# A Polynomial-Time Solution for Max-Affine Fixed-Point Equations

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**Abstract**

We consider systems of max-affine fixed-point equations and propose a polynomial-time algorithm to compute their solution set. The algorithm iteratively solves linear programs and removes constraints that cannot participate in any solution, ensuring correctness and polynomial complexity. This approach applies to several well-known problems in theoretical computer science, including stochastic games, mean-payoff games, parity games, and linear complementarity problems with P-matrices. The results are preliminary, have not been peer-reviewed and may contain errors or important gaps; feedback and corrections are welcome.

**Disclaimer:** *This preprint proposes a simple solution to some open problems in computer science. It has not been peer-reviewed and may contain errors or important gaps. Feedback and corrections are welcome.*

We consider a system of fixed-point equations of a max-affine operator on $\mathbb{R}^n$:

$$\forall i, \quad x_i = \max_{k \in K_i} [T^{(k)} x]_i \tag{1}$$

where for each $k$ and any $y \in \mathbb{R}^n$, $T^{(k)} y$ is a vector in $\mathbb{R}^n$ whose $i$-th coordinate is:

$$[T^{(k)} y]_i = \sum_j a_{ij}^{(k)} y_j + b_i^{(k)}.$$

We describe an iterative algorithm indexed by $t$ to find the set $X^*$ (possibly empty) of solutions to equation (1).

At $t = 0$, for each $i$, we set $K_i^{(0)} = K_i$.

At each step $t$, we consider the optimization problem

$$\min_{x \in \mathbb{R}^n} \max_{i, k \in K_i^{(t)}} \quad x_i - [T^{(k)} x]_i \tag{2}$$
$$\text{subject to: } \forall i, \quad \forall k \in K_i^{(t)}, \quad x_i \geq [T^{(k)} x]_i.$$

It is easy to see that this is a linear program.

Let $z^{(t)}$ be the optimum of problem (2), and $(x^{(t)}, i^{(t)}, k^{(t)})$ a tuple of indices corresponding to an optimal solution.

- If $z^{(t)} = 0$, the algorithm terminates, and the set $X^*$ is the set of $x$ satisfying

$$\forall i, \quad \forall k \in K_i^{(t)}, \quad x_i = [T^{(k)} x]_i,$$

  which defines either a singleton or a polytope.

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- If $z > 0$, we update the sets as follows:

$$\begin{aligned} K_{i^{(t)}}^{(t+1)} &= K_{i^{(t)}}^{(t)} \setminus \{k^{(t)}\} \\ \forall i \neq i^{(t)}, \quad K_i^{(t+1)} &= K_i^{(t)} \end{aligned}$$

  If $K_{i^{(t)}}^{(t+1)}$ is empty, then the algorithm terminates and $X^*$ is empty. Otherwise we go on with the next iteration.

This algorithm stops after at most $\sum_i |K_i| - n$ iterations and linear program resolutions. By rewriting system (1) using $n \sum_i \lceil \log_2 K_i \rceil$ variables and maxes over two parameters, the number of linear programs to solve can be reduced to $(n-1) \sum_i \lceil \log_2 K_i \rceil$.

The correctness of the algorithm follows from the fact (true at iteration 0 and inherited at each subsequent iteration) that as long as the constraint set of the linear program (2) contains the solution set $X^*$, for any $x^* \in X^*$, at each step $t$ we have

$$x_{i^{(t)}}^* - [T^{(k^{(t)})} x^*]_{i^{(t)}} \geq x_{i^{(t)}}^{(t)} - [T^{(k^{(t)})} x^{(t)}]_{i^{(t)}} = z > 0.$$

In other words, the constraint corresponding to indices $(i^{(t)}, k^{(t)})$ does not participate in the characterization of $X^*$, and can safely be removed.

A linear program can be solved in time $\tilde{O}(n^3 L)$. Therefore, the algorithm described here has polynomial complexity $\tilde{O}(n^4 L)$. In particular, it allows solving in polynomial time several problems that can be expressed in the form (1):

- Turn-based stochastic games on a graph with discount factor $\gamma$ [3] (with a dependence on $\log \frac{1}{1-\gamma}$);
- Mean payoff games with discount factor $\gamma$ [4] (with a dependence on $\log \frac{1}{1-\gamma}$);
- Mean payoff games (by reduction to the mean payoff games with discount factor $\gamma = 1 - \frac{1}{4n^3 W}$, where $W$ is a bound on the integer cost [4]);
- Parity games with $d$ priorities (by reduction to mean payoff games with maximum cost $W$ in $n^d$ [2]);
- Linear complementarity problems with a P-matrix (without dependence on the condition number of the matrix [1]).

## References


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