1 Threshold procedure

BOF BOF

We consider here the following situation:

Assumption 1. *Both players can make sure that the parities by which games are won are smaller than p.*

Let $i \equiv p \pmod 2$ be the player that can win a game with parity p. We want to compute the set W_i for which Player i can win with parity p and its complementary W_{1-i} for which he cannot (i.e. for which Player $1-i$ can force Player i to get a parity stricly smaller than p).

For $j \in \{0, 1\}$, consider the following set of vertices:

$$\begin{aligned}
L_i &= \{\ x\ ;\ p(x) = p\ \}, \\
L_{1-i} &= \{\ x\ ;\ p(x) < p\ \}, \\
L &= L_i \sqcup L_{1-i} = \{\ x\ ;\ p(x) \leq p\ \}, \\
H_j &= \{\ x\ ;\ p(x) > p \text{ and } p(x) \equiv j \pmod 2\ \}, \\
H &= H_i \sqcup H_j = \{\ x\ ;\ p(x) > p\ \}.
\end{aligned}$$

With these notations, Assumption 1 translates formally to

$$\begin{aligned}
W_0 &= \mathrm{Attr}_{G,0}(W_0 \cap L), \\
W_1 &= \mathrm{Attr}_{G,1}(W_1 \cap L).
\end{aligned}$$

Consider the set

$$A = \mathrm{Attr}_i(L_i) \backslash H_{1-i}.$$

Player i can ensure that every play that visits A infinitely often is won with priority p. If $A = V$, then we know that $W_i = V$ and $W_{1-i} = \emptyset$.

If $A \neq V$, consider the game G' restricted to $V' = V \backslash A$. Since $(L_i \cup H_i) \cap H_{1-i} = \emptyset$ and $L_i \cup H_i \subset \mathrm{Attr}_i(L_i \cup H_i)$, we deduce that $L_i \cup H_i \subset A$, and therefore all the priorities in G' are either strictly smaller than p or equal to $1-i$ modulo 2, and since Player i cannot escape from G' (since A an i-attractor), Player $1-i$ can prevent Player i to get a priority p on V', and therefore also on

$$B = \mathrm{Attr}_{G',1-i}(V').$$

If $B = V$, then we known that $W_i = \emptyset$ and $W_{1-i} = V$.

If $B \neq V$, we recursively call this procedure on the game G'' restricted to $V'' = V \backslash B$ and obtain the sets W''_i and W''_{1-i}. It follows that $W_i = W''_i$ and $W_{1-i} = B \cup W''_{1-i}$.