A Polynomial-Time Solution for Max-Affine Fixed-Point Equations

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Abstract

We consider systems of max-affine fixed-point equations and propose a polynomial-time algorithm to compute the solution set. The algorithm solves a sequence of linear programs and removes constraints that cannot participate in the solution, ensuring correctness and polynomial complexity. This approach applies to several well-known problems in theoretical computer science, including stochastic games, mean-payoff games, parity games, and linear complementarity problems with P-matrices.

Disclaimer: *This preprint proposes a simple solution to some open problems in computer science. It has not been peer-reviewed and may contain errors or important gaps. Feedback and corrections are welcome.*

Let n and m be integers. For any $x \in \mathbb{R}^n$, we shall write x_i its *ith*-coordinate. Let I be the set $\{1, 2, \ldots, n\}$. Let K be the set $\{1, 2, \ldots, m\}$. For any $(i, j) \in I^2$ and $k \in K$, let $a_{ij}^{(k)}$ and $b_i^{(k)}$ be real numbers.

We consider the following system of equations with unknown variable $x \in \mathbb{R}^n$:

$$\forall i \in \{1, 2, \ldots, n\}, \quad x_i = \max_{k \in K} \sum_j a_{ij}^{(k)} x_j + b_i^{(k)}. \tag{1}$$

For any $x \in \mathbb{R}^n$, write $T^{(k)}x$ and Tx the vectors such that for all $i \in I$,

$$[T^{(k)}x]_i = \sum_j a_{ij}^{(k)} x_j + b_i^{(k)},$$

$$[Tx]_i = \max_{k \in K}[T^{(k)}x]_i.$$

For each $k \in K$, $T^{(k)}$ is an affine operator. With these notations, the system of equations 1 is equivalent to the fixed point equation

$$x = Tx.$$

Let

$$X^* = \{x \; ; \; x = Tx\}$$

denote the set (possibly empty) of solutions of this system.

We shall describe an iterative algorithm indexed by t, to compute X^*.

At the initial iteration $t = 1$ of the algorithm, for each $i \in I$, we set $K_i^{(0)} = K$.

At each step t, we consider the subset of $\mathbb{R}^n$

$$\mathcal{C}^{(t)} = \{x \; ; \; x \geq Tx\} = \{x \; ; \; \forall i \in I, \; \forall k \in K_i^{(t)}, \; x_i \geq [T^{(k)}x]_i\}.$$

Since the operators $T^{(k)}$ are affine, this set is convex.

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Remark 1. *We have $X^* \subset \mathcal{C}^{(1)}$.*

Consider the optimization problem

$$\min_{x \in \mathcal{C}^{(t)}} \max_{i \in I} \max_{k \in K_i^{(t)}} \ x_i - [T^{(k)}x]_i. \tag{2}$$

Remark 2. *Observe that Problem 2 is* different from *the residual minimization formulation of the fixed point equation 1:*

$$\min_{x \in \mathcal{C}^{(t)}} \max_{i \in I} \ x_i - [Tx]_i$$

which can be reformulated as

$$\min_{x \in \mathcal{C}^{(t)}} \max_{i \in I} \min_{k \in K_i^{(t)}} \ x_i - [T^{(k)}x]_i.$$

In Problem 2, we do *maximize over the variable k.*

We can rewrite Problem 2 as follows:

$$\min_{x \in \mathcal{C}^{(t)}, z \in \mathcal{D}^{(t)}} z \tag{3}$$

with set:

$$\mathcal{D}^{(t)} = \{z \in \mathbb{R} \; ; \; \forall i \in I, \ \forall k \in K_i^{(t)}, \ z \geq x_i - [T^{(k)}x]_i\}.$$

The optimization problem 3 is a linear program, whose minimal value $z^{(t)}$ is non-negative. Let $(x^{(t)}, i^{(t)}, k^{(t)})$ be (any) parameter values corresponding to the optimal value $z^{(t)}$.

- If $z^{(t)} = 0$, the algorithm terminates, and X^* is set of solutions of the linear system

$$\forall i, \ \ \forall k \in K_i^{(t)}, \ \ x_i = [T^{(k)}x]_i.$$

- If $z^{(t)} > 0$, we update the sets as follows:

$$K_{i^{(t)}}^{(t+1)} = K_{i^{(t)}}^{(t)} \setminus \{k^{(t)}\}$$
$$\forall i \neq i^{(t)}, \ \ K_i^{(t+1)} = K_i^{(t)}$$

If $K_{i^{(t)}}^{(t+1)}$ is empty, then the algorithm terminates and we know that $X^* = \emptyset$. Otherwise we go on with the next iteration.

This algorithm stops after at most $(n-1)m$ iterations and linear program resolutions.

Remark 3. *By rewriting system (1) using $n\lceil \log_2 m \rceil$ variables and maxes over two parameters, the number of linear programs to solve can be reduced to $2(n\lceil \log_2 m \rceil - 1)$.*

The correctness of the algorithm follows from the fact, true at iteration $t = 1$ and inherited at each subsequent iteration, that as long as the constraint set $\mathcal{C}^{(t)}$ of the linear program (3) contains the whole set X^*, we have

$$\forall \mathbf{x} \in X^*, \ \ \mathbf{x}_{i^{(t)}} - [T^{(k^{(t)})}\mathbf{x}]_{i^{(t)}} \ \geq \ x_{i^{(t)}}^{(t)} - [T^{(k^{(t)})}x^{(t)}]_{i^{(t)}} \ = \ z^{(t)} \ > \ 0.$$

In other words, the constraint corresponding to indices $(i^{(t)}, k^{(t)})$ does not participate in the characterization of X^*, and can safely be removed.

A linear program with n variables and an input encoding of size L can be solved in time $\tilde{O}(n^3 L)$. Therefore, the algorithm described here has polynomial complexity $\tilde{O}(n^4 L)$. In particular, it allows solving in polynomial time several problems that can be expressed in the form (1):

- Turn-based stochastic games on a graph with discount factor γ [3] (with a dependence on $\log \frac{1}{1-\gamma}$);

- Mean payoff games with discount factor γ [4] (with a dependence on $\log \frac{1}{1-\gamma}$);

- Mean payoff games (by reduction to the mean payoff games with discount factor $\gamma = 1 - \frac{1}{4n^3 W}$, where W is a bound on the integer cost [4]);

- Parity games with d priorities (by reduction to mean payoff games with maximum cost W in n^d [2]);

- Linear complementarity problems with a P-matrix (without dependence on the condition number of the input matrix [1]).

References

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