Problem set 4 Spectral Theory

Bruno Seefeld

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1

(1)

Take v in the ideal I, by polar decomposition we have $v = u|v|$, with $|v|$ positive, we can write $|v| = u^*v$, this implies $|v| = |v|^* \in I$, therefore $v^* = |v|^*u^* \in I$.

(2)

Let $I = \{fz|f \in C(\mathbb{Z})\}$ where z is the identity on $\mathbb{D}$. We prove that z^* that does $\mapsto \overline{z}$ is not in I. Suppose there is $f \in I$ with $fz = z^*$, for $x \in \mathbb{R}$ we have $f(x)x = x$ therefore $f(x) = 1$, and $f(ix)(ix) = -ix$ and $f(ix) = -1$, taking $x \to 0$ we see that f is not continuous at 0, a contradiction. Therefore the ideal is not self adjoint.

2

5

Suppose p, q are Murray-von Neumann equivalent, with $p = vv^*$, $q = v^*v$. Let $v = u|v|$ be the polar decomposition of v, with u unitary and $|v|$ self adjoint. We have $p = p^2 = u|v|^2u^*$ and $q = |v|u * u|v| = |v|^2$ clearly $p = uqu^*$.

Now suppose p, q, $1-p, 1-q$ are unitarlly equivalent. Say $1-p = u(1-q)u^* = uu^* - uqu^*$, we have $p = uqu^*$ so we can suppose these pairs are conjugated by the same unitary transformation. Let's calculate $p = p^2 = (uqu^*)(uqu^*) = (uq)(qu^*) = (uq)(uq)^*$, take $v = uq$. we have $p = vv^*$ and $v^*v = qu^*uq = q$, therefore they are Murray-von Neumann equivalent.