A polynomial algorithm for the parity game

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Abstract

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Parity and mean-payoff games Given an arena $\mathcal{G} = (X = [1,n] = X_0 \sqcup X_1, E = [1,m], p)$ with $(n,m) \in \mathbb{N}^2$, a *parity game* is a game played by two players, ODD and EVEN. (X,E) is a directed graph. X is a set of n nodes and E a set of m directed edges such that each node has a least one outgoing edge. The set of nodes X is partitioned into a set of states X_1 belonging to ODD and a set of nodes X_0 belonging to EVEN. $p : X \to [1,d]$, known as a priority function, assigns an integer label to each node of the graph. A play is an infinitely long trajectory $(x_0, x_1, \dots)$ generated from some starting state x_0: at any time step t, the player to which the node x_t belongs chooses x_{t+1} among the adjacent nodes from x_t (following any of the outgoing edges of E starting from x_t). The winner of the game is decided from the infinte sequence of priorities $(p(x_0), p(x_1), \dots)$ occuring through the play: if the highest priority occurring infinitely often is odd, then ODD wins. Otherwise (if it is even), EVEN wins.

A *mean-payoff game* is a game played by two players, Max and Min, on a arena $\mathcal{G} = (X = [1,n] = X_1 \sqcup X_0, E = [1,m], w)$ similar to that of a parity game ; the only difference is that the priority function is replaced by a cost function $w : X \to [-W, W]$ where $W \in \mathbb{N}$. The dynamics of the game is the same as above. On potential plays $(x_0, x_1, \dots)$ induced by the players' choices, Max wants to maximize $\liminf_{t\to\infty} \frac{1}{t}\sum_{i=1}^t w(x_i)$ while Min wants to minimize $\limsup_{t\to\infty} \frac{1}{t}\sum_{i=1}^t g(x_i)$. Ehrenfeucht and Mycielski [1979] have shown that for each starting node x_0, such a game has a value $\nu(x_0)$, the optimal mean-payoff from x_0, such Max has a strategy to ensure that $\limsup_{t\to\infty} \frac{1}{t}\sum_{i=0}^t w(x_i) \geq \nu(x_0)$ and Min has a strategy to ensure that $\limsup_{t\to\infty} \frac{1}{t}\sum_{i=0}^t g(x_i) \leq \nu(x_0)$.

For both games, it is known (cf. Zielonka [1998] and Ehrenfeucht and Mycielski [1979]) that there exist optimal strategies that are positional (i.e. that are mapping from nodes to outgoing edges). In particular, when both players follow these positional strategies from some state x_0, the play follows a (potentially empty) path followed by an infinitely-repeated cycle, in other words an optimal cycle $(x_1^*, \dots, x_c^*)$.

1 The optimal priority of a parity game

Puri [1996] has introduced the following reduction of any parity game $\mathcal{G} = (X = [1,n] = X_1 \sqcup X_0, E = [1,m], p)$ to a mean-payoff game $\mathcal{G}' = (X = [1,n] = X_1 \sqcup X_0, E = [1,m], w)$ that involved the exact same graph (X,E) and the weight function:

$$\forall x, \quad w(x) = (-n)^{p(x)}.$$

Indeed, consider an optimal cycle $(x_1^*, \dots, x_c^*)$ of the mean-payoff game (using the above-mentionned positional strategies). Let $p = \max_{1 \leq i \leq c} p(x_i^*)$ be the maximal priority obtained in this cycle. If p is even

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then

$$0 < n^p - (n-1)n^{p-1} \leq \sum_{i=1}^{c} (-n)^{p(x_i^*)}.$$

If p is odd, we similarly have:

$$0 > -n^p + (n-1)n^{p-1} \geq \sum_{i=1}^{c} (-n)^{p(x_i^*)}.$$

As a consequence, from any starting node x_0, EVEN (resp. ODD) wins the parity game if the value $v(x_0)$ of the mean-payoff game is positive (resp. negative). Furthermore, an optimal pair of strategies for the mean-payoff game is also optimal for the parity game (note that the opposite is in general not true).

We shall consider a slight variation of Puri's reduction that consists in choosing the alternative weight function:

$$\forall x, \quad w(x) = (-K)^{p(x)}$$

with any K such that $K - (n-1) > n^2$ (for instance one may take $K = (n+1)^2$).

Consider an optimal cycle $(x_1^*, \ldots, x_c^*)$ of this mean-payoff game. Let $p = \max_{1 \leq i \leq c} p(x_i^*)$ be the maximal priority obtained in this cycle. If p is even then

$$n^2 K^{p-1} < (K-(n-1))K^{p-1} = K^p - (n-1)K^{p-1} \leq \sum_{i=1}^{c} (-K)^{p(x_i^*)} \leq nK^p,$$

and the value $v(x_0)$ from any state x_0 that reaches this cycle is such that

$$nK^{p-1} \leq \frac{n^2 K^{p-1}}{c} < v(x_0) \leq \frac{n}{c} K^p \leq nK^p.$$

When p is odd, we have

$$-n^2 K^{p-1} > -(K-(n-1))K^{p-1} = -K^p + (n-1)K^{p-1} \geq \sum_{i=1}^{c} (-K)^{p(x_i^*)} \geq -nK^p,$$

and the value $v(x_0)$ from any state x_0 that reaches this cycle is such that

$$-nK^{p-1} \geq \frac{-n^2 K^{p-1}}{c} > v(x_0) \geq -\frac{n}{c} K^p \geq -nK^p.$$

From any starting node x_0, we shall say that the *optimal priority* p is the unique value p such that $nK^{p-1} < v(x_0) \leq nK^p$ or $-nK^{p-1} > v(x_0) \geq -nK^p$. If this priority is even (resp. odd), EVEN (resp. ODD) wins the parity game from x_0.

Through this slightly modified reduction, one makes the parity game more precise: from any starting state, each player that loses tries to make the priority with which the game is won by the other player as low as possible.

2 An algorithm for computing the optimal parity

We now describe a recursive algorithm that computes *optimal parity* $p_*(x)$ for all states x.

Terminal condition: If the parity game only contains one priority p, then we know that for all states, the optimal parity is p.

Recursion When there are at least two priorities, let p be the maximal parity. For concreteness, let us assume that p is even. Let us consider the sub-problem whether EVEN can force ODD to win a game with priority p or whether ODD can force EVEN to cycle in states with priorities (strictly) smaller than p (in the original game, ODD may win or lose, but if he loses, it will be with a parity smaller than p). This sub-problem can be cast as a mean payoff game with weight function:

$$\forall x,\ w(x) = \mathbb{1}_{p(x)=p}.$$

Indeed, writing v the optimal value of this mean-payoff game,

Consider the finite k-horizon solution to this problem for $k = 1, 2, \ldots$: starting with $v_0(x) = 0$, we have

$$\forall x \in X_0, \quad v_{k+1}(x) = w(x) + \max_{y;(x,y)\in E} v_k(y),$$
$$\forall x \in X_1, \quad v_{k+1}(x) = w(x) + \min_{y;(x,y)\in E} v_k(y).$$

It is well known that $\frac{v_k}{k}$ tends to v when k tends to ∞. As we are going to see,

$$A = \{\ x\ ;\ v_N(x) \geq n\ \}$$
$$B = \{\ x\ ;\ 0 < v_N(x) < n\ \}$$
$$C = \{\ x\ ;\ v_N(x) = 0\}.$$

Lemma 1. *The infinite-horizon mean payoff game is won by EVEN on A and by ODD on $B \cup C$. On an optimal play, none of the states $x \in B$ appears on a cycle.*

We recursively solve the parity game restricted to the set C, a game which only contains priorities (strictly) smaller than p, i.e. obtain for each node $x \in B$ its optimal parity $p_*(x)$. From this, we can propagate this optimal parity from B to A by iterating (at most n times):

$$\forall x \in X_0 \cap A, \quad p_*(x) = \max_{y;(x,y)\in E} p_*(y),$$
$$\forall x \in X_1 \cap A, \quad p_*(x) = \min_{y;(x,y)\in E} p_*(y),$$

where the max and min operators above use the order relation $\preceq$ on priorities:

$$p \prec p' \Leftrightarrow (-2)^p < (-2)^{p'}.$$

As there is only one recursive call, and as the maximal priority necessarily decreases at each iteration, the above procedure takes at most d iterations, and

Theorem 1. *A parity game can be solved in polynomial time.*

References

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