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Ramsey-type problems in orientations of graphs

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Abstract

The Ramsey number $R(H)$ of a graph H is the minimum number n such that there exists a graph G on n vertices with the property that every two-coloring of its edges contains a monochromatic copy of H. In this work we devise some numerical bounds for a few variants of this notion when H is an acyclic oriented graph and restrictions are imposed on the graph G. In particular, we study the *oriented Ramsey problem* for an acyclic oriented graph $\vec{H}$, in which we require that every orientation $\vec{G}$ of the graph G contains a copy of $\vec{H}$. We also study the threshold function for this problem in random graphs. Finally, we consider the *isometric* case, in which we require the copy to be isometric, by which we mean that, for every two vertices $x, y \in V(\vec{H})$ and their respective copies x', y' in $\vec{G}$, the distance between x and y is equal to the distance between x' and y'. Our approach to these problems makes use of the hypergraph container method applied to random graphs.

Keywords: Ramsey theory, random graphs, directed graphs, orientations of graphs, container method.

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Chapter 1

Introduction

A classical problem of combinatorics consists of estimating the Ramsey number $R(H)$ of a graph H, which is defined as the smallest number n such that there exists a graph G on n vertices with the property that every two-coloring of its edges contains a monochromatic copy of H. Widely known results of Erdös [10] and Erdös and Szekeres [12] state that

$$2^{n/2} \leqslant R(K_n) \leqslant 2^{2n}.$$

Despite many recent improvements on these bounds, they have remained almost unchanged. However, there has been more success on determining $R(H)$ for some other specific choices of H, e.g.: cycles and paths.

In Chapter 2, we study the following variant of this problem for directed graphs. Given an oriented graph $\vec{H}$, we define $\vec{R}(\vec{H})$ as the smallest number n such that there exists a graph G on n vertices with the property that every orientation of its edges contains a copy of $\vec{H}$. Observe that $\vec{H}$ must be acyclic, since every undireced graph has an acyclic orientation. In the aforementioned chapter, we survey some known bounds for $\vec{R}(\vec{H})$ for some choices of $\vec{H}$, and devise some new bounds of our own. Some of our results depend on the notion of Ramsey numbers of *ordered graphs*, studied independently by Conlon, Fox, Lee, and Sudakov [9] and Balko, Cibulka, Král and Kynčl [2].

In Chapter 3, we study the oriented Ramsey problem in the binomial random graph $G(n, p)$, which is the random graph in which each edge appears with probability p, independently of each other edge. Making use of the hypergraph container method of Balogh, Morris and Samotij [3] and Saxton and Thomason [23], and inspired by some ideas from Nenadov and Steger [17] and Hàn, Retter, Rödl and Schacht [14], we prove a version for acyclic oriented graphs of the following celebrated result of Rödl and Ruciński [19].

Theorem (Rödl and Ruciński [19])**.** Let $r \geqslant 2$ and H be a graph. There exists a constant $C = C(H, r)$ such that, if $p \geqslant Cn^{-1/m_2(H)}$, then

$$\lim_{n\to\infty} \mathbb{P}[G(n, p) \to (H)_r] = 1,$$

where $G(n, p) \to (H)_r$ denotes that every two-coloring of the edges of $G(n, p)$ contains a monochromatic copy of H.

Finally, in Chapter 4, we introduce the *isometric* oriented Ramsey problem, in which we want every orientation to contain not only a copy, but an *isometric* copy, which means that for every two vertices $x, y \in V(\vec{H})$ and their respective copies x', y' in $\vec{G}$, the distance between x and y is equal to the distance between x' and y'. Moreover, the distance is taken with respect to the underlying undirected graphs. Using the same techniques of Chapter 3, we prove an upper bound on the isometric Ramsey number of acyclic orientations of cycles.

Chapter 2

The Oriented Ramsey Problem

2.1 Definitions

An **ordered graph** G is a pair $G = (G', <_G)$ where G' is a graph and $<_G$ is a total ordering of the vertices of G'. For convenience we write $V(G) := V(G')$ and $E(G) := E(G')$. When a graph G is equipped with a total ordering of its vertices, we will simply refer to G as an ordered graph without further qualifications.

An ordered graph G is said to **contain** an ordered graph H if there exists a function $\phi : V(H) \to V(G)$ such that, for every $x, y \in V(H)$, we have $\phi(x) <_G \phi(y)$ if and only if $x <_H y$, and $\{i, j\}$ is an edge of H only if $\{\phi(i), \phi(j)\}$ is an edge of G. In this case, we call ϕ a **monotone embedding**.

A **directed graph** or **digraph** $\vec{G}$ is a pair $\vec{G} = (V, E)$ where V is a set of vertices and E is a set such that $E \subseteq (V \times V) \setminus \{(v, v) : v \in V\}$. Just as in the case of undirected graphs, an element of E is called an **edge**; however, it may also be called an **arc** to differ from the undirected case. An **oriented graph** $\vec{G} = (V, E)$ is a digraph where $(u, v) \in E$ implies $(v, u) \notin E$ for every $u, v \in V$. Moreover, an oriented graph $\vec{G} = (V_1, E_1)$ is said to be an **orientation** of a graph $G = (V_2, E_2)$ if $V_1 = V_2$ and, for every $u, v \in V_1 = V_2$, we have $\{u, v\} \in E_2$ if and only if $(u, v) \in E_1$ or $(v, u) \in E_1$. In this case, we say that G is the **underlying undirected graph** of $\vec{G}$. Furthermore, when $\vec{G}$ is an oriented graph, we write G to denote the underlying undirected graph of $\vec{G}$. To avoid confusion, we will always denote a digraph by a capital letter with $\to$. Finally, we call an orientation of a complete graph a **tournament**.

Given graphs H and G, we write $G \to H$ if every two-coloring of the edges of G contains a monochromatic copy of H. If the graphs H and G are ordered graphs, we write $G \xrightarrow{\text{ord}} H$ to denote that the monochromatic copy is *ordered*. The **Ramsey number** $R(H)$ of a graph H is defined as

$$R(H) := \inf \{n \in \mathbb{N} : K_n \to H\}.$$

When the graph H is equipped with a total ordering, the **ordered Ramsey number** $R_<(H)$ can be defined analogously.

Given an oriented graph $\vec{H}$ and a graph G, we write $G \to \vec{H}$ if every orientation of G has an oriented copy of $\vec{H}$. The **oriented Ramsey number** $\vec{R}(\vec{H})$ is defined as

$$\vec{R}(\vec{H}) := \inf \left\{n \in \mathbb{N} : K_n \to \vec{H}\right\}.$$

2.2 Bounds for specific graphs

In this section we derive bounds for the oriented Ramsey number of some specific classes of graphs.

It is well-known, and not difficult to prove, that a tournament is acyclic if and only if it is transitive, and that there is exactly one transitive tournament on n vertices up to isomorphism (see e.g. Section 4.2 of [6]). Therefore, we can denote by $\vec{K}_k$ the acyclic tournament on k verties. To our knowledge, the following is the first bound to appear of the oriented Ramsey number of an oriented graph.

Theorem 2.1 (Erdős and Moser [11])**.** Let $\vec{K}_k$ be the acyclic orientation of K_k for some positive integer k. We have

$$2^{(k-1)/2} \leqslant \vec{R}(\vec{K}_k) \leqslant 2^{k-1}.$$

We remark that the lower bound above can be proved by a standard application of the probabilistic method (see e.g.: Theorem 1 of [11] or Proposition 1.1.1 of [1]), and the upper bound can be proved by induction on k, observing that every acyclic oriented graph has a topological ordering.

Since clearly $\vec{R}(\vec{H}) \leqslant \vec{R}(\vec{K}_h)$ for every acyclic oriented graph $\vec{H}$ on h vertices, we obtain the following corollary.

Corollary 2.2. Let $\vec{H}$ be an acyclic oriented graph on h vertices. We have $\vec{R}(\vec{H}) \leqslant 2^{h-1}$. In particular, the oriented Ramsey number $\vec{R}(\vec{H})$ is finite.

Definition 2.3. We denote by $\vec{P}_k$ the **directed path** of length k, which is the oriented graph with vertex set $V(\vec{P}_k) := [k+1]$ and edge set $E(\vec{P}_k) := \{(i, i+1) : i \in [k]\}$.

The following theorem is a known result of Gallai and Roy (see, for example, Theorem 14.5 of Bondy and Murty [5]).

Theorem 2.4 (Gallai-Roy Theorem)**.** If G is a graph such that $\chi(G) = k+1$, then $G \to \vec{P}_k$.

Proof sketch. Consider an arbitrary orientation $\vec{G}$ of G. Color each vertex $v \in V(G)$ with the number of vertices contained in the largest directed path in $\vec{G}$ which begins in v. Observe that this is a proper coloring. Therefore, the largest directed path contained in $\vec{G}$ has at least $\chi(G) = k$ vertices. $\square$

Since $\chi(K_{k+1}) = k+1$, we have thus completely determined the oriented Ramsey number of $\vec{P}_k$.

Corollary 2.5. For every $k \in \mathbb{N}$, we have $\vec{R}(\vec{P}_k) = k+1$.

We now give a bound for the oriented Ramsey number of $\vec{H}$ depending on the Ramsey number of H. Our proof will be inspired in the proof of Theorem 2.1 of [4], but, in reality, this idea already appeared in Cochand and Duchet [7] and in Rödl and Winkler [21]. First, we need a bound for the Ramsey number of ordered graphs.

Theorem 2.6 (Conlon, Fox, Lee, and Sudakov [9])**.** There exists a constant c such that, for every ordered graph H on n vertices, we have

$$R_{<}(H) \leqslant R(H)^{c \log^2 n}.$$

More precise bounds for $R_{<}(H)$ for specific classes of ordered graphs can be found in Conlon, Fox, Lee, and Sudakov [9] and Balko, Cibulka, Král and Kynčl [2].

Theorem 2.7. There exists a constant c such that the following holds. Let $\vec{H}$ be an acyclic oriented graph with h vertices and H its underlying undirected graph. There exists orderings $<_0$ and $<_1$ of the vertices of H such that, for $H_0 = (H, <_0)$ and $H_1 = (H, <_1)$, we have

$$\vec{R}(\vec{H}) \leqslant R_<(H_0) + R_<(H_1) \leqslant 2R(H)^{c \log^2(h)}.$$

Proof. Let $\vec{F}$ be the oriented graph formed by two disjoint copies of $\vec{H}$, in which one has reversed edges. More formally, let $\vec{F}$ be the oriented graph with vertex set

$$V(\vec{F}) := V(\vec{H}) \times \{0, 1\}$$

and edge set

$$E(\vec{F}) := \left\{((u,0),(v,0)), ((v,1),(u,1)) : (u,v) \in E(\vec{H})\right\}.$$

Since $\vec{H}$ is acyclic, the oriented graph $\vec{F}$ is also acyclic. Therefore, there exists an ordering $<$ of the vertices of $\vec{F}$ such that $u < v$ if $(u,v) \in E(\vec{F})$. Let F be the (ordered) underlying undirected graph of $\vec{F}$ equipped with the ordering $<$. Let $<_0$ be an ordering of the vertices of H such that, for $x, y \in V(H)$, we have $x <_0 y$ if and only if $(x, 0) < (y, 0)$. Define $<_1$ analogously. Let $H_0 := (H, <_0)$ and $H_1 := (H, <_1)$. Clearly, we have

$$R_<(F) \leqslant R_<(H_0) + R_<(H_1).$$

Let $\prec$ be an arbitrary ordering of the vertices of K_N. We thus consider K_N to be an ordered complete graph. By Theorem 2.6, there exists a number N such that

$$N = R_<(F) \leqslant R_<(H_0) + R_<(H_1) \leqslant 2R(H)^{c \log^2(h)}$$

and $K_N \xrightarrow{\text{ord}} F$.

Now it suffices to prove that $K_N \to \vec{H}$. Let $\vec{K}$ be an arbitrary orientation of K_N. Color the edges of K_N in the following way: an edge $\{u, v\} \in E(K_N)$ with $u \prec v$ is colored blue if $(u, v) \in E(\vec{K})$ and red otherwise. By the choice of N, there exists an ordered monochromatic copy of F in K_N. Let $\phi : V(F) \to V(K_N)$ be the monotone embedding of this copy. If the copy of F in K_N is blue, then the set of vertices $\left\{\phi((v,0)) : v \in V(\vec{H})\right\}$ induces a directed copy of $\vec{H}$ in $\vec{K}$ with the color blue. Otherwise, if the copy is red, then the set of vertices $\left\{\phi((v,1)) : v \in V(\vec{H})\right\}$ induces a copy with the color red. In either case we have proved $K_N \to \vec{H}$, as desired. □

Remark 2.8. The proof of Theorem 2.7 shows that the orderings $<_0$ and $<_1$ of $V(\vec{H})$ can be taken to be the topological ordering of $\vec{H}$ and the reverse topological ordering of $\vec{H}$, respectively.

The following theorem gives an exact formula for the classical Ramsey number of a cycle C_k on k vertices. One can find this result as Theorem 2 of a survey from Radziszowski [18].

Theorem 2.9 (Rosta [22], Faudree and Schelp [13])**.** We have

$$R(C_k) = \begin{cases} 6, & \text{if } k = 3 \text{ or } k = 4 \\ 2k - 1, & \text{if } k \geqslant 5 \text{ is odd} \\ 3k/2 - 1 & \text{if } k \geqslant 6 \text{ is even.} \end{cases}$$

Therefore, it clearly holds that $R(C_k) \leqslant 2k$ for every $k \geqslant 3$. We now get the following corollary.

Corollary 2.10. There exists a constant c such that the following holds. Let $k \geqslant 3$ and let $\vec{H}$ be an acyclic orientation of the cycle on k vertices C_k. We have

$$\vec{R}(\vec{H}) \leqslant 2(2k)^{c\log^2(k)}.$$

Chapter 3

An Oriented Ramsey Theorem for Random Graphs

3.1 Introduction

For a graph H, oriented or not, we denote by $m_2(H)$ its 2-**density**, defined as

$$m_2(H) := \max_{F \subseteq H, v(F) \geqslant 3} \frac{e(F) - 1}{v(F) - 2}.$$

The following is a famous result of Rödl and Ruciński [19], which determines, for an undirected graph H, the threshold function for $G(n,p) \to (H)_r$. Here we state only the 1-statement.

Theorem 3.1 (Rödl and Ruciński [19])**.** Let $r \geqslant 2$ and H be a graph. There exists a constant $C = C(H,r)$ such that, if $p \geqslant Cn^{-1/m_2(H)}$, then

$$\lim_{n \to \infty} \mathbb{P}[G(n,p) \to (H)_r] = 1.$$

In this chapter, we prove the following theorem, adapting the arguments of Nenadov and Steger [17], who gave a short proof of Theorem 3.1 using the method of containers, developed independently by Balogh, Morris and Samotij [3] and Saxton and Thomason [23]. The technique of using hypergraph containers in random graphs for Ramsey problems was further developed by Hàn, Retter, Rödl and Schacht [14], Rödl, Ruciński and Schacht [20] and Conlon, Dellamonica, La Fleur, Rödl and Schacht [8]. Our approach is also inspired by theirs. Some resemblance to their arguments is to be expected.

Theorem 3.2. Let $\vec{H}$ be an acyclic oriented graph. There exists a constant $C = C(\vec{H})$ such that, if $p \geqslant Cn^{-1/m_2(\vec{H})}$, then

$$\lim_{n \to \infty} \mathbb{P}\left[G(n,p) \to \vec{H}\right] = 1.$$

3.2 Saturation results for oriented graphs

First we need to prove a simple quantitative strengthening of Theorem 2.7.

Theorem 3.3. For every $\varepsilon > 0$ and every acyclic oriented graph $\vec{H}$ on h vertices, there exists a number $\delta = \delta(\vec{H}, \varepsilon)$ such that, for every $n \geqslant \vec{R}(\vec{H})$, the following holds. For every

$F \subseteq E(K_n)$, if there exists an orientation $\vec{F}$ of F such that $\vec{F}$ has at most $\varepsilon\binom{n}{h}$ copies of $\vec{H}$, then

$$|E(K_n) \setminus F| \geqslant \delta n^2.$$

Proof. Set $R := \vec{R}(\vec{H})$ and let $n_0 := R$. Fix $n \geqslant n_0$. Let $F \subseteq E(K_n)$ be such that there exists an orientation $\vec{F}$ of F with at most $\varepsilon\binom{n}{h}$ copies of $\vec{H}$. Let $\vec{K}$ be an orientation of K_n which agrees with the orientation $\vec{F}$ of F. Let

$$\mathcal{S} := \left\{ S \in \binom{V(\vec{K})}{R} : E(\vec{K}[S]) \subseteq \vec{F} \right\}.$$

By definition of R, every R-element subset of the vertices of $\vec{K}$ contains at least one copy of $\vec{H}$. Moreover, every copy of $\vec{H}$ in $\vec{K}$ is contained in at most $\binom{n-h}{R-h}$ R-element subsets. Therefore, double-counting on the pairs $(S, \vec{H}')$ where $S \in \mathcal{S}$ and $\vec{H}'$ is a copy of $\vec{H}$ contained in S yields

$$|\mathcal{S}| \leqslant \varepsilon\binom{n-h}{R-h}\binom{n}{h}.$$

This implies that the set $\overline{\mathcal{S}}$ defined as

$$\overline{\mathcal{S}} := \binom{V(\vec{K})}{R} \setminus \mathcal{S}$$

satisfies

$$\begin{aligned} |\overline{\mathcal{S}}| &= \binom{n}{R} - |\mathcal{S}| \\ &\geqslant \binom{n}{R} - \varepsilon\binom{n-h}{R-h}\binom{n}{h} \\ &= \binom{n}{R}\left(1 - \varepsilon\frac{\binom{n-h}{R-h}}{\binom{n}{R}}\binom{n}{h}\right) \\ &= \binom{n}{R}\left(1 - \varepsilon\binom{R}{h}\right). \end{aligned}$$

Observe that, by definition of $\overline{\mathcal{S}}$, every set $S \in \overline{\mathcal{S}}$ induces at least one edge $e \in E(\vec{K}) \setminus \vec{F}$. Moreover, every edge $e \in E(\vec{K}) \setminus \vec{F}$ is contained in at most $\binom{n-2}{R-2}$ R-element subsets. Now, double-counting on the pairs (S, e) where $S \in \overline{\mathcal{S}}$ and $e \in E(\vec{K}[S])$ we get

$$\left|E(\vec{K}) \setminus \vec{F}\right| \geqslant \frac{|\overline{\mathcal{S}}|}{\binom{n-2}{R-2}} \geqslant \left(1 - \varepsilon\binom{R}{h}\right)\frac{\binom{n}{R}}{\binom{n-2}{R-2}} > \left(1 - \varepsilon\binom{R}{h}\right)\frac{1}{R^2}n^2 = \delta n^2,$$

by setting $\delta := \left(1 - \varepsilon\binom{R}{h}\right)\frac{1}{R^2}$.

The desired result now follows by observing $|E(K_n) \setminus F| = \left|E(\vec{K}) \setminus \vec{F}\right|$. $\square$

Remark 3.4. The proof of Theorem 3.3 shows that δ can be taken as

$$\delta := \left(1 - \varepsilon\binom{R}{h}\right)\frac{1}{R^2}.$$

Moreover, if $\varepsilon \leqslant (1/2)\binom{R}{h}^{-1}$, then $\delta \geqslant 1/(2R^2)$. Finally, n_0 can be taken as $n_0 = R$.

3.3 A container lemma for digraphs

Let $\mathcal{H}$ be a ℓ-uniform hypergraph. For a set $J \subseteq V(\mathcal{H})$, we define the **degree of** J by

$$d(J) := |e \in E(\mathcal{H}) : J \subseteq e| \, .$$

For a vertex $v \in V(\mathcal{H})$, we let $d(v) := d(\{v\})$. For $j \in [\ell]$, we also define the **maximum** j**-degree** of a vertex $v \in V(\mathcal{H})$ by

$$d^{(j)}(v) := \max\left\{ d(J) : v \in J \in \binom{V(\mathcal{H})}{j} \right\}.$$

We denote the average of $d^{(j)}(v)$ for all $v \in V(\mathcal{H})$ by

$$d_j := \frac{1}{v(\mathcal{H})} \sum_{v \in V(\mathcal{H})} d^{(j)}(v).$$

Note that d_1 is the average degree of $\mathcal{H}$. Finally, for $\tau > 0$, we define δ_j as

$$\delta_j := \frac{d_j}{d_1 \tau^{j-1}}$$

and the **co-degree function** $\delta(\mathcal{H}, \tau)$ by

$$\delta(\mathcal{H}, \tau) := 2^{\binom{\ell}{2}-1} \sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}} \delta_j.$$

We now state a condensed version of the Container Lemma, as expressed in Saxton and Thomason [23]. This version can be found as Theorem 2.1 in [14].

Theorem 3.5 ([23], Corollary 3.6)**.** Let $0 < \varepsilon, \tau < 1/2$. Let $\mathcal{H} = (V, E)$ be a ℓ-uniform hypergraph. Suppose that τ satisfies $\delta(\mathcal{H}, \tau) \leqslant \varepsilon/12\ell!$. Then for integers $K = 800\ell(\ell!)^3$ and $s = \lfloor K \log(1/\varepsilon) \rfloor$ the following holds.

For every independent set $I \subseteq V$ in $\mathcal{H}$ there exists a s-tuple $T = (T_1, \ldots, T_s)$ of subsets of V and a subset $C = C(T) \subseteq V$ depending only on S such that

(a) $\bigcup_{i \in [s]} T_i \subseteq I \subseteq C$,

(b) $e(C) \leqslant \varepsilon \cdot e(\mathcal{H})$, and

(c) for every $i \in [s]$ we have $|T_i| \leqslant K\tau\,|V|$.

Here we prove a version of the container lemma for $\vec{H}$-free orientations of graphs. First, we need the following definitions.

Definition 3.6. Let $\vec{H}$ be an oriented graph and let $n \in \mathbb{N}$. Denote by $\vec{D}_n$ the digraph with vertex set $[n]$ and edge set

$$E(\vec{D}_n) := ([n] \times [n]) \setminus \{(v, v) : v \in [n]\} \, .$$

Definition 3.7 ([16], Definition 3.5)**.** Let $\vec{H}$ be an oriented graph with ℓ edges and let $n \in \mathbb{N}$. The hypergraph $\mathcal{D}(n, \vec{H}) = (\mathcal{V}, \mathcal{E})$ is a ℓ-uniform hypergraph with vertex set $\mathcal{V} := E(\vec{D}_n)$ and edge set

$$\mathcal{E} := \left\{ B \in \binom{\mathcal{V}}{\ell} : \text{the edges of } B \text{ form a digraph isomorphic to } \vec{H} \right\}.$$

Definition 3.8. Let $\vec{H}$ be an oriented graph with h vertices. In what follows, we denote by $\mathrm{emb}_{\vec{H}} := e(\mathcal{D}(h, \vec{H}))$ the number of copies of $\vec{H}$ in $\vec{D}_h$.

Our container lemma for $\vec{H}$-free orientations of graphs is as follows. We give a more general statement than needed for this chapter only because we are going to need this result also in Chapter 4.

Theorem 3.9 (Container lemma for $\vec{H}$-free orientations)**.** Let $0 < \varepsilon < 1/2$ and let $\vec{H}$ be an acyclic oriented graph with ℓ edges. There exists positive integers s and K and a real number $\delta > 0$ such that, for every $n \geqslant \vec{R}(\vec{H})$, the following holds.

Suppose $0 < \tau < 1/2$ satisfies $\delta(\mathcal{D}(n, \vec{H}), \tau) \leqslant \varepsilon/(12\ell!\,)$. For every graph G on n vertices such that $G \not\to \vec{H}$ there exists a s-tuple $T = (T_1, \ldots, T_s) \subseteq E(G)$ and a set $C = C(T) \subseteq E(K_n)$ depending only on T such that

(a) $\bigcup_{i \in [s]} T_i \subseteq E(G) \subseteq C$,

(b) $|E(K_n) \setminus C| \geqslant \delta n^2$, and

(c) $\left| \bigcup_{i \in [s]} T_i \right| \leqslant sK\tau n^2$.

Proof. Let $\vec{H}$ be an acyclic oriented graph with ℓ edges. Let n_0 and δ be as given by Theorem 3.3 for $\varepsilon_0 := \varepsilon \cdot \mathrm{emb}_{\vec{H}}$ and $\vec{H}$. By Remark 3.4, we can take $n_0 = \vec{R}(\vec{H})$. Fix $n \geqslant n_0$ and set $\mathcal{H} := \mathcal{D}(n, \vec{H})$. Since $\delta(\mathcal{H}, \tau) \leqslant \varepsilon_0/12\ell!$, Theorem 3.5 gives us numbers s and K for $\mathcal{H}$, ε and τ. Let G be a graph on n vertices such that $G \not\to \vec{H}$. There exists an orientation $\vec{G}$ of G such that $\vec{G}$ contains no copy of $\vec{H}$. Therefore, the set $E(\vec{G})$ is an independent set of $\mathcal{H}$. Let $\vec{T} = (\vec{T}_1, \ldots, \vec{T}_s)$ be a s-tuple of oriented edges and $\vec{C} = \vec{C}(\vec{T})$ such as Theorem 3.5 gives for $E(\vec{G})$. For $i \in [s]$, let T_i be the underlying set of undirected edges of $\vec{T}_i$. Define C analogously for $\vec{C}$. By item (a) of Theorem 3.5, we have

$$\bigcup_{i \in [s]} T_i \subseteq E(G) \subseteq C.$$

Observe now that $\mathrm{emb}_{\vec{H}}$ counts the number of copies of $\vec{H}$ in any subset of h vertices of $\vec{D}_n$, whence it follows that

$$e(\mathcal{H}) = \binom{n}{h} \mathrm{emb}_{\vec{H}}\,. \tag{3.1}$$

Therefore, by item (b) of Theorem 3.5 we conclude that $\vec{C}$ has at most $\varepsilon e(\mathcal{H}) = \varepsilon_0 \binom{n}{h}$ copies of $\vec{H}$. By the choice of $\delta = \delta(\vec{H}, \varepsilon_0)$, Theorem 3.3 now gives

$$|E(K_n) \setminus C| \geqslant \delta n^2.$$

Finally, we get by item (c)

$$\left|\bigcup_{i\in[s]} T_i\right| \leqslant sK\tau v(\mathcal{H}) \leqslant sK\tau n^2.$$

Therefore, there exists a s-tuple T and a set C as promised. This finishes the proof. $\square$

Remark 3.10. In light of Remark 3.4, we see that in Theorem 3.9 the value of δ can be taken as

$$\delta := \left(1 - \varepsilon \cdot \mathrm{emb}_{\vec{H}} \binom{R}{h}\right) \frac{1}{R^2},$$

where $R := \vec{R}(\vec{H})$. Moreover, if $\varepsilon \leqslant \left(2\,\mathrm{emb}_{\vec{H}} \binom{R}{h}\right)^{-1}$, then $\delta \geqslant 1/(2R^2)$. Finally, the values of s and K are just as in Theorem 3.5.

To apply Theorem 3.9, it is necessary to prove a bound on $\delta(\mathcal{D}(n,\vec{H}),\tau)$ for a suitable value of τ. This is done by the following lemma.

Lemma 3.11. Let $\vec{H}$ be an oriented graph with h vertices and $\ell \geqslant 2$ edges. Let also $D_\tau \geqslant 1$ and write $\tau := D_\tau n^{-1/m_2(\vec{H})}$. We have

$$\delta(\mathcal{D}(n,\vec{H}),\tau) \leqslant 2^{\binom{\ell}{2}} h^{h-2} D_\tau^{-1}.$$

Proof. For convenience, set $\mathcal{H} := \mathcal{D}(n,\vec{H})$. Let $J \subseteq V(\mathcal{H})$. Define

$$V_J := \bigcup_{(a,b)\in J} \{a,b\} \subseteq [n].$$

Note that (V_J, J) is the subdigraph of $\vec{D}_n$ induced by the set of edges J. For a set $S \subseteq [n]\setminus V_J$ such that $|S| = h - |V_J|$, let $\mathrm{emb}_{\vec{H}}(J,S)$ denote the number of copies $\vec{F}$ of $\vec{H}$ such that $V(\vec{F}) = V_J \cup S$ and $J \subseteq E(\vec{F})$. Since $\mathrm{emb}_{\vec{H}}(J,S)$ is the same number for any choice of S as above, we can write only $\mathrm{emb}_{\vec{H}}(J)$ to refer to this number.

Recall that $d(J)$ is the number of copies of $\vec{H}$ in $\vec{D}_n$ which contain the set J. Observe now that

$$d(J) = \binom{n-|V_J|}{h-|V_J|} \mathrm{emb}_{\vec{H}}(J). \tag{3.2}$$

For every $j \in [\ell]$, let

$$f(j) := \min_{\vec{H}'\subseteq\vec{H},\, e(\vec{H}')=j} v(H'). \tag{3.3}$$

It follows from (3.2) that

$$d(J) = \binom{n-|V_J|}{h-|V_J|} \mathrm{emb}_{\vec{H}}(J) \leqslant \binom{n-f(j)}{h-f(j)} \mathrm{emb}_{\vec{H}}(J).$$

Note now that, for every $e \in V(\mathcal{H})$, we have $d^{(1)}(e) = d(e) = \binom{n-2}{h-2} \mathrm{emb}_{\vec{H}}(\{e\})$. Therefore, the average d_1 of all $d^{(1)}(e)$ satisfies $d_1 = \binom{n-2}{h-2} \mathrm{emb}_{\vec{H}}(\{e\})$, for some fixed $e \in V(\mathcal{H})$. It follows that

$$\frac{d(J)}{d_1} \leqslant \frac{\binom{n-f(j)}{h-f(j)} \mathrm{emb}_{\vec{H}}(J)}{\binom{n-2}{h-2} \mathrm{emb}_{\vec{H}}(\{e\})} \leqslant \frac{\binom{n-f(j)}{h-f(j)}}{\binom{n-2}{h-2}} = \frac{(h-2)(h-3)\dots(h-f(j)+1)}{(n-2)(n-3)\dots(n-f(j)+1)} \leqslant \left(\frac{h}{n}\right)^{f(j)-2}.$$

Therefore, we have $d^{(j)}(v)/d_1 \leqslant h^{f(j)-2}n^{2-f(j)}$. Since $f(j) \leqslant h$, this gives us

$$\frac{d_j}{d_1} = \frac{1}{v(\mathcal{H})} \sum_{v \in V(\mathcal{H})} \frac{d^{(j)}(v)}{d_1} \leqslant \frac{1}{v(\mathcal{H})} \sum_{v \in V(\mathcal{H})} h^{f(j)-2}n^{2-f(j)} = h^{f(j)-2}n^{2-f(j)} \leqslant h^{h-2}n^{2-f(j)}.$$

We furthermore obtain

$$\delta_j = \frac{d_j}{d_1 \tau^{j-1}} \leqslant h^{h-2}n^{2-f(j)}\tau^{1-j} \leqslant h^{h-2}n^{2-f(j)+(j-1)/m_2(\vec{H})}D_\tau^{1-j}. \tag{3.4}$$

Observe now that, by definition of $m_2(\vec{H})$, we have $m_2(\vec{H}) \geqslant (j-1)/(f(j)-2)$. From this we may derive $2 - f(j) + (j-1)/m_2(\vec{H}) \leqslant 0$. Therefore, we can conclude from (3.4) that

$$\delta_j \leqslant h^{h-2}D_\tau^{1-j} \leqslant h^{h-2}D_\tau^{-1}. \tag{3.5}$$

Now we can finally bound the co-degree function $\delta(\mathcal{H}, \tau)$ by observing that

$$\delta(\mathcal{H}, \tau) = 2^{\binom{\ell}{2}-1} \sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}} \delta_j \leqslant 2^{\binom{\ell}{2}-1} h^{h-2} D_\tau^{-1} \sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}} \leqslant 2^{\binom{\ell}{2}} h^{h-2} D_\tau^{-1}.$$

This finishes the proof. □

3.4 Proof of Theorem 3.2

Now it remains to see how the methods developed in Section 3.3 can be used to prove Theorem 3.2. First, we need to prove one more lemma. For convenience, given numbers n, s and t, define

$$\mathcal{T}(n, s, t) := \left\{ (T_1, \ldots, T_s) : \bigcup_{i \in [s]} T_i \subseteq E(K_n) \text{ and } \left| \bigcup_{i \in [s]} T_i \right| \leqslant t \right\}.$$

Lemma 3.12. Let $0 < \varepsilon < 1/2$ and let $\vec{H}$ be an acyclic oriented graph with ℓ edges. There exists positive integers s and K and a real number $\delta > 0$ such that, for every $n \geqslant \vec{R}(\vec{H})$, the following holds. For every $0 < \tau < 1/2$ satisfying $\delta(\mathcal{D}(n, \vec{H}), \tau) \leqslant \varepsilon/(12\ell!)$ and for any choice of $p \in (0, 1)$, we have

$$\mathbb{P}[G(n,p) \not\to \vec{H}] \leqslant \exp(-\delta n^2 p) \left(1 + \sum_{k=1}^{t} \left(\frac{e 2^{s-1} n^2 p}{k} \right)^k \right),$$

where $t := sK\tau n^2$.

Proof. Let s, K and δ be as given by Theorem 3.9 for ε, τ and $\vec{H}$. If a graph G satisfies $G \not\to \vec{H}$, by Theorem 3.9 there exists a s-tuple $T = (T_1, \ldots, T_s) \in \mathcal{T}(n, s, t)$ and a set $C(T) \subseteq E(K_n)$ such that

$$\bigcup_{i \in [s]} T_i \subseteq E(G) \subseteq C(T) \tag{3.6}$$

and

$$D(T) := |E(K_n) \setminus C(T)| \geqslant \delta n^2.$$

Moreover, since $E(G) \subseteq C(T)$, we have

$$E(G) \cap D(T) = \emptyset. \tag{3.7}$$

Let $\mathcal{G}$ be the family of all graphs G on n vertices such that $G \not\to \vec{H}$. For a s-tuple $T = (T_1, \ldots, T_s) \in \mathcal{T}(n,s,t)$, let

$$\mathcal{G}_T' := \{G = G^n : T_i \subseteq E(G)\ \forall i \in [s]\},$$

and let

$$\mathcal{G}_T'' := \{G = G^n : E(G) \cap D(T) = \emptyset\}.$$

Observations (3.6) and (3.7) show that

$$\mathcal{G} \subseteq \bigcup_{T \in \mathcal{T}(n,s,t)} \mathcal{G}_T' \cap \mathcal{G}_T''.$$

As the sets T_i and $D(T)$ have empty intersection for every $i \in [s]$, it follows that the events $[G(n,p) \in \mathcal{G}_T']$ and $[G(n,p) \in \mathcal{G}_T'']$ are independent. We conclude

$$\mathbb{P}[G(n,p) \in \mathcal{G}] \leqslant \sum_{T \in \mathcal{T}(n,s,t)} \mathbb{P}\left[G(n,p) \in \mathcal{G}_T'\right] \cdot \mathbb{P}\left[G(n,p) \in \mathcal{G}_T''\right].$$

Since $D(T) \geqslant \delta n^2$ for every $T \in \mathcal{T}(n,s,t)$, we have

$$\mathbb{P}\left[G(n,p) \in \mathcal{G}_T''\right] \leqslant (1-p)^{\delta n^2} \leqslant \exp(-\delta n^2 p).$$

Moreover, we also have

$$\sum_{T \in \mathcal{T}(n,s,t)} \mathbb{P}\left[G(n,p) \in \mathcal{G}_T'\right] \leqslant \sum_{T \in \mathcal{T}(n,s,t)} p^{\left|\bigcup_{i\in[s]} T_i\right|}.$$

It follows that

$$\mathbb{P}[G(n,p) \in \mathcal{G}] \leqslant \exp(-\delta n^2 p) \cdot \sum_{T \in \mathcal{T}(n,s,t)} p^{\left|\bigcup_{i\in[s]} T_i\right|}. \tag{3.8}$$

We now proceed to bound the sum in (3.8). For every integer k such that $0 \leqslant k \leqslant t$, define

$$S(k) := \left\{T \in \mathcal{T}(n,s,t) : \left|\bigcup_{i\in[s]} T_i\right| = k\right\}.$$

Observe that $|S(k)| = \binom{\binom{n}{2}}{k}(2^s)^k$. Indeed, there are $\binom{\binom{n}{2}}{k}$ ways of choosing k edges from $E(K_n)$, and $(2^s)^k$ ways of assigning these edges to the sets of the s-tuples, which gives the desired equation. Therefore,

$$\sum_{T \in \mathcal{T}(n,s,t)} p^{\left|\bigcup_{i\in[s]} T_i\right|} = \sum_{k=0}^{t} |S(k)|\, p^k \leqslant \sum_{k=0}^{t} \binom{\binom{n}{2}}{k}(2^s)^k p^k \leqslant 1 + \sum_{k=1}^{t}\left(\frac{e 2^{s-1} n^2 p}{k}\right)^k.$$

Because of (3.8), this finishes the proof. □

Remark 3.13. The same quantitative remarks of Remark 3.10 hold for Lemma 3.12.

We now have all the necessary elements to prove Theorem 3.2, which we now do.

Proof of Theorem 3.2. Let ε be sufficiently small. Suppose $n \geqslant \vec{R}(\vec{H})$. In Lemma 3.11, set

$$D_\tau := \frac{12\ell!\, 2^{\binom{\ell}{2}} h^{h-2}}{\varepsilon},$$

and let $\tau := D_\tau n^{-1/m_2(\vec{H})}$. By Lemma 3.11, this yields $\delta(\mathcal{D}(n,\vec{H}),\tau) \leqslant \varepsilon/(12\ell!\,)$, where $\ell := e(\vec{H})$. We are, therefore, in the conditions of Lemma 3.12. Let s, K and δ be as in Lemma 3.12 for ε, τ and $\vec{H}$. Set $c := sKD_\tau$ and $p := Cn^{-1/m_2(\vec{H})}$, for some constant C sufficiently large with respect to c. By Lemma 3.12, we have

$$\mathbb{P}[G(n,p) \not\to \vec{H}] \leqslant \exp(-\delta n^2 p)\left(1 + \sum_{k=1}^{t}\left(\frac{e2^{s-1}n^2p}{k}\right)^k\right),$$

where $t := sK\tau n^2 = cn^{2-1/m_2(\vec{H})} = cn^2p/C$.

Let $f(k)$ be the function which maps k to $(eb/k)^k$, where $b = 2^{s-1}n^2p$. Note that this is the function in the final sum above. Since $2^{s-1}n^2p \geqslant cn^2p/C$ for C sufficiently large with respect to s and c, Fact A.2 yields

$$\begin{aligned}
1 + \sum_{k=1}^{cn^2p/C}\left(\frac{e2^{s-1}n^2p}{k}\right)^k &\leqslant 1 + \frac{cn^2p}{C}\left(\frac{Ce2^{s-1}n^2p}{cn^2p}\right)^{cn^2p/C} && C \text{ sufficiently large}\\
&\leqslant n^2\left(\frac{Ce2^{s-1}}{c}\right)^{cn^2p/C} && C \text{ sufficiently large}\\
&= n^2\exp\left(\frac{cn^2p}{C}(\log C + 1 + (s-1)\log 2 - \log c)\right)\\
&= n^2\exp\left(n^2p\frac{c(\log C + 1 + (s-1)\log 2 - \log c)}{C}\right)\\
&\leqslant n^2\exp\left(n^2p\frac{\delta}{3}\right) && C \text{ sufficiently large}\\
&\leqslant \exp\left(\frac{\delta n^2p}{2}\right) && n \text{ sufficiently large.}
\end{aligned}$$

We may now conclude

$$\mathbb{P}[G(n,p) \in \mathcal{G}] \leqslant \exp(-\delta n^2p)\exp\left(\frac{\delta n^2p}{2}\right) = \exp\left(-\frac{\delta n^2p}{2}\right) = o(1),$$

as desired. $\square$

Chapter 4

The Isometric Oriented Ramsey Number

4.1 Introduction

For an undirected graph G, we denote by $d_G(u,v)$ the distance between two vertices $u,v \in V(G)$. Given two oriented graphs $\vec{H}$ and $\vec{F}$, we say that a copy $f : V(\vec{H}) \to V(\vec{F})$ of $\vec{H}$ in $\vec{F}$ is an **isometric copy** if $d_H(x,y) = d_F(f(x), f(y))$ for every $x, y \in V(\vec{H})$. Note that the distance is taken with respect to the underlying undirected graphs.

Given an oriented graph $\vec{H}$ and a graph G, we write $G \xrightarrow{\text{iso}} \vec{H}$ if every orientation of G has an isometric oriented copy of $\vec{H}$. The **isometric oriented Ramsey number** $\vec{R}_{\text{iso}}(\vec{H})$ is defined as

$$\vec{R}_{\text{iso}}(\vec{H}) := \inf\left\{ n \in \mathbb{N} : \exists G = G^n \text{ such that } G \xrightarrow{\text{iso}} \vec{H} \right\}.$$

The following result states that the isometric oriented Ramsey number of acyclic oriented graphs is always finite.

Theorem 4.1 ([4], Theorem 2.1)**.** For every acyclic oriented graph $\vec{H}$, the isometric oriented Ramsey number $\vec{R}_{\text{iso}}(\vec{H})$ is finite.

The problem of estimating $\vec{R}_{\text{iso}}(\vec{H})$ for acyclic oriented graphs $\vec{H}$ first appeared in Banakh, Idzik, Pikhurko, Protasov and Pszczoła [4]. In this chapter we give an upper bound on $\vec{R}_{\text{iso}}(\vec{H})$ when $\vec{H}$ is an acyclic orientation of the cycle on k vertices C_k. In particular, we prove the following theorem.

Theorem 4.2. There exists a positive constant c such that the following holds. Let $\vec{H}$ be an acyclic orientation of C_k and set $R := \vec{R}(\vec{H})$. Then

$$\vec{R}_{\text{iso}}(\vec{H}) \leqslant ck^{12k^3} R^{8k^2}. \tag{4.1}$$

Remark 4.3. In light of Corollary 2.10 and Theorem 4.2, one readily sees that there exists constants c_1 and c_2 such that, for any acyclic orientation $\vec{H}$ of the cycle C_k, we have

$$\vec{R}_{\text{iso}}(\vec{H}) \leqslant c_1 k^{c_2 k^3}.$$

The approach employed in this chapter to prove Theorem 4.2 is inspired by the proof of Theorem 1.1 in Hàn, Retter, Rödl, and Schacht [14]. In what follows, we will use the notation already developed in Chapter 3.

4.2 Proof of Theorem 4.2

We begin by observing that, for every orientation $\vec{H}$ of the cycle C_k, we have

$$m_2(\vec{H}) = m_2(C_k) = \frac{k-1}{k-2}. \tag{4.2}$$

This will justify the choice of constants we will make in the rest of this section.

We now prove the following Lemma, which is a slightly improved version of Lemma 3.11 adjusted for orientations of cycles. Our proof makes uses of some arguments and results of the proof of Lemma 3.11. The reader is recommended to read first that proof if some steps in the following proof are unclear.

Lemma 4.4. Let $\vec{H}$ be an orientation of the cycle C_k. Let also $D_\tau \geqslant 1$ and define τ as $\tau := D_\tau n^{-(k-2)/(k-1)}$. For every $n \geqslant D_\tau^{(k-1)^2}$, we have

$$\delta(\mathcal{D}(n,\vec{H}),\tau) \leqslant 2^{\binom{k}{2}} k^{k-2} D_\tau^{-(k-1)}.$$

Proof. Fix $j \in [k]$. Let $f(j)$ be as defined in (3.3). Since $\vec{H}$ is an orientation of the cycle on k vertices, we have $f(j) = j+1$ for every $j \in [k-1]$ and $f(k) = k$. Furthermore, by (3.4) we obtain

$$\delta_j \leqslant k^{k-2} n^{2-f(j)+(j-1)(k-2)/(k-1)} D_\tau^{1-j}.$$

Therefore, for $j \in [k-1]$ we have

$$\begin{aligned} \delta_j &\leqslant k^{k-2} n^{1-j+(j-1)(k-2)/(k-1)} D_\tau^{1-j} \\ &= k^{k-2} n^{-(j-1)/(k-1)} D_\tau^{1-j} \\ &\leqslant k^{k-2} k^{-1/(k-1)} D_\tau^{-1} \\ &\leqslant k^{k-2} n^{-1/(k-1)}. \end{aligned} \tag{4.3}$$

Moreover, we obtain from (3.5) that

$$\delta_k \leqslant k^{k-2} D_\tau^{-(k-1)}. \tag{4.4}$$

Since, by assumption, we have $n \geqslant D_\tau^{(k-1)^2}$, inequalities (4.3) and (4.4) now give us

$$\max_{j \in [k]} \delta_j = \delta_k.$$

We therefore conclude

$$\delta(\mathcal{H},\tau) = 2^{\binom{\ell}{2}-1} \sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}} \delta_j \leqslant 2^{\binom{\ell}{2}-1} k^{k-2} D_\tau^{-(k-1)} \sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}} \leqslant 2^{\binom{\ell}{2}} k^{k-2} D_\tau^{-(k-1)},$$

as promised. $\square$

We may now proceed to the proof of Theorem 4.2. The proof will be as follows. We will consider the random graph $G(n,p)$ and, imitating the proof of Theorem 3.2, we will prove that, with positive probability, we have $G(n,p) \xrightarrow{\text{iso}} \vec{H}$, for a number n that satisfies (4.18) and a suitable choice of p. Our strategy will be to prove that the graph $G(n,p)$ has girth at least k and satisfies $G(n,p) \to \vec{H}$ for an acyclic orientation $\vec{H}$ of C_k, which implies $G(n,p) \xrightarrow{\text{iso}} \vec{H}$.

A difference with regards to the proof of Theorem 3.2 is that, instead of taking $p = Cn^{-(k-2)/(k-1)}$ for a constant C, we will make $C = C(n)$ depend on n, which will allow us to get a not so large value for n.

Proof of Theorem 4.2. We begin by setting the following numbers we are going to use in the proof:

$$\begin{aligned}
\varepsilon &= \frac{1}{2R^k}, && && (4.5)\\
D_\tau &= \frac{4\cdot 2^{k/2}\cdot k^2}{\varepsilon^{1/(k-1)}} && \leqslant 8R^2k^{k+2}, && (4.6)\\
K &= 800k(k!\,)^3 && \leqslant 800k^{3k+1}, && (4.7)\\
s &= \lfloor K\log(1/\varepsilon)\rfloor && \leqslant 1600k^{3k+2}R, && (4.8)\\
D_p &= KD_\tau s^2 10R^2\log(5R^2), && && (4.9)\\
n &= D_p^{k^2}, && && (4.10)\\
\tau &= D_\tau n^{-\frac{k-2}{k-1}}, && && (4.11)\\
p &= D_p n^{-\frac{k-2}{k-1}}. && && (4.12)
\end{aligned}$$

Observe that, for some positive constant $c > 0$, we have

$$D_p \leqslant c\cdot k^{10k+7}R^8 \leqslant k^{12k}R^8,$$

which implies

$$n \leqslant ck^{12k^3}R^{8k^2}.$$

Let us first prove the following claim. The proof goes just as in the proof of Claim 3.1 of [14].

Claim 4.5. We have $\mathbb{P}[\mathrm{girth}(G(n,p)) \geqslant k] \geqslant \exp(-kD_p^{k-1}n)$.

Proof of Claim 4.8. Let $\mathcal{C}(n,k)$ be the set of all cycles $C \subseteq E(K_n)$ of length at most $k-1$. Let

$$X := |\{C \in \mathcal{C}(n,k) : C \subseteq E(G(n,p))\}|$$

be the random variable counting the number of cycles of length at most $k-1$ in $G(n,p)$. For each cycle $C \subseteq E(K_n)$ of length at most $k-1$, let X_C be the indicator function of the event $E_C := \{C \subseteq E(G(n,p))\}$. Clearly, X is the sum of all such C. Therefore,

$$\mathbb{E}[X] = \sum_{C\in\mathcal{C}(n,k)} p^{|C|} = \sum_{j=3}^{k-1}\frac{(j-1)!}{2}\binom{n}{j}p^j \leqslant \sum_{j=3}^{k-1}\frac{(pn)^j}{2j} \leqslant \frac{k}{6}(pn)^{k-1} = \frac{k}{6}D_p^{k-1}n.$$

Moreover, the set of all graphs G on n vertices such that $C \nsubseteq E(G)$ is a monotone decreasing property. Therefore, using the FKG inequality (Corollary A.4), and applying inequality (A.2), we get

$$\mathbb{P}[\mathrm{girth}(G(n,p)) \geqslant k] = \prod_{C\in\mathcal{C}(n,k)}(1-p^{|C|}) \geqslant \prod_{C\in\mathcal{C}(n,k)}\exp\left(-\frac{p^{|C|}}{1-p^{|C|}}\right) \geqslant \exp\left(-\frac{\mathbb{E}[X]}{1-p^3}\right).$$

One may now easily check that

$$1 - p^3 = 1 - n^{-(k-2)/(k-1)+1/k^2} > 1/6,$$

since $n > 11$, and the claim follows. □

We now prove the following claim. Our proof will be similar to that of Theorem 3.2, with the difference that the calculations will be more involved.

Claim 4.6. We have

$$\mathbb{P}[G(n,p) \to \vec{H}] \geqslant 1 - \exp\left(-\frac{n^2 p}{4R^2}\right).$$

Proof of Claim 4.9. We want to apply Theorem 3.9. We begin by observing that our choice of D_τ implies

$$D_\tau^{k-1} = \left(\frac{4 \cdot 2^{k/2} \cdot k^2}{\varepsilon^{1/(k-1)}}\right)^{k-1} = \frac{4^{k-1} \cdot 2^{\binom{k}{2}} \cdot k^{2(k-1)}}{\varepsilon} \geqslant \frac{12 \cdot 2^{\binom{k}{2}} \cdot k^{k-2} \cdot k!}{\varepsilon}.$$

Hence, since clearly $n \geqslant D_\tau^{(k-1)^2}$, Lemma 4.4 now yields $\delta(\mathcal{D}(n,\vec{H}),\tau) \leqslant \varepsilon/(12k!\,)$. Observe, moreover, that

$$\varepsilon = \frac{1}{2R^k} \leqslant \frac{1}{2k!\,\binom{R}{k}} \leqslant \frac{1}{2\,\mathrm{emb}_{\vec{H}}\,\binom{R}{k}}.$$

Now Lemma 3.12, together with Remark 3.10 and Remark 3.13, gives us

$$\mathbb{P}[G(n,p) \not\to \vec{H}] \leqslant \exp\left(-\frac{n^2 p}{2R^2}\right)\left(1 + \sum_{j=1}^{sK\tau n^2}\left(\frac{e2^{s-1}n^2 p}{j}\right)^j\right). \tag{4.13}$$

We now proceed to bound the sum in (4.27). Let $f(k)$ be the function which maps j to $(eb/j)^j$, where $b = 2^{s-1}n^2 p$. Note that this is the function in the final sum above. Observe moreover that

$$2^{s-1}n^2 p = 2^{s-1}D_p n^{-\frac{k-2}{k-1}} n^2 \geqslant sKD_\tau n^{-\frac{k-2}{k-1}} n^2 = sK\tau n^2,$$

whence it follows by Fact A.2 that

$$1 + \sum_{k=1}^{sK\tau n^2}\left(\frac{e2^{s-1}n^2 p}{k}\right)^k \leqslant 1 + sK\tau n^2\left(\frac{e2^{s-1}n^2 p}{sK\tau n^2}\right)^{sK\tau n^2} = 1 + sK\tau n^2\left(\frac{e2^{s-1}D_p}{sKD_\tau}\right)^{sK\tau n^2}.$$

Moreover, since

$$sK\tau = sKD_\tau n^{-(k-2)/(k-1)} = sKD_\tau D_p^{-k^2(k-2)/(k-1)} \leqslant sKD_\tau D_p^{-1} < 1,$$

we obtain

$$
\begin{aligned}
1 + sK\tau n^2 \left(\frac{e2^{s-1}D_p}{sKD_\tau}\right)^{sK\tau n^2} &\leqslant n^2 \left(\frac{e2^{s-1}D_p}{sKD_\tau}\right)^{sK\tau n^2} \\
&= n^2 \exp\left(sK\tau n^2 \cdot \log \frac{e2^{s-1}D_p}{sKD_\tau}\right) \\
&= n^2 \exp\left(n^2 p \cdot \frac{sKD_\tau}{D_p} \cdot \log \frac{e2^{s-1}D_p}{sKD_\tau}\right) \\
&= n^2 \exp\left(n^2 p \cdot \frac{sKD_\tau}{D_p} \left(\log(e2^{s-1}) + \log \frac{D_p}{sKD_\tau}\right)\right).
\end{aligned}
$$

Observe now that

$$
\frac{sKD_\tau}{D_p} \log(e2^{s-1}) = \frac{\log(e2^{s-1})}{s10R^2 \log(5R^2)} \leqslant \frac{1}{10R^2 \log(5R^2)} \leqslant \frac{1}{10R^2}. \tag{4.14}
$$

Let now $x := D_p/(sKD_\tau)$ and set $y := x/s$. Since the function $\log(x)/x$ is decreasing for $x > e$, we have $\log(x)/x \leqslant \log(y)/y$. Note also that $y = 10R^2 \log(5R^2) \leqslant (5R^2)^2$ by inequality (A.3). Therefore, applying (A.3) once again, we obtain $\log y \leqslant \log(5R^2)$. These observations allow us to conclude that

$$
\frac{\log(D_p/(sKD_\tau))}{D_p/(sKD_\tau)} = \frac{\log x}{x} \leqslant \frac{\log y}{y} \leqslant \frac{\log(5R^2)}{10R^2 \log(5R^2)} = \frac{1}{10R^2}. \tag{4.15}
$$

Hence, by inequalities (4.28) and (4.29) we obtain

$$
n^2 \exp\left(n^2 p \cdot \frac{sKD_\tau}{D_p} \left(\log(e2^{s-1}) + \log \frac{D_p}{sKD_\tau}\right)\right) \leqslant n^2 \exp\left(\frac{n^2 p}{5R^2}\right).
$$

Observe now that

$$
\frac{n^2 p}{\log n} = \frac{D_p^{2k^2} D_p D_p^{-k^2(k-2)/(k-1)}}{k^2 \log(D_p)} \geqslant \frac{D_p^{2k^2} D_p D_p^{-k^2}}{k^2 D_p} = \frac{D_p^{k^2}}{k^2} \geqslant \frac{D_p}{k^2} \geqslant \frac{10R^2 D_\tau}{k^2} \geqslant 40R^2.
$$

From this we obtain

$$
2 \log n \leqslant \frac{n^2 p}{20R^2} = \frac{n^2 p}{4R^2} - \frac{n^2 p}{5R^2},
$$

which implies

$$
n^2 \exp\left(\frac{n^2 p}{5R^2}\right) \leqslant \exp\left(\frac{n^2 p}{4R^2}\right).
$$

All our work so far therefore implies

$$
1 + \sum_{j=1}^{sK\tau n^2} \left(\frac{e2^{s-1} n^2 p}{j}\right)^j \leqslant \exp\left(\frac{n^2 p}{4R^2}\right),
$$

which, in view of (4.27), yields

$$
\mathbb{P}[G(n,p) \not\to \vec{H}] \leqslant \exp\left(-\frac{n^2 p}{4R^2}\right).
$$

This finishes the proof of the claim. □

Now, in view of Claim 4.8 and Claim 4.9, we can deduce

$$\begin{aligned}\mathbb{P}[\operatorname{girth} G(n,p) \geqslant k \text{ and } G(n,p) \to \vec{H}] &\geqslant \mathbb{P}[\operatorname{girth}(G(n,p)) \geqslant k] + \mathbb{P}[G(n,p) \to \vec{H}] - 1 \\ &\geqslant \exp(-kD_p^{k-1}n) - \exp\left(-\frac{n^2p}{4R^2}\right).\end{aligned} \tag{4.16}$$

Since we also have

$$\frac{n^2p}{4R^2} = n \cdot \frac{D_p^{k^2+1-k^2(1-1/(k-1))}}{4R^2} > n \cdot \frac{D_p^{k+1}}{4R^2} > kD_p^{k-1}n,$$

we may now conclude from (4.30) that

$$\mathbb{P}[\operatorname{girth} G(n,p) \geqslant k \cap G(n,p) \to \vec{H}] > 0,$$

which finishes the proof. □

4.3 Isometric Oriented Ramsey Number of the Oriented Path

We begin by observing that, for every orientation $\vec{H}$ of the path P_k, we have

$$m_2(\vec{H}) = m_2(P_k) = 1. \tag{4.17}$$

In this section we prove the following theorem.

Theorem 4.7. There exists a positive constant c such that, for every $k \geqslant 1$, we have

$$\vec{R}_{\mathrm{iso}}(\vec{H}) \leqslant ck^{12k^2}R^{8k}, \tag{4.18}$$

where $R := \vec{R}(\vec{P}_k)$.

Proof of Theorem 4.2. We begin by setting the following numbers we are going to use in the proof:

$$\begin{aligned}
\varepsilon &= \frac{1}{2R^{k+1}}, && && (4.19)\\
D_\tau &= \frac{(k+1)^{k-1} \cdot 2^{\binom{k}{2}} \cdot 12k!}{\varepsilon} && \leqslant 24k^{k^2/\log k+k+1}R^{k+1}, && (4.20)\\
K &= 800k(k!)^3 && \leqslant 800k^{3k+1}, && (4.21)\\
s &= \lfloor K\log(1/\varepsilon)\rfloor && \leqslant 3200k^{3k+2}R, && (4.22)\\
D_p &= KD_\tau s^2 10R^2\log(5R^2), && && (4.23)\\
n &= D_p^k, && && (4.24)\\
\tau &= D_\tau n^{-1}, && && (4.25)\\
p &= D_p n^{-1}. && && (4.26)
\end{aligned}$$

Observe that, for some positive constant $c > 0$, we have

$$D_p \leqslant c \cdot k^{10k+7} R^8 \leqslant k^{12k} R^8,$$

which implies

$$n \leqslant ck^{12k^2} R^{8k}.$$

Let us first prove the following claim. The proof goes just as in the proof of Claim 3.1 of [14].

Claim 4.8. We have $\mathbb{P}[\mathrm{girth}(G(n,p)) \geqslant k] \geqslant \exp(-kD_p^{k-1})$.

Proof of Claim 4.8. Let $\mathcal{C}(n,k)$ be the set of all cycles $C \subseteq E(K_n)$ of length at most $k-1$. Let

$$X := |\{C \in \mathcal{C}(n,k) : C \subseteq E(G(n,p))\}|$$

be the random variable counting the number of cycles of length at most $k-1$ in $G(n,p)$. For each cycle $C \subseteq E(K_n)$ of length at most $k-1$, let X_C be the indicator function of the event $E_C := \{C \subseteq E(G(n,p))\}$. Clearly, X is the sum of all such C. Therefore,

$$\mathbb{E}[X] = \sum_{C \in \mathcal{C}(n,k)} p^{|C|} = \sum_{j=3}^{k-1} \frac{(j-1)!}{2} \binom{n}{j} p^j \leqslant \sum_{j=3}^{k-1} \frac{(pn)^j}{2j} \leqslant \frac{k}{6}(pn)^{k-1} = \frac{k}{6} D_p^{k-1} n.$$

Moreover, the set of all graphs G on n vertices such that $C \not\subseteq E(G)$ is a monotone decreasing property. Therefore, using the FKG inequality (Corollary A.4), and applying inequality (A.2), we get

$$\mathbb{P}[\mathrm{girth}(G(n,p)) \geqslant k] = \prod_{C \in \mathcal{C}(n,k)} (1 - p^{|C|}) \geqslant \prod_{C \in \mathcal{C}(n,k)} \exp\left(-\frac{p^{|C|}}{1 - p^{|C|}}\right) \geqslant \exp\left(-\frac{\mathbb{E}[X]}{1 - p^3}\right).$$

One may now easily check that

$$1 - p^3 = 1 - n^{-(k-2)/(k-1)+1/k^2} > 1/6,$$

since $n > 11$, and the claim follows. $\square$

We now prove the following claim. Our proof will be similar to that of Theorem 3.2, with the difference that the calculations will be more involved.

Claim 4.9. We have

$$\mathbb{P}[G(n,p) \to \vec{H}] \geqslant 1 - \exp\left(-\frac{n^2 p}{4R^2}\right).$$

Proof of Claim 4.9. We want to apply Theorem 3.9. We begin by observing that our choice of D_τ implies

$$D_\tau^{k-1} = \left(\frac{4 \cdot 2^{k/2} \cdot k^2}{\varepsilon^{1/(k-1)}}\right)^{k-1} = \frac{4^{k-1} \cdot 2^{\binom{k}{2}} \cdot k^{2(k-1)}}{\varepsilon} \geqslant \frac{12 \cdot 2^{\binom{k}{2}} \cdot k^{k-2} \cdot k!}{\varepsilon}.$$

Hence, since clearly $n \geqslant D_\tau^{(k-1)^2}$, Lemma 4.4 now yields $\delta(\mathcal{D}(n, \vec{H}), \tau) \leqslant \varepsilon/(12k!\,)$. Observe, moreover, that

$$\varepsilon = \frac{1}{2R^k} \leqslant \frac{1}{2k!\binom{R}{k}} \leqslant \frac{1}{2\,\mathrm{emb}_{\vec{H}}\binom{R}{k}}.$$

Now Lemma 3.12, together with Remark 3.10 and Remark 3.13, gives us

$$\mathbb{P}[G(n,p) \not\to \vec{H}] \leqslant \exp\left(-\frac{n^2p}{2R^2}\right)\left(1+\sum_{j=1}^{sK\tau n^2}\left(\frac{e2^{s-1}n^2p}{j}\right)^j\right). \tag{4.27}$$

We now proceed to bound the sum in (4.27). Let $f(k)$ be the function which maps j to $(eb/j)^j$, where $b = 2^{s-1}n^2p$. Note that this is the function in the final sum above. Observe moreover that

$$2^{s-1}n^2p = 2^{s-1}D_pn^{-\frac{k-2}{k-1}}n^2 \geqslant sKD_\tau n^{-\frac{k-2}{k-1}}n^2 = sK\tau n^2,$$

whence it follows by Fact A.2 that

$$1+\sum_{k=1}^{sK\tau n^2}\left(\frac{e2^{s-1}n^2p}{k}\right)^k \leqslant 1+sK\tau n^2\left(\frac{e2^{s-1}n^2p}{sK\tau n^2}\right)^{sK\tau n^2} = 1+sK\tau n^2\left(\frac{e2^{s-1}D_p}{sKD_\tau}\right)^{sK\tau n^2}.$$

Moreover, since

$$sK\tau = sKD_\tau n^{-(k-2)/(k-1)} = sKD_\tau D_p^{-k^2(k-2)/(k-1)} \leqslant sKD_\tau D_p^{-1} < 1,$$

we obtain

$$\begin{aligned} 1+sK\tau n^2\left(\frac{e2^{s-1}D_p}{sKD_\tau}\right)^{sK\tau n^2} &\leqslant n^2\left(\frac{e2^{s-1}D_p}{sKD_\tau}\right)^{sK\tau n^2} \\ &= n^2\exp\left(sK\tau n^2\cdot\log\frac{e2^{s-1}D_p}{sKD_\tau}\right) \\ &= n^2\exp\left(n^2p\cdot\frac{sKD_\tau}{D_p}\cdot\log\frac{e2^{s-1}D_p}{sKD_\tau}\right) \\ &= n^2\exp\left(n^2p\cdot\frac{sKD_\tau}{D_p}\left(\log(e2^{s-1})+\log\frac{D_p}{sKD_\tau}\right)\right). \end{aligned}$$

Observe now that

$$\frac{sKD_\tau}{D_p}\log(e2^{s-1}) = \frac{\log(e2^{s-1})}{s10R^2\log(5R^2)} \leqslant \frac{1}{10R^2\log(5R^2)} \leqslant \frac{1}{10R^2}. \tag{4.28}$$

Let now $x := D_p/(sKD_\tau)$ and set $y := x/s$. Since the function $\log(x)/x$ is decreasing for $x > e$, we have $\log(x)/x \leqslant \log(y)/y$. Note also that $y = 10R^2\log(5R^2) \leqslant (5R^2)^2$ by inequality (A.3). Therefore, applying (A.3) once again, we obtain $\log y \leqslant \log(5R^2)$. These observations allow us to conclude that

$$\frac{\log(D_p/(sKD_\tau))}{D_p/(sKD_\tau)} = \frac{\log x}{x} \leqslant \frac{\log y}{y} \leqslant \frac{\log(5R^2)}{10R^2\log(5R^2)} = \frac{1}{10R^2}. \tag{4.29}$$

Hence, by inequalities (4.28) and (4.29) we obtain

$$n^2\exp\left(n^2p\cdot\frac{sKD_\tau}{D_p}\left(\log(e2^{s-1})+\log\frac{D_p}{sKD_\tau}\right)\right) \leqslant n^2\exp\left(\frac{n^2p}{5R^2}\right).$$

Observe now that

$$\frac{n^2p}{\log n} = \frac{D_p^{2k^2} D_p D_p^{-k^2(k-2)/(k-1)}}{k^2 \log(D_p)} \geqslant \frac{D_p^{2k^2} D_p D_p^{-k^2}}{k^2 D_p} = \frac{D_p^{k^2}}{k^2} \geqslant \frac{D_p}{k^2} \geqslant \frac{10R^2 D_\tau}{k^2} \geqslant 40R^2.$$

From this we obtain

$$2\log n \leqslant \frac{n^2p}{20R^2} = \frac{n^2p}{4R^2} - \frac{n^2p}{5R^2},$$

which implies

$$n^2 \exp\left(\frac{n^2p}{5R^2}\right) \leqslant \exp\left(\frac{n^2p}{4R^2}\right).$$

All our work so far therefore implies

$$1 + \sum_{j=1}^{sK\tau n^2} \left(\frac{e2^{s-1}n^2p}{j}\right)^j \leqslant \exp\left(\frac{n^2p}{4R^2}\right),$$

which, in view of (4.27), yields

$$\mathbb{P}[G(n,p) \not\to \vec{H}] \leqslant \exp\left(-\frac{n^2p}{4R^2}\right).$$

This finishes the proof of the claim. □

Now, in view of Claim 4.8 and Claim 4.9, we can deduce

$$\begin{aligned}\mathbb{P}[\operatorname{girth} G(n,p) \geqslant k \text{ and } G(n,p) \to \vec{H}] &\geqslant \mathbb{P}[\operatorname{girth}(G(n,p)) \geqslant k] + \mathbb{P}[G(n,p) \to \vec{H}] - 1 \\ &\geqslant \exp(-kD_p^{k-1}n) - \exp\left(-\frac{n^2p}{4R^2}\right).\end{aligned} \tag{4.30}$$

Since we also have

$$\frac{n^2p}{4R^2} = n \cdot \frac{D_p^{k^2+1-k^2(1-1/(k-1))}}{4R^2} > n \cdot \frac{D_p^{k+1}}{4R^2} > kD_p^{k-1}n,$$

we may now conclude from (4.30) that

$$\mathbb{P}[\operatorname{girth} G(n,p) \geqslant k \cap G(n,p) \to \vec{H}] > 0,$$

which finishes the proof. □

Chapter 5

Conclusions

In this work, we have seen some bounds for the oriented Ramsey number of acyclic oriented graphs (Chapter 2), and have shown how to apply the hypergraph container method to study the oriented Ramsey problem in random graphs (Chapter 3). Moreover, we introduced the concept of isometric oriented Ramsey number, and we showed how the container method applied to random graphs can be used to prove actual bounds on the isometric oriented Ramsey number of concrete graphs (Chapter 4).

We think our work leaves some interesting problems open for further research. Firstly, it would be interesting if better bounds were found for the oriented Ramsey number of concrete graphs. It is not clear how far from optimal are the bounds given by using ordered Ramsey numbers.

Secondly, one could also consider not only orientations of graphs, but also orientations *and* colorings of edges, and require the oriented copy to be monochromatic. We believe our techniques can easily handle this case, and we are already working on this.

Moreover, one could also try to apply the techniques of Chapter 4 to derive bounds for the isometric Ramsey number of other graphs, like paths and Moore graphs. Finally, in light of Theorem 2.4, one may ask whether the techniques of Chapter 4 can be used to find graphs with high girth and high chromatic number. We are also working on both of these directions.

Appendix A

Inequalities and Probability

In this appendix we describe some results involving inequalities and probability theory.

Fact A.1. The following inequalities hold.

$$1 + x \leqslant e^x \qquad \forall x \in \mathbb{R}, \tag{A.1}$$

$$1 - x \geqslant \exp\left(\frac{-x}{1-x}\right) \qquad \forall x \in [0,1), \tag{A.2}$$

$$\log x \leqslant \frac{x}{2} \qquad \forall x > 0. \tag{A.3}$$

One can easily check the following fact by taking derivatives.

Fact A.2. Let $c > 0$ be a constant and define the function $f(x) := (ec/x)^x$ for $x > 0$. The function $f(x)$ achieves its maximum value at $x = c$, and is monotonically increasing for $x \leqslant c$ and monotonically decreasing for $x \geqslant c$.

Let V be a finite set. A **property of graphs** with respect to V is a subset of the set of all graphs with vertex set V, closed under isomorphism. A property P with respect to V is said to be **monotone increasing** if, for every two graphs $H \in P$ and G with vertex set V and such that H is a subgraph of G, we have $G \in P$. Moreover, such a property P is said to be **monotone decreasing** if, for every two graphs H and $G \in P$ with vertex set V and such that H is a subgraph of G, we have $H \in P$.

The following theorem is a simplified version of what is known as FKG inequality. The interested reader is pointed to Chapter 6 of [1] or Section 2.2 of [15] to learn more.

Theorem A.3 (FKG Inequality, Theorem 6.3.3 [1])**.** Let P_1, P_2, Q_1 and Q_2 be graph properties, where P_1 and P_2 are monotone increasing and Q_1 and Q_2 are monotone decreasing. We have

$$\mathbb{P}[G(n,p) \in P_1 \cap P_2] \geqslant \mathbb{P}[G(n,p) \in P_1] \cdot \mathbb{P}[G(n,p) \in P_2],$$
$$\mathbb{P}[G(n,p) \in Q_1 \cap Q_2] \geqslant \mathbb{P}[G(n,p) \in Q_1] \cdot \mathbb{P}[G(n,p) \in Q_2].$$

By induction, one easily gets the following corollary.

Corollary A.4. Let P_1, $P_2, \ldots, P_n$ and Q_1, $Q_2, \ldots, Q_n$ be graph properties, where P_1,

$P_2, \ldots, P_n$ are monotone increasing and Q_1, $Q_2, \ldots, Q_n$ are monotone decreasing. We have

$$\mathbb{P}\left[G(n,p) \in \bigcap_{i=1}^{n} P_i\right] \geqslant \prod_{i=1}^{n} \mathbb{P}[G(n,p) \in P_i]$$

$$\mathbb{P}\left[G(n,p) \in \bigcap_{i=1}^{n} Q_i\right] \geqslant \prod_{i=1}^{n} \mathbb{P}[G(n,p) \in Q_i]$$

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