Bruno Scherrer, bruno.scherrer@inria.fr, January 25, 2023.

Consider a parity game G between EVEN (or Player 0) and ODD (or Player 1) on a state space $X = \{1, \dots, n\}$ with priority function $\Omega : X \to \{1, \dots, d\}$. On infinite plays, ODD wants that among the priorities that appears infinitely often, the maximal one is odd, while EVEN wants it to be even.

I describe below a recursive procedure to solve this problem. Then I argue that the number of recursive calls is at most n.

Suppose for concreteness that we have a game G of which the maximal priority p is even (the odd case is similar). Consider the payoff function

$$\forall x, \quad g(x) = \mathbb{1}_{\Omega(x)=p},$$

and the Bellman operators associated to the corresponding payoff problem: for all $v \in \mathbb{R}^X$,

$$\begin{aligned} T_{\mu,\nu} v &= g + P_{\mu,\nu} v, \\ T_\mu v &= \min_\nu T_{\mu,\nu} v, \\ \tilde{T}_\nu v &= \max_\mu T_{\mu,\nu} v, \\ T v &= \max_\mu \min_\nu T_{\mu,\nu} v = \max_\mu T_\mu v = \min_\nu \tilde{T}_\nu v, \end{aligned}$$

where $P_{\mu,\nu}$ is the deterministic transition matrix when players use the (positional) strategies μ and ν.

The introduction of the payoff game is motivated by the following observations:

Lemma 1. *In the parity game G,*

1. *The set of states from which EVEN can force ODD to visit states with priority p infinitely often (and thus win) is $\{x \; ; \; T^{n(n+1)}0(x) \geq n\}$;*
2. *The set of states from which ODD can force EVEN to* only *visit states with priorities lower than p is $\{x \; ; \; T^k 0(x) = 0\}$ for any $k \geq n$.*
3. *From any state x such that $T^k 0(x) > 0$, EVEN has a strategy that allows to visit at least one state with priority p in the next k steps (and consequently, if ODD can force EVEN to visit states with priorities p only a finite number times, then the play must reach C in finite time and stay in C forever).*

Proof. Let us begin by the first item. If from state x, EVEN can force ODD to visit states with priority p infinitely often, EVEN can force to visit such states at least once every n steps; on a trajectory of length $n(n+1)$, then states with priority p are visited at least n times. And thus on the payoff game, $T^{n(n+1)}0(x) \geq n$. If from state x, ODD can prevent EVEN from visiting states with priority p infinitely often, this means that ODD has a strategy such that against any policy of EVEN, there is a path of length at most $n-1$ followed by a trajectory that only visits states with priority smaller than p. On the payoff game, this means that $T^{n(n+1)}0(x) < n$.

Now let us consider the second item. First (the easier part), assume that for some state x, ODD has a strategy ν such that whatever EVEN does, the trajectory only visits states with priorities smaller than p. Then, in the m-horizon payoff game for any $m \geq n$, this strategy forces the trajectory to only visit states with payoff 0, and thus:

$$0 \leq [T^m 0](x) \leq [(\tilde{T}_\nu)^m 0](x) \leq 0,$$

i.e. $T^m 0(x) = 0$.

Now, assume that we have a state x such that $T^m(x) = 0$ with $m \geq n$. Let $(\nu_1, \dots, \nu_m)$ be any sequence of stratregies such that

$$T^m 0 = \tilde{T}_{\nu_1} \dots \tilde{T}_{\nu_m} 0.$$

Against any (positional) strategy μ of EVEN, consider the trajectory $(x_0 = x, x_1, \dots, x_n)$ induced if ODD plays with $\nu_1, \dots, \nu_n$. There necessarily exists $0 \leq i < j \leq n$ and a state $y \in X$ such that $x_i = x_j = y$. We have

$$[T_{\mu,\nu_1} \dots T_{\mu,\nu_{i-1}} T_{\mu,\nu_i} \dots T_{\mu,\nu_{j-1}} T^{m-j} 0](x) \leq [T^m 0](x) = 0.$$

Since all the payoffs are non-negative, we deduce that

$$\begin{aligned}
[T^{m-j} 0](y) &= 0, \\
[T_{\mu,\nu_i} \dots T_{\mu,\nu_{j-1}} 0](y) &= 0, \\
[T_{\mu,\nu_1} \dots T_{\mu,\nu_{i-1}} 0](x) &= 0.
\end{aligned}$$

Thus, if ODD plays the infinite-horizon policy $\nu_1, \dots, \nu_{i-1} (\nu_i \dots \nu_{j-1})^\infty$ against EVEN playing μ, then the infinite-horizon payoff of this play is 0. In G, this means that from x, the priorities visited are all smaller than p.

The final item is obvious by considering the strategy $\mu_1, \dots, \mu_k$ such that

$$T^k 0 = T_{\mu_1} \dots T_{\mu_k} 0.$$

$\square$

Step 1. Compute the set

$$\begin{aligned}
A &= \{x \; ; \; T^n 0(x) \geq n\}, \\
B = A^c &= \{x \; ; \; v(x) < n\}, \\
C &= \{x \; ; \; v(x) = 0\}
\end{aligned}$$

From any state of A, we know that the game is won by EVEN. If $A = X$, EVEN wins from all states, and we can terminate. Otherwise define $B = X \backslash A$ and proceed to Step 2.

Step 2. Consider the game G_B restricted to B. For this game and its associated payoff game, compute

$$C = \{x \; ; \; v(x) = 0\}.$$

On G_B, EVEN cannot visit states with priorities p infinitely often. Thus, $C \neq \emptyset$. Consider G_C restriced to C. Solve recursively G_C (observe that this game only has priorities smaller than p).

Let $D \subset C$ be the states of G_C that are won by ODD.

If $D = \emptyset$, then the game G_B is also completely known by EVEN (indeed, either ODD allows to force EVEN to reach and stay forever in C and loses, or the play visits states of $B \backslash C$ infinitely often, from which EVEN can always get). In such a case, we deduce that EVEN wins in G_B from all states, and we can terminate.

If $D \neq \emptyset$, compute $E = 1 - \mathit{Attr}(D)$ in the game G_B. We know that in G_B, ODD can win from any state of E.

If $E = B$, we can terminate. Otherwise, if $E \neq B$, then we loop to the beginning of Step 2 with B replaced by $B \backslash E$ (observe that from states of $E \neq B$, it is the choices of EVEN that prevent ODD from reaching D, so ODD can still prevent EVEN from visiting states with priority p infinitely often).

To conclude, let us bound the number of calls $C(n)$ for a game of size n. Writing

$$I_c = \{1 \leq k_1, \dots, k_c \leq n \ ; \ \sum_{i=1}^{c} k_c = n\},$$

We have $C(1) = 1$ and

$$C(n) \leq \max_{c} \max_{k_1,\dots,k_c} \sum_{c} C(k_c).$$

It is easy to prove by induction on n that $C(n) \leq n$.