A Polynomial-Time Solution for Max-Affine Fixed-Point Equations

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Abstract

We consider systems of max-affine fixed-point equations with at most one solution and propose a polynomial-time algorithm to compute the solution or decide there is none. The algorithm solves a sequence of linear programs and removes constraints that cannot participate in the solution, ensuring correctness and polynomial complexity. This approach applies to several well-known problems in theoretical computer science, including stochastic games, mean-payoff games, parity games, and linear complementarity problems with P-matrices. The results are preliminary, have not been peer-reviewed and may contain errors or important gaps; feedback and corrections are welcome.

Disclaimer: *This preprint proposes a simple solution to some open problems in computer science. It has not been peer-reviewed and may contain errors or important gaps. Feedback and corrections are welcome.*

We consider a system of fixed-point equations of a max-affine operator on $\mathbb{R}^n$:

$$\forall i, \quad x_i = \max_{k \in K_i} [T^{(k)}x]_i \tag{1}$$

where for each k and any $y \in \mathbb{R}^n$, $T^{(k)}y$ is a vector in $\mathbb{R}^n$ whose i-th coordinate is:

$$[T^{(k)}y]_i = \sum_j a_{ij}^{(k)} y_j + b_i^{(k)}.$$

We assume that there is either one solution x^* or none to this system.

We describe an iterative algorithm indexed by t to compute x^* or decide there is no solution.

At $t = 0$, for each i, we set $K_i^{(0)} = K_i$.

At each step t, we consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \max_{i, k \in K_i^{(t)}} \quad & x_i - [T^{(k)}x]_i \\ \text{subject to: } \forall i, \quad \forall k \in K_i^{(t)}, \quad & x_i \geq [T^{(k)}x]_i. \end{aligned} \tag{2}$$

It is easy to see that this is a linear program.

Let $z^{(t)}$ be the optimum of problem (2), and $(x^{(t)}, i^{(t)}, k^{(t)})$ a tuple of indices corresponding to an optimal solution.

- If $z^{(t)} = 0$, the algorithm terminates, and x^* is the solution of the linear system

$$\forall i, \quad \forall k \in K_i^{(t)}, \quad x_i = [T^{(k)}x]_i.$$

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- If $z > 0$, we update the sets as follows:

$$K_{i^{(t)}}^{(t+1)} = K_{i^{(t)}}^{(t)} \setminus \{k^{(t)}\}$$
$$\forall i \neq i^{(t)}, \quad K_i^{(t+1)} = K_i^{(t)}$$

  If $K_{i^(t)}^{(t+1)}$ is empty, then the algorithm terminates and we know that there is no solution. Otherwise we go on with the next iteration.

This algorithm stops after at most $\sum_i |K_i| - n$ iterations and linear program resolutions. By rewriting system (1) using $n \sum_i \lceil \log_2 K_i \rceil$ variables and maxes over two parameters, the number of linear programs to solve can be reduced to $(n-1) \sum_i \lceil \log_2 K_i \rceil$.

The correctness of the algorithm follows from the fact (true at iteration 0 and inherited at each subsequent iteration) that as long as the constraint set of the linear program (2) contains the solution x^*, at each step t we have

$$x^*_{i^{(t)}} - [T^{(k^{(t)})} x^*]_{i^{(t)}} \geq x^{(t)}_{i^{(t)}} - [T^{(k^{(t)})} x^{(t)}]_{i^{(t)}} = z > 0.$$

In other words, the constraint corresponding to indices $(i^{(t)}, k^{(t)})$ does not participate in the characterization of x^*, and can safely be removed.

A linear program with n variables and an input encoding of size L can be solved in time $\tilde{O}(n^3 L)$. Therefore, the algorithm described here has polynomial complexity $\tilde{O}(n^4 L)$. In particular, it allows solving in polynomial time several problems that can be expressed in the form (1):

- Turn-based stochastic games on a graph with discount factor γ [3] (with a dependence on $\log \frac{1}{1-\gamma}$);
- Mean payoff games with discount factor γ [4] (with a dependence on $\log \frac{1}{1-\gamma}$);
- Mean payoff games (by reduction to the mean payoff games with discount factor $\gamma = 1 - \frac{1}{4n^3 W}$, where W is a bound on the integer cost [4]);
- Parity games with d priorities (by reduction to mean payoff games with maximum cost W in n^d [2]);
- Linear complementarity problems with a P-matrix (without dependence on the condition number of the matrix [1]).

References

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