Limited Principles of Omniscience in Constructive Type Theory

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Abstract

The Limited Principle of Omniscience (LPO) is often enough to prove theorems of classical mathematics. LPO is an instance of the Law of Excluded Middle (LEM) which states that Σ^0_1 propositions P (i.e. existential quantification over a decidable predicate on $\mathbb{N}$) are classical (i.e. $P \vee \neg P$ holds). It implies Markov's Principle (MP), stating that Σ^0_1 propositions are stable under double negation. Several variants of MP, varying in the definition of decidability, have been introduced and used in the literature, and we have shown in previous work that two of these variants can be separated. We further show here how to separate three variants (stated over (1) a decidable predicate; (2) a Boolean-valued function; and (3) a primitive recursive Boolean-valued function), and extend those results to LPO. Furthermore, we for the first time give these separations (formalized in Agda[1]) for Martin-Löf Type Theory (MLTT), which is at the heart of many dependent type theories.

Definitions In previous work we investigated three variants of Markov's Principle (MP) [18, 4, 14, 5] and discussed [7] how to separate the the last two in constructive type theory:

$$\begin{array}{lll}
\mathsf{MP}_{\mathbb{P}} & := \forall A : \mathbb{N} \to \mathbb{P}.\ (\forall n.\ An \vee \neg An) \to & \neg\neg(\exists n.\ An) \to (\exists n.\ An) \\
\mathsf{MP}_{\mathbb{B}} & := \forall f : \mathbb{N} \to \mathbb{B}. & \neg\neg(\exists n.\ fn = \mathsf{true}) \to (\exists n.\ fn = \mathsf{true}) \\
\mathsf{MP}_{\mathsf{PR}} & := \forall f : \mathbb{N} \to \mathbb{B}.\ \mathsf{primitive\text{-}recursive}\ f \to & \neg\neg(\exists n.\ fn = \mathsf{true}) \to (\exists n.\ fn = \mathsf{true})
\end{array}$$

We list these in order of their strength, as $\mathsf{MP}_{\mathbb{P}}$ implies $\mathsf{MP}_{\mathbb{B}}$, which in turn implies $\mathsf{MP}_{\mathsf{PR}}$. For the reverse directions, with the axiom of unique choice we can show that $\mathsf{MP}_{\mathbb{B}}$ implies $\mathsf{MP}_{\mathbb{P}}$ and under Church's Thesis $\mathsf{MP}_{\mathsf{PR}}$ implies $\mathsf{MP}_{\mathbb{B}}$.

Similarly, we can define three variants of the Limited Principle of Omniscience (LPO) [1]:

$$\begin{array}{lll}
\mathsf{LPO}_{\mathbb{P}} & := \forall A : \mathbb{N} \to \mathbb{P}.\ (\forall n.\ An \vee \neg An) \to & (\exists n.\ An) \vee \neg(\exists n.\ An) \\
\mathsf{LPO}_{\mathbb{B}} & := \forall f : \mathbb{N} \to \mathbb{B}. & (\exists n.\ fn = \mathsf{true}) \vee \neg(\exists n.\ fn = \mathsf{true}) \\
\mathsf{LPO}_{\mathsf{PR}} & := \forall f : \mathbb{N} \to \mathbb{B}.\ \mathsf{primitive\text{-}recursive}\ f \to & (\exists n.\ fn = \mathsf{true}) \vee \neg(\exists n.\ fn = \mathsf{true})
\end{array}$$

Similarly to MP, $\mathsf{LPO}_{\mathbb{P}}$ implies $\mathsf{LPO}_{\mathbb{B}}$ and with the axiom of unique choice the converse is true. $\mathsf{LPO}_{\mathbb{B}}$ also implies $\mathsf{LPO}_{\mathsf{PR}}$ and with Church's Thesis this implication becomes an equivalence.

Notice that each variant of LPO implies its corresponding variant of MP since in general any classical proposition is double negation stable. The converse implication does not hold, as it would require that for every Σ^0_1 proposition P, the proposition $P \vee \neg P$ is also Σ^0_1. This is not the case in general due to the $\neg P$.

MLTT and $\mathbf{TT}^{\square}_{\mathcal{C}}$ We separate the above variants of MP and LPO for MLTT [13] using: (1) a translation of MLTT to $\mathrm{TT}^{\square}_{\mathcal{C}}$; and (2) separations of those variants for $\mathrm{TT}^{\square}_{\mathcal{C}}$. $\mathrm{TT}^{\square}_{\mathcal{C}}$ [3] is a family of effectful type theories parameterized by: (1) a choice operator $\mathcal{C}$, which is used to implement effectful computations; and (2) a $\square$ modality to give meaning to effectful computations. In particular, in [7] we instantiated $\mathcal{C}$ with choice sequences [10, 19, 17, 16, 11, 20, 12] and $\square$ with Beth coverings [21, 9, 6] to separate $\mathsf{MP}_{\mathbb{B}}$ and $\mathsf{MP}_{\mathsf{PR}}$ by exhibiting models of $\mathrm{TT}^{\square}_{\mathcal{C}}$ that falsify

[1] github.com/vrahli/opentt/blob/77ec8765dcee83c061d567241b991a5ce261a5a8/types_lpo.lagda

$\mathsf{MP}_{\mathbb{B}}$ while satisfying $\mathsf{MP}_{\mathsf{PR}}$. A choice sequence can be seen as a reference to a list of values that can only be modified by extending it with further values. $\mathrm{TT}_{\mathcal{C}}^{\square}$'s semantics is given in terms of a poset $\mathcal{W}$ of worlds, where a world can be seen as a list of choice sequences along with their values so far. While a given world w might not contain the n-th value of a choice sequence δ, Beth coverings allow giving meaning to δ by only requiring that choices are "eventually" available, i.e., given any infinite sequence of worlds (w.r.t. $\mathcal{W}$'s ordering relation) starting from w, there is a world in that sequence where δ's n-th choice is defined.

Separating $\mathsf{MP}_{\mathsf{PR}}$ and $\mathsf{MP}_{\mathbb{B}}$ As explained in [7], instantiating $\mathcal{C}$ with choice sequences of Booleans and $\square$ with a Beth modality yields a model in which $\mathsf{MP}_{\mathsf{PR}}$ holds, while $\mathsf{MP}_{\mathbb{B}}$ does not.

Separating $\mathsf{MP}_{\mathbb{B}}$ and $\mathsf{MP}_{\mathbb{P}}$ To separate $\mathsf{MP}_{\mathbb{B}}$ and $\mathsf{MP}_{\mathbb{P}}$, we again instantiate $\square$ with a Beth modality, but $\mathcal{C}$ with choice sequences of propositions, i.e. the empty $\mathbb{0}$ and unit $\mathbb{1}$ types. As a result, using a similar argument to the one used to negate $\mathsf{MP}_{\mathbb{B}}$ in [7] using Boolean choices, we now obtain that $\neg\mathsf{MP}_{\mathbb{P}}$ holds in this model. To see that $\mathsf{MP}_{\mathbb{B}}$, while not holding with Boolean choice sequences, holds with propositional choice sequences, we must show that terms that compute to Booleans cannot make use of a proposition in an essential way. This is done by way of a bisimulation on $\mathrm{TT}_{\mathcal{C}}^{\square}$ terms which features congruence rules, as well as a rule relating the terms $\mathbb{0}$ and $\mathbb{1}$. Note that this result has appeared together with the contributions of [7] as [2].

Separating $\mathsf{LPO}_{\mathsf{PR}}$ and $\mathsf{LPO}_{\mathbb{B}}$ For the purposes of LPO, and in particular to falsify $\mathsf{LPO}_{\mathbb{B}}$, we once again require a Beth modality for instantiating $\square$, and choice sequences for instantiating $\mathcal{C}$. In this setup, to negate $\mathsf{LPO}_{\mathbb{B}}$ we must show it does not hold in any extension w_1 of the current world w. To prove that $\mathsf{LPO}_{\mathbb{B}}$ holding at w_1 leads to a contradiction, it is enough to show that it does so in some extension w_2 of w_1. We pick w_2 to be w_1 extended with a currently empty choice sequence δ, which inhabits $\mathbb{N} \to \mathbb{B}$, and instantiate $\mathsf{LPO}_{\mathbb{B}}$ with δ. We must then prove that either of $(\exists n.\ \delta\ n)$ or $\neg(\exists n.\ \delta\ n)$ holding at w_2 leads to a contradiction. Assuming that $(\exists n.\ \delta\ n)$ holds at w_2 we obtain a contradiction by showing that there is a path from w_2 where δ's entries are always false. Assuming that $\neg(\exists n.\ \delta\ n)$ holds at w_2, i.e. $\exists n.\ \delta\ n$ does not hold at any extension of w_2, we obtain a contradiction by showing that there is an extension w_3 of w_2 where an entry of δ is set to true (for $\neg A$ to hold at w_0, it must be that A does not hold at any extension of w_0). The same reasoning does not work for $\mathsf{LPO}_{\mathsf{PR}}$. To see why, recall that primitive recursive functions are encoded by (pure) natural numbers. As a result, any f which is primitive recursive must be equal to some f_{pure} which does not use choice sequences at all, and the model can prove LPO for such f_{pure}, assuming LPO in the metatheory.

Separating $\mathsf{LPO}_{\mathbb{B}}$ and $\mathsf{LPO}_{\mathbb{P}}$ As for MP, $\mathsf{LPO}_{\mathbb{B}}$ and $\mathsf{LPO}_{\mathbb{P}}$ can be separated by instantiating $\mathcal{C}$ with choice sequences of propositions ($\mathbb{1}$ and $\mathbb{0}$) instead of Booleans (true and false).

Choice Sequences vs. References While [3] uses both choice sequences and references to a single Boolean to falsify LEM, the same cannot be done to falsify $\mathsf{MP}_{\mathbb{B}}$ or $\mathsf{LPO}_{\mathbb{B}}$. In this development, a reference to a single Boolean can be modified in further extensions of a world. Crucially, at any point, a reference can be made immutable, fixing its value in all future worlds, allowing us to falsify LEM similarly to the proof sketch of $\neg\mathsf{LPO}_{\mathbb{B}}$ (where the immutable choice true is used to obtain a contradiction from $\neg(\exists n.\ \delta\ n)$). References can be used to falsify LEM because in that case there is no need to prove that they inhabit a type that comes with a dependent elimination principle. As explained in [15], observational effects and unrestricted dependent elimination cannot coexist. For references, this manifests as the fact that they do not inhabit $\mathbb{B}$. As opposed to choice sequences whose entries are fixed once generated, reference cells' contents can change, precluding them from inhabiting types that come with dependent elimination principles, and therefore from using them to falsify $\mathsf{MP}_{\mathbb{B}}$ or $\mathsf{LPO}_{\mathbb{B}}$.

Concluding Remarks The above discussion naturally leads to two main questions: (1) is it possible to falsify $\mathsf{MP}_{\mathbb{B}}$ and $\mathsf{LPO}_{\mathbb{B}}$ using other forms of effectful computations than choice sequences; and (2) what effectful computations that could be used to separate MP and LPO. Herbelin gives such a separation based on an exception mechanism [8].

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