

Consider a parity game G between EVEN (or Player 0) and ODD (or Player 1) on a state space $X = \{1, \dots, n\}$ with priority function $\Omega : X \rightarrow \{1, \dots, d\}$. On infinite plays, ODD wants that among the priorities that appears infinitely often, the maximal one is odd, while EVEN wants it to be even.

I describe below a recursive procedure to solve this problem. Then I argue that the number of recursive calls is at most n .

Suppose for concreteness that we have a game G of which the maximal priority p is even (the odd case is similar). Consider the payoff function

$$\forall x, \quad g(x) = \mathbb{1}_{\Omega(x)=p},$$

and the Bellman operators associated to the corresponding payoff problem: for all $v \in \mathbb{R}^X$,

$$\begin{aligned} T_{\mu,\nu}v &= g + P_{\mu,\nu}v, \\ T_\mu v &= \min_\nu T_{\mu,\nu}v, \\ \tilde{T}_\nu v &= \max_\mu T_{\mu,\nu}v, \\ Tv &= \max_\mu \min_\nu T_{\mu,\nu}v = \max_\mu T_\mu v = \min_\nu \tilde{T}_\nu v, \end{aligned}$$

where $P_{\mu,\nu}$ is the deterministic transition matrix when players use the (positional) strategies μ and ν .

The introduction of the payoff game is motivated by the following observations:

Lemma 1. *In the parity game G ,*

1. *The set of states from which EVEN can force ODD to visit states with priority p infinitely often (and thus win) is $\{x ; [T^{n^2}0](x) \geq n\}$;*
2. *The set of states from which ODD can force EVEN to only visit states with priorities lower than p is $\{x ; [T^k0](x) = 0\}$ for any $k \geq n$.*
3. *From any state x such that $T^k0(x) > 0$, EVEN has a strategy that allows to visit at least one state with priority p in the next k steps (and consequently, if ODD can force EVEN to visit states with priorities p only a finite number times, then the play must reach C in finite time and stay in C forever).*

Proof. Let us begin by the first item. If from state x , EVEN can force ODD to visit states with priority p infinitely often, EVEN can force to visit such states at least once every n steps; on a trajectory of length n^2 , then states with priority p are visited at least n times. And thus on the payoff game, $[T^{n^2}0](x) \geq n$. If from state x , ODD can prevent EVEN from visiting states with priority p infinitely often, this means that ODD has a strategy such that against any policy of EVEN, there is a path of length at most $n - 1$ followed by a trajectory that only visits states with priority smaller than p . On the payoff game, this means that $[T^{n^2}0](x) < n$.

Now let us consider the second item. First (the easier part), assume that for some state x , ODD has a strategy ν such that whatever EVEN does, the trajectory only visits states with priorities smaller than p . Then, in the m -horizon payoff game for any $m \geq n$, this strategy forces the trajectory to only visit states with payoff 0, and thus:

$$0 \leq [T^m0](x) \leq [(\tilde{T}_\nu)^m0](x) \leq 0,$$

i.e. $T^m 0(x) = 0$.

Now, assume that we have a state x such that $T^m(x) = 0$ with $m \geq n$. Let (ν_1, \dots, ν_m) be any sequence of strategies such that

$$T^m 0 = \tilde{T}_{\nu_1} \dots \tilde{T}_{\nu_m} 0.$$

Against any (positional) strategy μ of EVEN, consider the trajectory $(x_0 = x, x_1, \dots, x_n)$ induced if ODD plays with ν_1, \dots, ν_n . There necessarily exists $0 \leq i < j \leq n$ and a state $y \in X$ such that $x_i = x_j = y$. We have

$$[T_{\mu, \nu_1} \dots T_{\mu, \nu_{i-1}} T_{\mu, \nu_i} \dots T_{\mu, \nu_{j-1}} T^{m-j} 0](x) \leq [T^m 0](x) = 0.$$

Since all the payoffs are non-negative, we deduce that

$$\begin{aligned} [T^{m-j} 0](y) &= 0, \\ [T_{\mu, \nu_i} \dots T_{\mu, \nu_{j-1}} 0](y) &= 0, \\ [T_{\mu, \nu_1} \dots T_{\mu, \nu_{i-1}} 0](x) &= 0. \end{aligned}$$

Thus, if ODD plays the infinite-horizon policy $\nu_1, \dots, \nu_{i-1}(\nu_i \dots \nu_{j-1})^\infty$ against EVEN playing μ , then the infinite-horizon payoff of this play is 0. In G , this means that from x , the priorities visited are all smaller than p .

The final item is obvious by considering the strategy μ_1, \dots, μ_k such that

$$T^k 0 = T_{\mu_1} \dots T_{\mu_k} 0.$$

□

Step 1. Compute the set

$$A = \{x ; [T^{n^2} 0](x) \geq n\}.$$

From any state of A , we know (by the Lemma) that the game is won by EVEN. If $A = X$, EVEN wins from all states, and we can terminate. Otherwise define $B = X \setminus A$ and proceed to Step 2.

Step 2. Consider the game G_B restricted to B . For this game and its associated payoff game, compute

$$C = \{x ; [T^n 0](x) = 0\}.$$

On G_B , ODD can prevent EVEN from visiting states with priorities p infinitely often. Thus, necessarily $C \neq \emptyset$. Consider G_C restricted to C . Solve recursively G_C (observe that this game only has priorities smaller than p).

Let $D \subset C$ be the states of G_C that are won by ODD.

If $D = \emptyset$, then the game G_B is also completely known by EVEN (indeed, either ODD allows to force EVEN to reach and stay forever in C , and loses, or the play visits states of $B \setminus C$ infinitely often, from which EVEN can always reach a state with priority p). In such a case, we deduce that EVEN wins in G_B from all states, and we can terminate.

If $D \neq \emptyset$, compute $E = 1 - Attr(D)$ in the game G_B . We know that in G_B , ODD can win from any state of E .

If $E = B$, we can terminate. Otherwise, if $E \neq B$, then we loop to the beginning of Step 2 with B replaced by $B \setminus E$ (observe that from states of $E \neq B$, it is the choices of EVEN that prevent ODD from reaching D , so ODD can still prevent EVEN from visiting states with priority p infinitely often).

To conclude, let us bound the number of calls $C(n)$ for a game of size n . Writing

$$I_c = \{1 \leq k_1, \dots, k_c \leq n ; \sum_{i=1}^c k_i = n\},$$

We have $C(1) = 1$ and

$$C(n) \leq \max_c \max_{k_1, \dots, k_c} \sum_c C(k_i).$$

It is easy to prove by induction on n that $C(n) \leq n$.