A polynomial algorithm for the parity game through the notion of optimal priority

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April 6, 2022

Abstract

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Parity and mean-payoff games For any integers $i \leq j$, write $[i,j]$ for the set of integers $\{i, i+1, \dots, j\}$. A parity game between two players, EVEN and ODD, can be described by a tuple $\mathcal{G} = (X = [1,n] = X_0 \sqcup X_1,\ E = [1,m],\ p : X \to [1,d])$ with $(n,m,d) \in \mathbb{N}^3$. (X,E) is a directed graph. X is a set of n nodes and E a set of m directed edges such that each node has at least one successor node. The set of nodes X is partitioned into a set of nodes X_0 belonging to EVE and a set of nodes X_1 belonging to ODD. The function $p : X \to [1,d]$, known as a priority function, assigns an integer label to each node of the graph. A play is an infinitely long trajectory $(x_0, x_1, \dots)$ generated from some starting node x_0: at any time step t, the player to which the node x_t belongs chooses x_{t+1} among any of the outgoing edges of E starting from x_t. The winner of the game is decided from the infinte sequence of priorities $(p(x_0), p(x_1), \dots)$ occuring through the play: if the highest priority occurring infinitely often is even (resp. odd), then EVE (resp. ODD) wins.

A *mean-payoff game* between two players, MAX and MIN, can be described by a tuple $\mathcal{G} = (X = [1,n] = X_1 \sqcup X_0,\ E = [1,m],\ w : X \to [-W,W])$ with $(n,m,W) \in \mathbb{N}^3$ similar to that of a parity game ; the only difference is that the priority function is replaced by a cost function $w : X \to [-W,W]$. The dynamics of the game is the same as above. On potential plays $(x_0, x_1, \dots)$ induced by the players' choices, MAX wants to maximize $\liminf_{t\to\infty} \frac{1}{t}\sum_{i=1}^t w(x_i)$ while MIN wants to minimize $\limsup_{t\to\infty} \frac{1}{t}\sum_{i=1}^t g(x_i)$. Ehrenfeucht and Mycielski [1979] have shown that for each starting node x_0, such a game has a value $\nu(x_0)$, the optimal mean-payoff from x_0, such MAX has a strategy to ensure that $\limsup_{t\to\infty} \frac{1}{t}\sum_{i=0}^t w(x_i) \geq \nu(x_0)$ and MIN has a strategy to ensure that $\limsup_{t\to\infty} \frac{1}{t}\sum_{i=0}^t g(x_i) \leq \nu(x_0)$.

For both games, it is known (cf. Zielonka [1998] and Ehrenfeucht and Mycielski [1979]) that there exist optimal strategies that are positional (i.e. that are mapping from nodes to outgoing edges). In particular, when both players follow these positional strategies from some node x_0, the play follows a (potentially empty) path followed by an infinitely-repeated cycle, in other words an optimal cycle $(x_1^*, \dots, x_{c(x_0)}^*)$ for some $c(x_0) \in [1,n]$.

1 The optimal priority of a parity game

Puri [1996] has introduced the following reduction of any parity game $\mathcal{G} = (X,\ E,\ p)$ to a mean-payoff game $\mathcal{G}' = (X,\ E,\ w)$ that involved the exact same graph (X,E) and the cost function:

$$\forall x, \quad w(x) = (-n)^{p(x)}.$$

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Indeed, consider an optimal cycle $(x_1^*, \ldots, x_c^*)$ in this mean-payoff game (using the above-mentionned positional strategies). Let $p = \max_{1 \le i \le c} p(x_i^*)$ be the maximal priority obtained in this cycle. If p is even then

$$0 < n^p - (n-1)n^{p-1} \le \sum_{i=1}^{c} (-n)^{p(x_i^*)}.$$

If p is odd, we similarly have:

$$0 > -n^p + (n-1)n^{p-1} \ge \sum_{i=1}^{c} (-n)^{p(x_i^*)}.$$

As a consequence, from any starting node x_0, EVEN (resp. ODD) wins the parity game if the value $v(x_0)$ of the mean-payoff game is positive (resp. negative). Furthermore, an optimal pair of strategies for the mean-payoff game is also optimal for the parity game (note that the opposite is in general not true).

We shall consider a slight variation of Puri's reduction that consists in choosing the alternative cost function:

$$\forall x, \quad w(x) = (-K)^{p(x)}$$

with any K such that $K - (n-1) > n^2$ (for instance one may take $K = (n+1)^2$).

Consider an optimal cycle $(x_1^*, \ldots, x_c^*)$ in this mean-payoff game. Let $p = \max_{1 \le i \le c} p(x_i^*)$ be the maximal priority obtained in this cycle. If p is even then

$$n^2 K^{p-1} < (K - (n-1))K^{p-1} = K^p - (n-1)K^{p-1} \le \sum_{i=1}^{c} (-K)^{p(x_i^*)} \le nK^p,$$

and the value $v(x_0)$ from any node x_0 that reaches this cycle is such that

$$nK^{p-1} \le \frac{n^2 K^{p-1}}{c} < v(x_0) \le \frac{n}{c} K^p \le nK^p.$$

When p is odd, we have

$$-n^2 K^{p-1} > -(K - (n-1))K^{p-1} = -K^p + (n-1)K^{p-1} \ge \sum_{i=1}^{c} (-K)^{p(x_i^*)} \ge -nK^p,$$

and the value $v(x_0)$ from any node x_0 that reaches this cycle is such that

$$-nK^{p-1} \ge \frac{-n^2 K^{p-1}}{c} > v(x_0) \ge -\frac{n}{c} K^p \ge -nK^p.$$

From any starting node x_0, we shall say that the *optimal priority* $p_*(x_0)$ is the value p such that $nK^{p-1} < v(x_0) \le nK^p$ or $-nK^{p-1} > v(x_0) \ge -nK^p$ (by our choice of K this value is indeed unique). If this priority is even (resp. odd), EVEN (resp. ODD) wins the parity game from x_0.

Through this slightly modified reduction, one makes the parity game more precise: from any starting node, each player that cannot win tries to make the priority with which the game is won by the other player as low as possible.

2 An algorithm for computing the optimal priority

We shall now describe a recursive algorithm that computes the *optimal priority* $p_* : X \to [1, d]$ of a game $\mathcal{G} = \{X, E, p\}$ that has some similarity with the original algorithm proposed by Zielonka [1998] for computing the winning regions of a parity game.

Terminal condition: If the game $\mathcal{G}$ only contains only one priority q, then we know that for all nodes, the optimal parity is q.

Recursion When the game $\mathcal{G}$ has at least two different priorities, let q be the maximal priority. For concreteness, let us assume that q is even (the other case is similar). Let us consider the sub-problem whether EVEN can win the game with priority q or whether ODD can force EVEN to cycle in nodes with priorities (strictly) lower than q (in $\mathcal{G}$, ODD may win or lose, but if he loses, it will be with a priority lower than q). This sub-problem can be cast as a mean payoff game $\mathcal{G}' = (X,\ E,\ w)$ with cost function with values in $\{0,1\}$:

$$\forall x,\ w(x) = \mathbb{1}_{p(x)=q}.$$

Indeed, writing $\overline{v}$ the optimal value of this mean-payoff game, all nodes x such that $\overline{v}(x) \geq \frac{1}{n}$ are won by EVEN with priority q, and all other nodes x, for which we necessarily have $\overline{v}(x) = 0$, are such that ODD can force EVEN to cycle to nodes with lower priorities. Consider the (finite) k-horizon solutions to this problem for $k = 1, 2, \ldots$: starting with $v_0 = 0$, the optimal k-horizon payoffs can be computed by induction as follows:

$$v_{k+1} = T v_k$$

where for all v, the operator T is defined as follows:

$$\forall x \in X_0, \quad [Tv](x) = w(x) + \max_{y;(x,y)\in E} v(y),$$
$$\forall x \in X_1, \quad [Tv](x) = w(x) + \min_{y;(x,y)\in E} v(y).$$

Zwick and Paterson [1996]) showed that for any mean payoff game with cost function $w : X \to [-W, W]$, $\frac{v_k}{k}$ tends to $\overline{v}$ when k tends to ∞, and $\overline{v}$ can be deduced by rounding from v_N with $N = 4n^3W$ iterations, but such an approach only computes $\overline{v}$ and requires an extra procedure to compute optimal policies. We shall use here a more refined analysis: compute v_N for $N = n^2$. Consider the following partition of X:

$$A = \{\ x\ ;\ v_N(x) = 0\ \},$$
$$B = \{\ x\ ;\ 0 < v_N(x) < n\ \},$$
$$C = \{\ x\ ;\ v_N(x) \geq n\ \}.$$

Lemma 1. *The following properties hold:*

1. *For any state $x \in A$, there exists a strategy for ODD such that for all strategies of EVEN, plays from x in $\mathcal{G}'$ will only visit nodes with cost 0.*

2. *For any state $x \in B$, there exists a strategy for ODD such that for all strategies of EVEN, plays from x in $\mathcal{G}'$ will reach A in at most n steps.*

3. *For any state $x \in C$, there exists a strategy for EVEN such that for all strategies of ODD, plays from x in $\mathcal{G}'$ will visit nodes with cost 1 infinitely often.*

Proof. First of all, since for all $v_0 \geq Tv_0$, and the operator T is monotone in the sense that

$$\forall v, v'\ , v \leq v' \Longrightarrow Tv \leq Tv',$$

the sequence (v_k) is non-decreasing.

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□

We recursively solve the parity game restricted to the set A, i.e. the game $\mathcal{G}\backslash(B \cup C)$, a game which only contains priorities (strictly) smaller than q, i.e. obtain for each node $x \in A$ its optimal parity $p_*(x)$. From this, we can propagate this optimal priority from A to B by iterating (at most n times):

$$\forall x \in X_0 \cap B, \quad p_*(x) = \max_{y;(x,y)\in E} p_*(y),$$
$$\forall x \in X_1 \cap B, \quad p_*(x) = \min_{y;(x,y)\in E} p_*(y),$$

where the max and min operators above use the order relation $\preceq$ on priorities:

$$p \prec p' \Leftrightarrow (-2)^p < (-2)^{p'}.$$

As there is only one recursive call, and as the maximal priority necessarily decreases at each iteration, the above procedure takes at most d iterations, and

Theorem 1. *A parity game can be solved in polynomial time.*

References

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