\right) - f_y\left(\dfrac{\partial h'\psi_R}{\partial x}\right)\right].\end{aligned} \tag{XII.3.5}$$

Taking into account that

$$h \le \min\left\{\omega^2, 2\omega_0|\omega|\right\}$$

and the properties (II.6.2), (II.6.5), and (II.6.6$_1$) of the function ψ_R, we find

$$\left|\int_{\Omega^\rho} h\Delta\psi_R\right| \le \frac{c}{R^2}\int_{\Omega^\rho}\omega^2 \le \frac{c}{R^2}|\boldsymbol{v}|_{1,2}^2$$

$$\left|\int_{\Omega^\rho} h\boldsymbol{v}\cdot\nabla\psi_R\right| \le 2\omega_0\int_{\Omega^\rho}|\omega||\boldsymbol{v}\cdot\nabla\psi_R| \le 2\omega_0\|\omega\|_{2,\Omega^\rho}\|\boldsymbol{v}\cdot\nabla\psi_R\|_{2,\Omega^\rho}$$
$$\le c_1\omega_0(\ln\ln R)^{-1/2}|\boldsymbol{v}|_{1,2}. \tag{XII.3.6}$$

Furthermore,

$$\left|\int_{\Omega^\rho}\psi_R\left(f_y\frac{\partial h'}{\partial x} - f_x\frac{\partial h'}{\partial y}\right)\right| \le \frac{1}{2}\int_{\Omega^\rho}|\psi_R h''|(|\nabla\omega|^2+|\boldsymbol{f}|^2) \tag{XII.3.7}$$

and so, collecting (XII.3.3) and (XII.3.5)–(XII.3.6) we obtain

$$\int_{\{\Omega^\rho;|\omega|\le\omega_0\}}\psi_R|\nabla\omega|^2 \le \frac{1}{2}\left|\int_{\partial B_\rho}\left(\frac{\partial\omega^2}{\partial n} - \boldsymbol{v}\cdot\boldsymbol{n}\omega^2\right)\right|$$
$$+ c_2|\boldsymbol{v}|_{1,2}[|v|_{1,2}R^{-2} + \omega_0(\ln\ln R)^{-1/2}]$$
$$+ c_3\left[\|\boldsymbol{f}\|_{2,\Omega^\rho}^2 + \int_{\Omega^\rho}|h'||\nabla\psi_R||\boldsymbol{f}|\right].$$

We then let $R\to\infty$ into this relation and use the monotone convergence criterion along with the assumption on $\boldsymbol{f}$ to deduce, for some positive constant c_4 independent of ω_0,

$$\int_{\{\Omega^\rho;|\omega|\le\omega_0\}}|\nabla\omega|^2 \le c_4$$

which, by the arbitrarity of ω_0, implies $\nabla\omega\in L^2(\Omega^\rho)$. Having established this, we come back to (XII.3.5) and let $R\to\infty$ and then $\omega_0\to\infty$, which completes the proof. □

An immediate consequence of the result just shown is the following theorem of the Liouville type.

Theorem XII.3.1 *Let $\boldsymbol{v}$ be a D-solution to* (XII.0.1) *in the whole of* $\mathbb{R}^2$ *corresponding to $\boldsymbol{f}\equiv 0$. Then $\boldsymbol{v}\equiv$ const.*

Proof. From Lemma XII.3.1 it follows that ω = const. and then, from (XII.3.2), $\Delta\boldsymbol{v}=0$ in $\mathbb{R}^2$. Since $\boldsymbol{v}\in D^{1,2}(\mathbb{R}^2)$, by Exercise II.11.11, we find $\boldsymbol{v}\equiv$ const. and the proof is complete. □

The next objective is to find a pointwise estimate on $\boldsymbol{v}$. To this end, we propose the following

Lemma XII.3.2 *Let $\boldsymbol{v}$ and $\boldsymbol{f}$ be as in* Lemma XII.3.1*. Then,*

$$\boldsymbol{v}\in D^{1,q}(\Omega^\rho)\quad \textit{for all } q\in[2,\infty).$$

Proof. By (XII.3.2) and by Lemma XII.3.1 we have

$$\Delta \boldsymbol{v} \in L^2(\Omega^\rho)$$

and so, from the scalar version of Theorem V.5.3 (cf. Remark V.5.3), we deduce

$$\boldsymbol{v} \in D^{2,2}(\Omega^\rho),$$

which implies

$$\nabla \boldsymbol{v} \in W^{1,2}(\Omega^\rho).$$

The result is then a consequence of this latter property and the embedding Theorem II.3.4.

□

From the result just shown we obtained the desired pointwise bound on $\boldsymbol{v}$. Specifically, we have the following lemma.

Lemma XII.3.3 *Let* $\boldsymbol{v}$ *and* $\boldsymbol{f}$ *be as in* Lemma XII.3.2. *Then*

$$\lim_{|x|\to\infty} \left(|\boldsymbol{v}(x)|/\sqrt{\log|x|}\right) = 0 \quad \textit{uniformly.}$$

Proof. It is an immediate consequence of Lemma XII.3.2 and Theorem II.9.1.

□

Concerning the summability of higher-order derivatives, we can prove the following result.

Lemma XII.3.4 *Let* $\boldsymbol{v}$ *be as in* Lemma XII.3.1. *If for some* $\rho > \delta(\Omega^c)$ *and* $m \geq 0$,

$$\boldsymbol{f} \in W^{m,2}(\Omega^\rho),$$

then we have

$$D^2\boldsymbol{v} \in W^{m,2}(\Omega^\rho).$$

Proof. The result is already known from Lemma XII.3.2 if $m = 0$. Let us begin to prove it for $m = 1$. We operate with D_k on both sides of (XII.3.1) to obtain

$$\Delta\omega_k = \boldsymbol{v}\cdot\nabla\omega_k + D_k\boldsymbol{v}\cdot\nabla\omega + F_k \tag{XII.3.8}$$

with

$$\omega_k = D_k\omega$$

$$F_k = D_k\left(\frac{\partial f_x}{\partial y} - \frac{\partial f_y}{\partial x}\right).$$

Multiplying both sides of (XII.3.8) by $\psi_R\omega_k$, with ψ_R chosen as in Lemma XII.3.1, and integrating by parts over Ω^ρ yields

$$\int_{\Omega^\rho}\psi_R|\nabla\omega_k|^2 = \frac{1}{2}\int_{\Omega^\rho}\left[\omega_k^2(\Delta\psi_R+\boldsymbol{v}\cdot\nabla\psi_R)+2\psi_R\omega_k D_k\boldsymbol{v}\cdot\nabla\omega\right] + \int_{\Omega^\rho}\psi_R\omega_k F_k + \mathsf{B}_\rho. \tag{XII.3.9}$$

In (XII.3.9), as in the remaining part of the proof, we denote by B_ρ the generic contribution of boundary integrals over ∂B_ρ, whose explicit value is not important to our purposes. By the properties of ψ_R and Lemma XII.3.3 we have

$$|\Delta\psi_R+\boldsymbol{v}\cdot\nabla\psi_R|\le c_1/R. \tag{XII.3.10}$$

We also have

$$\left|\int_{\Omega^\rho}\psi_R\omega_k F_k\right|\le\frac{1}{2}\int_{\Omega^\rho}\psi_R|\nabla\omega_k|^2+c_2\left(|\omega|_{1,2}^2+|\boldsymbol{f}|_{1,2}^2\right). \tag{XII.3.11}$$

Moreover, using $\nabla\cdot\boldsymbol{v}=0$, we find

$$\begin{aligned}\psi_R\omega_k D_k\boldsymbol{v}\cdot\nabla\omega &= \frac{1}{2}D_k[\psi_R D_k\boldsymbol{v}\cdot\nabla\omega^2]-\nabla\cdot[\frac{1}{2}\psi_R\omega^2 D_k^2\boldsymbol{v}]\\ &\quad+\omega^2\nabla\psi_R\cdot D_k^2\boldsymbol{v}-\frac{1}{2}D_k\psi_R D_k\boldsymbol{v}\cdot\nabla\omega^2\\ &\quad-\psi_R\omega D_k\boldsymbol{v}\cdot\nabla\omega_k\end{aligned}$$

and so we deduce that

$$\left|\int_{\Omega^\rho}\psi_R\omega_k D_k\boldsymbol{v}\cdot\nabla\omega\right|\le c_3|\boldsymbol{v}|_{2,2}|\boldsymbol{v}|_{1,4}^2+\frac{1}{4}\int_{\Omega^\rho}\psi_R|\nabla\omega_k|^2+|\boldsymbol{v}|_{1,4}^2+\mathsf{B}_\rho.$$

In view of Lemma XII.3.2, we may conclude, for a constant c_4 independent of R, that

$$\left|\int_{\Omega^\rho}\psi_R\omega_k D_k\boldsymbol{v}\cdot\nabla\omega\right|\le c_4+\frac{1}{4}\int_{\Omega^\rho}\psi_R|\nabla\omega_k|^2. \tag{XII.3.12}$$

Replacing (XII.3.10)–(XII.3.12) in (XII.3.9) furnishes

$$\int_{\Omega^\rho}\psi_R|\nabla\omega_k|^2\le c_5$$

with $c_5=c_5(\boldsymbol{v},\boldsymbol{f})$. Thus, letting $R\to\infty$ into this inequality, we deduce that

$$D^2\omega\in L^2(\Omega^\rho). \tag{XII.3.13}$$

Recalling (XII.3.2) and that $\nabla\omega\in L^2(\Omega^\rho)$, from (XII.3.13), Lemma XII.3.2, and Remark V.5.3 we infer that

$$D^2 \boldsymbol{v} \in W^{1,2}(\Omega^\rho), \tag{XII.3.14}$$

which proves the lemma in the case where $m = 1$. It is now easy to extend the result to arbitrary $m > 1$. Actually, by (XII.3.14), Theorem II.3.4, and Lemma XII.3.2 we have

$$\begin{gathered} D^2 \boldsymbol{v} \in L^q(\Omega^\rho) \quad \text{for all } q \in [2, \infty) \\ \nabla \boldsymbol{v} \in L^\infty(\Omega^\rho). \end{gathered} \tag{XII.3.15}$$

Differentiating (XII.3.8) one more time we find that

$$\begin{aligned} \Delta \omega_{sk} &= \boldsymbol{v} \cdot \nabla \omega_{sk} + D_s \boldsymbol{v} \cdot \nabla \omega_k + D_k \boldsymbol{v} \cdot \nabla \omega_s + D_s D_k \boldsymbol{v} \cdot \nabla \omega + F_{sk} \\ &\equiv \boldsymbol{v} \cdot \nabla \omega_{sk} + \mathcal{F}_{sk} + F_{sk} \end{aligned} \tag{XII.3.16}$$

with

$$\omega_{sk} = D_s \omega_k$$

$$F_{sk} = D_s F_k.$$

We multiply both sides of (XII.3.16) by $\psi_R \omega_{sk}$ and integrate over Ω^ρ. Then we treat the term

$$\int_{\Omega^\rho} \psi_R \omega_{sk} \boldsymbol{v} \cdot \nabla \omega_{ks}$$

as we did for the analogous term in the case where $m = 1$. On the other hand, making use of (XII.3.15) we can show directly (without integration by parts) that

$$\left| \int_{\Omega^\rho} \mathcal{F}_{sk} \omega_{sk} \right| \le c_6,$$

with $c_6 = c_6(\boldsymbol{v}, \boldsymbol{f})$. We then conclude that

$$\nabla \omega_{sk} \in L^2(\Omega^\rho);$$

reasoning as before we obtain

$$D^2 \boldsymbol{v} \in W^{2,2}(\Omega^\rho).$$

Then, by Theorem II.3.4, it follows that

$$\begin{gathered} D^3 \boldsymbol{v} \in L^q(\Omega^\rho) \quad \text{for all } q \in [2, \infty) \\ D^2 \boldsymbol{v} \in L^\infty(\Omega^\rho). \end{gathered}$$

Iterating such a procedure as many times as needed, we then complete the proof of the lemma. □

We are now in a position to show a first result on the pointwise convergence on the derivatives of $\boldsymbol{v}$ and p.

Theorem XII.3.2 *Let $\boldsymbol{v}$ be a D-solution to* (XII.0.1). *If for some $\rho > \delta(\Omega^c)$ and some $m \geq 1$*

$$\boldsymbol{f} \in W^{m,2}(\Omega^\rho),$$

then we have

(i) $\lim_{|x|\to\infty} |D^\alpha \boldsymbol{v}(x)| = 0$ *uniformly,* $1 \leq |\alpha| \leq m.$

If, in addition,

$$\boldsymbol{f} \in W^{m+1,q}(\Omega^\rho), \quad \textit{some} \quad q \in (1,2),$$

we also have

(ii) $\lim_{|x|\to\infty} |D^\alpha p(x)| = 0$ *uniformly,* $1 \leq |\alpha| \leq m.$

Proof. Let $\boldsymbol{v}_\alpha \equiv D^\alpha \boldsymbol{v}$. By assumption and Lemma XII.3.4, we have

$$\boldsymbol{v}_\alpha \in W^{2,2}(\Omega^\rho), \quad 1 \leq |\alpha| \leq m.$$

Thus, by the embedding Theorem II.3.4, it follows that

$$\boldsymbol{v}_\alpha \in W^{1,r}(\Omega^\rho), \quad \text{for all } r \in [2,\infty),$$

so that (i) becomes a consequence of Theorem II.9.1. To show (ii), we observe that, taking the divergence of both sides of $(\text{XII.0.1})_1$ we can deduce that

$$\Delta p = 2\left(\frac{\partial v_x}{\partial x}\frac{\partial v_y}{\partial y} - \frac{\partial v_x}{\partial y}\frac{\partial v_y}{\partial x}\right) + \nabla\cdot\boldsymbol{f}. \tag{XII.3.17}$$

By hypothesis and Lemma XII.3.2, it follows that

$$\Delta p \in L^q(\Omega^\rho), \tag{XII.3.18}$$

where q is specified in the statement of the theorem. Moreover, since by (XII.0.1)

$$\begin{aligned}\frac{\partial p}{\partial x} &= \frac{\partial \omega}{\partial y} - v_x\frac{\partial v_x}{\partial x} - v_y\frac{\partial v_x}{\partial y} + f_x\\ \frac{\partial p}{\partial y} &= -\frac{\partial \omega}{\partial x} - v_x\frac{\partial v_y}{\partial x} - v_y\frac{\partial v_y}{\partial y} + f_y\end{aligned} \tag{XII.3.19}$$

with the help of Lemma XII.3.1 and Lemma XII.3.3, we immediately obtain

$$\int_{\Omega^\rho} \frac{|\nabla p|^2}{\log|x|} < \infty \tag{XII.3.20}$$

and so, from (XII.3.18), (XII.3.20) we may assert (cf. Exercise XII.3.1)

$$D^2 p \in L^q(\Omega^\rho). \tag{XII.3.21}$$

Since $q < 2$, we apply Theorem II.6.1 to ∇p to deduce the existence of a constant p_1 such that

$$\begin{gathered} \nabla p - p_1 \in L^s(\Omega^\rho), \quad s = 2q/(2-q), \\ \lim_{|x|\to\infty} \int_{S_2} |\nabla p(x) - p_1| = 0. \end{gathered} \tag{XII.3.22}$$

However, because of (XII.3.20), we must have $p_1 = 0$, and (XII.3.22) implies

$$\nabla p \in L^s(\Omega^\rho), \quad s = 2q/(2-q). \tag{XII.3.23}$$

Operating with D^α, $1 \le |\alpha| \le m$, on both sides of (XII.3.17) and using the results of Lemma XII.3.2 and Lemma XII.3.4, we obtain

$$\Delta p \in W^{m,q}(\Omega^\rho).$$

This latter property together, with (XII.3.23) and Remark V.5.3, allows us to infer that

$$D^2 p \in W^{m,q}(\Omega^\rho). \tag{XII.3.24}$$

By (XII.3.24) and the embedding Theorem II.3.4 we recover (at least)

$$D^\alpha p \in L^s(\Omega^\rho) \cap D^{1,q}(\Omega^\rho), \quad 1 \le |\alpha| \le m, \tag{XII.3.25}$$

with s given in (XII.3.23). From (XII.3.23), (XII.3.25), and Theorem II.9.1 we then obtain the pointwise convergence (ii). The theorem is proved. □

Exercise XII.3.1 Let p be a smooth field satisfying (XII.3.18) and (XII.3.20) for some $q \in (1,\infty)$. Show that p obeys (XII.3.21). *Hint:* Let $\pi = \varphi p$ where φ is a "cut-off" function that is zero in Ω_ρ and one in $\Omega^{2\rho}$. Derive the equation for π in the whole of $\mathbb{R}^2$, and use the results of Exercise II.11.9 and Exercise II.11.11.

We shall now draw attention to the behavior at infinity of $\boldsymbol{v}$ and p. We begin to show that the pressure p has a pointwise limit at infinity. This will be achieved through a number of intermediate results due to Gilbarg & Weinberger (1978, §4), which we are now going to derive. For a given (vector or scalar) function f in Ω we denote by $\overline{f} = \overline{f}(r)$ its average over the unit circle, that is,

$$\overline{f} = \frac{1}{2\pi} \int_0^{2\pi} f(r,\theta) d\theta.$$

Lemma XII.3.5 *Let $\boldsymbol{v}$ be a D-solution to* (XII.0.1). *If for some $\rho > \delta(\Omega^c)$*

$$\frac{f_r}{r} \in L^1(\Omega^\rho),$$

then there is a $p_0 \in \mathbb{R}$ such that

$$\lim_{r\to\infty} \overline{p}(r) = p_0.$$

Proof. Multiplying $(\text{XII.3.19})_1$ by $\cos\theta$ and $(\text{XII.3.19})_2$ by $\sin\theta$, adding up, and using (0.1_2), we find

$$\frac{\partial p}{\partial r} = \frac{1}{r}\left(\frac{\partial \omega}{\partial \theta} + v_x \frac{\partial v_y}{\partial \theta} - v_y \frac{\partial v_x}{\partial \theta}\right) + f_r. \tag{XII.3.26}$$

Integrating this equation from 0 to 2π, dividing by 2π, and observing that

$$\int_0^{2\pi} \overline{v}_x \frac{\partial v_y}{\partial \theta} d\theta = \int_0^{2\pi} \overline{v}_y \frac{\partial v_x}{\partial \theta} d\theta = 0,$$

we deduce that

$$\frac{\partial \overline{p}}{\partial r} = \frac{1}{2\pi r}\int_0^{2\pi}\left[(v_x - \overline{v}_x)\frac{\partial v_y}{\partial \theta} - (v_y - \overline{v}_y)\frac{\partial v_x}{\partial \theta}\right] d\theta + \overline{f}_r$$

and so, integrating from r_1 to r_2, squaring, and applying the Schwarz inequality, it follows that

$$\begin{aligned} 2\pi^2|\overline{p}(r_2) - \overline{p}(r_1)|^2 \leq & \int_{r_1}^{r_2}\int_0^{2\pi} \frac{|\boldsymbol{v} - \overline{\boldsymbol{v}}|^2}{r} dr d\theta \int_{r_1}^{r_2}\int_0^{2\pi} \frac{1}{r}\left|\frac{\partial \boldsymbol{v}}{\partial \theta}\right|^2 dr d\theta \\ & + \left(\int_{\Omega_{r_1,r_2}} \left|\frac{f_r}{r}\right|\right)^2. \end{aligned} \tag{XII.3.27}$$

By the Wirtinger inequality (II.5.17) and Exercise II.5.12,

$$\int_0^{2\pi} |\boldsymbol{v} - \overline{\boldsymbol{v}}|^2 d\theta \leq \int_0^{2\pi} \left|\frac{\partial \boldsymbol{v}}{\partial \theta}\right|^2 d\theta, \tag{XII.3.28}$$

and since $|\partial \boldsymbol{v}/\partial\theta| \leq r|\nabla \boldsymbol{v}|$, from (XII.3.27) we recover

$$2\pi^2|\overline{p}(r_2) - \overline{p}(r_1)|^2 \leq |\boldsymbol{v}|^4_{1,2,\Omega_{r_1,r_2}} + \|f_r/r\|^2_{1,\Omega_{r_1,r_2}}.$$

The result is proved. □

Lemma XII.3.6 *Let* $\boldsymbol{v}$ *and* $\boldsymbol{f}$ *be as in* Lemma XII.3.5. *Then there exists a sequence* $\{R_k\}$ *such that*

$$R_k \in (2^{2^k}, 2^{2^{k+1}})$$

$$\lim_{k\to\infty}\int_0^{2\pi} |p(R_k,\theta) - \overline{p}(R_k)|^2 d\theta = 0.$$

Proof. From the Wirtinger inequality (XII.3.28) applied to p and the integral theorem of the mean, for each $k \in \mathbb{N}$ there is an R_k, as stated in the lemma, such that (with $\rho_k = 2^{2^k}$)

$$\begin{aligned}\log 2\int_0^{2\pi} |p(R_k,\theta)-\overline{p}(R_k)|^2 d\theta &= \int_{\rho_k}^{\rho_{k+1}}\int_0^{2\pi} \frac{|p(r,\theta)-\overline{p}(r)|^2}{r\log r} d\theta dr\\ &\le \int_{\rho_k}^{\rho_{k+1}}\int_0^{2\pi} \left|\frac{\partial p}{\partial\theta}\right|^2 (r\log r)^{-1} d\theta dr\\ &\le \int_{\Omega_{\rho_k,\rho_{k+1}}} \frac{|\nabla p|^2}{\log|x|}.\end{aligned}$$

The result then follows from this inequality and (XII.3.20).

□

In the next lemma we shall prove that p converges strongly to some $p_0 \in \mathbb{R}$ in $L^2(S^1)$. To this end, we need a particular representation of $p-\overline{p}$ in suitable regions. Let $\{R_k\}$ be the sequence determined in the previous lemma and let A_{km}be the annulus

$$A_{km} = \left\{x \in \mathbb{R}^2 : \; R_k < |x| < R_m\right\}, \quad k < m.$$

It is well known that Green's function (of the first kind) for the Laplace operator in the domain A_{km} is given by

$$\begin{aligned}G(r,\theta;\rho,\varphi) = &\sum_{s=1}^{\infty} \frac{(r^s - R_k^{2s}/r^s)(\rho^s - R_m^{2s}/\rho^s)}{2\pi s(R_m^{2s}-R_k^{2s})}\cos\left(s(\theta-\varphi)\right)\\ &- \frac{\log(r/R_m)\log(R_k/\rho)}{2\pi\log(R_k/R_m)} \equiv G_1 + G_2, \quad r<\rho,\end{aligned} \tag{XII.3.29}$$

and with r and ρ interchanged if $r > \rho$, cf. Weinberger (1965, p.140, Problem 2 with answer on p.417). (The notation $(r,\theta;\rho,\varphi)$ means the pair $(\boldsymbol{x};\boldsymbol{y})$ expressed in polar coordinates.) Setting

$$\Pi = \Pi(\rho,\theta) \equiv p(r,\theta) - \overline{p}(r),$$

and observing that for all $r \in \overline{A}_{km}$

$$\int_0^{2\pi} G_2(r)\Delta\Pi(r,\theta)d\theta = \int_0^{2\pi} \frac{\partial G_2(r)}{\partial r}\Pi(r,\theta)d\theta = 0,$$

from (III.1.33) we deduce the following representation for Π in A_{km}

$$\begin{aligned}\Pi(r,\theta) = &\int_{R_k}^{R_m} \rho\int_0^{2\pi} G_1(r,\theta;\rho,\varphi)\Delta\Pi(\rho,\varphi)d\rho d\varphi\\ &+ \int_0^{2\pi} \frac{\partial G_1(r,\theta;R_m,\varphi)}{\partial\rho}\Pi(R_m,\varphi)R_m d\varphi\\ &- \int_0^{2\pi} \frac{\partial G_1(r,\theta;R_k,\varphi)}{\partial\rho}\Pi(R_k,\varphi)R_k d\varphi.\end{aligned} \tag{XII.3.30}$$

Formula (XII.3.30) allows us to prove the following.

Lemma XII.3.7 *Let* $\boldsymbol{v}$, $\boldsymbol{f}$, *and* p_0 *be as in* Lemma XII.3.5. *Assume, further, that for some* $\rho > \delta(\Omega^c)$

$$\boldsymbol{f} \in L^2(\Omega^\rho), \quad \nabla \cdot \boldsymbol{f} \in L^1(\Omega^\rho).$$

Then

$$\lim_{r\to\infty} \int_0^{2\pi} |p(r,\theta) - p_0|^2 = 0.$$

Proof. Squaring both sides of (XII.3.30) and integrating over θ from 0 to 2π, we obtain

$$\frac{1}{3}\int_0^{2\pi} \Pi^2(r,\theta)d\theta \le I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_{R_k}^{R_m} d\rho_1 \int_{R_k}^{R_m} d\rho_2 \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \{\Delta\Pi(\rho_1,\varphi_1)\Delta\Pi(\rho_2,\varphi_2) \\ &\qquad \times [\int_0^{2\pi} G_1(r,\theta;\rho_1,\varphi_1)G_1(r,\theta;\rho_2,\varphi_2)d\theta]\} \\ I_2 &= 2\pi\int_0^{2\pi} d\theta \left\{\int_0^{2\pi} \left|\frac{\partial G_1(r,\theta;R_m,\varphi)}{\partial\rho}\right|^2 R_m^2 d\varphi\right\} \cdot \int_0^{2\pi} |\Pi(R_m,\varphi)|^2 d\varphi \\ I_3 &= 2\pi\int_0^{2\pi} d\theta \left\{\int_0^{2\pi} \left|\frac{\partial G_1(r,\theta;R_k,\varphi)}{\partial\rho}\right|^2 R_k^2 d\varphi\right\} \cdot \int_0^{2\pi} |\Pi(R_k,\varphi)|^2 d\varphi. \end{aligned} \tag{XII.3.31}$$

By using the following orthogonality conditions holding for all $s, s' \in \mathbb{N} - \{0\}$,

$$\begin{aligned} &\int_0^{2\pi} \cos(s\theta)\cos(s'\theta)d\theta = \int_0^{2\pi} \sin(s\theta)\sin(s'\theta)d\theta = \pi\delta_{ss'} \\ &\int_0^{2\pi} \sin(s\theta)\cos(s'\theta)d\theta = 0 \end{aligned}$$

from (XII.3.29) we find

$$\begin{aligned} &\int_0^{2\pi} G_1(r,\theta;\rho_1,\varphi_1)G_1(r,\theta;\rho_2,\varphi_2)d\theta \\ &= \sum_{s=1}^{\infty} \frac{(r^s - R_k^{2s}/r^s)^2(\rho_1^s - R_m^{2s}/\rho_1^s)(\rho_2^s - R_m^{2s}/\rho_2^s)}{4\pi s^2(R_m^{2s} - R_k^{3s})^2}\cos[s(\varphi_1 - \varphi_2)], \end{aligned} \tag{XII.3.32}$$

when $r < \rho_1$, $r < \rho_2$, and with similar expression otherwise. By a straightforward calculation one shows that the right-hand side of (XII.3.32) attains its maximum at $\rho_1 = \rho_2 = r = (R_m R_k)^{1/2}$ and $\varphi_1 = \varphi_2$; therefore

$$
\begin{aligned}
\left| \int_0^{2\pi} G_1(r,\theta;\rho_1,\varphi_1) G_1(r,\theta;\rho_2,\varphi_2) d\theta \right| &\le \sum_{s=1}^{\infty} \frac{(R_m^s - R_k^s)^2}{4\pi s^2 (R_m^s + R_k^s)^2} \\
&\le \frac{1}{4\pi} \sum_{s=1}^{\infty} s^{-2} \equiv c_1 .
\end{aligned}
$$

Inserting this information into $(\text{XII.3.31})_1$ yields

$$
I_1 \le c_1 \left(\int_{A_{km}} |\Delta \Pi| \right)^2 . \tag{XII.3.33}
$$

In addition, since

$$
\left. \frac{\partial G_1}{\partial \rho} \right|_{\rho = R_m} = \sum_{s=1}^{\infty} \frac{(r^s - R_k^{2s}/r^s) R_m^{s-1}}{\pi (R_m^{2s} - R_k^{2s})} \cos[s(\theta - \varphi)],
$$

recalling that $R_k \in (2^{2^k}, 2^{2^{k+1}})$ it follows that, for $R_k < r \le R_{m-2}$,

$$
\begin{aligned}
\int_0^{2\pi} \left| \frac{\partial G_1(r,\theta;R_m,\varphi)}{\partial \rho} \right|^2 R_m^2 d\varphi &= \sum_{s=1}^{\infty} \frac{(r^s - R_k^{2s}/r^s)^2 R_m^{2s}}{\pi (R_m^{2s} - R_k^{2s})^2} \\
&\le \sum_{s=1}^{\infty} \frac{(R_{m-2}/R_m)^{2s}}{\pi [1 - (R_k/R_m)^{2s}]^2} \\
&\le \frac{\pi \sum_{s=1}^{\infty} 2^{-2^m s}}{\pi [1 - (R_k/R_m)^2]^2} \le c_2 .
\end{aligned} \tag{XII.3.34}
$$

Likewise, we show that if $R_{k+2} \le r < R_m$,

$$
\int_0^{2\pi} \left| \frac{\partial G_1(r,\theta;R_k,\varphi)}{\partial \rho} \right|^2 R_k^2 d\varphi \le c_3 . \tag{XII.3.35}
$$

Using (XII.3.34) and (XII.3.35) in $(\text{XII.3.31})_2$ and $(\text{XII.3.31})_3$, respectively, and taking into account (XII.3.33), we conclude for all $r \in [R_{k+2}, R_{m-2}]$, $m \ge k+5$,

$$
\begin{aligned}
\frac{1}{3} \int_0^{2\pi} \Pi^2(r,\theta) d\theta \le c_1 \left(\int_{A_{km}} |\Delta \Pi| \right)^2 &+ 2\pi c_2 \int_0^{2\pi} \Pi^2(R_m,\theta) d\theta \\
&+ 2\pi c_3 \textstyle\int_0^{\pi} \Pi^2(R_k,\theta) d\theta .
\end{aligned} \tag{XII.3.36}
$$

Now, by (XII.3.17) and assumption,

$$
\Delta \Pi \in L^1(\Omega^\rho), \tag{XII.3.37}
$$

and, therefore, the lemma follows from (XII.3.37) and Lemma XII.3.5 and Lemma XII.3.6 by letting m and k to infinity in (XII.3.36). □

We are now in a position to establish the pointwise convergence of the pressure at infinity.

Theorem XII.3.3 *Let $\boldsymbol{v}$ be a D-solution to* (XII.0.1). *Then, if for some $\rho > \delta(\Omega^c)$*

$$\frac{f_r}{r} \in L^1(\Omega^\rho), \quad \boldsymbol{f} \in L^2(\Omega^\rho), \quad \nabla\cdot\boldsymbol{f} \in L^1(\Omega^\rho),$$

there is a $p_0 \in \mathbb{R}$ such that

$$\lim_{|x|\to\infty} p(x) = p_0 \quad \textit{uniformly.}$$

Proof. We indicate by $p_1(x)$ the difference $p(x) - p_0$ where p_0 is defined in Lemma XII.3.5. Clearly, $\boldsymbol{v}$, p_1 is still a solution to (XII.0.1) and, in particular, p_1 verifies (XII.3.26). Let $x = (2R, \theta)$. Denoting by (r', θ') a polar coordinate system with the origin at x, from (XII.3.26) we have, after integration over r' and θ',

$$\begin{aligned} p_1(x) = \overline{p}_1(r') &+ \frac{1}{2\pi}\int_0^{r'} \overline{f}_r(\rho)d\rho \\ &+\frac{1}{2\pi}\int_0^{r'}\int_0^{2\pi}\frac{1}{\rho}\left\{[v_x(\rho,\theta') - \overline{v}_x(\rho)]\frac{\partial v_y(\rho,\theta')}{\partial\theta'}\right. \\ &\left.-[v_y(\rho,\theta') - \overline{v}_y(\rho)]\frac{\partial v_x(\rho,\theta')}{\partial\theta'}\right\} d\rho d\theta', \end{aligned} \tag{XII.3.38}$$

where the average is now meant with respect to the angle θ'. From the Wirtinger inequality (XII.3.28) and the Schwarz inequality, it follows for $r' \le R$ that

$$\begin{aligned} &\left|\int_0^{r'}\frac{1}{\rho}\left\{\int_0^{2\pi}[v_x(\rho,\theta') - \overline{v}_x(\rho)]\frac{\partial v_y(\rho,\theta')}{\partial\theta'}d\theta'\right\}d\rho\right| \\ &\qquad\qquad \le \int_0^R\int_0^{2\pi}|\nabla\boldsymbol{v}(\rho,\theta')|^2\rho d\rho d\theta' \le |\boldsymbol{v}|^2_{1,2,\Omega_{R,3R}}\,. \end{aligned} \tag{XII.3.39}$$

Likewise, one shows

$$\left|\int_0^{r'}\frac{1}{\rho}\left\{\int_0^{2\pi}[v_y(\rho,\theta') - \overline{v}_y(\rho)]\frac{\partial v_x(\rho,\theta')}{\partial\theta'}d\theta'\right\}d\rho\right| \le |\boldsymbol{v}|^2_{1,2,\Omega_{R,3R}}. \tag{XII.3.40}$$

Moreover,

$$\left|\int_0^{r'}\overline{f}_r(\rho)d\rho\right| \le \|f_r/r\|_{1,\Omega_{R,3R}} \tag{XII.3.41}$$

and so, by (XII.3.38)–(XII.3.41), we recover

$$2\pi|p_1(x)| \le 2\pi|\overline{p}_1(r')| + 2|\boldsymbol{v}|^2_{1,2,\Omega_{R,3R}} + \|f_r/r\|_{1,\Omega_{R,3R}}.$$

Multiplying both sides of this latter inequality by r' and integrating over $r' \in [0, R]$ and $\theta' \in [0, 2\pi)$ we find that

$$\begin{aligned}
2\pi|p_1(x)| &\le \frac{4\pi}{R^2}\int_0^R \int_0^{2\pi} |p_1(r',\theta')r'dr'd\theta' + 2|\boldsymbol{v}|^2_{1,2,\Omega_{R,3R}} + \|f_r/r\|_{1,\Omega_{R,3R}} \\
&\le \frac{4\pi}{R^2}\int_{\Omega_{R,3R}} |p_1| + 2|\boldsymbol{v}|^2_{1,2,\Omega_{R,3R}} + \|f_r/r\|_{1,\Omega_{R,3R}} \\
&\le 16\pi \max_{R\le r\le 3R}\int_0^{2\pi} |p_1(r,\theta)|d\theta + 2|\boldsymbol{v}|^2_{1,2,\Omega_{R,3R}} + \|f_r/r\|_{1,\Omega_{R,3R}}.
\end{aligned}$$

Passing to the limit $R \to \infty$ in this estimate and using Lemma XII.3.7, we then complete the proof of the theorem.

□

It remains to investigate the behavior at infinity of the velocity field $\boldsymbol{v}$. In this respect, Lemma XII.3.3 ensures that $\boldsymbol{v}$ cannot grow too fast at infinity. Under the simplifying assumption $\boldsymbol{f} = 0$ (or, more generally, $\boldsymbol{f}$ of bounded support in Ω) we shall show that, in fact, $\boldsymbol{v}$ is uniformly bounded. To reach the objective, we notice that, defining the *total head pressure field* Φ:

$$\Phi := p + \frac{1}{2}|\boldsymbol{v}|^2, \tag{XII.3.42}$$

by a simple calculation based on (XII.0.1) with $\boldsymbol{f} = 0$ one shows

$$\Delta\Phi - \boldsymbol{v}\cdot\nabla\Phi = \omega^2. \tag{XII.3.43}$$

Consider (XII.3.43) in Ω_{ρ_1,ρ_2}, for arbitrary ρ_1, ρ_2 with $\rho_2 > \rho_1 > \rho_0$ and some fixed $\rho_0 > \delta(\Omega^c)$. Since $\boldsymbol{v} \in L^\infty_{loc}(\Omega^\rho)$, we may apply to it the classical maximum principle of Hopf (1952) and Oleĭnik (1952) (cf. Protter & Weinberger 1967, pp. 61-64) to obtain that Φ cannot attain a maximum in Ω_{ρ_1,ρ_2} unless it is a constant. It also follows that

$$\max_{\theta\in[0,2\pi)} \Phi(r,\theta)$$

has no maximum in Ω_{ρ_1,ρ_2} and hence it must be monotonous in Ω^ρ, for sufficiently large ρ. Thus, we deduce that

$$\lim_{r\to\infty} \max_{\theta\in[0,2\pi)} \left[p(r,\theta) + \frac{1}{2}|\boldsymbol{v}(r,\theta)|^2\right] = A,$$

implying, by Theorem XII.3.3,[1] that

[1] We can assume, without loss, that $p_0 = 0$.

$$\lim_{r\to\infty} \max_{\theta\in[0,2\pi)} |\boldsymbol{v}(r,\theta)| = \sqrt{2A} \equiv L. \tag{XII.3.44}$$

However, we don't know if L is finite or infinite. As a consequence, the maximum principle is not enough to obtain the boundedness of $\boldsymbol{v}$, and we need more information about the function Φ. In the particular case $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv 0$ this is achieved through the following profound result, due to C. J. Amick (1986, Theorem 4(a); 1988, Theorem 11), which we shall state without proof.

Lemma XII.3.8 *Let $\boldsymbol{v}$ be a D-solution to* (XII.0.1) *corresponding to $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv 0$. Then there exists a continuous, non-intersecting curve*

$$\gamma : t \in [0,1) \to \gamma(t) \in \overline{\Omega}^\rho$$

such that

(i) $\gamma(0) \in \partial\Omega^\rho$;
(ii) $|\gamma(t)| \to \infty$ *as* $t \to 1$;

in addition, the function Φ is monotonically decreasing along γ, namely,

$$\Phi(\gamma(t)) < \Phi(\gamma(s)), \quad \text{for all } s,t \in [0,1), \;\; s < t. \tag{XII.3.45}$$

With this result in hand, we can show the following one (cf. Amick 1986, Theorem 4(b) and 1988, Theorem 12).

Lemma XII.3.9 *Let $\boldsymbol{v}$ and $\boldsymbol{f}$ be as in* Lemma XII.3.8. *Then*

$$\boldsymbol{v} \in L^\infty(\Omega^\rho),$$

and there is an $L \in [0,\infty)$ such that

$$\lim_{|x|\to\infty} \max_{\theta\in[0,2\pi)} |\boldsymbol{v}(x)| = L \quad \text{uniformly.} \tag{XII.3.46}$$

Proof. Since $p(x)$ tends to zero for large $|x|$, by (XII.3.45) we deduce that

$$|\boldsymbol{v}(\gamma(t))| \le c, \quad \text{for all } t \in [0,1), \tag{XII.3.47}$$

with c independent of t. Since $\boldsymbol{v} \in D^{1,2}(\Omega)$, we have

$$\int_{2^k}^{2^{k+1}} \int_0^{2\pi} \frac{1}{r} \left| \frac{\partial \boldsymbol{v}}{\partial \theta} \right|^2 d\theta dr \to 0, \quad \text{as } k \to \infty,$$

implying

$$\int_0^{2\pi} \left| \frac{\partial \boldsymbol{v}(r_k,\theta)}{\partial \theta} \right|^2 d\theta \to 0, \quad \text{as } k \to \infty, \tag{XII.3.48}$$

for some sequence $\{r_k\}$ with $r_k \in (2^k, 2^{k+1})$. Since, by properties (iii) and (ii), γ is connected and extends to infinity, for any $k \in \mathbb{N}$ we can find at least one $t_k \in [0, 1)$ such that

$$\gamma(t_k) = (r_k, \theta_k),$$

for some $\theta_k \in [0, 2\pi)$. Thus, in view of (XII.3.47), it follows that

$$|\boldsymbol{v}(r_k, \theta_k)| \leq c, \quad \text{for all } k \in \mathbb{N}. \tag{XII.3.49}$$

From the identity

$$\boldsymbol{v}(r_k, \theta) = \boldsymbol{v}(r_k, \theta_k) - \int_\theta^{\theta_k} \frac{\partial \boldsymbol{v}(r_k, \tau)}{\partial \tau} d\tau,$$

(XII.3.48), and (XII.3.49) we find

$$\max_{x \in \partial B_{r_k}} |\boldsymbol{v}(x)| \leq c_1, \quad \text{for all } k \in \mathbb{N}, \tag{XII.3.50}$$

with c_1 independent of k. We next apply the maximum principle to (XII.3.43) in the annulus $\Omega_{r_k, r_{k+1}}$ to find

$$\max_{x \in \Omega_{r_k, r_{k+1}}} \Phi(x) \equiv \max_{x \in \Omega_{r_k, r_{k+1}}} \left\{ p(x) + \frac{1}{2}|\boldsymbol{v}(x)|^2 \right\} \leq \max_{x \in \partial B_{r_k} \cup \partial B_{r_{k+1}}} \Phi(x). \tag{XII.3.51}$$

However, by (XII.3.50) and Theorem XII.3.3 we deduce

$$\max_{x \in \partial B_{r_k} \cup \partial B_{r_{k+1}}} \Phi(x) \leq c_2, \quad \text{for all } k \in \mathbb{N}$$

so that, again Theorem XII.3.3 and (XII.3.51) deliver

$$\max_{x \in \Omega_{r_k, r_{k+1}}} |\boldsymbol{v}(x)|^2 \leq c_3, \quad \text{for all } k \in \mathbb{N},$$

with a constant c_3 independent of k. Therefore, $\boldsymbol{v} \in L^\infty(\Omega^\rho)$. Since the second part of the lemma is an immediate consequence of the first and (XII.3.44), the proof is complete. $\square$

We shall next investigate if the velocity approaches some vector $\boldsymbol{v}_0$ at infinity. We have the following two possibilities:

(i) the number L in (XII.3.44) is zero;
(ii) the number L in (XII.3.44) belongs to $(0, \infty]$.[2]

In case (i) we have

$$\lim_{|x| \to \infty} \boldsymbol{v}(x) = 0, \quad \text{uniformly},$$

and we deduce at once that $\boldsymbol{v}_0 = 0$. On the other hand, if $L > 0$, using the ideas of Gilbarg & Weinberger (1978, §5), we proceed as follows. First of all, we need two preliminary lemmas.

[2] Of course, by Lemma XII.3.9, if $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv 0$ it follows that $L < \infty$.

Lemma XII.3.10 *Let $\boldsymbol{v}$ and $\boldsymbol{f}$ be as in* Lemma XII.3.8. *Then*

(i) $\displaystyle\lim_{r\to\infty}\int_0^{2\pi}|\boldsymbol{v}(r,\theta)-\overline{\boldsymbol{v}}(r)|^2 d\theta = 0,$

(ii) $\displaystyle\lim_{r\to\infty}|\overline{\boldsymbol{v}}(r)| = L,$

where L is defined in (XII.3.44).

Proof. By the Wirtinger inequality (XII.3.28) and the Cauchy inequality we have

$$\begin{aligned}\left|\frac{d}{dr}\int_0^{2\pi}|\boldsymbol{v}(r,\theta)-\overline{\boldsymbol{v}}(r)|^2 d\theta\right| &= \left|2\int_0^{2\pi}(\boldsymbol{v}-\overline{\boldsymbol{v}})\cdot\frac{\partial\boldsymbol{v}}{\partial r}d\theta\right| \\ &\le \int_0^{2\pi}\left[r|\nabla\boldsymbol{v}|^2+\frac{|\boldsymbol{v}-\overline{\boldsymbol{v}}|^2}{r^2}r\right]d\theta \\ &\le c\int_0^{2\pi} r|\nabla\boldsymbol{v}|^2 d\theta;\end{aligned}$$

therefore,

$$\lim_{r\to\infty}\int_0^{2\pi}|\boldsymbol{v}(r,\theta)-\overline{\boldsymbol{v}}(r)|^2 = \ell\in[0,\infty).$$

However, again by (XII.3.28), we have

$$\int_\rho^\infty\frac{1}{r}\left(\int_0^{2\pi}|\boldsymbol{v}(r,\theta)-\overline{\boldsymbol{v}}(r)|^2 d\theta\right)dr<\infty,$$

which implies $\ell = 0$, and (i) is proved. To show (ii), we observe that (XII.3.46) implies that, given any sequence $\{r_k\}\subset\mathbb{R}_+$ with $r_k\to\infty$, there is a corresponding sequence $\{\theta_k\}\subset[0,2\pi)$ such that

$$\lim_{r_k\to\infty}|\boldsymbol{v}(r_k,\theta_k)| = L. \tag{XII.3.52}$$

However, as in the proof of Lemma XII.3.8, we show the existence of a sequence $r_k\in(2^k,2^{k+1})$ such that (XII.3.48) holds. Since

$$|\boldsymbol{v}(r_k,\theta)-\boldsymbol{v}(r_k,\theta_k)|^2\le 2\pi\int_0^{2\pi}\left|\frac{\partial\boldsymbol{v}(r_k,\tau)}{\partial\tau}\right|^2 d\tau,$$

by (XII.3.52) and the triangle inequality we find that

$$\lim_{r_k\to\infty}|\boldsymbol{v}(r_k,\theta)| = L,\quad\text{uniformly in }\theta. \tag{XII.3.53}$$

Moreover, since

$$\int_0^{2\pi}(\boldsymbol{v}(r_k,\theta)-\overline{\boldsymbol{v}}(r_k))d\theta = 0\ \text{ for all } k\in\mathbb{N},$$

it follows that

$$|\boldsymbol{v}(r_k,\theta)-\overline{\boldsymbol{v}}(r_k)|^2 \le 2\pi \int_0^{2\pi} \left|\frac{\partial \boldsymbol{v}(r_k,\tau)}{\partial \tau}\right|^2 d\tau$$

which together with (XII.3.48) and (XII.3.53) allows us to conclude that

$$\lim_{k\to\infty} |\overline{\boldsymbol{v}}(r_k)| = L. \tag{XII.3.54}$$

Now, for $r \in (r_k, r_{k+1})$ we have

$$\begin{aligned}|\overline{\boldsymbol{v}}(r)-\overline{\boldsymbol{v}}(r_k)|^2 &= \left|\frac{1}{2\pi}\int_{r_k}^{r}\int_0^{2\pi}\frac{\partial \boldsymbol{v}}{\partial r}dr d\theta\right|^2 \\ &\le \frac{1}{(2\pi)^2}\int_{r_k}^{r_{k+1}}\int_0^{2\pi}\frac{1}{r}dr d\theta \int_{\Omega^{r_k}}|\nabla \boldsymbol{v}|^2 \le \frac{1}{2\pi}\log 2 |\boldsymbol{v}|^2_{1,2,\Omega^{r_k}},\end{aligned}$$

and so, in view of (XII.3.54), the property (ii) follows by letting $k\to\infty$ in this inequality.

□

Lemma XII.3.11 *Let* $\boldsymbol{v}$ *be as in* Lemma XII.3.2 *and assume that the number* L *in* (XII.3.44) *is finite. Then, if for some* $\rho > \delta(\Omega^c)$

$$r^{1/2}\boldsymbol{f} \in L^2(\Omega^\rho),$$

it follows that

$$r^{1/2}\nabla\omega \in L^2(\Omega^\rho).$$

Proof. From the identity (XII.3.5) with $h \equiv \omega^2$ and ψ_R replaced by $\eta_R = r\psi_R$, we deduce that

$$\begin{aligned}\int_{\Omega^\rho}\eta_R|\nabla\omega|^2 &= \frac{1}{2}\int_{\Omega^\rho}\omega^2(\Delta\eta_R+\boldsymbol{v}\cdot\nabla\eta_R)+\frac{1}{2}\int_{\partial B_\rho}\left(\frac{\partial\omega^2}{\partial n}-\boldsymbol{v}\cdot\boldsymbol{n}\omega^2\right) \\ &\quad +\int_{\Omega^\rho}\left[f_y\left(\frac{\partial\omega\eta_R}{\partial x}\right)-f_x\left(\frac{\partial\omega\eta_R}{\partial y}\right)\right].\end{aligned} \tag{XII.3.55}$$

By the properties (II.7.2) of ψ_R and Lemma XII.3.3, it follows that

$$|\nabla\eta_R|+|\boldsymbol{v}\cdot\nabla\eta_R|+|\Delta\eta_R| \le c \tag{XII.3.56}$$

for some constant c independent of R. Furthermore,

$$\left|\int_{\Omega^\rho}\left[f_y\left(\frac{\partial\omega\eta_R}{\partial x}\right)-f_x\left(\frac{\partial\omega\eta_R}{\partial y}\right)\right]\right| \le \frac{1}{2}\int_{\Omega^\rho}\left(\eta_R|\nabla\omega|^2+c_1\omega^2+c_2 r|\boldsymbol{f}|^2\right) \tag{XII.3.57}$$

so that by (XII.3.56) and (XII.3.57), identity (XII.3.55) gives

$$\int_{\Omega^\rho} \eta_R |\nabla \omega|^2 \le C,$$

for a constant C independent of R. Letting $R \to \infty$ and using the monotone convergence theorem completes the proof. $\square$

We are now in a position to prove the following result on the behavior of a D-solution at infinity.

Theorem XII.3.4 *Let $\boldsymbol{v}$ be a D-solution to* (XII.0.1) *with $\boldsymbol{f}$ of bounded support in Ω and let $L \in [0, \infty]$ be the number defined in* (XII.3.44). *Then, if $L < \infty$ (this certainly happens whenever $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv 0$), there is a $\boldsymbol{v}_0 \in \mathbb{R}^2$ such that*

$$\lim_{r\to\infty} \int_0^{2\pi} |\boldsymbol{v}(r,\theta) - \boldsymbol{v}_0|^2 d\theta = 0. \tag{XII.3.58}$$

Furthermore, if $\boldsymbol{v}_0 = 0$, we have

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = 0 \quad \textit{uniformly}. \tag{XII.3.59}$$

Finally, if $L = \infty$,

$$\lim_{r\to\infty} \int_0^{2\pi} |\boldsymbol{v}(r,\theta)|^2 d\theta = \infty. \tag{XII.3.60}$$

Proof. Let $\psi = \psi(r)$ be the argument of $\overline{\boldsymbol{v}}(r)$, that is,

$$\begin{aligned} \overline{v}_x(r) &= |\overline{\boldsymbol{v}}(r)| \cos\psi(r) \\ \overline{v}_y(r) &= |\overline{\boldsymbol{v}}(r)| \sin\psi(r) \end{aligned} \qquad \psi \in [0, 2\pi). \tag{XII.3.61}$$

Clearly, we have

$$\psi'(r) = \frac{\overline{v}_x \overline{v}_y' - \overline{v}_x' \overline{v}_y}{|\boldsymbol{v}|^2} \tag{XII.3.62}$$

where the prime means differentiation. Multiplying $(\text{XII.3.19})_1$ by $\sin\theta$, $(\text{XII.3.19})_2$ by $\cos\theta$, and adding up, for sufficiently large $|x|$ we find that

$$\frac{\partial \omega}{\partial r} + v_x \frac{\partial v_y}{\partial r} - v_y \frac{\partial v_x}{\partial r} + \frac{1}{r}\frac{\partial p}{\partial \theta} = 0. \tag{XII.3.63}$$

We take the average over θ of both sides of (XII.3.63) to deduce that

$$\begin{aligned} \frac{\partial \omega}{\partial r} + \overline{v}_x \overline{v}_y' - \overline{v}_x' \overline{v}_y + \frac{1}{2\pi}\int_0^{2\pi} \Big\{ &[v_x(r,\theta) - \overline{v}_x(r)] \frac{\partial v_y(r,\theta)}{\partial r} \\ &-[v_y(r,\theta) - \overline{v}_y(r)] \frac{\partial v_x(r,\theta)}{\partial r} \Big\} d\theta = 0. \end{aligned} \tag{XII.3.64}$$

From Lemma XII.3.10 we know that $|\overline{\boldsymbol{v}}(r)|$ converges to $L \geq 0$. Assume, for a while, that $L > 0$. Then we may find $\overline{\rho} > \delta(\Omega^c)$ such that

$$|\overline{\boldsymbol{v}}(r)| > L/2, \quad \text{for all } r > \overline{\rho}. \tag{XII.3.65}$$

We then divide both sides of (XII.3.64) by $|\overline{\boldsymbol{v}}(r)|^2$ and integrate over $\theta \in [0, 2\pi)$ and over $r \in (r_1, r_2)$, $r_2 > r_1 > \overline{\rho}$, to obtain

$$\psi(r_2) - \psi(r_1) = -\frac{1}{2\pi}\int_{r_1}^{r_2}\int_0^{2\pi} \frac{1}{|\overline{\boldsymbol{v}}(r)|^2}\left[\frac{\partial \omega}{\partial r} + (v_x - \overline{v}_x)\frac{\partial v_y}{\partial r} - (v_y - \overline{v}_y)\frac{\partial v_x}{\partial r}\right] dr d\theta.$$

Using (XII.3.65) and the Schwarz and the Wirtinger inequalities we see that

$$|\psi(r_2) - \psi(r_1)| \leq \frac{2}{\pi L^2}[\|r^{1/2}\omega\|_{2,\Omega_{r_1,r_2}} + |\boldsymbol{v}|_{1,2,\Omega_{r_1,r_2}}] \times [|\boldsymbol{v}|_{1,2,\Omega_{r_1,r_2}} + \left(\int_{r_1}^{r_2}\int_0^{2\pi}\frac{dr}{r^2}\right)^{1/2}]$$

and, therefore, letting $r_1, r_2 \to \infty$ and recalling Lemma XII.3.11, we obtain

$$\lim_{r\to\infty} \psi(r) = \psi_0 \tag{XII.3.66}$$

for some $\psi_0 \in [0, 2\pi]$. For $L \geq 0$ we define the vector

$$\boldsymbol{v}_0 = (L\cos\psi_0, L\sin\psi_0).$$

If $L \in (0, \infty)$, from Lemma XII.3.10(ii), (XII.3.61), and (XII.3.66) we conclude that

$$\lim_{r\to\infty} \overline{\boldsymbol{v}}(r) = \boldsymbol{v}_0,$$

which along with Lemma XII.3.10(i) implies (XII.3.58). If $L = 0$, we have $\boldsymbol{v}_0 = 0$ and (XII.3.59) follows from (XII.3.44). Finally, if $L = \infty$, (XII.3.60) follows directly from Lemma XII.3.10. Notice that, in view of Lemma XII.3.9, this circumstance can not occur if $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv 0$. The theorem is proved. □

Remark XII.3.1 The vector $\boldsymbol{v}_0$ determined in Theorem XII.3.4 need not be the vector $\boldsymbol{v}_\infty$ prescribed in (XII.0.2). *The problem of the coincidence of $\boldsymbol{v}_0$ and $\boldsymbol{v}_\infty$ therefore remains open.* Furthermore, if $\boldsymbol{v}_0 = 0$ and $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv 0$, we can conclude that $\boldsymbol{v} \equiv 0$ only in the trivial case $\Omega = \mathbb{R}^2$; see Theorem XII.3.1). ■

Remark XII.3.2 Another question that is open is to ascertain if, in the case $\boldsymbol{v}_0 \neq \boldsymbol{0}$, (XII.3.58) holds pointwise:

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = \boldsymbol{v}_0. \tag{XII.3.67}$$

C.J. Amick has shown the validity of (XII.3.67) in the class of symmetric flows. Precisely, let us call a solution $\boldsymbol{v}$, p to (XII.0.1) *symmetric around the x-axis*, or, more simply, *symmetric*, if p and v_x are even in y and v_y is odd in y. It can be shown that if the x-axis is of symmetry for $\mathring{\Omega}^c$, and $\boldsymbol{v}_*$ and $\boldsymbol{f}$ possess the same symmetry as $\boldsymbol{v}$ does, under suitable regularity conditions on the data, the class of symmetric D-solutions is not empty. If, in particular, $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv \boldsymbol{0}$, then such solutions satisfy (XII.3.67) uniformly; cf. Amick (1988, Theorem 27). If the flow is not symmetric, the best one can say, so far, is that (XII.3.67) holds in suitable large sectors. Namely, taking $\boldsymbol{v}_0 = (1,0)$, for all $\varepsilon \in (0,\pi/2)$, we have

$$\lim_{r\to\infty} \max_{|\theta|\in[\varepsilon,\pi-\varepsilon]} |\boldsymbol{v}(r,\theta) - \boldsymbol{v}_0| = 0;$$

provided that $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv 0$; cf. Amick (1988, Theorem 19). In any case, one can show that the modulus of $\boldsymbol{v}$ tends pointwise to the modulus of $\boldsymbol{v}_0$; see Amick (1988, Theorem 21).[3] ■

XII.4 Asymptotic Decay of the Vorticity and its Relevant Consequences

Theorem XII.3.2, Theorem XII.3.3, and Theorem XII.3.4 are all silent about the rate of decay of $\boldsymbol{v}$ and p and their derivatives at large distances. As a matter of fact, in Section XII.8 we shall show that, if $\boldsymbol{v}_0 \neq 0$ and (XII.3.67) holds uniformly, then the fields $\boldsymbol{v}$ and p present the *same* asymptotic structure of the Oseen fundamental tensor.[1] The key point in assessing this property is a detailed study of the pointwise rate of decay of the vorticity, ω, in the general case and, in particular, when $\boldsymbol{v}_0 \neq \boldsymbol{0}$. This investigation will lead to the important consequence that, if $\boldsymbol{v}_0 \neq \boldsymbol{0}$, then every D-solution corresponding to $\boldsymbol{v}_0$ and to $\boldsymbol{f}$ of bounded support must satisfy the following property

$$\int_{\Omega^\rho} \frac{|\boldsymbol{v}-\boldsymbol{v}_0|^2}{|x|^{1+\varepsilon}} < \infty\,, \quad \text{for all } \varepsilon > 0\,,$$

for sufficiently large ρ. Observe that a priori a D-solution only satisfies the *weaker* property

[3] In Galdi (2004, Theorem 3.4) it is stated that the pointwise limit (XII.3.67) also holds for non-symmetric flow. However, the validity of this result is based on Lemma 3.10 in the same paper, whose proof is not correct as presented.

[1] Notice that, in contrast, if $\boldsymbol{v}_0 = \boldsymbol{0}$, the example furnished in (XII.2.7) excludes, in general, for the corresponding velocity field, a uniform decay in a negative power of $|x|$.

$$\int_{\Omega^\rho} \frac{|\boldsymbol{v}-\boldsymbol{v}_0|^2}{|x|^2 \ln |x|^2} < \infty\,;$$

see (II.6.14).

With this in mind, we begin to establish a simple consequence of the two-sided maximum principle, namely, that, if $\boldsymbol{v}$ is bounded (as it happens when $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv 0$), then

$$\lim_{|x|\to\infty} |x|^{3/4}|\omega(x)| = 0, \quad \text{uniformly;} \tag{XII.4.1}$$

(cf. Gilbarg & Weinberger 1978, Theorem 5). Actually, using Lemma XII.3.11, we find that

$$\int_{2^k}^{2^{k+1}} \frac{1}{r} \int_0^{2\pi} \left(r^2\omega^2 + 2r^{3/2} \left| \omega \frac{\partial \omega}{\partial \theta} \right| \right) drd\theta < \infty, \quad \text{for all } k \in \mathbb{N},$$

which implies the existence of $r_k \in (2^k, 2^{k+1})$ such that

$$\int_0^{2\pi} \left(r_k^2 \omega^2(r_k,\theta) + 2r_k^{3/2} \left| \omega(r_k,\theta) \frac{\partial \omega(r_k,\theta)}{\partial \theta} \right| \right) d\theta \to 0, \text{ as } k \to \infty. \tag{XII.4.2}$$

However,

$$\omega^2(r_k,\theta) \le \frac{1}{2\pi} \int_0^{2\pi} \omega^2(r_k,\varphi)d\varphi + \frac{1}{\pi} \int_0^{2\pi} |\omega(r_k,\varphi)| \left| \frac{\partial \omega(r_k,\varphi)}{\partial \varphi} \right| d\varphi$$

which, by (XII.4.2), implies

$$r_k^{3/2}|\omega^2(r_k,\theta)| \to 0, \quad \text{as } k \to \infty, \text{ uniformly in } \theta.$$

The decay estimate (XII.4.1) then follows by applying the two-sided maximum principle (Protter & Weinberger 1967, pp. 61-64) to (XII.3.1) with $\boldsymbol{f} \equiv 0$ in the annuli $A_k = \{x \in \Omega : r_k < |x| < r_{k+1}\}$.

A first consequence of (XII.4.1) is the following pointwise decay for the gradient of $\boldsymbol{v}$:

$$\lim_{|x|\to\infty} \frac{|x|^{3/4}}{\log |x|} |\nabla \boldsymbol{v}(x)| = 0, \quad \text{uniformly}. \tag{XII.4.3}$$

We shall not prove this property here, since it is irrelevant to our further purposes, and refer instead the interested reader to Gilbarg & Weinberger (1978, Theorem 7) for a proof.[2]

Once (XII.4.1) has been established, it is easy to show, by means of local elliptic estimates, that the derivatives of arbitrary order (≥ 1) of ω decay, at least, like in (XII.4.1); see Exercise XII.4.1. Moreover, by combining (XII.4.1) and (XII.4.3), an analogous property can be proved also for $\boldsymbol{v}$ and p; see Exercise XII.4.1.

[2] Actually, in this paper a weaker decay for $\nabla \boldsymbol{v}$ is given, due to the circumstance that the authors do not assume $\boldsymbol{v}$ to be bounded but, rather, that it satisfies the growth condition of Lemma XII.3.3.

Exercise XII.4.1 Let $\boldsymbol{v}$ be a D-solution to (XII.0.1) corresponding to $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv \boldsymbol{0}$. Show that, as $|x| \to \infty$,

$$|D^\alpha \omega(x)| = o(|x|^{-3/4}), \quad \text{all } |\alpha| \geq 1. \tag{XII.4.4}$$

Hint: Combining (XII.3.1), Theorem XII.3.2 and the interior estimates for the Laplace operator (see Exercise IV.4.4) we have, for sufficiently large $|x|$,

$$|\omega|_{\ell+2,B_1(x)} \leq c\,\|\omega\|_{2,B_2(x)} \quad \text{all } \ell \geq -1, \tag{XII.4.5}$$

with c depending on ℓ and $\boldsymbol{v}$.

Moreover, show that

$$|D^\alpha \boldsymbol{v}(x)| = o(|x|^{-3/4} \log |x|), \quad \text{all } |\alpha| \geq 1. \tag{XII.4.6}$$

Hint: Equation $(\text{XII.0.1})_1$, with $\boldsymbol{f} \equiv \boldsymbol{0}$, can be rewritten as follows

$$\Delta \boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{v} + \nabla \Phi,$$

where $\boldsymbol{v} = (v_1, v_2)$, $\boldsymbol{\omega} := \omega \boldsymbol{e}_3$, and Φ is defined in (XII.3.42), so that

$$\Delta(D_k \boldsymbol{v}) = (D_k \boldsymbol{\omega}) \times \boldsymbol{v} + \boldsymbol{\omega} \times (D_k \boldsymbol{v}) + \nabla(D_k \Phi).$$

From Theorem IV.4.1, Theorem IV.4.4, Theorem XII.3.2, and Remark IV.4.1 it follows that

$$\|D_k \boldsymbol{v}\|_{m+2,2,B_2(x)} \leq c\,(\|\omega\|_{m+1,2,B_2(x)} + \|D_k \boldsymbol{v}\|_{2,B_2(x)}).$$

The desired estimate is a consequence of this latter, and of (XII.4.4), (XII.4.3).

Finally, denoted by $p = p(x)$ the pressure field associated to $\boldsymbol{v}$, show that

$$|D^\alpha p(x)| = o(|x|^{-3/4} \log |x|), \quad \text{all } |\alpha| \geq 1. \tag{XII.4.7}$$

Our next objective is to show that if (XII.3.67) holds uniformly, for some $\boldsymbol{v}_0 \neq \boldsymbol{0}$, then the vorticity ω and all its derivatives decay exponentially fast outside any sector that excludes the line $\{x \in \mathbb{R}^2 : \boldsymbol{x} = \lambda \boldsymbol{v}_0,\ \lambda > 0\}$. To this end, assume, without loss of generality, $\boldsymbol{v}_0 = -\boldsymbol{e}_1$, and, with the origin of coordinates in $\mathring{\Omega}^c$, set

$$\Omega^{R_0,\sigma} := \Big\{x = \big(x_1 = r\cos\theta, x_2 = r\sin\theta\big) \in \Omega :\ r \geq R_0,\ |\pi - \theta| \geq \sigma\Big\},$$

where $\sigma > 0$ and $R_0 > \delta(\Omega^c)$.

We have the following result, due, basically, to Amick (1988, §2.5).

Theorem XII.4.1 *Let $\boldsymbol{v}$ be a D-solution to* (XII.0.1) *with $\boldsymbol{f}$ of bounded support. Furthermore, suppose that $\boldsymbol{v}$ satisfies* (XII.3.67) *uniformly, with $\boldsymbol{v}_0 = -\boldsymbol{e}_1$. Then, for any $\sigma > 0$ there exist positive numbers R_0, C and γ depending on σ and $\boldsymbol{v}$, such that*[3]

[3] A much more detailed pointwise decay estimate for the vorticity will be provided in Theorem XII.8.4.

$$|\omega(x)| \le C \exp(-\gamma|x|)\,, \quad \textit{for all } x \in \Omega^{R_0,\sigma}\,. \tag{XII.4.8}$$

Therefore, from (XII.4.5), (XII.4.8) *and the embedding* Theorem II.3.4 *it follows, in addition,*

$$|D^\alpha \omega(x)| \le C_1 \exp(-\gamma|x|)\,, \quad \textit{for all } |\alpha| \ge 0 \textit{ and all } x \in \Omega^{R_0,\sigma}\,, \tag{XII.4.9}$$

where, this time, C_1 depends also on α.

Proof. In order to prove (XII.4.8), it is sufficient to prove its validity in the following three sector-like regions:

$$\begin{aligned} S_1 &= \{x \in \Omega: \ |x_2| \le \kappa_1 x_1,\ x_1 > M_1\} \\ S_2 &= \{x \in \Omega: \ |x_1| \le \kappa_2 x_2,\ x_2 > M_2\} \\ S_3 &= \{x \in \Omega: \ |x_1| \le \kappa_3(-x_2),\ x_2 < -M_3\} \end{aligned}$$

with $\kappa_i > 0$ arbitrarily large, and $M_i > 0$ sufficiently large, $i = 1, 2, 3$. We begin with the region S_1. Consider (XII.3.1) in the domain Ω^ρ, with ρ so large that supp $(\boldsymbol{f}) \subset B_\rho$. Then, multiplying both sides of (XII.3.1) in Ω^ρ by ω, we find

$$(\omega^2)_{x_1x_1} + (\omega^2)_{x_2x_2} = (v_1\omega^2)_{x_1} + (v_1\omega^2)_{x_2} + 2|\Delta\omega|^2\,, \tag{XII.4.10}$$

where we used the notation $(\cdot)_\xi := \partial(\cdot)/\partial\xi$, $(\cdot)_{\xi\eta} := \partial^2(\cdot)/\partial\xi\partial\eta$. Integrating the previous relation first over x_1 between $x_1(> M_1)$ and ∞, then over x_2 in $\mathbb{R}$, and taking into account Theorem XII.3.2 and Lemma XII.3.9, we get

$$\begin{aligned} -\frac{d}{dx_1}\int_{\mathbb{R}} \omega^2(x_1,x_2)dx_2 &= -\int_{\mathbb{R}} v_1\omega^2(x_1,x_2)dx_2 + 2\int_{x_1}^\infty\int_{\mathbb{R}}|\Delta\omega|^2 \\ &\ge -\int_{\mathbb{R}} v_1\omega^2(x_1,x_2)dx_2\,. \end{aligned}$$

In view of the assumptions on $\boldsymbol{v}$, we can find $M_1 > 0$ sufficiently large, such that $|v_1(x)+1| < 1/2$, for all $x_1 > M_1$ and all $x_2 \in \mathbb{R}$. Therefore, the preceding inequality furnishes

$$\frac{d}{dx_1}\int_{\mathbb{R}} \omega^2(x_1,x_2)dx_2 \le -\tfrac{1}{2}\int_{\mathbb{R}} \omega^2(x_1,x_2)dx_2\,, \quad \text{for all } x_1 > M_1\,,$$

which, in turn, delivers

$$\int_{\mathbb{R}} \omega^2(x_1,x_2)dx_2 \le c_1 \exp(-x_1/2) \ \ \text{for all } x_1 > M_1\,,$$

where $c_1 = c_1(\omega)$. Integrating this latter over the interval $(x_1 - 2, x_1 + 2)$, we deduce, in particular,

$$\|\omega\|_{2,B_2(x)} \le c_2 \exp(-x_1/2) \ \ \text{for all } x_1 > M_1\,,$$

from which, observing that for $x \in S_1$ we have $|x| \le x_1\sqrt{\kappa_1^2+1}$, the validity of (XII.4.8) in S_1 follows with the help of (XII.4.5) and the embedding Theorem II.3.4. We shall next prove (XII.4.8) in the region S_2. We begin to observe that, in view of the assumption on $\boldsymbol{v}$, for any $\varepsilon > 0$ we can find $M_2 = M_2(\varepsilon) > 0$ such that

$$|\boldsymbol{v}(x) + \boldsymbol{e}_1| < \varepsilon \quad \text{for all } x_1 \in \mathbb{R},\ x_2 > M_2\,. \tag{XII.4.11}$$

For $\beta > 0$, we introduce the new variables

$$\xi_1 = x_2 + \beta x_1\,, \quad \xi_2 = x_1\,,$$

so that (XII.4.10) furnishes, in particular,

$$(1+\beta^2)(\omega^2)_{\xi_1\xi_1} + (\omega^2)_{\xi_2\xi_2} + 2\beta(\omega^2)_{\xi_1\xi_2} \ge [(\beta v_1 + v_2)\omega^2]_{\xi_1} + (v_2\omega^2)_{\xi_2}\,.$$

Integrating both sides of this relation first over ξ_1 between $\xi_1 (> M_2)$ and ∞, then over ξ_2 in $\mathbb{R}$, and taking into account Theorem XII.3.2 and Lemma XII.3.9, we obtain

$$\begin{aligned}
-(1+\beta^2)\frac{d}{d\xi_1}\int_{\mathbb{R}}\omega^2(\,\xi_1,\xi_2)d\xi_2 &\ge -\int_{\mathbb{R}}(\beta v_1 + v_2)\omega^2(\xi_1,\xi_2)d\xi_2\\
&= \beta\int_{\mathbb{R}}\omega^2(\xi_1,\xi_2)d\xi_2 - \int_{\mathbb{R}}[\beta(v_1+1) + v_2]\omega^2(\xi_1,\xi_2)d\xi_2\,.
\end{aligned}$$

Using (XII.4.11), we deduce

$$-\frac{d}{d\xi_1}\int_{\mathbb{R}}\omega^2(\xi_1,\xi_2)d\xi_2 \ge \frac{\beta-(\beta+1)\varepsilon}{1+\beta^2}\int_{\mathbb{R}}\omega^2(\xi_1,\xi_2)d\xi_2\,, \quad \text{for all } \xi_1 > M_2\,,$$

which, upon integration, allows us to conclude

$$\int_{\mathbb{R}}\omega^2(\xi_1,\xi_2)d\xi_2 \le c_2\exp(-\gamma\xi_1)\,, \quad \gamma := \frac{\beta-(\beta+1)\varepsilon}{1+\beta^2}\,,\ \xi_1 > M_2 \tag{XII.4.12}$$

where $c_2 = c_2(\omega) > 0$ and $\gamma > 0$ for sufficiently small ε. Fix $x = (x_1, x_2)$ with $x_2 > M_2$, and, for $\delta > 0$ small enough, set

$$C_\delta(x) = \{y \in \mathbb{R}^2;\, |y_1 - x_1| < \delta,\ |y_2 - x_2| < \delta\}\,.$$

From (XII.4.12) we thus have, with $\xi_1 = x_2 + \beta x_1$,

$$\begin{aligned}
\|\omega\|_{2,B_\delta(x)} \le \|\omega\|_{2,C_\delta(x)} &\le \int_{\xi_1-(1+\beta)\delta}^{\xi_1+(1+\beta)\delta}\int_{x_1-\delta}^{x_1+\delta}\omega^2(\eta_1,\eta_2)d\eta_1 d\eta_2\\
&\le c_3\exp(-\gamma\xi_1) = c_3\exp[-\gamma(x_2+\beta x_1)]\,, \quad x_2 > M_2\,,\ x_1 \in \mathbb{R}.
\end{aligned} \tag{XII.4.13}$$

We now observe that $x_2 + \beta x_1 \ge x_2 - \beta|x_1|$, and since in S_2 it is $|x_1| \le \kappa_2 x_2$, with the choice $\beta = 1/(2\kappa_2)$, we infer

$$|x| \le x_2\sqrt{1+\kappa_2^2} \le 2(x_2+\beta x_1)\sqrt{1+\kappa_2^2}\,, \quad x \in S_2\,.$$

Coupling this information with (XII.4.13), (XII.4.5) and the embedding Theorem II.3.4, we conclude the validity of the decay estimate in (XII.4.8) for all $x \in S_2$. The proof that the same estimate holds in S_3 is completely analogous to that just furnished for S_2 and it is left to the reader. The theorem is therefore completely proved. □

An immediate and important consequence of Theorem XII.4.1 is described in the following.

Corollary XII.4.1 *Let the assumptions of* Theorem XII.4.1 *be satisfied. Then, the function* $\widetilde{\Phi} := \Phi + \frac{1}{2}$, *with* Φ *defined in* (XII.3.42) *satisfies the following pointwise decay*

$$|\widetilde{\Phi}(x)| \le C_1\, \exp(-C_2|x|) \;\; \textit{for all } x \in \Omega^{R_0,\sigma}\,,$$

with C_i, $i = 1, 2$, *independent of* x.

Proof. By assumption and by Theorem XII.3.3 (with $p_0 = 0$, for simplicity and without loss of generality) we have

$$\lim_{|x|\to\infty} \widetilde{\Phi}(x) = 0\,, \quad \text{uniformly}\,. \tag{XII.4.14}$$

Moreover, by an elementary calculation that uses $(\text{XII.0.1})_1$ and (XII.3.2) we find in Ω^{R_0}

$$\begin{aligned} \frac{\partial\widetilde{\Phi}}{\partial x_1} &= \frac{\partial\omega}{\partial x_2} - v_2\omega \\ \frac{\partial\widetilde{\Phi}}{\partial x_2} &= -\frac{\partial\omega}{\partial x_1} + v_1\omega\,. \end{aligned} \tag{XII.4.15}$$

Consequently, from Theorem XII.4.1 we obtain

$$|\nabla\widetilde{\Phi}(x)| \le c_1\, \exp(-c_2|x|)\,, \quad \text{for all } x \in \Omega^{R_0,\sigma}, \tag{XII.4.16}$$

with c_i, $i = 1, 2$, independent of x. Now, for every $x = (\theta, r)$, and $y = (\theta, r_1)$ in $\Omega^{R_0,\sigma}$, with $r_1 > r$, we have

$$|\widetilde{\Phi}(x)| \le |\widetilde{\Phi}(y)| + \int_r^{r_1} \left|\frac{\partial\widetilde{\Phi}}{\partial\rho}\right| d\rho\,,$$

so that, by (XII.4.16), we find

$$|\widetilde{\Phi}(x)| \le |\widetilde{\Phi}(y)| + c_3\, \exp(-c_2|x|)\,, \quad \text{for all } x \in \Omega^{R_0,\sigma}\,.$$

The result then follows by letting $|y| \to \infty$ in this relation and employing (XII.4.14).

□

Exercise XII.4.2 Let the assumptions of Corollary XII.4.1 hold. Show that

$$|D^\alpha \widetilde{\Phi}(x)| \le c_1 \exp(-c_2|x|) \ \text{ for all } \alpha| \ge 0,\, x \in \Omega^{R_0,\sigma}\,,$$

with c_1 and c_2 independent of x. *Hint*: Use (XII.4.15) along with Theorem XII.3.2 and Theorem XII.4.1.

We shall next furnish for the function $\widetilde{\Phi}$ a weighted-L^2 property holding in the *whole* of Ω^{R_0}. Precisely, we have the following result due to Sazonov (1999, §4).

Theorem XII.4.2 *Let the assumption of* Corollary XII.4.1 *hold. Then, for all* $\varepsilon \in (0,1)$,[4]

$$\int_{\Omega^{R_0}} \frac{|\widetilde{\Phi}|^2}{|x|^{1+\varepsilon}} < \infty\,. \tag{XII.4.17}$$

Proof. We begin to observe that, in order to show (XII.4.17), it is enough to show the following

$$\int_{\Omega^{R_0}\cap\{x_1<a\}} \frac{|\widetilde{\Phi}|^2}{|x_1|^{1+\varepsilon}} < \infty\,, \quad \text{some } a<0\,. \tag{XII.4.18}$$

In fact, since for all x in the sector-like region $S_a := \{x \in \Omega : |x_2| \le c|x_1|,\ x_1 < a\}$ it is $|x| \le |x_1|\sqrt{1+c^2}$, inequality (XII.4.18) implies

$$\int_{\Omega^{R_0}\cap S_a} \frac{|\widetilde{\Phi}|^2}{|x|^{1+\varepsilon}} < \infty\,,$$

which, in turn, in view of Corollary XII.4.1, secures (XII.4.17). In order to prove (XII.4.18), we begin to consider the sequence of problems ($N \in \mathbb{N}$, $N \ge N_0 > R_0$)

$$\begin{gathered} \Delta\widetilde{\Phi}_N - \boldsymbol{v}\cdot\nabla\widetilde{\Phi}_N = \omega^2 \ \text{ in } \Omega_{(N)} := \Omega_{R_0,N} \\ \widetilde{\Phi}_N\big|_{\partial\Omega_{R_0}} = \widetilde{\Phi}\big|_{\partial\Omega_{R_0}}\,, \quad \widetilde{\Phi}_N\big|_{\partial B_N} = 0 \end{gathered} \tag{XII.4.19}$$

It is not hard to show that, for each $N \ge N_0$, problem (XII.4.19) has a unique solution $\widetilde{\Phi} \in C^\infty(\overline{\Omega_{(N)}})$; see Exercise XII.4.3. Setting $\phi_N := \widetilde{\Phi} - \widetilde{\Phi}_N$, from (XII.3.43) and (XII.4.19) it follows that

$$\begin{gathered} \Delta\phi_N - \boldsymbol{v}\cdot\nabla\phi_N = 0 \ \text{ in } \Omega_{(N)} \\ \phi_N\big|_{\partial\Omega_{R_0}} = 0\,, \quad \phi_N\big|_{\partial B_N} = \widetilde{\Phi} \end{gathered} \tag{XII.4.20}$$

[4] Actually, the result holds for all $\varepsilon > 0$, but it will be used (and useful) only for ε in the stated range.

Therefore, applying the two-sided maximum principle (Protter & Weinberger 1967, pp. 61-64) we find

$$m_1 := \min\left\{\min_{\partial B_N} \widetilde{\Phi}, 0\right\} \le \zeta_N(x) \le \max\left\{\max_{\partial B_N} \widetilde{\Phi}, 0\right\} := m_2\,, \quad x \in \Omega_{(N)}\,. \tag{XII.4.21}$$

In view of (XII.4.14), for any $\varepsilon > 0$, we can find $\overline{N} = \overline{N}(\varepsilon) \in \mathbb{N}$ such that (XII.4.21) holds with $m_1 \ge -\varepsilon$, $m_2 \le \varepsilon$, and all $N \ge \overline{N}$. Consequently, extending $\widetilde{\Phi}_N$ to zero outside B_N, and continuing to denote by $\widetilde{\Phi}_N$ such extension, we conclude

$$\begin{aligned} &\|\widetilde{\Phi}_N\|_{\infty,\Omega^{R_0}} \le M\,, \\ &\lim_{N\to\infty} \widetilde{\Phi}_N(x) = \widetilde{\Phi}(x)\,, \quad \text{uniformly in } x \in \Omega^{R_0}\,, \end{aligned} \tag{XII.4.22}$$

where $M > 0$ is independent of N. We next write $\widetilde{\Phi}_N = \zeta_N + \psi\widetilde{\Phi}$, where ψ is a smooth "cut-off" function that is 1 for $|x| \le R_0 + \delta$ and is 0 for $|x| \ge R_0 + 2\delta$, with δ sufficiently small. Taking into account that $\widetilde{\Phi}$ satisfies (XII.3.42), we may rewrite (XII.4.19) as follows

$$\left.\begin{aligned} \Delta\zeta_N - \boldsymbol{v}\cdot\nabla\zeta_N = g \ \text{ in } \Omega_{(N)} \\ \zeta_N\big|_{\partial\Omega_{R_0}} = \zeta_N\big|_{\partial B_N} = 0 \end{aligned}\right.\,, \tag{XII.4.23}$$

where g is a smooth function of compact support independent of N. If we multiply through both sides of (XII.4.23)$_1$ by ζ_N, integrate by parts over $\Omega_{(N)}$, use (XII.4.23)$_2$, and recall that $\nabla\cdot\boldsymbol{v} = 0$, we at once obtain

$$|\zeta_N|_{1,2}^2 = -(g, \zeta_N)\,,$$

so that, by (XII.4.22)$_1$ and by the fact that $g \in L^1(\Omega^{R_0})$, we deduce

$$|\zeta_N|_{1,2} \le C\,, \tag{XII.4.24}$$

with C independent of N. Next, for $a < 0$, let

$$f(x) = \begin{cases} \dfrac{1}{\ln(-a)} & \text{if } x_1 \ge a \\[2ex] \dfrac{1}{\ln(-x_1)} & \text{if } x_1 \le a \end{cases}\,.$$

Multiplying through both sides of (XII.4.23)$_1$ by $f\,\zeta_N$, integrating by parts over $\Omega_{(N)}$, and taking into account (XII.4.23)$_2$ and $\nabla\cdot\boldsymbol{v} = 0$, we easily find

$$\begin{aligned} \|f\nabla\zeta_N\|_2^2 + \tfrac{1}{2}\int_{\Omega_{(N)}\cap\{x_1<a\}} \frac{|\zeta_N|^2}{|x_1|\ln^2|x_1|} \le (\zeta_N\nabla\zeta_N, \nabla f) + (g, \zeta_N) \\ + \tfrac{1}{2}\int_{\Omega_{(N)}\cap\{x_1<a\}} \frac{|\zeta_N|^2|\boldsymbol{v}+\boldsymbol{e}_1|}{|x_1|\ln^2|x_1|}\,. \end{aligned} \tag{XII.4.25}$$

By the Cauchy inequality (II.2.5), the properties of f and (XII.4.24) we have

$$\begin{aligned}(\zeta_N \nabla \zeta_N, \nabla f) &\leq \tfrac{1}{8}\int_{\Omega_{(N)}\cap\{x_1<a\}} \frac{|\zeta_N|^2}{|x_1|\ln^2|x_1|} + c_1|\zeta_N|^2_{1,2} \\ &\leq \tfrac{1}{8}\int_{\Omega_{(N)}\cap\{x_1<a\}} \frac{|\zeta_N|^2}{|x_1|\ln^2|x_1|} + c_2\,,\end{aligned}$$

with c_2 independent of N. Moreover, in view of $(\text{XII.4.22})_1$,

$$(g, \zeta_N) \leq c_3$$

with c_2 independent of N. Finally, by virtue of the assumptions made on $\boldsymbol{v}$, we can choose $|a|$ (and, consequently, N) so large as

$$\tfrac{1}{2}\int_{\Omega_{(N)}\cap\{x_1<a\}} \frac{|\zeta_N|^2|\boldsymbol{v}+\boldsymbol{e}_1|}{|x_1|\ln^2|x_1|} \leq \tfrac{1}{8}\int_{\Omega_{(N)}\cap\{x_1<a\}} \frac{|\zeta_N|^2|}{|x_1|\ln^2|x_1|}\,.$$

Employing all the above information into (XII.4.25), we recover

$$\int_{\Omega_{(N)}\cap\{x_1<a\}} \frac{|\zeta_N|^2}{|x_1|\ln^2|x_1|} \leq c_4$$

where c_4 does not depend on N. Recalling the definition of ζ_N, this latter inequality delivers at once

$$\int_{\Omega_{(N)}\cap\{x_1<a\}} \frac{|\widetilde{\Phi}_N|^2}{|x_1|\ln^2|x_1|} \leq c_5 \tag{XII.4.26}$$

with c_5 independent of N. Fix $r > R_0$ and take $N > r$. Then (XII.4.26) implies

$$\int_{\Omega_{R_0,r}\cap\{x_1<a\}} \frac{|\widetilde{\Phi}_N|^2}{|x_1|\ln^2|x_1|} \leq c_5$$

so that, letting $N \to \infty$ in this relation and using (XII.4.22), by the Lebesgue dominated convergence theorem we deduce

$$\int_{\Omega_{R_0,r}\cap\{x_1<a\}} \frac{|\widetilde{\Phi}|^2}{|x_1|\ln^2|x_1|} \leq c_5\,,$$

which, in turn, by the arbitrariness of r (and the regularity of $\widetilde{\Phi}$) implies (XII.4.18). The proof of the theorem is thus complete. □

Exercise XII.4.3 Show that, for each $N \geq N_0$, problem (XII.4.19) has one and only one solution $\widetilde{\Phi}_N \in C^\infty(\overline{\Omega_{(N)}})$. *Hint*: Use Galerkin method along with the local regularity estimates for the Poisson equation given in Exercise IV.4.4 and Exercise IV.5.3.

The weighted-L^2 summability property of $\widetilde{\Phi}$ that we have just proved allows us to deduce an analogous property for $\boldsymbol{v}+\boldsymbol{e}_1$. Notice that, being $\boldsymbol{v}$ a D-solution, we have (*cf.* (II.6.14))

$$\int_{\Omega^\rho} \frac{|\boldsymbol{v}+\boldsymbol{e}_1|^2}{|x|^2 \ln |x|^2} < \infty\,,$$

where $\rho > \delta(\Omega^c)$. However, we are able to obtain a much stronger summability property, as shown in the following theorem due to Sazonov (1999, Lemma 4).

Theorem XII.4.3 *Let the assumption of* Theorem XII.4.1 *hold. Then, for all* $\varepsilon \in (0,1)$,[5]

$$\int_{\Omega^{R_0}} \frac{|\boldsymbol{v}+\boldsymbol{e}_1|^2}{|x|^{1+\varepsilon}} < \infty\,. \tag{XII.4.27}$$

Proof. The proof is based on a suitable representation of the velocity field $\boldsymbol{v}$ in terms of the total head pressure $\widetilde{\Phi}$. This latter is more simply (and elegantly) obtained if we rewrite the linear momentum equation (XII.0.1)$_1$ in terms of the complex variable formalism. Let $z = x_1 + i\,x_2$, $u(z) := v_1(x_1,x_2) + i\,v_2(x_1,x_2)$ ($i := \sqrt{-1}$), and define the operators

$$\frac{\partial}{\partial z} := \tfrac{1}{2}\left(\frac{\partial}{\partial x_1} + \frac{1}{i}\frac{\partial}{\partial x_2}\right)\,, \qquad \frac{\partial}{\partial \overline{z}} := \tfrac{1}{2}\left(\frac{\partial}{\partial x_1} - \frac{1}{i}\frac{\partial}{\partial x_2}\right)\,.$$

Since

$$\begin{aligned} 4\frac{\partial^2 u}{\partial z \partial \overline{z}} &= \Delta v_1 + i\,\Delta v_2 \\ 2\frac{\partial \widetilde{\Phi}}{\partial \overline{z}} &= \frac{\partial \widetilde{\Phi}}{\partial x_1} + i\,\frac{\partial \widetilde{\Phi}}{\partial x_2} \\ \frac{\partial}{\partial z}(u^2 - 1) &= v_2 \omega - i\,v_1 \omega\,, \end{aligned}$$

from (XII.4.15) and (XII.3.2) we deduce that (XII.0.1)$_1$ can be rewritten, in Ω^{R_0}, as follows

$$\frac{\partial}{\partial z}\left[4\frac{\partial u}{\partial \overline{z}} - (u^2 - 1)\right] = 2\frac{\partial \widetilde{\Phi}}{\partial \overline{z}}\,. \tag{XII.4.28}$$

Pick $y = (y_1, y_2) \in \Omega^{R_0}$, and set $\zeta = y_1 + i\,y_2$. We recall the following generalized Cauchy integral formula

$$w(\zeta) = -\frac{1}{2\pi i}\int_{\partial A} \frac{w(z)}{\overline{z} - \overline{\zeta}}\, d\overline{z} - \frac{1}{\pi}\int_A \frac{\partial w}{\partial z}\,\frac{1}{\overline{z} - \overline{\zeta}}\, dx_1 dx_2\,, \tag{XII.4.29}$$

where A is a bounded subdomain of Ω^{R_0}, $\zeta \in A$, and $w \in C^1(\overline{A})$. The proof of (XII.4.29) is based on the (conjugate form of the) well-known Stokes formula:

[5] See footnote 4.

$$\int_{\partial A} w(z)\, dz = 2i \int_A \frac{\partial w}{\partial \overline{z}}\, dx_1 dx_2 \tag{XII.4.30}$$

and on the fact that $1/\overline{z}$ is analytic for $z \neq 0$; see, e.g., Hörmander (1966, Theorem I.2.1) for details. Choosing in (XII.4.29) $w = 4\partial u/\partial \overline{z} - (u^2 - 1)$ and $A = \Omega_{R_0,r}$, we obtain

$$\begin{aligned}\frac{1}{\pi}\int_{\Omega_{R_0,r}} \frac{\partial}{\partial z}\big[4\frac{\partial u}{\partial \overline{z}} - (u^2-1)\big](z)\, \frac{1}{\overline{z}-\overline{\zeta}}\, dx_1 dx_2 &= (u^2(\zeta)-1) - 4\frac{\partial u}{\partial \overline{z}}(\zeta)\\ &\quad - \frac{1}{2\pi i}\int_{\partial\Omega_{R_0,r}} \frac{4\partial u/\partial\overline{z}(z) - (u^2-1)(z)}{\overline{z}-\overline{\zeta}}\, d\overline{z}\,.\end{aligned} \tag{XII.4.31}$$

Moreover, using (XII.4.30) with $A = \Omega_{R_0,r,\eta} := \Omega_{R_0,r} - B_\eta(y)$ and $w = \widetilde{\Phi}(z)/(\overline{z}-\overline{\zeta})$, we find

$$\begin{aligned}\frac{1}{\pi}\int_{\Omega_{R_0,r,\eta}} \frac{(\partial\widetilde{\Phi}/\partial\overline{z})(z)}{\overline{z}-\overline{\zeta}} dx_1 dx_2 &= \frac{1}{\pi}\int_{\Omega_{R_0,r,\eta}} \frac{\widetilde{\Phi}(z)}{(\overline{z}-\overline{\zeta})^2} dx_1 dx_2\\ &\quad + \frac{1}{2\pi i}\int_{\partial\Omega_{R_0,r}} \frac{\widetilde{\Phi}(z)}{\overline{z}-\overline{\zeta}} dz + \frac{1}{2\pi i}\int_{|\overline{\zeta}-\overline{z}|=\eta} \frac{\widetilde{\Phi}(z)}{\overline{z}-\overline{\zeta}} dz\end{aligned} \tag{XII.4.32}$$

We would like to let $\eta \to 0$ in this latter relation. We begin to notice that, setting $\xi = \xi_1 + i\xi_2$, it is

$$\frac{1}{(\overline{\xi})^2} = \frac{2}{|\xi|^2}\left\{\frac{\xi_2^2 - \xi_1^2}{|\xi|^2} - 2i\frac{\xi_1\xi_2}{|\xi|^2}\right\}, \tag{XII.4.33}$$

which shows that $1/(\overline{\xi})^2$ is a singular kernel, and, therefore, by Theorem II.11.4, we have

$$\lim_{\eta\to 0}\int_{\Omega_{R_0,r,\eta}} \frac{\widetilde{\Phi}(z)}{(\overline{z}-\overline{\zeta})^2}\, dx_1 dx_2 = \int_{\Omega_{R_0,r}} \frac{\widetilde{\Phi}(z)}{(\overline{z}-\overline{\zeta})^2}\, dx_1 dx_2$$
$$\text{in the P.V. sense, for a.a. } \zeta \in \Omega_{R_0,r}\,. \tag{XII.4.34}$$

Furthermore, with the parameterization $z = \zeta + \eta\, e^{i\theta}$, $\theta \in [0, 2\pi]$, we find, in view of the continuity of $\widetilde{\Phi}$,

$$\int_{|\overline{\zeta}-\overline{z}|=\eta} \frac{\widetilde{\Phi}(z)}{\overline{z}-\overline{\zeta}} dz = \int_0^{2\pi} \widetilde{\Phi}(\zeta + \eta\, e^{i\theta})\, e^{2i\theta} d\theta = o(1) \quad \text{as } \eta \to 0\,.$$

From this equation, (XII.4.32) and (XII.4.33) we thus deduce, for a.a. $\zeta \in \Omega_{R_0,r}$,

$$\frac{1}{\pi}\int_{\Omega_{R_0,r}}\frac{(\partial\widetilde{\Phi}/\partial\overline{z})(z)}{\overline{z}-\overline{\zeta}}dx_1dx_2=\frac{1}{\pi}\int_{\Omega_{R_0,r}}\frac{\widetilde{\Phi}(z)}{(\overline{z}-\overline{\zeta})^2}dx_1dx_2 + \frac{1}{2\pi i}\int_{\partial\Omega_{R_0,r}}\frac{\widetilde{\Phi}(z)}{\overline{z}-\overline{\zeta}}dz\,, \tag{XII.4.35}$$

where the first integral on the right-hand side of (XII.4.34) is understood in the P.V. sense. Combining (XII.4.28), (XII.4.31) and (XII.4.35) we infer

$$u^2(\zeta)-1=4\frac{\partial u}{\partial\overline{z}}(\zeta)+\frac{2}{\pi}\int_{\Omega_{R_0,r}}\frac{\widetilde{\Phi}(z)}{(\overline{z}-\overline{\zeta})^2}dx_1dx_2 + \frac{1}{2\pi i}\int_{\partial\Omega_{R_0,r}}\frac{4\partial u/\partial\overline{z}(z)-(u^2-1)(z)}{\overline{z}-\overline{\zeta}}\,d\overline{z}+\frac{1}{\pi i}\int_{\partial\Omega_{R_0,r}}\frac{\widetilde{\Phi}(z)}{\overline{z}-\overline{\zeta}}dz \tag{XII.4.36}$$

We finally let $r\to\infty$ in (XII.4.36). Employing the properties

$$\boldsymbol{v}(x)+\boldsymbol{e}_1\,,\quad \nabla\boldsymbol{v}(x)\,,\quad \widetilde{\Phi}(x)\ \to 0\,,\quad \text{uniformly as } |x|\to\infty$$

and taking into account that $1/|\overline{z}-\overline{\zeta}|=O(|z|^{-1})$ as $|z|\to\infty$ for each fixed $\zeta\in\Omega^{R_0}$, by letting $r\to\infty$ into (XII.4.36) we at once reach the desired representation:

$$\begin{aligned}u^2(\zeta)-1&=\frac{2}{\pi}\int_{\Omega^{R_0}}\frac{\widetilde{\Phi}(z)}{(\overline{z}-\overline{\zeta})^2}dx_1dx_2+4\frac{\partial u}{\partial\overline{z}}(\zeta)\\&\quad+\frac{1}{2\pi i}\int_{\partial\Omega^{R_0}}\frac{4\partial u/\partial\overline{z}(z)-(u^2-1)(z)}{\overline{z}-\overline{\zeta}}\,d\overline{z}+\frac{1}{\pi i}\int_{\partial\Omega_{R_0}}\frac{\widetilde{\Phi}(z)}{\overline{z}-\overline{\zeta}}dz\,,\\&:=\frac{2}{\pi}\int_{\Omega^{R_0}}\frac{\widetilde{\Phi}(z)}{(\overline{z}-\overline{\zeta})^2}dx_1dx_2+\mathscr{F}(u)(\zeta)\end{aligned} \tag{XII.4.37}$$

for a.a. $\zeta\in\Omega^{R_0}$ and where, again, the integral on the right-hand side of the last line in (XII.4.37) is interpreted in a suitable sense that will be made clear next. In fact, in view of the properties (XII.4.33) of the kernel $1/(\overline{\xi})^2$, Theorem XII.4.2 and Stein's Theorem II.11.5, the integral transform

$$\mathscr{S}(\widetilde{\Phi}):=\int_{\Omega^{R_0}}\frac{\widetilde{\Phi}(z)}{(\overline{z}-\overline{\zeta})^2}dx_1dx_2$$

is well-defined (in the P.V. sense) and, for all $\varepsilon\in(0,1)$, it satisfies

$$|x|^{(1+\varepsilon)/2}\mathscr{S}(\widetilde{\Phi})\in L^2(\Omega^{R_0})\,. \tag{XII.4.38}$$

Since, for each fixed $z\in\partial\Omega^{R_0}$, $1/|\overline{z}-\overline{\zeta}|=O(|\zeta|^{-1})$ as $|\zeta|\to\infty$, and since $\boldsymbol{v}$ is a D-solution, we easily establish

$$|x|^{(1+\varepsilon)/2}\,\mathscr{F}(u)\in L^2(\Omega^R)\,,\quad R>R_0\,,\tag{XII.4.39}$$

so that, by (XII.4.37)–(XII.4.39), we conclude

$$|x|^{(1+\varepsilon)/2}\,(u^2-1)\in L^2(\Omega^R)\,.$$

Let us denote by $L^2_w(\Omega^R)$ the (weighted) space of functions g with the property $|x|^{(1+\varepsilon)/2}\,g\in L^2(\Omega^R)$. Observing that $u^2-1=v_1^2-1-v_2^2+2i\,v_1v_2$, we deduce: (i) $-v_2+(v_1+1)v_2=g$, and (ii) $v_1^2-1-v_2^2=h$, with $g,h\in L^2_w(\Omega^R)$. Thus, recalling that: (iii) $v_1(x)+1=o(1)$ as $|x|\to\infty$, from (i) it follows, for sufficiently large $|x|$,

$$|v_2(x)|\le|g(x)|+|v_1(x)+1|\,|v_2(x)|\le|g(x)|+\tfrac{1}{2}|v_2(x)|\,,$$

that is, $v_2\in L^2_w(\Omega^{R_0})$.[6] Replacing this information back in (ii), we find $v_1^2-1=f$, with $f\in L^2_w(\Omega^R)$. Since $2(v_1+1)=(v_1+1)(v_1+1)-v_1^2+1$, by (iii) we recover, for sufficiently large $|x|$,

$$2|v_1(x)+1|\le|f(x)|+|v_1(x)+1|\,|v_1(x)+1|\le|f(x)|+|v_1(x)+1|$$

that is $v_1+1\in L^2_w(\Omega^{R_0})$, and the proof of the theorem is completed.

□

XII.5 Existence and Uniqueness of Solutions for Small Data and $\boldsymbol{v}_\infty\neq 0$

Objective of this section is to show that the two-dimensional exterior problem admits a unique solution –at least when the velocity $\boldsymbol{v}_\infty$ is not zero and the data are sufficiently small.

The leading idea is the same we used in similar circumstances, namely, to couple suitably the L^q-estimates derived for the *linearized* approximation together with a contraction mapping argument. It is important to emphasize, however, that this procedure is not going to work unless $\boldsymbol{v}_\infty\neq 0$ and, even in this situation, its success is by no means evident a priori, as we are about to illustrate. To begin, let us take first $\boldsymbol{v}_\infty=0$. As we know from the linearized Stokes theory of Section V.3 and Section V.5, in such a case only two types of existence results with corresponding L^q-estimates are available: those of Theorem V.4.6 and those of Theorem V.5.1. Now, by those of Theorem V.4.6 we are not able to control the behavior of the solution at infinity, while the results of Theorem V.5.1 require for their validity a (necessary and sufficient) compatibility condition on the data, and we cannot use them to prove existence unless we show that such a condition is also necessary in the nonlinear case. We may then conclude that the L^q theory developed for the

[6] Recall that $\boldsymbol{v}$ is $C^\infty(\overline{\Omega^{R_0}})$.

Stokes problem, as it stands, cannot be used to obtain existence for the full nonlinear Navier–Stokes problem. In fact, existence of solutions of the exterior two-dimensional Navier–Stokes problem corresponding to $\boldsymbol{v}_\infty = 0$ remains an open question, even for small data.[1]

On the other hand, the L^q theory for the linearized Oseen problem derived in Theorem VII.5.1 does not suffer from these drawbacks and can be used, at least in principle, to show existence to the nonlinear problem when $\boldsymbol{v}_\infty \neq 0$, $\boldsymbol{v}_\infty = \boldsymbol{e}_1$, say. However, employing these estimates, together with a contraction-mapping technique, we have to face other problems. Actually, we wish to find a fixed point (in a subset X of an appropriate Banach space) of the mapping

$$L : \boldsymbol{u} \in X \to L(\boldsymbol{u}) = \boldsymbol{v} \in X$$

where $\boldsymbol{v}$ solves the problem

$$\left.\begin{aligned} &\Delta \boldsymbol{v} + \mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1} = \mathcal{R}\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\pi + \boldsymbol{f} \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\boldsymbol{v} = \boldsymbol{v}_* + \boldsymbol{e}_1 \text{ at } \partial\Omega \tag{XII.5.1}$$
$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = 0.$$

In the limit of small data, i.e., $\mathcal{R} \to 0$, as is usually requested by a contraction argument, there will be a competition between the linear term

$$\mathcal{R}\frac{\partial \boldsymbol{v}}{\partial x_1} \tag{XII.5.2}$$

and the nonlinear one

$$\mathcal{R}\boldsymbol{u}\cdot\nabla\boldsymbol{u}. \tag{XII.5.3}$$

If, in the range of vanishing $\mathcal{R}$, the contribution of the former were negligible with respect to that of the latter, it would be very unlikely to prove existence, because the linear part in (XII.5.1) would then approach the Stokes system for which, as we noticed, the procedure is not working. Fortunately, what happens is that (XII.5.2) prevails on (XII.5.3) and the machinery produces nonlinear existence. Nevertheless, the proof of this fact is by no means trivial and, in fact, it relies on the following two crucial circumstances:

(a) The validity of suitable, new a priori estimates for solutions to the linearized Oseen problem where, unlike those derived in Theorem VII.7.1 (cf. (VII.7.29)), the constant c entering the estimates themselves does not depend on $\mathcal{R} \in (0, 1]$; cf. Lemma XII.5.1. Because of the Stokes paradox, the new estimates have to be weaker than (VII.7.29); cf. Remark VII.7.2.

[1] See the Notes for this Chapter regarding some existence results when $\boldsymbol{v}_\infty = 0$.

(b) The component v_2 of the velocity field $\boldsymbol{v}$ in the solution to the Oseen problem, presents no "wake region" and, as a consequence, it has at large distances a behavior "better" than that exhibited by the component v_1.

For simplicity, we shall restrict ourselves to the case when the domain Ω is exterior to a single "body" (the closure of a bounded, simply-connected domain), leaving to the reader the task of considering more general situations. In addition, in order to render the notation less heavy, *throughout this section the Reynolds number* $\mathcal{R}$ *will be denoted by* λ.

We next introduce some other notations that will be frequently used in the sequel. For $q \in (1, 6/5]$, we indicate by $\mathcal{C}_q$ *the class of those vector functions* $\boldsymbol{u} = (u_1, u_2)$ *defined in* Ω *such that*:

$$u_2 \in L^{2q/(2-q)}(\Omega) \cap D^{1,q}(\Omega)$$

$$\boldsymbol{u} \in L^{3q/(3-2q)}(\Omega) \cap D^{2,q}(\Omega).$$

For $\boldsymbol{u} \in \mathcal{C}_q$ and $\lambda > 0$, we put

$$\langle \boldsymbol{u} \rangle_{\lambda,q} \equiv \lambda(\|u_2\|_{2q/(2-q)} + |u_2|_{1,q}) + \lambda^{2/3}\|\boldsymbol{u}\|_{3q/(3-2q)} + \lambda^{1/3}|\boldsymbol{u}|_{1,3q/(3-q)} \tag{XII.5.4}$$

and, as usual, we write $\langle \boldsymbol{u} \rangle_{\lambda,q,\Omega}$ when we need to specify the domain where the norm (XII.5.4) is defined.

Remark XII.5.1 For $\boldsymbol{u} \in \mathcal{C}_q$, it holds that

$$\lim_{|x|\to\infty} \boldsymbol{u}(x) = 0 \quad \text{uniformly.}$$

Actually, by Theorem II.6.1, $\boldsymbol{u} \in D^{1,2q/(2-q)}(\Omega)$. Since $2q/(2-q) > 2$ and $\boldsymbol{u} \in L^{3q/(3-2q)}(\Omega)$, the property follows from Theorem II.9.1. If $\boldsymbol{u}$ has only finite norm (XII.5.4), then

$$\lim_{|x|\to\infty} \int_0^{2\pi} |\boldsymbol{u}(|x|, \theta)| d\theta = 0. \tag{XII.5.5}$$

In fact, if $q \in (1, 6/5)$ then $\boldsymbol{u} \in D^{1,r}(\Omega)$ for some $r < 2$ and so from Lemma II.6.3 and the condition $\boldsymbol{u} \in L^{3q/(3-2q)}(\Omega)$ we find (XII.5.5). If $q = 6/5$, we have $\boldsymbol{u} \in D^{1,2}(\Omega) \cap L^6(\Omega)$, and we proceed as follows. Set

$$f(r) = \int_0^{2\pi} |\boldsymbol{u}(r, \theta)|^2 d\theta, \quad r = |x|.$$

Then there exists a sequence $\{r_n\} \subset \mathbb{R}_+$ accumulating at infinity such that

$$\lim_{r_n\to\infty} f(r_n) = 0.$$

For all sufficiently large r we may find $n \in \mathbb{N}$ such that

$$f(r) = f(r_n) + 2\int_{r_n}^{r}\int_0^{2\pi} \boldsymbol{u}(\rho,\theta)\cdot\frac{\partial \boldsymbol{u}(\rho,\theta)}{\partial\rho}d\rho d\theta.$$

Applying the Hölder inequality on the right-hand side of this relation, it follows that

$$|f(r)| \le |f(r_n)| + 2\|\boldsymbol{u}\|_{6,\Omega_{r_n}}|\boldsymbol{u}|_{1,2,\Omega_{r_n}}\left(\int_{r_n}^{\infty}\rho^{-2}d\rho\right)^{1/3},$$

which again proves (XII.5.5). ■

We shall now give some preparatory results. Our first objective is to prove a fundamental estimate for the linearized Oseen problem corresponding to zero body force.

Lemma XII.5.1 *Let $\Omega \subset \mathbb{R}^2$ be an exterior domain of class C^2 and let*

$$\boldsymbol{u}_* \in W^{2-1/q,q}(\partial\Omega), 1 < q \le 6/5.$$

Then, for any $\lambda > 0$ there is a unique solution to the Oseen problem

$$\left.\begin{aligned}\Delta\boldsymbol{u} + \lambda\frac{\partial\boldsymbol{u}}{\partial x_1} &= \nabla\pi\\ \nabla\cdot\boldsymbol{u} &= 0\end{aligned}\right\}\ \text{in } \Omega$$
$$\boldsymbol{u} = \boldsymbol{u}_* \text{ at } \partial\Omega \tag{XII.5.6}$$
$$\lim_{|x|\to\infty}\boldsymbol{u}(x) = 0$$

such that $\boldsymbol{u} \in \mathcal{C}_q$, $\pi \in D^{1,q}(\Omega)$. Moreover, if $q \in (1, 6/5)$ there is a $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0]$

$$\langle\boldsymbol{u}\rangle_{\lambda,q} \le c\lambda^{2(1-1/q)}|\log\lambda|^{-1}\|\boldsymbol{u}_*\|_{2-1/q,q(\partial\Omega)} \tag{XII.5.7}$$

with $c = c(\Omega, q, \lambda_0)$.

Proof. From Theorem VII.5.1 we know that there is a unique solution to problem (XII.5.6) with $\langle\boldsymbol{u}\rangle_{\lambda,q}$ finite and such that

$$\boldsymbol{u} \in D^{2,q}(\Omega^R), \quad \pi \in D^{1,q}(\Omega^R). \tag{XII.5.8}$$

On the other hand, from Theorem IV.4.1 and Theorem IV.5.1 we obtain

$$\begin{aligned}\|\boldsymbol{u}\|_{2,q,\Omega_R} + \|\pi\|_{1,q,\Omega_R} \le c_1\big(&\lambda\|D_1\boldsymbol{u}\|_{q,\Omega_{2R}} + \|\boldsymbol{u}\|_{1,q,\Omega_{2R}}\\ &+\|\pi\|_{q,\Omega_{2R}} + \|\boldsymbol{u}_*\|_{2-1/q,q(\partial\Omega)}\big).\end{aligned} \tag{XII.5.9}$$

Since, by Theorem VII.5.1 and (VII.7.14), $\boldsymbol{u}$ and π satisfy for all $R > \delta(\Omega^c)$

$$\|\boldsymbol{u}\|_{2,\Omega_R} + \|\pi\|_{2,\Omega_R} + |\boldsymbol{u}|_{1,2} \le c_2(1+\lambda)\|\boldsymbol{u}_*\|_{1/2,2(\partial\Omega)}, \tag{XII.5.10}$$

from (XII.5.9), (XII.5.10) it follows that

$$\|\boldsymbol{u}\|_{2,q,\Omega_R} + \|\pi\|_{1,q,\Omega_R} \leq c_3\big(\|\boldsymbol{u}_*\|_{2-1/q,q(\partial\Omega)} + \|\boldsymbol{u}_*\|_{1/2,2(\partial\Omega)}\big). \tag{XII.5.11}$$

However, the trace Theorem II.4.4 implies that

$$\|\boldsymbol{u}_*\|_{1/2,2(\partial\Omega)} \leq c\|\boldsymbol{u}_*\|_{2-1/q,q(\partial\Omega)}$$

and from (XII.5.11) we conclude that

$$\|\boldsymbol{u}\|_{2,q,\Omega_R} + \|\pi\|_{1,q,\Omega_R} \leq c_4\|\boldsymbol{u}_*\|_{2-1/q,q(\partial\Omega)} \tag{XII.5.12}$$

with c_4 independent of $\lambda \in (0, B]$, arbitrary $B > 0$. Thus, in particular, (XII.5.9) and (XII.5.13) tell us

$$\boldsymbol{u} \in \mathcal{C}_q, \quad \pi \in D^{1,q}(\Omega),$$

completing the proof of the first part of the lemma. We now pass to the proof of (XII.5.7). Without loss of generality, we assume that $\Omega^c \subset B_{1/2}$, with the origin of coordinates taken in $\mathring{\Omega}^c$. From the embedding Theorem II.3.4,

$$\|u_2\|_{2q/(2-q),\Omega_1} + |u_2|_{1,q,\Omega_1} + \|\boldsymbol{u}\|_{3q/(3-2q),\Omega_1} + |\boldsymbol{u}|_{1,3q/(3-q),\Omega_1} \leq c_5|\boldsymbol{u}|_{2,q,\Omega_1} \tag{XII.5.13}$$

and so, since $2(1-1/q) < 1/3$ for $q \in (1, 6/5)$, from (XII.5.12), (XII.5.13), and all $\lambda \in (0,1]$ we find for some c_6 independent of λ

$$\langle \boldsymbol{u} \rangle_{\lambda,q,\Omega_1} \leq c_6 \lambda^{2(1-1/q)+\varepsilon}\|\boldsymbol{u}_*\|_{2-1/q,q(\partial\Omega)} \tag{XII.5.14}$$

where $\varepsilon = 1/3 - 2(1-1/q) > 0$. Let us denote by $\boldsymbol{E}(x;\lambda)$ the Oseen tensor corresponding to the Reynolds number λ. From Exercise VII.3.5 we know that $\boldsymbol{E}$ obeys the homogeneity condition

$$\boldsymbol{E}(x;\lambda) = \boldsymbol{E}(\lambda x; 1). \tag{XII.5.15}$$

Moreover, from the representation Theorem VII.6.2 we deduce for $j = 1, 2$ that

$$\begin{aligned} u_j = &-\boldsymbol{\mathcal{T}}(\boldsymbol{u}) \cdot \boldsymbol{E}_j(x;\lambda) \\ &+ \int_{\partial\Omega} \boldsymbol{u}_*(z) \cdot \boldsymbol{T}[\boldsymbol{E}_j(x-z;\lambda), e_j(x-z)] \cdot \boldsymbol{n}(z) d\sigma_z \\ &- \lambda \int_{\partial\Omega} \boldsymbol{u}_*(z) \cdot \boldsymbol{E}_j(x-z;\lambda) n_1(z) d\sigma_z \\ &- \int_{\partial\Omega} [\boldsymbol{E}_j(x-z;\lambda) - \boldsymbol{E}_j(x;\lambda)] \cdot \boldsymbol{T}(\boldsymbol{u},\pi) \cdot \boldsymbol{n}(z) d\sigma_z \end{aligned} \tag{XII.5.16}$$

with

$$\boldsymbol{\mathcal{T}}(\boldsymbol{u}) = \int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{u},\pi)\cdot\boldsymbol{n}$$

and

$$\begin{aligned}\boldsymbol{E}_1 &= (E_{11}, E_{12})\\ \boldsymbol{E}_2 &= (E_{12}, E_{22}).\end{aligned} \tag{XII.5.17}$$

Recalling that $\Omega^c \subset B_{1/2}$ from (XII.5.16) and the mean value theorem applied to $D^\alpha(\boldsymbol{E}_j(x-z;\lambda) - \boldsymbol{E}_j(x;\lambda))$, $\alpha = 0,1$, we derive for all $x \in \Omega^1$

$$\begin{aligned}|u_j(x)| \le\, & |\boldsymbol{\mathcal{T}}(\boldsymbol{u})||\boldsymbol{E}_j(x;\lambda)|\\ & +D\{\lambda \sup_{z\in\Omega_{1/2}} |\boldsymbol{E}_j(x-z;\lambda)|\\ & + \sup_{z\in\Omega_{1/2}} [|\boldsymbol{e}(x-z)| + |\nabla_x \boldsymbol{E}_j(x-z;\lambda)|]\}\\ |\nabla u_j(x)| \le\, & |\boldsymbol{\mathcal{T}}(\boldsymbol{u})||\nabla\boldsymbol{E}_j(x;\lambda)| + D\{\lambda \sup_{z\in\Omega_{1/2}} |\nabla_x\boldsymbol{E}_j(x-z;\lambda)|\\ & + \sup_{z\in\Omega_{1/2}} [|\nabla_x\boldsymbol{e}(x-z)| + |D_x^2\boldsymbol{E}_j(x-z;\lambda)|]\}\end{aligned} \tag{XII.5.18}$$

where

$$D = \|\nabla\boldsymbol{u}\|_{1,\partial\Omega} + \|\pi\|_{1,\partial\Omega} + \|\boldsymbol{u}_*\|_{1,\partial\Omega}. \tag{XII.5.19}$$

Taking into account (XII.5.15), from (XII.5.18)$_1$ we find, with $y = \lambda x$,

$$\begin{aligned}|u_j(x)| &\le |\boldsymbol{\mathcal{T}}(\boldsymbol{u})||\boldsymbol{E}_j(y;1)| + \lambda D\{\sup_{z\in\Omega_{1/2}} [|\boldsymbol{E}_j(y-\lambda z;1)|\\ &\qquad + |\boldsymbol{e}(y-\lambda z)| + |\nabla_y\boldsymbol{E}_j(y-\lambda z;1)|]\}\\ &\le |\boldsymbol{\mathcal{T}}(\boldsymbol{u})||\boldsymbol{E}_j(y;1)| + \lambda D\{\sup_{|z|\le\lambda/2} [|\boldsymbol{E}_j(y-z;1)|\\ &\qquad + |\boldsymbol{e}(y-z)| + |\nabla_y\boldsymbol{E}_j(y-z;1)|]\}\end{aligned}$$

and so

$$\begin{aligned}\|u_j\|^t_{t,\Omega^1} \le\, & 2^t\lambda^{-2}\Big\{|\boldsymbol{\mathcal{T}}(\boldsymbol{u})|^t\|\boldsymbol{E}_j(y;1)\|^t_{t,\mathbb{R}^2}\\ & + \lambda^t D^t \int_{|y|\ge\lambda} \{\sup\nolimits_{|z|\le\lambda/2} [|\boldsymbol{E}_j(y-z;1)|\\ & + |\boldsymbol{e}(y-z)| + |\nabla_y\boldsymbol{E}_j(y-z;1)|]\}^t dy\Big\}.\end{aligned} \tag{XII.5.20}$$

To estimate the integral on the right-hand side of (XII.5.20) we observe that

$$\begin{aligned}\int_{|y|\ge\lambda} \sup_{|z|\le\lambda/2} |\mathbf{E}_j(y-z;1)|^t dy \le\, & \int_{2\ge|y|\ge\lambda} \sup_{|z|\le\lambda/2} |\mathbf{E}_j(y-z;1)|^t dy\\ & + \int_{|y|\ge 2} \sup_{|z|\le 1/2} |\boldsymbol{E}_j(y-z;1)|^t dy.\end{aligned} \tag{XII.5.21}$$

From (VII.3.36) we have

$$|\boldsymbol{E}(y-z;1)| \le c(|\log|y-z|| + 1), \ \ |y|, |z| \le 2$$

and since

$$|z| \le \lambda/2, |y| \ge \lambda \ \text{ implies } \ |y-z| \ge \tfrac{1}{2}|y|, \tag{XII.5.22}$$

it follows that

$$\int_{2\ge|y|\ge\lambda} \sup_{|z|\le\lambda/2} |\boldsymbol{E}_j(y-z;1)|^t dy \le c_7 \tag{XII.5.23}$$

with c_7 independent of $\lambda \in (0,1]$. Furthermore, since by the mean value theorem,

$$|(E_{ij}(y-z) - E_{ij}(y))| = |z_l \frac{\partial E_{ij}(y-\beta z)}{\partial z_l}|, \quad \beta \in (0,1),$$

from (VII.3.37) and (VII.3.45) it follows for all $|y|$ sufficiently large ($\ge |y_0|$, say) that

$$\int_{|y|\ge|y_0|} \sup_{|z|\le 1/2} |\boldsymbol{E}_j(y-z;1)|^t dy \le c \int_{|y|\ge|y_0|} (|\boldsymbol{E}_j(y;1)|^t + |y|^{-tr_j}) dy$$

where

$$r_j = \begin{cases} 1 & \text{if } j=1 \\ 3/2 & \text{if } j=2. \end{cases}$$

From the local regularity of $\boldsymbol{E}$, and from (VII.3.42), (VII.3.43) we conclude that

$$\int_{|y|\ge 2} \sup_{|z|\le 1/2} |\boldsymbol{E}_j(y-z;1)|^t dy \le c_8 \tag{XII.5.24}$$

with c_8 independent of $\lambda \in (0,1]$ for all values of t such that

$$\boldsymbol{E}_j(y;1) \in L^t(|y| \ge |y_0|). \tag{XII.5.25}$$

Collecting (XII.5.21), (XII.5.23), and (XII.5.24), it follows that

$$\int_{|y|\ge\lambda} \sup_{|z|\le\lambda/2} |\boldsymbol{E}_j(y-z;1)|^t dy \le c_9 \tag{XII.5.26}$$

with c_9 independent of $\lambda \in (0,1]$ and for all values of t for which (XII.5.25) holds. In addition, from (VII.3.17) and (XII.5.22) we infer for all $t > 2$ that

$$\begin{aligned} \int_{|y|\ge\lambda} \sup_{|z|\le\lambda/2} |\boldsymbol{e}(y-z;1)|^t dy &\le c_{10}\Big\{\int_{2\ge|y|\ge\lambda} |y|^{-t} dy \\ &\quad + \int_{|y|\ge 2} \sup_{|z|\le 1/2} |y-z|^{-t} dy\Big\} \\ &\le c_{11}(\lambda^{2-t}+1). \end{aligned} \tag{XII.5.27}$$

Likewise,

$$\int_{|y|\geq\lambda} \sup_{|z|\leq\lambda/2} |\nabla_y \boldsymbol{E}_j(y-z;1)|^t dy \leq$$
$$\int_{2\geq|y|\geq\lambda} \sup_{|z|\leq\lambda/2} |\nabla_y \boldsymbol{E}_j(y-z;1)|^t dy + \int_{|y|\geq 2} \sup_{|z|\leq 1/2} |\nabla_y \boldsymbol{E}_j(y-z;1)|^t dy. \tag{XII.5.28}$$

From (VII.3.36) we have

$$|\nabla \boldsymbol{E}(y-z;1)| \leq c_{12}|y-z|^{-1}, \quad |y|,|z| \leq 2$$

and so, by (XII.5.22), it follows that

$$\int_{2\geq|y|\geq\lambda} \sup_{|z|\leq\lambda/2} |\nabla_y \mathbf{E}_j(y-z;1)|^t dy \leq c_{13} \int_{2\geq|y|\geq\lambda} |y|^{-t} dy \leq c_{14}(1+\lambda^{2-t}). \tag{XII.5.29}$$

Also, employing the asymptotic properties of $\nabla \boldsymbol{E}_j(y;1)$ (cf. (VII.3.46) and (VII.3.47)) together with the following ones on the second derivatives,

$$|D^2 \boldsymbol{E}_j(y)| \leq c|y|^{-s_j}$$

where[2]

$$s_j = \begin{cases} 3/2 \text{ for } j=1 \\ 2 \quad\ \text{ for } j=2 \end{cases}$$

we are able to establish, exactly as we did for (XII.5.24), the following estimate:

$$\int_{|y|\geq 2} \sup_{|z|\leq 1/2} |\nabla_y \boldsymbol{E}_j(x-z;1)|^t dy \leq c_{15} \tag{XII.5.30}$$

for those values of t such that

$$\nabla \boldsymbol{E}_j(y;1) \in L^t(|y| \geq |y_0|). \tag{XII.5.31}$$

Thus, from (XII.5.28)–(XII.5.30) we recover

$$\int_{|y|\geq\lambda} \sup_{|z|\leq\lambda/2} |\nabla_y \boldsymbol{E}_j(y-z;1)|^t dy \leq c_{16}(1+\lambda^{2-t}), \tag{XII.5.32}$$

with c_{16} independent of λ and t satisfying (XII.5.31). From (XII.5.20), (XII.5.26), (XII.5.27), and (XII.5.32) we find

$$\|u_j\|_{t,\Omega^1} \leq c_{17} \left(\lambda^{-2/t}|\boldsymbol{T}(\boldsymbol{u}))| + (1+\lambda^{1-2/t})D\right)$$

[2] These bounds on $D^2\boldsymbol{E}_j$ are obtained directly by a formal differentiation of (VII.3.37).

for all $\lambda \in (0,1]$, for all t for which (XII.5.25) and (XII.5.31) hold, and with a constant c_{17} independent of λ. Since $t > 2$, from this latter inequality, we deduce that

$$\|u_j\|_{t,\Omega^1} \le 2c_{17}\left(\lambda^{-2/t}|\boldsymbol{\mathcal{T}}(\boldsymbol{u}))| + D\right). \tag{XII.5.33}$$

Estimate (XII.5.33) furnishes, in particular,

$$\lambda\|u_2\|_{2q/(2-q),\Omega^1} + \lambda^{2/3}\|\boldsymbol{u}\|_{3q/(3-2q),\Omega^1} \le c\left(\lambda^{2(1-1/q)}|\boldsymbol{\mathcal{T}}(\boldsymbol{u}))| + \lambda^{2/3}D\right) \tag{XII.5.34}$$

with c independent of $\lambda \in (0,1]$.[3] In a completely analogous way, starting with (XII.5.18)$_2$ we show

$$\begin{aligned}|u_j|^\tau_{1,\tau,\Omega^1} \le 2^\tau\lambda^{\tau-2}\Big\{&|\boldsymbol{\mathcal{T}}(\boldsymbol{u})|^\tau\|\nabla\boldsymbol{E}_j(y;1)\|^\tau_{\tau,\mathbb{R}^2}\\ &+\lambda^\tau D^\tau\int_{|y|\ge\lambda}\{\sup_{|z|\le\lambda/2}[|\nabla_y\boldsymbol{E}_j(y-z;1)|\\ &+|\nabla_y\boldsymbol{e}(y-z)|+|D^2_y\boldsymbol{E}_j(y-z;1)|]\}^\tau dy\Big\}.\end{aligned}$$

Therefore, noting that, by (VII.3.17),

$$|\nabla_y\boldsymbol{e}(y-z)| \le c_{18}|y-z|^{-2},$$

and that by (VII.3.21),

$$\begin{aligned}\int_{2\ge|y|\ge\lambda}\sup_{|z|\le\lambda/2}|D^2_y\boldsymbol{E}_j(y-z;1)|^\tau dy &\le c_{19}\int_{2\ge|y|\ge\lambda}|y|^{-2\tau}dy\\ &\le c_{20}(1+\lambda^{2(1-\tau)}),\end{aligned}$$

using the same procedure employed to obtain (XII.5.33), we arrive at

$$|u_j|_{1,\tau,\Omega^1} \le c_{21}\left(\lambda^{1-2/\tau}|\boldsymbol{\mathcal{T}}(\boldsymbol{u})| + D\right) \tag{XII.5.35}$$

for all values of $\lambda \in (0,1]$ and all values of τ such that

$$\begin{gathered}\nabla\boldsymbol{E}_j(y;1) \in L^\tau(|y| \ge |y_0|),\\ D^2\boldsymbol{E}_j \in L^\tau(|y| \ge |y_0|).\end{gathered}$$

Thus, from (VII.3.46), (VII.3.49) and (XII.5.35), and observing that $\tau > 1$, we derive, in particular,

$$\lambda|u_2|_{1,q,\Omega^1} + \lambda^{1/3}|\boldsymbol{u}|_{1,3q/(3-q),\Omega^1} \le c\left(\lambda^{2(1-1/q)}|\boldsymbol{\mathcal{T}}(\boldsymbol{u})| + \lambda^{1/3}D\right) \tag{XII.5.36}$$

[3] Observe that $\lambda \le \lambda^{2/3}$ for $0 \le \lambda \le 1$.

for some $c = c(\Omega, q)$ and with $q \in (1, 6/5)$. The next step is to increase D (defined in (XII.5.19)) in terms of the boundary norm $\|\boldsymbol{u}_*\|_{2-1/q,q(\partial\Omega)}$. Using the trace Theorem II.4.4 we have

$$D \le c_{22} \left(\|\boldsymbol{u}\|_{2,q,\Omega_1} + \|\pi\|_{1,q,\Omega_1} + \|\boldsymbol{u}_*\|_{2-1/q,q(\partial\Omega)} \right)$$

and so this inequality, together with (XII.5.12), implies

$$D \le c_{23} \|\boldsymbol{u}_*\|_{2-1/q,q(\partial\Omega)}.$$

Observing that $1/3 > 2(1 - 1/q)$ for $q \in (1, 6/5)$, from (XII.5.14), (XII.5.34), and (XII.5.36) we obtain

$$\langle \boldsymbol{u} \rangle_{\lambda,q} \le c_{24} \lambda^{2(1-1/q)} \left(|\boldsymbol{\mathcal{T}}(\boldsymbol{u})| + \lambda^{\varepsilon} \|\boldsymbol{u}_*\|_{2-1/q,q(\partial\Omega)} \right) \tag{XII.5.37}$$

with $\varepsilon = 1/3 - 2(1 - 1/q) > 0$. From Theorem VII.8.1 we know that there is a $\lambda_1 > 0$ such that

$$|\boldsymbol{\mathcal{T}}(\boldsymbol{u})| \le c_{25} |\log \lambda|^{-1} \|\boldsymbol{u}_*\|_{2-1/q,q(\partial\Omega)}$$

for all $\lambda \in (0, \lambda_1]$ and with $c_{25} = c_{25}(\Omega, q, \lambda_1)$. Thus, replacing this estimate in (XII.5.37), we recover (XII.5.7) with $\lambda_0 = \min\{1, \lambda_1\}$, and the lemma is proved. $\square$

Remark XII.5.2 By using the same line of proof, one can show that if $q = 6/5$, the solutions of the previous lemma satisfy the weaker estimate

$$\langle \boldsymbol{u} \rangle_{\lambda,q} \le c \lambda^{2(1-1/q)} \|\boldsymbol{u}_*\|_{2-1/q,q(\partial\Omega)}$$

for all $\lambda \in (0, 1]$. $\blacksquare$

The next result shows an estimate for solutions to the Oseen problem with zero boundary data. It is somehow weaker than that shown in Theorem VII.7.1, cf. (VII.7.29), but with the advantage that, in the present case, the constant c that enters the estimate can be rendered independent of $\lambda \in (0, 1]$.

Lemma XII.5.2 *Let Ω be as in* Lemma XII.5.1. *Then, given*

$$\boldsymbol{f} \in L^q(\Omega), \quad 1 < q < 6/5,$$

there is at least one solution to the Oseen problem:

$$\left.\begin{aligned} &\Delta \boldsymbol{w} + \lambda \frac{\partial \boldsymbol{w}}{\partial x_1} = \nabla \tau + \boldsymbol{f} \\ &\nabla \cdot \boldsymbol{w} = 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\boldsymbol{w} = 0 \text{ at } \partial\Omega$$
$$\lim_{|x|\to\infty} \boldsymbol{w}(x) = 0 \tag{XII.5.38}$$

such that

$$\boldsymbol{w} \in \mathcal{C}_q, \quad \tau \in D^{1,q}(\Omega).$$

This solution satisfies the estimate

$$\langle \boldsymbol{w} \rangle_{\lambda,q} \leq c \|\boldsymbol{f}\|_q \tag{XII.5.39}$$

for all $\lambda \in (0, \lambda_0]$ (λ_0 *given in* Lemma XII.5.1) *and with* $c = c(\Omega, q, \lambda_0)$. *Moreover, if* $\boldsymbol{w}_1$, τ_1 *is another solution to* (XII.5.38) *corresponding to the same data and with* $\langle \boldsymbol{w}_1 \rangle_{\lambda,q}$ *finite,*[4] *then* $\boldsymbol{w} = \boldsymbol{w}_1$.

Proof. Extend $\boldsymbol{f}$ to zero outside Ω and continue to denote by $\boldsymbol{f}$ this extension. We look for a solution $\boldsymbol{w}$, τ of the form

$$\boldsymbol{w} = \boldsymbol{v} + \boldsymbol{u}, \quad \tau = p + \pi \tag{XII.5.40}$$

where

$$\left.\begin{aligned} &\Delta \boldsymbol{v} + \lambda \frac{\partial \boldsymbol{v}}{\partial x_1} = \nabla p + \boldsymbol{f} \\ &\nabla \cdot \boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^2$$

while $\boldsymbol{u}$, π solve problem (XII.5.6) with $\boldsymbol{u}_* = -\boldsymbol{v}$ at $\partial\Omega$. From Theorem VII.4.1 we know that there is a unique solution $\boldsymbol{v}$, p satisfying the estimate

$$\langle \boldsymbol{v} \rangle_{\lambda,q} + |\boldsymbol{v}|_{2,q} + |p|_{1,q} \leq c_1 \|\boldsymbol{f}\|_q \tag{XII.5.41}$$

with c_1 independent of $\lambda \in (0, 1]$. We need another estimate for a suitable norm of $\boldsymbol{v}$ on the unit ball B centered at the origin. To this end, we write $\boldsymbol{v}$ as the Oseen volume potential $(\text{VII.3.50})_1$[5]

$$\boldsymbol{v}(x) = \int_{\mathbb{R}^2} \boldsymbol{E}(x-y; \lambda) \cdot \boldsymbol{f}(y) dy = \int_{\mathbb{R}^2} \boldsymbol{E}(\lambda(x-y); 1) \cdot \boldsymbol{f}(y) dy$$

where the homogeneity property (XII.5.15) has been employed. By the Hölder inequality we have (with $q' = q/(q-1)$)

$$|\boldsymbol{v}(x)|^{q'} \leq \left(\int_{\mathbb{R}^2} |\boldsymbol{E}(\lambda(x-y); 1)|^{q'} dy \right) \|\boldsymbol{f}\|_q^{q'}.$$

Since

$$\int_{\mathbb{R}^2} |\boldsymbol{E}(\lambda(x-y); 1)|^{q'} dy \leq \lambda^{-2} \int_{\mathbb{R}^2} |\boldsymbol{E}(z; 1)|^{q'} dz = c\lambda^{-2},$$

it follows that

$$\|\boldsymbol{v}\|_{q',B} \leq 2\pi c \lambda^{-2(1-1/q)} \|\boldsymbol{f}\|_q, \tag{XII.5.42}$$

[4] Of course, $\boldsymbol{w}_1$ and τ_1 satisfy (XII.5.38) in the sense of Definition VII.1.1(v).

[5] This is easily shown by a standard approximating procedure that starts with $\boldsymbol{f} \in C_0^\infty(\Omega)$ and uses (XII.5.41).

which is the inequality we wanted to show. We now pass to the solution $\boldsymbol{u}$, π. From Lemma XII.5.1 we know that

$$\boldsymbol{u} \in \mathcal{C}_q, \quad \pi \in D^{1,q}(\Omega)$$

and that for all $\lambda \in (0, \lambda_0]$

$$\langle \boldsymbol{u} \rangle_{\lambda,q} \leq c_2 \lambda^{2(1-1/q)} \|\boldsymbol{v}\|_{2-1/q,q(\partial\Omega)}. \tag{XII.5.43}$$

Our task is to increase the right-hand side of (XII.5.43) in terms of $\boldsymbol{f}$. To this end, we observe that by the trace Theorem II.4.4, the Ehrling inequality (II.5.20), and the Hölder inequality,[6]

$$\|\boldsymbol{v}\|_{2-1/q,q(\partial\Omega)} \leq c_3 \|\boldsymbol{v}\|_{2,q,\Omega_1} \leq c_4 \left(\|\boldsymbol{v}\|_{q,\Omega_1} + |\boldsymbol{v}|_{2,q,\Omega_1} \right)$$

with c_5 depending only on Ω. Since $q' > q$, from this latter inequality and (XII.5.41)–(XII.5.43) we conclude, for all $\lambda \in (0, \lambda_0]$, that

$$\langle \boldsymbol{u} \rangle_{\lambda,q} \leq c_5 \|\boldsymbol{f}\|_q \tag{XII.5.44}$$

with $c_6 = c_6(\Omega, q, \lambda_0)$. Estimate (XII.5.39) then becomes a consequence of (XII.5.40), (XII.5.41), and (XII.5.44). To show the result completely we have to prove the uniqueness part. However, this follows from Exercise VII.6.2 once we notice that $\boldsymbol{w}$ and $\boldsymbol{w}_1$ are $3q/(3-q)$-generalized solutions to the Oseen problem corresponding to the same data (cf. Remark XII.5.1). □

Remark XII.5.3 Lemma XII.5.2 can be extended to the case $q = 6/5$. In such a situation we may take $\lambda_0 = 1$.

Combining Lemma XII.5.1 and Lemma XII.5.2, we immediately obtain the following result. ■

Lemma XII.5.3 *Let Ω be as in* Lemma XII.5.1. *Then, given*

$$\boldsymbol{f} \in L^q(\Omega), \quad \boldsymbol{u}_* \in W^{2-1/q,q}(\partial\Omega), \quad 1 < q < 6/5,$$

there is a unique solution to the Oseen problem

$$\left.\begin{aligned} &\Delta \boldsymbol{u} + \lambda \frac{\partial \boldsymbol{u}}{\partial x_1} = \nabla \pi + \boldsymbol{f} \\ &\nabla \cdot \boldsymbol{u} = 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\boldsymbol{u} = \boldsymbol{u}_* \text{ at } \partial\Omega$$
$$\lim_{|x|\to\infty} \boldsymbol{u}(x) = 0$$

such that $\boldsymbol{u} \in \mathcal{C}_q$, $\pi \in D^{1,q}$. Moreover, there is a $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0]$, this solution satisfies

[6] We may assume, without loss, that Ω^c is contained in $B_{1/2}$.

$$\langle \boldsymbol{u} \rangle_{\lambda,q} \le c \left[\lambda^{2(1-1/q)} |\log \lambda|^{-1} \|\boldsymbol{u}_*\|_{2-1/q,q(\partial\Omega)} + \|\boldsymbol{f}\|_q \right],$$

with $c = c(\Omega, q, \lambda_0)$.

Finally, we need the following result concerning functions having a finite $\langle \cdot \rangle_{\lambda,q}$-norm.

Lemma XII.5.4 *Let* Ω *be an arbitrary domain in* $\mathbb{R}^2$ *and let* $\boldsymbol{v}$, $\boldsymbol{w}$ *be two solenoidal vector functions in* Ω *for which the norm* (XII.5.4), *with* $1 < q \le 6/5$, *is finite. Then the following inequality holds for all* $\lambda > 0$

$$\|\boldsymbol{v} \cdot \nabla \boldsymbol{w}\|_q \le 4\lambda^{-1-2(1-1/q)} \langle \boldsymbol{v} \rangle_{\lambda,q} \langle \boldsymbol{w} \rangle_{\lambda,q}.$$

Proof. Taking into account that $\boldsymbol{v}$ and $\boldsymbol{w}$ are divergence-free, we obtain

$$\boldsymbol{v} \cdot \nabla \boldsymbol{w} = \left(-v_1 \frac{\partial w_2}{\partial x_2} + v_2 \frac{\partial w_1}{\partial x_2} \right) \boldsymbol{e}_1 + \left(-v_1 \frac{\partial w_2}{\partial x_1} + v_2 \frac{\partial w_2}{\partial x_2} \right) \boldsymbol{e}_2$$

and so, by the Hölder inequality and (XII.5.4),

$$\begin{aligned} \|\boldsymbol{v} \cdot \nabla \boldsymbol{w}\|_q &\le \left(\|v_1\|_{3q/(3-2q)} |w_2|_{1,3/2} + \|v_2\|_3 |\boldsymbol{w}|_{1,3q/(3-q)} \right) \\ &\le \left(\lambda^{-2/3} |w_2|_{1,3/2} \langle \boldsymbol{v} \rangle_{\lambda,q} + \lambda^{-1/3} \|v_2\|_3 \langle \boldsymbol{w} \rangle_{\lambda,q} \right). \end{aligned} \tag{XII.5.45}$$

From the interpolation inequality (II.2.10) we find (with $q' = q/(q-1)$) that

$$|w_2|_{1,3/2} \le |w_2|_{1,q}^{3/q'} |w_2|_{1,3q/(3-q)}^{1-3/q'} \le \lambda^{-2/q'-1/3} \langle \boldsymbol{w} \rangle_{\lambda,q}$$

$$\|v_2\|_3 \le \|v_2\|_{2q/(2-q)}^{6/q'} \|\boldsymbol{v}\|_{3q/(3-2q)}^{1-6/q'} \le \lambda^{-2/q'-2/3} \langle \boldsymbol{v} \rangle_{\lambda,q}$$

and the lemma becomes a consequence of this relation and (XII.5.45). □

We are now in a position to prove an existence and uniqueness result for the nonlinear problem. We shall prove it when Ω is an exterior domain, leaving to the reader the (much simpler) case where $\Omega = \mathbb{R}^2$ in Exercise XII.5.1.

Theorem XII.5.1 *Let* $\Omega \subset \mathbb{R}^2$ *be an exterior domain of class* C^2 *and let*

$$\boldsymbol{f} \in L^q(\Omega), \quad \boldsymbol{v}_* \in W^{2-1/q,q}(\partial\Omega), \quad \boldsymbol{v}_\infty = \boldsymbol{e}_1$$

with $q \in (1, 6/5)$. *There exists a positive constant* $\lambda_0 > 0$ *such that if for some* $\lambda \in (0, \lambda_0]$,

$$|\log \lambda|^{-1} \|\boldsymbol{v}_* + \boldsymbol{e}_1\|_{2-1/q,q(\partial\Omega)} + \lambda^{2/q-1} \|\boldsymbol{f}\|_q < 1/32c^2, \tag{XII.5.46}$$

with c *given in* Lemma XII.5.3, *then problem* (XII.0.1), (XII.0.2) *(with* $\mathcal{R} \equiv \lambda$*) has at least one solution* $\boldsymbol{v}$, p *such that*

$$
\begin{aligned}
v_2 &\in L^{2q/(2-q)}(\Omega) \cap D^{1,q}(\Omega)\\
\boldsymbol{v} + \boldsymbol{e}_1 &\in L^{3q/(3-2q)}(\Omega) \cap D^{1,3q/(3-q)}(\Omega) \cap D^{2,q}(\Omega)\\
p &\in D^{1,q}(\Omega).
\end{aligned}
$$

Furthermore, this solution satisfies the estimate

$$
\langle \boldsymbol{v} + \boldsymbol{e}_1 \rangle_{\lambda,q} \le c^2 \left(\lambda^{2(1-1/q)} |\log \lambda|^{-1} \|\boldsymbol{v}_* - \boldsymbol{e}_1\|_{2-1/q,q(\partial\Omega)} + \lambda \|\boldsymbol{f}\|_q \right) \tag{XII.5.47}
$$

where $\langle \cdot \rangle_{\lambda,q}$ *is defined in* (XII.5.4). *Finally, if* $\boldsymbol{v}_1, p_1$ *is another solution to problem* (XII.0.1), (XII.0.2) *corresponding to the same data such that*

$$
c\lambda^{-2(1-1/q)} \langle \boldsymbol{v}_1 + \boldsymbol{e}_1 \rangle_{\lambda,q} < 13/64, \tag{XII.5.48}
$$

then $\boldsymbol{v} \equiv \boldsymbol{v}_1$ *and* $p \equiv p_1 + \text{const.}$

Proof. We look for a solution $\boldsymbol{v}$ to (XII.0.1), (XII.0.2) of the form

$$
\boldsymbol{v} = \boldsymbol{u}_0 + \lambda^{2(1-1/q)} \boldsymbol{u} + \boldsymbol{e}_1, \qquad p = p_0 + \lambda^{2(1-1/q)} \pi, \tag{XII.5.49}
$$

where

$$
\begin{gathered}
\left.
\begin{aligned}
&\Delta \boldsymbol{u}_0 + \lambda \frac{\partial \boldsymbol{u}_0}{\partial x_1} = \nabla p_0 + \lambda \boldsymbol{f}\\
&\nabla \cdot \boldsymbol{u}_0 = 0
\end{aligned}
\right\} \text{ in } \Omega\\
\boldsymbol{u}_0 = \boldsymbol{v}_* + \boldsymbol{e}_1 \text{ at } \partial\Omega\\
\lim_{|x|\to\infty} \boldsymbol{u}(x) = 0
\end{gathered} \tag{XII.5.50}
$$

and

$$
\begin{gathered}
\left.
\begin{aligned}
&\Delta \boldsymbol{u} + \lambda \frac{\partial \boldsymbol{u}}{\partial x_1} = \lambda [\lambda^{2(1-1/q)} \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \lambda^{2(1-1/q)} \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_0\\
&\qquad\qquad\qquad + \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}_0] + \nabla \pi\\
&\nabla \cdot \boldsymbol{u} = 0
\end{aligned}
\right\} \text{ in } \Omega\\
\boldsymbol{u} = 0 \text{ at } \partial\Omega\\
\lim_{|x|\to\infty} \boldsymbol{u}(x) = 0.
\end{gathered} \tag{XII.5.51}
$$

A solution to (XII.5.50) is determined by Lemma XII.5.3. For all $\lambda \in (0, \lambda_0]$ such a solution obeys the estimate

$$
\langle \boldsymbol{u}_0 \rangle_{\lambda,q} \le c \left(\lambda^{2(1-1/q)} |\log \lambda|^{-1} \|\boldsymbol{u}_*\|_{2-1/q,q(\partial\Omega)} + \lambda \|\boldsymbol{f}\|_q \right) \equiv D. \tag{XII.5.52}
$$

A solution to (XII.5.51) is likewise obtained from Lemma XII.5.1 with the help of the contraction-mapping theorem. To this end, for $\lambda \in (0, \lambda_0]$, set

$$\mathcal{B}_{\lambda,q} = \left\{ \boldsymbol{w} : \Omega \to \mathbb{R}^2 : \langle \boldsymbol{w} \rangle_{\lambda,q} < \infty;\ \nabla \cdot \boldsymbol{w} = 0 \text{ in } \ \Omega;\ \boldsymbol{w} = 0 \text{ at } \partial\Omega \right\}.$$

Clearly, $\mathcal{B}_{\lambda,q}$ is a Banach space endowed with the norm $\langle \cdot \rangle_{\lambda,q}$. For $\delta > 0$, let

$$X_{\lambda,q,\delta} = \{ \boldsymbol{w} \in \mathcal{B}_{\lambda,q} :\ \langle \boldsymbol{w} \rangle_{\lambda,q} \le \delta \}.$$

Consider the mapping

$$L : \boldsymbol{w} \in X_{\lambda,q,\delta} \to L(\boldsymbol{w}) = \boldsymbol{u} \in \mathcal{B}_{\lambda,q}, \tag{XII.5.53}$$

where $\boldsymbol{u}$ solves

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) = -\lambda(\frac{\partial \boldsymbol{u}}{\partial x_1}, \nabla \boldsymbol{\varphi}) - (\mathcal{F}(\boldsymbol{w}), \boldsymbol{\varphi}) \tag{XII.5.54}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$, with

$$\mathcal{F}(\boldsymbol{w}) = \lambda \left(\lambda^{2(1-1/q)} \boldsymbol{w} \cdot \nabla \boldsymbol{w} + \lambda^{-2(1-1/q)} \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_0 + \boldsymbol{u}_0 \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{u}_0 \right). \tag{XII.5.55}$$

By Lemma XII.5.4 it follows that

$$\|\mathcal{F}(\boldsymbol{w})\|_q < \infty,$$

and so, by Lemma XII.5.2, we derive that the mapping (XII.5.53) is well defined and that the solution $\boldsymbol{u}$ verifies

$$\boldsymbol{u} \in D^{2,q}(\Omega), \quad \pi \in D^{1,q}(\Omega) \tag{XII.5.56}$$

where π is the pressure field associated to $\boldsymbol{u}$ by Lemma VII.1.1. Thus, in particular, $\boldsymbol{u}(x)$ tends to zero uniformly as $|x| \to \infty$; cf. Remark XII.5.1. We next show that, given $\lambda \in (0, \lambda_0]$ there is a $\delta = \delta(\lambda) > 0$ such that L is a contraction in $X_{\lambda,q,\delta}$, provided the data obey condition (XII.5.46). From Lemma XII.5.4, (XII.5.52), and (XII.5.55) we find

$$\|\mathcal{F}(\boldsymbol{w})\|_q \le 4 \left(\langle \boldsymbol{w} \rangle_{\lambda,q}^2 + \lambda^{-4(1-1/q)} D^2 + 2\lambda^{-2(1-1/q)} D \langle \boldsymbol{w} \rangle_{\lambda,q} \right)$$

and therefore, in view of Lemma XII.5.3, we obtain that for any $\boldsymbol{w} \in X_{\lambda,q,\delta}$ there is a unique solution $\boldsymbol{u}$ to (XII.5.54)–(XII.5.55) satisfying (XII.5.56) such that

$$\langle \boldsymbol{u} \rangle_{\lambda,q} = \langle L(\boldsymbol{w}) \rangle_{\lambda,q} \le 4c \left(\langle \boldsymbol{w} \rangle_{\lambda,q}^2 + \lambda^{-4(1-1/q)} D^2 + 2\lambda^{-2(1-1/q)} D \langle \boldsymbol{w} \rangle_{\lambda,q} \right). \tag{XII.5.57}$$

Since $\langle \boldsymbol{w} \rangle_{\lambda,q} \le \delta$, the preceding inequality yields

$$\langle L(\boldsymbol{w}) \rangle_{\lambda,q} \le 4c \left(\delta^2 + \lambda^{-4(1-1/q)} D^2 + 2\lambda^{-2(1-1/q)} D\delta \right). \tag{XII.5.58}$$

Thus, taking (for instance)

$$\delta \equiv \lambda^{-2(1-1/q)} D < (32c)^{-1}, \tag{XII.5.59}$$

from (XII.5.57) it follows for all $\lambda \in (0, \lambda_0]$ that

$$\langle L(\boldsymbol{w})\rangle_{\lambda,q} \le 4c \cdot \frac{\delta}{8c} = \frac{\delta}{2}, \tag{XII.5.60}$$

which furnishes that the mapping (XII.5.53) transforms $X_{\lambda,q,\delta}$ into itself, with δ given in (XII.5.59). It is now simple to show that, in fact, L is a contraction in $X_{\lambda,q,\delta}$. Actually, performing the same kind of reasoning leading to (XII.5.57), for any $\boldsymbol{w}_1$, $\boldsymbol{w}_2 \in X_{\lambda,q,\delta}$ we show

$$\begin{aligned}\langle L(\boldsymbol{w}_1) - L(\boldsymbol{w}_2)\rangle_{\lambda,q} \le 4c[2\lambda^{-2(1-1/q)}\langle \boldsymbol{u}_0\rangle_{\lambda,q}\langle \boldsymbol{w}_1 - \boldsymbol{w}_2\rangle_{\lambda,q} \\ + \left(\langle \boldsymbol{w}_1\rangle_{\lambda,q} + \langle \boldsymbol{w}_2\rangle_{\lambda,q}\right)\langle \boldsymbol{w}_1 - \boldsymbol{w}_2\rangle_{\lambda,q}]\end{aligned}$$

and so, by (XII.5.52) and (XII.5.59) we deduce that

$$\langle L(\boldsymbol{w}_1 - L(\boldsymbol{w}_2)\rangle_{\lambda,q} \le 8c\left(\lambda^{-2(1-1/q)} D + \delta\right)\langle \boldsymbol{w}_1 - \boldsymbol{w}_2\rangle_{\lambda,q} \le \frac{1}{2}\langle \boldsymbol{w}_1 - \boldsymbol{w}_2\rangle_{\lambda,q},$$

which proves that L is a contraction in $X_{\lambda,q,\delta}$. We may thus conclude that, under the assumption (XII.5.59) (that is, (XII.5.46)) on $\boldsymbol{v}_*$, $\boldsymbol{f}$, and λ, problem (XII.5.51) admits at least one solution $\boldsymbol{u}$ with $\langle \boldsymbol{u}\rangle_{\lambda,q}$ finite. In fact, in view of Lemma XII.5.2, it follows that $\boldsymbol{u}$, π also satisfy condition (XII.5.56) with π pressure field associated to $\boldsymbol{u}$. As a consequence, the fields (XII.5.49) constitute a solution to (XII.0.1), (XII.0.2). Furthermore, by (XII.5.52), (XII.5.59), and (XII.5.60) we find that $\boldsymbol{v}$ also satisfies estimate (XII.5.47). It remains to show uniqueness. Setting

$$\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v}_1, \ \pi = p - p_1,$$

we have

$$\left.\begin{aligned}&\Delta \boldsymbol{w} + \lambda \frac{\partial \boldsymbol{w}}{\partial x_1} = \nabla \pi + \boldsymbol{g} \\ &\nabla \cdot \boldsymbol{w} = 0\end{aligned}\right\} \text{ in } \Omega$$

$$\boldsymbol{w} = 0 \text{ at } \partial\Omega$$

$$\lim_{|x|\to\infty} \boldsymbol{w}(x) = 0.$$

where

$$\boldsymbol{g} := \lambda\left(\boldsymbol{w} \cdot \nabla \boldsymbol{u}_1 + \boldsymbol{u} \cdot \nabla \boldsymbol{w}\right).$$

From Lemma XII.5.3 and Lemma XII.5.4 it follows that

$$\langle \boldsymbol{w}\rangle_{\lambda,q} \le 4c\lambda^{-2(1-1/q)}\langle \boldsymbol{w}\rangle_{\lambda,q}\left(\langle \boldsymbol{u}_1\rangle_{\lambda,q} + \langle \boldsymbol{u}\rangle_{\lambda,q}\right). \tag{XII.5.61}$$

By a direct computation that makes use of (XII.5.57)–(XII.5.59) we find

$$4c\lambda^{-2(1-1/q)}\left(\langle \boldsymbol{u}_1\rangle_{\lambda,q} + \langle \boldsymbol{u}\rangle_{\lambda,q}\right) < 1$$

so that (XII.5.61) implies $\boldsymbol{w} \equiv 0$, thus completing the proof of the theorem. □

Remark XII.5.4 It is worth observing that Theorem XII.5.1 does not require that the datum $\boldsymbol{v}_*$ satisfies the zero-outflux condition:

$$\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} = 0.$$

■

Remark XII.5.5 We wish to emphasize that Theorem XII.5.1 furnishes existence (and uniqueness) for the physically significant problem obtained by setting $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv \boldsymbol{0}$, and describing the (plane) steady flow of a viscous liquid around a cylinder translating with constant speed. Let us denote by $\mathscr{P}$ this problem. For problem $\mathscr{P}$, the assumption (XII.5.46) reduces, in fact, to $|\log\lambda| > c$, for some $c = c(\Omega) > 0$, which is certainly satisfied by taking λ sufficiently small, which means, small translating speed. It is also interesting to observe that the logarithmic factor, that is crucial to prove such a result, comes from the estimate of the total force, $\boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{T}}(\lambda)$, exerted by the liquid on the cylinder in the Oseen approximation, and provided in Theorem VII.8.1. As we know, in the class of generalized solutions to the Oseen approximation $\boldsymbol{\mathcal{T}}(\lambda)$ is not zero for all $\lambda > 0$, while it becomes zero in the Stokes approximation, that corresponds to $\lambda = 0$ (Stokes paradox, see Remark V.3.5). So, we are expecting $\boldsymbol{\mathcal{T}}(\lambda) \to 0$ as $\lambda \to 0$, and Theorem VII.8.1 gives us an estimate of the rate at which this happens. Consequently, *if there were no Stokes paradox, we would have not been able to prove existence for problem $\mathscr{P}$!* ■

Remark XII.5.6 Solutions determined in Theorem XII.5.1 are unique in the class of those solutions verifying (XII.5.48). This result is much weaker than the analogous one proved for the three-dimensional case. In fact, for the situation at hand, we must require that both $\boldsymbol{v}$ and $\boldsymbol{v}_1$ are small in suitable norms, and it is not known if a solution $\boldsymbol{v}$ obeying (XII.5.46) is unique in the class of those solutions $\boldsymbol{v}_1$ that merely satisfy the condition $\langle \boldsymbol{v}_1\rangle_{\lambda,q} < \infty$; cf. also Section XII.2. ■

Remark XII.5.7 Since $q < 6/5$, solutions determined in Theorem XII.5.1 belong to $D^{1,s}(\Omega)$, for some $s < 2$. Moreover, they are generalized solutions in the sense of Definition IX.1.1. In fact, they obviously satisfy conditions (ii)-(v). Furthermore, since $\boldsymbol{v} \in D^{2,q}(\Omega) \cap D^{1,3q/(3-q)}(\Omega)$, $1 < q < 6/5$, by Theorem II.6.1, it follows that $\boldsymbol{v} \in D^{1,2q/(2-q)}(\Omega)$ and so, noticing that

$$3q/(3-q) < 2 < 2q/(2-q)$$

we conclude, by interpolation, that $\boldsymbol{v} \in D^{1,2}(\Omega)$ and so also issue (i) of Definition X.1.1 is verified. ■

Remark XII.5.8 Solutions determined in Theorem XII.5.1 satisfy the energy equality; cf. Corollary XII.7.1. ■

Exercise XII.5.1 Let $\boldsymbol{f} \in L^q(\mathbb{R}^2)$, $1 < q \le 6/5$. Show that there is an $M > 0$ such that if

$$\lambda^{2/q-1} \|\boldsymbol{f}\|_q < M,$$

problem (XII.0.1), (XII.0.2) in $\Omega = \mathbb{R}^2$ with $\boldsymbol{v}_\infty = \boldsymbol{e}_1$ admits at least one solution $\boldsymbol{v}, p$ satisfying the following properties:

$$\langle \boldsymbol{v} \rangle_{\lambda,q,\mathbb{R}^2} < \infty, \quad \boldsymbol{v} \in D^{2,q}(\mathbb{R}^2), \quad p \in D^{1,q}(\mathbb{R}^2).$$

Discuss also the uniqueness of these solutions.

XII.6 A Necessary Condition for Non-Existence with Arbitrary Large Data

In the previous section we have shown existence of solutions corresponding to $\boldsymbol{v}_\infty \neq \boldsymbol{0}$, provided the size of the data is suitably restricted (depending on the Reynolds number). On the other side, we know from Section XII.2 and Section XII.3 that, for data of arbitrary size (and sufficiently smooth),[1] we can construct a corresponding pair $(\boldsymbol{v}, p)$ satisfying (XII.0.1). As far as the behavior at infinity is concerned, namely, (XII.0.2), at least in the very significant physical case $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv \boldsymbol{0}$, we know that such a $\boldsymbol{v}$ tends, as $|x| \to \infty$, to a vector $\boldsymbol{v}_0$, in a suitable sense, and, in the case of symmetric flow, even uniformly pointwise; see Theorem XII.3.4 and Remark XII.3.2. However, we do not know if $\boldsymbol{v}_0 = \boldsymbol{v}_\infty$.

At this point, it appears natural to investigate whether solutions corresponding to large data do exist. This question has been taken up by Galdi (1999b) in the case when $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv \boldsymbol{0}$ and Ω possesses an axis of symmetry, that we may take coincident with the x_1-axis (say). Under such assumption on the data and after suitable scaling, we may rewrite the original problem (XII.0.1)–(XII.0.2) as follows

$$\left.\begin{aligned} &\Delta \boldsymbol{v} = \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla p \\ &\nabla \cdot \boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\boldsymbol{v}|_{\partial\Omega} = 0 \tag{XII.6.1}$$
$$\lim_{|x| \to \infty} (\boldsymbol{v}(x) + \lambda \boldsymbol{e}_1) = \boldsymbol{0}\,,$$

where, as in the previous section, we have denoted by λ the Reynolds number.

As we mentioned in the Introduction, (XII.6.1) describes one of the most studied problems in fluid dynamics, namely, the plane, steady-state motion of

[1] We are thinking of boundary data with zero or even "small" flux, in the sense specified in (X.4.47).

a viscous liquid around a cylinder of symmetric cross-section (e.g., a circle), that translates with constant speed λ in the positive $\boldsymbol{e}_1$-direction orthogonal to its axis. The existence problem for *arbitrary* data is then equivalent, in this situation, to finding solutions to (XII.6.1) for *any* prescribed value of $\lambda > 0$.[2] Since the smoothness of Ω is irrelevant, we may assume Ω of class C^∞ (e.g., the exterior of a circle), so that any solution $(\boldsymbol{v}, p)$ to (XII.6.1) with $\boldsymbol{v} \in W^{1,2}(\Omega_R)$, for all large $R > 0$, is $C^\infty(\overline{\Omega})$; see Theorem X.1.1. Thus, given $\lambda > 0$, we denote by $\mathscr{C}(\lambda)$ the class of solutions to (XII.6.1) with $\boldsymbol{v} \in W^{1,2}(\Omega_R)$, all $R > \delta(\Omega^c)$, and such that

$$\lim_{r\to\infty} \int_0^{2\pi} |\boldsymbol{v}(r,\theta) - \lambda \boldsymbol{e}_1| d\theta = 0\,.$$

The following result holds (Galdi 1999b, Theorem 4.1; Galdi 2004, §4.3).

Theorem XII.6.1 *Let Ω be smooth and symmetric around the x_1-axis. Suppose there exists $\overline{\lambda} > 0$ such that problem* (XII.6.1) *has no solution in the class $\mathscr{C}(\lambda)$ for all $\lambda \geq \overline{\lambda}$. Then, the following homogeneous problem*

$$\left.\begin{aligned} &\Delta \boldsymbol{u} = \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\tau \\ &\nabla\cdot\boldsymbol{u} = 0 \end{aligned}\right\} \ \text{in } \Omega$$
$$\boldsymbol{u}|_{\partial\Omega} = 0 \tag{XII.6.2}$$
$$\lim_{|x|\to\infty} \boldsymbol{u}(x) = \boldsymbol{0} \ \ \textit{uniformly pointwise}\,,$$

must have a non-zero *solution $(\boldsymbol{u}, \tau)$ such that:*

(i) $(\boldsymbol{u}, \tau)$ *is symmetric around the x_1-axis, in the sense specified in* Remark XII.3.2*;*
(ii) $\boldsymbol{u}, \tau \in C^\infty(\overline{\Omega})$*;*
(iii) $\nabla\boldsymbol{u}, \nabla\tau \in W^{m,q}(\Omega)$, *for all $m \geq 0$ and $q \geq 2$;*
(iv) *For all $|\alpha| \geq 0$, $\lim_{|x|\to\infty} D^\alpha \boldsymbol{u}(x) = \boldsymbol{0}$, $\lim_{|x|\to\infty} D^\alpha \tau(x) = 0$, uniformly pointwise.*

We shall not prove this result here and refer the interested reader to Galdi, *loc. cit.*. Instead, we limit ourselves to observe that the proof is given by contradiction, namely, if (XII.6.2) has *only* the zero solution, $\boldsymbol{u} \equiv \nabla\tau \equiv \boldsymbol{0}$, in the class (i)–(iv), then problem (XII.6.1) has at least a solution in the class $\mathscr{C}(\lambda)$ for all λ belonging to a suitable *unbounded* set.

We would like to make some comments about the above result. A sufficient condition to show that (XII.6.2) has only the zero solution is the existence of an unbounded sequence of numbers $\{R_m\}_{m\in\mathbb{N}}$, such that

[2] From Theorem XII.7.2, we know that existence holds for all $\lambda \in (0, \lambda_0]$, for some $\lambda_0 > 0$.

$$\lim_{R_m\to\infty}\int_0^{2\pi} \left(\tfrac{1}{2}u^2(R_m,\theta)+\tau(R_m,\theta)\right)\boldsymbol{u}(R_m,\theta)\cdot\boldsymbol{x}_m = 0, \tag{XII.6.3}$$

where $\boldsymbol{x}_m = (R_m,\theta)$. Actually, multiplying both sides of (XII.6.2)$_1$ by $\boldsymbol{u}$ and integrating by parts over Ω_R, we find

$$\int_{\Omega_R}|\nabla\boldsymbol{u}|^2 = \int_{|x|=R}\left(\boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial n}-(\tfrac{1}{2}u^2+\phi)\boldsymbol{u}\cdot\boldsymbol{n}\right). \tag{XII.6.4}$$

Since $\nabla\boldsymbol{u}\in W^{1,2}(\Omega)$, we deduce

$$\lim_{R\to\infty}\int_{|x|=R}\left|\frac{\partial\boldsymbol{u}}{\partial n}\right| = 0.$$

Therefore, from (XII.6.4) and the fact that $\boldsymbol{u}$ is bounded, we obtain

$$\int_{\Omega_R}|\nabla\boldsymbol{u}|^2 = -\lim_{R\to\infty}\int_{|x|=R}\left(\tfrac{1}{2}u^2+\phi\right)\boldsymbol{u}\cdot\boldsymbol{n},$$

and so, if (XII.6.3) is satisfied, we conclude $\boldsymbol{u}\equiv 0$, by "classical" uniqueness method.

We also notice that in the case when $\Omega=\mathbb{R}^2$ (an unrealistic assumption in our present situation) the uniqueness of the zero solution to (XII.6.2) follows from Theorem XII.3.1, and it appears very intriguing the idea that the introduction of a compact boundary may derail this property.

Finally, the possibility that (XII.6.2) may admit a non-zero smooth solution is very questionable on physical grounds, and the occurrence of such a situation would give less credibility to the Navier–Stokes model, or, at the very least, would cast serious doubts on the meaning of the two-dimensional assumption.

XII.7 Global Summability of Generalized Solutions when $\boldsymbol{v}_\infty \neq 0$

In this and the next section we shall analyze the asymptotic properties of generalized solutions corresponding to $\boldsymbol{v}_\infty\neq 0$. Specifically, we shall prove a result analogous to that obtained in Section X.5 for three-dimensional motions, namely, that every generalized solution tending at infinity to a nonzero velocity field $\boldsymbol{v}_\infty$ shows asymptotically the same behavior as the Oseen fundamental solution.

On the other hand, if $\boldsymbol{v}_\infty=0$, the structure of a generalized solution at large distances is a completely open question. In this regard, it should be emphasized that, when $\boldsymbol{v}_\infty=0$, generalized solutions that are regular in a neighborhood of infinity need not be represented there by an expansion in "reasonable" functions of $r\equiv|x|$ with coefficients independent of r. Actually, the fields given in (XII.2.7) for $\mathcal{R} \in (1,2)$ provide examples of generalized solutions that decay more slowly than any negative power of r.

In the current section we shall prove some preliminary results based on the work of Galdi and Sohr (1995) and Sazonov (1999). In particular, we deduce that every generalized solution corresponding to $\boldsymbol{v}_\infty \equiv (1,0)$[1] has the same global summability properties of the Oseen fundamental tensor, provided $\boldsymbol{f}$ is of bounded support.[2] This will imply, in particular, that the velocity field and the associated pressure of these generalized solutions satisfy the energy equality. Moreover, we obtain representation formulas identical to those derived in Section X.5 for the three-dimensional case.

We begin by showing some existence and uniqueness results for a suitable linearization of (XII.0.1), (XII.0.2). Specifically, let us consider the following problem

$$\begin{aligned} \Delta \boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial x_1} &= a \frac{\partial \boldsymbol{u}}{\partial x_1} + \boldsymbol{A} u_2 + \nabla \pi + \boldsymbol{G} \\ \nabla \cdot \boldsymbol{u} &= g, \end{aligned} \qquad \text{(XII.7.1)}$$

where a, $\boldsymbol{A}$, $\boldsymbol{G}$ and g are prescribed functions.

Lemma XII.7.1 *Let*

$$\begin{gathered} \boldsymbol{G} \in L^q(\mathbb{R}^2), \quad g \in W^{1,q}(\mathbb{R}^2), \quad q \in (1, 3/2) \\ \boldsymbol{A} \in L^2(\mathbb{R}^2), \quad a \in L^\infty(\mathbb{R}^2). \end{gathered}$$

Moreover, let $\boldsymbol{u}, \pi$ *be any solution to* (XII.7.1) *such that for some* $\overline{q} \in (1,2)$,

$$u_2 \in L^{2\overline{q}/(2-\overline{q})}(\mathbb{R}^2), \quad D^2 \boldsymbol{u}, \frac{\partial \boldsymbol{u}}{\partial x_1} \in L^{\overline{q}}(\mathbb{R}^2),$$

and

$$\lim_{|x| \to \infty} u_1(x) = 0.$$

Then, there exists a positive constant $c = c(q, \overline{q})$ *(*cf. (XII.7.4) *and* (XII.7.11)*) such that if*

$$\|a\|_\infty + \|\boldsymbol{A}\|_2 < c,$$

we have for some $\pi_0 \in \mathbb{R}$

$$\begin{aligned} &\boldsymbol{u} \in D^{2,q}(\mathbb{R}^2) \cap D^{1,3q/(3-q)}(\mathbb{R}^2) \cap L^{3q/(3-2q)}(\mathbb{R}^2), \\ &u_2 \in D^{1,q}(\mathbb{R}^2) \cap L^{2q/(2-q)}(\mathbb{R}^2) \\ &(\pi - \pi_0) \in D^{1,q}(\mathbb{R}^2) \cap L^{2q/(2-q)}(\mathbb{R}^2). \end{aligned}$$

[1] Of course, this choice causes no loss of generality whenever $\boldsymbol{v}_\infty \neq 0$.

[2] More generally, we may assume $\boldsymbol{f}$ to satisfy suitable decay properties in a neighborhood of infinity.

Proof. Let X_q, $1 < q < 3/2$, denote the Banach space of solenoidal functions $\boldsymbol{w} \in L^1_{loc}(\mathbb{R}^2)$ such that the norm

$$\begin{aligned}\|\boldsymbol{w}\|_{X_q} := \|w_2\|_{2q/(2-q)} + \|\nabla w_2\|_q + \left\|\frac{\partial \boldsymbol{w}}{\partial x_1}\right\|_q + \|\nabla \boldsymbol{w}\|_{3q/(3-q)}\\ + \|D^2 \boldsymbol{w}\|_q + \|\boldsymbol{w}\|_{3q/(3-2q)}\end{aligned}$$

is finite. Denote by $B_q^{(\delta)}$ the ball in X_q of radius δ (> 0) and consider the map

$$\mathcal{L} : \boldsymbol{w}' \in B_q^{(\delta)} \to \boldsymbol{w} \in X_q,$$

where $\boldsymbol{w}$ solves the problem

$$\left.\begin{aligned}\Delta \boldsymbol{w} + \frac{\partial \boldsymbol{w}}{\partial x_1} &= a\frac{\partial \boldsymbol{w}'}{\partial x_1} + \boldsymbol{A} w_2' + \nabla \tau + \boldsymbol{G}\\ \nabla \cdot \boldsymbol{w} &= g.\end{aligned}\right\} \tag{XII.7.2}$$

Since for all $q \in (1, 2)$, by the Hölder inequality, we have

$$\left\| a\frac{\partial \boldsymbol{w}'}{\partial x_1} + \boldsymbol{A} w_2' \right\|_q \leq \|a\|_\infty \left\|\frac{\partial \boldsymbol{w}'}{\partial x_1}\right\|_q + \|\boldsymbol{A}\|_2 \|w_2'\|_{2q/(2-q)}, \tag{XII.7.3}$$

by the hypotheses made on $\boldsymbol{G}$ and g, the map $\mathcal{L}$ is well defined for all $q \in (1, 3/2)$. Furthermore, by Theorem VII.4.1 and (XII.7.3), for all $\boldsymbol{w}' \in B_q^{(\delta)}$ we find

$$\|\boldsymbol{w}\|_{X_q} \leq c_1 \left[(\|a\|_\infty + \|\boldsymbol{A}\|_2) \|\boldsymbol{w}'\|_{X_q} + \|\boldsymbol{G}\|_q + \|g\|_{1,q} \right]$$

for some $c_1 = c_1(q)$. Thus, assuming (for instance)

$$\|a\|_\infty + \|\boldsymbol{A}\|_2 < \frac{1}{2c_1}, \tag{XII.7.4}$$

and choosing $\delta \geq 2c_1(\|\boldsymbol{G}\|_q + \|g\|_{1,q})$, it follows that $\mathcal{L}$ transforms $B_q^{(\delta)}$ into itself. Moreover, from (XII.7.2) with $\boldsymbol{G} \equiv g \equiv 0$, and by (XII.7.3), (XII.7.4) we obtain

$$\|\boldsymbol{w}\|_{X_q} \leq \tfrac{1}{2} \|\boldsymbol{w}'\|_{X_q},$$

and so the existence of a solution $\boldsymbol{w}$, τ to (XII.7.1) with $\boldsymbol{w} \in X_q$ follows from the contraction mapping theorem. Moreover, $\tau \in D^{1,q}(\mathbb{R}^2)$ and, therefore, by Theorem II.6.1, $\tau - \tau_0 \in L^{2q/(2-q)}(\mathbb{R}^2)$, for some $\tau_0 \in \mathbb{R}$. We shall now show that $\boldsymbol{u} = \boldsymbol{w}$, $\tau = \pi$+const *a.e.* in $\mathbb{R}^2$. To this end, letting

$$\overline{\boldsymbol{w}} = \boldsymbol{w} - \boldsymbol{u}, \quad s = \tau - \pi,$$

it follows that

$$\left.\begin{aligned}\Delta \overline{\boldsymbol{w}} + \frac{\partial \overline{\boldsymbol{w}}}{\partial x_1} &= a\frac{\partial \overline{\boldsymbol{w}}}{\partial x_1} + \boldsymbol{A} \overline{w}_2 + \nabla s\\ \nabla \cdot \overline{\boldsymbol{w}} &= 0.\end{aligned}\right\} \tag{XII.7.5}$$

It is easy to show that

$$\overline{w}_2 \in L^{2\overline{q}/(2-\overline{q})}(\mathbb{R}^2), \quad \frac{\partial \overline{\boldsymbol{w}}}{\partial x_1} \in L^{\overline{q}}(\mathbb{R}^2).$$

To this end, it is enough to prove that

$$w_2 \in L^{2\overline{q}/(2-\overline{q})}(\mathbb{R}^2), \quad \frac{\partial \boldsymbol{w}}{\partial x_1} \in L^{\overline{q}}(\mathbb{R}^2). \tag{XII.7.6}$$

Assume $q < \overline{q}$ (the other case $q > \overline{q}$ can be treated likewise by interchanging the role of q and $\overline{q}$). Since $\boldsymbol{w} \in D^{2,q}(\mathbb{R}^2)$, $q < 2$, by Theorem II.6.1 we have

$$w_2 \in D^{1,2q/(2-q)}(\mathbb{R}^2) \tag{XII.7.7}$$

and so, since $w_2 \in L^{2q/(2-q)}(\mathbb{R}^2)$, from this, from (XII.7.7), and from Theorem II.9.1 we obtain

$$w_2 \in L^{\infty}(\mathbb{R}^2),$$

which proves the first relation in (XII.7.6). Furthermore, by the properties

$$\boldsymbol{w} \in D^{2,q}(\mathbb{R}^2), \quad \frac{\partial \boldsymbol{w}}{\partial x_1} \in L^{q}(\mathbb{R}^2),$$

and Lemma II.3.3 we find that

$$\frac{\partial \boldsymbol{w}}{\partial x_1} \in L^{s}(\mathbb{R}^2), \quad \text{for all } s \in [q, 2]$$

and also the second relation in (XII.7.6) follows. From (XII.7.4) with $\boldsymbol{w}'$ replaced by $\overline{\boldsymbol{w}}$ we may thus conclude that

$$\boldsymbol{F} \equiv a\frac{\partial \overline{\boldsymbol{w}}}{\partial x_1} + \boldsymbol{A}\overline{w}_2 \in L^{\overline{q}}(\mathbb{R}^2).$$

Therefore, in view of Theorem VII.4.1, the problem

$$\begin{gathered} \Delta \boldsymbol{z} + \frac{\partial \boldsymbol{z}}{\partial x_1} = \boldsymbol{F} + \nabla \sigma \\ \nabla \cdot \boldsymbol{z} = 0 \end{gathered}$$

admits at least one solution $\boldsymbol{w}^*$, s^*, such that

$$\begin{aligned} \|D^2\boldsymbol{w}^*\|_{\overline{q}} + \|w_2^*\|_{2\overline{q}/(2-\overline{q})} + \left\|\frac{\partial \boldsymbol{w}^*}{\partial x_1}\right\|_{\overline{q}} &\le c_2\|\boldsymbol{F}\|_{\overline{q}} \\ &\le c_2\left(\|a\|_\infty + \|\boldsymbol{A}\|_2\right)\left(\left\|\frac{\partial \overline{\boldsymbol{w}}}{\partial x_1}\right\|_{\overline{q}} + \|\overline{w}_2\|_{2\overline{q}/(2-\overline{q})}\right) \end{aligned} \tag{XII.7.8}$$

with $c_2 = c_2(\overline{q})$. We shall now show that

$$D^2(\boldsymbol{w}^* - \overline{\boldsymbol{w}}) \equiv \frac{\partial(\boldsymbol{w}^* - \overline{\boldsymbol{w}})}{\partial x_1} \equiv 0, \quad w_2^* \equiv \overline{w}_2. \tag{XII.7.9}$$

Actually, setting $\boldsymbol{v} = \boldsymbol{w}^* - \overline{\boldsymbol{w}}$, $p = s^* - s$, it follows that

$$\begin{gathered} \Delta \boldsymbol{v} + \frac{\partial \boldsymbol{v}}{\partial x_1} = \nabla p \\ \nabla \cdot \boldsymbol{v} = 0. \end{gathered}$$

We may now use a local representation for $\boldsymbol{v}$ in terms of the Oseen-Fujita truncated fundamental tensor (Section VII.6):

$$\begin{aligned} D^\alpha v_j(x) &= -\int_{B_R(x)} \mathcal{H}_{ij}(x-y) D^\alpha v_i(y) dy \\ &:= -\int_{B_R(x)} \mathcal{H}_{ij}^{(R)}(x-y) D^\alpha (w_i^*(y) - w_i(y) - u_i(y)) dy \end{aligned} \tag{XII.7.10}$$

where

$$|\mathcal{H}_{ij}^{(R)}(x-y)| \leq C R^{-3/2}$$

for all sufficiently large R and with C independent of R. Recalling the summability properties of $\boldsymbol{w}^*$, $\boldsymbol{w}$ and $\boldsymbol{u}$ and using this latter inequality and the Hölder inequality at the right hand side of (XII.7.10) for various values of α, we can easily show the validity of (XII.7.9). For instance, with $|\alpha| = 2$ we find that

$$|D^2 v_j(x)| \leq C_1 \left[R^{-3/2} R^{2(1-1/\overline{q})} (|\boldsymbol{v}|_{2,\overline{q}} + |\boldsymbol{u}|_{2,\overline{q}}) + R^{-3/2} R^{2(1-1/q)} |\boldsymbol{w}|_{2,q} \right]$$

and so, noticing that $-3/2 + 2(1 - 1/s) < 0$ for all $s < 4$, we prove the first relation in (XII.7.9) by letting $R \to \infty$ in this last inequality. The other relations in (XII.7.9) follow in a similar manner. From (XII.7.9) and (XII.7.8) we then obtain

$$\begin{aligned} &\|D^2 \overline{\boldsymbol{w}}\|_{\overline{q}} + \|\overline{w}_2\|_{2\overline{q}/(2-\overline{q})} + \left\| \frac{\partial \overline{\boldsymbol{w}}}{\partial x_1} \right\|_{\overline{q}} \\ &\qquad \leq c_2 \left(\|a\|_\infty + \|\boldsymbol{A}\|_2 \right) \left(\left\| \frac{\partial \overline{\boldsymbol{w}}}{\partial x_1} \right\|_{\overline{q}} + \|\overline{w}_2\|_{2\overline{q}/(2-\overline{q})} \right) \end{aligned}$$

and so, if

$$\|a\|_\infty + \|\boldsymbol{A}\|_2 < \frac{1}{2c_2}, \tag{XII.7.11}$$

we conclude

$$D^2(\boldsymbol{w} - \boldsymbol{u}) \equiv \frac{\partial(\boldsymbol{w} - \boldsymbol{u})}{\partial x_1} \equiv 0, \quad w_2 \equiv u_2$$

and the lemma follows from the properties of $\boldsymbol{w}$ and the fact that u_1 tends to zero as $|x|$ tends to infinity.

□

Remark XII.7.1 We do not know if the result just shown continues to hold under the following conditions on $\boldsymbol{u}$:

$$D^2\boldsymbol{u},\ \frac{\partial \boldsymbol{u}}{\partial x_1} \in L^2(\mathbb{R}^2),\ \lim_{|x|\to\infty} \boldsymbol{u}(x) = 0.$$

Nevertheless, we can treat the case $\overline{q} = 2$ if we suppose $\boldsymbol{A} \equiv 0$, as shown in the following. ■

Lemma XII.7.2 *Let $\boldsymbol{u}, \pi$ be a solution to* (XII.7.1) *with $\boldsymbol{A} \equiv 0$, such that*

$$\boldsymbol{u} \in D^{2,2}(\mathbb{R}^2),\ \frac{\partial \boldsymbol{u}}{\partial x_1} \in L^2(\mathbb{R}^2).$$

Suppose, further,

$$\boldsymbol{G} \in L^r(\mathbb{R}^2), \quad g \in W^{1,r}(\mathbb{R}^2), \quad \textit{for some } r \in (1,2).$$

Then, there exists a positive constant $c = c(r)$ such that if

$$\|a\|_\infty < c,$$

we have

$$D^2\boldsymbol{u},\ \frac{\partial \boldsymbol{u}}{\partial x_1} \in L^r(\mathbb{R}^2).$$

Proof. Reasoning exactly as in the proof of Lemma XII.7.1, we can show the existence of a solution $\boldsymbol{w}, \tau$ to the problem (XII.7.1) with $\boldsymbol{A} \equiv 0$, satisfying

$$\boldsymbol{w} \in D^{2,r}(\mathbb{R}^2),\ \tau \in D^{1,r}(\mathbb{R}^2),\ \frac{\partial \boldsymbol{w}}{\partial x_1} \in D^{1,r}(\mathbb{R}^2).$$

Letting $\overline{\boldsymbol{w}} = \boldsymbol{w} - \boldsymbol{u}$, $s = \tau - \pi$, it follows that

$$\begin{aligned} \Delta\overline{\boldsymbol{w}} + \frac{\partial \overline{\boldsymbol{w}}}{\partial x_1} &= a\frac{\partial \overline{\boldsymbol{w}}}{\partial x_1} - \nabla s \\ \nabla \cdot \overline{\boldsymbol{w}} &= 0. \end{aligned}$$

Again as in the proof of Lemma XII.7.1, we may use Lemma II.3.3 to show that

$$a\frac{\partial \overline{\boldsymbol{w}}}{\partial x_1} \in L^2(\mathbb{R}^2)$$

and so, by Theorem VII.4.1, and by means of the same procedure used in Lemma XII.7.1, we obtain

$$\|D^2\overline{\boldsymbol{w}}\|_2 + \left\|\frac{\partial \overline{\boldsymbol{w}}}{\partial x_1}\right\|_2 \le c\|a\|_\infty \left\|\frac{\partial \overline{\boldsymbol{w}}}{\partial x_1}\right\|_2.$$

Therefore, if $\|a\|_\infty$ is sufficiently small, we find that

$$D^2\overline{\boldsymbol{w}} \equiv \frac{\partial \overline{\boldsymbol{w}}}{\partial x_1} \equiv 0,$$

and the lemma is proved. □

We need another preparatory result.

Lemma XII.7.3 *Let $\boldsymbol{v}$ be a D-solution to* (XII.0.1) *with $\boldsymbol{f}$ of bounded support and satisfying* (XII.3.67) *uniformly, with $\boldsymbol{v}_0 \neq \boldsymbol{0}$. Then, for sufficiently large ρ,*

$$(\boldsymbol{v}-\boldsymbol{v}_0) \in L^s(\Omega^\rho) \quad \textit{for all } s > 16\,.$$

Proof. Without loss of generality, we take $\boldsymbol{v}_0 = -\boldsymbol{e}_1$ and set $\boldsymbol{u} := \boldsymbol{v} + \boldsymbol{e}_1$. We take $R > R_0$ with R_0 given in Theorem XII.4.3, and pick x with $|x| \geq 2R$. Taking into account (XII.3.2), we may use the representation formula (V.3.14) with $u = u_1, u_2$ and $f = \partial\omega/\partial x_2, -\partial\omega/\partial x_1$, respectively. After integrating by parts this formula, with the help of (V.3.9) and (V.3.13), we easily deduce the following inequality

$$|\boldsymbol{u}(x)| \leq c_1 \left(R^{-2+\eta} \int_{\beta_R(x)} |\boldsymbol{u}(y)| dy + \int_{B_R(x)} \frac{|\omega(y)|}{|x-y|} dy \right), \tag{XII.7.12}$$

where $\beta_R(x) := B_R(x) - B_{R/2}(x)$, and $\eta > 0$ is arbitrary. From (XII.5.1) we know that there exists $\overline{R} > R_0$ such that $|\omega(y)| \leq c_2|y|^{-3/4}$, for all $|y| \geq \overline{R}$. Since for $y \in B_R(x)$ it is $|y| \geq |x| - R$, and since $|x| \geq 2R$, by choosing $R > \overline{R}$, we find

$$|\omega(y)| \leq c_2|y|^{-3/4} \leq 2c_2|x|^{-3/4}, \quad y \in B_R(x)\,.$$

Replacing this information back into (XII.7.12) we deduce

$$\begin{aligned} |\boldsymbol{u}(x)| &\leq c_3 \left(R^{-2+\eta} \int_{\beta_R(x)} |\boldsymbol{u}(y)| dy + |x|^{-3/4} \int_0^R dr \right), \\ &= c_3 \left(R^{-2+\eta} \int_{\beta_R(x)} |\boldsymbol{u}(y)| dy + |x|^{-3/4} R \right). \end{aligned} \tag{XII.7.13}$$

Furthermore, by Schwarz inequality, Theorem XII.4.3, and recalling $|x| \geq 2R$, we obtain

$$\begin{aligned} \int_{\beta_R(x)} |\boldsymbol{u}(y)| dy &\leq \left(\int_{\Omega_{R_0}} \frac{|\boldsymbol{u}|^2}{|y|^{1+\varepsilon}} \right)^{1/2} \left(\int_{B_R} |y|^{1+\varepsilon} dy \right)^{1/2} \leq c_4 R^{(1+\varepsilon)/2+1} \\ &\leq c_5\, |x|^{(1+\varepsilon)/2} R\,, \end{aligned}$$

where ε is arbitrary in $(0,1)$ Substituting this latter into (XII.7.13) and choosing $\eta = \varepsilon/2$ it follows that

$$|\boldsymbol{u}(x)| \leq c_6 \left(|x|^{1/2+\varepsilon} R^{-1} + |x|^{-3/4} R \right), \quad |x| \geq 2R\,,$$

so that, minimizing with respect to R, we conclude $|\boldsymbol{u}(x)| \leq c|x|^{(-1+4\varepsilon)/8}$ for sufficiently large $|x|$ and arbitrary $\varepsilon \in (0,1)$, which concludes the proof of the lemma.

□

With these lemmas in hand, we are now able to establish the main result of this section, which is the two-dimensional counterpart of the result given in Theorem X.6.4 for the three-dimensional case.

Theorem XII.7.2 *Let $\boldsymbol{v}, p$ be a solution to the Navier–Stokes system* (XII.0.1) *in Ω of class C^2, with*

$$\boldsymbol{f} \in L^q(\Omega) \text{ with bounded support}, \quad \boldsymbol{v}_* \in W^{2-1/q_0,q_0}(\partial\Omega),$$

for some $q_0 > 2$, all $q \in (1, q_0]$, and such that

$$\begin{aligned} &\boldsymbol{v} \in D^{1,2}(\Omega), \\ &\lim_{|x|\to\infty} (\boldsymbol{v}(x) + \boldsymbol{e}_1) = \mathbf{0}. \end{aligned} \tag{XII.7.14}$$

Then, we have

$$\begin{aligned} &(v_1 + 1) \in L^{t_1}(\Omega) \quad \text{for all } t_1 > 3, \\ &v_2 \in L^{t_2}(\Omega) \quad \text{for all } t_2 > 2, \\ &\frac{\partial v_1}{\partial x_2} \in L^{t_3}(\Omega) \quad \text{for all } t_3 > 3/2, \\ &\frac{\partial v_1}{\partial x_1}, \nabla v_2 \in L^{t_4}(\Omega) \quad \text{for all } t_4 > 1, \\ &(p - p_0) \in L^{t_5}(\Omega) \quad \text{for all } t_5 > 2, \end{aligned}$$

where p_0 is a constant.

Proof. From (XII.7.14)$_1$ and from the regularity results near the boundary for the solutions to the Stokes problem (Theorem IV.5.1), we easily deduce

$$\boldsymbol{v} \in W^{2,q_0}(\Omega_r), \quad p \in W^{1,q_0}(\Omega_r), \tag{XII.7.15}$$

for all $r > \delta(\Omega^c)$. Then, by Lemma XII.3.2 it follows that

$$\nabla \boldsymbol{v} \in W^{1,2}(\Omega). \tag{XII.7.16}$$

From the assumption, (XII.7.15), (XII.7.16), and (XII.0.1) we also have

$$\nabla p \in L^2(\Omega). \tag{XII.7.17}$$

Now, for $R \geq \rho$, let ψ_R be a smooth "cut-off" function defined by

$$\psi_R(x) = \begin{cases} 0 \text{ if } |x| < R/2 \\ 1 \text{ if } |x| \geq R. \end{cases}$$

Setting

$$\boldsymbol{u} = \psi_R(\boldsymbol{v} - \boldsymbol{e}_1) \equiv \psi_R \overline{\boldsymbol{v}}, \quad \pi = \psi_R p,$$

from (XII.0.1) we deduce that $\boldsymbol{u}, \pi$ satisfy the following system in $\mathbb{R}^2$

$$\begin{aligned} &\Delta \boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial x_1} = (\psi_{R/2}\overline{v}_1)\frac{\partial \boldsymbol{u}}{\partial x_1} + \left(\psi_{R/2}\frac{\partial \boldsymbol{v}}{\partial x_2}\right)u_2 + \nabla\pi + \boldsymbol{G}_1 \\ &\qquad \nabla\cdot\boldsymbol{u} = g, \end{aligned} \tag{XII.7.18}$$

where

$$\begin{aligned} &\boldsymbol{G}_1 = \psi_R\boldsymbol{f} + 2\nabla\psi_R\cdot\nabla\boldsymbol{v} + \Delta\psi_R\overline{\boldsymbol{v}} - \frac{\partial\psi_R}{\partial x_1}\overline{\boldsymbol{v}} - \overline{v}_1\boldsymbol{v}\frac{\partial\psi_R}{\partial x_1} - p\nabla\psi_R \\ &g = \overline{\boldsymbol{v}}\cdot\nabla\psi_R \end{aligned}$$

Clearly, we have

$$\boldsymbol{G}_1 \in L^q(\mathbb{R}^2),\ g \in W^{1,q}(\mathbb{R}^2)\ \text{ for all } q \in (1, q_0].$$

Moreover, we observe that in view of the assumption, by taking R sufficiently large, the quantities

$$\|\psi_{R/2}\overline{v}_1\|_\infty, \quad \left\|\psi_{R/2}\frac{\partial \boldsymbol{v}}{\partial x_2}\right\|_2$$

can be made less than any prescribed constant. Setting

$$\overline{q} = \frac{2s}{2+s} \quad (<2),$$

with s given in Lemma XII.7.3, by the Hölder inequality and assumption we find that

$$u_2\frac{\partial \boldsymbol{v}}{\partial x_2} \in L^{\overline{q}}(\mathbb{R}^2),$$

and so by Lemma XII.7.2 with $a = \psi_{R/2}\overline{v}_1$, $\boldsymbol{G} = \boldsymbol{G}_1 + u_2\dfrac{\partial \boldsymbol{v}}{\partial x_2}$, $r = \overline{q}$, and by (XII.7.16), (XII.7.18), and by the properties of ψ_R, we deduce

$$D^2\boldsymbol{v},\ \frac{\partial \boldsymbol{v}}{\partial x_1} \in L^{\overline{q}}(\Omega^R).$$

From this and (XII.7.15) we find

$$u_2 \in L^{2\overline{q}/(2-\overline{q})}(\mathbb{R}^2), \quad D^2\boldsymbol{u},\ \frac{\partial \boldsymbol{u}}{\partial x_1} \in L^{\overline{q}}(\mathbb{R}^2). \tag{XII.7.19}$$

We next apply Lemma XII.7.1 with $a = \psi_{R/2}\overline{v}_1$, $\boldsymbol{A} = \psi_{R/2}\dfrac{\partial \boldsymbol{v}}{\partial x_1}$ and $\boldsymbol{G} = \boldsymbol{G}_1$. In view of (XII.7.14)$_2$, (XII.7.18), (XII.7.19), and the properties of ψ_R we find that for any given $q \in (1, 3/2)$ there exists a sufficiently large R such that

$$
\begin{aligned}
&(v_1 + 1) \in L^{3q/(3-2q)}(\Omega^R)\\
&v_2 \in L^{2q/(2-q)}(\Omega^R)\\
&\frac{\partial v_1}{\partial x_2} \in L^{3q/(3-q)}(\Omega^R)\\
&\frac{\partial v_1}{\partial x_1},\ \nabla v_2 \in L^q(\Omega^R)\\
&(p - p_0) \in L^{2q/(2-q)}(\Omega^R)
\end{aligned}
$$

These conditions along with (XII.7.15), (XII.7.16)–(XII.7.17), and Theorem II.3.4, allow us to conclude the validity of the summability properties stated in the theorem. □

Remark XII.7.2 By the same method of proof, we can also show for all $s \in (1, 3/2)$ that

$$\boldsymbol{v} \in D^{2,s}(\Omega), \quad p \in D^{1,s}(\Omega).$$

Therefore, if, in addition to the assumptions of Theorem XII.7.2, we suppose Ω of class C^3 and

$$\boldsymbol{v}_* \in W^{3-1/r,r}(\partial\Omega), \quad \text{some } r > 2, \tag{XII.7.20}$$

from Theorem X.1.1 and Theorem XII.3.3 we obtain

$$\boldsymbol{v} \in D^{2,\tau}(\Omega), \quad p \in D^{1,\tau}(\Omega), \quad \text{for all } \tau > 1. \tag{XII.7.21}$$

■

We shall next draw some interesting consequences of Theorem XII.7.2. First of all, we have the following theorem.

Theorem XII.7.3 *Let the assumptions of* Theorem XII.7.2 *be satisfied. Then* $\boldsymbol{v}$ *and the corresponding pressure field* p *obey the energy equation* (X.2.29).

Proof. By Theorem XII.7.2,

$$\boldsymbol{v} + \boldsymbol{v}_\infty \in L^4(\Omega) \cap L^q(\Omega), \quad \text{for all } q > 3. \tag{XII.7.22}$$

In addition,

$$\frac{\partial \boldsymbol{v}}{\partial x_1} \in L^s(\Omega), \quad \text{for all } s > 1,$$

and so, in particular,

$$\frac{\partial \boldsymbol{v}}{\partial x_1} \in L^{q'}(\Omega). \tag{XII.7.23}$$

Therefore, the result follows from (XII.7.22), (XII.7.23), Theorem X.2.2 and Exercise X.2.2. □

Theorem XII.7.3 at once gives the following corollary.

Corollary XII.7.1 *The solution $\boldsymbol{v}$, p determined in* Theorem XII.5.1 *satisfies the energy equality* (X.2.29).

Another consequence of Theorem XII.7.2 is that $\boldsymbol{v}$ and p can be represented in a way analogous to that determined for the three-dimensional case. Specifically, we have the following.

Theorem XII.7.4 *Let the assumption of* Theorem XII.7.2 *be satisfied. Assume, further, Ω of class C^2 and*

$$\boldsymbol{f} \in L^t(\Omega), \quad \boldsymbol{v}_* \in W^{2-1/r,r}(\partial\Omega) \quad t \in (1,\infty), \; r \in (2,\infty).$$

Then, setting $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{v}_\infty$, the following representations hold for all $x \in \Omega$

$$\begin{aligned} u_j(x) = \mathcal{R}\int_\Omega & E_{ij}(x-y)(f_i(y) + u_l(y)D_l u_i(y))dy \\ & + \int_{\partial\Omega} [u_i(y)T_{il}(\boldsymbol{w}_j, e_j)(x-y) \\ & - E_{ij}(x-y)T_{il}(\boldsymbol{u},p)(y) - \mathcal{R}u_i(y)E_{ij}(x-y)\delta_{1l}]n_l d\sigma_y \end{aligned} \tag{XII.7.24}$$

and

$$\begin{aligned} u_j(x) = \mathcal{R}\int_\Omega & E_{ij}(x-y)f_i(y)dy - \mathcal{R}\int_\Omega u_l(y)u_i(y)D_l E_{ij}(x-y)dy \\ & + \int_{\partial\Omega} [u_i(y)T_{il}(\boldsymbol{w}_j, e_j)(x-y) - E_{ij}(x-y)(T_{il}(\boldsymbol{u},p)(y) \\ & - \mathcal{R}u_l(y)u_i(y)) - \mathcal{R}u_i(y)E_{ij}(x-y)\delta_{1l}]n_l d\sigma_y \end{aligned} \tag{XII.7.25}$$

and

$$\begin{aligned} p(x) = p_0 - \mathcal{R}\int_\Omega & e_i(x-y)(f_i(y) + u_l(y)D_l u_i(y))dy \\ & + \int_{\partial\Omega} \{e_i(x-y)T_{il}(\boldsymbol{u},p)(y) - 2u_i(y)\frac{\partial}{\partial x_l}e_i(x-y) \\ & - \mathcal{R}[e_1(x-y)u_l(y) - u_i(y)e_i(x-y)\delta_{1l}]n_l d\sigma_y . \end{aligned} \tag{XII.7.26}$$

In these relations $\boldsymbol{E}$, $\boldsymbol{e}$ is the Oseen fundamental solution, while $\boldsymbol{w}_j$ is defined in (VII.6.3) *and p_0 is a constant. All volume integrals in* (XII.7.24)–(XII.7.26) *are absolutely convergent.*

Proof. The proof is, in fact, completely analogous to (and, in some steps, even simpler than) that given in Theorem X.5.2 for the three-dimensional case; it is left to the reader as an exercise. □

From Theorem XII.7.4 and Theorem VII.6.2 we at once obtain the following.

Theorem XII.7.5 *Let the assumptions of* Theorem XII.7.4 *be satisfied. Then setting* $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{v}_\infty$, *the following asymptotic representation formulas hold for all sufficiently large* $|x|$

$$\begin{aligned} u_j(x) &= \mathcal{M}_i E_{ij} + \mathcal{R}\int_\Omega E_{ij}(x-y)u_l(y)D_l u_i(y)dy + s_j^{(1)}(x) \\ u_j(x) &= m_i E_{ij}(x) - \mathcal{R}\int_\Omega u_l(y)u_i(y)D_l E_{ij}(x-y)dy + s_j^{(2)}(x) \\ p(x) &= p_0 - \mathcal{M}_i^* e_i(x) - \mathcal{R}\int_\Omega e_i(x-y)u_l(y)D_l u_i(y)dy + h(x) \end{aligned} \tag{XII.7.27}$$

where $p_0 \in \mathbb{R}$,

$$\begin{aligned} \mathcal{M}_i &= -\int_{\partial\Omega}[T_{il}(\boldsymbol{u},p) + \mathcal{R}\delta_{1l}u_i]n_l + \mathcal{R}\int_\Omega f_i \\ m_i &= -\int_{\partial\Omega}[T_{il}(\boldsymbol{u},p) + \mathcal{R}(\delta_{1l}u_i - u_i u_l)]n_l + \mathcal{R}\int_\Omega f_i \\ \mathcal{M}_i^* &= -\int_{\partial\Omega}[T_{il}(\boldsymbol{u},p) + \mathcal{R}(\delta_{1l}u_i - \delta_{1i}u_l)]n_l + \mathcal{R}\int_\Omega f_i \end{aligned} \tag{XII.7.28}$$

$i = 1, 2$, *and, for* $|\alpha| \geq 0$, $j = 1, 2$, *and* $|x| \to \infty$,

$$\begin{aligned} D^\alpha s_j^{(k)}(x) &= O(|x|^{-(2+|\alpha|)/2}), \\ D^\alpha h(x) &= O(|x|^{-2-|\alpha|}). \end{aligned} \tag{XII.7.29}$$

Remark XII.7.3 A proof of (XII.7.24)–(XII.7.26) under the following assumption on the behavior at infinity of $\boldsymbol{u}$

$$\int_\Omega \frac{|\boldsymbol{u}|^2}{|x|^2} < \infty$$

has been given by Smith (1965, Theorems 2 and 3). However, the volume integrals appearing in the representations are to be understood as a limit of integrals over the intersection of Ω with concentric discs whose radii tend to infinity. ■

XII.8 The Asymptotic Structure of Generalized Solutions when $\boldsymbol{v}_\infty \neq 0$

In this section we will prove that the asymptotic behavior of every generalized solution satisfying the assumptions of Theorem XII.7.2 is governed by the fundamental solution of Oseen. To reach this goal, we shall follow, more or less, the same argument employed for the analogous three-dimensional result. In what follows, we shall set, without loss, $\boldsymbol{v}_\infty = \boldsymbol{e}_1$.

We need some preliminary lemmas.

Lemma XII.8.1 *Let $\boldsymbol{v}$ satisfy the assumptions of* Theorem XII.7.2, *and let the support of $\boldsymbol{f}$ be contained in Ω_ρ. Then, for all $R \geq \rho$, the following estimate holds:*

$$\int_{\Omega^R} \nabla \boldsymbol{v} : \nabla \boldsymbol{v} \leq c R^{-1/3+\varepsilon}$$

where ε is an arbitrary positive number and c is independent of R.[1]

Proof. Multiplying (XII.0.1) by $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{v}_\infty$ and integrating by parts over Ω_{R,R_*}, $\rho \leq R < R_*$, we find

$$\int_{\Omega_{R,R_*}} \nabla \boldsymbol{u} : \nabla \boldsymbol{u} = F(R) + F(R_*) \tag{XII.8.1}$$

where

$$F(r) = \int_{\partial B_r} \left\{ \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial n} - \tfrac{1}{2} \boldsymbol{u}^2 \boldsymbol{v} \cdot \boldsymbol{n} - p(\boldsymbol{u} \cdot \boldsymbol{n}) \right\}.$$

Using the summability properties of $\boldsymbol{u}, p$, given in Theorem XII.7.2, one can show that there exists at least one sequence $\{R_k\}_{k \in \mathbb{N}}$ with $R_k \to \infty$ as $k \to \infty$, along which $F(R_k)$ goes to zero. Thus, replacing R_* by R_k in (XII.8.1) and letting $k \to \infty$ we find

$$G(R) = F(R) \tag{XII.8.2}$$

where

$$G(R) \equiv \int_{\Omega^R} \nabla \boldsymbol{u} : \nabla \boldsymbol{u}.$$

Taking into account that, by Theorem XII.7.2,

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} \in L^1(\Omega^\rho),$$

we find

$$g_1(R) \equiv \int_{\partial B_R} \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial n} \in L^1(\rho, \infty). \tag{XII.8.3}$$

Furthermore, recalling that $\boldsymbol{v} \in L^\infty(\Omega^\rho)$, by Young's inequality we obtain for $\alpha > 0$

$$R^{-\alpha} g_2(R) \equiv R^{-\alpha} \int_{\partial B_R} |\boldsymbol{u}^2 \boldsymbol{v} \cdot \boldsymbol{n}| \leq c_1 \left\{ R^{-\alpha q' + 1} + \int_{\partial B_R} \boldsymbol{u}^{2q} \right\}$$

and so, by Theorem XII.7.2,

[1] Of course, the constant c –as all the other constants, given in subsequent proofs, that will enter similar estimates– depends on ε in such a way that $c \to \infty$ as $\varepsilon \to 0$.

$$R^{-\alpha} g_2(R) \in L^1(\rho, \infty) \quad \text{for all } \alpha > 2/3. \tag{XII.8.4}$$

Finally, again by Young's inequality,

$$R^{-\alpha} g_3(R) \equiv R^{-\alpha} \int_{\partial B_R} |p\boldsymbol{u} \cdot \boldsymbol{n}| \le c_2 \left\{ R^{-\alpha s'+1} + \int_{\partial B_R} |p|^s |\boldsymbol{u}|^s \right\}.$$

Since, by Theorem XII.7.2, $p\boldsymbol{u} \in L^s(\Omega^\rho)$ for all $s > 6/5$[2] we deduce

$$R^{-\alpha} g_3(R) \in L^1(\rho, \infty) \quad \text{for all } \alpha > 1/3. \tag{XII.8.5}$$

Observing that

$$R^{-\alpha} G(R) \le R^{-\alpha} \left(g_1(R) + g_2(R) + g_3(R) \right),$$

from (XII.8.3)–(XII.8.5) we find

$$R^{-\alpha} G(R) \in L^1(\rho, \infty) \quad \text{for all } \alpha > 2/3$$

and since

$$G'(R) = -\int_{\partial B_R} \nabla \boldsymbol{u} : \nabla \boldsymbol{u} < 0,$$

we conclude from Lemma X.8.1 that

$$G(R) \le c R^{-1+\alpha}$$

which proves the estimate. □

Using this result we can show the following.

Lemma XII.8.2 *Let $\boldsymbol{v}$ satisfy the assumptions of* Lemma XII.8.1. *Then, for all large $|x|$,*

$$\boldsymbol{v}(x) + \boldsymbol{v}_\infty = O(|x|^{-1/2+\varepsilon}),$$

where ε is a positive number arbitrarily close to zero.

Proof. We collect some asymptotic properties of the Oseen tensor $\boldsymbol{E}$ which will be frequently used during the proof. In fact, from (VII.3.42), (VII.3.43) and (VII.3.46) we have

$$\left. \begin{aligned} |\boldsymbol{E}(x)| &\le c|x|^{-1/2} \\ |\nabla \boldsymbol{E}_2(x)| &\le c|x|^{-2} \end{aligned} \right\} \quad \text{for all } x \in \mathcal{A} \tag{XII.8.6}$$

and

[2] We may take, without loss, the constant p_0 in Theorem XII.7.2 to be zero.

$$\begin{aligned}
&\boldsymbol{E}_1 \in L^q(\mathbb{R}^2), \quad \text{for all } q > 3 \\
&\boldsymbol{E}_2 \in L^q(\mathbb{R}^2), \quad \text{for all } q > 2, \\
&\frac{\partial \boldsymbol{E}_i}{\partial x_1} \in L^q(\mathbb{R}^2) \ \text{ for all } q \in (1,2), \quad i = 1,2 \\
&\frac{\partial \boldsymbol{E}_i}{\partial x_2} \in L^q(\mathbb{R}^2) \ \text{ for all } q \in (3/2,2), \ \ i = 1,2,
\end{aligned} \tag{XII.8.7}$$

where $\mathcal{A}$ is the exterior of a ball of sufficiently large radius and $\boldsymbol{E}_1, \boldsymbol{E}_2$ are defined in (XII.5.17). We begin to show that

$$\boldsymbol{v}(x) + \boldsymbol{v}_\infty = O(|x|^{-1/3+\varepsilon}). \tag{XII.8.8}$$

In view of $(\text{XII.7.27})_1$ and (XII.7.29), to show (XII.8.8) it is enough to prove that

$$N_j(x) \equiv \int_\Omega E_{ij}(x-y) u_l(y) D_l u_i(y) dy = O(|x|^{-1/3-\varepsilon}),$$

where, we recall, $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{v}_\infty$. To this end, setting $|x| = 2R$ (sufficiently large), we split $\boldsymbol{N}$ as follows:

$$\begin{aligned}
N_j(x) &= \int_{\Omega_R} E_{ij}(x-y) u_l(y) D_l u_i(y) dy + \int_{\Omega^R} E_{ij}(x-y) u_l(y) D_l u_i(y) dy \\
&\equiv N_j^{(1)} + N_j^{(2)}.
\end{aligned} \tag{XII.8.9}$$

Since for $y \in \Omega_R$ it is $|x-y| \geq R = |x|/2$, by (XII.8.6) we find

$$|N_j^{(1)}| \leq \frac{c}{|x|^{1/2}} \int_{\Omega_R} |\boldsymbol{u} \cdot \nabla \boldsymbol{u}|.$$

Therefore, taking into account that by Theorem XII.7.2 we have $\boldsymbol{u} \cdot \nabla \boldsymbol{u} \in L^1(\Omega)$, we conclude that

$$|N_j^{(1)}| \leq \frac{c_1}{|x|^{1/2}}. \tag{XII.8.10}$$

By the Hölder inequality we obtain

$$|N_j^{(2)}| \leq \|\boldsymbol{E}\|_{q,\mathbb{R}^2} \|\boldsymbol{u}\|_{s,\Omega_R} \|\nabla \boldsymbol{u}\|_{2,\Omega_R} \tag{XII.8.11}$$

where

$$\frac{1}{s} + \frac{1}{q} = \frac{1}{2}. \tag{XII.8.12}$$

Choosing, for instance, $s = q = 4$ and using Theorem XII.7.2, Lemma XII.8.1, and $(\text{XII.8.7})_{1,2}$ we deduce

$$|N_j^{(2)}| \leq \frac{c_2}{|x|^\gamma},$$

for arbitrary $\gamma < 1/6$. From this condition, $(\text{XII.7.27})_1$, (XII.8.9), and (XII.8.10) we then recover

$$|\boldsymbol{u}(x)| \leq \frac{c_3}{|x|^\gamma}, \quad \text{for arbitrary } \gamma < 1/6. \tag{XII.8.13}$$

We now use (XII.8.13) to improve the uniform bound on $\boldsymbol{N}^{(2)}$. To this end, we observe that, by (XII.8.7)$_{1,2}$, we can take the exponent q in (XII.8.12) to be any number greater than 3 which, in turn, by Theorem XII.7.2, implies that we can choose s arbitrarily in the interval $(3,6)$. Thus, writing

$$|\boldsymbol{u}|^s = |\boldsymbol{u}|^{6-\varepsilon-\sigma}|\boldsymbol{u}|^\sigma, \tag{XII.8.14}$$

with arbitrary small positive ε, and σ arbitrarily close to $3-\varepsilon$, from Lemma XII.8.1, (XII.8.11) and (XII.8.13) we deduce that

$$|N_j^{(2)}| \leq c_2 \frac{1}{|x|^\gamma}\frac{1}{|x|^{\gamma\sigma/s}} = \frac{c_2}{|x|^{\gamma(1+\sigma/s)}}.$$

This condition together with (XII.7.27)$_1$, (XII.8.9) and (XII.8.10) then gives

$$|\boldsymbol{u}(x)| \leq \frac{c_3}{|x|^{\gamma(1+\sigma/s)}}. \tag{XII.8.15}$$

Using this estimate, we can give a further improvement on the bound for $\boldsymbol{N}^{(2)}$ which will eventually lead to (XII.8.8). We again use (XII.8.11), (XII.8.14), and (XII.8.15) to deduce that

$$|N_j^{(2)}| \leq c_2 \frac{1}{|x|^\gamma}\frac{1}{|x|^{\gamma(1+\sigma/s)}} = c_2|x|^{-\gamma\left(1+\sigma/s+(\sigma/s)^2\right)},$$

which in turn furnishes

$$|\boldsymbol{u}(x)| \leq c_3|x|^{-\gamma\left(1+\sigma/s+(\sigma/s)^2\right)}.$$

Iterating this procedure as many times as we please, we thus conclude the validity of the following bound for $\boldsymbol{u}$

$$|\boldsymbol{u}(x)| \leq \frac{c}{|x|^{\gamma\ell}}$$

where

$$\ell = \sum_{k=0}^{\infty}\left(\frac{\sigma}{s}\right)^k$$

Recalling that σ/s and γ can be taken arbitrarily less then 1/2 and 1/6 respectively, we deduce $\gamma\ell = 1/3 - \varepsilon$, which proves (XII.8.8). Next, using (XII.8.8), we shall show that u_2 satisfies the following improved estimate

$$u_2 = O(|x|^{-1/2+\varepsilon}). \tag{XII.8.16}$$

To this end, we observe that from (XII.8.9) and the Hölder inequality, we have

$$\left|N_2^{(2)}(x)\right| = \left|\int_{\Omega_R} E_{i2}(x-y)u_l(y)D_l u_i(y)dy\right| \leq \|\boldsymbol{E}_2\|_{q,\mathbb{R}^2}\|\boldsymbol{u}\|_{s,\Omega_R}\|\nabla\boldsymbol{u}\|_{2,\Omega_R} \tag{XII.8.17}$$

where s and q satisfy (XII.8.12). For $0 < \alpha < s$, from (XII.8.8), Lemma XII.8.1, and (XII.8.17) we obtain

$$\left|N_2^{(2)}(x)\right| \leq \frac{c}{|x|^{1/6+\alpha/3s-\eta_1}}\|\boldsymbol{E}_2\|_{q,\mathbb{R}^2}\|\boldsymbol{u}\|_{s-\alpha,\Omega_R}^{(s-\alpha)/s}, \tag{XII.8.18}$$

for a positive η_1 arbitrarily close to zero. If we choose

$$\frac{\alpha}{s} = 1 - \frac{3}{s} - \eta_2, \quad \eta_2 > 0 \tag{XII.8.19}$$

we find $s - \alpha > 3$, which in turn, by Theorem XII.7.2, implies that the right-hand side of (XII.8.18) is finite for all those values of q such that $\boldsymbol{E}_2 \in L^q(\mathcal{A})$. As we know from $(\text{XII.8.7})_2$, we may take $q = 2 + \eta_3$, for a positive η_3 arbitrarily close to zero. Thus, from (XII.8.12) and (XII.8.19) it follows that

$$\frac{\alpha}{s} = 1 - 3\left(\frac{1}{2} - \frac{1}{q}\right) - \eta_2 = 1 - \eta_4$$

where η_4 is positive and arbitrarily close to zero. Therefore, from (XII.8.17) we obtain

$$\left|N_2^{(2)}(x)\right| \leq \frac{c}{|x|^{1/2-\varepsilon}},$$

which along with $(\text{XII.7.27})_1$, $(\text{XII.7.29})_1$, (XII.8.9), and (XII.8.10) proves (XII.8.16). Finally, we shall prove that also u_1 satisfies the following improved estimate

$$u_1(x) = O(|x|^{-1/2+\varepsilon}). \tag{XII.8.20}$$

To reach this goal, we observe that, from $(\text{XII.7.27})_2$, $(\text{XII.7.29})_2$ and (XII.8.6), it is enough to show that

$$n(x) \equiv \int_{\Omega} u_l(y)u_i(y)D_l E_{i2}(x-y)dy = O(|x|^{-1/2+\varepsilon}). \tag{XII.8.21}$$

As before, we split n into two integrals as follows

$$\begin{aligned} n(x) = &\int_{\Omega_R} u_l(y)u_i(y)D_l E_{i2}(x-y)dy \\ &+ \int_{\Omega_R} u_l(y)u_i(y)D_l E_{i2}(x-y)dy \\ \equiv\ & n^{(1)} + n^{(2)}, \end{aligned}$$

where $2R = |x|$. Recalling that for $y \in \Omega_R$ it is $|x-y| \geq R$, from $(\text{XII.8.6})_2$ and the Hölder inequality we find

$$|n^{(1)}(x)| \le cR^{2/q}\|u\|_{2q}^2.$$

Since, by Theorem XII.7.2, we may take $2q = 3+\eta_1$, for arbitrary $\eta_1 > 0$, the previous inequality furnishes

$$|n^{(1)}(x)| \le c|x|^{-2/3+\eta}, \quad \text{arbitrary } \eta > 0. \tag{XII.8.22}$$

Concerning $n^{(2)}$, we notice that

$$\begin{aligned} n^{(2)} = \int_{\Omega^R} & \left\{u_1^2(y) D_1 E_{11}(x-y) + u_1(y)u_2(y)\left(D_1 E_{21}(x-y)\right.\right. \\ & \left.\left. + D_2 E_{11}(x-y)\right) + u_2^2(y) D_2 E_{21}(x-y)\right\} dy \\ \equiv \sum_{i=1}^{4} & I_i. \end{aligned} \tag{XII.8.23}$$

From the Hölder inequality we have, for any $\eta \in (0,2)$

$$|I_1| \le \max_{x\in\Omega_R} |\boldsymbol{u}(x)|^{2-\eta} \|\boldsymbol{u}\|_{\eta s,\Omega_R} \|D_1 E_{11}\|_{s',\mathbb{R}^2},$$

and so, taking $s = 3/\eta + \varepsilon_1$ with ε_1 positive and arbitrarily close to zero, from (XII.8.7)$_3$, (XII.8.8) and this latter inequality we find

$$|I_1| \le c|x|^{-2/3+\varepsilon}. \tag{XII.8.24}$$

Likewise, we show that

$$|I_2| \le c|x|^{-2/3+\varepsilon}. \tag{XII.8.25}$$

To estimate I_3, we observe that, again from the Hölder inequality, for any $\eta \in (0,1)$ we have that

$$|I_3| \le \max_{x\in\Omega^R} |u_2(x)|^{1-\eta} \|u_2\|_{\eta q_2,\Omega^R} \|u_1\|_{q_1,\Omega^R} \|D_2 E_{11}\|_{q_3,\mathbb{R}^2}, \tag{XII.8.26}$$

where

$$\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1. \tag{XII.8.27}$$

Since, by Theorem XII.7.2 and (XII.8.7)$_4$, we may choose q_1 and q_2 arbitrarily close to 3 and $3/2$, respectively, from (XII.8.27) we deduce that, for given $\eta > 0$ we may select q_2 in such a way that $q_2\eta > 2$. Thus, from (XII.8.26), Theorem XII.7.2, and (XII.8.15) we find that

$$|I_3| \le c|x|^{-1/2+\varepsilon} \tag{XII.8.28}$$

for a positive ε arbitrarily close to zero. In a similar way we show that

$$|I_4| \le c|x|^{-1/2+\varepsilon},$$

and therefore, from this inequality, (XII.8.22)–(XII.8.25), and (XII.8.28) we infer (XII.8.20). The lemma then follows from (XII.8.16) and (XII.8.20). $\square$

We need another preliminary result.

Lemma XII.8.3 *Let $g(y)$ be a nonnegative function satisfying the estimate*

$$g(y) \le C_1 \min\left\{1, |y|^{-1+\varepsilon}\right\},$$

for arbitrarily small $\varepsilon > 0$ and with a positive constant C_1 independent of y. Then, setting $|x| = R$, for all sufficiently large R, we have

$$\int_{\Omega_{R/2}} g(y)\left(g(y) + |\boldsymbol{E}_1(y)|\right)|\nabla E_{11}(x-y)|dy \le C_2|x|^{-1+\varepsilon},$$

where $\boldsymbol{E} = \{E_{ij}\}$ is the Oseen fundamental tensor and $\boldsymbol{E}_1 = (E_{11}, E_{12})$.

Proof. From (VII.3.45) we have

$$|\nabla E_{11}(x-y)| \le c_1|x-y|^{-1}.$$

Let $\overline{R}$ (> 1) be a positive number such that the representations (VII.3.37) are valid for all $|y| \ge \overline{R}$. Then, for all $R > 2\overline{R}$,

$$\begin{aligned}
&\int_{\Omega_{R/2}} g(y)[g(y) + |\boldsymbol{E}_1(y)|]|\nabla E_{11}(x-y)|dy \\
&\qquad \le c_2|x|^{-1}\left(\int_{\Omega_{\overline{R}}} g(y)|\boldsymbol{E}_1(y)|dy + \int_{\Omega_{\overline{R},R/2}} g(y)|\boldsymbol{E}_1(y)| + \int_{\Omega_{R/2}} g^2(y)dy\right) \\
&\qquad \le c_3|x|^{-1}\left(1 + \int_{\Omega_{\overline{R},R/2}} g(y)|\boldsymbol{E}_1(y)|dy + \int_{\Omega_{R/2}} g^2(y)dy\right).
\end{aligned}$$

Now, from (VII.3.37$_{1,2}$) with $r = |y|$ it follows that (for simplicity, we put $\lambda = \mathcal{R}/2 = 1$)

$$\begin{aligned}
E_{11}(y) &= c_4\frac{e^{-r(1+\cos\varphi)}}{\sqrt{r}}(1-\cos\varphi) + O(r^{-1}) \\
E_{22}(y) &= O(r^{-1}).
\end{aligned}$$

Therefore

$$\int_{\Omega_{\overline{R},R/2}} g(y)|\boldsymbol{E}_1(y)|dy$$

$$\le c_5\left\{\int_1^{R/2}\left[r^{-1+\varepsilon} + r^{-1/2+\varepsilon}\left(\int_0^{2\pi} e^{-r(1+\cos\varphi)}(1-\cos\varphi)d\varphi\right)\right]dr\right\}.$$

However, from the inequality given before (VII.3.43) we know that

$$\int_0^{2\pi} e^{-r(1+\cos\varphi)}(1-\cos\varphi)d\varphi \le c_6 r^{-1/2}.$$

As a consequence,

$$\int_{\Omega_{\overline{R},R/2}} g(y)|\boldsymbol{E}_1(y)|dy \le c_7|x|^{\varepsilon}.$$

Finally, from the Hölder inequality,

$$\int_{\Omega_{R/2}} g^2 \le |\Omega_{R/2}|^{1/q'}\|g\|_{2q}^2$$

and since $g \in L^r(\Omega)$ for all $r > 2$, we can take q' as large as we please to conclude that

$$\int_{\Omega_{R/2}} g^2 \le c_8|x|^{\varepsilon},$$

and the lemma follows. □

We shall next derive the asymptotic structure of the velocity field.

Theorem XII.8.1 *Let the assumptions of* Theorem XII.7.2 *be satisfied. Then for all sufficiently large* $|x|$, *we have*

$$\boldsymbol{v}(x) + \boldsymbol{v}_\infty = \boldsymbol{m}\cdot\boldsymbol{E}(x) + \boldsymbol{\mathcal{V}}(x)$$

where $(i = 1, 2)$

$$m_i = -\int_{\partial\Omega}[T_{il}(\boldsymbol{u},p) + \mathcal{R}(\delta_{1l}u_i - u_iu_l)]n_l + \mathcal{R}\int_\Omega f_i\,, \tag{XII.8.29}$$

$\boldsymbol{E}$ *is the Oseen fundamental tensor,* $\boldsymbol{u} = \boldsymbol{v}+\boldsymbol{v}_\infty$ *and* $\boldsymbol{\mathcal{V}}(x)$ *verifies the estimate*

$$\boldsymbol{\mathcal{V}}(x) = O(|x|^{-1+\varepsilon}) \tag{XII.8.30}$$

for arbitrarily small $\varepsilon > 0$.

Proof. From the representation $(\text{XII.7.27})_2$, $(\text{XII.7.28})_2$, $(\text{XII.7.29})_1$ it follows that, to show the result, it is enough to prove that the nonlinear term:

$$n_i[\boldsymbol{u}(x)] = \int_\Omega \boldsymbol{u}(y)\cdot\nabla\boldsymbol{E}_i(x-y)\cdot\boldsymbol{u}(y)dy, \quad i = 1, 2,$$

with $\boldsymbol{E}_i$ defined in (XII.5.17), verifies the estimate

$$\boldsymbol{n}(x) = O(|x|^{-1+\varepsilon}). \tag{XII.8.31}$$

Writing down the integrand in components we find

$$
\begin{aligned}
n_i[\boldsymbol{u}(x)] = \int_\Omega [u_1^2(y) D_1 E_{1i}(x-y) &+ u_1(y)u_2(y)(D_2 E_{1i}(x-y) \\
&+ D_1 E_{2i}(x-y)) + u_2^2(y) D_2 E_{2i}(x-y)]dy \\
:= \sum_{k=1}^{4} I_{i,k}\,. &
\end{aligned}
\tag{XII.8.32}
$$

We set $R = |x|$ and split each integral $I_{i,k}$ into the sum of two integrals over $\Omega_{R/2}$ and $\Omega^{R/2}$ and denote these latter by $I_{i,k,R/2}$ and $I_{i,k}^{R/2}$, respectively. We begin to estimate $I_{i,k,R/2}$. From (VII.3.45) we have (at least)

$$
|D_1 E_{1i}(x-y)| \le c|x-y|^{-3/2}, \quad y \in \Omega_{R/2},\ i = 1,2,
$$

and so, by the Hölder inequality, it follows that

$$
|I_{i,1,R/2}| \le c_1 R^{-1} |\Omega_{R/2}|^{1/s} \|D_1 E_{1i}\|_{q/3,\mathbb{R}^2}^3 \|u_1\|_{2r,\Omega_{R/2}}^2 \tag{XII.8.33}
$$

with

$$
1/q + 1/r + 1/s = 1. \tag{XII.8.34}
$$

By Theorem XII.7.2, we may take r any number strictly greater than $3/2$, while $(\text{XII.8.7})_3$ implies

$$
D_1 E_{1i} \in L^\lambda(\mathbb{R}^2) \ \text{ for all } \lambda \in (1,2). \tag{XII.8.35}
$$

Thus, choosing $q = 3+\eta$, $\eta > 0$ and arbitrarily small, we can take s as large as we please. Consequently, (XII.8.33) furnishes for any positive small ε

$$
I_{i,1,R/2} = O(|x|^{-1+\varepsilon}),\ i = 1,2. \tag{XII.8.36}
$$

Moreover, since by (VII.3.45),

$$
|D_2 E_{1i}(x-y)| \le c|x-y|^{-t_i}, \quad y \in \Omega_{R/2},\ i = 1,2, \tag{XII.8.37}
$$

where

$$
t_i = \begin{cases} 1 & \text{if } i = 1 \\ 3/2 & \text{if } i = 2, \end{cases}
$$

again by the Hölder inequality, it follows that

$$
|I_{2,2,R/2}| \le c_1 R^{-1} \|D_2 E_{12}\|_{q/3,\mathbb{R}^2}^3 \|u_1 u_2\|_{q',\Omega_{R/2}}. \tag{XII.8.38}
$$

Now, by Theorem XII.7.2,

$$
u_1 u_2 \in L^r(\Omega) \ \text{ for all } r > 6/5, \tag{XII.8.39}
$$

while $(\text{XII.8.7})_4$ implies

$$
D_2 E_{1i} \in L^\sigma(\mathbb{R}^2) \ \text{ for all } \sigma \in (\tau_i, 2),\ i = 1,2, \tag{XII.8.40}
$$

with

$$\tau_i = \begin{cases} 3/2 \text{ if } i = 1 \\ 1 \quad\ \text{if } i = 2. \end{cases}$$

Thus, choosing $q \in (3, 6)$, from (XII.8.38), (XII.8.39), and (XII.8.40) we find

$$I_{2,2,R/2} = O(|x|^{-1}). \tag{XII.8.41}$$

Likewise, for all $\alpha \in (0, 1)$,

$$|I_{1,2,R/2}| \le c_1 R^{-1+\alpha} \|D_2 E_{11}\|^{\alpha}_{\alpha q,\mathbb{R}^2} \|u_1 u_2\|_{q',\Omega_{R/2}} \tag{XII.8.42}$$

and so, selecting for arbitrarily small $\varepsilon > 0$

$$\alpha = 1/4 + \varepsilon, \quad 3/2\alpha < q < 6,$$

in view of (XII.8.39), (XII.8.40) and (XII.8.42) we obtain

$$I_{1,2,R/2} = O(|x|^{-3/4+\varepsilon}). \tag{XII.8.43}$$

The estimates for $I_{i,3,R/2}$ and $I_{i,4,R/2}$ are somewhat simpler. In fact, from (VII.3.45) we have

$$|\nabla E_{2i}(x - y)| \le c|x - y|^{-3/2}, \quad y \in \Omega_{R/2},\ i = 1, 2,$$

and so, for $i = 1, 2$,

$$\begin{aligned} |I_{i,3,R/2}| &\le cR^{-3/2} |\Omega_{R/2}|^{1/q'} \|u_1 u_2\|_{q,\Omega_{R/2}} \\ |I_{i,4,R/2}| &\le cR^{-3/2} |\Omega_{R/2}|^{1/s'} \|u_2\|^2_{2s,\Omega_{R/2}}. \end{aligned}$$

By (XII.8.39) we can take q' arbitrarily less than 6 while, by Theorem XII.7.2, s is an arbitrary number greater than 1. Therefore,

$$I_{i,k,R/2} = O(|x|^{-1}), \quad i = 1, 2,\ k = 3, 4. \tag{XII.8.44}$$

It remains to estimate the integrals $I^{R/2}_{i,k}$. By the Hölder inequality and Lemma XII.8.2, for arbitrarily small $\eta > 0$ we have

$$\begin{aligned} |I^{R/2}_{i,1}| &\le \|u_1\|^2_{2q,\Omega^R} \|D_1 E_{1i}\|_{q',\mathbb{R}^2} \\ &\le c_1 \|D_1 E_{1i}\|_{q',\mathbb{R}^2} \left(\int_{R/2}^{\infty} r^{-q+1+\eta} dr \right)^{1/q} \\ &\le c_2 R^{-1+(2+\eta)/q} \|D_1 E_{1i}\|_{q',\mathbb{R}^2}. \end{aligned}$$

Since, by (XII.8.35), we can take q' as close to 1 as we please, we deduce

$$I_{i,1}^{R/2} = O(|x|^{-1+\varepsilon}),\ i = 1, 2. \tag{XII.8.45}$$

In a similar way, one shows

$$I_{i,3}^{R/2} = O(|x|^{-1+\varepsilon}),\ i = 1, 2. \tag{XII.8.46}$$

To estimate $I_{i,k}^{R/2}$, $i = 1, 2$, $k = 2, 4$ we argue as follows. For $\alpha \in (0, 1)$, by Lemma XII.8.2 and the Hölder inequality we have for arbitrarily small $\eta > 0$

$$|I_{i,2}^{R/2}| \leq c_4 R^{-1+\alpha+\eta} \|D_2 E_{1i}\|_{q',\mathbb{R}^2} \|u_1 u_2\|_{\alpha q}^{\alpha}. \tag{XII.8.47}$$

In view of (XII.8.39) and (XII.8.40), for $i = 2$ we may take α arbitrarily close to zero to deduce, for any $\varepsilon > 0$, that

$$I_{2,2}^{R/2} = O(|x|^{-1+\varepsilon}). \tag{XII.8.48}$$

On the other hand, if $i = 1$, we choose $\alpha = 2/5 + \delta$, arbitrarily small $\delta > 0$, and $q \in (6/5\alpha, 3)$, so that, again by (XII.8.39) and (XII.8.40), the inequality (XII.8.47) delivers

$$I_{1,2}^{R/2} = O(|x|^{-3/5+\varepsilon}). \tag{XII.8.49}$$

In a completely analogous way we obtain

$$I_{2,4}^{R/2} = O(|x|^{-1+\varepsilon}) \tag{XII.8.50}$$

and

$$I_{1,4}^{R/2} = O(|x|^{-3/5+\varepsilon}). \tag{XII.8.51}$$

We shall now show that estimates (XII.8.49), (XII.8.51) can, in fact, be improved. Actually, from (XII.7.27)–(XII.7.29), (XII.8.32), (XII.8.36), (XII.8.41), (XII.8.43)–(XII.8.46), and (XII.8.48)–(XII.8.51) we deduce the following asymptotic uniform bounds for $\boldsymbol{u}$:

$$u_1(x) = O(|x|^{-1/2}),\quad u_2(x) = O(|x|^{-1+\varepsilon}). \tag{XII.8.52}$$

Let us now split the integral $I_{1,2}^{R/2}$ as the sum of two integrals: one over $\Omega_{R/2,2R}$ and the other over Ω^{2R}. Let us denote them by $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. In view of (XII.8.52), for arbitrarily small $\varepsilon > 0$, we have

$$|\mathcal{I}_1| \leq c R^{-3/2+\varepsilon} \|\nabla E_{11}\|_{1,B_{4R}(x)}$$

and so the estimate of Exercise VII.3.4 implies

$$|\mathcal{I}_1| \leq c_1 R^{-3/2+\varepsilon} \left(\int_1^{4R} r^{-1/2} dr + 1 \right),$$

which, in turn, gives

$$\mathcal{I}_1 = O(|x|^{-1+\varepsilon}). \tag{XII.8.53}$$

Consider, next, $\mathcal{I}_2$. For $|y| \geq 2R = 2|x|$, by the triangle inequality it follows that

$$2|y| \geq |x-y|$$

and so, with the aid of (XII.8.52) and Exercise VII.3.4, we find for arbitrarily small $\varepsilon > 0$ that

$$\begin{aligned}|\mathcal{I}_2| &\leq c \int_{\Omega^{2R}} |x-y|^{-3/2+\varepsilon} |\nabla E_{11}(x-y)| dy \\ &\leq c \int_{B^R(x)} |x-y|^{-3/2+\varepsilon} |\nabla E_{11}(x-y)| dy \\ &\leq c_1 \int_R^\infty \rho^{-2+\varepsilon} d\rho.\end{aligned}$$

Therefore

$$\mathcal{I}_2 = O(|x|^{-1+\varepsilon}). \tag{XII.8.54}$$

From (XII.8.53), (XII.8.54) we infer

$$I_{1,2}^{R/2} = O(|x|^{-1+\varepsilon}). \tag{XII.8.55}$$

In an entirely analogous way we show

$$I_{1,4}^{R/2} = O(|x|^{-1+\varepsilon}). \tag{XII.8.56}$$

To complete the proof of the theorem we need to improve the estimate on the term $I_{1,2,R/2}$ given in (XII.8.43). From what we have proved so far we know that $\boldsymbol{u}$ admits the following asymptotic estimates

$$\begin{aligned}u_1(x) &= \boldsymbol{m} \cdot \boldsymbol{E}_1(x) + n_1[\boldsymbol{u}(x)] + O(|x|^{-1}) \\ u_2(x) &= O(|x|^{-1+\varepsilon}),\end{aligned} \tag{XII.8.57}$$

where

$$n_1 = I_{1,2,R/2} + O(|x|^{-1+\varepsilon}). \tag{XII.8.58}$$

Recalling the expression of $I_{1,2,R/2}$ and using (XII.8.43), (XII.8.57), and (XII.8.58) we deduce that

$$|I_{1,2,R/2}| \leq c_1 \int_{\Omega_{R/2}} g(y) \left(|n_1(y)| + |\boldsymbol{E}_1(y)| + g(y)\right) |D_2 E_{11}(x-y)| dy,$$

where g satisfies the assumptions of Lemma XII.8.1. Using this lemma along with the Hölder inequality, from the preceding inequality we deduce

$$|I_{1,2,R/2}| \leq c_2 \int_{\Omega_{R/2}} g(y) |n_1(y)| |D_2 E_{11}(x-y)| dy + O(|x|^{-1+\varepsilon}). \tag{XII.8.59}$$

In view of (XII.8.58), again employing Lemma XII.8.1 we have[3]

$$|I_{1,2,R/2}| \le c_3 \int_{\Omega_{R/2}} g(y)|I_{1,2,R/2}(y)||D_2 E_{11}(x-y)|dy + O(|x|^{-1+\varepsilon}).$$

Now assume that for some $\beta \in [3/4, 1)$,

$$|I_{1,2,R/2}(x)| \le c_4 |x|^{-\beta+\varepsilon}. \tag{XII.8.60}$$

Recalling that g satisfies the assumptions of Lemma XII.8.1 we find

$$|I_{1,2,R/2}| \le c_5 R^{-1+\alpha} \|D_2 E_{11}\|^{\alpha}_{\alpha q} [R^{-1-\beta+2/q'+\varepsilon} + 1] + O(|x|^{-1+\varepsilon}). \tag{XII.8.61}$$

We want the quantity in square brackets to be bounded in R and so we require

$$1 + \beta - 2/q' > 0.$$

Recalling that q is a number strictly greater than $3/2\alpha$ which can be taken arbitrarily close to $3/2\alpha$ (cf. (XII.8.40), (XII.8.42)), we thus find

$$1 - \beta = 4\alpha/3 - \eta, \tag{XII.8.62}$$

for arbitrarily small positive η. With this restriction for α, from (XII.8.61) we obtain a new value for β, that is,

$$\beta_1 = 1 - \alpha,$$

for which (XII.8.60) holds. We may iterate this procedure along sequences $\{\beta_k\}$, $\{\alpha_k\}$ which, by virtue of this last condition, (XII.8.43), and (XII.8.62), satisfy

$$1 - \beta_k = 4\alpha_k/3 - \eta, \quad \beta_{k+1} = 1 - \alpha_k, \quad \beta_0 = 3/4.$$

Solving for β_k we see

$$\beta_{k+1} = 1/4 + 3\beta_k/4 - 3\eta/4.$$

Since $\{\beta_k\}$ is increasing, by the arbitrariness of η, from this relation we find $\beta_k \to 1$ as $k \to \infty$, and from (XII.8.60) we conclude

$$I_{1,2,R/2} = O(|x|^{-1+\varepsilon}),$$

which completes the proof of the theorem. □

Remark XII.8.1 As in the three-dimensional situation, from Theorem XII.8.1 we obtain that the component v_1 parallel to the velocity at infinity $\boldsymbol{v}_\infty = (1, 0)$ exhibits a paraboloidal wake behavior in the direction of $\boldsymbol{v}_\infty$. Actually, for all sufficiently large $|x|$ we obtain the uniform estimate

[3] Throughout the rest of the proof, the symbol ε in the various formulas need not be the same. However, it will denote, as usual, an arbitrarily small positive number.

$$v_1(x) + 1 = O(|x|^{-1/2}).$$

Moreover, let φ be the angle made by a ray starting from the origin (in Ω^c, say) with the negatively directed x_1-axis, and define the parabolic-like region:[4]

$$\mathscr{P}_\sigma := \left\{(x,\varphi) \in \mathbb{R}^2 \times (-\pi/2, \pi/2) : (1+\cos\varphi) \le |x|^{-1+2\sigma},\ \sigma \in (0, 1/2]\right\}. \tag{XII.8.63}$$

Then, in view of (VII.3.40), (VII.3.41), we have the faster decay

$$v_1(x) + 1 = O(|x|^{-1/2-\alpha}), \quad x \notin \mathscr{P}_\sigma, \tag{XII.8.64}$$

with $\alpha = \min(2\sigma, 1/2 - \varepsilon)$, arbitrary small $\varepsilon > 0$. Estimate (XII.8.64) can be slightly improved, due to the fact that estimate (XII.8.31) is not the best possible. A detailed study in this direction has been performed by Smith (1965), who shows sharp bounds for the nonlinear term and, therefore, for the remainder $\boldsymbol{\mathcal{V}}$ defined in (XII.8.30). In particular, (XII.8.30) can be improved to the following:

$$\boldsymbol{\mathcal{V}}(x) = O\left(|x|^{-1}\log^2|x|\right).$$

Sharper estimates for $\mathcal{V}_1$ can also be obtained in the region (XII.8.63), cf. Smith (1965, Theorem 5). Specifically, in this region and for large $|x|$, one can prove

$$\mathcal{V}_1(x) = O\left(|x|^{-1-\sigma}\log|x|(1+|x|^{-\sigma}\log|x|)\right).$$

On the other hand, unlike the three-dimensional case, the component v_2 of $\boldsymbol{v}$ orthogonal to $\boldsymbol{v}_\infty$ "essentially" exhibits no wake region. In fact, from Theorem XII.8.1 we have that for all sufficiently large $|x|$, inside and outside the wake region,

$$v_2(x) = O(|x|^{-1+\varepsilon}).$$

It should be observed that this *uniform* estimate can be slightly improved to the following:

$$v_2(x) = O\left(|x|^{-1}\log|x|\right),$$

as shown by Smith (1965, formula (28b)). Sharper estimates for the remainder $\mathcal{V}_2$ can be proved in the region (XII.8.63), cf. Smith (1965, Theorem 5), where we have for $|x|$ large enough,

$$\mathcal{V}_2(x) = O\left(|x|^{-1-\sigma}(1+|x|^{-\sigma}\log|x|)\right).$$

An asymptotic expansion for $\boldsymbol{v}$ up to $|x|^{-3/2}$ has been given by Babenko (1970, Theorem 6.1). ■

Remark XII.8.2 The uniform estimate

$$\boldsymbol{v}(x) + \boldsymbol{v}_\infty = O(|x|^{-1/2})$$

is sharp in the sense that the condition

[4] See Remark VII.3.1.

$$\boldsymbol{v}(x) + \boldsymbol{v}_\infty = o(|x|^{-1/2}) \tag{XII.8.65}$$

holds if and only if we impose some restriction on the flow. Specifically, we have that (XII.8.65) is valid if and only if the first component of the vector $\boldsymbol{m}$ defined in (XII.8.29) is vanishing. The proof follows the same lines as that given in Corollary X.8.1 for the three-dimensional case and is therefore left to the reader as an exercise.[5] This result differs from the corresponding one in three dimensions where $\boldsymbol{v}(x) + \boldsymbol{v}_\infty = o(|x|^{-1})$ if and only if all components of $\boldsymbol{m}$ are zero. The fact that in two dimensions only m_1 needs to be zero is a consequence of the property that, in such a case, the component E_{12} of the Oseen fundamental tensor exhibits no wake. ■

Our next (and final) task in this section is to study the behavior at infinity of the first derivatives of $\boldsymbol{v}$ and of the pressure field p. This study is performed along the same lines as the proof of Theorem XII.8.1. From the representation formulas (XII.7.27)$_1$–(XII.7.29) we obtain for $i, k = 1, 2$ and all sufficiently large $|x|$

$$D_k v_i(x) = \boldsymbol{\mathcal{M}} \cdot D_k \boldsymbol{E}_i(x) + \mathcal{R} N_{i,k}[\boldsymbol{u}(x)] + D_k s_i^{(2)}(x), \tag{XII.8.66}$$

with $\boldsymbol{E}_i$ defined in (XII.5.17) and

$$N_{i,k}[\boldsymbol{u}(x)] = \int_\Omega \boldsymbol{u}(y) \cdot \nabla \boldsymbol{u}(y) \cdot D_k \boldsymbol{E}_i(x-y) dy. \tag{XII.8.67}$$

As in Theorem XII.8.1, the asymptotic behavior of $\nabla \boldsymbol{v}$ is determined once we establish that for $N_{i,k}$. It is expected that the behavior of $D_k \boldsymbol{v}$ will be different for different values of k, as a consequence of the unequal behavior at large distances of $D_k \boldsymbol{E}$, cf. (VII.3.45) and (VII.3.46). However, the method of proof is essentially the same for both $k = 1, 2$ and, therefore, we shall restrict our attention to $k = 1$, limiting ourselves to state the result for $k = 2$ in Theorem XII.8.2. Setting $|x| = R$, we split $N_{i,1}$ as the sum of two integrals $I_{i,R/2}$ and $I_i^{R/2}$ on the domains $\Omega_{R/2}$ and $\Omega^{R/2}$, respectively. By (VII.3.45) we have, for $i = 1, 2$,

$$|D_1 \boldsymbol{E}_i(x-y)| \le c|x-y|^{-3/2}, \quad \text{as } |x-y| \to \infty,$$

with $\boldsymbol{E}_i$ defined in (XII.5.17). On the other hand, by Theorem XII.7.2 we also have

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} \in L^q(\Omega), \quad \text{for all } q > 1, \tag{XII.8.68}$$

so that we deduce

$$|I_{i,R/2}| \le c|x|^{-3/2} |\Omega_{R/2}|^{-1/q'} \|\boldsymbol{u} \cdot \nabla \boldsymbol{u}\|_q \le c_1 |x|^{-3/2} |\Omega_{R/2}|^{-1/q'}.$$

Taking q' arbitrarily large, we infer

[5] See also Smith (1965, Theorem 11).

$$I_{i,R/2} = O(|x|^{-3/2+\varepsilon}), \quad i = 1, 2, \tag{XII.8.69}$$

where $\varepsilon > 0$ is arbitrarily small. We now pass to estimate $I_i^{R/2}$. From Theorem XII.8.1 and with the help of the local representation (IX.8.33) we show

$$D_k \boldsymbol{v}(x) = O(|x|^{-1/2}), \quad k = 1, 2. \tag{XII.8.70}$$

We have

$$\begin{aligned} I_i^{R/2} = & \int_{\Omega^{R/2}} [u_1(y) D_1 \boldsymbol{u}(y) \cdot D_1 \boldsymbol{E}_i(x-y) \\ & \qquad + u_2(y) D_2 \boldsymbol{u}(y) \cdot D_1 \boldsymbol{E}_i(x-y)] dy \\ & \equiv \mathcal{I}_1 + \mathcal{I}_2. \end{aligned}$$

From Theorem XII.8.1 and (XII.8.69) it follows for $q < 2$ that

$$\begin{aligned} |\mathcal{I}_1| & \le c_2 \int_{\Omega^{R/2}} |y|^{-1} |D_1 \boldsymbol{E}_i(x-y)| dy \\ & \le c_3 R^{-1+2/q'} \|D_1 \boldsymbol{E}_i\|_{q,\mathbb{R}^2} \\ |\mathcal{I}_2| & \le c_4 \int_{\Omega^{R/2}} |y|^{-3/2+\varepsilon} |D_1 \boldsymbol{E}_i(x-y)| dy \\ & \le c_5 R^{-3/2+(2+\varepsilon)/q'} \|D_1 \boldsymbol{E}_i\|_{q,\mathbb{R}^2}. \end{aligned}$$

In view of (XII.8.35), we may take in these relations q' arbitrarily large to find

$$\begin{aligned} \mathcal{I}_1 &= O(|x|^{-1+\varepsilon}) \\ \mathcal{I}_2 &= O(|x|^{-3/2+\varepsilon}). \end{aligned} \tag{XII.8.71}$$

From (XII.8.66), (XII.8.67), (XII.8.69), and (XII.8.71) we obtain

$$D_1 \boldsymbol{v}(x) = O(|x|^{-1+\varepsilon}), \quad k = 1, 2,$$

which improves on (XII.8.69) with $k = 1$. We may then use this latter estimate in bounding the integral $\mathcal{I}_1$. If we do this and proceed as before, we find

$$|\mathcal{I}_1| \le c_6 \int_{\Omega^{R/2}} |y|^{-3/2+\varepsilon} |D_1 \boldsymbol{E}_i(x-y)| dy$$

which, together with the Hölder inequality and (XII.8.35), in turn, implies

$$\mathcal{I}_1 = O(|x|^{-3/2+\varepsilon}).$$

We may then conclude

$$N_{i,1}[\boldsymbol{u}(x)] = O(|x|^{-3/2+\varepsilon}), \quad i = 1, 2, \tag{XII.8.72}$$

for arbitrary $\varepsilon > 0$. In a completely analogous way we show

$$N_{i,2}[\boldsymbol{u}(x)] = O(|x|^{-1+\varepsilon}), \quad i = 1, 2. \tag{XII.8.73}$$

We thus have proved the following.

Theorem XII.8.2 *Let* $\boldsymbol{v}$ *be as in* Theorem XII.8.1. *Then as* $|x| \to \infty$,

$$D_k \boldsymbol{v}(x) = \boldsymbol{\mathcal{M}} \cdot D_k \boldsymbol{E}(x) + \boldsymbol{\mathcal{T}}_k(x), \quad k = 1, 2,$$

holds, where $(i = 1, 2)$

$$\mathcal{M}_i = -\int_{\partial\Omega} [T_{il}(\boldsymbol{u}, p) + \mathcal{R}\delta_{1l} u_i] n_l + \mathcal{R} \int_{\Omega} f_i,$$

$\boldsymbol{E}$ *is the Oseen fundamental tensor, and*

$$\boldsymbol{\mathcal{T}}_k(x) = O(|x|^{-\alpha_k+\varepsilon}),$$

with $\alpha_1 = 3/2$ *and* $\alpha_2 = 1$.

Remark XII.8.3 Slightly improved uniform estimates can be given for the remainder $\boldsymbol{\mathcal{T}}_k(x)$; see Smith (1965). Specifically, for large $|x|$, one has the uniform estimate

$$\begin{aligned} \mathcal{T}_{12}(x) &= O(|x|^{-1} \log^2 |x|) \\ \mathcal{T}_{11}(x),\ \mathcal{T}_{2,2} &= O(|x|^{-3/2} \log^2 |x|) \\ \mathcal{T}_{21}(x) &= O(|x|^{-3/2}). \end{aligned}$$

Outside the wake region (XII.8.63), one can prove the following sharper estimates

$$\begin{aligned} \mathcal{T}_{12}(x) &= O(|x|^{-1-2\sigma} \log^2 |x|) \\ \mathcal{T}_{11}(x),\ \mathcal{T}_{2,2} &= O(|x|^{-3/2-\sigma} \log^2 |x|) \\ \mathcal{T}_{21}(x) &= O(|x|^{-3/2-\sigma}). \end{aligned}$$

For details, we refer the interested reader to Theorem 6 of Smith (1965). ∎

Our next objective is to derive an asymptotic formula for the pressure field. To this end, we start with the representation $(\text{XII.7.27})_3$, which can be written as

$$p(x) = p_0 - \mathcal{M}_i^* e_i(x) - \mathcal{R} \sum_{k=1}^{3} P_k(x) + h(x) \tag{XII.8.74}$$

where

$$P_1(x) = \int_{\Omega_{R/2}} e_i(x-y)u_l(y)D_lu_i(y)dy$$
$$P_2(x) = \int_{\Omega_{R/2,2R}} e_i(x-y)u_l(y)D_lu_i(y)dy \qquad \text{(XII.8.75)}$$
$$P_3(x) = \int_{\Omega^{2R}} e_i(x-y)u_l(y)D_lu_i(y)dy$$

and $R = |x|$. Furthermore, the quantities $\boldsymbol{\mathcal{M}}^*$, h are defined in (XII.7.23) and (XII.7.24). Using the uniform estimate

$$|\boldsymbol{e}(x-y)| \le c|x-y|^{-1}$$

together with (XII.8.68), Theorem XII.8.1, Theorem XII.8.2, and Lemma II.9.1, we deduce, with ε meaning an arbitrary positive number, that

$$\begin{aligned}
|P_1(x)| &\le c_1R^{-1}|\Omega_{R/2}|^{1/q'}\|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_q \le c_2|x|^{-1+\varepsilon}\\
|P_2(x)| &\le \int_{\Omega_{R/2,2R}} \left(|u_1(y)D_1(y)| + |u_2(y)D_2(y)D_2\boldsymbol{u}(y)|\right)|\boldsymbol{e}(x-y)|dy\\
&\le c_3|x|^{2-\varepsilon}\int_{\Omega_{R/2,2R}} |\boldsymbol{e}(x-y)|dy \le c_4|x|^{-1+\varepsilon}\\
|P_3(x)| &\le c_5\int_{\Omega^{2R}} |y|^{2-\varepsilon}|x-y|^{-1}dy \le c_6|x|^{-1+\varepsilon}.
\end{aligned} \qquad \text{(XII.8.76)}$$

Collecting $(\text{XII.7.24})_2$, and (XII.8.74)–(XII.8.76) furnishes the following.

Theorem XII.8.3 *Let $\boldsymbol{v}$ be as in* Theorem XII.8.1. *Then there is a $p_0 \in \mathbb{R}$ such that as $|x| \to \infty$,*

$$p(x) = p_0 - \mathcal{M}_i^* e_i(x) + \mathcal{P}(x),$$

where $(i = 1, 2)$

$$\mathcal{M}_i^* = -\int_{\partial\Omega} \{T_{il}(\boldsymbol{u}, p) + \mathcal{R}[\delta_{1l}u_i - \delta_{1i}u_l]\}n_l + \mathcal{R}\int_\Omega f,$$

$\boldsymbol{e}(x)$ is the pressure associated to the Oseen fundamental tensor $\boldsymbol{E}$, and

$$\mathcal{P}(x) = O(|x|^{-1+\varepsilon})$$

for arbitrary $\varepsilon > 0$.

Remark XII.8.4 A minor improvement on the behavior of the term $\mathcal{P}(x)$ can be obtained starting with a representation formula slightly different from (XII.8.74); cf. Smith (1965, Theorem 7). ■

Finally, we wish to mention some properties regarding the asymptotic structure of the vorticity field:

$$\omega = \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \tag{XII.8.77}$$

associated to a solution $\boldsymbol{v}$ to (XII.0.1), (XII.0.2) in the case $\boldsymbol{v}_\infty \neq 0$. This problem has been studied in full detail by Babenko (1970) and Clark (1971). The main result states, essentially, that if $\boldsymbol{f}$ is of bounded support and if $\boldsymbol{v}$ satisfies an estimate of the form

$$\boldsymbol{v}(x) + \boldsymbol{v}_\infty = O(|x|^{-1/4-\varepsilon}) \tag{XII.8.78}$$

for some $\varepsilon > 0$, then ω decays exponentially fast in the region outside the paraboloidal wake region. Now, as we obtain from Lemma XII.8.2, every generalized solution that tends pointwise and uniformly to $\boldsymbol{v}_\infty (\neq 0)$ at infinity obeys (XII.8.78), and so the vorticity field of every such generalized solution corresponding satisfies the above mentioned property. In particular, this holds for solutions determined in Theorem XII.5.1. Thus from Lemma XII.8.2 and a theorem of Babenko (1970, Theorem 8.1) (cf. also Clark 1971, Theorem 3.5′), one can show the following result, whose proof will be omitted.

Theorem XII.8.4 *Let the assumptions of* Theorem XII.8.1 *be satisfied. Then the vorticity field* (XII.8.77) *obeys the following representation for all sufficiently large* $|x|$[6]

$$\omega(x) = \nabla\Psi(x) \times \boldsymbol{m} + \mathcal{B}(x),$$

where

$$\Psi(x) \equiv e^{\mathcal{R}x_1/2} K_0\left(\mathcal{R}|x|/2\right),$$

K_0 *is the modified Bessel function of the second type of order zero* (cf. (VII.3.10), *and* (VII.3.15)), $\boldsymbol{m}$ *is defined in* (XII.8.29), *and*

$$\mathcal{B}(x) = O(e^{-s(x)}|x|^{-3/2}\log|x|), \quad s(x) := (|x| + x_1).$$

XII.9 Limit of Vanishing Reynolds Number: Transition to the Stokes Problem

In this section we shall investigate the behavior of solutions constructed in Theorem XII.5.1, in the limit of vanishing Reynolds number. We shall prove, in particular, that they converge to a uniquely determined solution of a suitable

[6] As usual, if $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2)$, by the notation $\boldsymbol{a} \times \boldsymbol{b}$ we mean the quantity $a_1 b_2 - a_2 b_1$.

Stokes problem. However, similarly to what we proved for the linear case in Section VII.8, this limiting process need not preserve the condition at infinity on the velocity field and, in fact, this condition is preserved if and only if the data are prescribed in a certain way.

For the sake of simplicity, we assume throughout that the body force $\boldsymbol{f}$ is identically vanishing. Furthermore, as in Section XII.5, we shall denote by λ the Reynolds number $\mathcal{R}$.

The first step is to show a uniform bound (independent of λ) for solutions determined in Theorem XII.5.1. This will be established in the following.

Lemma XII.9.1 *Let the assumptions of* Theorem XII.5.1 *be satisfied and let* $\boldsymbol{f} \equiv 0$. *Then, there is a* $c = c(\Omega, q, \boldsymbol{v}_*, \lambda_0) > 0$ *such that*

$$\int_{\Omega} \nabla \boldsymbol{v} : \nabla \boldsymbol{v} \le c.$$

Proof. We suppose, as usual, the origin of coordinates in Ω^c. Set

$$\boldsymbol{\sigma} = \Phi \nabla \log |x|$$

with m such that

$$\int_{\partial\Omega} (\boldsymbol{v}_* - \boldsymbol{\sigma}) \cdot \boldsymbol{n} = 0.$$

In view of this latter property, by the results of Exercise III.3.5, there is a field $\boldsymbol{V}$ satisfying

(i) $\nabla \cdot \boldsymbol{V} = 0$ in Ω,
(ii) $\boldsymbol{V} = \boldsymbol{v}_* - \boldsymbol{\sigma}$ at $\partial\Omega$,
(iii) $\boldsymbol{V} \in W^{1,2}(\Omega)$,
(iv) $\boldsymbol{V} \equiv 0$ in Ω^R, for some $R > \delta(\Omega^c)$.

Setting

$$\boldsymbol{v} = \boldsymbol{w} - \boldsymbol{e}_1 + \boldsymbol{\sigma} + \boldsymbol{V}, \tag{XII.9.1}$$

from the property of $\boldsymbol{v}$, $\boldsymbol{\sigma}$, and $\boldsymbol{V}$, and with the help of the embedding Theorem II.3.4, we deduce that the field $\boldsymbol{w}$ verifies the following properties

$$\begin{gathered} \langle \boldsymbol{w} \rangle_{\lambda,q} < \infty \\ \boldsymbol{w} = 0 \ \text{ at } \ \partial\Omega \\ \lim_{|x|\to\infty} \boldsymbol{w}(x) = 0. \end{gathered} \tag{XII.9.2}$$

By Corollary XII.7.1, $\boldsymbol{v}$ satisfies the energy equality (X.2.7) and so, by virtue of (XII.9.1), (XII.9.2), and by an integration by parts that uses the relation

$$\frac{\partial \sigma_i}{\partial x_j} = \frac{\partial \sigma_j}{\partial x_i} \quad i, j = 1, 2,$$

we find

$$|\boldsymbol{w}|_{1,2}^2 = \int_\Omega \left\{\nabla \boldsymbol{w} : \nabla \boldsymbol{V} - \lambda[\boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot(\boldsymbol{\sigma}+\boldsymbol{V}) + \boldsymbol{V}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{V} + \boldsymbol{V}\cdot\frac{\partial \boldsymbol{w}}{\partial x_1}]\right\}. \quad \text{(XII.9.3)}$$

By the properties of $\boldsymbol{V}$ and Theorem II.3.4, we have

$$\|\boldsymbol{V}\|_r < \infty \quad \text{for all } r \in [1,\infty).$$

Therefore, by using the Schwarz inequality along with inequality (II.2.5) in (XII.9.3), we show

$$|\boldsymbol{w}|_{1,2}^2 \le \lambda \left| \int_\Omega \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot(\boldsymbol{\sigma}+\boldsymbol{V})\right| + c_1 \quad \text{(XII.9.4)}$$

where c_1 depends on $\boldsymbol{V}$, Ω, and λ_0 but not on λ. We now observe that, by the Hölder inequality, it follows that

$$\left| \int_\Omega \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot(\boldsymbol{\sigma}+\boldsymbol{V})\right| \le \|\boldsymbol{w}\|_{3q/(3-2q)} |\boldsymbol{w}|_{1,3q/(3-q)} \|\boldsymbol{\sigma}+\boldsymbol{V}\|_{q/(2q-2)}$$

and so, since $q/(2q-2) > 2$, from (XII.9.4) and (XII.5.4) we recover

$$|\boldsymbol{w}|_{1,2}^2 \le c_2 \langle \boldsymbol{w}\rangle_{\lambda,q}^2 + c_1 \quad \text{(XII.9.5)}$$

with c_2 independent of $\lambda \in (0,\lambda_0]$. From the estimate (XII.5.47) and (XII.9.1) we then obtain, at once,

$$\langle \boldsymbol{w}\rangle_{\lambda,q} \le c_3$$

with c_3 independent of $\lambda \in (0,\lambda_0]$. As a consequence, the lemma follows from this latter inequality, (XII.9.5), and (XII.9.1). □

The next lemma shows other uniform bounds for solutions on every compact set in $\overline{\Omega}$.

Lemma XII.9.2 *Let the assumptions of* Lemma XII.9.1 *be satisfied. Then there exists a constant* $c = c(\Omega, R, q, \lambda_0, \mathbf{v}_*)$ *such that for all* $R > \delta(\Omega)^c$

$$\|\boldsymbol{v}\|_{2,q,\Omega_R} + \|p\|_{1,q,\Omega_R} \le c$$

.

Proof. Using Theorem IV.4.1 and Theorem IV.5.1 in equation (XII.0.1), we find

$$\begin{aligned}\|\boldsymbol{v}\|_{2,q,\Omega_R} + \|p\|_{1,q,\Omega_R} \le c_1 \,(\lambda\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{q,\Omega_{R_1}} + \|\boldsymbol{v}\|_{1,q,\Omega_{R_1}} \\ + \|p\|_{q,\Omega_{R_1}} + \|\boldsymbol{v}_*\|_{2-1/q,q(\partial\Omega)})\end{aligned} \quad \text{(XII.9.6)}$$

for all $R_1 > R > \delta(\Omega^c)$ and with $c = c(\Omega, q, R, R_1)$. From Theorem II.3.4 and (II.5.18) we find for any $s \in [1,\infty)$ and any $R > \delta(\Omega^c)$

$$\|\boldsymbol{v}\|_{s,\Omega_R} \le c_2(\|\boldsymbol{v}\|_{2,\Omega_R} + |\boldsymbol{v}|_{1,2,\Omega_R})$$
$$\le c_3(|\boldsymbol{v}|_{1,2} + \|\boldsymbol{v}_*\|_{2-1/q,q(\partial\Omega)})$$

and so, by Lemma XII.9.1,

$$\|\boldsymbol{v}\|_{s,\Omega_R} \le c_4, \quad s \ge 1,\ R > \delta(\Omega^c) \tag{XII.9.7}$$

with $c_4 = c_4(\Omega, R, s)$. Since

$$\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{q,\Omega_{R_1}} \le \|\boldsymbol{v}\|_{2q/(2-q),\Omega_{R_1}}|\boldsymbol{v}\|_{1,2,\Omega_{R_1}},$$

coupling (XII.9.6) and (XII.9.7) with the help of Lemma XII.9.1,

$$\|\boldsymbol{v}\|_{2,q,\Omega_R} + \|p\|_{1,q,\Omega_R} \le c_5(\|p\|_{q,\Omega_{R_1}} + 1) \tag{XII.9.8}$$

with c_5 independent of $\lambda \in (0, \lambda_0]$. To estimate the pressure term on the right-hand side of (XII.9.8), we use Lemma IV.1.1 to obtain (after the possible modification of p by adding a constant)

$$\|p\|_{q,\Omega_{R_1}} \le c_6(\lambda\|\boldsymbol{v}\|^2_{2q,\Omega_{R_1}} + |\boldsymbol{v}|_{1,2}),$$

which, by (XII.9.7) and Lemma XII.9.1, in turn implies

$$\|p\|_{q,\Omega_{R_1}} \le c_7,$$

with c_7 independent of $\lambda \in (0, \lambda_0]$. The lemma then follows from this latter inequality and (XII.9.8). □

The next result shows that solutions of Theorem XII.5.1 tend, in the limit $\lambda \to 0$, to solutions of a suitable Stokes problem.

Lemma XII.9.3 *Let the assumptions of* Lemma XII.9.1 *be satisfied. Denote by* $\boldsymbol{w}$, π *the unique solution to the following Stokes problem*

$$\left.\begin{aligned} \Delta\boldsymbol{w} &= \nabla\pi \\ \nabla\cdot\boldsymbol{w} &= 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\boldsymbol{w} = \boldsymbol{v}_* \text{ at } \partial\Omega \tag{XII.9.9}$$
$$|\boldsymbol{w}|_{1,2} < \infty.$$

Then as $\lambda \to 0$, *the solutions* $\boldsymbol{v}$, p *constructed in* Theorem XII.5.1 *satisfy*

$$\begin{aligned} \nabla\boldsymbol{v} &\overset{w}{\to} \nabla\boldsymbol{w} \quad \text{in } L^2(\Omega) \\ \boldsymbol{v} &\overset{w}{\to} \boldsymbol{w} \quad \text{in } W^{2,q}(\Omega_R) \\ p &\overset{w}{\to} \pi \quad \text{in } W^{1,q}(\Omega_R) \end{aligned} \tag{XII.9.10}$$

for any $R > \delta(\Omega^c)$.

Proof. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be any sequence converging to zero. From Lemma XII.9.1 and Lemma XII.9.2 it readily follows that, along a subsequence at least, conditions (XII.9.10) hold. However, the field $\boldsymbol{w}$ satisfying (XII.9.9) is uniquely determined, whatever the sequence may be (cf. Theorem V.2.1) and, therefore, the result is proved. □

We are now in a position to show the following main result.

Theorem XII.9.1 *Let the assumptions of* Theorem XII.5.1 *hold and let* $(\boldsymbol{v}, p)$ *be the solution constructed in that theorem corresponding to* $\boldsymbol{f} \equiv 0$. *Then, denoting by* $\boldsymbol{w}$, π *the solution to the Stokes problem* (XII.9.9) *we have that, as* $\lambda \to 0$, $\boldsymbol{v}$, p *tend to* $\boldsymbol{w}$, π *in the sense specified by* Lemma XII.9.3. *Moreover, there is a* $\boldsymbol{w}_0 \in \mathbb{R}^2$ *such that*

$$\lim_{|x|\to\infty} \boldsymbol{w}(x) = \boldsymbol{w}_0 \tag{XII.9.11}$$

and we have

$$\boldsymbol{w}_0 + \boldsymbol{e}_1 = \frac{1}{4\pi} \lim_{\lambda\to 0} \boldsymbol{\mathcal{T}}(\boldsymbol{v})|\log\lambda| \tag{XII.9.12}$$

where

$$\boldsymbol{\mathcal{T}}(\boldsymbol{v}) = \int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{v}, p)\cdot\boldsymbol{n}.$$

Finally, the limit process preserves the prescription at infinity, i.e., $\boldsymbol{w}_0 = -\boldsymbol{e}_1$ *if and only if the data satisfy the conditions*

$$\int_{\partial\Omega} (\boldsymbol{v}_* + \boldsymbol{e}_1)\cdot\boldsymbol{T}(\boldsymbol{h}_i, \pi_i)\cdot\boldsymbol{n} = 0 \quad i{=}1{,}2 \tag{XII.9.13}$$

where $\{\boldsymbol{h}_i, \pi_i\}$ *are the "exceptional" solutions to the homogeneous Stokes system constructed in* Lemma V.5.1. *In the particular case where* Ω *is exterior to a unit circle,* (XII.9.13) *reduces to*

$$\int_{\partial\Omega} (v_{*i} + \delta_{1i}) = 0 \quad i{=}1{,}2.$$

Proof. The first part of the statement follows from Lemma XII.9.3. Moreover, from Theorem V.3.2 we know that $\boldsymbol{w}$ satisfies the representation

$$w_j = w_{0j} + \int_{\partial\Omega} [v_{*i}T_{il}(\boldsymbol{u}_j, q_j)(x-y) - U_{ij}(x-y)T_{il}(\boldsymbol{w},\pi)(y)]n_l(y)d\sigma_y \tag{XII.9.14}$$

for some $\boldsymbol{w}_0 \in \mathbb{R}^2$, which, by the regularity of $\boldsymbol{v}_*$, proves (XII.9.11). On the other hand, by Theorem XII.7.4, we have

$$\begin{aligned} u_j(x) = \lambda\int_\Omega E_{ij}(x-y)u_l(y)D_l u_i(y)dy + \int_{\partial\Omega} [u_i(y)T_{il}(\boldsymbol{w}_j, e_j)(x-y) \\ -E_{ij}(x-y)T_{il}(\boldsymbol{u}, p)(y) - \lambda u_i(y)E_{ij}(x-y)\delta_{1l}]n_l d\sigma_y, \end{aligned} \tag{XII.9.15}$$

where $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{e}_1$. We wish to take $\lambda \to 0$ into (XII.9.14). From the Hölder inequality, (VII.3.41), (VII.3.43), (XII.5.4), and (XII.5.15) it follows for $q \in (1, 6/5)$ that

$$\begin{aligned}|\int_{\Omega} E_{ij}(x-y)u_l D_l u_i(y)dy| &\le \|\boldsymbol{u}\|_{3q/(3-2q)}|\boldsymbol{u}|_{1,3q/(3-q)}\|\boldsymbol{E}\|_{q/(2q-2)} \\ &\le \lambda^{-1}\lambda^{-4(1-1/q)}\langle\boldsymbol{u}\rangle^2_{\lambda,q}\end{aligned}$$

where $c_1 = c_1(q, \Omega)$. However, the solutions of Theorem XII.5.1 verify the estimate (XII.5.47) and so, from the preceding inequality, we obtain

$$\lambda\left|\int_{\Omega} E_{ij}(x-y)u_l(y)D_l u_i(y)dy\right| \le c_2|\log\lambda|^{-2}, \tag{XII.9.16}$$

where c_2 is independent of $\lambda \in (0, \lambda_0]$. Now, integrating by parts over the region Ω^c, we find

$$\int_{\partial\Omega}\left[T_{il}(\boldsymbol{w}_j, e_j(x-y)) - \lambda E_{ij}(x-y)\delta_{1l}\right] n_l(y)d\sigma_y = 0. \tag{XII.9.17}$$

Moreover, by Lemma XII.9.2 and the compact embedding theorems of trace at the boundary (cf. Theorem II.4.1 and Theorem II.5.2; cf. also the Notes for Chapter II), it follows that

$$\|\boldsymbol{v} - \boldsymbol{w}\|_{1,q,\partial\Omega} + \|p - \pi\|_{q,\partial\Omega} \to 0 \quad \text{as } \lambda \to 0. \tag{XII.9.18}$$

With the aid of (XII.9.17), (XII.9.18), and the asymptotic formula (VII.3.36) we thus obtain

$$\begin{aligned}\lim_{\lambda\to 0}\int_{\partial\Omega}\Big[u_i(y)T_{il}(\boldsymbol{w}_j, e_j)(x-y) - &\lambda u_i(y)E_{ij}(x-y)\delta_{1l}\Big]n_l d\sigma_y \\ &= \int_{\partial\Omega} v_{*i}(y)T_{il}(\boldsymbol{u}_j, \boldsymbol{q}_j)(x-y)d\sigma_y.\end{aligned} \tag{XII.9.19}$$

Finally, again from (VII.3.36) and (XII.9.18) we find

$$\begin{aligned}-\lim_{\lambda\to 0}\int_{\partial\Omega} &E_{ij}(x-y)\; T_{il}(\boldsymbol{u}, p)(y)n_l(y)d\sigma_y \\ &= \frac{1}{4\pi}\lim_{\lambda\to 0}\Big[\int_{\partial\Omega} T_{il}(\boldsymbol{u}, p)n_l\Big]|\log\lambda| \\ &\quad -\int_{\partial\Omega} U_{ij}(x-y)T_{il}(\boldsymbol{w}, \pi)(y)n_l(y)d\sigma_y.\end{aligned} \tag{XII.9.20}$$

Collecting (XII.9.14)–(XII.9.16), (XII.9.19), and (XII.9.20) we obtain (XII.9.12). The last part of the theorem is proved as in Section VII.8. Precisely, we have $\boldsymbol{w}_0 = -\boldsymbol{e}_1$ if and only if $\boldsymbol{u}_0 \equiv \boldsymbol{w} + \boldsymbol{e}_1$ is a solution to the problem

$$\left.\begin{aligned} \Delta \boldsymbol{u}_0 &= \nabla \pi \\ \nabla \cdot \boldsymbol{u}_0 &= 0 \end{aligned}\right\} \text{ in } \Omega$$

$$\boldsymbol{u}_0 = \boldsymbol{v}_* + \boldsymbol{e}_1 \text{ at } \partial\Omega$$

$$\lim_{|x|\to\infty} \boldsymbol{u}_0(x) = 0.$$

However, as we know from the results of Section V.7, such a solution exists if and only if (XII.9.13) is satisfied. The theorem is therefore proved.

□

Remark XII.9.1 An interesting consequence of Theorem XII.9.1 is the derivation of an asymptotic formula (in the limit of vanishing Reynolds number) for the force, $\boldsymbol{\mathcal{F}}(\boldsymbol{v}) := -\boldsymbol{\mathcal{T}}(\boldsymbol{v})$ exerted by the liquid on a body moving in it with constant velocity $\boldsymbol{e}_1$. Specifically, taking $\boldsymbol{v}_* \equiv 0$, from Theorem XII.9.1 we have that the limit solution $\boldsymbol{w}$ is identically zero so that from (XII.9.12) it easily follows in the limit $\lambda \to 0$

$$\boldsymbol{\mathcal{F}}(\boldsymbol{v}) = \Big(-4\pi \boldsymbol{e}_1 + \boldsymbol{o}(1)\Big)|\log\lambda|^{-1} \qquad \text{(XII.9.21)}$$

where $\boldsymbol{o}(1)$ denotes a vector quantity tending to zero with λ. This formula tells us that, in the limit of vanishingly small Reynolds number, the total force exerted from the liquid on the body is determined entirely by the velocity at infinity $\boldsymbol{e}_1$ and that it is directed along the line of this vector, namely, it produces only "drag" and no "lift". Surprisingly enough, it does not depend on the shape of the body.

Equation (XII.9.21) was obtained for the first time by Finn & Smith (1967b, Theorem 5.4).[1] However, due to the lack of suitable uniqueness theory, the solutions $\boldsymbol{v}$, p used by these authors differ a priori from those in Theorem XII.5.1, for which (XII.9.21) was derived. ■

Remark XII.9.2 Condition (XII.9.13) is satisfied if the body moves by self-propulsion; see Galdi (1999a, 2004 §1.2.2). ■

XII.10 Notes for the Chapter

Section XII.1. The first systematic treatment of the two-dimensional exterior problem for the Navier–Stokes equations traces back to the work of Goldstein (1933a, 1933b) and Leray (1933). In particular, the latter author proved existence of a solution to (XII.0.1) with finite Dirichlet integral. Leray's solution is obtained as a suitable limit of a sequence of solutions $\{\boldsymbol{v}_k, p_k\}$ of

[1] Due to a different definition of $\boldsymbol{\mathcal{T}}(\boldsymbol{v})$, the left-hand side of formula (XII.7.4) of Finn and Smith (1967b) differs from the left-hand side of (XII.9.21) by a factor λ^{-1}.

suitable problems $\mathcal{P}_k$ defined on a family $\{\Omega_{R_k}\}$ of bounded domains with $R_k \to \infty$ as $k \to \infty$. For example, in the physically relevant case when $\boldsymbol{v}_* \equiv \boldsymbol{f} \equiv 0$ and $\boldsymbol{v}_\infty \neq 0$ (steady plane motion of a viscous liquid past a translating long cylinder), the generic $\mathcal{P}_k$ is given by

$$\left.\begin{aligned} &\Delta \boldsymbol{v}_k = \boldsymbol{v}_k \cdot \nabla \boldsymbol{v}_k + \nabla p_k \\ &\nabla \cdot \boldsymbol{v}_k = 0 \end{aligned}\right\} \text{ in } \Omega_{R_k}$$
$$\boldsymbol{v}_k = 0 \;\text{ at } \partial\Omega \tag{$*$}$$
$$\boldsymbol{v}_k = -\boldsymbol{v}_\infty \;\text{ at } \partial B_{R_k},$$

where, for simplicity, we have put $\mathcal{R} = 1$. Leray proved the existence of a uniform bound for the Dirichlet integral of $\boldsymbol{v}_k$, that is,

$$\int_{\Omega_{R_k}} \nabla \boldsymbol{v}_k : \nabla \boldsymbol{v}_k \leq M \tag{$**$}$$

with M independent of k. Passing to the limit $k \to \infty$, he showed that (at least along a subsequence) $\boldsymbol{v}_k$, p_k converges to $\boldsymbol{v}_L$, p_L with $\boldsymbol{v}_L$, p_L solving (XII.0.1). The limit solution $\boldsymbol{v}_L$ still has a finite Dirichlet integral. However, Leray was not able to show that $\boldsymbol{v}_L$ also verifies (XII.0.2); cf. *loc. cit.* pp. 54-55. The study of the asymptotic behavior of Leray's solution was initiated by Gilbarg & Weinberger (1974). These authors obtained, in particular, that $\boldsymbol{v}_L$ converges (in the mean) to some vector, $\boldsymbol{v}_0$, which, as we have noticed in the general context of Section XII.3, need not coincide with $-\boldsymbol{v}_\infty$.

However, a more fundamental question concerning Leray's solution is the following one (cf. Finn 1970, p. 88): *Is $\boldsymbol{v}_L$ nontrivial?* Actually, we are not assured, a priori that $\boldsymbol{v}_L$ is nonidentically zero. As a matter of fact, Leray's construction in the linear case would lead to an identically vanishing solution $\boldsymbol{v}_L^{(s)}$ as a consequence of the Stokes paradox.To see this, let us disregard in $(*)$ the nonlinear term $\boldsymbol{v}_k \cdot \nabla \boldsymbol{v}_k$ for each $k \in \mathbb{N}$. Applying Leray's procedure, we would obtain that $\boldsymbol{v}_L^{(s)}$ solves the Stokes problem with zero boundary data and that, in view of $(**)$, $\boldsymbol{v}_L^{(s)}$ has a finite Dirichlet integral. Therefore, by Theorem V.2.2 we infer $\boldsymbol{v}_L^{(s)} \equiv 0$.

It is worth emphasizing that the above question arises also if, instead of a Leray's solution, we consider a generalized solution to (XII.0.1), (XII.0.2) constructed via Galerkin's method. Actually, in such a case, we look for $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{V}$, where $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ and $\boldsymbol{V}$ is a solenoidal extension of $-\boldsymbol{v}_\infty$, such that $\boldsymbol{V}(x) = -\boldsymbol{v}_\infty$, for all sufficiently large $|x|$, and $\boldsymbol{V} = \boldsymbol{0}$ at $\partial\Omega$; see Remark X.4.2. Therefore, as we know from Theorem II.7.6(ii) and Theorem III.5.1, $\boldsymbol{V} \in \mathcal{D}_0^{1,2}(\Omega)$, for Ω locally Lipschitz (for example). As a consequence, we can not exclude $\boldsymbol{u} = -\boldsymbol{V}$, namely, we can not exclude $\boldsymbol{v} \equiv \boldsymbol{0}$.

In the general nonlinear case, the answer to the question is still unknown. However, for symmetric flow (cf. Remark XII.3.2) Amick (1988, §4.2) has shown that $\boldsymbol{v}_L$ is nontrivial; see also Galdi (1999b, Theorem 3.1). On the other hand, it is very likely that if $\boldsymbol{v}_0 = 0$, then $\boldsymbol{v}_L \equiv 0$, but no proof is available yet; see Section XII.6. The validity of this latter condition has the following important consequence. In fact, if it is true, we would obtain, in particular, that, for symmetric flow, $\boldsymbol{v}_L$ must converge to a non-zero limit at infinity. Since, as observed in Remark XII.3.2, this convergence is uniformly pointwise, from Theorem XII.8.1 we could then conclude that every symmetric solution $\boldsymbol{v}_L, p_L$ has at large distances the same asymptotic structure of the Oseen fundamental solution. Moreover, one could prove $\boldsymbol{v}_0 = \alpha\boldsymbol{v}_\infty$, for some $\alpha \in (0, 1]$; see Galdi (1998, Section 3).

Section XII.2. The counterexample given here is the same that appears in Ladyzhenskaya's book (1969, pp. xi-xii).

Section XII.3. Theorem XII.3.1 and Lemma XII.3.3 were obtained for the first time by Gilbarg & Weinberger (1978). Their proof of the theorem is different from ours since it relies on the maximum principle for the vorticity field, which only holds in dimension two. Likewise, the proof of the lemma given by these authors is based on the Cauchy integral formula of complex functions (*cf.* (XII.4.29)) which, of course, is applicable only to plane flow. On the other hand, our proof relies on a general theorem concerning pointwise behavior of functions in spaces $D^{1,q}$. Theorem XII.3.2 is due to me.

Section XII.4. The question of the pointwise rate of decay of higher order derivatives for $\nabla\boldsymbol{v}$ and p of the type considered in Exercise XII.4.1 is treated in Russo (2010a). However this author's estimates turn out to be more conservative than those given in (XII.4.6) and (XII.4.7).

Section XII.5. Existence with $\boldsymbol{v}_\infty \neq 0$ was first shown by Finn & Smith (1967b), and it relies on their work for the analogous linear problem, cf. Finn & Smith (1967a). However, this result is obtained under somewhat more restrictive assumptions on the body force and the smoothness of Ω and $\boldsymbol{v}_*$ than those required in Theorem XII.5.1, which is taken from Galdi (1993). Moreover, the method of Finn and Smith is completely different than that of Galdi. Another approach to existence with $\boldsymbol{v}_\infty \neq \mathbf{0}$ is provided in Galdi (2004, §2.1).

As we already mentioned, in the case $\boldsymbol{v}_\infty = \mathbf{0}$, to date, no general existence theory has been developed, and few results are available only under suitable symmetry assumptions on the data. In particular, we refer the reader to the work by Galdi (2004, §3.3) and to the more recent one by Yamazaki (2009).

More precisely, the former author assumes the domain $\mathring{\Omega}^c$ to have two orthogonal axes of symmetry, that we may take coinciding with the x_1 and x_2 directions. Moreover, denote by $\mathscr{S}$ the class of vector functions $\boldsymbol{w} = (w_1, w_2)$ such that

$$w_1(x_1, x_2) = -w_1(-x_1, x_2) = w_1(x_1, -x_2)\,,$$

$$w_2(x_1, x_2) = w_2(-x_1, x_2) = -w_2(x_1, -x_2)\,.$$

Then, in Galdi (2004, Theorem 3.2) it is shown that for every boundary data $\boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega) \cap \mathscr{S}$, with flux through $\partial\Omega$ sufficiently small,[1] there exists

[1] By a mere oversight, this latter assumption is not mentioned in Galdi, *loc. cit.*

at least one generalized solution to (XII.0.1)–(XII.0.2) with $\boldsymbol{f} = \mathbf{0}$.[2] This solution, which is, in fact, of class $C^\infty(\Omega)$ along with the associated pressure field p, is also in the class $\mathscr{S}$. The simple (but crucial) observation for the proof of this result consists in the fact that a vector field $\boldsymbol{w} \in D_0^{1,2}(\Omega) \cap \mathscr{S}$ obeys the following inequality

$$\int_\Omega \frac{|\boldsymbol{w}|^2}{|x|^2} \leq c \int_\Omega |\nabla \boldsymbol{w}|^2 ,$$

and this, in turn, implies

$$\lim_{r\to\infty} \int_0^{2\pi} |\boldsymbol{w}(r,\theta)|^2 d\theta = 0 ;$$

see Galdi, *loc. cit.*, for details. As a matter of fact, this latter condition in conjunction with Theorem XII.3.4 implies the stronger property for the generalized solution:

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = \mathbf{0} , \quad \text{uniformly} .$$

However, we are not expecting, in general, to give a specific order of decay for such solutions in terms of negative powers of $|x|$. The reason is because, when Ω is the exterior of a circle, they belong to the same class where solutions (XII.2.7) belong, for which, as we know, no order of decay of the above type can be given.

In the paper of Yamazaki (2009) the case $\Omega = \mathbb{R}^2$ is considered, and it is assumed that $\boldsymbol{f} = \left(\dfrac{\partial F}{\partial x_2}, -\dfrac{\partial F}{\partial x_1}\right)$, where F belongs to the "antisymmetry class" $\mathscr{A}$ defined by the following conditions

$$\begin{aligned} &w(x_1,x_2) = -w_1(-x_1,x_2) , \; w_(x_1,-x_2) = -w(x_1,x_2) , \\ &w(x_1,x_2) = -w(x_2,x_1) , \; w(-x_1,-x_2) = -w(x_2,x_1) . \end{aligned}$$

Under the further assumption that $F = O(|x|^{-2})$ as $|x| \to \infty$, and its magnitude is suitably restricted, in Theorem 2.1 of Yamazaki, *loc. cit.*, it is shown the existence of a corresponding generalized solution which satisfies further summability properties. Moreover, the vorticity decays like $|x|^{-2}$.

I conjecture that the two-dimensional, plane exterior problem corresponding to $\boldsymbol{v}_\infty = \mathbf{0}$ is, generically, not solvable. In the spirit of the approach followed by Galdi (2009) where an analogous result is proved in the three-dimensional case, a way of showing this conjecture could be by assessing that the relevant nonlinear Navier–Stokes operator, properly defined, is Fredholm of negative index.

[2] The case $\boldsymbol{f} \neq \mathbf{0}$ can be easily handled, provided $\boldsymbol{f} \in \mathscr{S}$ and decays suitably fast at large distances.

Section XII.7. The result of Galdi & Sohr (1995) on the summability properties of generalized solutions was later rediscovered by Sazonov (1999), under more stringent assumptions.

The "cut-off" technique employed in Theorem XII.7.2 has been extended to dimension greater than two by Farwig and Sohr (1995, 1998).

Section XII.8. Though inspired by the work of Smith (1965), the method presented here is due to me.

Employing the results of Smith (1965), Amick (1991) proved an analog of Theorem XII.8.1–Theorem XII.8.3 for the case of *symmetric flows*.

Section XII.9. The main result of this section is from the work of Galdi (1993).

XIII

Steady Navier–Stokes Flow in Domains with Unbounded Boundaries

Ὅν οί ϑεοὶ φιλοῦσιν, ἀποϑνήσκει νέος.

MENANDROS

Introduction

Let us consider a steady Navier–Stokes flow of a liquid filling a domain with two unbounded "outlets." The relevant region of flow is thus a domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ such that

$$\Omega = \bigcup_{i=0}^{2} \Omega_i$$

where Ω_0 is a compact set of $\mathbb{R}^n$ while, in possibly different coordinate systems,

$$\Omega_i = \{x \in \mathbb{R}^n : x_n > 0,\ x' \equiv (x_1, \ldots, x_{n-1}) \in \Sigma_i(x_n)\} \quad i = 1, 2,$$

and $\Sigma_i = \Sigma_i(x_n)$ are domains of $\mathbb{R}^{n-1}$ smoothly varying with x_n. The difficulties one encounters in studying the mathematical properties of such a flow have already been discussed at some length in the Introduction to Chapter VI and, therefore, they will not be repeated here. In this respect, we wish only to recall that, even in the linearized Stokes approximation, many fundamental problems continue to be open when the cross sections Σ_i remain either bounded or unbounded. As expected, in the Navier–Stokes case, in addition to these unsolved questions, we have others that are merely due to the nonlinear character of the equations.

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The objective of the present chapter is to study two characteristic problems (one with bounded cross section, the other with an unbounded one) which, though completely solved in the linearized case, when analyzed in the nonlinear context present several basic aspects that are still far from clear.

The first one is the so-called Leray's problem; cf. also Section VI.1. In this case Ω is a "distorted channel" of $\mathbb{R}^n$, $n = 2, 3$, that is, Σ_i are bounded and independent of x_n, so that each outlet Ω_i reduces to a semi-infinite straight cylinder for $n = 3$ and to a semi-infinite strip for $n = 2$. One has to study steady flow that corresponds to a given velocity flux Φ through the cross section of Ω and whichin each Ω_i tends to the Poiseuille flow $\boldsymbol{v}_0^{(i)}$ corresponding to Φ. As we know, these fields satisfy

$$\begin{gathered}\boldsymbol{v}_0^{(i)} = v_0^{(i)}(x')\mathbf{e}_n \\ \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j^2} v_0^{(i)}(x') = -C_i \ \text{ in } \Sigma_i \\ v_0^{(i)}(x') = 0 \ \text{ at } \partial\Sigma_i\end{gathered} \tag{XIII.0.1}$$

with C_i constants uniquely related to Φ; see Exercise VI.0.1.

The second is a problem introduced by Heywood (1976); cf. also Section VI.5. Here Ω is an "aperture domain" of $\mathbb{R}^3$, namely,

$$\Omega = \{x \in \mathbb{R}^3 : x_3 \neq 0 \text{ or } x' \in S\}$$

with S a two-dimensional bounded domain. Therefore, the cross sections Σ_i coincide with the whole of $\mathbb{R}^2$. The question is to determine a flow corresponding to a given velocity flux Φ through S and whose velocity field tends to zero at large distances.

In both cases we wish to analyze existence, uniqueness, and asymptotic behavior of corresponding solutions. To solve these questions we shall use an approach that is similar in principle to that employed in the linear case. Of course, we now have the further complication of the nonlinear term. This complication manifests itself in several ways, which we shall now briefly describe.

As in the linear case, we look for a generalized solution $\boldsymbol{v}$ in the form

$$\boldsymbol{v} = \boldsymbol{u} + \mathbf{a}.$$

In this relation $\boldsymbol{a}$ is a flux carrier, i.e., a smooth solenoidal field in Ω that vanishes on $\partial\Omega$, tends to the prescribed velocity field at large distances, and satisfies

$$\int_\Sigma \boldsymbol{a} \cdot \boldsymbol{n} = \Phi$$

with $\boldsymbol{n}$ unit normal to Σ, whereas

$$\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega).$$

As we have already seen in the case of a region of flow with a compact boundary, in the case at hand in order to guarantee the existence of a solution it is enough to prove an a priori estimate for the Dirichlet integral of $\boldsymbol{u}$. Moreover, again as in the case of a compact boundary (cf. Sections VII.4 and IX.4), to show existence without restrictions on $|\Phi|$, we have to show that, for each $\alpha > 0$ there is a flux carrier $\boldsymbol{a} = \boldsymbol{a}(\Phi; \alpha)$ such that

$$-\int_{\Omega} \boldsymbol{u} \cdot \nabla \boldsymbol{a} \cdot \boldsymbol{u} \leq \alpha |\boldsymbol{u}|_{1,2}^2, \quad \text{for all} \ \ \boldsymbol{u} \in \mathcal{D}(\Omega). \tag{XIII.0.2}$$

However, if the exits Ω_i are cylindrical (or, more generally, have bounded cross section) the existence of flux carriers satisfying (XIII.0.2) is not yet known. Nevertheless, one can construct, in such a case, fields $\boldsymbol{a}$ verifying the following condition:

$$\left| \int_{\Omega} \boldsymbol{u} \cdot \nabla \boldsymbol{a} \cdot \boldsymbol{u} \right| \leq c |\Phi| |\boldsymbol{u}|_{1,2}^2, \quad \text{for all} \ \ \boldsymbol{u} \in \mathcal{D}(\Omega),$$

for some $c = c(\Omega, n)$. The consequence of this fact is that, unlike the linear case, so far, one is able to produce existence of solutions to Leray's problem only for small values of $|\Phi|$ (compared to the coefficient of kinematical viscosity ν). The question of whether Leray's problem is solvable for any value of the flux therefore remains open. On the other hand, if the outlets Ω_i contain a semi-infinite cone, Ladyzhenskaya & Solonnikov (1977) have shown that there are vector fields $\boldsymbol{a}$ verifying condition (XIII.0.2) and, as a consequence, for domains with this type of outlets it is possible to show existence "in the large," that is, for arbitrary values of the flux Φ. Thus, in particular, this kind of existence holds for an "aperture domain."

Once a solution has been determined, the next task is to investigate its asymptotic structure. In the case of Leray's problem, one shows that, again if $|\Phi|$ is sufficiently small, all generalized solutions (together with their derivatives of arbitrary order) must tend to the corresponding Poiseuille velocity field exponentially fast. Similarly, for Heywood's problem, one is able to give a detailed asymptotic expansion, which resembles that given for the linear case, provided $|\Phi|$ is sufficiently small. If these problems can be solved for arbitrary values of the flux, it remains an open question.[1]

Another point that we would like to emphasize is the two-dimensional version of the flow through an aperture. The situation is in a sense similar to the plane exterior flow which we have analyzed in the preceding chapter. Specifically, in the case at hand we can prove, with no restriction on the flux, existence of vector fields $\boldsymbol{v}$ which solve the momentum equation, satisfy the flux and boundary conditions and such that

$$\int_{\Omega} \nabla \boldsymbol{v} : \nabla \boldsymbol{v} < \infty. \tag{XIII.0.3}$$

[1] It is likely that in the case of the (three-dimensional) aperture flow the restriction on Φ can be removed, but no proof is known.

The difficulty is to show that (XIII.0.3) guarantees that $\boldsymbol{v}$ vanishes at large distances. We shall not investigate this question here, and refer the reader to the paper by Galdi, Padula, & Passerini (1995), where it is shown that $\boldsymbol{v}$ tends to zero uniformly pointwise. Moreover, if the aperture S is symmetric around the x_2-axis,[2] and if $|\Phi|$ is sufficiently small, it can be shown that the solution behaves at large distances as a suitable Jeffery–Hamel solution, see Galdi, Padula, & Solonnikov (1996); see also Remark XIII.9.5. If S is not symmetric, the question of the asymptotic behavior is open.

Finally, we wish to remark that, even though obtained for particular regions of flow, most of the results we find could be extended without conceptual difficulties to more general situations. For example, we could show existence of generalized solutions (with no restriction on the flux) in domains whose outlets contain and are contained in suitable semi-infinite cones or for domains of the type considered in Section VI.3. Likewise, we could furnish a complete asymptotic description (for small values of the flux) of generalized solutions in domains whose outlets contain the body of revolution $\{|x'| < x_n^\alpha,\ \alpha > 1\}$. For other results concerning steady flow in domains with unbounded boundaries, we refer the reader to the Notes for this Chapter.

XIII.1 Leray's Problem: Generalized Solutions and Related Properties

Let us consider the steady flow of a viscous liquid moving in a smooth infinite "distorted channel." We shall thus assume that the relevant region of flow is a domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, of class C^∞[1] with two cylindrical ends, namely,

$$\Omega = \bigcup_{i=0}^{2} \Omega_i \qquad \text{(XIII.1.1)}$$

where Ω_0 is a compact subset of Ω and Ω_i, $i = 1, 2$, are disjoint domains which, in possibly different coordinate systems, are given by

$$\Omega_1 = \left\{x \in \mathbb{R}^n :\ x_n < 0,\ x' \in \Sigma_1\right\}$$

$$\Omega_2 = \left\{x \in \mathbb{R}^n :\ x_n > 0,\ x' \in \Sigma_2\right\}.$$

Here, $x' = (x_1, \ldots, x_{n-1})$ and Σ_i, $i = 1, 2$, are C^∞–smooth simply connected domains of the plane if $n = 3$, while, if $n = 2$ (the case of plane flow), $\Sigma_i = (-d_i, d_i)$, for some $d_i > 0$. Denote by Σ a cross section of Ω, that is, any bounded intersection of Ω with an $(n-1)$–dimensional plane that in Ω_i

[2] Orthogonal to S.

[1] Here applies the same remark we made for the linear case in Footnote 1 of Section VI.1.

reduces to Σ_i, and by $\boldsymbol{n}$ a unit vector orthogonal to Σ and oriented from Ω_1 toward Ω_2 (say).

The main objective of this and of the next three sections is to study the solvability of the following *Leray's problem*:[2] *Given $\Phi \in \mathrm{IR}$, to find a pair $\boldsymbol{v}, p$ such that*

$$\left.\begin{aligned} \nu\Delta\boldsymbol{v} &= \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p \\ \nabla\cdot\boldsymbol{v} &= 0 \end{aligned}\right\} \quad \text{in } \Omega \tag{XIII.1.2}$$

with

$$\begin{gathered} \boldsymbol{v} = 0 \quad \text{at } \partial\Omega \\ \int_{\Sigma} \boldsymbol{v}\cdot n = \Phi \end{gathered} \tag{XIII.1.3}$$

and

$$\boldsymbol{v} \to \boldsymbol{v}_0^{(i)} \quad \text{as } |x| \to \infty \text{ in } \Omega_i \tag{XIII.1.4}$$

where $\boldsymbol{v}_0^{(i)}$, $i = 1, 2$, are the velocity fields (XIII.0.1)*, of the Poiseuille flow in Ω_i, corresponding to the flux Φ.*

We begin to give a generalized formulation of this problem, which parallels that furnished for the linearized case in Section VI.1. Specifically, multiplying $(\text{XIII.1.2})_1$ by $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and integrating by parts, we deduce

$$\nu\big(\nabla\boldsymbol{v}, \nabla\boldsymbol{\varphi}\big) = \big(\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{v}\big), \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega). \tag{XIII.1.5}$$

We thus have the following definition

Definition XIII.1.1. A vector field $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ is called a *weak* (or *generalized*) *solution to Leray's problem* (XIII.1.2)–(XIII.1.4) if and only if

(i) $\boldsymbol{v} \in W^{1,2}_{loc}(\overline{\Omega})$;
(ii) $\boldsymbol{v}$ satisfies (XIII.1.5);
(iii) $\boldsymbol{v}$ is (weakly) divergence-free in Ω;
(iv) $\boldsymbol{v}$ satisfies (XIII.1.3) in the trace sense;
(v) $(\boldsymbol{v} - \boldsymbol{v}_0^{(i)}) \in W^{1,2}(\Omega_i)$, $\quad i = 1, 2$.

Remark XIII.1.1 As shown in Section VI.1, for all $\boldsymbol{v}$ with $\boldsymbol{v} - \boldsymbol{v}_0^{(i)}(x') \in W^{1,2}(\Omega_i)$ one has

$$\int_{\Sigma_i} |\boldsymbol{v}(x', x_n) - \boldsymbol{v}_0^{(i)}(x')|^2 dx' \to 0 \quad \text{as } |x_n| \to \infty \text{ in } \Omega_i.$$

Therefore, condition (v) of Definition XIII.1.1 is the generalized version of (XIII.1.4). ■

[2] For simplicity, we shall assume that no body forces are acting on the liquid. Obvious generalizations are left to the reader as an exercise.

Remark XIII.1.2 Since

$$|\big(\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi},\boldsymbol{v}\big)| \le \sup_{\sigma}|\nabla\boldsymbol{\varphi}|\|\boldsymbol{v}\|_{2,\sigma}$$

with $\sigma = \operatorname{supp}\boldsymbol{\varphi}$, identity (XIII.1.5) is well defined for a generalized solution. ■

The following result shows that to every weak solution we can associate a corresponding pressure field.

Lemma XIII.1.1 *Let $\boldsymbol{v}$ be a generalized solution to Leray's problem* (XIII.1.2)–(XIII.1.4). *Then there exists $p \in L^2_{loc}(\Omega)$ such that*

$$\nu\big(\nabla\boldsymbol{v},\nabla\boldsymbol{\psi}\big) = \big(\boldsymbol{v}\cdot\nabla\boldsymbol{\psi},\boldsymbol{v}\big) + \big(p\nabla\cdot\boldsymbol{\psi}\big) \quad \textit{for all } \boldsymbol{\psi}\in C_0^\infty(\Omega). \tag{XIII.1.6}$$

Proof. In view of (i) of Definition XIII.1.1 and Corollary III.5.2, it suffices to show that

$$\mathcal{F}(\boldsymbol{\psi}) \equiv \big(\boldsymbol{v}\cdot\nabla\boldsymbol{\psi},\boldsymbol{v}\big)\ , \quad \boldsymbol{\psi}\in D_0^{1,2}(\Omega')$$

defines a bounded linear functional on $D_0^{1,2}(\Omega')$, with Ω' any bounded domain of Ω. However, by the Hölder inequality and the embedding Theorem II.3.4, we have the following.

$$|\mathcal{F}(\boldsymbol{\psi})| \le |\boldsymbol{\psi}|_{1,2,\Omega'}\|\boldsymbol{v}\|^2_{4,\Omega'} \le c|\boldsymbol{\psi}|_{1,2,\Omega'}\|\boldsymbol{v}\|^2_{1,2,\Omega'}$$

with $c = c(\Omega', n)$, and the lemma follows. □

We end this section by establishing the differentiability properties of generalized solutions. Specifically, we have the following

Theorem XIII.1.1 *Let $\boldsymbol{v}$ be a generalized solution to Leray's problem* (XIII.1.2)–(XIII.1.4) *and let p be the corresponding pressure field associated to $\boldsymbol{v}$ by* Lemma XIII.1.1. *Then*

$$\boldsymbol{v},\ p \in C^\infty(\overline{\Omega'}) \tag{XIII.1.7}$$

for all bounded domains Ω' with $\Omega' \subset \Omega$.

Proof. We show the proof for $n = 3$; the case where $n = 2$ is treated analogously. Since $\boldsymbol{v} \in W^{1,2}_{loc}(\overline{\Omega})$, we have

$$\boldsymbol{v}\cdot\nabla v \in L^{3/2}_{loc}(\overline{\Omega}) \tag{XIII.1.8}$$

and, therefore, from Theorems IV.4.1 and IV.5.1 it follows that

$$\boldsymbol{v}\in W^{2,3/2}_{loc}(\overline{\Omega}), \quad p\in W^{1,3/2}_{loc}(\overline{\Omega}).$$

Thus $\boldsymbol{v}$ and p have a better regularity than assumed at the outset and, with the help of Theorem II.2.4 we infer

$$\boldsymbol{v} \cdot \nabla v \in L^q_{loc}(\overline{\Omega}) \quad \text{for all} \;\; q < 3,$$

which improves on (XIII.1.8). Again using Theorem IV.4.1 and Theorem IV.5.1, we find

$$\boldsymbol{v} \in W^{2,q}_{loc}(\overline{\Omega}), \;\; p \in W^{1,q}_{loc}(\overline{\Omega}), \quad \text{for all} \;\; q < 3.$$

We then obtain a further improvement on the (local) summability of the solution and iterating this procedure, we prove, by induction, the validity of (XIII.1.7). □

Remark XIII.1.3 If the domain Ω is not of class C^∞, we can prove a weaker version of (XIII.1.7), namely,

$$\boldsymbol{v}, \; p \in C^\infty(\Omega')$$

only for every bounded Ω' with $\overline{\Omega}' \subset \Omega$. The regularity up to the boundary will then depend on the assumed regularity of Ω, as specified by the assumptions of Theorem IV.5.1; cf. Footnote 1. ■

XIII.2 On the Uniqueness of generalized Solutions to Leray's Problem

In this section we shall prove a general uniqueness result for generalized solutions to Leray's problem (XIII.1.2)–(XIII.1.4). To this end, we need to study in some detail the continuity properties of the trilinear form $\big(\boldsymbol{v} \cdot \nabla \boldsymbol{u}, \boldsymbol{w}\big)$.

We begin to show some embedding inequalities in the domains Ω_i, $i = 1, 2$. For $t \geq 0$, $t_2 > t_1 > 0$, we set

$$\begin{aligned} \Omega_1^t &= \big\{x \in \Omega_1 : \; x_n < -t\big\} \\ \Omega_2^t &= \big\{x \in \Omega_2 : \; x_n > t\big\} \\ \Omega_{0,t} &= \Omega - \big[\overline{\Omega}_1^t \cup \overline{\Omega}_2^t\big] \\ \Omega_{i,t_1,t_2} &= \Omega_i^{t_1} \cap \Omega_i^{t_2}, \quad i = 1, 2. \end{aligned} \tag{XIII.2.1}$$

Lemma XIII.2.1 *Let $\Omega \equiv \Omega_i$, and $\Sigma \equiv \Sigma_i$, $i = 1, 2$, and let $u \in W^{m,q}(\Omega_{t,t+1})$, $m \geq 0$, $q \geq 1$. Then the embedding inequalities* (II.3.17)–(II.3.18) *hold in $\Omega_{t,t+1}$ with constants c_1, c_2, and c_3 independent of t. Moreover, for all $u \in D^{1,2}(\Omega)$ with $u = 0$ at $\partial\Omega - \{x_n = 0\}$ we have*

$$\begin{aligned} \|u\|_{4,\Omega_{t,t+1}} &\leq \kappa |u|_{1,2,\Omega_{t,t+1}} \\ \|u\|_{6,\Omega_{t,t+1}} &\leq K |u|_{1,2,\Omega_{t,t+1}} \end{aligned} \tag{XIII.2.2}$$

where[1]

$$\kappa = \begin{cases} \left(\frac{1}{4}|\Sigma| + \frac{1}{\sqrt{2}}|\Sigma|^{1/2}\right)^{1/4} & \textit{if } n = 3 \\ \left[\dfrac{8d^2}{\pi}\left(\dfrac{d}{\pi}+1\right)\right]^{1/4} & \textit{if } n = 2 \end{cases}$$

and

$$K = \begin{cases} \left[\frac{9}{8}(1 + \frac{1}{\sqrt{2}}|\Sigma|^{1/2})\right]^{1/3} & \textit{if } n = 3 \\ (4\kappa d^2)^{1/6} \quad \textit{if } n = 2. \end{cases}$$

Proof. The first part of the lemma is an immediate consequence of Theorem II.3.4. Actually, once we establish, from this theorem, inequalities (II.3.17)–(II.3.18) for t=0 with constants c_i, $i = 1, 2, 3$, independent of t, by virtue of the translational invariance $x_n \to x_n - t$ they remain established for all $t > 0$, with the same constants c_i. We shall now show the second part of the lemma. From (II.3.9) (in the case $n = 3$) and the elementary inequality

$$|u(x_1, x_2)|^2 \le \int_{-d}^{d} |u(\xi_1, x_2)| \left|\frac{\partial u(\xi_1, x_2)}{\partial \xi_1}\right| d\xi_1 \qquad x_1 \in (-d, d)$$

(in the case $n = 2$), we have

$$\|u\|^4_{4,\Sigma} \le \lambda \|u\|^2_{2,\Sigma}\ |u|^2_{1,2,\Sigma}, \tag{XIII.2.3}$$

where

$$\lambda = \begin{cases} \frac{1}{2} \ \text{ if } n = 3 \\ 2d \text{ if } n = 2. \end{cases}$$

Furthermore, from (II.5.3) we have

$$\|u\|^2_{2,\Sigma} \le \mu |u|^2_{1,2,\Sigma}, \tag{XIII.2.4}$$

where μ is the *Poincaré constant* for Σ. An upper bound for μ is obtained from Exercise II.5.2 and (II.5.5), and we have

$$\mu \le \begin{cases} \dfrac{|\Sigma|}{2} & \text{if } n = 3 \\ \dfrac{(2d)^2}{\pi^2} & \text{if } n = 2. \end{cases}$$

Integrating (XIII.2.3) over the variable x_3, we derive

$$\|u\|^4_{4,\Omega_{t,t+1}} \le \lambda \int_t^{t+1} \|u\|^2_{2,\Sigma} \tag{XIII.2.5}$$

[1] Recall that $|\Sigma| = 2d$ if $n = 2$.

On the other hand, for all $x' \in \Sigma$, $\xi_3 \in (t, t+1)$ and $q \geq 1$,

$$|u(x',t)|^q \leq |u(x',\xi_3)|^q + q \int_t^{t+1} |u(x',\xi)|^{q-1} \, |\nabla u(x',\xi)| d\xi, \qquad \text{(XIII.2.6)}$$

and so, integrating this latter inequality with $q = 2$ over x' and ξ_3, with the help of the Schwarz inequality we find

$$\|u\|_{2,\Sigma}^2 \leq \|u\|_{2,\Omega_{t,t+1}}^2 + 2\|u\|_{2,\Omega_{t,t+1}} |u|_{1,2,\Omega_{t,t+1}}.$$

Using (XIII.2.4) in this relation furnishes

$$\|u\|_{2,\Sigma}^2 \leq (\mu + 2\sqrt{\mu}) |u|_{1,2,\Omega_{t,t+1}}^2.$$

Inequality $(\text{XIII.2.2})_1$ becomes then a consequence of the latter and of (XIII.2.5). The case where $n = 2$ is treated in a completely similar way. Finally, we show $(\text{XIII.2.2})_2$. We consider the case where $n = 3$, leaving to the reader the two-dimensional case as an exercise. From (II.3.9) we have

$$\|u\|_{6,\Sigma}^6 \leq \frac{9}{8} \|u\|_{4,\Sigma}^4 |u|_{1,2,\Sigma}^2$$

and so, integrating between t and $t+1$ we find

$$\frac{8}{9} \|u\|_{6,\Omega_{t,t+1}}^6 \leq \int_t^{t+1} \|u\|_{4,\Sigma}^4 |u|_{1,2}^2.$$

Using (XIII.2.6) with $q = 4$ and integrating over $x' \in \Sigma$ and $\xi_3 \in [t, t+1]$, it follows that

$$\|u\|_{4,\Sigma}^4 \leq \|u\|_{4,\Omega_{t,t+1}}^4 + 4 \int_{\Omega_{t,t+1}} |u|^3 |\nabla u|$$

and so, observing that from the Hölder inequality and (XIII.2.4),

$$\|u\|_{4,\Omega_{t,t+1}}^4 \leq \|u\|_{6,\Omega_{t,t+1}}^3 \|u\|_{2,\Omega_{t,t+1}} \leq \sqrt{\mu} \|u\|_{6,\Omega_{t,t+1}}^3 |u|_{1,2,\Omega_{t,t+1}}$$

$$\int_{\Omega_{t,t+1}} |u|^3 |\nabla u| \leq \|u\|_{6,\Omega_{t,t+1}}^3 |u|_{1,2,\Omega_{t,t+1}},$$

from the last four displayed inequalities and recalling the value of μ, we conclude that

$$\|u\|_{6,\Omega_{t,t+1}}^6 \leq \frac{9}{8} \left(1 + \frac{|\Sigma|^{1/2}}{\sqrt{2}} \right) \|u\|_{6,\Omega_{t,t+1}}^3 |u|_{1,2,\Omega_{t,t+1}}^3,$$

and $(\text{XIII.2.2})_2$ follows. The lemma is proved. □

Next we shall investigate the continuity of the trilinear form $(\boldsymbol{v} \cdot \nabla \boldsymbol{u}, \boldsymbol{w})$ in suitable function spaces. To this end, set

$$\mathcal{C}_\Phi = \left\{ \boldsymbol{v} \in W_{loc}^{1,2}(\overline{\Omega}) : \ |\boldsymbol{v} - \boldsymbol{v}_0^{(i)}|_{1,2,\Omega_i} < \infty, \quad i = 1, 2 \right\}$$

where, we recall, $\boldsymbol{v}_0^{(i)}$ is the (uniquely determined) velocity field of the Poiseuille flow in Ω_i corresponding to the flux Φ.

Remark XIII.2.1 Every weak solution to Leray's problem (corresponding to the flux Φ) is in the class $\mathcal{C}_\Phi$. ∎

Lemma XIII.2.2 *Let Ω be as in* (XIII.1.1) *and let $\boldsymbol{u}, \boldsymbol{w} \in D_0^{1,2}(\Omega)$. Then if $\boldsymbol{v} \in D_0^{1,2}(\Omega)$ we have*

$$|(\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{w})| \le c_1|\boldsymbol{v}|_{1,2}|\boldsymbol{u}|_{1,2}|\boldsymbol{w}|_{1,2} \tag{XIII.2.7}$$

with $c_1 = c_1(\Omega, n)$. Moreover, if $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega)$, we have

$$(\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{w}) = -(\boldsymbol{v}\cdot\nabla\boldsymbol{w}, \boldsymbol{u}) \tag{XIII.2.8}$$

so that

$$(\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{u}) = 0. \tag{XIII.2.9}$$

Finally, if $\boldsymbol{v} \in \mathcal{C}_\Phi$, and

$$\mathcal{A}_i \equiv |\boldsymbol{v} - \boldsymbol{v}_0^{(i)}|_{1,2,\Omega_i},$$

there is $c_i = c_i(\Omega, n) > 0$, $i = 2, 3$ such that

$$\begin{aligned}
|(\boldsymbol{u}\cdot\nabla\boldsymbol{v}, \boldsymbol{w})| &\le c_2\left(\textstyle\sum_{i=1}^2 \mathcal{A}_i + |\boldsymbol{v}|_{1,2,\Omega_0} + |\Phi|\right)|\boldsymbol{u}|_{1,2}|\boldsymbol{w}|_{1,2}\\
|(\boldsymbol{v}\cdot\nabla\boldsymbol{w}, \boldsymbol{u})| &\le c_2\left(\textstyle\sum_{i=1}^2 \mathcal{A}_i + |\boldsymbol{v}|_{1,2,\Omega_0} + |\Phi|\right)|\boldsymbol{u}|_{1,2}|\boldsymbol{w}|_{1,2}\\
|(\boldsymbol{v}\cdot\nabla\boldsymbol{v}, \boldsymbol{u})| &\le c_3\left(\textstyle\sum_{i=1}^2 \mathcal{A}_i^2 + |\boldsymbol{v}|_{1,2,\Omega_0}^2 + |\Phi|^2\right)|\boldsymbol{u}|_{1,2}.
\end{aligned} \tag{XIII.2.10}$$

If, in addition, $\nabla\cdot\boldsymbol{v} = 0$, then

$$(\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{u}) = 0. \tag{XIII.2.11}$$

Proof. We split $(\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{w})$ as the sum of three integrals $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_0$ over the regions Ω_1, Ω_2, and Ω_0, respectively. We have, by the Hölder inequality,

$$|\mathcal{I}_1| \le \|\boldsymbol{v}\|_{4,\Omega_1}\|\boldsymbol{w}\|_{4,\Omega_1}|\boldsymbol{u}|_{1,2}. \tag{XIII.2.12}$$

However, from Lemma XIII.2.1, it follows that

$$\|u\|_{4,\Omega_1} \le \kappa_i|u|_{1,2} \quad \text{for all } u \in D_0^{1,2}(\Omega),\ \ i = 1, 2, \tag{XIII.2.13}$$

and so (XIII.2.12) yields

$$|\mathcal{I}_1| \le \kappa_1^2|\boldsymbol{v}|_{1,2}|\boldsymbol{w}|_{1,2}|\boldsymbol{u}|_{1,2}. \tag{XIII.2.14}$$

Likewise, we prove

$$|\mathcal{I}_2| \le \kappa_2^2|\boldsymbol{v}|_{1,2}|\boldsymbol{w}|_{1,2}|\boldsymbol{u}|_{1,2}. \tag{XIII.2.15}$$

From the embedding Theorem II.3.4 it follows that[2]

$$\begin{aligned}|\mathcal{I}_0| &\le \|\boldsymbol{v}\|_{4,\Omega_0}\|\boldsymbol{w}\|_{4,\Omega_0}|\boldsymbol{u}|_{1,2}\\ &\le c\|\boldsymbol{v}\|_{1,2,\Omega_0}\|\boldsymbol{w}\|_{1,2,\Omega_0}|\boldsymbol{u}|_{1,2}.\end{aligned}$$

However, since $\boldsymbol{v}$ and $\boldsymbol{w}$ vanish (in the trace sense) at the boundary $\partial\Omega$, from (II.5.18) we find that $\|\cdot\|_{1,2,\Omega_0}$ and $|\cdot|_{1,2,\Omega_0}$ are equivalent norms so that the latter inequality furnishes, in particular,

$$|\mathcal{I}_0| \le c|\boldsymbol{v}|_{1,2}|\boldsymbol{w}|_{1,2}|\boldsymbol{u}|_{1,2}. \tag{XIII.2.16}$$

Relation (XIII.2.7) is a consequence of (XIII.2.14)–(XIII.2.16). To show (XIII.2.8) we notice that it is trivially verified (by integration by parts) if $\boldsymbol{w}, \boldsymbol{v} \in C_0^\infty(\Omega)$. Under the assumptions on $\boldsymbol{w}$, $\boldsymbol{v}$ stated in the theorem, condition (XIII.2.8) follows from the continuity property (XIII.2.7) and the density of $C_0^\infty(\Omega)$ into $D_0^{1,2}(\Omega)$. Of course, (XIII.2.8) implies (XIII.2.9). Let us now prove (XIII.2.10). We split, as before, $\big(\boldsymbol{u}\cdot\nabla\boldsymbol{v}, \boldsymbol{w}\big)$ as the sum of $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_0$. We have

$$|\mathcal{I}_1| \le |\big(\boldsymbol{u}\cdot\nabla(\boldsymbol{v}-\boldsymbol{v}_0^{(i)}), \boldsymbol{w}\big)_{\Omega_1}| + |\big(\boldsymbol{u}\cdot\nabla\boldsymbol{v}_0^{(i)}, \boldsymbol{w}\big)_{\Omega_1}|.$$

Thus, by (XIII.2.13) and the Hölder inequality, we deduce that

$$|\mathcal{I}_1| \le \kappa_1^2\mathcal{A}_1|\boldsymbol{u}|_{1,2}|\boldsymbol{w}|_{1,2} + \int_1^\infty |\boldsymbol{v}_0^{(1)}|_{1,2,\Sigma}\|\boldsymbol{u}\|_{4,\Sigma}\|\boldsymbol{w}\|_{4,\Sigma},$$

where integration is performed over the x_3 variable. Because of Exercise VI.0.1, (XIII.2.3), and (XIII.2.4), it readily follows that

$$|\mathcal{I}_1| \le c(\mathcal{A}_1 + |\Phi|)|\boldsymbol{u}|_{1,2}|\boldsymbol{w}|_{1,2}. \tag{XIII.2.17}$$

Likewise, we show

$$|\mathcal{I}_2| \le c(\mathcal{A}_2 + |\Phi|)|\boldsymbol{u}|_{1,2}|\boldsymbol{w}|_{1,2}. \tag{XIII.2.18}$$

Concerning $\mathcal{I}_0$, we have

$$|\mathcal{I}_0| \le c|\boldsymbol{v}|_{1,2,\Omega_0}\|\boldsymbol{u}\|_{1,2,\Omega_0}\|\boldsymbol{w}\|_{1,2,\Omega_0},$$

where use has been made of the Hölder inequality and the embedding Theorem II.3.4. However, as already remarked, $\|\cdot\|_{1,2,\Omega_0}$ and $|\cdot|_{1,2,\Omega_0}$ are equivalent norms for functions from $D_0^{1,2}(\Omega_0)$ and so

$$|\mathcal{I}_0| \le c|\boldsymbol{v}|_{1,2,\Omega_0}|\boldsymbol{u}|_{1,2}|\boldsymbol{w}|_{1,2}, \tag{XIII.2.19}$$

and $(\text{XIII.2.10})_1$ becomes a consequence of (XIII.2.17)–(XIII.2.19). The proof of $(\text{XIII.2.10})_2$ is very much the same as that just furnished for $(\text{XIII.2.10})_1$

[2] Throughout the rest of the proof, the symbol c will denote a quantity depending (at most) on Ω and n.

and, therefore, it will be omitted. Also the proof of $(\text{XIII}.2.10)_3$ is obtained in a similar fashion, once we notice that

$$\boldsymbol{v}_0^{(i)} \cdot \nabla \boldsymbol{v}_0^{(i)} \equiv 0 \ \text{ in } \Omega_i \ \ i = 1, 2.$$

To show the last property (XIII.2.11), we observe that, since by $(\text{XIII}.2.10)_2$

$$\boldsymbol{v} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u} \in L^1(\Omega),$$

it follows that

$$\int_\Omega \boldsymbol{v} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u} = \lim_{R\to\infty} \int_{\Omega_{0,R}} \boldsymbol{v} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u} \tag{XIII.2.20}$$

with $\Omega_{0,R}$ is defined in (XIII.2.1). By integration by parts, for all $R > 0$,

$$\int_{\Omega_{0,R}} \boldsymbol{v} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u} = \frac{1}{2} \left[\int_{\Sigma_1(R)} \boldsymbol{v} \cdot \boldsymbol{n} u^2 + \int_{\Sigma_2(R)} \boldsymbol{v} \cdot \boldsymbol{n} u^2 \right] \tag{XIII.2.21}$$

where $\Sigma_1(R)$ [respectively $\Sigma_2(R)$] denotes the cross section Σ_1 [respectively Σ_2] calculated at $x_n = -R$ [respectively $x_n = R$] in the system of coordinates that Ω_1 [respectively Ω_2] is referred to. Since

$$\mathcal{I}_R \equiv \int_{\Sigma_1(R)} \boldsymbol{v} \cdot \boldsymbol{n} u^2 = \int_{\Sigma_1(R)} (\boldsymbol{v} - \boldsymbol{v}_0^{(1)}) \cdot \boldsymbol{n} u^2 + \int_{\Sigma_1(R)} \boldsymbol{v}_0^{(1)} \cdot \boldsymbol{n} u^2,$$

it follows that

$$|\mathcal{I}_R| \leq \big(\|\boldsymbol{v} - \boldsymbol{v}_0^{(1)}\|_{2,\Sigma_1(R)} + \|\boldsymbol{v}_0^{(1)}\|_{2,\Sigma_1(R)} \big) \|\boldsymbol{u}\|^2_{4,\Sigma_1(R)}.$$

By Remark XIII.1.1 and Exercise VI.0.1 we have

$$\big(\|\boldsymbol{v} - \boldsymbol{v}_0^{(1)}\|_{2,\Sigma_1(R)} + \|\boldsymbol{v}_0^{(1)}\|_{2,\Sigma_1(R)} \big) \leq c_1$$

for some c_1 independent of R. Therefore,

$$|\mathcal{I}_R| \leq \|\boldsymbol{u}\|^2_{4,\Sigma_1(R)}. \tag{XIII.2.22}$$

By the trace inequality of Theorem II.4.1 and (XIII.2.4) we readily see that

$$\|\boldsymbol{u}\|_{4,\Sigma_1(R)} \leq c \|\boldsymbol{u}\|_{1,2,\Omega_1^R} \leq c\sqrt{\mu} |\boldsymbol{u}|_{1,2,\Omega_1^R},$$

which, along with (XIII.2.22) and the condition $\boldsymbol{u} \in D_0^{1,2}(\Omega)$, implies

$$\lim_{R\to\infty} \mathcal{I}_R = 0.$$

Likewise, one shows

$$\lim_{R\to\infty} \int_{\Sigma_2(R)} \boldsymbol{v} \cdot \boldsymbol{n} u^2 = 0$$

so that, by this relation, (XIII.2.20), and (XIII.2.21) we conclude the validity of (XIII.2.11). The lemma is proved. $\square$

Remark XIII.2.2 Explicit values for the constants c_i, $i = 1, 2, 3$ appearing in the statement of Lemma XIII.2.2 can be of some interest. In particular, c_2 is related to uniqueness conditions for generalized solutions; see Theorem XIII.2.1. Even though it appears difficult to give an estimate valid for a general domain Ω, this is actually possible if Ω is of special shape. For instance, if Ω is an infinite straight cylinder of circular cross section Σ, one can show

$$c_2 = 2 \max \left\{ \kappa, \frac{c_P}{\sqrt{2}} |\Sigma|^{1/2} \right\}$$

where κ is the constant given in (XIII.2.2) while c_P is the Poiseuille constant defined in Exercise VI.0.1. ■

We are now in a position to show the main result of this section.

Theorem XIII.2.1 *Let* $\boldsymbol{v}$ *be a generalized solution to Leray's problem* (XIII.1.2), (XIII.1.4) *corresponding to the flux* Φ. *If*

$$\sum_{i=1}^{2} |\boldsymbol{v} - \boldsymbol{v}_0^{(i)}|_{1,2,\Omega_i} + |\boldsymbol{v}|_{1,2,\Omega_0} + |\Phi| < \frac{\nu}{c_2},$$

with c_2 *given in* $(\text{XIII.2.10})_1$, *then* $\boldsymbol{v}$ *is the only generalized solution corresponding to* Φ.

Proof. Let $\boldsymbol{v}_1$ be another generalized solution corresponding to Φ. Setting

$$\boldsymbol{u} = \boldsymbol{v}_1 - \boldsymbol{v}$$

from (XIII.1.5), it follows that

$$\nu\big(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}\big) + \big(\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{\varphi}\big) + \big(\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi}\big) + \big(\boldsymbol{v} \cdot \nabla \boldsymbol{u}, \boldsymbol{\varphi}\big) = 0 \qquad \text{(XIII.2.23)}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. It is readily shown that

$$\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega). \qquad \text{(XIII.2.24)}$$

Actually, in Ω_i, $i = 1, 2$,

$$\boldsymbol{u} = (\boldsymbol{v}_1 - \boldsymbol{v}_0^{(i)}) - (\boldsymbol{v} - \boldsymbol{v}_0^{(i)})$$

and, as a consequence of (i), (iv), and (v) of Definition XIII.1.1 and in view of Exercise VI.1.1, it follows that

$$\boldsymbol{u} \in D_0^{1,2}(\Omega).$$

Since $\boldsymbol{u}$ is solenoidal, we have, in fact

$$\boldsymbol{u} \in \widehat{\mathcal{D}}_0^{1,2}(\Omega);$$

cf. Section III.5. However, by Exercise III.5.1,

$$\widehat{\mathcal{D}}_0^{1,2}(\Omega) = \mathcal{D}_0^{1,2}(\Omega)$$

and (XIII.2.24) is proved. By virtue of Lemma XIII.2.2 we can now extend (XIII.2.23) to all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and, because of (XIII.2.24), we may take $\boldsymbol{\varphi} = \boldsymbol{u}$. By properties (XIII.2.9) and (XIII.2.11),

$$\big(\boldsymbol{v} \cdot \nabla \boldsymbol{u}, \boldsymbol{u}\big) = \big(\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{u}\big) = 0,$$

and so (XIII.2.23) with $\boldsymbol{\varphi} = \boldsymbol{u}$ delivers

$$\nu |\boldsymbol{u}|_{1,2}^2 + \big(\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{u}\big) = 0. \tag{XIII.2.25}$$

Employing $(\text{XIII.2.10})_1$, we find

$$|(\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{u})| \le c_2 \left(\sum_{i=1}^{2} |\boldsymbol{v} - \boldsymbol{v}_0^{(i)}|_{1,2,\Omega_i} + |\boldsymbol{v}|_{1,2,\Omega_0} + |\Phi| \right) |\boldsymbol{u}|_{1,2}^2,$$

and the theorem follows from this inequality and (XIII.2.25). □

Remark XIII.2.3 In the case when Ω is an infinite straight cylinder an estimate from above for the constant c_2 is given in Remark XIII.2.1. ■

XIII.3 Existence and Uniqueness of Solutions to Leray's Problem

Existence will be proved by the same Galerkin technique employed in Chapters IX–XI for analogous questions in domains with compact boundaries. To this end, we need, as in the linear case, a suitable extension of the Poiseuille velocity fields $\boldsymbol{v}_0^{(i)}$, $i = 1, 2$. However, since in the case at hand the equations are nonlinear, we have to face the same problem we already encountered in Section IX.4 and Section X.4. Actually, let us denote by $\boldsymbol{a}$ an extension of $\boldsymbol{v}_0^{(i)}$, $i = 1, 2$, that is, a sufficiently smooth solenoidal vector field in Ω that vanishes on $\partial\Omega$, equals $\boldsymbol{v}_0^{(i)}$ at large distances in Ω_i, $i = 1, 2$, and such that

$$\int_{\Sigma} \boldsymbol{a} \cdot \boldsymbol{n} = \Phi.$$

We then look for a solution to (XIII.1.2)–(XIII.1.4) of the form

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}, \quad \boldsymbol{u} \in \mathcal{D}_0^{1,2}.$$

As we have learned from the Galerkin technique, for such a solution to exist we need an a priori bound for $|\boldsymbol{u}|_{1,2}$ depending only on the data. Replacing formally this $\boldsymbol{v}$ into (XIII.1.2) we find

$$\left.\begin{aligned}&\nu\Delta\boldsymbol{u} = \boldsymbol{u}\cdot\nabla\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{a}+\boldsymbol{a}\cdot\nabla\boldsymbol{u}+\nabla p-\nu\Delta\boldsymbol{a}+\boldsymbol{a}\cdot\nabla\boldsymbol{a}\\&\nabla\cdot\boldsymbol{u}=0\end{aligned}\right\}\quad\text{in } \Omega$$

$$\boldsymbol{u}=0 \quad \text{at } \partial\Omega$$

$$\int_\Sigma \boldsymbol{u}\cdot\boldsymbol{n}=0$$

$$\lim_{|x|\to\infty}\boldsymbol{u}=0 \quad \text{in } \Omega_i.$$

Dot-multiplying the first equation by $\boldsymbol{u}$, which, according to the Galerkin method, can be assumed to be a member of $\mathcal{D}(\Omega)$, integrating by parts and employing the boundary conditions we formally derive the following identity

$$\nu\int_\Omega \nabla\boldsymbol{u}:\nabla\boldsymbol{u} = -\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{a}\cdot\boldsymbol{u}-\nu\int_\Omega\nabla\boldsymbol{a}:\nabla\boldsymbol{u}-\int_\Omega\boldsymbol{a}\cdot\nabla\boldsymbol{a}\cdot\boldsymbol{u}.$$

Using the results of Lemma XIII.2.2 along with the properties of $\boldsymbol{a}$, it is not difficult to show that the latter identity leads to the following estimate[1]

$$\nu\int_\Omega \nabla\boldsymbol{u}:\nabla\boldsymbol{u}\le-\int_\Omega \boldsymbol{u}\cdot\nabla a\cdot u + C\left(\nu\int_\Omega\nabla\boldsymbol{u}:\nabla\boldsymbol{u}\right)^{1/2}$$

with C depending only on the data. Thus, exactly as in the case of a region of flow with a compact boundary, if we want to prove existence without restrictions from below on the kinematical viscosity ν, we should show that for any $\alpha\in(0,\nu)$ there exists $\mathbf{a}(x;\alpha)$ such that

$$-\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{a}\cdot\boldsymbol{u}<\alpha\int_\Omega\nabla\boldsymbol{u}:\nabla\boldsymbol{u}\quad\text{for all } \boldsymbol{u}\in\mathcal{D}(\Omega).$$

However, the existence of such an extension is not known and, perhaps, in view of what we have seen in the case of a bounded region of flow, it may not hold. Nevertheless, in place of the preceding property, one does prove the existence of an extension $\boldsymbol{a}$ verifying

$$\left|\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{a}\cdot\boldsymbol{u}\right|<c^{-1}|\Phi|\int_\Omega\nabla\boldsymbol{u}:\nabla\boldsymbol{u}\quad\text{for all } \boldsymbol{u}\in\mathcal{D}(\Omega)$$

for some $c=c(\Omega,n)$, which, therefore, furnishes the desired bound on $|\boldsymbol{u}|_{1,2}$ provided

$$|\Phi|<c\nu.$$

As a consequence, the question of existence of solutions to Leray's problem for arbitrary values of ν or (equivalently) for arbitrary values of the flux Φ remains open.

We shall now pass to the construction of the field $\boldsymbol{a}$ that will satisfy this property.

[1] See also the proof of Theorem XIII.3.2.

Lemma XIII.3.3 *There exists a field* $\boldsymbol{a} : \Omega \to \mathbb{R}^n$ *such that*

(i) $\boldsymbol{a} \in W^{2,2}_{loc}(\overline{\Omega})$;
(ii) $\nabla \cdot \boldsymbol{a} = 0$ *in* Ω;
(iii) $\boldsymbol{a} = 0$ *at* $\partial\Omega$;

and, for some $R > 0$,

(iv) $\boldsymbol{a} = \boldsymbol{v}_0^{(i)}$ *in* Ω_i^R;
(v) $|\boldsymbol{a}|_{1,2,\Omega_{0,R}} \le c|\Phi|$;

with Ω_i^R *and* $\Omega_{0,R}$ *defined in* (XIII.2.1) *and with* $c = c(\Omega, n, R)$.

Proof. The field $\boldsymbol{a}$ is constructed exactly as in Section VI.1. The only thing to prove is condition (v). However, we know that in $\Omega_{0,R}$ $\boldsymbol{a}$ satisfies

$$|\boldsymbol{a}|_{1,2,\Omega_{0,R}} \le c \sum_{i=1}^{2} |\boldsymbol{v}_0^{(i)}|_{1,2,\Sigma_i},$$

with $c = c(\Omega_{0,R}, n)$, so that (v) follows from this inequality and Exercise VI.0.1. □

The preceding lemma allows us to show the following existence result.

Theorem XIII.3.2 *There is a constant* $c = c(\Omega, n) > 0$ *such that if*

$$|\Phi| < c\nu, \tag{XIII.3.1}$$

Leray's problem (XIII.1.2)–(XIII.1.4) *admits at least one generalized solution* $\boldsymbol{v}$. *Moreover, for some positive* $C = C(\Omega, n)$,

$$\sum_{i=1}^{2} |\boldsymbol{v} - \boldsymbol{v}_0^{(i)}|_{1,2,\Omega_i} + |\boldsymbol{v}|_{1,2,\Omega_0} \le C(1 + \frac{1}{\nu})(|\Phi| + |\Phi|^2). \tag{XIII.3.2}$$

Proof. We look for a solution of the form

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}$$

with $\boldsymbol{a}$ given in Lemma XIII.3.3 and $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ satisfying

$$\begin{aligned} \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) - (\boldsymbol{u} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{u}) = -(\boldsymbol{u} \cdot \nabla \boldsymbol{a}, \boldsymbol{\varphi}) + (\boldsymbol{a} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{u}) \\ -(\boldsymbol{a} \cdot \nabla \boldsymbol{a}, \boldsymbol{\varphi}) + \nu(\Delta \boldsymbol{a}, \boldsymbol{\varphi}). \end{aligned} \tag{XIII.3.3}$$

It is clear that $\boldsymbol{v}$ satisfies all the requirements (i)-(v) of Definition XIII.1.1. A solution to (XIII.3.3) is determined via the Galerkin method. Thus, let

$$\{\boldsymbol{\varphi}_k\} \subset \mathcal{D}(\Omega)$$

denote a sequence whose linear hull is dense in $\mathcal{D}_0^{1,2}(\Omega)$. By Lemma VII.2.1 and the embedding Theorem II.3.4, we can take $\{\boldsymbol{\varphi}_k\}$ satisfying the following conditions:

(i) $(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j) = \delta_{ij}$;
(ii) Given $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and $\varepsilon > 0$, there are a positive integer $m = m(\varepsilon)$ and real numbers $\gamma_1, \dots, \gamma_m$ such that

$$\|\boldsymbol{\varphi} - \sum_{i=1}^{m} \gamma_i \boldsymbol{\varphi}\|_{C^1} < \varepsilon.$$

A sequence of "approximating solutions" $\{\boldsymbol{u}_m\}$ to (XIII.3.3) is then sought of the type

$$\begin{aligned} \boldsymbol{u}_m &= \sum_{k=1}^{m} \xi_{km} \boldsymbol{\varphi}_k \\ \nu\big(\nabla \boldsymbol{u}_m, \nabla \boldsymbol{\varphi}_k\big) - \big(\boldsymbol{u}_m \cdot \nabla \boldsymbol{\varphi}_k, \boldsymbol{u}_m\big) &= -\big(\boldsymbol{u}_m \cdot \nabla \boldsymbol{a}, \boldsymbol{\varphi}_k\big) + \big(\boldsymbol{a} \cdot \nabla \boldsymbol{\varphi}_k, \boldsymbol{u}_m\big) \\ &\quad -\big(\boldsymbol{a} \cdot \nabla \boldsymbol{a}, \boldsymbol{\varphi}_k\big) + \nu\big(\Delta \boldsymbol{a}, \boldsymbol{\varphi}_k\big), \end{aligned} \tag{XIII.3.4}$$

with $k = 1, \dots, m$. Existence to (XIII.3.4), for each $m \in \mathbb{N}$, can be established exactly as in Theorem IX.3.1 and Theorem X.4.1 (see Lemma IX.3.2), provided we show a suitable bound for $|\boldsymbol{u}_m|_{1,2}$. To obtain this bound, we multiply $(\text{XIII.3.4})_2$ by ξ_{km} and sum over k from 1 to m. Recalling that, by Lemma XIII.2.2, for all $m \in \mathbb{N}$,

$$\big(\boldsymbol{u}_m \cdot \nabla \boldsymbol{u}_m, \boldsymbol{u}_m\big) = \big(\boldsymbol{a} \cdot \nabla \boldsymbol{u}_m, \boldsymbol{u}_m\big) = 0, \tag{XIII.3.5}$$

we find

$$\nu |\boldsymbol{u}_m|_{1,2}^2 = -\big(\boldsymbol{u}_m \cdot \nabla \boldsymbol{a}, \boldsymbol{u}_m\big) + \big(\boldsymbol{a} \cdot \nabla \boldsymbol{a}, \boldsymbol{u}_m\big) + \nu\big(\Delta \boldsymbol{a}, \boldsymbol{u}_m\big). \tag{XIII.3.6}$$

From Lemma XIII.2.2 and Lemma XIII.3.3 it follows that

$$\begin{aligned} |\big(\boldsymbol{u}_m \cdot \nabla \boldsymbol{a}, \boldsymbol{u}_m\big)| &\le c_1 |\Phi|\ |\boldsymbol{u}_m|_{1,2}^2 \\ |\big(\boldsymbol{a} \cdot \nabla \boldsymbol{a}, \boldsymbol{u}_m\big)| &\le c_1 |\Phi|^2\ |\boldsymbol{u}_m|_{1,2} \end{aligned} \tag{XIII.3.7}$$

with $c_1 = c_1(\Omega, n)$. Furthermore, from (VI.1.9) and (VI.1.10) we have

$$|\big(\Delta \boldsymbol{a}, \boldsymbol{u}_m\big)| = |\big(\nabla \boldsymbol{a}, \nabla \boldsymbol{u}_m\big)_{\Omega_{0,R}}|$$

and so, again by Lemma XIII.3.3

$$|\big(\Delta \boldsymbol{a}, \boldsymbol{u}_m\big)| \le c_2 |\Phi|\ |\boldsymbol{u}_m|_{1,2}, \tag{XIII.3.8}$$

with $c_2 = c_2(\Omega, n)$. Thus, if we take, for instance,

$$|\Phi| < \frac{1}{2c_1}\nu,$$

using (XIII.3.5)–(XIII.3.8) and Lemma IX.3.2 we show existence to problem (XIII.3.4) for all $m \in \mathbb{N}$. Furthermore,

$$|\boldsymbol{u}_m|_{1,2} \le \frac{2c_3}{\nu}(|\Phi| + |\Phi|^2). \tag{XIII.3.9}$$

Using (XIII.3.9) and the weak compactness property of the spaces $\dot{D}^{1,2}(\Omega)$, we may find a subsequence $\{\boldsymbol{u}_{m'}\}$ and a vector field $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ such that

$$\begin{gathered}\boldsymbol{u}_{m'} \to \boldsymbol{u} \ \text{ weakly in } D_0^{1,2}(\Omega)\\ \boldsymbol{u}_{m'} \to \boldsymbol{u} \ \text{ strongly in } L^2(\Omega'), \text{ for all bounded } \Omega' \subset \Omega.\end{gathered}$$

Reasoning as in Theorem IX.3.1 and Theorem X.4.1 and taking advantage of condition (ii) on $\{\boldsymbol{\varphi}_k\}$, we then show that $\boldsymbol{u}$ satisfies (XIII.3.3) for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. Since in view of (XIII.3.9) and the property of weak limits, we have

$$|\boldsymbol{u}|_{1,2} \le \frac{2c_3}{\nu}(|\Phi| + |\Phi|^2),$$

with the help of Lemma XIII.3.3 and Exercise VI.0.1 we also have

$$\begin{aligned}|\boldsymbol{v} - \boldsymbol{v}_0^{(i)}|_{1,2,\Omega_i} &\le |\boldsymbol{a} - \boldsymbol{v}_0^{(i)}|_{1,2,\Omega_i} + |\boldsymbol{u}|_{1,2}\\ &\le |\boldsymbol{a}|_{1,2,\Omega_{0,R}} + |\boldsymbol{v}_0^{(i)}|_{1,2,\Omega_i\cap\Omega_{0,R}} + \frac{2c_3}{\nu}(|\Phi| + |\Phi|^2)\\ &\le c_4\left(1 + \frac{1}{\nu}\right)(|\Phi| + |\Phi|^2).\end{aligned} \tag{XIII.3.10}$$

Likewise, again by Lemma XIII.3.3 and Exercise VI.0.1, we find

$$\begin{aligned}|\boldsymbol{v}|_{1,2,\Omega_0} &\le |\boldsymbol{a}|_{1,2,\Omega_0} + |\boldsymbol{u}|_{1,2}\\ &\le c_5\left(1 + \frac{1}{\nu}\right)(|\Phi| + |\Phi|^2).\end{aligned}$$

Therefore, estimate (XIII.3.2) follows from this inequality and (XIII.3.10), and the theorem is proved. □

Remark XIII.3.4 Theorem XIII.3.2 states, in particular, that, unlike the linearized case, the nonlinear Leray's problem is solvable under the restriction (XIII.3.1) for the flux Φ. The investigation of whether this restriction can be removed is, undoubtedly, one of the most challenging problems in theoretical fluid dynamics. In this respect, it should be observed that if we relax requirement (v) of Definition XIII.1.1, that is, if we do not impose a priori that the solution converges to the corresponding Poiseuille flows in the outlets Ω_i, then one can show existence of solutions for arbitrary values of Φ. Specifically, Ladyzhenskaya & Solonnikov (1980, Theorems 3.1, 3.2) have shown[2] that for any $\Phi \in \mathbb{R}$ there exists a pair $\boldsymbol{v}$, p obeying (XIII.1.2)–(XIII.1.4). Concerning the behavior at infinity, this solution satisfies the following conditions:[3]

[2] Actually, in a class of domains larger (for shape) than that considered by us here.
[3] See (XIII.2.1) for the definition of the domains involved in (XIII.3.11).

$$\begin{aligned}&\int_{\Omega_i^t} \nabla \boldsymbol{v} : \nabla \boldsymbol{v} \le c_1 t, \quad \text{for all } t > 0\\&\int_{\Omega_{t,t+1}} \nabla \boldsymbol{v} : \nabla \boldsymbol{v} \le c_2 \quad \text{for all } t > 0,\end{aligned} \tag{XIII.3.11}$$

with c_1 and c_2 independent of t. Moreover, $\boldsymbol{v}$, p is unique if $|\Phi|$ is "sufficiently small." These results resemble in a sense those for D-solutions of the two-dimensional nonlinear exterior problem, since in the case at hand the main problem also remains the investigation of the asymptotic behavior, starting with a certain regularity at large distances, here expressed by (XIII.3.11). In particular, denoting by $\mathcal{S}_i = \mathcal{S}_i(\Phi, \Sigma_i)$, $i = 1, 2$, the "limit sets," i.e., the set constituted by those vector fields that the solution $\boldsymbol{v}$ satisfying (XIII.3.11) tends to eventually as $|x| \to \infty$ in Ω_i, one should investigate if $\mathcal{S}_i = \{\boldsymbol{v}_0^{(i)}\}$. ■

Remark XIII.3.5 Also in view of what was observed in the preceding remark, it appears of a certain interest to determine an explicit and possibly sharp value for the constant c entering condition (XIII.3.1). In this respect we have a result due to Amick (1977) that ensures that, if Ω_0 is simply connected, c depends only on Ω_i (through their sections Σ_i) and not on Ω_0; see Amick, *loc. cit.* Theorem 3.6. Moreover, c can be determined in an "optimal" way by solving a suitable variational problem strictly related to the nonlinear stability property of Poiseuille flow and, if the cross sections Σ_i are of special shape, c can be explicitly evaluated. For instance, if Σ_i is a circle of radius R_i, we have

$$c = 127.9 \min\{R_1, R_2\};$$

see Amick *loc. cit.* §3.4. ■

Exercise XIII.3.1 (*Generalization of* Theorem XIII.3.2 *to domains with more than two cylindrical ends*). Assume that instead of two exits to infinity, Ω_1 and Ω_2, the domain Ω has $m \ge 3$ exits $\Omega_1', \ldots, \Omega_l'$, where $\Omega_1', \ldots, \Omega_j'$ can be represented as Ω_1 ("upstream" exits) and $\Omega_{j+1}', \ldots, \Omega_l'$ as Ω_2 ("downstream" exits). Assume also that

$$\Omega - \bigcup_{i=1}^{l} \Omega_i'$$

is bounded and that Ω is of class C^∞. Denote by Φ_i the fluxes in Ω_i'. Then show that for every choice of Φ_i satisfying the compatibility condition of zero total flux, i.e.,

$$\sum_{i=1}^{j} \Phi_i = \sum_{i=j+1}^{l} \Phi_i,$$

there is a $c = c(\Omega, n) > 0$ such that if

$$\sum_{i=1}^{l} |\Phi_i| < c\nu$$

Leray's problem is solvable in Ω.

With the help of Theorem XIII.2.1 we can readily obtain conditions under which solutions determined in Theorem XIII.3.2 are unique. Actually, combining Theorem XIII.2.1 with (XIII.3.2) we immediately find the following result.

Theorem XIII.3.3 *Let $\boldsymbol{v}$ be a generalized solution to Leray's problem constructed in* Theorem XIII.3.2, *corresponding to flux Φ. If*

$$\left[C\left(1+\frac{1}{\nu}\right)+1\right]|\Phi|+C\left(1+\frac{1}{\nu}\right)|\Phi|^2<\frac{\nu}{c_2}$$

with c_2 and C given in Theorem XIII.2.1 *and* Theorem XIII.3.2, *respectively, then $\boldsymbol{v}$ is the only generalized solution corresponding to Φ.*

XIII.4 Decay Estimates for Steady Flow in a Semi–Infinite Straight Channel

Our objective in this section is to establish the rate at which solutions determined in Theorem XIII.3.2 decay to the corresponding Poiseuille velocity fields. We shall show that, as $|x| \to \infty$, they decay pointwise and exponentially fast. As in the linear case, this will follow as a corollary to a more general result holding for a class of solutions wider than that determined in Theorem XIII.3.2. More generality regards, essentially, the behavior at infinity, while a restriction on Φ of the type (XIII.3.1) is always needed.

We shall restrict our attention to flows occurring in the straight cylinder

$$\Omega = \{x_n > 0\} \times \Sigma,$$

where the cross section Σ is a C^∞–smooth, bounded, and simply connected domain in $\mathbb{R}^{n-1}$ ($n = 2, 3$). However, some of the results we find can be extended to cover more general situations. The cross section at distance a from the origin will be denoted by $\Sigma(a)$, despite all cross sections having the same shape and size. Let $\boldsymbol{v}_0 = \boldsymbol{v}_0(x')$ be the vector field associated with the Poiseuille flow in Ω and corresponding to flux Φ. Further, let $\boldsymbol{u}$, τ be a smooth solution[1] to the following boundary-value problem:

$$\left.\begin{aligned}&\nu\Delta\boldsymbol{u} = \boldsymbol{u}\cdot\nabla u + \boldsymbol{u}\cdot\nabla v_0 + \boldsymbol{v}_0\cdot\nabla\boldsymbol{u} + \nabla\tau\\ &\nabla\cdot\boldsymbol{u} = 0\end{aligned}\right\}\ \text{in } \Omega$$
$$\boldsymbol{u} = 0 \quad \text{at } \partial\Omega - \{x_n = 0\} \tag{XIII.4.1}$$
$$\int_\Sigma u_n = 0.$$

[1] For simplicity, we assume $\boldsymbol{u}$, τ smooth, that is, indefinitely differentiable in the closure of any bounded subset of Ω. We note, however, that the same conclusions may be reached merely by assuming that $\boldsymbol{u}$ and τ possess a priori the same regularity of generalized solutions and then employing Theorem XIII.1.1.

Remark XIII.4.1 If $\boldsymbol{v}$ is a generalized solution to Leray's problem, then $\boldsymbol{u} \equiv \boldsymbol{v} - \boldsymbol{v}_0^{(i)}$ satisfies (XIII.4.1) with $\Omega \equiv \Omega_i$, $\Sigma \equiv \Sigma_i$ and $\boldsymbol{v}_0 \equiv \boldsymbol{v}_0^{(i)}$. ■

We begin to show some local estimates for problem (XIII.4.1). To this end, for $R > 0$ we let
$$\Omega^R = \{x \in \Omega : x_n > R\}.$$

Lemma XIII.4.1 *Let* $\boldsymbol{u}$, τ *be a smooth solution to* (XIII.4.1) *with*
$$|\boldsymbol{u}|_{1,2} \leq M < \infty.$$
Then for every $m \geq 0$ *and* $R \geq 0$*, the following estimate holds*
$$\|\boldsymbol{u}\|_{m+2,2,\Omega^{R+1}} + \|\nabla\tau\|_{m,2,\Omega^{R+1}} \leq c\|\boldsymbol{u}\|_{1,2,\Omega^R} \tag{XIII.4.2}$$
where $c = c(n, m, M, \Sigma, \Phi, \nu)$.

Proof. Throughout the proof, the symbol c_i, $i = 1, 2, \ldots$, denotes a generic quantity depending, at most, on m, M, Σ, and Φ. Let
$$0 < \varepsilon < \frac{1}{m+2}. \tag{XIII.4.3}$$
By $(\text{XIII.2.2})_2$ and assumption, for all $k = 0, 1, 2, \ldots,$ and all $R_1 \geq 0$ we derive[2]
$$\begin{aligned}\|\boldsymbol{u} \cdot \nabla u\|_{3/2,\Omega_{R_1+k,R_1+k+1}} &\leq \|\boldsymbol{u}\|_{6,\Omega_{R_1+k,R_1+k+1}}|\boldsymbol{u}|_{1,2,\Omega_{R_1+k,R_1+k+1}} \\ &\leq c_1|\boldsymbol{u}|_{1,2,\Omega_{R_1+k,R_1+k+1}}\end{aligned}$$
and so, summing over k we obtain, in particular,
$$\|\boldsymbol{u} \cdot \nabla u\|_{3/2,\Omega^{R_1}} \leq c_1|\boldsymbol{u}|_{1,2,\Omega^{R_1}} \qquad \text{for all } R_1 \geq 0.$$
In a similar fashion, from the Hölder inequality, the regularity of $\boldsymbol{v}_0$, and inequality (XIII.2.4) we readily find
$$\|\boldsymbol{v}_0 \cdot \nabla\boldsymbol{u} + \boldsymbol{u} \cdot \nabla\boldsymbol{v}_0\|_{3/2,\Omega^{R_1}} \leq c_2|\boldsymbol{u}|_{1,2,\Omega^{R_1}} \qquad \text{for all } R_1 \geq 0.$$
Thus, setting
$$\boldsymbol{f} = \boldsymbol{u} \cdot \nabla u + \boldsymbol{v}_0 \cdot \nabla\boldsymbol{u} + \boldsymbol{u} \cdot \nabla\boldsymbol{v}_0,$$
it follows that
$$\|\boldsymbol{f}\|_{3/2,\Omega^{R_1}} \leq c_3|\boldsymbol{u}|_{1,2,\Omega^{R_1}} \ , \qquad \text{for all } R_1 \geq 0. \tag{XIII.4.4}$$

[2] We give a proof that applies equally to both cases $n = 2$ and 3. However, for $n = 2$ the proof could be simplified.

Employing (VI.1.13) with $s = R_1 + k$, $k = 0, 1, 2, \ldots$, $\delta = \varepsilon$ and taking into account (XIII.4.4), after summing over k we find

$$\|\boldsymbol{u}\|_{2,3/2,\Omega^{R_2}} \le c_4 \|\boldsymbol{u}\|_{1,2,\Omega^{R_2-\varepsilon}} \qquad \text{for all } R_2 \ge 1.$$

From this relation and the first part of Lemma XIII.2.1 it then follows that

$$\sup_{\Omega^{R_2}} |\boldsymbol{u}| \le c_5. \tag{XIII.4.5}$$

Thus,

$$\|\boldsymbol{u} \cdot \nabla \boldsymbol{u}\|_{2,\Omega^{R_2}} \le c_6 \|\boldsymbol{u}\|_{1,2,\Omega^{R_2}}$$

which, by the regularity properties of $\boldsymbol{v}_0$, in turn implies

$$\|\boldsymbol{f}\|_{2,\Omega^{R_2}} \le c_7 |\boldsymbol{u}|_{1,2,\Omega^{R_2}}. \tag{XIII.4.6}$$

We next employ (XIII.4.5) together with (VI.1.13) calculated for $s = R_3 + k$, $k \in \mathbb{N}$, and $\delta = \varepsilon$. Summing over k and taking into account (XIII.4.5) we deduce

$$\|\boldsymbol{u}\|_{2,2,\Omega^{R_3}} + \|\nabla \tau\|_{2,\Omega^{R_3}} \le c_8 |\boldsymbol{u}|_{1,2,\Omega^{R_3-\varepsilon}}, \tag{XIII.4.7}$$

for all R_3 such that

$$R_3 \ge R_2 + \varepsilon. \tag{XIII.4.8}$$

We now choose

$$R_2 = R + \varepsilon\,, \quad R_3 = R + 1. \tag{XIII.4.9}$$

Since ε obeys (XIII.4.3) (with $m = 0$), it follows that (XIII.4.7) is satisfied and that $R_3 - \varepsilon = R + 1 - \varepsilon \ge R$. Therefore, (XIII.4.7)–(XIII.4.9) prove (XIII.4.2) for $m = 0$. Iterating this method, it is possible to show the validity of (XIII.4.2) for all $m \ge 0$. We show this for $m = 1$, leaving to the reader the proof of the general iterative procedure. By assumption, the first part of Lemma XIII.2.1, and (XIII.4.5) it readily follows that

$$\begin{aligned}|\boldsymbol{u} \cdot \nabla u|_{1,2,\Omega^{R_3}} &\le |\boldsymbol{u}|^2_{1,4,\Omega^{R_3}} + c_5 |\nabla \boldsymbol{u}|_{1,2,\Omega^{R_3}} \\ &\le c_9 (\|\boldsymbol{u}\|^2_{2,2,\Omega^{R_3}} + \|\boldsymbol{u}\|_{2,2,\Omega^{R_3}}).\end{aligned}$$

In view of (XIII.4.7) and the assumptions on $\boldsymbol{u}$ we deduce

$$|\boldsymbol{u} \cdot \nabla u|_{1,2,\Omega^{R_3}} \le c_{10} \|\boldsymbol{u}\|_{1,2,\Omega^{R_3-\varepsilon}}.$$

Likewise, from the regularity properties of $\boldsymbol{v}_0$ and (XIII.4.7) we find

$$\|\boldsymbol{v}_0 \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{v}_0\|_{1,2,\Omega^{R_3}} \le c_{11} \|\boldsymbol{u}\|_{1,2,\Omega^{R_3-\varepsilon}}.$$

As a consequence, we infer that

$$\|\mathbf{f}\|_{1,2,\Omega^{R_3}} \le c_{12} \|\boldsymbol{u}\|_{1,2,\Omega^{R_3-\varepsilon}}. \tag{XIII.4.10}$$

We now employ (VI.1.13) with $s = R_4 + k$, $k \in \mathrm{IN}$ and $\delta = \varepsilon$. Summing over k and taking into account (XIII.4.10), it follows that

$$\|\boldsymbol{u}\|_{3,2,\Omega^{R_4}} + \|\nabla\tau\|_{1,2,\Omega^{R_4}} \le c_{13}|\boldsymbol{u}|_{1,2,\Omega^{R_3-\varepsilon}}, \tag{XIII.4.11}$$

for all R_4 such that

$$R_4 \ge R_3 + \varepsilon \tag{XIII.4.12}$$

where R_3 satisfies (XIII.4.8) with $R_2 \ge 1$. If we choose

$$R_2 = R + \varepsilon\,, \quad R_3 = R + 2\varepsilon\,, \quad R_4 = R + 1\,,$$

we see that, by (XIII.4.3) with $m = 1$, conditions (XIII.4.8) and (XIII.4.12) are satisfied and, furthermore, $R_3 - \varepsilon = R + \varepsilon > R$. We then conclude the validity of (XIII.4.2) with $m = 1$. The proof of the lemma can be then considered accomplished. □

The conclusions of Lemma XIII.4.1 can be derived under much weaker assumptions on the summability of $\nabla\boldsymbol{u}$, provided that the flux Φ is "sufficiently small." To show this, we need a preliminary result concerning a differential inequality.

Lemma XIII.4.2 *Let $y \in C^1(\mathrm{IR}_+)$ be nonnegative with a nonnegative first derivative. Assume for all $t > 0$*

$$ay(t) \le a_1 y'(t) + a_2(y'(t))^{3/2} + b \tag{XIII.4.13}$$

where a is a positive constant while a_1, a_2, and b are nonnegative constants. Then if

$$\liminf_{t\to\infty} t^{-3}y(t) = 0$$

we have

$$y(t) \le 2\frac{b+1}{a}, \quad \text{for all } t > 0. \tag{XIII.4.14}$$

Proof. From Young's inequality (II.2.7) it follows that there is a $c = c(a_1)$ such that for all $t > 0$

$$a_1 y'(t) \le c(y'(t))^{3/2} + b_1$$

where $b_1 = b + 1$. As a consequence, (XIII.4.13) yields

$$ay(t) \le d(y'(t))^{3/2} + b_1 \tag{XIII.4.15}$$

with $d = a_2 + c$. Assume (XIII.4.14) is false, We can then find $t_0 > 0$ such that

$$y(t_0) > 2\frac{b_1}{a}$$

and, since $y'(t) \geq 0$,

$$y(t) > 2\frac{b_1}{a} \quad \text{for all } t \geq t_0. \tag{XIII.4.16}$$

Employing (XIII.4.16) in (XIII.4.15) we obtain

$$A(y(t))^{2/3} \leq y'(t), \quad \text{for all } t \geq t_0$$

with[3]

$$A^{2/3} \equiv \frac{a}{2d} > 0.$$

Integrating this inequality from t and $t_1 > t$, it follows that

$$\frac{y^{1/3}(t_1)}{t_1} - \frac{y^{1/3}(t)}{t_1} \geq \frac{A}{3}\left(1 - \frac{t}{t_1}\right)$$

and so taking the lim inf as $t_1 \to \infty$ of both sides of this inequality we derive

$$\liminf_{t_1 \to \infty} t_1^{-3} y(t_1) \geq \left(\frac{A}{3}\right)^3 > 0,$$

which contradicts the assumption. The lemma is therefore proved. $\square$

The result just shown allows us to prove the following.

Lemma XIII.4.3 *Assume* $\boldsymbol{u}, \tau$ *is a smooth solution to* (XIII.4.1) *such that*

$$\liminf_{x_n \to \infty} x_n^{-3} \int_0^{x_n} \left(\int_{\Sigma(t)} |\nabla \boldsymbol{u}(x', t)|^2 d\Sigma\right) dt = 0.$$

Assume, further, that the flux Φ *associated with the Poiseuille flow* $\boldsymbol{v}_0$ *satisfies*

$$|\Phi| < \frac{\nu}{(c_P \lambda \mu)^{1/2}}, \tag{XIII.4.17}$$

where c_P *is the constant given in* Exercise VI.0.1 *and* λ, μ *are given in* (XIII.2.3) *and* (XIII.2.4), *respectively. Then*

$$\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u} < \infty.$$

Proof. Multiplying $(\text{XIII.4.1})_1$ by $\boldsymbol{u}$ and integrating by parts over $(0, x_n) \times \Sigma$, we find

[3] Notice that we can always take $d > 0$.

$$
\begin{aligned}
\nu\,\mathcal{G}(x_n) &\equiv \nu\int_0^{x_n}\left[\int_{\Sigma(t)}\nabla \boldsymbol{u}:\nabla \boldsymbol{u}\,d\Sigma\right]dt \\
&= \int_{\Sigma(x_n)}\left[-\tau u_n + \tfrac{\nu}{2}\frac{\partial u^2}{\partial x_n} - \frac{1}{2}u^2(u_n+v_0)\right] \\
&\quad -\int_{\Sigma(0)}\left[-\tau u_n + \frac{\nu}{2}\frac{\partial u^2}{\partial x_n} - \frac{1}{2}u^2(u_n+v_0)\right] \\
&\quad -\int_0^{x_n}\left[\int_{\Sigma(t)}\boldsymbol{u}\cdot\nabla \boldsymbol{v}_0\cdot \boldsymbol{u}\,d\Sigma\right]dt
\end{aligned}
\tag{XIII.4.18}
$$

where $\boldsymbol{v}_0 = v_0\boldsymbol{e}_n$. By using the Schwarz inequality, (XIII.2.3), and (XIII.2.4) we obtain

$$
\begin{aligned}
\left|\int_0^{x_n}\int_{\Sigma(t)}\boldsymbol{u}\cdot\nabla \boldsymbol{v}_0\cdot \boldsymbol{u}\,d\Sigma dt\right| &\le \left(\int_{\Sigma(t)}|\nabla \boldsymbol{v}_0|^2\right)^{1/2}\int_0^{x_n}\left(\int_{\Sigma(t)}u^4 d\Sigma\right)^{1/2} dt \\
&\le (\lambda\mu)^{1/2}\left(\int_{\Sigma(t)}|\nabla \boldsymbol{v}_0|^2\right)^{1/2}\mathcal{G}(x_n) \\
&= (\lambda\mu c_P)^{1/2}|\Phi|\mathcal{G}(x_n),
\end{aligned}
\tag{XIII.4.19}
$$

where, in the last step, we have used the results of Exercise VI.0.1. Assuming, now, the validity of (XIII.4.17) and setting

$$
\gamma = \nu - (\lambda\mu c_P)^{1/2}|\Phi|\ (>0), \tag{XIII.4.20}
$$

from (XIII.4.18) and (XIII.4.19), it follows that

$$
\gamma\mathcal{G}(x_n) \le \int_{\Sigma(x_n)}\left[-\tau u_n + \frac{\nu}{2}\frac{\partial u^2}{\partial x_n} - \frac{1}{2}u^2(u_n+v_0)\right] + b
$$

with b a nonnegative quantity independent of x_n. Integrating both sides of this relation from t to $t+1$ furnishes

$$
\gamma\int_t^{t+1}\mathcal{G}(x_n)dx_n \le \int_{\Omega_{t,t+1}}\left[-\tau u_n + \frac{\nu}{2}\frac{\partial u^2}{\partial x_n} - \frac{1}{2}u^2(u_n+v_0)\right]+b. \tag{XIII.4.21}
$$

We wish to estimate the term involving τ. To this end, we proceed as in the linear case; see Theorem VI.2.1. Specifically, denoting by $\boldsymbol{\omega}$ a solution to (VI.2.14), from (XIII.4.1) we find

$$
\begin{aligned}
\int_{\Omega_{t,t+1}}\tau u_n &= -\int_{\Omega_{t,t+1}}\nabla\tau\cdot\boldsymbol{\omega} \\
&= \int_{\Omega_{t,t+1}}[\nu\nabla \boldsymbol{u}:\nabla\boldsymbol{\omega} - \boldsymbol{u}\cdot\nabla\boldsymbol{\omega}\cdot u \\
&\quad -\boldsymbol{u}\cdot\nabla\boldsymbol{\omega}\cdot\boldsymbol{v}_0 - \boldsymbol{v}_0\cdot\boldsymbol{\omega}\cdot\boldsymbol{u}]\,.
\end{aligned}
\tag{XIII.4.22}
$$

Taking into account the properties of $\boldsymbol{\omega}$, from Lemma XIII.2.1 and (XIII.2.4) we find

$$\begin{aligned}
\left|\int_{\Omega_{t,t+1}} \nabla \boldsymbol{u} : \nabla \boldsymbol{\omega}\right| &\le c_0 |\boldsymbol{u}|_{1,2,\Omega_{t,t+1}} \|\boldsymbol{u}\|_{2,\Omega_{t,t+1}} \\
&\le c_0 \sqrt{\mu} |\boldsymbol{u}|^2_{1,2,\Omega_{t,t+1}} \\
\left|\int_{\Omega_{t,t+1}} \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} \cdot u\right| &\le c_0 \|\boldsymbol{u}\|^2_{4,\Omega_{t,t+1}} \|\boldsymbol{u}\|_{2,\Omega_{t,t+1}} \\
&\le c_0 \kappa^2 \sqrt{\mu} |\boldsymbol{u}|^2_{1,2,\Omega_{t,t+1}} \\
\left|\int_{\Omega_{t,t+1}} \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} \cdot \boldsymbol{v}_0\right| &\le c_0 \widehat{v} \|\boldsymbol{u}\|_{2,\Omega_{t,t+1}} |\boldsymbol{u}|_{1,2,\Omega_{t,t+1}} \\
&\le c_0 \widehat{v} \sqrt{\mu} |\boldsymbol{u}|^2_{1,2,\Omega_{t,t+1}} \\
\left|\int_{\Omega_{t,t+1}} \boldsymbol{v}_0 \cdot \boldsymbol{\omega} \cdot \boldsymbol{u}\right| &\le c_0 \widehat{v} \sqrt{\mu} |\boldsymbol{u}|^2_{1,2,\Omega_{t,t+1}}
\end{aligned} \tag{XIII.4.23}$$

with

$$\widehat{v} = \max_{\Sigma} |v_0(x')|.$$

Putting

$$c_1 = c_0 \sqrt{\mu} (\nu + \kappa^2 + 2\widehat{v}),$$

from (XIII.4.21)–(XIII.4.23) we deduce

$$\gamma \int_t^{t+1} \mathcal{G}(x_n) dx_n \le c_1 |\boldsymbol{u}|^2_{1,2,\Omega_{t,t+1}} - \frac{1}{2} \int_{\Omega_{t,t+1}} \left[\nu \frac{\partial u^2}{\partial x_n} - u^2 (u_n + v_0)\right] + b. \tag{XIII.4.24}$$

We now observe that, again from (XIII.2.4) and Lemma XIII.2.1, it follows that

$$\begin{aligned}
\left|\int_{\Omega_{t,t+1}} \frac{\partial u^2}{\partial x_n}\right| &\le 2 \|\boldsymbol{u}\|_{2,\Omega_{t,t+1}} |\boldsymbol{u}|_{1,\Omega_{t,t+1}} \le 2\sqrt{\mu}\, |\boldsymbol{u}|^2_{1,\Omega_{t,t+1}} \\
\left|\int_{\Omega_{t,t+1}} u^2 v_0\right| &\le \widehat{v} \mu |\boldsymbol{u}|^2_{1,\Omega_{t,t+1}} \\
\left|\int_{\Omega_{t,t+1}} u^2 u_n\right| &\le \|\boldsymbol{u}\|^2_{4,\Omega_{t,t+1}} \|\boldsymbol{u}\|_{2,\Omega_{t,t+1}} \le \kappa^2 \sqrt{\mu}\, |\boldsymbol{u}|^3_{1,2,\Omega_{t,t+1}}.
\end{aligned} \tag{XIII.4.25}$$

Replacing these latter inequalities into (XIII.4.24), we find

$$\gamma \int_t^{t+1} \mathcal{G}(x_n) dx_n \le c_2 |\boldsymbol{u}|^2_{1,2,\Omega_{t,t+1}} + c_3 |\boldsymbol{u}|^3_{1,2,\Omega_{t,t+1}} + b \tag{XIII.4.26}$$

where

$$c_2 = c_1 + \sqrt{\mu}(\nu + \tfrac{1}{2}\widehat{v}\sqrt{\mu})$$

$$c_3 = \tfrac{1}{2}\kappa^2\sqrt{\mu}.$$

Setting

$$y(t) = \int_t^{t+1} \mathcal{G}(x_n)$$

and recalling the definition of $\mathcal{G}(x_n)$, it follows that

$$y'(t) = |\boldsymbol{u}|^2_{1,2,\Omega t,t+1}.$$

Therefore, inequality (XIII.4.25) yields

$$\gamma y(t) \le c_2 y'(t) + c_3(y'(t))^{3/2} + b$$

which, in turn, by Lemma XIII.4.2, implies

$$y(t) \le 2\frac{b+1}{\gamma}.$$

The result then follows from this estimate and an argument entirely analogous to that employed at the end of the proof of Theorem VI.2.1. □

The next result establishes an exponential decay property of the type of de Saint-Venant.

Lemma XIII.4.4 *Let the assumptions of* Lemma XIII.4.3 *be satisfied. Then, there are positive constants* $\sigma_i = \sigma_i(\Sigma, n, \|\boldsymbol{u}\|_{1,2,\Omega}, \Phi, \nu)$, $i = 1, 2$, *such that*

$$\|\boldsymbol{u}\|_{1,2,\Omega^R} \le \sigma_1 \|\boldsymbol{u}\|_{1,2,\Omega} \exp(-\sigma_2 R), \quad \textit{for all } R > 0.$$

Proof. We notice that, in view of Lemma XIII.4.3, $\|\boldsymbol{u}\|_{1,2,\Omega}$ is finite. Multiplying $(\text{XIII.4.1})_1$ by $\boldsymbol{u}$ and integrating by parts over $(R, x_n) \times \Sigma \equiv \Omega_{R,x_n}$ we find

$$\begin{aligned}\nu\int_{\Omega_{R,x_n}} |\nabla \boldsymbol{u}|^2 = &\int_{\Sigma(x_n)} \left[-\tau u_n + \frac{\nu}{2}\frac{\partial u^2}{\partial x_n} - \frac{1}{2}u^2(u_n + v_0)\right] \\ &-\int_{\Sigma(R)} \left[-\tau u_n + \frac{\nu}{2}\frac{\partial u^2}{\partial x_n} - \frac{1}{2}u^2(u_n + v_0)\right] \\ &-\int_{\Omega_{R,x_n}} \boldsymbol{u}\cdot\nabla \boldsymbol{v}_0 \cdot \boldsymbol{u}\,.\end{aligned} \tag{XIII.4.27}$$

From Lemma XIII.4.1 and the embedding Theorem II.5.2 it follows, in particular, that

$$\sup_{x' \in \Sigma} \left(|\boldsymbol{u}(x', x_n)| + |\tau(x', x_n)| \right) \to 0 \quad \text{as } x_n \to \infty,$$

and so, reasoning as in the proof of Theorem VI.2.2, we can show that, in the limit $x_n \to \infty$, the surface integral over $\Sigma(x_n)$ is vanishing. Thus, in this limit, (XIII.4.27) yields

$$\nu \mathcal{H}(R) = -\int_{\Sigma(R)} \left[-\tau u_n + \frac{\nu}{2} \frac{\partial u^2}{\partial x_n} - \frac{1}{2} u^2 (u_n + v_0) \right] - \int_{\Omega^R} \boldsymbol{u} \cdot \nabla \boldsymbol{v}_0 \cdot \boldsymbol{u}, \tag{XIII.4.28}$$

with

$$\nu \mathcal{H}(R) \equiv \nu |\boldsymbol{u}|^2_{1,2,\Omega^R}.$$

Proceeding as in (XIII.4.19), we can show

$$\left| \int_{\Omega^R} \boldsymbol{u} \cdot \nabla \boldsymbol{v}_0 \cdot \boldsymbol{u} \right| \leq (\lambda \mu c_P)^{1/2} |\Phi| \mathcal{H}(R),$$

which, in view of (XIII.4.17), once replaced in (XIII.4.28) furnishes

$$\gamma \mathcal{H}(\mathcal{R}) \leq -\int_{\Sigma(R)} \left[-\tau u_n + \frac{\nu}{2} \frac{\partial u^2}{\partial x_n} - \frac{1}{2} u^2 (u_n + v_0) \right]$$

with γ given in (XIII.4.20). We next integrate both sides of this relation between $t+l$ and $t+l+1$, $l = 0, 1, 2, \ldots$, to obtain

$$\begin{aligned} \gamma \int_{t+l}^{t+l+1} \mathcal{H}(R) \leq & -\frac{\nu}{2} \int_{\Sigma(t+l+1)} u^2 + \frac{\nu}{2} \int_{\Sigma(t+l)} u^2 \\ & + \int_{\Omega_{t+l,t+l+1}} \left[\tau u_n + \frac{1}{2} u^2 (u_n + v_0) \right]. \end{aligned} \tag{XIII.4.29}$$

The volume integral on the right-hand side of (XIII.4.29) can be increased exactly as in (XIII.4.22), (XIII.4.23), and $(\text{XIII.4.25})_{2,3}$ and so we deduce

$$\left| \int_{\Omega_{t+l,t+l+1}} \tau u_n + \frac{1}{2} u^2 (u_n + v_0) \right| \leq c_1 |\boldsymbol{u}|^2_{1,2,\Omega_{t+l,t+l+1}} (1 + |\boldsymbol{u}|_{1,2,\Omega_{t+l,t+l+1}}), \tag{XIII.4.30}$$

with $c_1 = c_1(n, \Sigma, \Phi)$. Collecting (XIII.4.29) and (XIII.4.30) we derive

$$\gamma \int_{t+l}^{t+l+1} \mathcal{H}(R) \leq c_2 |\boldsymbol{u}|^2_{1,2,\Omega_{t+l,t+l+1}} - \frac{\nu}{2} \int_{\Sigma(t+l+1)} u^2 + \frac{\nu}{2} \int_{\Sigma(t+l)} u^2 \tag{XIII.4.31}$$

where c_2 also depends on $|\boldsymbol{u}|_{1,2,\Omega}$. Summing both sides of (XIII.4.31) from $l = 0$ to $l = \infty$ and taking into account that, by Remark XIII.1.1

$$\lim_{t \to \infty} \int_{\Sigma(t)} u^2(x', t) d\Sigma = 0,$$

we find

$$\gamma \int_t^\infty \mathcal{H}(R) \le c_2 \mathcal{H}(t) + \frac{\nu}{2} \int_{\Sigma(t)} u^2. \tag{XIII.4.32}$$

However, by (XIII.2.4),

$$\int_{\Sigma(t)} u^2 \le \mu \int_{\Sigma(t)} |\nabla \boldsymbol{u}|^2 = -\mu \mathcal{H}'(t)$$

and so from (XIII.4.32) we arrive at

$$\gamma \int_t^\infty \mathcal{H}(R) \le c_2 \mathcal{H}(t) - \frac{\mu\nu}{2} \mathcal{H}'(t).$$

Using Lemma VI.2.2 into this inequality furnishes the desired result and the proof of the lemma is complete. □

From Lemma XIII.4.1, Lemma XIII.4.3, and Lemma XIII.4.4, and with the help of Lemma XIII.2.1, we are able to deduce at once the following main result.

Theorem XIII.4.1 *Let* $\boldsymbol{u}$, τ *be a smooth solution to* (XIII.4.1) *satisfying*

$$\liminf_{x_n \to \infty} x_n^{-3} \int_0^{x_n} \Big(\int_{\Sigma(t)} |\nabla \boldsymbol{u}(x', t)|^2 d\Sigma \Big) dt = 0.$$

Then, if Φ *satisfies* (XIII.4.17), *it follows that*

$$\|\boldsymbol{u}\|_{1,2,\Omega} < \infty.$$

Moreover, there is a positive constant $c_1 = c_1(\Sigma, n, \Phi, \|\boldsymbol{u}\|_{1,2,\Omega}, \nu)$ *such that*

$$|D^\alpha \boldsymbol{u}(x)| + |D^\alpha \nabla \tau(x)| \le c_1 \exp(-\sigma_2 x_n) \tag{XIII.4.33}$$

for every $x \in \Omega$ *with* $x_n \ge 1$ *and all* $|\alpha| \ge 0$, *and where* σ_2 *is given in* Lemma XIII.4.4.

This theorem, along with the uniqueness Theorem XIII.3.3 and the help of Remark XIII.4.1, immediately produces the following general result concerning the asymptotic behavior of weak solutions to Leray's problem.

Corollary XIII.4.1 *There exists a positive constant* $c = c(\Omega, n)$ *such that, if*

$$|\Phi| < c\nu,$$

all generalized solutions $\boldsymbol{v}$ *to Leray's problem* (XIII.1.2)–(XIII.1.4) *satisfy the decay property* (XIII.4.33) *in each outlet* Ω_i, $i = 1, 2$ *with* $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_0^{(i)}$ *and* $\tau = p - C_i$.

Remark XIII.4.2 Another equivalent way of stating Theorem XIII.4.1 is to say that, under condition (XIII.4.17) for Φ, every (smooth) solution $\boldsymbol{u}$ to (XIII.4.1) either verifies

$$\liminf_{x_n\to\infty} x_n^{-3}\int_0^{x_n}\Big(\int_{\Sigma(t)}|\nabla\boldsymbol{u}(x',t)|^2 d\Sigma\Big)dt > 0$$

or decays pointwise and exponentially fast with all its derivatives of arbitrary order. ■

Remark XIII.4.3 Reasoning as in Remark VI.2.1, we can show that $\tau(x)$ tends to a constant τ_0 exponentially fast. ■

Remark XIII.4.4 Corollary XIII.4.1 is similar to the analogous result determined for the linearized Stokes approximation in Section VI.2, with the addition of the flux restriction (XIII.4.17). ■

XIII.5 Flow in an Aperture Domain. Generalized Solutions and Related Properties

We shall now focus our attention on the investigation of existence and uniqueness of flows in certain domains with "exits" having an unbounded cross section. Even though our method carries over to more general situations, we shall restrict ourselves to the special case where the region of motion Ω is a three-dimensional "aperture domain," that is (cf. also Section III.4.3 and Section VI.5),

$$\Omega = \big\{x \in \mathbb{R}^3 : x_3 \neq 0 \text{ or } x' \equiv (x_1, x_2) \in S\big\} \tag{XIII.5.1}$$

with S (the aperture) a bounded locally Lipschitz domain of $\mathbb{R}^2$ that contains a disk of finite radius; Heywood (1976). The reason for this choice is because the technical details simplify to an extent in such a way that the analysis becomes formally simpler. However, whenever the case dictates, we shall explicitly mention possible extensions of our results to more general domains. Moreover, again for the sake of simplicity, we shall assume no body force acting on the liquid, leaving the obvious generalization to the interested reader.

We wish to solve the following *Heywood's problem*: Given $\Phi \in \mathbb{R}$ (the flux through the aperture), to determine a pair $\boldsymbol{v}$, p defined in Ω given by (XIII.5.1), such that

$$\left.\begin{aligned}\nu\Delta \boldsymbol{v} &= \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p\\ \nabla\cdot\boldsymbol{v} &= 0\end{aligned}\right\}\ \text{in } \Omega$$
$$\boldsymbol{v} = 0 \ \text{ at } \partial\Omega$$
$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = 0 \tag{XIII.5.2}$$
$$\int_S v_3 = \Phi.$$

The reader will immediately recognize that (XIII.5.2) is the fully nonlinear counterpart of the problem studied within the linear approximation in Section VI.5.

We begin to give a weak formulation of problem (XIII.5.2). To this end, multiplying $(\text{XIII.5.2})_1$ by $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and formally integrating by parts we find

$$\nu(\nabla\boldsymbol{v}, \nabla\boldsymbol{\varphi}) = (\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{v}), \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega). \tag{XIII.5.3}$$

Thus, also in light of what we did for the linear case in Section VI.3 and Section VI.5, we give the following definition

Definition XIII.5.1. A vector field $\boldsymbol{v} : \Omega \to \mathbb{R}^3$ with Ω given in (XIII.5.1) is said to be a *weak* (or *generalized*) *solution* to problem (XIII.5.2) if and only if

(i) $\boldsymbol{v} \in \widehat{\mathcal{D}}_0^{1,2}(\Omega)$;
(ii) $\boldsymbol{v}$ satisfies (XIII.5.3);
(iii) $\boldsymbol{v}$ satisfies $(\text{XIII.5.2})_5$ in the trace sense.

Remark XIII.5.1 In view of Lemma II.6.3, it is easy to show that every function satisfying (i) also satisfies

$$\lim_{|x|\to\infty} \int_{S_3} |\boldsymbol{v}(|x|,\omega)|d\omega = 0. \tag{XIII.5.4}$$

Actually, let ψ be a function that equals zero in a neighborhood of $\overline{S}$ and one far from S. Then $\boldsymbol{w} \equiv \psi\boldsymbol{v} \in D^{1,2}(\mathbb{R}^3_\pm)$, $\boldsymbol{w} = 0$ at $\{x_3 = 0\}$, and the assertion follows from the reasonings preceding Theorem II.6.3. With the help of (XIII.5.4) it is at once recognized that conditions (i)-(iii) of Definition XIII.5.1 translate (XIII.5.2) into a weak form. ■

Our next objective is to associate to every weak solution a suitable pressure field. Since the "aperture" S is bounded and contains a disk of finite radius, we shall suppose without loss of generality that

$$\left\{x' \in \mathbb{R}^2 : |x'| < 1\right\} \subset \overline{S} \subset \left\{x' \in \mathbb{R}^2 : |x'| < 2\right\}.$$

We have the following lemma.

Lemma XIII.5.1 *Let $\boldsymbol{v}$ be a generalized solution to problem* (XIII.5.2). *Then there is a $p \in L^2_{loc}(\overline{\Omega})$ such that*

$$\nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) = (\boldsymbol{v} \cdot \nabla \boldsymbol{\psi}, \boldsymbol{v}) + (p, \nabla \cdot \boldsymbol{\psi}) \ \ \textit{for all } \boldsymbol{\psi} \in C_0^\infty(\Omega). \tag{XIII.5.5}$$

Moreover, there are constants $p_+, p_- \in \mathbb{R}$ *such that*

$$\begin{aligned} p - p_+ &\in L^3(\mathbb{R}^3_+ - B_3) \\ p - p_- &\in L^3(\mathbb{R}^3_- - B_3). \end{aligned}$$

Proof. Let $\boldsymbol{\psi} \in D_0^{1,2}(\Omega')$ where Ω' is any bounded domain of Ω. Since

$$|(\boldsymbol{v} \cdot \nabla \boldsymbol{\psi}, \boldsymbol{v})| \leq \|\boldsymbol{v}\|^2_{4,\Omega'} |\boldsymbol{\psi}|_{1,2},$$

by the Sobolev inequality (II.3.7) we find

$$|(\boldsymbol{v} \cdot \nabla \boldsymbol{\psi}, \boldsymbol{v})| \leq c_1 \|\boldsymbol{v}\|_6^2 |\boldsymbol{\psi}|_{1,2} \leq c_2 |\boldsymbol{v}|^2_{1,2} |\boldsymbol{\psi}|_{1,2},$$

and so

$$(\boldsymbol{v} \cdot \nabla \boldsymbol{\psi}, \boldsymbol{v})$$

defines a bounded linear functional in $D_0^{1,2}(\Omega')$. Therefore, the first part of the lemma is a consequence of Lemma IV.1.1. Let $\chi = \chi(|x|)$ be a smooth function that equals zero for $|x| \leq 3$ and one for $|x| \geq 4$ and set $\boldsymbol{w} = \chi \boldsymbol{v}$, $\pi = \chi p$. Using Definition XIII.5.1 and the identity (XIII.5.5), it is not hard to show that $\boldsymbol{w}$, π is a weak solution to the following Stokes problem in $\mathbb{R}^3_+$ (an analogous property holding in $\mathbb{R}^3_-$)

$$\left.\begin{aligned} \nu \Delta \boldsymbol{w} &= \nabla \pi + \mathbf{F} \\ \nabla \cdot \boldsymbol{w} &= g \end{aligned}\right\} \ \text{in } \mathbb{R}^3_+ \qquad \boldsymbol{w} = 0 \ \ \text{at } x_3 = 0 \tag{XIII.5.6}$$

where

$$\begin{aligned} \boldsymbol{F} &= \boldsymbol{v} \cdot \nabla \boldsymbol{w} + 2 \nabla \chi \cdot \nabla \boldsymbol{v} + \boldsymbol{v} \Delta \chi - \boldsymbol{v} \cdot \nabla \chi \boldsymbol{v} + p \nabla \chi \\ g &= \boldsymbol{v} \cdot \nabla \chi. \end{aligned} \tag{XIII.5.7}$$

Recalling that $p \in L^2_{loc}(\overline{\Omega})$ and using the properties of χ and condition (i) of Definition XIII.5.1, we find for all $\boldsymbol{\psi} \in C_0^\infty(\mathbb{R}^3_+)$

$$\begin{aligned} |(2\nabla \chi \cdot \nabla \boldsymbol{v}, \boldsymbol{\psi}) + &(\boldsymbol{v} \Delta \chi, \boldsymbol{\psi}) - (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\psi}) + (p \nabla \chi, \boldsymbol{\psi})| \\ &\leq c \left(\|\boldsymbol{v}\|_{3,\sigma} + \|\boldsymbol{v}\|^2_{3,\sigma} + \|p\|_{2,\sigma} \right) |\boldsymbol{\psi}|_{1,3/2} \end{aligned} \tag{XIII.5.8}$$

where $\sigma = \text{supp}\,(\chi)$. As a consequence, (XIII.5.8) implies

$$2 \nabla \chi \cdot \nabla \boldsymbol{v} + \boldsymbol{v} \Delta \chi - \boldsymbol{v} \cdot \nabla \chi \boldsymbol{v} + p \nabla \chi \in D_0^{-1,3}(\mathbb{R}^3_+). \tag{XIII.5.9}$$

Moreover, by Theorem II.6.3, by (XIII.5.4) and (i) of Definition XIII.5.1, it is

$$\boldsymbol{v} \in L^6(\mathbb{R}^3_+)$$

and so we find

$$|(\boldsymbol{v}\cdot\nabla\boldsymbol{w},\boldsymbol{\psi})| = |(\boldsymbol{v}\otimes\boldsymbol{w},\nabla\boldsymbol{\psi})| \le \|\boldsymbol{v}\|^2_{6,\mathbb{R}^3_+}|\boldsymbol{\psi}|_{1,3/2}. \tag{XIII.5.10}$$

Finally,

$$\boldsymbol{v}\cdot\nabla\chi \in L^3(\mathbb{R}^3_+). \tag{XIII.5.11}$$

From (XIII.5.7)–(XIII.5.11), it follows that

$$\boldsymbol{F} \in D_0^{-1,3}(\mathbb{R}^3_+)\,, \quad g \in L^3(\mathbb{R}^3_+)\,. \tag{XIII.5.12}$$

Since, by assumption, $\boldsymbol{w}$ is a generalized solution to (XIII.5.6) (cf. Section IV.3), from (XIII.5.12) and Theorem IV.3.3 we conclude, in particular, that

$$\pi - p_+ \in L^3(\mathbb{R}^3_+)$$

for some constant $p_+ \in \mathbb{R}^3$. Recalling that $\pi = p$ in $\mathbb{R}^3_+ - B_3$, we obtain the desired result. □

Remark XIII.5.2 The preceding lemma shows, among other things, that the pressure field tends in each outlet to a definite constant in the sense of L^3. This observation, along with the analogy to the linearized case (cf. Remark VI.4.7), suggests a different formulation of problem (XIII.5.2), where one prescribes, in place of the flux condition $(\text{XIII.5.2})_5$, the following ones:

$$\lim_{|x|\to\infty,\ x_3>0} p(x) = p_+$$

$$\lim_{|x|\to\infty,\ x_3<0} p(x) = p_-,$$

where $p_\pm$ are *prescribed* constants.[1] This view, which originates with the work of Heywood (1976), has been taken by several authors. We refer the reader to the papers of Solonnikov (1981, 1983) and the references cited therein. ■

The last result of this section concerns the differentiability of weak solutions to problem (XIII.5.2). Since the aperture S has no "thickness," we can prove that these solutions are regular everywhere in Ω except at the boundary ∂S. Of course, if S had "thickness" so that Ω would become smooth, the corresponding generalized solutions would be equally smooth. In the current situation, we have the following result whose proof, which patterns that furnished in Theorem XIII.1.1, is left to the reader as an exercise.

Theorem XIII.5.1 *Let $\boldsymbol{v}$ be a generalized solution to problem* (XIII.5.2) *and let p be the corresponding pressure field associated to $\boldsymbol{v}$ by* Lemma XIII.5.1. *Then*

$$\boldsymbol{v},\ p \in C_0^\infty(\overline{\Omega}')$$

where Ω' is any bounded domain of Ω that does not contain ∂S.

[1] Of course, one of the two constants can be taken to be zero.

Remark XIII.5.3 We wish to observe that some of the concepts and results introduced so far, can be recovered when Ω is a two-dimensional aperture domain:

$$\{x \in \mathbb{R}^2 : x_2 \neq 0 \ \text{ or } \ x_1 \in (-d, d)\},$$

for some $d > 0$. Of course, in this case we can also give a weak formulation of problem (XIII.5.2). In these regards, we notice that the reasoning developed in Remark XIII.5.1 cannot be repeated here, because of the lack of the Sobolev inequality. However, one can show the validity of the following inequality

$$\int_\Omega \frac{\boldsymbol{v}^2}{(|x|+1)^2} \leq c \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{v}$$

where $c > 0$ and $\boldsymbol{v} \in \widehat{\mathcal{D}}_0^{1,2}(\Omega)$; see Galdi, Padula, & Passerini (1995). Using this inequality, it can be proved that (XIII.5.4) is valid also in the two-dimensional case and that, in fact, $\boldsymbol{v}(x)$ tends to zero uniformly pointwise as $|x| \to \infty$. Furthermore, the first part of Lemma XIII.5.1 continues to hold and the pressure field tends uniformly pointwise to a suitable constant p_+ [respectively p_-] at large distances in $\mathbb{R}^2_+$ [respectively $\mathbb{R}^2_-$]. Finally, Theorem XIII.5.1 holds also in the two-dimensional case. For detail we refer the reader to the paper of Galdi, Padula, & Passerini. ■

XIII.6 Energy Equation and Uniqueness for Flows in an Aperture Domain

The main objective of this section is to formulate conditions under which a generalized solution is unique. As in the case of the three-dimensional exterior problem, however, we are not able to furnish such conditions merely in the class of generalized solutions but, rather in a subclass obeying a suitable energy inequality. In fact, as we shall prove later, this subclass is certainly nonempty.

Let us derive formally the energy equation for solutions to problem (XIII.5.2). To this end, we multiply $(\text{XIII.5.2})_1$ by $\boldsymbol{v}$ and integrate over $\Omega_R \equiv \Omega \cap B_R$ for sufficiently large R to obtain

$$\begin{aligned}-\nu \int_{\Omega_R} \nabla \boldsymbol{v} : \nabla \boldsymbol{v} + \nu \int_{\partial B_R} \boldsymbol{v} \cdot \frac{\partial \boldsymbol{v}}{\partial n} = \frac{1}{2} \int_{\partial B_R} v^2 \boldsymbol{v} \cdot \boldsymbol{n} + \int_{\partial B_R^+} (p - p_+) \boldsymbol{v} \cdot \boldsymbol{n} \\ + \int_{\partial B_R^-} (p - p_-) \boldsymbol{v} \cdot \boldsymbol{n} + (p_+ - p_-) \int_S v_3\end{aligned}$$

where p_+ and p_- are the limits to which p tends as $|x| \to \infty$ in $\mathbb{R}^3_+$ and $\mathbb{R}^3_-$, respectively, cf. Lemma XIII.5.1, and $B_R^\pm = B_R \cap \mathbb{R}^3_\pm$. Letting $R \to \infty$ and assuming that all surface integrals tend to zero (at least along a sequence), we find

$$\int_{\Omega} \nabla \boldsymbol{v} : \nabla \boldsymbol{v} = -\frac{p_* \Phi}{\nu} \qquad \text{(XIII.6.1)}$$

where Φ is the flux of $\boldsymbol{v}$ through S and $p_* = p_+ - p_-$. Relation (XIII.6.1) is the energy equation for problem (XIII.5.2) and describes the balance between the energy dissipated by the liquid and the work done on it. Notice that, in agreement with physical intuition, a positive flux implies $p_+ < p_-$, while $p_- < p_+$ otherwise.

As the reader may have noticed, in obtaining (XIII.6.1) we have assumed that the solution has a certain degree of regularity at large spatial distances that a priori need not be matched by a generalized solution. We may thus wonder which extra conditions we must impose on a generalized solution so that it will obey the energy equation. An answer is furnished by the following.

Theorem XIII.6.1 *Let $\boldsymbol{v}$ be a generalized solution to problem* (XIII.5.2). *Then, if*

$$\boldsymbol{v} \in L^3(\Omega),$$

$\boldsymbol{v}$ *obeys the energy equation* (XIII.6.1), *where $p_* = p_+ - p_-$ and $p_\pm$ are the constants associated to p by* Lemma XIII.5.1.

Proof. Let $\psi_R = \psi_R(|x|)$ be a smooth "cut-off" function such that

$$\psi_R(|x|) = \begin{cases} 1 \text{ if } |x| \le R \\ 0 \text{ if } |x| \ge 2R \end{cases}$$

$$|\nabla \psi_R(|x|)| \le M, \quad \text{for some } M \text{ independent of } R.$$

In view of Lemma XIII.5.1, identity (XIII.5.5) continues to hold for all $\boldsymbol{\psi} \in W_0^{1,2}(\Omega')$, where Ω' is an arbitrary bounded domain with $\overline{\Omega'} \subset \Omega$. Thus, we may take $\boldsymbol{\psi} \equiv \psi_R \boldsymbol{v}$ to obtain

$$\nu(\psi_R \nabla \boldsymbol{v}, \nabla \boldsymbol{v}) = (\psi_R \boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{v}) + (p, \nabla \cdot (\psi_R \boldsymbol{v})) + (\boldsymbol{v} \cdot \nabla \psi_R, \boldsymbol{v} \cdot \boldsymbol{v}). \quad \text{(XIII.6.2)}$$

Recalling the properties of ψ_R and the assumptions on $\boldsymbol{v}$, we deduce at once that

$$\begin{aligned} \lim_{R\to\infty} (\psi_R \nabla \boldsymbol{v}, \nabla \boldsymbol{v}) &= |\boldsymbol{v}|_{1,2}^2 \\ \lim_{R\to\infty} (\boldsymbol{v} \cdot \nabla \psi_R, \boldsymbol{v} \cdot \boldsymbol{v}) &= 0. \end{aligned} \qquad \text{(XIII.6.3)}$$

Moreover,

$$\begin{aligned} (p, \nabla \cdot (\psi_R \boldsymbol{v})) = & \int_{\mathbb{R}^3_+} (p - p_+) \boldsymbol{v} \cdot \nabla \psi_R + \int_{\mathbb{R}^3_-} (p - p_-) \boldsymbol{v} \cdot \nabla \psi_R \\ & + p_* \int_{\Omega} \nabla \cdot (\psi_R \boldsymbol{v}) \\ & = \int_{\mathbb{R}^3_+} (p - p_+) \boldsymbol{v} \cdot \nabla \psi_R + \int_{\mathbb{R}^3_-} (p - p_-) \boldsymbol{v} \cdot \nabla \psi_R - p_* \Phi, \end{aligned}$$

where $p_\pm$ are the constants associated to p by Lemma XIII.5.1. Taking into account that, again by Lemma XIII.5.1,

$$p - p_+ \in L^3(\mathbb{R}^3_+ - B_3),$$

and setting $\Omega_{R,2R} = \Omega \cap B_{2R} \cap B_R$, we thus find

$$\begin{aligned}\left|\int_{\mathbb{R}^3_+} (p-p_+)\boldsymbol{v}\cdot\nabla\psi_R\right| &\le \|p-p_+\|_{3,\Omega_{R,2R}}\|\boldsymbol{v}\|_3\|\nabla\psi_R\|_{3,\Omega_R}\\ &\le c\|\boldsymbol{v}\|_3\|p-p_+\|_{3,\Omega_{R,2R}}\int_R^{2R}\frac{dr}{r}\\ &= c\log 2\|\boldsymbol{v}\|_3\|p-p_+\|_{3,\Omega_{R,2R}}.\end{aligned}$$

Therefore,

$$\lim_{R\to\infty}\left|\int_{\mathbb{R}^3_+} (p-p_+)\boldsymbol{v}\cdot\nabla\psi_R\right| = 0.$$

Employing a similar argument for the integral over $\mathbb{R}^3_-$, we may then conclude that

$$\lim_{R\to\infty}(p, \nabla\cdot(\psi_R\boldsymbol{v})) = -p_*\Phi. \tag{XIII.6.4}$$

Furthermore, setting as before $B_R^\pm = \mathbb{R}^3_\pm \cap B_R$, after integration by parts we obtain

$$(\psi_R\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{v}) = \int_{B_R^+}\psi_R\boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{v} + \int_{B_R^-}\psi_R\boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{v} = \tfrac{1}{2}\int_\Omega \boldsymbol{v}\cdot\nabla\psi_R\boldsymbol{v}\cdot\boldsymbol{v},$$

so that, recalling the assumption on $\boldsymbol{v}$ and the properties of ψ_R, it follows that

$$\lim_{R\to\infty}(\psi_R\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{v}) = 0. \tag{XIII.6.5}$$

Equation (XIII.6.1) becomes a consequence of (XIII.6.2)–(XIII.6.5).

Let $\boldsymbol{w}$ be a generalized solution to (XIII.5.2) and let π be the pressure field associated to $\boldsymbol{w}$ by Lemma XIII.5.1. We shall say that $\boldsymbol{w}$ satisfies the *energy inequality* if

$$|\boldsymbol{w}|^2_{1,2} \le -\frac{\pi_*\Phi}{\nu} \tag{XIII.6.6}$$

where $\pi_* = \pi_+ - \pi_-$ and $\pi_\pm$ are the constants associated to π by Lemma XIII.5.1. The following uniqueness result holds. □

Theorem XIII.6.2 *Let $\boldsymbol{v}$ be a generalized solution to problem* (XIII.5.2) *such that*

$$\|\boldsymbol{v}\|_3 < \frac{\sqrt{3}}{2}\nu.$$

Then $\boldsymbol{v}$ is unique in the class of all generalized solutions $\boldsymbol{w}$ corresponding to the same flux Φ and satisfying (XIII.6.6).

Proof. Let ψ_R be the "cut-off" function introduced in the proof of Theorem XIII.6.1. As in that case, we may take $\boldsymbol{\psi} = \psi_R \boldsymbol{w}$ in the identity (XIII.5.5) to obtain

$$\nu(\psi_R \nabla \boldsymbol{v}, \nabla \boldsymbol{w}) = -(\psi_R \boldsymbol{v} \cdot \nabla \boldsymbol{w}, \boldsymbol{v}) - (\boldsymbol{v} \cdot \boldsymbol{w}, \boldsymbol{v} \cdot \nabla \psi_R) + (p, \nabla \cdot (\psi_R \boldsymbol{w})). \tag{XIII.6.7}$$

Using the summability properties of $\boldsymbol{v}$, $\boldsymbol{w}$ it is not difficult to show the following relations

$$\begin{aligned} \lim_{R\to\infty} (\psi_R \nabla \boldsymbol{v}, \nabla \boldsymbol{w}) &= (\nabla \boldsymbol{v}, \nabla \boldsymbol{w}) \\ \lim_{R\to\infty} (\psi_R \boldsymbol{v} \cdot \nabla \boldsymbol{w}, \boldsymbol{v}) &= (\boldsymbol{v} \cdot \nabla \boldsymbol{w}, \boldsymbol{v}). \end{aligned} \tag{XIII.6.8}$$

Moreover, with the aid of (II.3.7), we find

$$\begin{aligned} |(\boldsymbol{v} \cdot \boldsymbol{w}, \boldsymbol{v} \cdot \nabla \psi_R)| &\le \|\boldsymbol{v}\|^2_{3,\Omega_{R,2R}} \|\boldsymbol{w}\|_{6,\Omega_{R,2R}} \|\nabla \psi_R\|_{6,\Omega_{R,2R}} \\ &\le \frac{c_2}{R} \|\boldsymbol{v}\|^2_{3,\Omega_{R,2R}} |\boldsymbol{w}|_{1,2}, \end{aligned}$$

and so

$$\lim_{R\to\infty} (\boldsymbol{v} \cdot \boldsymbol{w}, \boldsymbol{v} \cdot \nabla \psi_R) = 0. \tag{XIII.6.9}$$

We next consider the following identity

$$(p, \nabla \cdot (\psi_R \boldsymbol{w})) = \int_{\mathbb{R}^3_+} (p - p_+) \boldsymbol{w} \cdot \nabla \psi_R + \int_{\mathbb{R}^3_-} (p - p_-) \boldsymbol{w} \cdot \nabla \psi_R - p_* \Phi, \tag{XIII.6.10}$$

where $p_\pm$ are the constants introduced in Lemma XIII.5.1 and $p_* = p_+ - p_-$. By the Hölder inequality, (II.3.7), and the properties of ψ_R, it follows that

$$\begin{aligned} \left| \int_{\mathbb{R}^3_+} (p - p_+) \boldsymbol{w} \cdot \nabla \psi_R \right| &\le \|p - p_+\|_{3/2,\Omega_R,2R} \|\boldsymbol{w}\|_{6,\Omega_{R,2R}} \|\nabla \psi_R\|_{6,\Omega_{R,2R}} \\ &\le \frac{c}{R} \|p - p_+\|_{3/2,\Omega_{R,2R}} |\boldsymbol{w}|_{1,2}. \end{aligned} \tag{XIII.6.11}$$

It is not difficult to see that

$$p - p_+ \in L^{3/2}(\mathbb{R}^3_\pm - B_3). \tag{XIII.6.12}$$

Actually, we know that $\chi \boldsymbol{v}$ and χp, with the χ "cut-off" function introduced in the proof of Lemma XIII.5.1, satisfy the Stokes system (XIII.5.6), (XIII.5.7) (and an analogous one in $\mathbb{R}^3_-$). Because of the assumption on $\boldsymbol{v}$, we have

$$\boldsymbol{v} \in L^3(\Omega), \ \ \boldsymbol{v} \cdot \nabla \boldsymbol{v} \in D_0^{-1,3/2}(\Omega),$$

from which it readily follows that

$$\boldsymbol{F} \in D_0^{-1,3/2}(\mathbb{R}^3_+)\,, \quad g \in L^{3/2}(\mathbb{R}^3_+)$$

with $\boldsymbol{F}$ and g given in (XIII.5.7). Thus, applying the results of Theorem IV.3.3 to (XIII.5.6), (XIII.5.7) and recalling that $\chi\boldsymbol{v}$ is a generalized solution to (XIII.5.6), (XIII.5.7), we find (XIII.6.12). As a consequence of (XIII.6.12), from (XIII.6.11) (and the analogous property in $\mathbb{R}^3_-$), we conclude that

$$\lim_{R\to\infty}(p, \nabla\cdot(\psi_R \boldsymbol{w})) = -p_*\Phi. \tag{XIII.6.13}$$

Therefore, (XIII.6.7)–(XIII.6.12) yield

$$-\nu(\nabla\boldsymbol{v}, \nabla\boldsymbol{w}) = -(\boldsymbol{v}\cdot\nabla\boldsymbol{w}, \nabla\boldsymbol{v}) + p_*\Phi. \tag{XIII.6.14}$$

Writing identity (XIII.5.5) with $\boldsymbol{w}$ in place of $\boldsymbol{v}$ and choosing $\psi_R\boldsymbol{v}$ for $\boldsymbol{\psi}$, by means of arguments very close to those just used one can show

$$-\nu(\nabla\boldsymbol{w}, \nabla\boldsymbol{v}) = -(\boldsymbol{w}\cdot\nabla\boldsymbol{v}, \nabla\boldsymbol{w}) + \pi_*\Phi. \tag{XIII.6.15}$$

We next observe that by Theorem XIII.6.1, $\boldsymbol{v}$ obeys the energy equality (XIII.6.1) while $\boldsymbol{w}$, by hypothesis, satisfies the energy inequality (XIII.6.6). Thus, adding the four displayed relations (XIII.6.1), (XIII.6.6), (XIII.6.13), and (XIII.6.14) and setting $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{w}$, we arrive at

$$\nu|\boldsymbol{u}|^2_{1,2} \le -(\boldsymbol{v}\cdot\nabla\boldsymbol{w}, \boldsymbol{v}) - (\boldsymbol{w}\cdot\nabla\boldsymbol{v}, \boldsymbol{w}). \tag{XIII.6.16}$$

The following relations are easily shown:

$$(\boldsymbol{w}\cdot\nabla\boldsymbol{v}, \boldsymbol{w}) = -(\boldsymbol{w}\cdot\nabla\boldsymbol{w}, \boldsymbol{v}), \quad (\boldsymbol{u}\cdot\nabla\boldsymbol{v}, \boldsymbol{v}) = 0. \tag{XIII.6.17}$$

Assuming, for a while, the validity of (XIII.6.17) and using it into (XIII.6.16) delivers

$$\begin{aligned}\nu|\boldsymbol{u}|^2_{1,2} &\le -(\boldsymbol{v}\cdot\nabla\boldsymbol{w}, \boldsymbol{v}) + (\boldsymbol{w}\cdot\nabla\boldsymbol{w}, \boldsymbol{v}) + (\boldsymbol{u}\cdot\nabla\boldsymbol{v}, \boldsymbol{v}) \\ &= (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v}).\end{aligned} \tag{XIII.6.18}$$

Employing the Schwarz inequality on the right-hand side of (XIII.6.18) along with the Sobolev inequality (II.3.7), we find

$$|(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v})| \le \|\boldsymbol{v}\|_3\|\boldsymbol{u}\|_6|\boldsymbol{u}|_{1,2} \le \frac{2}{\sqrt{3}}\|\boldsymbol{v}\|_3|\boldsymbol{u}|^2_{1,2}.$$

If we combine this inequality with (XIII.6.18), we arrive at

$$\left(\nu - \frac{2}{\sqrt{3}}\|\boldsymbol{v}\|_3\right)|\boldsymbol{u}|^2_{1,2} \le 0,$$

which, in turn, furnishes $\boldsymbol{u} \equiv 0$ under the stated assumption on $\boldsymbol{v}$. To show the theorem completely, it remains to prove the identities (XIII.6.17). Setting $B_R^+ = \mathbb{R}^3_+ \cap B_R$, we have

$$\int_{B_R^+} \boldsymbol{w}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{w} = -\int_{B_R^+} \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{v} + \int_{\partial B_R\cap\mathbb{R}^3_+} \boldsymbol{w}\cdot\boldsymbol{n}\boldsymbol{v}\cdot\boldsymbol{w} - \int_S w_3\boldsymbol{v}\cdot\boldsymbol{w}. \quad \text{(XIII.6.19)}$$

By the Hölder inequality, it follows that

$$\left|\int_{\partial B_R\cap\mathbb{R}^3_+} \boldsymbol{w}\cdot\boldsymbol{n}\boldsymbol{v}\cdot\boldsymbol{w}\right| \le cR^{2/3}\|\boldsymbol{w}\|^2_{6,\partial B_R\cap\mathbb{R}^3_+}\|\boldsymbol{v}\|_{3,\partial B_R\cap\mathbb{R}^3_+}. \quad \text{(XIII.6.20)}$$

However, since

$$\int_0^\infty \left[\int_{\partial B_R^+} \left(|\boldsymbol{w}|^6 + |\boldsymbol{v}|^3\right) dB_R\right] dR < \infty,$$

we have, at least along a sequence,

$$\|\boldsymbol{w}\|^2_{6,\partial B_R\cap\mathbb{R}^3_+} + \|\boldsymbol{v}\|_{3,\partial B_R\cap\mathbb{R}^3_+} = o(R^{-2/3})$$

and therefore, from (XIII.6.20) and (XIII.6.19) we conclude that

$$\int_{\mathbb{R}^3_+} \boldsymbol{w}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{w} = -\int_{\mathbb{R}^3_+} \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{v} - \int_S w_3\boldsymbol{v}\cdot\boldsymbol{w}. \quad \text{(XIII.6.21)}$$

Likewise,

$$\int_{\mathbb{R}^3_+} \boldsymbol{w}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{w} = -\int_{\mathbb{R}^3_+} \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{v} + \int_S w_3\boldsymbol{v}\cdot\boldsymbol{w} \quad \text{(XIII.6.22)}$$

and $(\text{XIII.6.17})_1$ follows from (XIII.6.21), (XIII.6.22). In a completely analogous way one can show $(\text{XIII.6.17})_2$, whose proof is thus left to the reader as an exercise. The theorem is proved. □

Remark XIII.6.1 In the case of a plane aperture domain, Theorem XIII.6.2 continues to hold, provided the assumption on the L^3-norm of $\boldsymbol{v}$ is replaced by the following one

$$\sup_{x\in\Omega} |\boldsymbol{v}(x)|\,|x| \le c\nu,$$

for a suitable $c > 0$; see Galdi, Padula, & Solonnikov (1996). ■

XIII.7 Existence and Uniqueness of Flows in an Aperture Domain

The objective of this section is to show existence and uniqueness of generalized solutions to problem (XIII.5.2). The main feature of these solutions is that, unlike the case of flow in domains with cylindrical ends (Leray's problem), they exist without restriction on the coefficient of kinematical viscosity ν or on the flux Φ. This nice circumstance is due to the fact that, in the case under consideration, for any $\eta > 0$ there is a solenoidal vector $\boldsymbol{a} = \boldsymbol{a}(x;\eta) \in \widehat{\mathcal{D}}_0^{1,2}(\Omega)$ carrying the flux Φ such that

$$\left| \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{a} \cdot \boldsymbol{u} \right| \leq \eta \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u}, \quad \text{for all } \boldsymbol{u} \in \mathcal{D}(\Omega). \tag{XIII.7.1}$$

The validity of (XIII.7.1), in turn, is made possible by the fact that the "outlets" to infinity for the domain (XIII.5.1) have an *unbounded* cross section. The existence of such fields $\boldsymbol{a}(x;\eta)$ was discovered by Ladyzhenskaya & Solonnikov (1977, §2). It should be remarked that the method used by these authors to construct the fields $\boldsymbol{a}(x;\eta)$ applies to domains whose outlets to infinity Ω_i are more general than those of domain (XIII.5.1); in fact, the only condition required is that each Ω_i contains a semi-infinite cone; cf. also Solonnikov & Pileckas (1977, Lemma 4), Solonnikov (1981), and Solonnikov (1983, §2.4)

The construction of the field $\boldsymbol{a}(x;\eta)$ is the object of the next lemma. As the reader will recognize, this construction resembles the one given in Lemma IX.4.2 for the case of a flow in a bounded domain.

Lemma XIII.7.1 *Let Ω be as in* (XIII.5.1) *and let $\eta > 0$. Then there exists a solenoidal vector field $\boldsymbol{a} = \boldsymbol{a}(x;\eta) \in C^\infty(\Omega)$ vanishing in a neighborhood of $\partial\Omega$ such that*

(i) $|\boldsymbol{a}(x)| \leq M|x|^{-2}$; $|\nabla \boldsymbol{a}(x)| \leq M|x|^{-3}$, *for all sufficiently large $|x|$;*
(ii) $\boldsymbol{a} \in \widehat{\mathcal{D}}_0^{1,2}(\Omega)$;
(iii) $\displaystyle\int_S a_3 = 1$;
(iv) $\boldsymbol{a}$ *verifies* (XIII.7.1).

Proof. Let

$$\boldsymbol{b} = \frac{1}{2\pi|x'|^2}\,(-x_2, x_1, 0)\,.$$

Clearly,

$$\left.\begin{aligned} \nabla \times \boldsymbol{b} = 0 \\ \nabla \cdot \boldsymbol{b} = 0 \end{aligned}\right\} \ \text{ in } \Omega - \{0\}.$$

We next introduce a system of cylindrical coordinates (r, θ, x_3) with the origin at the center of the unit disk $\mathcal{C} = \{|x'| < 1\}$ which, without loss, we have assumed to be strictly contained in S. The three unit vectors will be denoted by $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_3$, respectively. By a simple computation based on the properties of $\boldsymbol{b}$ we find

$$\int_{\partial S} \boldsymbol{b} \times \boldsymbol{n} \cdot \boldsymbol{e}_3 = -1, \tag{XIII.7.2}$$

where $\boldsymbol{n}$ is the exterior unit normal to ∂S.[1] In fact, introducing

[1] Since S is locally Lipschitz, $\boldsymbol{n}$ exists *a.e.* on ∂S; cf. Lemma II.4.1.

$$\boldsymbol{w} = (b_2, -b_1, 0)$$

we have

$$\nabla' \cdot \boldsymbol{w} = 0 \quad \text{in } \omega = S - \mathcal{C},$$

where ∇' is the divergence operator restricted to the variables x'. Thus, by the divergence theorem,

$$\int_{\partial S} \boldsymbol{w} \cdot \boldsymbol{n} = \int_{\partial \mathcal{C}} \boldsymbol{w} \cdot \boldsymbol{e}_r.$$

Since for any vector $\boldsymbol{s} = (s_1, s_2, 0)$, $\boldsymbol{w} \cdot \boldsymbol{s} = -\boldsymbol{b} \times \boldsymbol{s} \cdot \boldsymbol{e}_3$, it follows that

$$-\int_{\partial S} \boldsymbol{b} \times \boldsymbol{n} \cdot \boldsymbol{e}_3 = -\int_{\partial \mathcal{C}} \boldsymbol{b} \times \boldsymbol{e}_r \cdot \boldsymbol{e}_3 = \frac{1}{2\pi} \int_0^{2\pi} (\sin^2 \theta + \cos^2 \theta) d\theta = 1,$$

which shows (XIII.7.2). Now, let $\rho = \rho(x)$ denote the regularized distance of x from $\partial \Omega$. By Lemma III.6.1 we obtain, in particular,

$$\begin{aligned} &\delta(x) \le \rho(x) \le k_1 \delta(x) \\ &|D^\alpha \rho(x)| \le k_2 \left[\delta(x)\right]^{1-|\alpha|}, \quad 0 \le |\alpha| \le 2 \end{aligned} \tag{XIII.7.3}$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ and k_1, k_2 are independent of x. Let γ denote the x_3-axis and set

$$d = \operatorname{dist}(\partial S, \gamma).$$

Let σ, ψ be smooth nondecreasing functions of the real variable t such that

$$\sigma(t) = \begin{cases} \dfrac{d}{2} \text{ if } t \le d/2 \\ t \ \text{ if } t \ge d \end{cases}$$

and

$$\psi(t) = \begin{cases} 0 \text{ if } t \le 0 \\ 1 \text{ if } t \ge 1. \end{cases}$$

Finally, we set

$$\zeta(x) = \psi\left(\varepsilon \ln \frac{\sigma(|x'|)}{\rho(x)}\right), \quad \varepsilon \in (0,1). \tag{XIII.7.4}$$

In view of the property of the function ψ, we at once recognize that

$$\zeta(x) = \begin{cases} 0 \text{ if } \sigma(|x'|) \le \rho(x) \\ 1 \text{ if } \sigma(|x'|) \ge \rho(x) e^{1/\varepsilon} \end{cases} \tag{XIII.7.5}$$

which implies, in particular,

$$\operatorname{supp}(\nabla\zeta) \subset \left\{x \in \Omega : \ \rho(x) \le \sigma(|x'|) \le \rho(x) e^{1/\varepsilon}\right\}. \tag{XIII.7.6}$$

Moreover, setting
$$C = \{x \in \Omega : |x'| < d/2\}$$
it is immediately seen that
$$\zeta(x) = 0, \quad \text{for all } x \in C. \tag{XIII.7.7}$$
In fact, for $x \in C$ we have
$$\delta(x) \geq d - |x'| \geq d/2$$
which, by (XIII.7.3), in turn yields
$$\rho(x) \geq d/2, \quad \text{for all } x \in C. \tag{XIII.7.8}$$
In addition, if $x \in C$, from the definition of the function σ we derive
$$\sigma(|x'|) = d/2, \quad \text{for all } x \in C. \tag{XIII.7.9}$$
Thus, from (XIII.7.8) and (XIII.7.9) we find
$$\rho(x) \geq \sigma(|x'|), \quad \text{for all } x \in C,$$
which, by (XIII.7.5), implies (XIII.7.7). Also, setting
$$N_\varepsilon = \left\{ x \in \Omega : \delta(x) \leq \frac{de^{-1/\varepsilon}}{2k_1} \right\},$$
we have
$$\zeta(x) = 1, \quad \text{for all } x \in N_\varepsilon. \tag{XIII.7.10}$$
In fact, since σ is nondecreasing,
$$\sigma(|x'|) \geq d/2,$$
and so, by (XIII.7.5), we have $\zeta = 1$ whenever
$$\rho(x) \leq \frac{d}{2} e^{-1/\varepsilon},$$
which, by (XIII.7.3) implies (XIII.7.10). The next task is to prove an estimate for $\nabla\zeta$ and $D^2\zeta$ at large distances. To this end, we observe that, by a straightforward computation, we obtain
$$\frac{\partial \zeta}{\partial x_k} = \varepsilon \psi' S_k, \quad \frac{\partial^2 \zeta}{\partial x_k \partial x_l} = \varepsilon^2 \psi'' S_k S_l + \varepsilon \psi' \frac{\partial S_k}{\partial x_l}$$
where the prime means differentiation with respect to the argument involved and

$$S_k = \sigma' \frac{\partial |x'|}{\partial x_k} \frac{1}{\sigma(|x'|)} - \frac{1}{\rho(x)} \frac{\partial \rho(x)}{\partial x_k}$$

$$\frac{\partial S_k}{\partial x_l} = \left[\frac{\sigma''}{\sigma(|x'|)} - \frac{(\sigma')^2}{\sigma^2(|x'|)} \right] \frac{\partial |x'|}{\partial x_k} \frac{\partial |x'|}{\partial x_l}$$
$$+ \frac{\sigma'}{\sigma(|x'|)} \frac{\partial^2 |x'|}{\partial x_k \partial x_l} + \frac{1}{\rho^2(x)} \frac{\partial \rho}{\partial x_k} \frac{\partial \rho}{\partial x_l} - \frac{1}{\rho(x)} \frac{\partial^2 \rho}{\partial x_k \partial x_l}.$$

Therefore, observing that

$$|\psi'|, |\psi''|, |\sigma'|, |\sigma''| \le M,$$

for some M independent of x, by virtue of (XIII.7.3), (XIII.7.6), and (XIII.7.7) we conclude that

$$\left.\begin{aligned} |\nabla \zeta(x)| &\le \varepsilon c_1 \delta^{-1}(x) \\ |D^2 \zeta(x)| &\le c_2 \delta^{-2}(x) \end{aligned}\right\} \quad \text{for all } x \in \Omega, \tag{XIII.7.11}$$

with c_1 and c_2 independent of x. The field $\boldsymbol{a}$ is introduced by the following relation

$$\boldsymbol{a} = \nabla \times (\zeta \boldsymbol{b}). \tag{XIII.7.12}$$

Clearly, $\boldsymbol{a}$ is solenoidal and, in view of (XIII.7.7) it is also in $C^\infty(\Omega)$. From the identity $\nabla \times (\zeta \boldsymbol{b}) = \nabla \zeta \times \boldsymbol{b} + \zeta \nabla \times \boldsymbol{b}$ and the properties of $\boldsymbol{b}$ we have

$$\boldsymbol{a} = \nabla \zeta \times \boldsymbol{b} \tag{XIII.7.13}$$

from which, and with the help of (XIII.7.10), it follows that

$$\boldsymbol{a}(x) = 0, \quad \text{for all } x \in N_\varepsilon. \tag{XIII.7.14}$$

Furthermore, we have

$$\int_S a_3 = \int_S (D_1(\zeta b_2) - D_2(\zeta b_1))$$

and so, by the divergence theorem, (XIII.7.2), and (XIII.7.10), we infer

$$\int_S a_3 = \int_{\partial S} \zeta (b_2 n_1 - b_1 n_2) = - \int_{\partial S} \boldsymbol{b} \times \boldsymbol{n} \cdot \boldsymbol{e}_3 = 1,$$

which proves property (iii). From (XIII.7.11) and (XIII.7.13) we find

$$\left.\begin{aligned} |\boldsymbol{a}(x)| &\le \frac{c_3}{\delta(x)|x'|} \\ |\nabla \boldsymbol{a}(x)| &\le c_4 \left(\frac{1}{\delta(x)|x'|^2} + \frac{1}{\delta^2(x)|x'|} \right) \end{aligned}\right\} \quad \text{all } x \in \operatorname{supp}(\boldsymbol{a}) \subset \operatorname{supp}(\nabla \zeta).$$

If we take $|x|$ sufficiently large, $|x| > R$ (say), from (XIII.7.6) and the definition of σ we have

$$\delta(x) \le c_5|x'| \le c_6\delta(x), \quad x \in \operatorname{supp}(\boldsymbol{a}).$$

Therefore, for all $x \in \operatorname{supp}(\boldsymbol{a})$ with $|x| > R$, we find

$$|\boldsymbol{a}(x)| \le \frac{c_3}{2}\left(\frac{1}{\delta^2(x)} + \frac{1}{|x'|^2}\right) = \frac{c_3}{2}\left(2 + \frac{|x'|^2}{\delta^2(x)} + \frac{\delta^2(x)}{|x'|^2}\right)\frac{1}{\delta^2(x) + |x'|^2}$$
$$\le \frac{c_7}{\delta^2(x) + |x'|^2}$$

and, likewise,

$$|\nabla \boldsymbol{a}(x)| \le \frac{c_8}{(\delta^2(x) + |x'|^2)^{3/2}}.$$

Observing that $\delta^2(x) \ge x_3^2$, property (i) follows from these latter inequalities. Using (i) and the fact that $\boldsymbol{a} \in C^\infty(\overline{\Omega}')$, for all bounded Ω' with $\overline{\Omega}' \subset \Omega$, we deduce, in particular, that

$$\int_\Omega |\nabla \boldsymbol{a}(x)|^2 < \infty.$$

Therefore,

$$\boldsymbol{a} \in D^{1,2}(\Omega)$$

and since $\boldsymbol{a}$ is solenoidal and by (XIII.7.14) it vanishes in a neighborhood of $\partial\Omega$, we may conclude, with the help of a standard "cut-off" argument, the validity of statement (ii) in the lemma. It remains to prove condition (iv). We begin to observe that, given $\eta > 0$, using integration by parts and the Schwarz inequality, for all $\boldsymbol{u} \in \mathcal{D}(\Omega)$,

$$\left|\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{a}\cdot\boldsymbol{u}\right| = \left|\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{a}\right| \le \frac{\eta}{2}\int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{u} + \frac{1}{2\eta}\int_\Omega u^2a^2$$

holds. As a consequence, to prove (iv), we will show that, given an arbitrary small $\lambda > 0$, we may select ε in such a way that

$$\int_\Omega u^2a^2 < \lambda\int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{u}. \tag{XIII.7.15}$$

To this end for a fixed $h > 0$, we split $\mathbb{R}^3_+$ into the following three regions (an analogous splitting holds for $\mathbb{R}^3_-$):

$$R_1 = \left\{x \in \mathbb{R}^3_+ : x_3 > h,\ x' \in S\right\}$$

$$R_2 = \left\{x \in \mathbb{R}^3_+ : x_3 < h,\ x' \in S\right\}$$

$$R_3 = \mathbb{R}^3_+ - \left\{\overline{R}_1 \cup \overline{R}_2\right\}$$

and denote by $\widetilde{R}_i$, $i=1,2,3$, the intersection of R_i with the support of $\boldsymbol{a}$. By virtue of $(\text{XIII.7.11})_1$, (XIII.7.13), and the definition of $\boldsymbol{b}$, we find

$$\int_{\widetilde{R}_i} u^2 a^2 \le c\varepsilon \int_{\widetilde{R}_i} \frac{u^2}{\delta^2(x)|x'|^2}, \quad i=1,2,3. \tag{XIII.7.16}$$

with c independent of ε. We wish to show the inequality

$$\int_{\widetilde{R}_i} \frac{u^2}{\delta^2(x)|x'|^2} \le c_1 \int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{u}, \quad i=1,2,3 \tag{XIII.7.17}$$

which, with the help of (XIII.7.16), implies

$$\int_{\mathbb{R}^3_+} u^2 a^2 \le 3cc_1\varepsilon \int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{u}.$$

Since an analogous reasoning holds for $\mathbb{R}^3_-$, we conclude the validity of (XIII.7.15) and complete the proof of the lemma. It remains to show the inequality (XIII.7.17). To this end, we notice that, for functions $\boldsymbol{u} \in \mathcal{D}(\Omega)$ the following estimate holds

$$\int_\Omega \frac{u^2}{x_3^2+1} \le c_2 \int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{u}, \tag{XIII.7.18}$$

with $c_2 = c_2(S)$; cf. Exercise XIII.7.1. In view of (XIII.7.7), we have

$$|x'| \ge d/2, \quad \text{for all } x \in \widetilde{R}_1. \tag{XIII.7.19}$$

Moreover, since

$$\delta(x) \ge x_3 \ge h \;\; \text{for all } x \in \widetilde{R}_1,$$

it follows that

$$\frac{1}{\delta^2(x)} \le \frac{c_3}{x_3^2+1}, \quad \text{for all } x \in \widetilde{R}_1 \tag{XIII.7.20}$$

with $c_3 = c_3(h)$. Thus, from (XIII.7.18)–(XIII.7.20) we infer that

$$\int_{\widetilde{R}_1} \frac{u^2}{\delta^2(x)|x'|^2} \le \frac{4c_2c_3}{d^2} \int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{u}. \tag{XIII.7.21}$$

Let us next consider the region $\widetilde{R}_3$. For all $x \in \widetilde{R}_3$,

$$\delta(x) = x_3$$

$$|x'| \ge d,$$

and so

$$\int_{\widetilde{R}_3} \frac{u^2}{\delta^2(x)|x'|^2} \le \frac{1}{d^2} \int_{\widetilde{R}_3} \frac{u^2}{x_3^2}.$$

However, using the elementary inequality[2]

$$\int_0^\infty \frac{f^2(t)}{t^2} \le 4 \int_0^\infty \left(\frac{df}{dt}\right)^2, \tag{XIII.7.22}$$

holding for (sufficiently smooth) functions vanishing in a neighborhood of zero and infinity, we at once obtain

$$\int_{\widetilde{R}_3} \frac{u^2}{x_3^2} \le 4 \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u}$$

and, as a consequence,

$$\int_{\widetilde{R}_3} \frac{u^2}{\delta^2(x)|x'|^2} \le \frac{4}{d^2} \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u}. \tag{XIII.7.23}$$

It remains to estimate the integral over the region $\widetilde{R}_2$. To this end, setting

$$\delta_1(y) = \operatorname{dist}(y, \partial S) \quad y \in S,$$

for all $x \in \widetilde{R}_2$ we find that

$$\delta^2(x) = x_3^2 + \delta_1^2(x')$$

$$|x'| \ge d/2$$

which, in turn furnishes

$$\int_{\widetilde{R}_2} \frac{u^2(x)}{\delta^2(x)|x'|^2} \le \frac{4}{d^2} \int_0^h dx_3 \int_S \frac{u^2(x', x_3)}{\delta_1^2(x')} dx'. \tag{XIII.7.24}$$

Since S is locally Lipschitz and $\boldsymbol{u} \in W_0^{1,2}(S)$, we may use Lemma III.6.3 to obtain

$$\int_S \frac{u^2(x', x_3)}{\delta_1^2(x')} dx' \le c \int_S \nabla \boldsymbol{u} : \nabla \boldsymbol{u} dx' \tag{XIII.7.25}$$

with $c = c(S)$. Integrating (XIII.7.25) over $x_3 \in [0, h)$ delivers

$$\int_0^h dx_3 \int_S \frac{u^2(x', x_3)}{\delta_1^2(x')} dx' \le c \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u}. \tag{XIII.7.26}$$

Using (XIII.7.26) in conjunction with (XIII.7.24) allows us to conclude

$$\int_{\widetilde{R}_2} \frac{u^2(x)}{\delta^2(x)|x'|^2} \le c_9 \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u}.$$

Estimate (XIII.7.17) then follows from this inequality, (XIII.7.21), and (XIII.7.23) and therefore, by what we observed, the proof of the lemma is complete. □

[2] For the (simple) demonstration of (XIII.7.22), see the proof of Lemma III.6.3.

Exercise XIII.7.1 Show inequality (XIII.7.18), for all $\boldsymbol{u} \in \mathcal{D}(\Omega)$. *Hint:* Establish (XIII.7.18) for Ω a half-space. Then use a "cut-off" argument together with inequality (II.5.18).

The result just shown permits us to obtain the following.

Theorem XIII.7.3 *Given arbitrary $\Phi \in \mathbb{R}$, problem* (XIII.5.2) *admits at least one corresponding generalized solution $\boldsymbol{v}$. This solution obeys the energy inequality:*

$$|\boldsymbol{v}|_{1,2}^2 \leq -\frac{p_* \Phi}{\nu}$$

where $p_* = p_+ - p_-$ and $p_\pm$ are the constants associated by Lemma XIII.5.1 *to the pressure field p corresponding to $\boldsymbol{v}$.*

Proof. We look for a solution of the form

$$\boldsymbol{v} = \boldsymbol{u} + \Phi \boldsymbol{a}$$

where $\boldsymbol{a} = \boldsymbol{a}(x;\eta)$ is the vector constructed in Lemma XIII.7.1 corresponding to a certain value of η that will be specified later, and $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ obeys

$$\begin{aligned} \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) - (\boldsymbol{u} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{u}) = &-\Phi(\boldsymbol{u} \cdot \nabla \boldsymbol{a}, \boldsymbol{\varphi}) + \Phi(\boldsymbol{a} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{u}) \\ &-\Phi^2(\boldsymbol{a} \cdot \nabla \boldsymbol{a}, \boldsymbol{\varphi}) - \nu \Phi(\nabla \boldsymbol{a}, \nabla \boldsymbol{\varphi}). \end{aligned} \tag{XIII.7.27}$$

It is clear that $\boldsymbol{v}$ satisfies all requirements of Definition XIII.5.1. As in analogous circumstances, a solution to (XIII.7.27) will be determined via the Galerkin method. Let $\{\boldsymbol{\varphi}_k\} \subset \mathcal{D}(\Omega)$ denote a sequence of functions whose linear hull is dense in $\mathcal{D}_0^{1,2}(\Omega)$ and satisfying the properties listed in Lemma VII.2.1. We set

$$\boldsymbol{u}_m = \sum_{k=1}^m \xi_{km} \boldsymbol{\varphi}_k$$

where the coefficients are required to satisfy the following system

$$\begin{aligned} \nu(\nabla \boldsymbol{u}_m, \nabla \boldsymbol{\varphi}_k) - (\boldsymbol{u}_m \cdot \nabla \boldsymbol{\varphi}_k, \boldsymbol{u}_m) = &-\Phi\big(\boldsymbol{u}_m \cdot \nabla \boldsymbol{a}, \boldsymbol{\varphi}_k\big) + \Phi\big(\boldsymbol{a} \cdot \nabla \boldsymbol{\varphi}_k, \boldsymbol{u}_m\big) \\ &-\Phi^2\big(\boldsymbol{a} \cdot \nabla \boldsymbol{a}, \boldsymbol{\varphi}_k\big) - \nu \Phi\big(\nabla \boldsymbol{a}, \boldsymbol{\varphi}_k\big) \end{aligned} \tag{XIII.7.28}$$

with $k = 1, \ldots, m$. Existence to the algebraic system (XIII.7.28) for each $m \in \mathbb{N}$ can be established exactly as in Theorem IX.3.1, provided we show a suitable bound for $|\boldsymbol{u}_m|_{1,2}$. To obtain such a bound, we multiply (XIII.7.28) by ξ_{km} and sum over the index k from one to m. Observing that

$$(\boldsymbol{u}_m \cdot \nabla \boldsymbol{u}_m, \boldsymbol{u}_m) = (\boldsymbol{a} \cdot \nabla \boldsymbol{u}_m, \boldsymbol{u}_m) = 0, \tag{XIII.7.29}$$

we find

$$\nu |\boldsymbol{u}_m|_{1,2}^2 = -\Phi(\boldsymbol{u}_m \cdot \nabla \boldsymbol{a}, \boldsymbol{u}_m) - \Phi^2(\boldsymbol{a} \cdot \nabla \boldsymbol{a}, \boldsymbol{u}_m) - \nu \Phi(\nabla \boldsymbol{a}, \nabla \boldsymbol{u}_m). \tag{XIII.7.30}$$

Using the Schwarz and Cauchy inequalities, we deduce for all $\eta > 0$

$$\begin{aligned} \Phi^2(\boldsymbol{a}\cdot\nabla\boldsymbol{a},\boldsymbol{u}_m) &= -\Phi^2(\boldsymbol{a}\cdot\nabla\boldsymbol{u}_m,\boldsymbol{a}) \le \tfrac{\Phi^4}{2\eta}\|\boldsymbol{a}\|_4^4 + \tfrac{\eta}{2}|\boldsymbol{u}_m|_{1,2}^2 \\ -\nu\Phi(\nabla\boldsymbol{a},\nabla\boldsymbol{u}_m) &\le \tfrac{\nu^2\Phi^2}{2\eta}|\boldsymbol{a}|_{1,2}^2 + \tfrac{\eta}{2}|\boldsymbol{u}_m|_{1,2}^2. \end{aligned} \tag{XIII.7.31}$$

Moreover, by Lemma XIII.7.1,

$$-\Phi(\boldsymbol{u}_m\cdot\nabla\boldsymbol{a},\boldsymbol{u}_m) \le \eta|\Phi||\boldsymbol{u}_m|_{1,2}^2. \tag{XIII.7.32}$$

Thus, choosing

$$\eta < \frac{\nu}{(1+|\Phi|)},$$

and recalling that $\boldsymbol{a} \in L^4(\Omega)\cap D^{1,2}(\Omega)$, we may use (XIII.7.29)–(XIII.7.32) and Lemma IX.3.2 to show existence to (XIII.7.28) for all $m \in \mathbb{N}$. Moreover, we also find that

$$|\boldsymbol{u}_m|_{1,2}^2 \le c_1(\Phi,\nu). \tag{XIII.7.33}$$

Employing (XIII.7.33) together with the weak compactness of the spaces $\mathcal{D}_0^{1,2}(\Omega)$ we may select from $\{\boldsymbol{u}_m\}$ a sequence, denoted again by $\{\boldsymbol{u}_m\}$, and a vector field $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ such that

$$\begin{aligned} &\boldsymbol{u}_m \xrightarrow{w} \boldsymbol{u} \ \text{ in } \mathcal{D}_0^{1,2}(\Omega) \\ &\boldsymbol{u}_m \to \boldsymbol{u} \ \text{ in } L^2(\Omega'), \end{aligned} \tag{XIII.7.34}$$

for any bounded domain $\Omega' \subset \Omega$. By these convergence properties and taking advantage of the properties of the functions $\{\boldsymbol{\varphi}_k\}$, we use a by-now-standard procedure to show that $\boldsymbol{u}$ solves (XIII.7.28) and that, as a consequence, $\boldsymbol{v}$ is a weak solution to (XIII.5.2). It remains to show that $\boldsymbol{v}$ satisfies the energy inequality. For this purpose, setting

$$\boldsymbol{v}_m = \boldsymbol{u}_m + \Phi\boldsymbol{a},$$

from (XIII.7.30), after a simple manipulation we find

$$\nu|\boldsymbol{v}_m|_{1,2}^2 = -\Phi(\boldsymbol{v}_m\cdot\nabla\boldsymbol{a},\boldsymbol{v}_m) + \nu\Phi(\nabla\boldsymbol{a},\nabla\boldsymbol{v}_m) \tag{XIII.7.35}$$

where use has been made of the property

$$(\boldsymbol{v}_m\cdot\nabla\boldsymbol{a},\boldsymbol{a}) = 0.$$

With the help of the convergence conditions (XIII.7.34), it is easily shown that

$$\lim_{m\to\infty}(\boldsymbol{v}_m\cdot\nabla\boldsymbol{a},\boldsymbol{v}_m) = (\boldsymbol{v}\cdot\nabla\boldsymbol{a},\boldsymbol{v}). \tag{XIII.7.36}$$

Actually, we have

$$\begin{aligned} |(\boldsymbol{v}_m\cdot\nabla\boldsymbol{a},\boldsymbol{v}_m) - (\boldsymbol{v}\cdot\nabla\boldsymbol{a},\boldsymbol{v})| &\le |((\boldsymbol{v}_m-\boldsymbol{v})\cdot\nabla\boldsymbol{a},\boldsymbol{v}_m)| \\ &\quad + |(\boldsymbol{v}\cdot\nabla\boldsymbol{a},(\boldsymbol{v}_m-\boldsymbol{v}))| \\ &\equiv I_1(m) + I_2(m). \end{aligned}$$

Recalling that for R sufficiently large

$$|\nabla \boldsymbol{a}(x)| \le c|x|^{-3}, \quad |x| > R, \tag{XIII.7.37}$$

we find

$$\begin{aligned} I_1(m) &\le \left(\int_\Omega |\nabla \boldsymbol{a}|^{4/3}|\boldsymbol{v}_m - \boldsymbol{v}|^2\right)^{1/2}\left(\int_\Omega |\nabla \boldsymbol{a}|^{2/3} v_m^2\right)^{1/2} \\ &\le c_1 \left(\int_\Omega \frac{v_m^2}{|x|^2}\right)^{1/2}\left(\int_\Omega |\nabla \boldsymbol{a}|^{4/3}|\boldsymbol{v}_m - \boldsymbol{v}|^2\right)^{1/2}. \end{aligned}$$

Thus, from this inequality, (II.6.10), and (XIII.7.33) it follows that

$$I_1(m) \le c_2 \left(\int_\Omega |\nabla \boldsymbol{a}|^{4/3}|\boldsymbol{v}_m - \boldsymbol{v}|^2\right)^{1/2} \tag{XIII.7.38}$$

with c_2 independent of m. Setting $\Omega_R = \Omega \cap B_R$, $\Omega^R = \Omega - \overline{\Omega}_R$, from (XIII.7.37) we have

$$\begin{aligned} \int_\Omega |\nabla \boldsymbol{a}|^{4/3}|\boldsymbol{v}_m - \boldsymbol{v}|^2 &\le \int_{\Omega_R} |\nabla \boldsymbol{a}|^{4/3}|\boldsymbol{v}_m - \boldsymbol{v}|^2 + \int_{\Omega^R} |\nabla \boldsymbol{a}|^{4/3}|\boldsymbol{v}_m - \boldsymbol{v}|^2 \\ &\le c_3 \int_{\Omega_R} |\boldsymbol{v}_m - \boldsymbol{v}|^2 + \frac{c^{4/3}}{R^2}\int_{\Omega^R} \frac{|\boldsymbol{v}_m - \boldsymbol{v}|^2}{|x|^2} \end{aligned}$$

and so, again by (II.6.10) and (XIII.7.33), we find

$$I_1(m) \le c_4 \int_{\Omega_R} |\boldsymbol{v}_m - \boldsymbol{v}|^2 + \frac{c_5}{R^2}$$

with c_4 and c_5 independent of m. Taking the lim sup as $m \to \infty$ at both sides of this latter relation and bearing in mind $(\text{XIII.7.34})_2$, by the arbitrarity of R we conclude that

$$\limsup_{m\to\infty} I_1(m) = 0,$$

that is,

$$\lim_{m\to\infty} I_1(m) = 0.$$

Likewise, we show

$$\lim_{m\to\infty} I_2(m) = 0$$

and (XIII.7.36) is completely established. Let us now take the lim inf as $m \to \infty$ of both sides of (XIII.7.35). Employing (XIII.7.36), $(\text{XIII.7.34})_1$, and Theorem II.2.4 we obtain

$$\nu|\boldsymbol{v}|_{1,2}^2 \le -\Phi(\boldsymbol{v}\cdot\nabla \boldsymbol{a}, \boldsymbol{v}) + \nu\Phi(\nabla \boldsymbol{a}, \nabla \boldsymbol{v}). \tag{XIII.7.39}$$

We wish to put the right-hand side of (XIII.7.39) into a different form. To this end, we recall that $\boldsymbol{v}$ obeys the identity (XIII.5.5). Denoting by ψ_R the "cut-off" function of Theorem XIII.6.1, we choose, as test function $\boldsymbol{\psi}$ into (XIII.5.5) the function $\Phi\psi_R\boldsymbol{a}$. We thus obtain

$$\begin{aligned}\Phi\nu(\psi_R\nabla\boldsymbol{v},\nabla\boldsymbol{a})-\Phi(\psi_R\boldsymbol{v}\cdot\nabla\boldsymbol{a},\boldsymbol{v}) &= -\Phi(p,\nabla\cdot(\psi_R\boldsymbol{a}))\\ &\quad -\Phi\nu(\nabla\boldsymbol{v},\boldsymbol{a}\otimes\nabla\psi_R)+\Phi(\boldsymbol{v}\cdot\boldsymbol{a}\otimes\nabla\psi_R,\boldsymbol{v}).\end{aligned} \tag{XIII.7.40}$$

Using the properties of the function $\boldsymbol{a}$ and recalling that $|\nabla\psi_R|\le MR^{-1}$ with M independent of R, we readily establish the following relations:

$$\begin{aligned}&\lim_{R\to\infty}(\psi_R\nabla\boldsymbol{v},\nabla\boldsymbol{a})=(\nabla\boldsymbol{v},\nabla\boldsymbol{a})\\ &\lim_{R\to\infty}(\psi_R\boldsymbol{v}\cdot\nabla\boldsymbol{a},\boldsymbol{v})=(\boldsymbol{v}\cdot\nabla\boldsymbol{a},\boldsymbol{v})\\ &\lim_{R\to\infty}(\nabla\boldsymbol{v},\boldsymbol{a}\otimes\nabla\psi_R)=\lim_{R\to\infty}(\boldsymbol{v}\cdot\boldsymbol{a}\otimes\nabla\psi_R,\boldsymbol{v})=0.\end{aligned} \tag{XIII.7.41}$$

Furthermore, reasoning exactly as in the proof of (XIII.6.4), we show

$$\lim_{R\to\infty}(p,\nabla\cdot(\psi_R\boldsymbol{a}))=-(p_+-p_-) \tag{XIII.7.42}$$

with $p_\pm$ constants defined in Lemma XIII.5.1. Collecting (XIII.7.40)–(XIII.7.42) we find

$$-\Phi(\boldsymbol{v}\cdot\nabla\boldsymbol{a},\boldsymbol{v})+\Phi(\boldsymbol{v}\cdot\nabla\boldsymbol{a},\boldsymbol{v})=-p_*\Phi,$$

which, in view of (XIII.7.39), proves that $\boldsymbol{v}$ obeys the energy inequality. The theorem is completely proved. □

Remark XIII.7.2 Existence for arbitrary value of the flux can be established for a wide class of domains with outlets Ω_i whose cross sections $\Sigma_i(x_n)$ become unbounded for large values of x_n. This class certainly includes domains for which Ω_i are bodies of rotation of the type considered in the linear case in Theorem VI.3.1. The proof of this result, which patterns that just given in Theorem XIII.7.3, can be found in Solonnikov & Pileckas (1977, Theorem 8); cf. also Ladyzhenskaya & Solonnikov (1980, Theorem 4.1). For existence in domains of more general types, we refer the reader to the work of Solonnikov (1983, §3.3). ■

Remark XIII.7.3 An existence result of the same type as Theorem XIII.7.3 can be proved also in dimension two; see Galdi, Padula, & Passerini (1995) and Remark XIII.5.3. ■

The remaining part of this section is devoted to analyze the uniqueness of generalized solutions constructed in Theorem XIII.7.3. To reach this goal, we need an intermediate result that ensures the existence of generalized solutions to (XIII.5.2) enjoying the additional property of being members of $L^3(\Omega)$. This result will be obtained under the assumption that the magnitude of the flux Φ is suitably restricted.

Lemma XIII.7.2 *Let c be the constant entering the estimate given in* Theorem VI.5.1. *Then, if*

$$|\Phi| < \frac{3\nu}{16c^2},$$

problem (XIII.5.2) *has at least one generalized solution $\boldsymbol{v}$ that satisfies*

$$\boldsymbol{v} \in \widehat{\mathcal{D}}_0^{1,3/2}(\Omega) \cap L^3(\Omega).$$

Moreover, the following estimate holds:

$$\|\boldsymbol{v}\|_3 \leq \frac{2c}{\sqrt{3}}|\Phi|. \tag{XIII.7.43}$$

Finally, denoting by p the pressure field corresponding to $\boldsymbol{v}$ according to Lemma XIII.5.1, *and by $p_\pm$ the constants associated to p by the same lemma, we have*

$$p - p_\pm \in L^2(\mathbb{R}^3_\pm) \cap L^{3/2}(\mathbb{R}^3_\pm). \tag{XIII.7.44}$$

Proof. Let

$$X = \widehat{\mathcal{D}}_0^{1,2}(\Omega) \cap \widehat{\mathcal{D}}_0^{1,3/2}(\Omega)$$
$$\|\cdot\|_X = |\cdot|_{1,2} + |\cdot|_{1,3/2},$$

and set

$$X_\delta = \{\boldsymbol{w} \in X : \|\cdot\|_X \leq \delta\}.$$

Clearly, X endowed with the norm $\|\cdot\|_X$ is a Banach space and X_δ is a closed subset of X. Consider the mapping

$$L : \boldsymbol{w} \in X_\delta \to L(\boldsymbol{w}) = \boldsymbol{v} \in X$$

where $\boldsymbol{v}$ solves

$$\begin{gathered}\nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) = -(\boldsymbol{w} \cdot \nabla \boldsymbol{w}, \boldsymbol{\varphi}), \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega) \\ \int_S v_3 = \Phi. \end{gathered} \tag{XIII.7.45}$$

The map L is well defined. Actually, by the Sobolev inequality (II.3.7),

$$|\boldsymbol{w} \cdot \nabla \boldsymbol{w}|_{-1,3/2} \leq \|\boldsymbol{w}\|_3^2 \leq \frac{1}{3}|\boldsymbol{w}|_{1,3/2}^2. \tag{XIII.7.46}$$

Moreover, by the interpolation inequality (II.2.7) and again by (II.3.7), for some $\theta \in (0,1)$,

$$|\boldsymbol{w}\cdot \nabla \boldsymbol{w}|_{-1,2} \leq \|\boldsymbol{w}\|_4^2 \leq \|\boldsymbol{w}\|_3^{2\theta}\|\boldsymbol{w}\|_6^{2(1-\theta)} \leq \frac{2^{2(1-\theta)}}{3}|\boldsymbol{w}|_{1,3/2}|\boldsymbol{w}|_{1,2}. \tag{XIII.7.47}$$

Thus,

$$\boldsymbol{w} \cdot \nabla \boldsymbol{w} \in \widehat{\mathcal{D}}_0^{-1,2}(\Omega) \cap \widehat{\mathcal{D}}_0^{-1,3/2}(\Omega),$$

and from Theorem VI.5.1 we infer that the solution $\boldsymbol{v}$ to (XIII.7.45) is in the space X, which proves that L is well defined. By the estimate given in Theorem VI.5.1, (XIII.7.46), and (XIII.7.47), we obtain

$$\|\boldsymbol{v}\|_X \le c\left(|\Phi| + \frac{4}{3\nu}\|\boldsymbol{w}\|_X^2\right). \tag{XIII.7.48}$$

Using this inequality, it is easy to show that, for suitably restricted $|\Phi|$ the map L transforms X_δ into itself, with δ appropriately chosen. In fact, for $\boldsymbol{w} \in X_\delta$, (XIII.7.48) delivers

$$\|\boldsymbol{v}\|_X \le \left(|\Phi| + \frac{4}{3\nu}\delta^2\right)$$

and so, choosing

$$\delta = 2c|\Phi|, \tag{XIII.7.49}$$

and recalling the assumption of the lemma, we obtain

$$\|\boldsymbol{v}\|_X \le \delta\left(\frac{1}{2} + \frac{1}{2}\right) = \delta, \tag{XIII.7.50}$$

thus proving the desired property of L. Furthermore, for all $\boldsymbol{w}_1$, $\boldsymbol{w}_2 \in X_\delta$, in virtue of Theorem VI.5.1, (XIII.7.46), and (XIII.7.47) it easily follows that

$$\begin{aligned}\|L(\boldsymbol{w}_1) - L(\boldsymbol{w}_2)\|_X &\le \frac{2c}{\nu}\left(\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_3\|\boldsymbol{w}_1\|_3 + \|\boldsymbol{w}_2 - \boldsymbol{w}_1\|_4\|\boldsymbol{w}\|_4\right)\\ &\le \frac{8c\delta}{3\nu}\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_X.\end{aligned}$$

Since, by assumption,

$$\frac{8c}{3\nu}\delta = \frac{16c^2|\Phi|}{3\nu} < 1,$$

we conclude that L is a contraction operator on X_δ and, therefore, it admits a fixed point in X_δ. The existence of a solution $\boldsymbol{v} \in X$ to problem (XIII.5.2) is thus established. Moreover, the Sobolev inequality (II.3.7), together with (XIII.7.49) and (XIII.7.50), implies

$$\|\boldsymbol{v}\|_3 \le \frac{1}{\sqrt{3}}|\boldsymbol{v}|_{1,3/2} \le \frac{2c}{\sqrt{3}}|\Phi|,$$

proving (XIII.7.43). Finally, the summability properties of the pressure field are at once established from Lemma XIII.5.1 and Theorem VI.5.1. The lemma is then completely proved. □

Combining Lemma XIII.7.1 with Theorem XIII.6.2 we obtain the following uniqueness result.

Theorem XIII.7.4 *Assume that Φ verifies the condition*

$$|\Phi| < m\nu,$$

with $m = \min\{3/4c, 3/16c^2\}$ *and where c is the constant introduced in* Lemma XIII.7.1. *Then the corresponding generalized solution* $\boldsymbol{v}$ *constructed in* Theorem XIII.7.3 *is unique in the class of generalized solutions* $\boldsymbol{w}$ *satisfying the energy inequality* (XIII.6.6).

Other simple but interesting consequences of Lemma XIII.7.1 and Theorem XIII.6.2, in light of Theorem XIII.7.4, are given in the following corollary.

Corollary XIII.7.1 *Let the assumption of* Theorem XIII.7.4 *be satisfied. Then every generalized solution* $\boldsymbol{v}$ *corresponding to* Φ *and verifying the inequality* (XIII.6.6) *satisfies the following summability conditions*

$$\boldsymbol{v} \in L^3(\Omega) \cap \widehat{\mathcal{D}}_0^{1,3/2}(\Omega).$$

In particular, $\boldsymbol{v}$ *satisfies the energy equation.*

XIII.8 Global Summability of Generalized Solutions for Flow in an Aperture Domain

The remaining part of this chapter is devoted to the investigation of the asymptotic structure of generalized solutions. As in the case of flows in exterior regions, this study will be performed in two different steps. In the first, we determine general summability properties of weak solutions of the type constructed in Theorem XIII.7.3. Successively, using these conditions, we will furnish a complete representation of the solution at large spatial distances. However, as we proved in Theorem XIII.7.4, in order for a weak solution to verify the conditions stated in Theorem XIII.7.3, a small value of the flux Φ is needed. As a consequence, the asymptotic structure of generalized solutions corresponding to arbitrary values of Φ remains open.

In this section we shall determine the summability properties of weak solutions corresponding to fluxes of suitably restricted size. Such a result will be achieved as a corollary to a more general one, which we are going to derive.

Notation. Unless the contrary is explicitly stated, in the sequel we shall denote by Ω the half-space $\mathbb{R}^3_+$.

Let $\varphi \in C^1(\mathbb{R})$ be a nonincreasing nonnegative function with $\varphi(t) = 1$ when $t \leq 1$ and $\varphi(t) = 0$ when $t \geq 2$. For $a > 0$ set

$$\varphi_a(x) = \varphi\left(\frac{|x|}{a}\right). \tag{XIII.8.1}$$

Clearly, the support of $\nabla\varphi_a$ is contained in the domain

$$S_a = \left\{x \in \mathbb{R}^3 : \ a < |x| < 2a\right\}$$

and

$$|\nabla\varphi_a(x)| \le \frac{c}{a}, \quad x \in \mathbb{R}^3, \tag{XIII.8.2}$$

with c (> 0) independent of a and x. We put

$$\Omega_{(a)} = \mathbb{R}^3_+ \cap S_a\,, \quad \Omega_a = \mathbb{R}^3_+ \cap B_a\,. \tag{XIII.8.3}$$

The following result holds.

Lemma XIII.8.1 *Let $\boldsymbol{v} \in L^6_{loc}(\overline{\Omega}) \cap L^3(\Omega)$ with $\nabla\cdot\boldsymbol{v} = 0$.*[1] *Then there is a sequence $\{\boldsymbol{u}_k\}$ of solenoidal*[2] *functions in Ω such that*

$$\begin{gathered}\boldsymbol{u}_k \in L^s(\Omega) \ \ \text{for all } s \in (3/2, 6], \text{ for all } k \in \mathbb{N}\\ \lim_{k\to\infty} \|\boldsymbol{u}_k - \boldsymbol{v}\|_3 = 0.\end{gathered} \tag{XIII.8.4}$$

In addition to (XIII.8.4), *given $\varepsilon > 0$, the sequence $\{\boldsymbol{u}_k\}$ can be chosen such that*

$$\begin{gathered}\boldsymbol{u}_k = \boldsymbol{u}_k^{(1)} + \boldsymbol{u}_k^{(2)}\\ \|\boldsymbol{u}_k^{(1)}\|_3 < \varepsilon\\ \operatorname{supp}(\boldsymbol{u}_k^{(2)}) \subset \overline{\Omega}_\rho\\ \|\boldsymbol{u}_k^{(2)}\|_{q,\Omega_\rho} \le \|\boldsymbol{v}\|_{q,\Omega_\rho}, \quad \text{for all } q \in (1,6),\end{gathered} \tag{XIII.8.5}$$

where the number ρ depends only on ε and $\boldsymbol{v}$.

Proof. Consider the problem:

$$\begin{gathered}\nabla\cdot\boldsymbol{z}_k = -\nabla\cdot(\varphi_k\boldsymbol{v}) = -\nabla\varphi_k\cdot\boldsymbol{v}\\ \boldsymbol{z}_k \in D^{1,r}_0(\Omega)\\ |\boldsymbol{z}_k|_{1,r} \le c\|\nabla\varphi_k\cdot\boldsymbol{v}\|_r.\end{gathered} \tag{XIII.8.6}$$

From Corollary V.3.1 and the assumption on $\boldsymbol{v}$ we know that there is a solution to problem (XIII.8.6) for any $r \in (1,6]$. From the Sobolev inequality (II.3.7) we have (see (XIII.8.3)$_1$)

$$\|\boldsymbol{z}_k\|_3 \le c|\boldsymbol{z}_k|_{1,3/2} \le c\|\nabla\varphi_k\|_{3,\Omega_{(k)}}\|\boldsymbol{v}\|_{3,\Omega_{(k)}}$$

and so, in view of (XIII.8.2),

$$\|\boldsymbol{z}_k\|_3 \le c_1\|\boldsymbol{v}\|_{3,\Omega_{(k)}} \tag{XIII.8.7}$$

[1] In the generalized sense.
[2] In the generalized sense.

with c_1 independent of k. Moreover, since $\nabla\varphi_k \cdot \boldsymbol{v} \in L^r(\Omega)$, $1 < r < 3$, for all $k \in \mathbb{N}$, again by the Sobolev inequality we deduce that

$$\boldsymbol{z}_k \in L^{3r/(3-r)}(\Omega), \quad 1 < r < 3,$$

that is,

$$\boldsymbol{z}_k \in L^s(\Omega), \quad \text{for all } s \in (3/2, \infty). \tag{XIII.8.8}$$

Given $\varepsilon > 0$, we choose $\rho = \rho(\varepsilon, \boldsymbol{v})$ such that

$$\|(1-\varphi_\rho)\boldsymbol{v}\|_3 < \varepsilon/2 \tag{XIII.8.9}$$

and set, for all sufficiently large k,

$$\begin{aligned} \boldsymbol{u}_k &= \varphi_k \boldsymbol{v} + \boldsymbol{z}_k \\ \boldsymbol{u}_k^{(1)} &= \varphi_k(1-\varphi_\rho)\boldsymbol{v} + \boldsymbol{z}_k \\ \boldsymbol{u}_k^{(2)} &= \boldsymbol{\varphi}_\rho \boldsymbol{v}. \end{aligned}$$

Since for k large enough

$$\varphi_k\varphi_\rho = \varphi_\rho,$$

it follows that

$$\boldsymbol{u}_k = \boldsymbol{u}_k^{(1)} + \boldsymbol{u}_k^{(2)}.$$

The field $\boldsymbol{u}_k$ is solenoidal for all $k \in \mathbb{N}$. Furthermore, by (XIII.8.8) and assumption, we deduce $(\text{XIII.8.4})_1$. By (XIII.8.7) and the properties of φ_k we also have

$$\|\boldsymbol{u}_k - \boldsymbol{v}\|_3 \leq \|(1-\varphi_k)\boldsymbol{v}\|_3 + \|\boldsymbol{z}_k\|_3 \to 0 \text{ as } k \to \infty$$

and we recover $(\text{XIII.8.4})_2$. Statements $(\text{XIII.8.5})_3$ and $(\text{XIII.8.5})_4$ are evident. Finally, by (XIII.8.7) and (XIII.8.9) for k large enough, it follows that

$$\|\boldsymbol{u}_k^{(1)}\|_3 \leq \|(1-\varphi_\rho)\boldsymbol{v}\|_3 + c_1\|\boldsymbol{v}\|_{3,\Omega_{(k)}} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The lemma is proved. □

Lemma XIII.8.2 *Let $\boldsymbol{v}$, p be a weak solution to the problem*

$$\left.\begin{aligned} \nu\Delta\boldsymbol{v} &= \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p \\ \nabla\cdot\boldsymbol{v} &= 0 \end{aligned}\right\} \quad \text{in } \Omega$$

$$\boldsymbol{v} = \boldsymbol{v}_* \quad \text{at } \partial\Omega,$$

where, for all $s \in (1, 3/2]$,

$$\boldsymbol{v}_* \in L^s(\partial\Omega)$$

and

$$\langle\langle \boldsymbol{v}_* \rangle\rangle_{1-1/s,s}, \quad \langle\langle \boldsymbol{v}_* \rangle\rangle_{1/2,2}$$

are finite.[3] *Then if*

$$\boldsymbol{v} \in L^s(\Omega)$$

it follows that

$$\boldsymbol{v} \in D^{1,q}(\Omega) \cap L^{3q/(3-q)}(\Omega), \quad \textit{for all } q \in (1,2].$$

Proof. For simplicity, we shall put $\nu = 1$. Consider the following sequence of problems

$$\left.\begin{aligned} \Delta \boldsymbol{w} &= \boldsymbol{u}_k \cdot \nabla \boldsymbol{w} + \nabla \pi \\ \nabla \cdot \boldsymbol{w} &= 0 \end{aligned}\right\} \text{ in } \Omega \qquad\qquad \text{(XIII.8.10)}$$
$$\boldsymbol{w} = \boldsymbol{v}_* \text{ at } \partial\Omega,$$

where $\{\boldsymbol{u}_k\}$ is the sequence of functions associated to $\boldsymbol{v}$ by Lemma XIII.8.1 and corresponding to a certain value of ε, which will be specified later.[4] It is simple to show for problem (XIII.8.10) the existence of a weak solution $\boldsymbol{w}_k$, for each $k \in \mathbb{N}$. Actually, we may extend the field $\boldsymbol{v}_*$ to some $\mathbf{U} \in D^{1,2}(\Omega)$; see Section II.10. $\boldsymbol{U}$ need not be solenoidal, but we may add to it a field $\boldsymbol{u}$ such that

$$\nabla \cdot \boldsymbol{u} = -\nabla \cdot \mathbf{U}$$

$$\boldsymbol{u} \in D_0^{1,2}(\Omega)$$

$$|\boldsymbol{u}|_{1,2} \leq c\|\nabla \cdot \mathbf{U}\|_2.$$

The field $\boldsymbol{u}$ exists by virtue of Corollary V.3.1. Thus,

$$\boldsymbol{V} = \boldsymbol{u} + \boldsymbol{U}$$

is a solenoidal extension of $\boldsymbol{v}_*$ satisfying, by the trace Theorem II.10.2 and the property of $\boldsymbol{u}$, the bound

$$|\boldsymbol{V}|_{1,2} \leq c\,\langle\langle \boldsymbol{v}_* \rangle\rangle_{1/2,2}. \qquad \text{(XIII.8.11)}$$

We look for a solution to (XIII.8.10) of the form $\boldsymbol{w} = \boldsymbol{z} + \boldsymbol{V}$,[5] where

$$\left.\begin{aligned} \Delta \boldsymbol{z} &= \boldsymbol{u}_k \cdot \nabla \boldsymbol{z} + \boldsymbol{u}_k \cdot \nabla \boldsymbol{V} - \Delta \boldsymbol{V} + \nabla \pi \\ \nabla \cdot \boldsymbol{z} &= 0 \end{aligned}\right\} \text{ in } \Omega \qquad\qquad \text{(XIII.8.12)}$$
$$\boldsymbol{z} = 0 \text{ at } \partial\Omega,$$

[3] These are the trace norms at the boundary of functions defined in a half-space; see Section II.10.

[4] Notice that, since $\boldsymbol{v} \in D^{1,2}(\Omega)$, by the embedding Theorem II.3.4, it follows that $\boldsymbol{v} \in L^6_{loc}(\Omega)$ so that the assumptions of Lemma XIII.8.1 are satisfied.

[5] For simplicity, we shall drop the subscript k in $\boldsymbol{w}_k$ and $\boldsymbol{z}_k$.

Multiplying $(\text{XIII.8.12})_1$ by $\boldsymbol{z}$, integrating by parts over Ω, and using $(\text{XIII.8.12})_1$ we formally obtain

$$-|\boldsymbol{z}|_{1,2}^2 = (\boldsymbol{u}_k \cdot \boldsymbol{V}, \boldsymbol{z}) + (\nabla \boldsymbol{V}, \nabla \boldsymbol{z}). \tag{XIII.8.13}$$

Using the Hölder inequality in the terms at the right-hand side of this relation, together with the Sobolev inequality (II.3.7), we deduce

$$|(\boldsymbol{u}_k \cdot \nabla \boldsymbol{V}, \boldsymbol{z})| \le \|\boldsymbol{u}_k\|_3 |\boldsymbol{V}|_{1,2} \|\boldsymbol{z}\|_6 \le \gamma \|\boldsymbol{u}_k\|_3 |\boldsymbol{V}|_{1,2} |\boldsymbol{z}|_{1,2}$$

$$|(\nabla \boldsymbol{V}, \nabla \boldsymbol{z})| \le |\boldsymbol{V}|_{1,2} |\boldsymbol{z}|_{1,2}.$$

These inequalities along with (XIII.8.11) and (XIII.8.13), yield

$$|\boldsymbol{z}|_{1,2} \le c_1 \langle\langle \boldsymbol{v}_* \rangle\rangle_{1/2,2},$$

and recalling that $\boldsymbol{w} = \boldsymbol{z} + \boldsymbol{V}$, we also have

$$|\boldsymbol{w}|_{1,2} \le c_2 \langle\langle \boldsymbol{v}_* \rangle\rangle_{1/2,2}. \tag{XIII.8.14}$$

The above bound on $\boldsymbol{z}$ allows us to determine, for all $k \in \mathbb{N}$, a generalized solution to (XIII.8.10) satisfying (XIII.8.14). Since

$$\boldsymbol{u}_k \in L^r(\Omega) \ \text{ for all } r \in (3/2, 6],$$

and $\nabla \cdot \boldsymbol{u}_k = 0$, by inequality (II.6.22) we find for all $q \in (6/5, 3/2]$

$$|\boldsymbol{u}_k \cdot \nabla \boldsymbol{w}|_{-1,q} \le \|\boldsymbol{u}_k \boldsymbol{w}\|_q \le \|\boldsymbol{u}_k\|_{6q/(6-q)} \|\boldsymbol{w}\|_6 \le c_1 \|\boldsymbol{u}_k\|_{6q/(6-q)} |\boldsymbol{w}|_{1,2} < \infty.$$

Thus, we may assert

$$\boldsymbol{F} \equiv \boldsymbol{u}_k \cdot \nabla \boldsymbol{w} \in D_0^{-1,q}(\Omega) \ \text{ for all } q \in (6/5, 3/2]. \tag{XIII.8.15}$$

Using (XIII.8.15) and Theorem V.3.3, it follows that

$$\boldsymbol{w} \in D^{1,q}(\Omega) \cap L^{3q/(3-q)}(\Omega) \ \text{ for all } q \in (6/5, 3/2]. \tag{XIII.8.16}$$

In view of (XIII.8.16), we may give a different estimate for $\boldsymbol{F}$. Specifically, by the Hölder inequality and $(\text{XIII.8.5})_{2,4}$ we have for all $q \in (6/5, 3/2]$

$$\begin{aligned} |\boldsymbol{F}|_{-1,q} \equiv |\boldsymbol{u}_k \cdot \nabla \boldsymbol{w}|_{-1,q} &\le \|\boldsymbol{u}_k^{(1)}\|_3 \|\boldsymbol{w}\|_{3q/(3-q)} + \|\boldsymbol{v}\|_{4,\Omega_\rho} \|\boldsymbol{w}\|_{4q/(4-q),\Omega_\rho} \\ &\le \varepsilon \|\boldsymbol{w}\|_{3q/(3-q)} + \|\boldsymbol{v}\|_{4,\Omega_\rho} \|\boldsymbol{w}\|_{4q/(4-q),\Omega_\rho}. \end{aligned} \tag{XIII.8.17}$$

From (XIII.8.17) and estimate (V.3.28) we deduce

$$\begin{aligned} (1 - \varepsilon c) \|\boldsymbol{w}\|_{3q/(3-q)} + |\boldsymbol{w}|_{1,q} + \|\pi\|_q \le c \big(&\langle\langle \boldsymbol{v}_* \rangle\rangle_{1-1/q,q} \\ &+ \|\boldsymbol{v}\|_{4,\Omega_\rho} \|\boldsymbol{w}\|_{4q/(4-q),\Omega_\rho} \big), \end{aligned} \tag{XIII.8.18}$$

where c denotes the constant entering (V.3.28). Thus, choosing $\varepsilon < 1/c$, from (XIII.8.14) and (XIII.8.18) we deduce, for all $q \in (6/5, 3/2]$, that

$$
\begin{aligned}
|\boldsymbol{w}|_{1,2} + \|\boldsymbol{w}\|_{3q/(3-q)} + |\boldsymbol{w}|_{1,q} + \|\pi\|_q \\
\le c_2 \big(\langle\langle \boldsymbol{v}_* \rangle\rangle_{1/2,2} + \langle\langle \boldsymbol{v}_* \rangle\rangle_{1-1/q,q} \\
+ \|\boldsymbol{w}\|_{4q/(4-q),\Omega_\rho}\big) .
\end{aligned}
\tag{XIII.8.19}
$$

Our next objective is to show the existence of a constant $c_3 = c_3(\rho, q, \boldsymbol{v})$ independent of k such that

$$
\|\boldsymbol{w}\|_{4q/(4-q),\Omega_\rho} \le c_3 \big(\langle\langle \boldsymbol{v}_* \rangle\rangle_{1/2,2} + \langle\langle \boldsymbol{v}_* \rangle\rangle_{1-1/q,q}\big) . \tag{XIII.8.20}
$$

Contradicting (XIII.8.20) means that there are sequences

$$
\{\boldsymbol{u}_m\}, \quad \{\boldsymbol{v}_{*m}\}
$$

such that if $\{\boldsymbol{w}_m, \pi_m\}$ are weak solutions to the problem

$$
\left.
\begin{aligned}
\Delta \boldsymbol{w}_m &= \boldsymbol{u}_m \cdot \nabla \boldsymbol{w}_m + \nabla \pi_m \\
\nabla \cdot \boldsymbol{w}_m &= 0
\end{aligned}
\right\} \text{ in } \Omega
\qquad
\boldsymbol{w}_m = \boldsymbol{v}_{*_m} \ \text{ at } \partial\Omega,
\tag{XIII.8.21}
$$

we have that

$$
\langle\langle \boldsymbol{v}_{*m} \rangle\rangle_{1/2,2} + \langle\langle \boldsymbol{v}_{*m} \rangle\rangle_{1-1/q,q} \to 0 \ \text{ as } m \to \infty, \tag{XIII.8.22}
$$

while

$$
\|\boldsymbol{w}_m\|_{4q/(4-q),\Omega_\rho} = 1. \tag{XIII.8.23}
$$

In view of (XIII.8.19) it follows, in particular, that

$$
|\boldsymbol{w}_m|_{1,2} + \|\boldsymbol{w}_m\|_{3q/(3-q)} + |\boldsymbol{w}_m|_{1,q} + \|\pi_m\|_q \le M_1,
$$

with M_1 independent of m. Therefore, by Theorem II.1.3, Exercise II.6.2, and Theorem II.5.2 there is $\overline{\boldsymbol{w}}$, $\overline{\pi}$ such that (at least along a subsequence)

$$
\begin{aligned}
&\boldsymbol{w}_m \xrightarrow{w} \overline{\boldsymbol{w}} \ \text{ in } \dot{D}^{1,q}(\Omega) \cap L^{3q/(3-q)}(\Omega) \cap \dot{D}^{1,2}(\Omega) \\
&\boldsymbol{w}_m \to \overline{\boldsymbol{w}} \ \text{ in } L^r(\Omega_R) \text{ for all } r \in (1, 3q/(3-q)),\ R > 0 \\
&\pi_m \xrightarrow{w} \overline{\pi} \ \text{ in } L^q(\Omega).
\end{aligned}
\tag{XIII.8.24}
$$

Because of (XIII.8.22) and the trace Theorem II.10.1, it follows that $\boldsymbol{w}$ has zero trace at $\partial\Omega$; also, $\nabla \cdot \overline{\boldsymbol{w}} = 0$ and so

$$
\overline{\boldsymbol{w}} \in \mathcal{D}_0^{1,q}(\Omega) \cap \mathcal{D}_0^{1,2}(\Omega). \tag{XIII.8.25}
$$

Moreover, owing to $(\text{XIII.8.5})_{2,4}$, we have

$$\|\boldsymbol{u}\|_3 \le M_2, \tag{XIII.8.26}$$

with M_2 independent of m, and we may select a subsequence (denoted again by $\{\boldsymbol{u}_m\}$) and find a field $\boldsymbol{\mathcal{U}} \in L^3(\Omega)$ such that

$$\boldsymbol{u}_m \to \boldsymbol{\mathcal{U}} \;\; \text{weakly in } L^3(\Omega). \tag{XIII.8.27}$$

By (XIII.8.24), for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}(\nabla \boldsymbol{w}_m, \nabla \boldsymbol{\psi}) &\to (\nabla \overline{\boldsymbol{w}}, \nabla \boldsymbol{\psi}) \\ (\pi_m, \nabla \cdot \boldsymbol{\psi}) &\to (\overline{\pi}, \nabla \cdot \boldsymbol{\psi}).\end{aligned} \tag{XIII.8.28}$$

Also, by (XIII.8.27), it follows that

$$\begin{aligned}(\boldsymbol{u}_m \cdot \nabla \boldsymbol{\psi}, \boldsymbol{w}_m) - (\boldsymbol{\mathcal{U}} \cdot \nabla \boldsymbol{\psi}, \overline{\boldsymbol{w}}) &= ((\boldsymbol{u}_m - \boldsymbol{\mathcal{U}}) \cdot \nabla \boldsymbol{\psi}, \overline{\boldsymbol{w}}) \\ &\quad -(\boldsymbol{u}_m \cdot \nabla \boldsymbol{\psi}, (\overline{\boldsymbol{w}} - \boldsymbol{w}_m)) \\ &\equiv \mathcal{I}_1(m) + \mathcal{I}_2(m).\end{aligned} \tag{XIII.8.29}$$

By the Hölder inequality,

$$\|\nabla \boldsymbol{\psi} \cdot \overline{\boldsymbol{w}}\|_{3/2} \le \|\nabla \boldsymbol{\psi}\|_{q/(q-1)} \|\overline{\boldsymbol{w}}\|_{3q/(3-q)}$$

and (XIII.8.24) and (XIII.8.27) imply

$$\lim_{m\to\infty} \mathcal{I}_1(m) = 0. \tag{XIII.8.30}$$

In addition,

$$|(\boldsymbol{u}_m \cdot \nabla \boldsymbol{\psi}, (\overline{\boldsymbol{w}} - \boldsymbol{w}_m))| \le \|\boldsymbol{u}_m\|_3 \|\boldsymbol{\psi}\|_{s'} \|\overline{\boldsymbol{w}} - \boldsymbol{w}_m\|_{s,\varrho}$$

where $\varrho = \operatorname{supp}(\boldsymbol{\psi})$ and $s < 3q/(3-q)$. Thus, by (XIII.8.24), we find

$$\lim_{m\to\infty} \mathcal{I}_2(m) = 0. \tag{XIII.8.31}$$

Collecting (XIII.8.28)–(XIII.8.31), taking into account that $\boldsymbol{w}_m$, π_m is a generalized solution to (XIII.8.21), and using (XIII.8.22), it follows that

$$\begin{aligned}0 &= \lim_{m\to\infty} \{(\nabla \boldsymbol{w}_m, \nabla \boldsymbol{\psi}) - (\boldsymbol{u}_m \cdot \nabla \boldsymbol{\psi}, \boldsymbol{w}_m) - (\pi_m, \nabla \cdot \boldsymbol{\psi})\} \\ &= (\nabla \overline{\boldsymbol{w}}, \nabla \boldsymbol{\psi}) - (\boldsymbol{\mathcal{U}} \cdot \nabla \boldsymbol{\psi}, \overline{\boldsymbol{w}}) - (\overline{\pi}, \nabla \cdot \boldsymbol{\psi}).\end{aligned}$$

Therefore, by (XIII.8.25), $\overline{\boldsymbol{w}}$ solves the homogeneous problem

$$\begin{aligned}(\nabla \overline{\boldsymbol{w}}, \nabla \boldsymbol{\psi}) = (\boldsymbol{\mathcal{U}} \cdot \nabla \boldsymbol{\psi}, \overline{\boldsymbol{w}}) + (\overline{\pi}, \nabla \cdot \boldsymbol{\psi}) \;\; &\text{for all } \boldsymbol{\psi} \in C_0^\infty(\Omega) \\ \overline{\boldsymbol{w}} \in \mathcal{D}_0^{1,q}(\Omega) \cap \mathcal{D}_0^{1,2}(\Omega).&\end{aligned} \tag{XIII.8.32}$$

We may take, in particular, $\boldsymbol{\psi} \in \mathcal{D}(\Omega)$ to obtain

$$
\begin{aligned}
(\nabla \overline{\boldsymbol{w}}, \nabla \boldsymbol{\psi}) &= (\boldsymbol{\mathcal{U}} \cdot \nabla \boldsymbol{\psi}, \overline{\boldsymbol{w}}) \;\; \text{for all } \boldsymbol{\psi} \in \mathcal{D}(\Omega) \\
\overline{\boldsymbol{w}} &\in \mathcal{D}_0^{1,2}(\Omega).
\end{aligned}
\tag{XIII.8.33}
$$

Using the density property of $\mathcal{D}(\Omega)$ into $\mathcal{D}_0^{1,2}(\Omega)$ along with Lemma X.2.1, we may replace $\boldsymbol{\psi}$ with $\overline{\boldsymbol{w}}$ in (XIII.8.32) to derive

$$
-\,|\overline{\boldsymbol{w}}|_{1,2} = (\boldsymbol{\mathcal{U}} \cdot \nabla \overline{\boldsymbol{w}}, \overline{\boldsymbol{w}}). \tag{XIII.8.34}
$$

However, $\boldsymbol{\mathcal{U}}$ is (weakly) divergence-free and, as a consequence,

$$
(\boldsymbol{\mathcal{U}} \cdot \nabla \boldsymbol{w}_r, \boldsymbol{w}_r) = 0
$$

along a sequence $\{\boldsymbol{w}_r\} \subset \mathcal{D}(\Omega)$ approximating $\overline{\boldsymbol{w}}$ in $\mathcal{D}_0^{1,2}(\Omega)$. Thus, again by Lemma X.2.1, we obtain

$$
(\boldsymbol{\mathcal{U}} \cdot \nabla \overline{\boldsymbol{w}}, \overline{\boldsymbol{w}}) = 0
$$

and $(\text{XIII.8.32})_2$, (XIII.8.34) imply $\overline{\boldsymbol{w}} \equiv 0$ a.e. in Ω. From $(\text{XIII.8.24})_2$ we then find

$$
\boldsymbol{w}_m \to 0 \;\; \text{strongly in } L^{4q/(4-q)}(\Omega_\rho),
$$

which contradicts (XIII.8.23). The validity of (XIII.8.20) is therefore established and (XIII.8.19) furnishes that, for all $k \in \mathbb{N}$, there exists a generalized solution $\boldsymbol{w}_k$, π_k to (XIII.8.10) satisfying the estimate

$$
\begin{aligned}
|\boldsymbol{w}_k|_{1,2} + \|\boldsymbol{w}_k\|_{3q/(3-q)} &+ |\boldsymbol{w}_k|_{1,q} + \|\pi_k\|_q \\
&\le c(\langle\langle \boldsymbol{v}_* \rangle\rangle_{1/2,2} + \langle\langle \boldsymbol{v}_* \rangle\rangle_{1-1/q,q})
\end{aligned}
\tag{XIII.8.35}
$$

with a constant c independent of k. As a consequence, from $\{\boldsymbol{w}_k, \pi_k\}$ we can select a subsequence, denoted again by $\{\boldsymbol{w}_k, \pi_k\}$, and find a pair $\{\overline{\boldsymbol{w}}, \overline{\pi}\}$ obeying the property (XIII.8.24). Taking into account $(\text{XIII.8.25})_2$ and reasoning as we did to recover (XIII.8.32) we show that $\{\overline{\boldsymbol{w}}, \overline{\pi}\}$ is a generalized solution to the problem

$$
\left.
\begin{aligned}
\Delta \boldsymbol{w} &= \boldsymbol{v} \cdot \nabla \boldsymbol{w} + \nabla \pi \\
\nabla \cdot \boldsymbol{w} &= 0
\end{aligned}
\right\} \;\; \text{in } \Omega
\qquad
\boldsymbol{w} = \boldsymbol{v}_* \;\; \text{at } \partial\Omega
\tag{XIII.8.36}
$$

such that

$$
\begin{aligned}
&\overline{\boldsymbol{w}} \in D^{1,q}(\Omega) \cap L^{3q/(3-q)}(\Omega) \cap D^{1,2}(\Omega) \\
&\overline{\pi} \in L^q(\Omega).
\end{aligned}
\tag{XIII.8.37}
$$

However, $\boldsymbol{v}$, p is also a generalized solution to (XIII.8.36) and by assumption, (XIII.8.36), and (XIII.8.37) we conclude that the differences

$$
\boldsymbol{w} \equiv \boldsymbol{v} - \overline{\boldsymbol{w}}, \quad \pi = p - \overline{\pi}
$$

solve the generalized form of (XIII.8.36) with $\boldsymbol{v}_* \equiv 0$. Since $\boldsymbol{w} \in \mathcal{D}_0^{1,2}(\Omega)$, reasoning as we did to show that (XIII.8.33) admits only the identically vanishing generalized solution, we prove $\boldsymbol{w} \equiv 0$, $\pi \equiv$ const. a.e. in Ω. We then conclude that

$$\boldsymbol{v} \in D^{1,q}(\Omega) \cap L^{3q/(3-q)}(\Omega), \quad \text{for all } q \in (6/5, 3/2]. \tag{XIII.8.38}$$

Since $3q/(3-q)\,(> 2)$ can be chosen as close to 2 as we please by picking $q\,(> 6/5)$ suitably close to $6/5$, and since

$$|\boldsymbol{v} \cdot \nabla \boldsymbol{v}|_{-1,r} \le \|\boldsymbol{v}\|_{2r}^2$$

we may assert that

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} \in D_0^{-1,q}(\Omega) \;\text{ for all } q \in (1, 3/2]$$

and, as a consequence, by Theorem V.3.3 it follows that

$$\boldsymbol{v} \in D^{1,q}(\Omega) \cap L^{3q/(3-q)}(\Omega) \;\text{ for all } q \in (1, 3/2]. \tag{XIII.8.39}$$

Since $\boldsymbol{v} \in D^{1,q}(\Omega)$, the lemma follows from (XIII.8.39) after application of the interpolation inequality (II.2.7). □

Combining Lemma XIII.8.1, Theorem XIII.7.4, and Corollary XIII.7.1, we obtain the (first part of the) following result concerning (global) summability properties of generalized solutions to problem (XIII.5.2) in the aperture domain Ω.

Theorem XIII.8.1 *Let the flux Φ satisfy the assumption of* Theorem XIII.7.4. *Then any corresponding generalized solution $\boldsymbol{v}$ to* (XIII.5.2) *that obeys the energy inequality* (XIII.6.6) *enjoys the following summability properties:*

$$\boldsymbol{v} \in D^{1,q}(\Omega) \cap L^{3q/(3-q)}(\Omega) \;\text{ for all } q \in (1, 2].$$

Moreover, denoting by p the pressure field associated to $\boldsymbol{v}$ by Lemma XIII.5.1 *we have*

$$p - p_{\pm} \in L^s(\mathbb{R}_{\pm}) \;\text{ for all } s \in (1, 3],$$

where the constants $p_{\pm}$ are defined in Lemma XIII.5.1.

Proof. We have to show only the summability of p. However, since

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} \in D_0^{-1,s}(\Omega) \;\text{ for all } s \in (1, 3],$$

the stated property follows at once from Theorem VI.5.1. □

XIII.9 Asymptotic Structure of Generalized Solutions for Flow in an Aperture Domain

Using the results on the Stokes problem in the half-space as given in Theorem IV.3.2 and Theorem IV.3.3, it is a straightforward exercise to show that every generalized solution to problem (XIII.5.2) satisfies the following asymptotic properties

$$\begin{gathered}\lim_{|x|\to\infty} D^\alpha \boldsymbol{v}(x) = 0\\ \lim_{|x|\to\infty, x\in\mathbb{R}^3_\pm} D^\alpha (p(x) - p_\pm) = 0,\end{gathered} \tag{XIII.9.1}$$

for all $|\alpha| \geq 0$; cf. Exercise XIII.9.1. However, nothing can be said about the order of decay, unless we put some suitable restriction on the magnitude of the flux Φ of the velocity field through the aperture S. Actually, under the hypothesis of Theorem XIII.7.4, it is a relatively simple task to determine the precise asymptotic structure of any corresponding generalized solution $\boldsymbol{v}$ and of the associated pressure field p. In showing this, we shall follow an approach entirely analogous to that employed in similar circumstances for flow in exterior domains; cf. Section X.8 and X.6.

First of all, we need a representation for $\boldsymbol{v}$. By Theorem XIII.7.4, any generalized solution corresponding to "small" Φ satisfies the condition

$$v_l v_i \in L^2(\Omega), \quad l, i = 1, 2, 3,$$

and so, by Lemma VI.5.1, it follows that for all $x \in \mathbb{R}^3_\pm$ $\boldsymbol{v}(x)$ admits the representation

$$\begin{aligned} v_j(x) = &-\frac{1}{\nu}\int_{\mathbb{R}^3_\pm} D_l G^\pm_{ij}(x,y) v_l(y) v_i(y) dy \\ &- \int_S [v_i(y) T_{il}(\boldsymbol{G}^\pm_j, g^\pm_j)(x,y) n_l(y) d\sigma_y, \end{aligned} \tag{XIII.9.2}$$

where $\boldsymbol{G}^\pm$ is the Green's tensor for the Stokes problem in $\mathbb{R}^3_\pm$ and $\mathbf{g}^\pm$ is the associated "pressure" field. Relation (XIII.9.2) is the starting point of our asymptotic analysis. Since what we shall say equally applies to $\mathbb{R}^3_+$ and $\mathbb{R}^3_-$, to fix the ideas we shall deal with $\mathbb{R}^3_+$. Setting, for simplicity, $\boldsymbol{G}^+ = \boldsymbol{G}$ and $\boldsymbol{g}^+ = \boldsymbol{g}$, we begin to notice that from (XIII.9.2) we derive, in particular, for all $x \in \mathbb{R}^3_+$

$$\begin{aligned} v_j(x) = &-\frac{1}{\nu}\int_{\mathbb{R}^3_+} D_l G_{ij}(x,y) v_l(y) v_i(y) dy + T_{i3}(\boldsymbol{G}_j, g_j)(x,0)\int_S v_i(y) d\sigma_y \\ &- \int_S [T_{il}(\boldsymbol{G}_j, g_j)(x,y) - T_{il}(\boldsymbol{G}_j, g_j)(x,0)]\, v_i(y) n_l(y) d\sigma_y. \end{aligned}$$

Recalling the estimates (IV.3.50), from the latter relation, it follows for all sufficiently large $x \in \mathbb{R}^3_+$ that

$$v_j(x) = -\frac{1}{\nu}\int_{\mathbb{R}^3_+} D_l G_{ij}(x,y) v_l(y) v_i(y) dy + B_i T_{i3}(\boldsymbol{G}_j, g_j)(x,0) + \varphi_i(x) \tag{XIII.9.3}$$

where

$$B_i := \int_S v_i\,, \quad \varphi_i(x) = O(|x|^{-3}). \tag{XIII.9.4}$$

Our next objective is to give an asymptotic estimate of the integral term in (XIII.9.3). Specifically, we shall show that this term can be split as the sum of two terms: the first is proportional to $D_l G_{ij}(x,0)$ and the second is $O(|x|^{-3+\delta})$, for arbitrary positive δ. To reach this goal, we need some preliminary results. For $R > r > \delta(S)$ and with the origin of coordinates in the interior of S, we set

$$\begin{aligned} \Omega_a &= \mathbb{R}^3_+ \cap B_a, \quad a = r, R \\ \Omega_{r,R} &= \Omega_R - \overline{\Omega}_r \\ \Omega^a &= \mathbb{R}^3_+ - \overline{\Omega}_a \quad a = r, R \end{aligned}$$

and

$$\mathcal{G}(R) = \int_{\Omega^R} \nabla \boldsymbol{v} : \nabla \boldsymbol{v}. \tag{XIII.9.5}$$

We have the following.

Lemma XIII.9.1 *Let the assumptions of* Theorem XIII.7.4 *be satisfied. Then the generalized solution* $\boldsymbol{v}$ *verifies*

$$\mathcal{G}(R) \le cR^{-1}$$

with c *independent of* R.

Proof. Without loss of generality, we set $p_+ = 0$. Multiply $(\text{XIII.5.2})_1$ by $\boldsymbol{v}$, integrate by parts over Ω_{R,R_1}, $R_1 > R$, and use $(\text{XIII.5.2})_3$. We thus obtain

$$\nu \int_{\Omega_{R,R_1}} \nabla \boldsymbol{v} : \nabla \boldsymbol{v} = \int_{\partial B_R \cup \partial B_{R_1}} \left[\nu \boldsymbol{v} \cdot \frac{\partial \boldsymbol{v}}{\partial n} - p\boldsymbol{v}\cdot\boldsymbol{n} - \frac{v^2}{2}\boldsymbol{v}\cdot\boldsymbol{n}\right]. \tag{XIII.9.6}$$

Since, by Theorem XIII.7.4,

$$\boldsymbol{v} \cdot \frac{\partial \boldsymbol{v}}{\partial n}, \quad p\boldsymbol{v}, \quad v^3 \in L^1(\mathbb{R}^3_+), \tag{XIII.9.7}$$

letting $R_1 \to \infty$ along a suitable sequence into (XIII.9.6), we find

$$\nu\mathcal{G}(R) = \int_{\partial B_R} \left[\nu \boldsymbol{v} \cdot \frac{\partial \boldsymbol{v}}{\partial n} - p\boldsymbol{v}\cdot\boldsymbol{n} - \frac{v^2}{2}\boldsymbol{v}\cdot\boldsymbol{n}\right].$$

In view of (XIII.9.7) we have

$$\mathcal{G} \in L^1(R,\infty), \quad \text{for all } R > \delta(S),$$

and since $\mathcal{G}'(R) < 0$, the result follows from Lemma X.8.1. □

Lemma XIII.9.2 *Let the assumptions of* Theorem XIII.7.4 *be fulfilled. Then the generalized solution* $\boldsymbol{v}$ *satisfies*

$$\boldsymbol{v}(x) = O(|x|^{-1}), \quad \textit{as } |x| \to \infty \textit{ in } \mathbb{R}^3_+.$$

Proof. Set $|x| = 2R$, for $R > \delta(S)$. From (XIII.9.3)–(XIII.9.4) and (IV.3.50) it follows that

$$|\boldsymbol{v}(x)| \le c\left[\int_{\Omega_R} \frac{v^2(y)}{|x-y|^2}dy + \int_{\Omega^R} \frac{v^2(y)}{|x-y|^2}dy\right] + O(|x|^{-3}). \tag{XIII.9.8}$$

Since, by Theorem XIII.7.4, $\boldsymbol{v} \in L^2(\Omega)$, we deduce

$$\int_{\Omega_R} \frac{v^2(y)}{|x-y|^2}dy \le c_1|x|^{-2}. \tag{XIII.9.9}$$

Furthermore, let $\delta(S) < \rho/2 < \rho < R$ and let ψ_ρ be a smooth function in $\mathbb{R}^3$ with

$$\psi_\rho(x) = \begin{cases} 0 \text{ if } |x| \le \rho/2 \\ 1 \text{ if } |x| \ge \rho. \end{cases}$$

Extending $\boldsymbol{v}(x)$ by zero for $x \in \mathbb{R}^3_-$ and applying inequality (II.6.20) to the function $\psi_\rho(x)\boldsymbol{v}(x)$, we obtain

$$\int_{\Omega^R} \frac{v^2(y)}{|x-y|^2}dy \le c_2 \int_{\Omega^R} \nabla\boldsymbol{v} : \nabla\boldsymbol{v}$$

with c_2 independent of R. Thus, from (XIII.9.5) and Lemma XIII.9.1 we deduce

$$\int_{\Omega^R} \frac{v^2(y)}{|x-y|^2}dy \le c_3|x|^{-1}$$

and the lemma follows from this inequality, (XIII.9.8), and (XIII.9.9). □

Lemma XIII.9.3 *Let the assumptions of* Theorem XIII.7.4 *be fulfilled. Then the generalized solution* $\boldsymbol{v}$ *satisfies*

$$\boldsymbol{v}(x) = O(|x|^{-2}), \quad \textit{as } |x| \to \infty \textit{ in } \mathbb{R}^3_+.$$

Proof. From (XIII.9.3)–(XIII.9.4) and (IV.3.50) we see that it is enough to show that for large $|x|$ in $\mathbb{R}^3_+$

$$N_j(x) = \int_{\mathbb{R}^3_+} D_l G_{ij}(x,y) v_l(y) v_i(y) dy = O(|x|^{-2}) \tag{XIII.9.10}$$

holds. From (IV.3.50) it follows, in particular, that

$$|D^\alpha G_{ij}(x,y)| \le \frac{c}{|x-y|^{1+|\alpha|}}, \quad 0 \le |\alpha| \le 2, \tag{XIII.9.11}$$

and so

$$|\mathbf{N}(x)| \le c \int_{\mathbb{R}^3_+} \frac{v^2(y)}{|x-y|^2}. \tag{XIII.9.12}$$

We split the region of integration $\mathbb{R}^3_+$ into three subregions Ω_R, $\Omega_{R,3R}$, and Ω^{3R}, where $R = |x|/2$, and denote by I_1, I_2, and I_3 the three corresponding integrals. Since

$$|x| \le 2|x-y|, \quad \text{for } y \in \Omega_R,$$

we find

$$I_1 \le \frac{2c}{|x|^2} \int_{\mathbb{R}^3_+} v^2 = \frac{c_1}{|x|^2}, \tag{XIII.9.13}$$

where, in the last step, we have used the property $\boldsymbol{v} \in L^2(\Omega)$; cf. Theorem XIII.7.4. Also,

$$|y| \le 3|x-y|, \quad \text{for } y \in \Omega^{3R},$$

and so

$$I_3 \le 3c \int_{\Omega^{3R}} \frac{v^2(y)}{|y|^2}.$$

Recalling that $\boldsymbol{v} \in L^2(\Omega)$, we conclude

$$I_3 \le \frac{c_2}{R^2} \int_\Omega v^2 = \frac{c_3}{|x|^2}. \tag{XIII.9.14}$$

Concerning the estimate for I_2, we have

$$I_2 = \int_{C_R} \frac{v^2(y)}{|x-y|^2} dy + \int_{B_1(x)} \frac{v^2(y)}{|x-y|^2} dy, \tag{XIII.9.15}$$

where

$$C_R = \Omega_{R,3R} - B_1(x).$$

By Lemma XIII.9.2,

$$\boldsymbol{v}(x) = O(|x|^{-1}) \tag{XIII.9.16}$$

and so

$$\int_{B_1(x)} \frac{v^2(y)}{|x-y|^2} dy \le \frac{c_4}{|x|^2} \int_{B_1(x)} \frac{dy}{|x-y|^2} = \frac{c_5}{|x|^2}. \tag{XIII.9.17}$$

Furthermore, again by (XIII.9.16), for any positive η we have

$$\int_{C_R} \frac{v^2(y)}{|x-y|^2} dy \le \frac{c_5}{|x|^{3/2-\eta}} \int_{C_R} \frac{v^{1/2+\eta}(y)}{|x-y|^2} dy$$

and so, employing the Hölder inequality,

$$\int_{C_R} \frac{v^2(y)}{|x-y|^2} dy \le \frac{c_5}{|x|^{3/2-\eta}} \left(\int_{C_R} \frac{dy}{|x-y|^{2q'}} \right)^{1/q'} \left(\int_{\Omega} v^{(\frac{1}{2}+\eta)q} \right)^{1/q}.$$

By Theorem XIII.7.4, it follows that $\boldsymbol{v} \in L^r(\Omega)$ for all $r > 3/2$ and arbitrarily close to $3/2$. Therefore, the preceding inequality yields

$$\int_{C_R} \frac{v^2(y)}{|x-y|^2} dy \le \frac{c_6}{|x|^{3/2-\delta}} \tag{XIII.9.18}$$

where $\delta > 0$ is arbitrarily close to zero. Collecting (XIII.9.10), (XIII.9.12)–(XIII.9.15), (XIII.9.17), and (XIII.9.18), with the help of (XIII.9.3)–(XIII.9.3) we derive

$$\boldsymbol{v}(x) = O(|x|^{-3/2+\delta}). \tag{XIII.9.19}$$

With this improved estimate on $\boldsymbol{v}$ we can give an improved estimate of the first integral on the right-hand side of (XIII.9.15). Actually, from (XIII.9.19) we find

$$\int_{C_R} \frac{v^2(y)}{|x-y|^2} dy \le \frac{c_7}{|x|^2} \int_{C_R} \frac{v^{(2-4\delta)/(3-2\delta)}(y)}{|x-y|^2} dy.$$

Thus, applying the Hölder inequality and recalling that $\boldsymbol{v} \in L^r(\Omega)$ for all $r \in (3/2, 6]$ we arrive at

$$\int_{C_R} \frac{v^2(y)}{|x-y|^2} \le \frac{c_8}{|x|^2},$$

and the proof of the lemma is complete. □

Lemma XIII.9.4 *Let the assumptions of* Theorem XIII.7.4 *be fulfilled. Then the generalized solution* $\boldsymbol{v}$ *satisfies as* $|x| \to \infty$ *in* $\mathbb{R}^3_+$

$$\int_{\mathbb{R}^3_+} [D_l G_{ij}(x,y) - D_l G_{ij}(x,0)]\, v_l(y) v_i(y) dy = O(|x|^{-3+\delta}),$$

with δ *an arbitrary small positive number.*

Proof. We denote by $\mathcal{I}$ the integral we want to estimate and write

$$\begin{aligned} \mathcal{I} = & \int_{\Omega_R} [D_l G_{ij}(x,y) - D_l G_{ij}(x,0)]\, v_l(y) v_i(y) dy \\ & + \int_{\Omega^R} [D_l G_{ij}(x,y) - D_l G_{ij}(x,0)]\, v_l(y) v_i(y) dy \\ \equiv & \, \mathcal{I}_1 + \mathcal{I}_2, \end{aligned} \tag{XIII.9.20}$$

where $|x| = 2R$. From the mean value theorem and (XIII.9.11) we deduce

$$|D_l G_{ij}(x,y) - D_l G_{ij}(x,0)| = |y_l D_k D_l G_{ij}(x,\beta y)| \le \frac{c|y|}{|x-\beta y|^3}, \quad \beta \in (0,1).$$

We have

$$|x| \le 2|x-\beta y|, \quad y \in \Omega_R,$$

and so

$$|\mathcal{I}_1| \le \frac{8c}{|x|^3}\int_{\Omega_R} v^2(y)|y|dy.$$

From Lemma XIII.9.3 and the Hölder inequality we find

$$|\mathcal{I}_1| \le \frac{c_1}{|x|^3}|\Omega_R|^{1/q'}\|\boldsymbol{v}\|^{3/2}_{3q/2,\Omega}$$

and since, by Theorem XIII.7.4, we can take q arbitrarily close to 1, we derive

$$|\mathcal{I}_1| \le \frac{c_2}{|x|^{3-\delta}} \tag{XIII.9.21}$$

with $\delta > 0$ arbitrarily close to zero. Moreover, from (XIII.9.11) it follows that

$$\begin{aligned}|\mathcal{I}_2| &\le \int_{\Omega^R} |D_l G_{ij}(x,y)v_l(y)v_i(y)|\,dy + |D_l G_{ij}(x,0)|\int_{\Omega^R} v^2(y)dy \\ &\le c\left\{\int_{\Omega^R}\frac{v^2(y)}{|x-y|^2}dy + \frac{1}{|x|^2}\int_{\Omega^R} v^2\right\}.\end{aligned} \tag{XIII.9.22}$$

Therefore, using the estimate on $\boldsymbol{v}$ given in Lemma XIII.9.3 and then applying Lemma II.9.2, we infer that

$$\int_{\Omega^R}\frac{v^2(y)}{|x-y|^2}dy \le \frac{c_3}{|x|^2}\int_{\mathbb{R}^3}\frac{dy}{|x-y|^2|y|^2} \le \frac{c_4}{|x|^3}. \tag{XIII.9.23}$$

Likewise, again from Lemma XIII.9.3 and the property $\boldsymbol{v} \in L^r(\Omega)$ for all $r \in (3/2, 6]$, we find for any positive δ sufficiently close to zero

$$\int_{\Omega^R} v^2(y)dy \le \frac{c_5}{|x|^{1-\delta}}\int_{\Omega} v^{3/2+\delta} = \frac{c_6}{|x|^{1-\delta}}. \tag{XIII.9.24}$$

The lemma then follows from (XIII.9.20)–(XIII.9.24). □

The representation (XIII.9.3)–(XIII.9.4), together with Lemma XIII.9.4, allows us to show the following result concerning the asymptotic structure of a generalized solution corresponding to a flux of suitably restricted size.

Theorem XIII.9.1 *Let the flux Φ satisfy the assumption of* Theorem XIII.7.4. *Then any corresponding generalized solution $\boldsymbol{v}$ to problem* (XIII.5.2) *that* *obeys the energy inequality* (XIII.6.6) *admits the following asymptotic representation as* $|x| \to \infty$, $x \in \mathbb{R}^3_\pm$:

$$v_j(x) = A^\pm_{il} D_l G^\pm_{ij}(x,0) + B^\pm_i T_{i3}(\boldsymbol{G}^\pm_j, g^\pm_j)(x,0) + \mathcal{V}_j(x), \quad j = 1,2,3. \tag{XIII.9.25}$$

Here $\boldsymbol{G}^\pm \equiv \{G^\pm_{ij}\}$, $\mathbf{g}^\pm \equiv \{g^\pm_j\}$ *is the Green's tensor-solution for the Stokes problem in* $\mathbb{R}^3_\pm$ *(see* (IV.3.46)-(IV.3.49)*)*, $\boldsymbol{G}^\pm_j \equiv (G^\pm_{1j}, G^\pm_{2j}, G^\pm_{3j})$, *and*

$$A^\pm_{il} := \frac{1}{\nu}\int_{\mathbb{R}^3_\pm} v_i v_l\,, \quad B^\pm_i := \pm\int_S v_i\,, \quad \mathcal{V}_j(x) = O(|x|^{-3+\delta}),$$

with δ *an arbitrary positive number.*

Remark XIII.9.1 From (XIII.9.25) and (IV.3.50) we obtain the uniform asymptotic estimate for $\boldsymbol{v}$ (see also Lemma XIII.9.3):

$$\boldsymbol{v}(x) = O(|x|^{-2}).$$

Moreover, comparing (XIII.9.25) with the analogous formula (VI.5.23) obtained in the linear case, we recognize in (XIII.9.25) the presence of the extra terms involving the quantities $A^\pm_{il}$. Therefore, unlike the case of an exterior domain (when $\boldsymbol{v}_\infty \neq 0$), the nonlinearities give an explicit contribution to the asymptotic expansion. ■

Remark XIII.9.2 Following the methods of Coscia & Patria (1992), one can show that the first derivatives of $\boldsymbol{v}$ obey the asymptotic estimate

$$D_k \boldsymbol{v}(x) = O(|x|^{-3}), \quad \text{as } |x| \to \infty.$$

We shall not perform this study here; we refer the reader to Theorem 12 of the paper by Coscia & Patria; cf. also Exercise XIII.9.2. ■

The final part of this section is devoted to the proof of an asymptotic representation for the pressure field p. As before, we shall show this in $\mathbb{R}^3_+$, an analogous formula holding in $\mathbb{R}^3_-$. Let $\varphi_R(x)$ be a smooth function that equals one for $|x| \geq 2R$ and zero for $|x| \leq R$ with $|\nabla\varphi_R| \leq MR^{-1}$, M independent of R. Setting

$$\boldsymbol{w} = \varphi_R \boldsymbol{v}, \quad \tau = \varphi_R(p - p_+),$$

with p, p_+ introduced in Lemma XIII.5.1, we easily establish that $\boldsymbol{w}$, τ satisfy the following problem

$$\left.\begin{aligned} \nu\Delta\boldsymbol{w} &= \nabla\tau + \boldsymbol{F} \\ \nabla\cdot\boldsymbol{w} &= g \end{aligned}\right\} \text{ in } \mathbb{R}^3_+ \qquad \boldsymbol{w} = \boldsymbol{v} \ \text{ at } x_3 = 0 \tag{XIII.9.26}$$

with

$$\begin{aligned}\boldsymbol{F} &= 2\nabla\varphi_R\cdot\nabla\boldsymbol{v}+\Delta\varphi_R\boldsymbol{v}+\varphi_R\boldsymbol{v}\cdot\nabla\boldsymbol{v}+(p-p_+)\cdot\nabla\varphi_R\\ g &= \nabla\varphi_R\cdot\boldsymbol{v}.\end{aligned}\tag{XIII.9.27}$$

Denote, further, by $\boldsymbol{h}$ a solution to the non-homogeneous Stokes problem

$$\left.\begin{aligned}\nu\Delta\boldsymbol{h} &= \nabla\sigma\\ \nabla\cdot\boldsymbol{h} &= -\nabla\varphi_R\cdot\boldsymbol{v}\end{aligned}\right\}\ \text{in } \mathbb{R}^3_+ \qquad \boldsymbol{h}=0 \ \text{ at } x_3=0.\tag{XIII.9.28}$$

As we know from Theorem IV.3.2, this problem admits a (unique) solution satisfying the estimate

$$|\boldsymbol{h}|_{2,q}+|\sigma|_{1,q}\le c\|\nabla\varphi_R\cdot\boldsymbol{v}\|_{1,q} \ \text{ for all } q>1,\tag{XIII.9.29}$$

with a constant c independent of R. Using Theorem XIII.7.4, together with the properties of φ_R, we easily show

$$\lim_{R\to\infty}\|D^\beta(\nabla\varphi_R)D^\chi\boldsymbol{v}\|_q=0 \ \text{ for all } q>1, \text{ and } |\beta|,\,|\chi|{=}0{,}1.\tag{XIII.9.30}$$

Choosing $q=q_1<3$ and modifying σ by the addition of an appropriate constant, from Theorem II.6.3 we find

$$\|\sigma\|_{3q_1/(3-q_1)}\le\gamma_1|\sigma|_{q_1}.\tag{XIII.9.31}$$

Thus, from (XIII.9.29) and (XIII.9.31), with the help of Theorem II.9.1, we find for all $x\in\mathbb{R}^3_+$

$$|\sigma(x)|\le\gamma_2\left(|\sigma|_{1,q_1}+|\sigma|_{1,q_2}\right)\quad q_2>3,$$

with γ_2 independent of R. From this relation, (XIII.9.29), and (XIII.9.30) we then conclude

$$\lim_{R\to\infty}\sigma(x)=0,\quad\text{for all } x\in\mathbb{R}^3_+.\tag{XIII.9.32}$$

Setting

$$\boldsymbol{u}=\boldsymbol{w}+\boldsymbol{h},$$

from (XIII.9.26)–(XIII.9.28) we find

$$\left.\begin{aligned}\nu\Delta\boldsymbol{u} &= \nabla\tau_1+\boldsymbol{F}\\ \nabla\cdot\boldsymbol{u} &= 0\end{aligned}\right\}\ \text{in } \mathbb{R}^3_+ \qquad \boldsymbol{u}=\boldsymbol{v} \ \text{ at } x_3=0\tag{XIII.9.33}$$

with

$$\tau_1=\tau+\sigma.\tag{XIII.9.34}$$

Taking into account that $\boldsymbol{F}$ is smooth and of compact support, reasoning as in the proof of Lemma VI.5.1(ii) from (XIII.9.33)–(XIII.9.34) we then find for all $x \in \mathbb{R}^3_+$

$$\tau_1(x) = -\int_{\mathbb{R}^3_+} g_i(x,y)F_i(y)dy + 2\nu \int_S v_i(y)\frac{\partial g_i(x,y)}{\partial x_3}d\sigma_y, \tag{XIII.9.35}$$

where, for simplicity, we set $\boldsymbol{g} = \boldsymbol{g}^+$. Our objective is to let $\varphi_R \to 1$ into (XIII.9.35) or, equivalently, $R \to \infty$. To this end, we recall that $\boldsymbol{g}$ enjoys the uniform estimate

$$|D^\alpha \boldsymbol{g}(x,y)| \le \frac{c_1}{|x-y|^{|\alpha|+2}}, \quad |\alpha| \ge 0; \tag{XIII.9.36}$$

cf. (IV.3.50), (IV.3.51). Therefore, for all $x \in \mathbb{R}^3_+$ we have

$$\int_{\mathbb{R}^3_+} |\boldsymbol{g}(x,y)\cdot \boldsymbol{v}(y)\Delta\varphi_R(y)|\,dy \le c_2 \left[\int_{B_d(x)} \frac{|\boldsymbol{v}(y)\Delta\varphi_R(y)|}{|x-y|^2}dy \right.$$
$$\left. + \int_{\mathbb{R}^3_+ - B_d(x)} \frac{|\boldsymbol{v}(y)\Delta\varphi_R(y)|}{|x-y|^2}dy\right]$$

where $d < x_3$. The Hölder inequality, with $r > 3$ and $s < 3/2$, yields

$$\begin{aligned}
\int_{B_d(x)} \frac{|\boldsymbol{v}(y)\Delta\varphi_R(y)|}{|x-y|^2}dy &\le \left(\int_{B_d(x)} |x-y|^{-2r'}dy\right)^{1/r'} \|\Delta\varphi_R \boldsymbol{v}\|_r \\
\int_{\mathbb{R}^3_+ - B_d(x)} \frac{|\boldsymbol{v}(y)\Delta\varphi_R(y)|}{|x-y|^2}dy &\le \left(\int_{\mathbb{R}^3_+ - B_d(x)} |x-y|^{-2s'}dy\right)^{1/s'} \|\Delta\varphi_R \boldsymbol{v}\|_s
\end{aligned} \tag{XIII.9.37}$$

and so, in view of (XIII.9.30) from (XIII.9.37) we conclude

$$\lim_{R\to\infty} \int_{\mathbb{R}^3_+} \boldsymbol{g}(x,y)\cdot \boldsymbol{v}(y)\Delta\varphi_R(y)dy = 0. \tag{XIII.9.38}$$

By similar reasonings, which we leave to the reader, we show

$$\begin{aligned}
&\lim_{R\to\infty}\int_{\mathbb{R}^3_+} \boldsymbol{g}(x,y)\cdot \nabla\boldsymbol{v}(y)\cdot\nabla\varphi_R(y)dy = 0 \\
&\lim_{R\to\infty}\int_{\mathbb{R}^3_+} \boldsymbol{g}(x,y)\cdot \nabla\varphi_R(y)(p(y)-p_+)(y)dy = 0 \\
&\lim_{R\to\infty}\int_{\mathbb{R}^3_+} \boldsymbol{v}(y)\cdot\nabla\boldsymbol{v}(y)\cdot\boldsymbol{g}(x,y)\varphi_R(y)dy = \int_{\mathbb{R}^3_+} \boldsymbol{v}(y)\cdot\nabla\boldsymbol{v}(y)\cdot\boldsymbol{g}(x,y)dy.
\end{aligned} \tag{XIII.9.39}$$

Collecting (XIII.9.27)$_1$, (XIII.9.32), (XIII.9.34), (XIII.9.38), and (XIII.9.39) we arrive at the following representation for p

$$p(x) = p_+ - \int_{\mathbb{R}^3_+} g_i(x,y) v_l(y) D_l v_i(y) dy + 2\nu \int_S v_i(y) \frac{\partial g_i(x,y)}{\partial x_3} d\sigma_y. \quad \text{(XIII.9.40)}$$

Using the mean value theorem, together with (XIII.9.36), from (XIII.9.40) we find

$$p(x) = p_+ - \int_{\mathbb{R}^3_+} g_i(x,y) v_l(y) D_l v_i(y) dy + 2\nu \frac{\partial g_i(x,0)}{\partial x_3} \int_S v_i + O(|x|^{-4}). \quad \text{(XIII.9.41)}$$

Our next task is to furnish an estimate of the volume integral $\mathcal{I}$ on the right-hand side of (XIII.9.41). To this end, we write

$$\mathcal{I} = \int_{\mathbb{R}^3_+ - B_1(x)} g_i(x,y) v_l(y) D_l v_i(y) dy + \int_{B_1(x)} g_i(x,y) v_l(y) D_l v_i(y) dy.$$

Integrating by parts the first integral, we find

$$\begin{aligned}\mathcal{I} = &- \int_{\mathbb{R}^3_+ - B_1(x)} D_l g_i(x,y) v_l(y) v_i(y) dy + \int_S g_i(x,y) v_i(y) v_l(y) n_l(y) d\sigma_y \\ &+ \int_{\partial B_1(x)} g_i(x,y) v_i(y) v_l(y) n_l(y) d\sigma_y + \int_{B_1(x)} g_i(x,y) v_l(y) D_l v_i(y) dy\end{aligned}$$

and so, employing the estimates of Theorem XIII.9.1 and Exercise XIII.9.2 we deduce that

$$\begin{aligned}\mathcal{I} = &- \int_{\mathbb{R}^3_+ - B_1(x)} D_l g_i(x,y) v_l(y) v_i(y) dy + \int_S g_i(x,y) v_i(y) v_l(y) n_l(y) d\sigma_y \\ &+ O(|x|^{-4}), \quad \text{as } |x| \to \infty.\end{aligned} \quad \text{(XIII.9.42)}$$

However, from (IV.3.46)$_{3,4}$ and (IV.3.48)$_{5,6}$, it follows that $\boldsymbol{g}(x,y)|_{y_3=0} = 0$ for all $x \in \mathbb{R}^3_+$ and from (XIII.9.42) we conclude that

$$\mathcal{I} = - \int_{\mathbb{R}^3_+ - B_1(x)} D_l g_i(x,y) v_l(y) v_i(y) dy + O(|x|^{-4}), \quad \text{as } |x| \to \infty. \quad \text{(XIII.9.43)}$$

Let us denote by $\mathcal{I}_1$ the integral in (XIII.9.43). We have

$$\begin{aligned}\mathcal{I}_1 = &D_l g_j(x,0) \int_{\mathbb{R}^3_+} v_l v_i + \int_{\mathbb{R}^3_+ - B_1(x)} \left[D_l g_j(x,y) - D_l g_i(x,0) \right] v_l(y) v_i(y) dy \\ &- D_l g_i(x,0) \int_{B_1(x)} v_l v_i\end{aligned}$$

and so, by Theorem XIII.9.1 and (XIII.9.36) we find

$$\mathcal{I}_1 = D_l g_j(x,0)A^+_{il} + \int_{\mathbb{R}^3_+ - B_1(x)} [D_l g_j(x,y) - D_l g_i(x,0)]\, v_l(y)v_i(y)dy + O(|x|^{-4}) \tag{XIII.9.44}$$

where A^+_{il} is defined in Theorem XIII.9.1. By (XIII.9.36) and the mean value theorem we have

$$|D_l g_i(x,y) - D_l g_i(x,0)| \le c\frac{|y|}{|x-\beta y|^4}.$$

As a consequence, setting $|x| = 2R$, it follows that

$$\int_{\Omega_R} |D_l g_j(x,y) - D_l g_i(x,0)|\, v^2(y)dy \le \frac{c_1}{|x|^4}\int_{\Omega_R} v^2(y)|y|dy.$$

Reasoning as in the proof of (XIII.9.21) we then show

$$\int_{\Omega_R} |D_l g_j(x,y) - D_l g_i(x,0)|\, v^2(y)dy = O(|x|^{-4+\delta}) \tag{XIII.9.45}$$

where $\delta > 0$ can be taken arbitrarily close to zero. Furthermore, again by (XIII.9.36) and Theorem XIII.9.1, we have

$$|D_l g_j(x,0)| \int_{\Omega^R} v^2 \le \frac{c_2}{|x|^{4-\delta}}\int_{\Omega^R} v^{3/2+\delta}(y)dy = O(|x|^{-4+\delta}) \tag{XIII.9.46}$$

and

$$\int_{\Omega^R - B_1(x)} |D_l g_j(x,y)|v^2(y)dy \le \frac{c_3}{|x|^{4-\delta}}\int_{|x-y|\ge 1} \frac{dy}{|x-y|^3|y|^\delta} = O(|x|^{-4+\delta}). \tag{XIII.9.47}$$

Collecting (XIII.9.41) and (XIII.9.43)–(XIII.9.47) we then obtain the following result.

Theorem XIII.9.2 *Let the flux Φ satisfy the assumption of* Theorem XIII.7.4. *Then the pressure field associated to any generalized solution to* (XIII.5.2) *that obeys the energy inequality* (XIII.6.6) *admits the following asymptotic representation as $|x| \to \infty$, $x \in \mathbb{R}^3_\pm$*

$$p(x) = p_\pm + D_l g^\pm_i(x,0)A^\pm_{il} + 2\nu D_3 g^\pm_i(x,0)B^\pm_i + P(x) \tag{XIII.9.48}$$

where $p_\pm$ and $A^\pm_{il}$, $B^\pm_i$ are defined in Lemma XIII.5.1 *and* Theorem XIII.9.1, *respectively, while $\boldsymbol{g}^\pm$ is the vector "pressure field" associated to the Green's tensor for the Stokes problem in the $\mathbb{R}^3_\pm$; see* (IV.3.46)–(IV.3.49). *Finally,*

$$P(x) = O(|x|^{-4+\delta}),$$

with δ a positive number that can be taken arbitrarily close to zero.

Remark XIII.9.3 From (XIII.9.36) and (XIII.9.48) we derive, in particular, the following uniform asymptotic estimate

$$p(x) - p_{\pm} = O(|x|^{-3}), \quad |x| \to \infty, \quad x \in \mathbb{R}^3_{\pm}.$$

■

Remark XIII.9.4 A consideration similar to that performed in Remark XIII.9.1 applies to the representation formula (XIII.9.48). ■

Exercise XIII.9.1 Let $\boldsymbol{v}$ be any generalized solution to problem (XIII.5.2). Show that $\boldsymbol{v}$ and the corresponding pressure field p obey the asymptotic conditions (9 .1). *Hint*: Consider (XIII.5.2) as a Stokes problem in $\mathbb{R}^3_{\pm}$ with a body force $-\boldsymbol{v}\cdot\nabla\boldsymbol{v}$. Then apply the estimate of Theorem IV.3.2, Theorem IV.3.3 together with the embedding inequalities of Theorem II.3.4.

Exercise XIII.9.2 Let the assumptions of Theorem XIII.9.1 be fulfilled. Show that the first derivatives of the velocity field $\boldsymbol{v}$ satisfy the following asymptotic estimate.

$$D_k\boldsymbol{v}(x) = O(|x|^{-2}), \quad |x| \to \infty, \quad k = 1, 2, 3.$$

Hint: From the representation formula (XIII.9.3)–(XIII.9.4) we have

$$v_j(x) = \frac{1}{\nu}\int_{\mathbb{R}^3_+} G_{ij}(x,y)v_l(y)D_l v_i(y)dy + B_i T_{i3}(\boldsymbol{G}_j, g_j)(x,0) + \varphi_i(x).$$

Differentiate both sides of this relation and argue as in the proof of (X.8.28). Finally, use the asymptotic estimates given in $(\text{IV}.3.50)_1$ and in Lemma XIII.9.3.

Remark XIII.9.5 In the case of a two-dimensional aperture domain, the asymptotic structure of any generalized solution $\boldsymbol{v}$ obeying the energy inequality and corresponding to small flux Φ is known if the aperture S is symmetric around the axis orthogonal to S, see Galdi, Padula, & Solonnikov (1996); see also Nazarov (1996). In such a case, $\boldsymbol{v}$ and the associated pressure p admit the following representation for large $|x|$

$$\left.\begin{aligned} \boldsymbol{v}(x) &= \boldsymbol{V}_{\pm}(x) + O(|x|^{-1-\alpha}) \\ p(x) &= P_{\pm}(x) + O(|x|^{-2-\alpha}) \end{aligned}\right. \quad x \in \mathbb{R}^2_{\pm}, \quad \alpha \in (0,1),$$

where $\boldsymbol{V}_{\pm}, P_{\pm}$ is a suitable Jeffery–Hamel solution for the half-plane $\mathbb{R}^2_{\pm}$, corresponding to Φ (Rosenhead, 1940). Specifically, in a polar coordinate system $(r, \theta_{\pm})$ with the origin at $x = 0$ and θ counted from the positive [respectively, negative] x_1-axis, we have

$$\begin{aligned} \boldsymbol{V}_{\pm}(x) &= \pm\Phi\frac{f_{\pm}(\theta)}{r}\boldsymbol{e}_r \\ P_{\pm}(x) &= \pm\Phi\frac{\mathcal{P}(f_{\pm})}{r} + \text{const.} \end{aligned}$$

Here, $\boldsymbol{e}_r$ is the unit vector in the r direction, $f_{\pm}(\theta)$ is a symmetric solution to a suitable second-order nonlinear ordinary differential problem with

$$\max_{\theta\in[0,2\pi]}\left\{|f_{\pm}(\theta)|,|f'_{\pm}(\theta)|\right\}$$

sufficiently small, and $\mathcal{P}$ a known function of $f_{\pm}$. ■

XIII.10 Notes for the Chapter

Section XIII.1. The first, significant contribution to the solvability of Leray's problem is due to Amick (1977, 1978)[1] who reduced the proof of existence to the resolution of a well-known variational problem relating to the stability of Poiseuille flow in a pipe; see Remark XIII.3.5. Our definition of generalized solution is essentially taken from his work. However, Amick left out the investigation of uniqueness while he examined the asymptotic structure of solutions only in some particular cases. A rich and detailed analysis of Leray's problem and, more generally, of the problem of flow in domains having outlets to infinity of bounded (not necessarily constant) cross sections is due to the St Petersburg School. We refer, in particular, to the papers of Ladyzhenskaya & Solonnikov (1980), Solonnikov (1983), Kapitanskiĭ (1982), Nazarov & Pileckas (1983, 1990), Kapitanskiĭ & Pileckas (1983) and the literature cited therein. One of the main features of this approach is, as we already observed, that condition $(\mathrm{XIII.1.4})_1$ is replaced with suitable "growth" assumptions at large distances.

Existence of solutions in two-dimensional, suitably symmetric channels with "islands" has been investigated by Morimoto & Fujita (2002). In this connection, we refer the reader to the other contributions by the same authors provided in the bibliography cited therein, and pertaining to more general flow regions, as well as to the review article of Morimoto (2007).

A different approach to existence (and uniqueness) in a two-dimensional infinite channel, based on the theory of semi-Fredholm operators, has been taken by Rabier (2002a, 2002b). More precisely, this author shows the existence of solutions "around" a Poiseuille flow corresponding to *any* given flux Φ and to symmetric data, whereas for data that are not necessarily symmetric, the same result holds provided $3|\Phi|/(4\nu) < 4647$.

The problem of existence and uniqueness of flow in curved pipes has been addressed by Galdi & Robertson (2005).

Section XIII.2. The method used here has been inspired by the work of Ladyzhenskaya & Solonnikov (1980). However, all main results, such as Theorem XIII.2.1, are due to me.

[1] A previous attempt of Ladyzhenskaya, unfortunately, failed, cf. *Dokl. Akad. Nauk SSSR*, **124** (1959) 551-553.

Section XIII.3. The methods and results of this section are due to me.

Section XIII.4. The main results of this section, Theorem XIII.4.1 and Corollary XIII.4.1, are due to me. They are obtained by coupling the ideas of Horgan & Wheeler (1978) with those of Ladyzhenskaya & Solonnikov (1980). In particular, Lemma XIII.4.2 and Lemma XIII.4.3 are special cases of analogous results of Ladyzhenskaya & Solonnikov. Somewhat weaker estimate can be obtained from the work of Horgan (1978) and Ames & Payne (1989). Spatial decay estimates in outlets with cross sections that reduce to a point at large distances have been obtained by Iosif'jan (1979).

In the case $n = 2$, the straight semi-infinite cylinder reduces to the semi-infinite channel $\{x_1 \in (0,d)\,, x_2 > 0\}$, where the decay of the solution (to the appropriate data at infinity) has been shown to be of exponential type, for small data at least. It is interesting to observe that if, instead, we consider the three-dimensional semi-infinite layer, $\{x_1 \in (0,d)\,, x_2 \in \mathbb{R}\,, x_3 > 0\}$, solutions only show a power-like decay (for small data), that turns out to be optimal; see Pileckas (2002), Nazarov & Pileckas (1999b).

Section XIII.5. The aperture flow problem was formulated by Heywood (1976, §6). To him is due the generalized formulation given in Definition XIII.5.1.

Properties of the pressure field in weighted spaces, analogous to those furnished in Lemma XIII.5.1 in L^3-spaces, are proved by Borchers, Galdi, & Pileckas (1993, Theorem 3.1).

Section XIII.6. Results of this section are due to me. The idea of exploiting the energy inequality in the study of uniqueness is taken from Borchers, Galdi, & Pileckas (1993).

Section XIII.7. Existence of generalized solutions for the aperture flow problem was first proved by Heywood (1976, Theorem 17) under the assumption of small flux. This restriction was successively removed by Ladyzhenskaya & Solonnikov (1977). Other significant results in this direction, in classes of solutions that are more regular than weak ones at large distances, are due to Borchers & Pileckas (1992) and Chang (1993); cf. also Coscia & Patria (1992), Chang (1992). In particular, Borchers & Pileckas (1992) prove existence for the formulation where the flux condition is replaced by the prescription of the pressure drop $p_+ - p_-$.

Lemma XIII.7.1 and Theorem XIII.7.4 are due to me. Similar results can be found in Borchers & Pileckas (1992) and Borchers, Galdi, & Pileckas (1993).

There is much literature concerning steady flow in domains with exits having unbounded cross sections, more general than aperture domains. In addition to the papers already quoted, we wish to mention the work of Pileckas (1981, 1984) and Galdi & Sohr (1992). Special regards are deserved by the contribution of Amick & Fraenkel (1980); cf. also Amick (1979). Here the authors analyze existence and pointwise asymptotic decay of plane flow in domains with two outlets Ω_i of unbounded cross section Σ_i, $i = 1, 2$, under very general assumptions on the way in which Σ_i may grow at large distances.

For instance, Ω_i can be a symmetric channel of the type

$$\Omega_i = \{x \in \mathbb{R}^2 :\ x_2 \in \Sigma_i(x_1),\ x_1 > 0\} \tag{$*$}$$

with

$$\Sigma_i = (-x_1^k, x_1^k), \quad k \geq 0. \tag{$**$}$$

Existence is proved either with small or arbitrary flux, depending on the rate at which Σ_i widens at infinity (that is, in case $(*)$, $(**)$, on the value of k). It is interesting to observe that, as expected (cf. Remark VI.4.6), if this rate is too slow, the pressure field becomes unbounded at infinity. Such a detailed study is made possible since, in the two-dimensional case, it is possible to use conformal-mapping techniques. Important questions, however, left out of this work are uniqueness and decay order of solutions.

The problem of existence, uniqueness and asymptotic behavior of solutions in domains Ω with $m \geq 2$ outlets Ω_i, $i = 1, \ldots m$, whose cross-section becomes unbounded in such a way that the Dirichlet integral of the velocity field is infinite, has been recently studied and solved in a series of papers by Pileckas (1996a, 1996b, 1996c, 1997) and Nazarov & Pileckas (1997), in the particular case when each Ω_i is a body of revolution of the type

$$\{x \in \mathbb{R}^n :\ x_n > 0,\ |x'| < g_i(x_n)\}, \quad n = 2, 3.$$

Here $g_i(x_n)$ are smooth functions satisfying the following "growth" conditions

(i) $g_i(t) \geq g_0 = \text{const.} > 0;\ \ |g_i(t_2) - g_i(t_1)| \leq M|t_2 - t_1|$

(ii) $\displaystyle \lim_{t\to\infty} \frac{dg_i}{dt} = 0;\ \ \left|\frac{dg_i}{dt}\right| \leq M$

(iii) $\displaystyle \int_0^\infty g_i^{-(n-1)(q-1)-q}(t)dt < \infty$, some $q > 2$; $\displaystyle \int_0^\infty g_i^{-(n+1)}(t)dt = \infty.$

Notice that (iii) implies that the velocity field has an infinite Dirichlet integral, see the Introduction to Chapter VI. Moreover, requirements (i) and (ii) exclude the case of aperture domains Then, for $n = 2$, if the total flux is sufficiently small one shows the existence of a unique solution and the validity of corresponding sharp asymptotic estimates. For $n = 3$ existence and asymptotic decay are shown for *arbitrary* values of the total flux, provided g_i satisfies the further conditions

$$\int_0^\infty g_i^{4/3}(t)dt = \infty$$

$$\left|g_i^{1/3}(t)\frac{dg_i}{dt}\right| \leq \gamma \ll 1, \quad \text{for sufficiently large } t.$$

As expected, uniqueness only holds for small flux. For detailed result and further references, we refer the reader to the review article of Pileckas (1996c).

Section XIII.8. The results of this section are due to me. Results similar to those proved in Lemma XIII.8.2 are given by Borchers & Pileckas (1992) under the following assumption on the velocity field $\boldsymbol{v}$

$$\int_{\Omega} |x|^{1+\varepsilon} \nabla \boldsymbol{v} : \nabla \boldsymbol{v} < \infty, \quad \text{for some } \varepsilon > 0.$$

Section XIII.9. Asymptotic representation formulas similar to (XIII.9.25) and (XIII.9.48) are furnished by Borchers & Pileckas (1992). Results of the same type as those showed in Theorem XIII.9.1 and Theorem XIII.9.2 can be derived from Borchers, Galdi, & Pileckas (1993, Theorem 4.1).

Existence, uniqueness, and asymptotic behavior of solutions to the plane aperture domain problem in the case of symmetric aperture have been given, independently, by Nazarov (1996) and Galdi, Padula, & Solonnikov (1996).

Finally, existence, uniqueness, and asymptotic behavior of solutions in two-dimensional domains constituted by a semi-infinite channel merging into a half-plane, have been addressed by Nazarov, Sequeira, & Videman (2001, 2002).

E 'a canzuncella
d' 'aruta nuvella,
guardate, vedite . . . fenesce accussi'.

S. DI GIACOMO, A' Testa d'Aruta, vv. 42-44

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