Ramsey-type problems in orientations of graphs

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Introduction

1. A bound for the *oriented Ramsey number* of an acyclic oriented graph, depending on the Ramsey number of its underlying undirected graph.
2. Threshold function for the oriented Ramsey problem in random graphs.
3. A bound for the *isometric Ramsey number* of an acyclic orientation of the cycle on k vertices, using the hypergraph container lemma.

Definitions

- Given a graph H, let
$$R(H) := \inf \{n \in \mathbb{N} : K_n \to (H)_2\},$$
where $K_n \to (H)_2$ means that every two-coloring of the edges of K_n contains a monochromatic copy of H.
- Given an oriented graph $\vec{H}$, let
$$\vec{R}(\vec{H}) := \inf \left\{n \in \mathbb{N} : K_n \xrightarrow{\text{or}} \vec{H}\right\},$$
where $K_n \xrightarrow{\text{or}} \vec{H}$ denotes that every orientation of K_n has an oriented copy of $\vec{H}$. Observe that $\vec{H}$ must be acyclic.
- Given an ordered graph $H_<$, let
$$R_<(H) := \inf \{n \in \mathbb{N} : K_n \to (H_<)_2\},$$
where $K_n \to (H_<)_2$ means that every two-coloring of the edges of K_n (with a fixed ordering) contains a monochromatic *ordered* copy of $H_<$.

Known bounds

- **Theorem 1** (Conlon, Fox, Lee, Sudakov [3]). There exists a constant c such that, for every ordered graph $H_<$ on h vertices, we have $R_<(H_<) \leqslant R(H)^{c \log^2 h}$.
- $\vec{K}_k$:= acyclic orientation of K_k for some positive integer k.
- **Theorem 2** (Erdős and Moser [4]).
$$2^{(k-1)/2} \leqslant \vec{R}(\vec{K}_k) \leqslant 2^{k-1}.$$
- Implication: $\vec{R}(\vec{H}) \leqslant 2^{h-1}$ if $\left|V(\vec{H})\right| = h$.
- $\vec{P}_k$:= directed path with k edges.
- **Theorem 3** (Corollary of Gallai and Roy).
$$\vec{R}(\vec{P}_k) = k + 1.$$

Our bound

- **Theorem 4**. There exists a universal constant c such that
$$\vec{R}(\vec{H}) \leqslant 2R(H)^{c \log^2(h)}.$$
- **Proof idea**. Let $\vec{F}$ be the oriented graph formed by two disjoint copies of $\vec{H}$, in which one has reversed edges. Let $F_<$ be the (ordered) underlying undirected graph of $\vec{F}$ equipped with a topological ordering of the vertices. We prove that, if $K_n \to F_<$, then $K_n \xrightarrow{\text{or}} \vec{H}$. The result follows by Theorem 1. □
- This proof is inspired in Theorem 2.1 of [2].
- **Corollary**. Let $\vec{H}$ be an acyclic orientation of the cycle on k vertices C_k. We have
$$\vec{R}(\vec{H}) \leqslant 2(2k)^{c \log^2(k)}.$$

The problem for random graphs

- **Theorem 5**. There exists a constant $C = C(\vec{H})$ such that, if $p \geqslant Cn^{-1/m_2(\vec{H})}$, then
$$\lim_{n \to \infty} \mathbb{P}\left[G(n,p) \xrightarrow{\text{or}} \vec{H}\right] = 1,$$
where $m_2(\vec{H})$ is defined as
$$m_2(H) := \max_{F \subseteq H, v(F) \geqslant 3} \frac{e(F) - 1}{v(F) - 2}.$$
- For the proof, we use the hypergraph container lemma of Saxton and Thomason [7] and Balogh, Morris and Samotij [1], adapting the arguments of Nenadov and Steger [6].

A proof via the container lemma

- We prove that, that, if a graph G on n vertices does not satisfy $G \xrightarrow{\text{or}} \vec{H}$, then there exists a s-tuple $\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s)$ of subsets of $E(G)$ and a set $C = C(\mathcal{T}) \subseteq E(K_n)$ depending only on $\mathcal{T}$ such that
  (i) The number s depends only on $\vec{H}$,
  (ii) $\bigcup_{i \in [s]} \mathcal{T}_i \subseteq E(G) \subseteq C$,
  (iii) $|\mathcal{T}_i|$ is "small" (in a precise technical sense) for every $i \in [s]$,
  (iv) $|C|$ is "small" in a precise technical sense.
- We can therefore bound the probability that $G(n,p) \not\xrightarrow{\text{or}} \vec{H}$ by the probability that there exists such a s-tuple $\mathcal{T}$ and a corresponding set $C(\mathcal{T})$. Since these sets are "small", this probability must also be "small".

Isometric Ramsey Number

- A copy of $\vec{H}$ in $\vec{G}$ is said to be *isometric* if, for every two vertices $x, y \in V(\vec{H})$ and their respective copies x', y' in $\vec{G}$, the distance between x and y is equal to the distance between x' and y'.
- The distance is taken with respect to the underlying undirected graphs.
- We write $G \xrightarrow{\text{iso}} \vec{H}$ when every orientation of the edges of G contains an oriented *isometric* copy of $\vec{H}$.
- Given an oriented graph $\vec{H}$, let
$$\vec{R}_{\text{iso}}(\vec{H}) := \inf \left\{n \in \mathbb{N} : \text{there exists a graph } G \text{ of order } n \text{ such that } G \xrightarrow{\text{iso}} \vec{H}.\right\}.$$
- The problem of estimating $\vec{R}_{\text{iso}}(\vec{H})$ for an acyclic oriented graph $\vec{H}$ first appeared in Banakh, Idzik, Pikhurko, Protasov and Pszczoła [2].

A bound for acyclic cycles

- **Theorem 6**. There exists a positive constant c such that, for every acyclic orientation $\vec{H}$ of the cycle on k vertices, we have
$$\vec{R}_{\text{iso}}(\vec{H}) \leqslant ck^{12k^3} R^{8k^2},$$
where $R := \vec{R}(\vec{H})$.
- Our strategy is to prove that, for a number $n \leqslant ck^{12k^3} R^{8k^2}$ and a suitable choice of p, the graph $G(n,p)$ has girth at least k and satisfies $G(n,p) \xrightarrow{\text{or}} \vec{H}$, which implies $G(n,p) \xrightarrow{\text{iso}} \vec{H}$.
- The proof uses the container lemma and adapts arguments from [5].
- The general argument is similar to that of the oriented Ramsey theorem for random graphs.

Further research

- Find better bounds for the oriented Ramsey number of specific graphs. Find bounds for the isometric Ramsey number of other oriented graphs, like orientations of paths.
- We hope to consider in the near future the *colored oriented Ramsey problem*, where we consider colorings and orientations of edges, and require the oriented copy to be monochromatic. We wish to prove corresponding results for random graphs and find bounds for the isometric case.
- According to the theorem of Gallai and Roy, for any graph G we have $\chi(G) = \max\left\{k + 1 : G \xrightarrow{\text{or}} \vec{P}_k\right\}$. Therefore, if $G \xrightarrow{\text{or}} \vec{P}_k$, then $\chi(G) \geqslant k + 1$. One might use this to obtain a better bound on the size of graphs of high girth and high chromatic number.

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Acknowledgements

The author was supported by FAPESP, Proc. 2015/26678-9