A Polynomial-Time Solution for Max-Affine Fixed-Point Equations

Bruno Scherrer*

December 28, 2025

Abstract

We consider systems of max-affine fixed-point equations and propose a polynomial-time algorithm to compute the solution set. The algorithm solves a sequence of linear programs and removes constraints that cannot participate in the solution, ensuring correctness and polynomial complexity. This approach applies to several well-known problems in theoretical computer science, including stochastic games, mean-payoff games, parity games, and linear complementarity problems with P-matrices.

Let n and m be integers. For any $x \in \mathbb{R}^n$, we shall write x_i its ith-coordinate. Let I be the set $\{1, 2, \ldots, n\}$. Let K be the set $\{1, 2, \ldots, m\}$. For any $(i, j) \in I^2$ and $k \in K$, let $a_{ij}^{(k)}$ and $b_i^{(k)}$ be real numbers.

We consider the following system of equations with unknown variable $x \in \mathbb{R}^n$:

$$\forall i \in \{1, 2, \ldots, n\}, \quad x_i = \max_{k \in K} \sum_j a_{ij}^{(k)} x_j + b_i^{(k)}. \tag{1}$$

For any $x \in \mathbb{R}^n$, write $T^{(k)}x$ and Tx the vectors such that for all $i \in I$,

$$[T^{(k)}x]_i = \sum_j a_{ij}^{(k)} x_j + b_i^{(k)},$$
$$[Tx]_i = \max_{k \in K} [T^{(k)}x]_i.$$

For each $k \in K$, $T^{(k)}$ is an affine operator. With these notations, the system of equations 1 is equivalent to the fixed point equation

$$x = Tx.$$

Let

$$X^* = \{x \; ; \; x = Tx\}$$

denote the set (possibly empty) of solutions of this system.

We shall describe an iterative algorithm indexed by t, to compute X^*.

At the initial iteration $t = 1$ of the algorithm, for each $i \in I$, we set $K_i^{(0)} = K$.

At each step t, we consider the subset of $\mathbb{R}^n$

$$\mathcal{C}^{(t)} = \{x \; ; \; x \geq Tx\} = \{x \; ; \; \forall i \in I, \; \forall k \in K_i^{(t)}, \; x_i \geq [T^{(k)}x]_i\}.$$

Since the operators $T^{(k)}$ are affine, this set is convex.

Remark 1. *We have $X^* \subset \mathcal{C}^{(1)}$.*

*INRIA, bruno.scherrer@inria.fr

If the set $\mathcal{C}^{(1)}$ is empty (which can be checked in polynomial time), it implies that $X^* = \emptyset$ and there is nothing else to do. We therefore assume for what follows that $\mathcal{C}^{(1)}$ is not empty.

Consider the optimization problem

$$\min_{x \in \mathcal{C}^{(t)}} \max_{i \in I} \max_{k \in K_i^{(t)}} \; x_i - [T^{(k)} x]_i. \tag{2}$$

Remark 2. *Observe that Problem 2 is* different from *the residual minimization formulation of the fixed point equation 1:*

$$\min_{x \in \mathcal{C}^{(t)}} \max_{i \in I} \; x_i - [Tx]_i$$

which can be reformulated as

$$\min_{x \in \mathcal{C}^{(t)}} \max_{i \in I} \min_{k \in K_i^{(t)}} \; x_i - [T^{(k)} x]_i.$$

In Problem 2, we do *maximize over the variable k.*

We can rewrite Problem 2 as follows:

$$\min_{x \in \mathcal{C}^{(t)}, z \in \mathcal{D}^{(t)}} z \tag{3}$$

with set:

$$\mathcal{D}^{(t)} = \{ z \in \mathbb{R} \; ; \; \forall i \in I, \; \forall k \in K_i^{(t)}, \; z \geq x_i - [T^{(k)} x]_i \}.$$

The optimization problem 3 is a linear program, whose minimal value $z^{(t)}$ is non-negative. Let $(x^{(t)}, i^{(t)}, k^{(t)})$ be (any) parameter values corresponding to the optimal value $z^{(t)}$.

- If $z^{(t)} = 0$, the algorithm terminates, and X^* is set of solutions of the linear system

$$\forall i, \;\; \forall k \in K_i^{(t)}, \;\; x_i = [T^{(k)} x]_i.$$

- If $z^{(t)} > 0$, we update the sets as follows:

$$\begin{aligned} K_{i^{(t)}}^{(t+1)} &= K_{i^{(t)}}^{(t)} \setminus \{k^{(t)}\} \\ \forall i \neq i^{(t)}, \;\; K_i^{(t+1)} &= K_i^{(t)} \end{aligned}$$

  If $K_{i^{(t)}}^{(t+1)}$ is empty, then the algorithm terminates and we know that $X^* = \emptyset$. Otherwise we go on with the next iteration.

This algorithm stops after at most $(n-1)m$ iterations and linear program resolutions.

Remark 3. *By rewriting system* (1) *using* $n\lceil \log_2 m \rceil$ *variables and maxes over two parameters, the number of linear programs to solve can be reduced to* $2(n\lceil \log_2 m \rceil - 1)$*.*

The correctness of the algorithm follows from the fact, true at iteration $t = 1$ and inherited at each subsequent iteration, that as long as the constraint set $\mathcal{C}^{(t)}$ of the linear program (3) contains the whole set X^*, we have

$$\forall \mathbf{x} \in X^*, \;\; \mathbf{x}_{i^{(t)}} - [T^{(k^{(t)})} \mathbf{x}]_{i^{(t)}} \;\geq\; x_{i^{(t)}}^{(t)} - [T^{(k^{(t)})} x^{(t)}]_{i^{(t)}} \;=\; z^{(t)} \;>\; 0.$$

In other words, the constraint corresponding to indices $(i^{(t)}, k^{(t)})$ does not participate in the characterization of X^*, and can safely be removed.

A linear program with n variables and an input encoding of size L can be solved in time $\tilde{O}(n^3 L)$. Therefore, the algorithm described here has polynomial complexity $\tilde{O}(n^4 L)$. In particular, it allows solving in polynomial time several problems that can be expressed in the form (1):

- Turn-based stochastic games on a graph with discount factor γ [3] (with a dependence on $\log \frac{1}{1-\gamma}$);
- Mean payoff games with discount factor γ [4] (with a dependence on $\log \frac{1}{1-\gamma}$);
- Mean payoff games (by reduction to the mean payoff games with discount factor $\gamma = 1 - \frac{1}{4n^3 W}$, where W is a bound on the integer cost [4]);
- Parity games with d priorities (by reduction to mean payoff games with maximum cost W in n^d [2]);
- Linear complementarity problems with a P-matrix (without dependence on the condition number of the input matrix [1]).

References

[1] Masakazu Kojima, Nimrod Megiddo, Toshihito Noma, and Akiko Yoshise. A unified approach to interior point algorithms for linear complementarity problems: A summary. *Oper. Res. Lett.*, 10:247–254, 1991.

[2] Anuj Puri. *Theory of Hybrid Systems and Discrete Event Systems*. PhD thesis, Berkeley, CA, USA, 1995. UMI Order No. GAX96-21326.

[3] Lloyd S. Shapley. Stochastic games. *Proceedings of the National Academy of Sciences*, page UC Berkeley Press, 1953.

[4] Uri Zwick and Mike Paterson. The complexity of mean payoff games on graphs. *Theor. Comput. Sci.*, 158(1&2):343–359, 1996.