Stochastic Simulations
Homework 1: Quasi-Monte Carlo Methods

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Problem statement: an Asian option (or average value option) is a special type of option contract. For Asian options the payoff is determined by the average underlying price over some preset period of time. Consider the problem of pricing an Asian option with maturity $T > 0$ based on the stock price S, which is given as the solution to the stochastic differential equation:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

S_0 is given and W_t is a standard one-dimensional Wiener process. The solution for such an equation is: $S_t = S_0 e^{X_t}$ with $X_t = (r - \sigma^2/2)t + \sigma W_t$. It follows that S_t has a log-normal distribution for any $t > 0$.

For $m \in \mathbb{N}$, let $t_i = i\Delta t$ denote the discrete observation times of the stock price S_t (e.g. daily at market closure). The payoff of an Asian call option is given by:

$$\Psi_j \left(S_{t_0}, S_{t_1}, ..., S_{t_m}\right) = P_j \left(\frac{1}{m}\Sigma_{i=1}^{m}(S_{t_i} - K)\right) \tag{1}$$

$K \leq S_0$ denotes the strike price.

Goal: Use Quasi-Monte Carlo to estimate the expected payoff $\mathbb{E}\left(\Psi_j \left(S_{t_0}, S_{t_1}, ..., S_{t_m}\right)\right)$. Estimate the QMC error using the CLT by estimating the variance with a randomized QMC. Plot both the estimated error based on random shifts as functions of the number of samples N, say, and estimate the convergence rate. Compare with the standard Monte Carlo error and explain the observed results for each payoff.

The low-discrepancy numbers w generated according to the Sobol sequence.

We report the results given the following parameters:

$m = 256, r = 0.5, \sigma = 0.3, T = 2, S_0 = 5, K = 10$

Besides, $P_j(z)$ and $\tilde{P}_j(z)$ for $j = 1, 2$.

$$P_1(z) = z_+ = max(z, 0) \tag{2}$$

$$P_2(z) = \frac{log(1 + e^{\beta z})}{\beta}, \beta > 0 \tag{3}$$

$$\tilde{P}_1(z) = 20 \times \mathbb{1}_{z \geq 0} \tag{4}$$

$$\tilde{P}_2(z) = 20(1 + e^{-2\gamma z})^{-1}, \gamma > 0 \tag{5}$$

Four values were chosen for β and γ as follows: $\beta_1, \gamma_1 = 0.02; \beta_2, \gamma_2 = 0.5; \beta_3, \gamma_3 = 2; \beta_4, \gamma_4 = 5$

The number of samples for each simulation was: $50 * 2^n$ for $n = 1, ..., 10$, which gives a total of 10 runs.

In all plots, the horizontal axis is the number of samples per run and the vertical axis is the error estimate. All plots are in logarithmic scale for both axes.

The mean value in all tables and plots represents the payoff of the Asian call option according to equation 1.

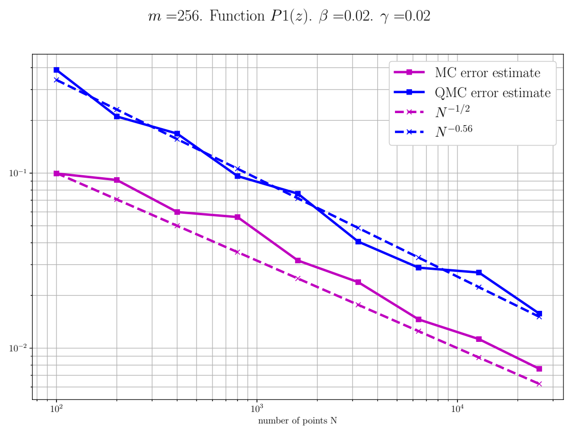
For all functions, firstly a chart summarizing the expected value and the estimate error associated to each method is given. Then four plots for different values of β and γ are shown. The Python code is provided together with this file.

Results for $P_1(z)$

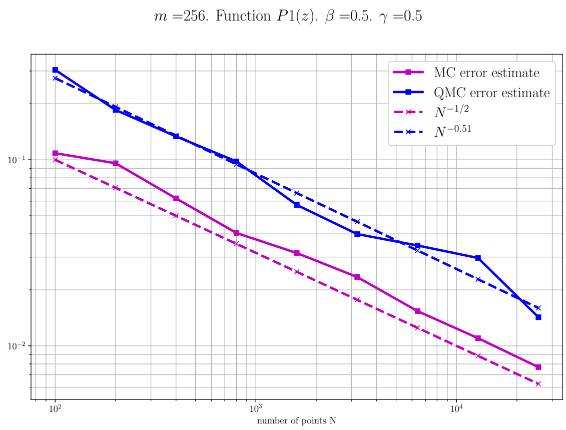
We obtained the following results for mean and error estimate given CMC and QMC:

mean & error	β_1	β_2	β_3	β_4
mean CMC	0.4791	0.4847	0.4718	0.4778
error CMC	0.0079	0.0080	0.0077	0.0078
mean QMC	0.4722	0.4655	0.4748	0.4756
error QMC	0.0178	0.0144	0.0115	0.0171

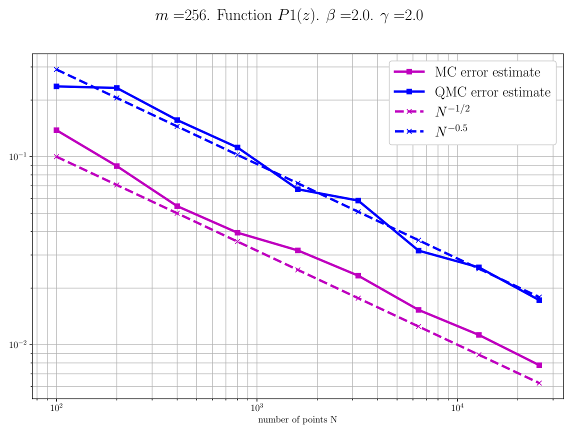
Table 1: Results for $P_1(z)$.



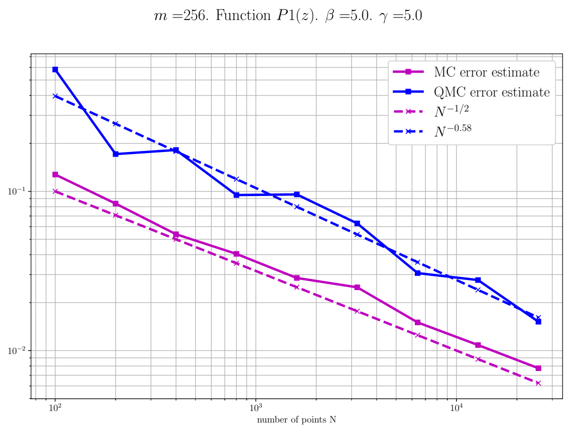
(a) β_1



(b) β_2



(c) β_3



(d) β_4

Figure 1: Plots for $P_1(z)$.

From the previous plots and in table **??**, we can see that the error estimate for the CMC method shows a asymptotic rate of $N^{-1/2}$, which translates into the slope of CMC error estimate being parallel to $N^{-1/2}$. This goes according to what we have studied during the lectures. This will be noted in all plots shown hereafter.

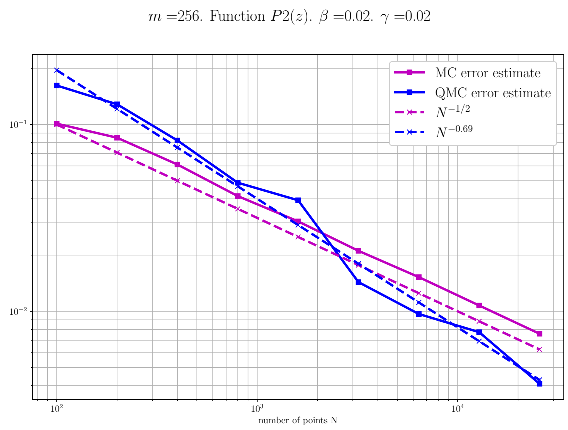
Regarding the QMC, we observed that for the four cases presented the rate of error estimate is between $N^{-0.50}$ and $N^{-0.6}$, which is really close to what is observed in CMC. Similarly, for β_1 and β_4 we obtained the fastest rates.

When β is close to 0, the CMC rate is almost the same as the one computed via QMC. We could state that the optimal value of β is between 2 and 5. This means that the β's which improve the most the rate with respect to $N^{-1/2}$ could be $\beta \leq 0.02$ $\beta \geq 5.0$. As a result, for $P_1(z)$ it makes sense to use QMC if β is taken as described and the parameters are the ones given by the problem statement, otherwise CMC is more suitable if we consider that computing the so-called Sobol sequence is time consuming. For β_2 and β_3 there is no difference between using CMC or QMC in regard to the error estimate rate.

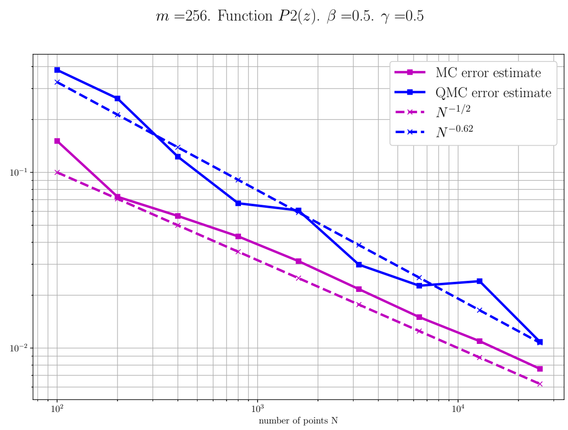
Results for $P_2(z)$
We obtained the following results for mean and error estimate given CMC and QMC:

mean & error	β_1	β_2	β_3	β_4
mean CMC	33.9803	1.1020	0.5212	0.4847
error CMC	0.0076	0.0076	0.0076	0.0078
mean QMC	33.9772	1.0975	0.5078	0.4835
error QMC	0.0062	0.0106	0.0177	0.0160

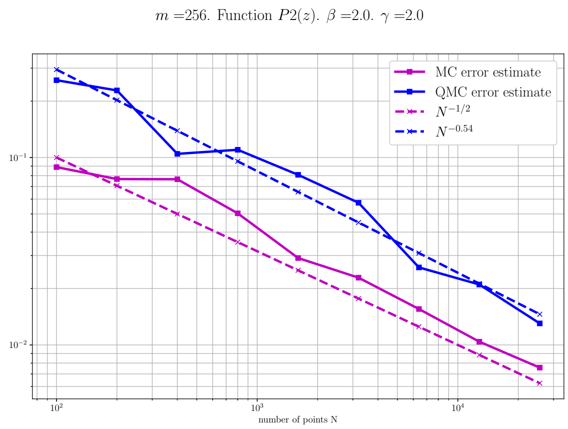
Table 2: Results for $P_2(z)$.



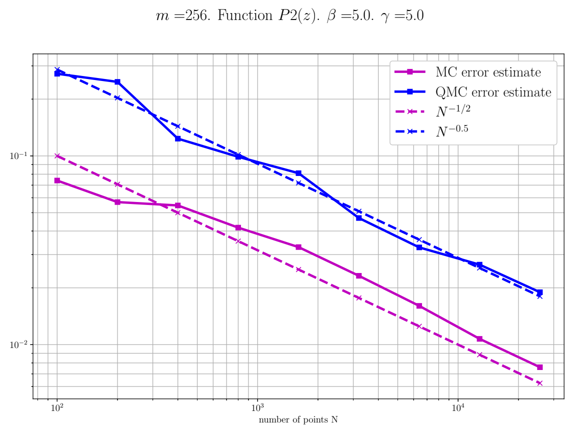
(a) β_1



(b) β_2



(c) β_3



(d) β_4

Figure 2: Plots for $P_2(z)$.

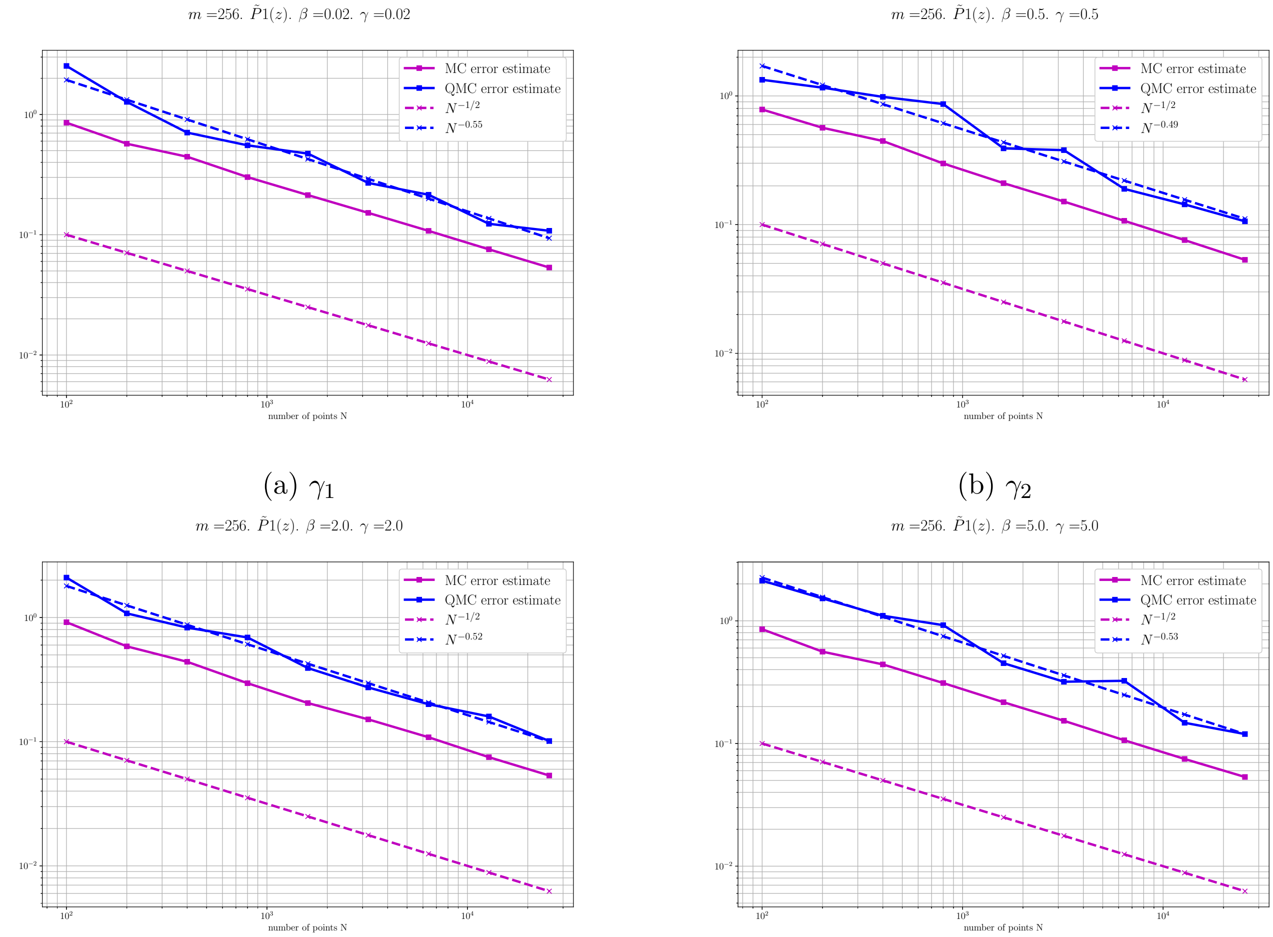
The analysis for $P_2(z)$ is similar to $P_1(z)$ but now the best rate of convergence happens when β is closest to 0. This becomes clear when we have a look at figure 2 (a) and (d). For the latter, the QMC rate is basically the same as the one of CMC. As a result, the closest β is to zero, the more the QMC rate improves with respect to $N^{-0.5}$.

Results for $\tilde{P}_1(z)$

We obtained the following results for mean and error estimate given CMC and QMC:

mean & error	γ_1	γ_2	γ_3	γ_4
mean CMC	4.8344	4.7617	4.7648	4.7469
error CMC	0.0535	0.0532	0.0533	0.0532
mean QMC	4.8063	4.9203	4.8203	4.8422
error QMC	0.1189	0.1054	0.1258	0.1379

Table 3: Results for $\tilde{P}_1(z)$.



(a) γ_1 (b) γ_2 (c) γ_3 (d) γ_4

Figure 3: Plots for $\tilde{P}_1(z)$.

We observe that the rate of convergence for QMC and CMC are practically equal to $N^{-0.5}$. One could say that there is no gain in using QMC considering this one is more expensive computationally.

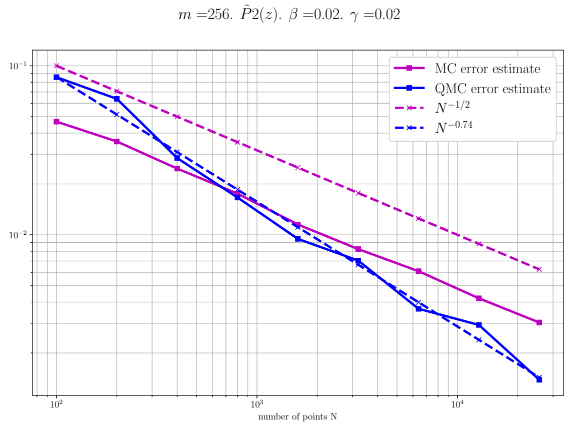
Results for $\tilde{P}_2(z)$

We obtained the following results for mean and error estimate given CMC and QMC:

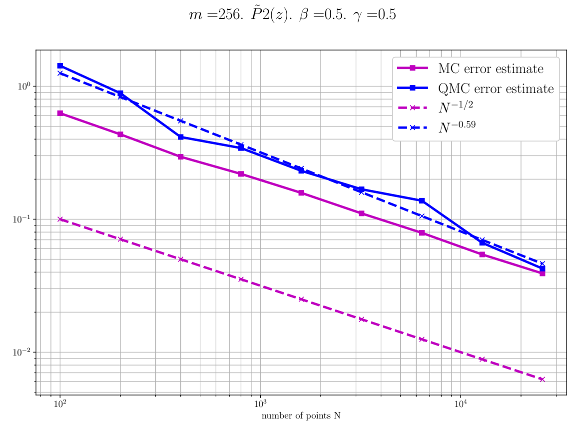
mean & error	γ_1	γ_2	γ_3	γ_4
mean CMC	9.7146	5.7432	4.8358	4.8364
error CMC	0.0030	0.0387	0.0494	0.0519
mean QMC	9.7183	5.7994	4.9250	4.8118
error QMC	0.0021	0.0485	0.0904	0.0632

Table 4: Results for $\tilde{P}_2(z)$.

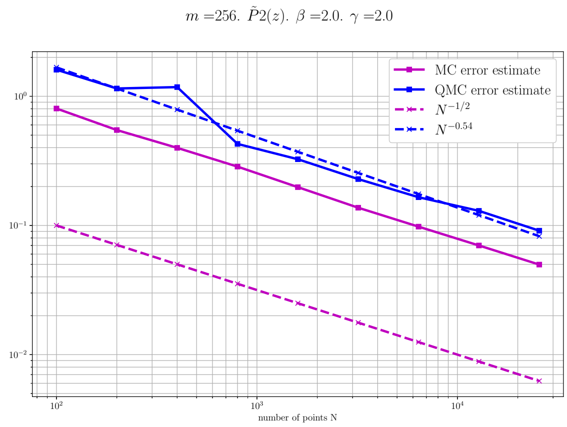
We observed that for values of γ close to zero, the expected valued is quite different from the ones we obtained for γ_2, γ_3 y γ_4. We can think of this fact as a stabilization process of the expected value. This is also interesting since for the other values of γ the expected value computed by CMC and QMC are alike.



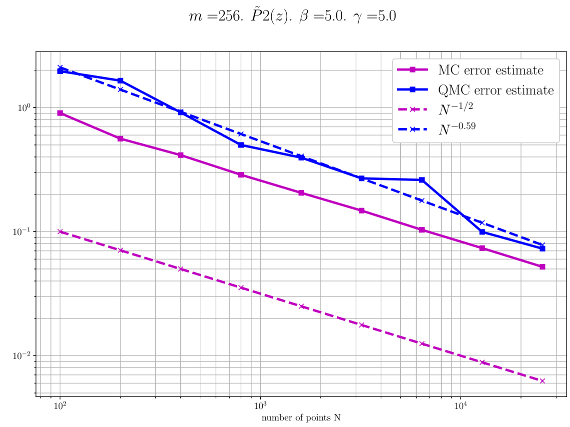
(a) γ_1



(b) γ_2



(c) γ_3



(d) γ_4

Figure 4: Plots for $\tilde{P}_2(z)$.

As with $P_2(z)$, the best rate is obtained when we considered γ as closest to zero as possible. In such a case, it makes sense to implement a QMC method. As γ becomes larger, the rate of QMC is close to the one we get using CMC. Besides, for all cases, there is improvement of the error estimate rate.
From what we have seen here, although the Quasi-Monte Carlo method is intent to improve the convergence of the Crude Monte Carlo method $N^{-0.5}$, this may not always happen, as seen in the previous results. If our problem depended on fewer variables S_{t_i} in equation 1, QMC would show a better rate than CMC. This shows two important points: CMC rate is independent of the problem dimension, as expected, and QMC rate of convergence may be slower that the theoretical result in higher dimensions; our problem was a 256-dimension one.