A polynomial algorithm for the parity game

Bruno Scherrer*

April 7, 2022

Abstract

...

For any integers $i \leq j$, write $[i,j]$ for the set of integers $\{i, i+1, \dots, j\}$.

A parity game between two players, Player 0 and Player 1, can be described by a tuple $\mathcal{G} = (X = [1,n] = X_0 \sqcup X_1,\ E = [1,m],\ \Omega : X \to [1,d])$ with $(n,m,d) \in \mathbb{N}^3$. (X,E) is a directed graph. X is a set of n nodes and E a set of m directed edges such that each node has at least one successor node. The set of nodes X is partitioned into a set of nodes X_0 belonging to Player 0 and a set of nodes X_1 belonging to Player 1. The function $\Omega : X \to [1,d]$ assigns an integer label called priority to each node of the graph. A play is an infinitely long trajectory $(x_0, x_1, \dots)$ generated from some starting node x_0: at any time step t, the player to which the node x_t belongs chooses x_{t+1} among any of the outgoing edges of E starting from x_t. The winner of the game is decided from the infinte sequence of priorities $(\Omega(x_0), \Omega(x_1), \dots)$ occuring through the play: if the highest priority occurring infinitely often is even (resp. odd), then Player 0 (resp. Player 1) wins.

It is known (cf. Zielonka [1998]) that there exist optimal strategies that are positional (i.e. that are mapping from nodes to outgoing edges). In particular, when both players follow these positional strategies from some node x_0, the play follows a (potentially empty) path followed by an infinitely-repeated cycle $(x_1, \dots, x_{c(x_0)})$ for some $c(x_0) \in [1,n]$.

The goal of this paper is to describe a polynomial algorithm for computing optimal strategies for both players.

1 An incremental procedure

Consider a parity game $\mathcal{G}$ that has at least two different priorities. Let p be the maximal priority and let $i \equiv p \mod 2$ be the corresponding player. In this section, we shall consider the sub-problem whether Player i can win the game with priority p or whether Player $1-i$ can force Player i to cycle in nodes with priorities (strictly) lower than p (in $\mathcal{G}$, Player $1-i$ may win or lose, but if he loses, it will be with a priority lower than p).

Consider the total payoff game $\mathcal{G}' = (X,\ E,\ g)$ with cost function

$$\forall x,\ g(x) = (-1)^p \mathbb{1}_{\Omega(x)=p},$$

in which Player 0 wants to maximize the total cost on induced infinitely-long trajectories while Player 1 wants to minimize it.

For any node x, define the possible transitions:

$$\mathrm{succ}(x) = \{\ y; (x,y) \in E\ \}.$$

Let N and M be the set of decision rules respectively for Player 0 and Player 1:

$$M = \{\ \mu : X_0 \to X\ ;\ \forall x \in X_0,\ \mu(x) \in \mathrm{succ}(x)\ \},$$
$$N = \{\ \mu : X_1 \to X\ ;\ \forall x \in X_1,\ \mu(x) \in \mathrm{succ}(x)\ \}.$$

*INRIA, Université de Lorraine, bruno.scherrer@inria.fr

For any pair of decision rules $\mu \in M$ and $\nu \in N$, let $P_{\mu,\nu}$ be the corresponding transition matrix

$$\forall x \in X_0,\ \forall y \in X,\ P_{\mu,\nu}(x,y) = \mathbb{1}_{y=\mu(x)},$$
$$\forall x \in X_1,\ \forall y \in X,\ P_{\mu,\nu}(x,y) = \mathbb{1}_{y=\nu(x)}.$$

Consider the Bellman operators associated to this total payoff game:

$$T_{\mu,\nu} v = g + P_{\mu,\nu} v,$$
$$T v = \max_{\mu \in M} \min_{\nu \in N} \tilde{T}_{\mu,\nu} v.$$

Consider the following functions

$$v_{\mu,\nu} = \lim_{k \to \infty} (T_{\mu,\nu})^k 0,$$
$$v_* = \lim_{k \to \infty} T^k 0.$$

...

The motivation for introducing the total payoff game $\mathcal{G}'$ is the following simple observation:

Lemma 1. *Player i can win the parity game $\mathcal{G}$ with priority p from some starting node x if and only if $|v_*(x)| = \infty$ in the total payoff game $\mathcal{G}'$.*

Corollary 1. *Player $1-i$ can force Player i to cycle on nodes with priorities lower than p from some starting node x if and only if $|v_*(x)| < \infty$ in the total payoff game $\mathcal{G}'$.*

To provide an efficient algorithm for parity games, we shall need to deepen the above observation. For concreteness, assume from now on that the maximal priority p is even (the odd case is similar). Therefore $i = 0$ and for all x, $g(x) \in \{0, 1\}$.

Let Z be the set of nodes from which Player 1 can ensure to visit only nodes with parity (strictly) smaller than p in $\mathcal{G}$, that is nodes with cost 0 in $\mathcal{G}'$.

$$Z = \{x \in X \ ;\ v_*(x) = 0\}\,.$$

It turns out that the Z can be computed in a simple way:

Lemma 2. *The set Z is equal to*

$$\{\ x \in X \ ;\ [T^n 0](x) = 0\ \}\,.$$

Proof. Call Z' the set just described in the lemma. First observe that the operator T is monotone and $0 \leq T0$ (since costs are non-negative). Therefore, the sequence of functions $(T^k 0)_{k \geq 0}$ is non-decreasing. An immediate consequence is that $Z' \subset Z$. Let us prove now that $Z \subset Z'$ by contradiction. Assume that some x is in Z but not in Z', i.e, that $[T^n 0](x) = 0$ and $v_*(x) > 0$. This last inequality implies that Player 0 has a strategy that can ensure, whatever the decisions of Player 1, at least a cost of 1 from x (on the infinite play). Then, necessarily, Player 0 can also obtain it in j steps with $j \in [1, n]$, and thus $[T^n 0](x) \geq [T^j 0](x) > 0$, which is a contradiction. $\square$

If Z is the empty set, then we know that the game $\mathcal{G}$ is won from all nodes with priority p.

From now on, assume that Z is non-empty. For every node belonging to Player 1, define the following set of transitions:

$$\forall x \in X_1 \cap Z,\ N_x = \{y \in \mathrm{succ}(x)\ ;\ y \in Z\},$$
$$\forall x \notin X_1 \cap Z,\ N_x = \emptyset.$$

Let $Y \supset Z$ be the set of nodes from which Player 1 can prevent Player 0 to cycle infinitely often in nodes with priority p:

$$Y = \{ \ x \in X \ ; \ v_*(x) < \infty \ \} = \{ \ x \in X \ ; \ \exists \nu \in N, \ \forall \mu \in M, \ v_{\mu,\nu}(x) < \infty \ \}$$

For any $x \in Y$, Let F_x be the set of by which Player 1 can prevent Player 0 to cycle infinitely often in nodes with priority p from some starting node:

$$F_x = \{ \ \nu \in N \ ; \ \forall \mu \in M, \ v_{\mu,\nu}(x) < \infty \ \} \, .$$

The set of strategies F_x and the set Y can be characterized as follows:

Lemma 3. *We have*

$$F_x = \{ \ \nu \in N \ ; \ \forall \mu \in M, \ \exists j \in [0,n], \ \exists y \in Z, \ \mathbb{1}_x (P_{\mu,\nu})^j = \mathbb{1}_y \ \},$$
$$Y = \{ \ x \in X \ ; \ \exists \nu \in N \ ; \ \forall \mu \in M, \ \exists j \in [0,n], \ \exists y \in Z, \ \mathbb{1}_x (P_{\mu,\nu})^j = \mathbb{1}_y \ \}.$$

Proof. The characterization of F_x is a consequence of the fact that after j steps for some $j \in [0,n]$, the trajectory induced by μ and ν has necessarily entered its limiting cycle, that necessarily has value 0 (otherwise it would contradict the definition of F_Y). The characterization of Y is a consequence of that of F_Y. $\square$

In the parity game literature, Y is called the 1-*attractor set of* Z. Given Z, Y and F_Y can be computed in n steps as follows:

...

In particular, we can observe that F_Y is a rectangular set.

2 An algorithm for the parity game

Terminal condition: If the game $\mathcal{G}$ only contains only one priority p, then we know that for all nodes, the optimal priority is q.

Recursion When the game $\mathcal{G}$ has at least two different priorities, let p be the maximal priority and let $i \equiv p \mod 2$ be the corresponding player. Let us consider the sub-problem whether Player i can win the game with priority p or whether Player $1-i$ can force Player i to cycle in nodes with priorities (strictly) lower than p (Player $1-i$ may win or lose, but if he loses, it will be with a priority lower than p). This sub-problem can be cast as a mean payoff game $\mathcal{G}' = (X, \ E, \ w)$ with cost function:

$$\forall x, \ g(x) = (-1)^p \mathbb{1}_{p(x)=p}.$$

We recursively solve the parity game restricted to the set A, i.e. the game $\mathcal{G} \backslash (B \cup C)$, a game which only contains priorities (strictly) lower than p, i.e. obtain for each node $x \in A$ its optimal priority $p_*(x)$. From this, we can propagate ? this optimal priority from A to B by iterating (at most n times):

$$\forall x \in X_0 \cap B, \ \ p_*(x) = \max_{y;(x,y)\in E} p_*(y),$$
$$\forall x \in X_1 \cap B, \ \ p_*(x) = \min_{y;(x,y)\in E} p_*(y),$$

where the max and min operators above use the order relation $\preceq$ on priorities:

$$p \prec p' \Leftrightarrow (-2)^p < (-2)^{p'}.$$

As there is only one recursive call, and as the maximal priority necessarily decreases at each iteration, the above procedure takes at most d iterations, and

Theorem 1. *A parity game can be solved in polynomial time.*

References

Andrzej Ehrenfeucht and Jan Mycielski. Positional strategies for mean payoff games. *International Journal of Game Theory*, 8:109–113, 1979.

Wiesław Zielonka. Infinite games on finitely coloured graphs with applications to automata on infinite trees. *Theor. Comput. Sci.*, 200(1–2):135–183, June 1998. ISSN 0304-3975. doi: 10.1016/S0304-3975(98)00009-7. URL `https://doi.org/10.1016/S0304-3975(98)00009-7`.