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Consider a parity game G between EVEN (or Player 0) and ODD (or Player 1) on a state space $X = \{1,\dots,n\}$ with priority function $\Omega : X \to \{1,\dots,d\}$. On infinite plays, ODD wants that among the priorities that appears infinitely often, the maximal one is odd, while EVEN wants it to be even.

Consider the lexicographic order relation on $\mathbb{N}^2$:

$$(a,b) \leq (a',b') \Leftrightarrow a \leq a' \text{ or } (a = a' \text{ and } b \leq b'),$$

the corresponding min/max/+/- operators.

Consider the cost functions:

$$g_p = \mathbb{1}_{\Omega(x)=p} * (-1)^{\Omega(x)}.$$

Consider the following Bellman operators:

$$T_{p,\mu,\nu}(v,w) = (v, g_p + P_{\mu,\nu} w)$$
$$T_p(v,w) = \max_\mu \min_\nu T_{p,\mu,\nu}(v,w)$$

Start with $v = (0,0,\dots,0)$.

- While there exists x such that $v(x) = 0$, do
  - let $A = X$.
  - Repeat:
    - Consider the game reduced to A and the highest priority p.
    - Compute
      $$B = \{x \; ; \; [(T_p)^n(v,0)](x) = (0,0)\}$$
    - If $B = \emptyset$, then set
      $$\forall x \in A, \; v(x) = p * (-1)^p$$
      and break the repeat loop
    - Otherwise, let $A \leftarrow B$.
- Return v

Idea: Consider priority p belonging to Player $i = p \mod 2$ and a play on the sub-game A_p. There exists of policy for Player i such that the only possibility for Player $1-i$ to avoid losing with priority p on A_p is to reach A_{p-1} and stay in it forever. When this set is empty, it means that Player i wins (with that policy) with priority p on the set A_p, so we update the value consequently. When this set is non-empty, we know that Player $1-i$ has a policy such that Player i cannot escape A_{p-1}, on which Player i cannot win with priority p (so from A_{p-1}, this is what Player $1-i$ wants to do).

Decouple v and w (only useful on the first step) ?:
Consider the following operators:

$$\begin{aligned}
T_{p,\mu,\nu} w &= g_p + P_{\mu,\nu} w \\
T_p w &= \max_{\mu} \min_{\nu} T_{p,\mu,\nu} w \\
U_{\mu,\nu} v &= P_{\mu,\nu} v \\
U v &= \max_{\mu} \min_{\nu} U_{\mu,\nu} v.
\end{aligned}$$

Start with $v = (0, 0, \dots, 0)$.

- Compute $w = U^n v$.
  - Let
    $$A = \{x \ ; \ w(x) = 0\}.$$
    If $A = \emptyset$, return w.
  - Repeat:
    * Consider the game reduced to A and the highest priority p in it.
    * Compute
      $$B = \{x \ ; \ [(T_p)^n 0)](x) = 0\}$$
    * If $B = \emptyset$, then set
      $$\forall x \in A, \ v(x) = p * (-1)^p$$
      and break the repeat loop
    * Otherwise, let $A \leftarrow B$ and iterate.