École polytechnique fédérale de Lausanne

MATH-450. Numerical integration of stochastic differential equations

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Mini project

Parameter inference for SDEs

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In this mini project, we consider the problem of inferring the parameters of a stochastic differential equation (SDE) given continuous-time observations of its solution. Let $T > 0$ be a final time and $X = (X(t), 0 \leq t \leq T)$ be the Ornstein-Uhlenbeck process which solves the Ito SDE:

$$dX(t) = \alpha X(t)dt + \sqrt{2\sigma}dW(t) \ , \ X(0) = X_0 \in \mathbb{R}, \tag{1}$$

where $\alpha > 0$ is called the drift coefficient and $\sigma > 0$ is the diffusion coefficient.

Q1. We solved and studied (1) during the exercise session 8. It corresponds to exercise 6. Consequently, we have that the solution to(1) is given by

$$X(t) = X_0 e^{-\alpha t} + \sqrt{2\sigma} \int_0^t e^{-\alpha(t-s)} dW(s). \tag{2}$$

Furthermore, $X(t)$ follows a normal distribution $\mu_t = \mathcal{N}\left(X_0 e^{-\alpha t}, \frac{\sigma}{\alpha}\left(1 - e^{-2\alpha t}\right)\right)$, and thus its probability density function ρ_t is given by (3).

$$\rho_t(x) = \sqrt{\frac{\alpha}{2\pi\sigma\left(1 - e^{-2\alpha t}\right)}} \exp\left(\frac{\alpha}{2\sigma\left(1 - e^{-2\alpha t}\right)}(x - X_0 e^{-\alpha t})^2\right). \tag{3}$$

Q2. We are given the generator $\mathcal{L}$ of the SDE (1):

$$\mathcal{L}\phi(x) = -\alpha x \phi'(x) + \sigma \phi''(x), \tag{4}$$

where $\phi : \mathbb{R} \to \mathbb{R}$. Then to compute the L^2- adjoint of the generator $\mathcal{L}$, denoted by $\mathcal{L}^\star$, we know that $\mathcal{L}^\star$ satisfies:

$$\int_{\mathbb{R}} v(x)\mathcal{L}u(x)dx = \int_{\mathbb{R}} u(x)\mathcal{L}^\star v(x)dx,$$

where $u, v : \mathbb{R} \to \mathbb{R}$ are smooth functions with compact support. Then we have that:

$$\int_{\mathbb{R}} v(x)\mathcal{L}u(x)dx = \int_{\mathbb{R}} \left(-\alpha x \frac{du(x)}{dx}\right) v(x)dx + \sigma \int_{\mathbb{R}} \frac{d^2u(x)}{dx^2} v(x)dx. \tag{5}$$

Based on [1], we use integration by parts to solve the first term on the right-hand side of ((5) using $m = xv(x)$, $dn = dv$ and $\int mdn = mn - \int ndm$ thus:

$$\begin{aligned}
\int_{\mathbb{R}} \left(-\alpha x \frac{du(x)}{dx}\right) v(x)dx &= -\alpha \left(v(x)x \int_{\mathbb{R}} du - \int_{\mathbb{R}} u(x)d\left(v(x)x\right)\right) \\
&= \alpha \left(\int_{\mathbb{R}} u(x)xdv + u(x)v(x)x\right) \\
&= \int_{\mathbb{R}} u(x)d\left(\alpha v(x)x\right)
\end{aligned}$$

Where we assume that the functions u, v vanish at infinity. Likewise, we can use similar arguments for the second term on the right-hand side of (5). As a result, we have:

$$\mathcal{L}^\star v(x) = \frac{d}{dx}\left(\alpha x v(x)\right) + \sigma \frac{d^2 v(x)}{dx^2} = \alpha v(x) + \alpha x v'(x) + \sigma v''. \tag{6}$$

Then we can compute the expected value and variance of $X(t)$ when $t \to \infty$ as follows:

$$\lim_{t\to+\infty} \mathbb{E}\left[X(t)\right] = \lim_{t\to+\infty} \mathbb{E}\left[X_0 e^{-\alpha t} + \sqrt{2\sigma}\int_0^t e^{-\alpha(t-s)} dW(s)\right] = 0,$$

and

$$\lim_{t\to+\infty} \mathbb{E}\left[\left(X(t) - \mathbb{E}\left[X(t)\right]\right)^2\right] = \frac{\sigma}{\alpha}.$$

Then, the invariant distribution is $\rho_\infty = \mathcal{N}\left(0, \dfrac{\sigma}{\alpha}\right)$. Consequently, we can make use of the Radon–Nikodym theorem to compute the invariant measure μ_∞ as follows:

$$\rho_\infty = \frac{d\mu_\infty}{dx}.$$

where dx is the Lebesgue measure. As a result, the invariant measure is:

$$d\mu_\infty(dx) = \sqrt{\frac{\alpha}{2\pi\sigma}} e^{-\frac{x^2}{2}\frac{\alpha}{\sigma}} dx \tag{7}$$

We now verify that ρ_∞ is the density function of the invariant measure by using the Foker-Planck equation.

$$\mathcal{L}^\star \rho_\infty = \frac{d}{dx}\left(\alpha x \rho_\infty\right) + \sigma \frac{d^2 \rho_\infty}{dx^2} = \alpha\rho_\infty\left(1 - \frac{x^2\alpha}{\sigma}\right) + \alpha\rho_\infty\left(-1 + \frac{x^2\alpha}{\sigma}\right) = 0,$$

and

$$\int_{\mathbb{R}} \sqrt{\frac{\alpha}{2\pi\sigma}} e^{-\frac{x^2}{2}\frac{\alpha}{\sigma}} dx = 1.$$

Q3. When we solve (1) using the Euler-Maruyama scheme for $T = 10^4$, $h = 10^{-2}$, $M = 10^4$ different realizations of the Brownian motion, and setting $\alpha = 1$ and $\sigma = 1$, we obtain figure 1. We observe that the final distribution computed numerically when solving (1) is in accordance with the invariant distribution μ_∞.

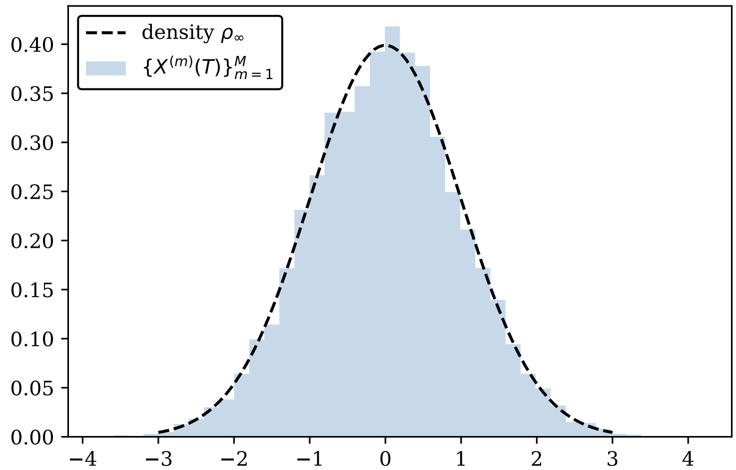


Figure 1: Distribution of $X^m(T)$.

Q4. To compute the covariance function when $s, t \to \infty$, from **Q2**, we know that the invariant distribution follows a normal distribution μ_∞ with zero mean and variance σ/α. Taking the solution to XX given in YYY, we have

$$
\begin{aligned}
\operatorname{Cov}(X(s), X(t)) = & \\
&= \mathrm{E}\left[(X(t) - \mathrm{E}[X(t)])\,(X(s) - \mathrm{E}[X(s)])\right] \\
&= \mathrm{E}\left[(X(t))\,(X(s))\right] \\
&= 2\sigma \mathrm{E}\left[\int_0^s e^{-\alpha(s-u)} dW(u) \int_0^t e^{-\alpha(t-v)} dW(v)\right] \\
&=^{*}\ 2\sigma e^{-\alpha(s+t)} \cdot \mathrm{E}\left[\int_0^s e^{2\alpha u} du\right] \text{ if } s < t \\
&= \frac{\sigma}{\alpha} e^{-\alpha(s+t)} \left(e^{2\alpha s} - 1\right) \\
&=^{**}\ 2\sigma e^{-\alpha(s+t)} \cdot \mathrm{E}\left[\int_0^t e^{2\alpha v} dv\right] \text{ if } t < s \\
&= \frac{\sigma}{\alpha} e^{-\alpha(s+t)} \left(e^{2\alpha t} - 1\right).
\end{aligned}
\tag{8}
$$

In $*$ and $**$ in (8), we use the Ito's isometry by noticing that if $s < t$ -the case for $t > s$ follows similarly-, we can split the stochastic integral into two terms. Due to independence of Brownian increments, we end with an integral of the form $\mathrm{E}\left[\int_0^s e^{\alpha(u)} dW(u) \int_0^s e^{\alpha(v)} dW(v)\right]$, then we apply Ito's isometry.

In either case $s < t$ or $t < s$ in (8), we obtain that

$$
\begin{aligned}
\mathrm{Cov}\left(X(s), X(t)\right) = & \\
&= \frac{\sigma}{\alpha} e^{-\alpha(s+t)} \left(e^{2\min\{s,t\}} - 1 \right) \\
&= \frac{\sigma}{\alpha} e^{-\alpha|t-s|},
\end{aligned} \tag{9}
$$

since $s, t \to \infty$. Defining $C(s,t) := \mathrm{Cov}\left(X(s), X(t)\right)$, we have the desired result.

Q5. To compute, we directly use the hint given in the problem sheet for a function $g\left(\widetilde{X}_n, \widetilde{X}_{n+1}\right) := \left(\widetilde{X}_n - \widetilde{X}_{n+1}\right)^2$. Thus we get the following

$$
\sigma_\infty^N = \lim_{N\to\infty} \hat{\sigma}_\infty^N = \frac{1}{2\Delta} \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left(\widetilde{X}_{n+1} - \widetilde{X}_n\right)^2 = \frac{1}{2\Delta} \mathbb{E}^{\mu_\infty} \left[\left(\widetilde{X}_0 - \widetilde{X}_\Delta\right)^2 \right].
$$

Using linearity of expectation, the covariance function computed in **Q4**, and the fact that $\widetilde{X}_0$ and $\widetilde{X}_\Delta$ are distributed according to the invariant distribution in **Q2**, we obtain the desired result as follows

$$
\sigma_\infty^N = \frac{1}{2\Delta} \mathbb{E}^{\mu_\infty} \left[\left(\widetilde{X}_0 - \widetilde{X}_\Delta\right)^2 \right] = \frac{1}{2\Delta} \left(2\frac{\sigma}{\alpha} - 2\frac{\sigma}{\alpha} e^{-\alpha|\Delta|} \right) = \frac{\sigma}{\alpha\Delta} \left(1 - e^{-\alpha|\Delta|} \right).
$$

Similarly for α_∞^N

$$
\alpha_\infty^N = \lim_{N\to\infty} \hat{\alpha}_\infty^N = -\frac{1}{\Delta} \lim_{N\to\infty} \frac{\frac{1}{N} \sum_{n=0}^{N-1} \widetilde{X}_n \left(\widetilde{X}_{n+1} - \widetilde{X}_n\right)}{\frac{1}{N} \sum_{n=0}^{N-1} \widetilde{X}_n^2}.
$$

Then we use the hint given in the problem sheet, linearity of expectation, the covariance function computed in **Q4**, and the fact that $\widetilde{X}_0$ and $\widetilde{X}_\Delta$ are distributed according to the invariant distribution in **Q2**; we have

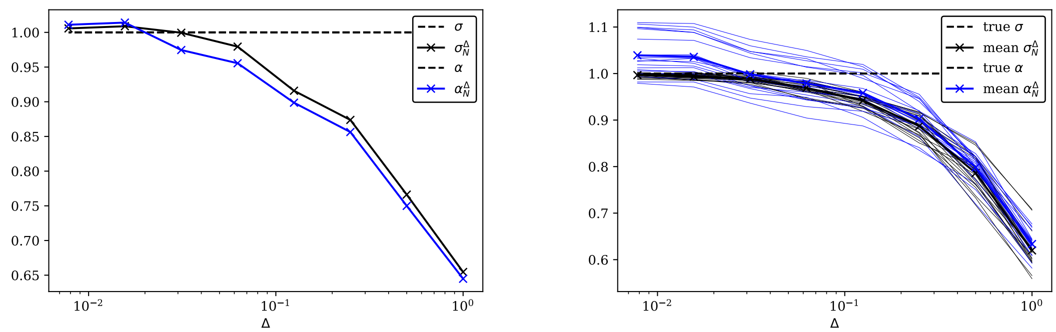
$$
\alpha_\infty^N = -\frac{1}{\Delta} \frac{\mathbb{E}^{\mu_\infty} \left[\left(\widetilde{X}_0 \widetilde{X}_\Delta - \widetilde{X}_0^2 \right) \right]}{\mathbb{E}^{\mu_\infty} \left[\widetilde{X}_0^2 \right]} = -\frac{\sigma}{\alpha\Delta} \left(\frac{e^{-\alpha|\Delta|} - 1}{\frac{\sigma}{\alpha}} \right) = -\frac{1}{\Delta} \left(e^{-\alpha|\Delta|} - 1 \right)
$$

.

Q6. From **Q6**, we shall recall that it can be proven that $\lim_{x\to 0} \dfrac{e^x - 1}{x} = 1$. Thus, we can make use of this result and observe that both $\lim_{\Delta\to 0} \sigma_\infty^N$ and $\lim_{\Delta\to 0} \alpha_\infty^N$ are of such a structure. Consequently, we have

$$
\sigma_\infty^N = \sigma \text{ and } \alpha_\infty^N = \alpha.
$$

Q7. We solve (1) for $T = 10^3$ using the Euler Maruyama scheme with $h = 10^{-3}$, $\alpha = 1$, $\sigma = 1$, and we consider different sampling rates $\Delta = 2^{-i}$ for $i = 0, 1, \ldots, 7$. For each Δ, we compute and plot its corresponding value of the estimators of drift coefficient $\hat{\alpha}_N^{\Delta}$ and the diffusion coefficient $\hat{\sigma}_N^{\Delta}$. This leads to figure 2.



(a) $\hat{\alpha}_N^{\Delta}$ and $\hat{\sigma}_N^{\Delta}$, single realization of BM.

(b) $\hat{\alpha}_N^{\Delta}$ and $\hat{\sigma}_N^{\Delta}$, 20 realizations of BM.

Figure 2: Estimator for $\hat{\alpha}_N^{\Delta}$ and $\hat{\sigma}_N^{\Delta}$.

In figure 2, on the left-hand plot, we have a single realization of the Brownian motion; on the right-hand side we have 20 independent realization the Brownian motion. From these 20 runs, we compute the mean value of $\hat{\alpha}_N^{\Delta}$ and $\hat{\sigma}_N^{\Delta}$ at each Δ, which is plotted using a thicker line width.

Q8. In the problem sheet we are given the generator $\mathcal{L}$ of the SDE. Then we directly express the eigenvalue $-\mathcal{L}_a \phi(x; a) = \lambda(a) \phi(x; a)$ problem as (10).

$$ax\phi^{'}(x; a) - \sigma\phi^{''}(x; a) = \lambda(a)\phi(x; a). \tag{10}$$

Q9. We are requested to prove that following recurrence relation for the eigenvalues $\{\lambda_j(a)\}_{j=0}^{\infty}$ and eigenfunctions $\{\phi_j(\cdot; a)\}_{j=0}^{\infty}$:

$$\begin{aligned} \lambda_j(a) &= ja, j \in \mathrm{N}. \\ \phi_0(x; a) &= 1, \\ \phi_1(x; a) &= x, \\ \phi_j(x; a) &= x\phi_{j-1}(x; a) - \frac{\sigma}{\alpha}(j-1)\phi_{j-2}(x; a), j \geq 2. \end{aligned} \tag{11}$$

We prove that (11) satisfies (12) for the eigenfunctions.

$$\phi_j'(x; a) = j\phi_{j-1}(x; a), j \geq 1. \tag{12}$$

We notice that for the base case $j = 1$, (12) holds $\phi_1'(x;a) = \phi_0(x;a) = 1$, which means $\phi_1(x;a) = x$. Similarly, for $j = 2$, we have $\phi_2'(x;a) = 2\phi_1(x;a) = 2x$; then, $\phi_2(x;a) = x^2 - \sigma/\alpha$ since σ/α is a constant.

We now assume that $\phi_k'(x;a) = k\phi_{k-1}(x;a)$ holds for $k \geq 2$ holds. Thus, by (12), we obtain

$$\begin{aligned}
\phi_{k+1}'(x;a) &= \phi_k(x;a) + x\phi_k'(x;a) - \frac{\sigma}{a}k\phi_{k-1}'(x;a) \\
&= \phi_k(x;a) + xk\phi_{k-1}(x;a) - k\frac{\sigma}{a}(k-1)\phi_{k-2}(x;a) \\
&= \phi_k(x;a) + k\left(x\phi_{k-1}(x;a) - \frac{\sigma}{a}(k-1)\phi_{k-2}(x;a)\right) \\
&= (k+1)\phi_k(x;a).
\end{aligned} \tag{13}$$

Then (12) holds. In addition, we prove $\lambda_j(a) = ja$. It is straightforward to prove that for $j = 0$ and $j = 1$, such a relation holds using (12) and the eigenvalue problem as stated in **Q8**. We assume that $\lambda_k(a) = ka$ holds and we try proving $\lambda_{k+1}(a) = (k+1)a$.

We can see that $\phi_k''(x;a) = k\phi_{k-1}'(x;a) = k(k-1)\phi_{k-2}'(x;a)$. Using **Q8**, we have

$$-\mathcal{L}_a\phi_{k+1}(x;a) = ax(k+1)\phi_k(x;a) - \sigma k(k+1)\phi_{k-1}(x;a) = \lambda_{k+1}(a)\phi_{k+1}(x;a).$$

Rewriting this, we have

$$(k+1)a\phi_{k+1}(x;a) = \lambda_{k+1}(a)\phi_{k+1}(x;a) \Rightarrow (k+1)a = \lambda_{k+1}(a).$$

Q10. We are given the estimation function (14).

$$G(a) = \frac{1}{N}\sum_{j=1}^{J}\sum_{n=0}^{N-1}\psi_j\left(\widetilde{X}_n\right)\left(\phi_j\left(\widetilde{X}_{n+1};a\right) - e^{-\lambda_j(a)\Delta}\phi_j\left(\widetilde{X}_n;a\right)\right). \tag{14}$$

We define the estimator $\widetilde{\alpha}_N^{\Delta}$ as the solution of the nonlinear equation (14), i.e., $\widetilde{\alpha}_N^{\Delta} := G(a) = 0$.

Now we take $J = 1$ and $\psi_1(x) = x$ and substitute these into (14). Furthermore, from **Q9**, we have that $\lambda_1(a) = a$ and $\phi_1(x;a) = x$; thus we obtain

$$G(a) = \frac{1}{N}\sum_{n=0}^{N-1}\widetilde{X}_n\left(\widetilde{X}_{n+1} - e^{-a\Delta}\widetilde{X}_n\right). \tag{15}$$

To compute $\mathcal{G}(a) = \lim_{N\to\infty} G(a)$, we proceed as in **Q5**; that is we follow the hint given for **Q5**, which leads us to

$$\mathcal{G}(a) = \mathbb{E}^{\mu_\infty}\left[\widetilde{X}_0\widetilde{X}_\Delta\right] - e^{-a\Delta}\mathbb{E}^{\mu_\infty}\left[\widetilde{X}_0^2\right] = \frac{\sigma}{\alpha}e^{-\alpha|\Delta|} - \frac{\sigma}{\alpha}e^{-a|\Delta|} = 0 \iff a = \alpha.$$

Q11. From **Q10**, we solve (15), $G(a) = 0$ for a. To recall from previous point, we refer to a such that $G(a) = 0$ as $\widetilde{\alpha}_N^\Delta$. Consequently, we obtain

$$e^{-\widetilde{\alpha}_N^\Delta \Delta} = \frac{\frac{1}{N}\sum_{n=0}^{N-1} \widetilde{X}_n \widetilde{X}_{n+1}}{\frac{1}{N}\sum_{n=0}^{N-1} \widetilde{X}_n^2} \Rightarrow \widetilde{\alpha}_N^\Delta = -\frac{1}{\Delta} \ln\left(\frac{\frac{1}{N}\sum_{n=0}^{N-1} \widetilde{X}_n \widetilde{X}_{n+1}}{\frac{1}{N}\sum_{n=0}^{N-1} \widetilde{X}_n^2}\right).$$

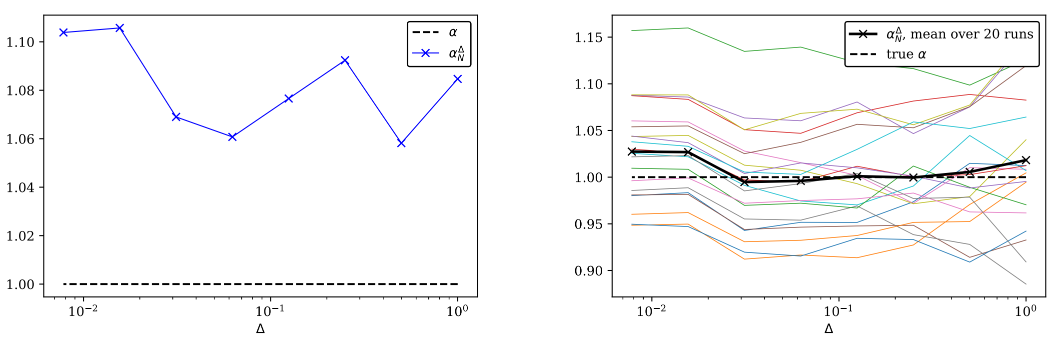
We now compute $\lim_{N\to\infty} \widetilde{\alpha}_N^\Delta$, we follow the hint given in **Q5**. As a result, we have the following expression.

$$\lim_{N\to\infty} \widetilde{\alpha}_N^\Delta = -\frac{1}{\Delta}\left[\lim_{N\to\infty} \ln\left(\frac{1}{N}\sum_{n=0}^{N-1} \widetilde{X}_n \widetilde{X}_{n+1}\right) - \lim_{N\to\infty} \ln\left(\frac{1}{N}\sum_{n=0}^{N-1} \widetilde{X}_n^2\right)\right].$$

To actually use the hint given in **Q5**, we first point out that we can bring the $\ln(\cdot)$ outside the limit since it can be shown that $\lim_{x\to\infty} \ln(f(x)) = \ln(\lim_{x\to\infty} f(x))$ for a continuous function $f(x)$. We then have:

$$\begin{aligned}
\lim_{N\to\infty} \widetilde{\alpha}_N^\Delta &= -\frac{1}{\Delta}\left[\ln\left(\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \widetilde{X}_n \widetilde{X}_{n+1}\right) - \ln\left(\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \widetilde{X}_n^2\right)\right] \\
&= -\frac{1}{\Delta}\left[\ln \mathbb{E}^{\mu_\infty}\left[\widetilde{X}_0 \widetilde{X}_\Delta\right] - \ln \mathbb{E}^{\mu_\infty}\left[\widetilde{X}_0\right]\right] \\
&= -\frac{1}{\Delta}\left[\ln\left(\frac{\sigma}{\alpha} e^{-\alpha|\Delta|}\right) - \ln\left(\frac{\sigma}{\alpha}\right)\right] \\
&= \alpha
\end{aligned}$$

Q12. We solve (1) for $T = 10^3$ using the Euler Maruyama scheme with $h = 10^{-3}$, $\alpha = 1$, $\sigma = 1$ and we consider different sampling rates $\Delta = 2^{-i}$ for $i = 0, 1, \ldots, 7$. For each Δ, we compute and plot its corresponding value of the drift coefficient α using **Q11**. This leads to figure 3.



(a) $\widetilde{\alpha}_N^\Delta$, single realization of BM.

(b) Mean $\widetilde{\alpha}_N^\Delta$, 20 realizations of BM.

Figure 3: Estimator for $\widetilde{\alpha}_N^\Delta$.

In figure 3, on the left-hand plot, we have a single realization of the Brownian motion; on the right-hand side we have 20 independent realization the Brownian motion. From these 20 runs, we compute the mean value of $\widetilde{\alpha}_N^{\Delta}$ at each Δ, which is plotted using a thicker line width.

We observe in figure 3 that are close to the true value of α for each Δ, which is quite different from the estimator plotted in **Q7** where we observe a stronger dependence on Δ.

Q13. We solve (1) for $T = 10^3$ using the Euler Maruyama scheme with $h = 10^{-2}$, $\alpha = 1$, $\sigma = 1$, $M = 10^4$ different realizations of the Brownian motion, and we consider a sampling rate $\Delta = 1$. For each realization of the Brownian motion, we compute and plot it corresponding value of the drift coefficient $\widetilde{\alpha}_N^{\Delta,(m)}$ using the expression given in **Q14**. This leads to figure 4.

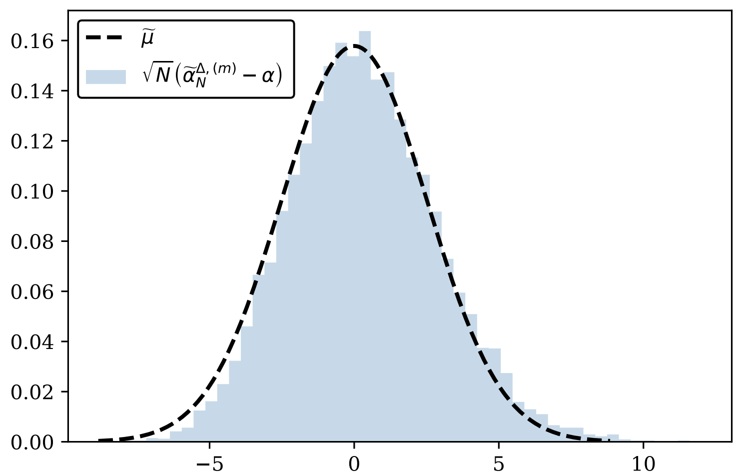


Figure 4: Distribution of several realizations of $\widetilde{\alpha}_N^{\Delta,(m)}$.

In figure 4, we have $\widetilde{\mu} = \mathcal{N}(0, \Sigma)$ where $\Sigma = \dfrac{e^{2\alpha\Delta} - 1}{\Delta^2}$. Based on the same figure, we can say that the distribution of the estimator $\widetilde{\alpha}_N^{\Delta,(m)}$ follows what the central limit theorem states.

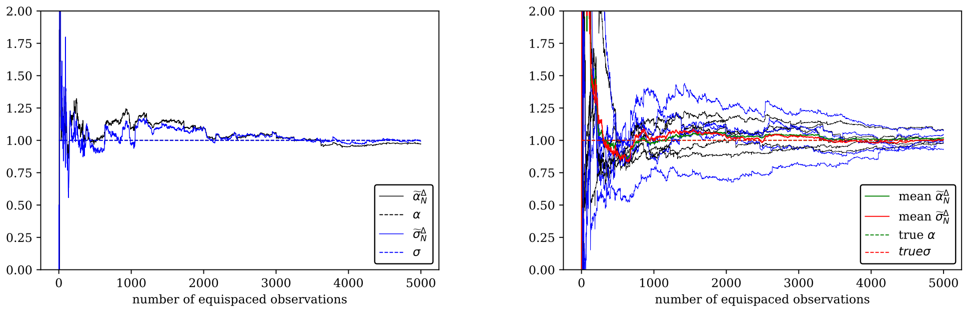
Q14. We are given a new estimation function to estimate both coefficients α and σ.

$$\mathbf{G}\left(a, s\right) = \frac{1}{N} \sum_{j=1}^{J} \sum_{n=0}^{N-1} \Psi_j \left(\widetilde{X}_n\right) \left(\phi_j \left(\widetilde{X}_{n+1}; a, s\right) - e^{-\lambda_j(a,s)\Delta} \phi_j \left(\widetilde{X}_n; a, s\right)\right), \qquad (16)$$

where we choose a set $\{\Psi_j\}_{j=1}^{J}$ of vector-valued smooth functions such that $\Psi_j : \mathbb{R} \to \mathbb{R}^2$. Setting $J = 2$, $\Psi_1(x) = \Psi_2(x) = (x^2 x)^T$, from **Q9**, $\lambda_1(a; s) = a$, $\lambda_2(a; s) = 2a$, $\phi_1\left(x; a, s\right) = x$, and $\phi_2\left(x; a, s\right) = x^2 - \dfrac{s}{a}$, and plugging these into (16), we obtain (17).

$$
\begin{aligned}
&\mathbf{G}\left(a, s\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\iff\ & \frac{1}{N} \sum_{n=0}^{N-1} \begin{pmatrix} \widetilde{X}_n^2 \\ \widetilde{X}_n \end{pmatrix} \left(\widetilde{X}_{n+1} - e^{-a\Delta} \widetilde{X}_n + \widetilde{X}_{n+1}^2 - \frac{s}{a} - e^{-2a\Delta} \left(\widetilde{X}_n^2 - \frac{s}{a} \right) \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned} \tag{17}
$$

Q15. We solve (1) for $T = 5 \cdot 10^3$ using the Euler Maruyama scheme with $h = 10^{-2}$, $\alpha = 1$, $\sigma = 1$, one realization of the Brownian motion, and we consider a sampling rate $\Delta = 1$. For each of the available observations of the solution to (1), we compute and plot its corresponding estimator for $\widetilde{\alpha}_N^\Delta$ and $\widetilde{\sigma}_N^\Delta$ solving the expression given in **Q14**. This leads to figure 5.



(a) $\widetilde{\alpha}_N^\Delta$ and $\widetilde{\sigma}_N^\Delta$, single realization of BM

(b) Mean of $\widetilde{\alpha}_N^\Delta$ and $\widetilde{\sigma}_N^\Delta$, 5 realizations of BM.

Figure 5: Estimators for $\widetilde{\alpha}_N^\Delta$ and $\widetilde{\sigma}_N^\Delta$.

References

[1] Grigorios A Pavliotis. *Stochastic processes and applications: diffusion processes, the Fokker-Planck and Langevin equations*. Vol. 60. Springer, 2014.