A polynomial algorithm for the parity game

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April 7, 2022

Abstract

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For any integers $i \leq j$, write $[i, j]$ for the set of integers $\{i, i+1, \dots, j\}$.

A parity game between two players, Player 0 and Player 1, can be described by a tuple $\mathcal{G} = (X = [1, n] = X_0 \sqcup X_1,\ E = [1, m],\ \Omega : X \to [1, d])$ with $(n, m, d) \in \mathbb{N}^3$. (X, E) is a directed graph. X is a set of n nodes and E a set of m directed edges such that each node has at least one successor node. The set of nodes X is partitioned into a set of nodes X_0 belonging to Player 0 and a set of nodes X_1 belonging to Player 1. The function $\Omega : X \to [1, d]$ assigns an integer label called priority to each node of the graph. A play is an infinitely long trajectory $(x_0, x_1, \dots)$ generated from some starting node x_0: at any time step t, the player to which the node x_t belongs chooses x_{t+1} among any of the outgoing edges of E starting from x_t. The winner of the game is decided from the infinte sequence of priorities $(\Omega(x_0), \Omega(x_1), \dots)$ occuring through the play: if the highest priority occurring infinitely often is even (resp. odd), then Player 0 (resp. Player 1) wins.

It is known (cf. Zielonka [1998]) that there exist optimal strategies that are positional (i.e. that are mapping from nodes to outgoing edges). In particular, when both players follow these positional strategies from some node x_0, the play follows a (potentially empty) path followed by an infinitely-repeated cycle $(x_1, \dots, x_{c(x_0)})$ for some $c(x_0) \in [1, n]$.

The goal of this paper is to describe a polynomial algorithm for computing optimal strategies for both players.

1 An incremental procedure

Consider a parity game $\mathcal{G}$ that has at least two different priorities. Let p be the maximal priority and let $i \equiv p \mod 2$ be the corresponding player. In this section, we shall consider the sub-problem whether Player i can win the game with priority p or whether Player $1-i$ can force Player i to cycle in nodes with priorities (strictly) lower than p (in $\mathcal{G}$, Player $1-i$ may win or lose, but if he loses, it will be with a priority lower than p).

Consider the total payoff game $\mathcal{G}' = (X,\ E,\ g)$ with cost function

$$\forall x,\ g(x) = (-1)^p \mathbb{1}_{\Omega(x)=p},$$

in which Player 0 wants to maximize the total cost on induced infinitely-long trajectories while Player 1 wants to minimize it.

Let N and M be the set of valid decision rules respectively for Player 0 and Player 1:

$$M = \{\ \mu : X_0 \to X\ ;\ \forall x \in X_0,\ \mu(x) \in \{\ y ; (x, y) \in E \}\ \},$$
$$N = \{\ \mu : X_1 \to X\ ;\ \forall x \in X_1,\ \mu(x) \in \{\ y ; (x, y) \in E \}\ \}.$$

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For any pair of decision rules $\mu \in M$ and $\nu \in N$, let $P_{\mu,\nu}$ be the corresponding transition matrix

$$\forall x \in X_0,\ \forall y \in X,\ P_{\mu,\nu}(x,y) = \mathbb{1}_{y=\mu(x)},$$
$$\forall x \in X_1,\ \forall y \in X,\ P_{\mu,\nu}(x,y) = \mathbb{1}_{y=\nu(x)}.$$

Consider the Bellman operators associated to this total payoff game:

$$T_{\mu,\nu} v = g + P_{\mu,\nu} v,$$
$$Tv = \max_{\mu \in M} \min_{\nu \in N} \tilde{T}_{\mu,\nu} v.$$

Consider the following functions

$$v_{\mu,\nu} = \lim_{k \to \infty} (T_{\mu,\nu})^k 0,$$
$$v_* = \lim_{k \to \infty} T^k 0.$$

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The motivation for introducing the above total payoff game is the following simple observation:

Lemma 1. *Player i can win the parity game $\mathcal{G}$ with priority p from some starting node x if and only if $|v_*(x)| = \infty$.*

Corollary 1. *Player $1-i$ can force Player i to cycle on nodes with priorities lower than p from some starting node x if and only if $|v_*(x)| < \infty$.*

To provide an efficient algorithm for parity games, we shall need to deepen the above observation. For concreteness, assume from now on that the maximal priority p is even (the odd case is similar). Therefore $i = 0$ and for all x, $g(x) \in \{0, 1\}$.

Let Y be the set of nodes from which Player 1 can prevent Player 0 to cycle infinitely often in nodes with parity p:

$$Y = \{\ x \in X\ ;\ v_*(x) < \infty\ \} = \{\ x \in X\ ;\ \exists \nu \in N,\ \forall \mu \in M,\ v_{\mu,\nu}(x) < \infty\ \}$$

Let F be the set of positional strategies by which Player 1 can prevent Player 0 to cycle infinitely often in nodes with parity p from some starting node:

$$F = \{\ \nu \in N\ ;\ \exists x \in X,\ \forall \mu \in M,\ v_{\mu,\nu}(x) < \infty\ \}.$$

Let Z be the set of nodes from which Player 1 only visits nodes with parity (strictly) smaller than p:

$$Z = \{x \in X\ ;\ v_*(x) = 0\}.$$

We shall provide an efficiently-computable characterization of the sets Y, F and Z. Indeed, the set of strategies F and the set Y can be characterized as follows:

Lemma 2. *We have*

$$F = \{\ \nu \in N\ ;\ \exists x \in X,\ \forall \mu \in M,\ \exists j \in [0,n],\ \exists y \in Z,\ \mathbb{1}_x (P_{\mu,\nu})^j = \mathbb{1}_y\ \},$$
$$Y = \{\ x \in X\ ;\ \exists \nu \in N\ ;\ \forall \mu \in M,\ \exists j \in [0,n],\ \exists y \in Z,\ \mathbb{1}_x (P_{\mu,\nu})^j = \mathbb{1}_y\ \}.$$

Proof. The characterization of F is a consequence of the fact that after j steps for some $j \in [0,n]$, the trajectory induced by μ and ν has necessarily entered its limiting cycle, that necessarily has value 0 (otherwise it would contradict the definition of F). The characterization of Y is a consequence of that of F. $\square$

In the parity game literature, Y is called the 1-*attractor set of* Z. Given Z, Y and F can be computed in n steps as follows:

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In particular, we can observe that F is a rectangular set.

Eventually, it turns out that the set Z can also be characterized in a simple way:

Lemma 3. *The set Z is equal to*

$$Z' = \{ \ x \in X \ ; \ [T^n 0](x) = 0 \ \} .$$

Proof. First observe that the operator T is monotone and $0 \leq T0$ (since costs are non-negative). Therefore, the sequence of functions $(T^k 0)_{k \geq 0}$ is non-decreasing. An immediate consequence is that $Z' \subset Z$. Let us prove now that $Z \subset Z'$. Assume that $v_*(x) > 0$ for some x. Then this implies that Player 0 has a strategy that can obtain at least a cost of 1 from x (on the infinite play). Then, necessarily, Player 0 can also obtain it in j steps with $j \in [1, n]$, and thus $[T^n 0](x) \geq [T^j 0](x) > 0$. ☐

2 An algorithm for the parity game

Terminal condition: If the game $\mathcal{G}$ only contains only one priority p, then we know that for all nodes, the optimal parity is q.

Recursion When the game $\mathcal{G}$ has at least two different priorities, let p be the maximal priority and let $i \equiv p \mod 2$ be the corresponding player. Let us consider the sub-problem whether Player i can win the game with priority p or whether Player $1-i$ can force Player i to cycle in nodes with priorities (strictly) lower than p (Player $1-i$ may win or lose, but if he loses, it will be with a priority lower than p). This sub-problem can be cast as a mean payoff game $\mathcal{G}' = (X, \ E, \ w)$ with cost function:

$$\forall x, \ g(x) = (-1)^p \mathbb{1}_{p(x)=p}.$$

We recursively solve the parity game restricted to the set A, i.e. the game $\mathcal{G} \backslash (B \cup C)$, a game which only contains priorities (strictly) lower than p, i.e. obtain for each node $x \in A$ its optimal parity $p_*(x)$. From this, we can propagate ? this optimal priority from A to B by iterating (at most n times):

$$\forall x \in X_0 \cap B, \quad p_*(x) = \max_{y;(x,y) \in E} p_*(y),$$
$$\forall x \in X_1 \cap B, \quad p_*(x) = \min_{y;(x,y) \in E} p_*(y),$$

where the max and min operators above use the order relation $\preceq$ on priorities:

$$p \prec p' \Leftrightarrow (-2)^p < (-2)^{p'}.$$

As there is only one recursive call, and as the maximal priority necessarily decreases at each iteration, the above procedure takes at most d iterations, and

Theorem 1. *A parity game can be solved in polynomial time.*

References

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