Problem set 4 Spectral Theory

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(1)

Take v in the ideal I, by polar decomposition we have $v = u|v|$, with $|v|$ positive, we can write $|v| = u^*v$, this implies $|v| = |v|^* \in I$, therefore $v^* = |v|^*u^* \in I$.

(2)

Let $I = \{fz|f \in C(\mathbb{Z})\}$ where z is the identity on $\mathbb{D}$. We prove that z^* that does $\mapsto \overline{z}$ is not in I. Suppose there is $f \in I$ with $fz = z^*$, for $x \in \mathbb{R}$ we have $f(x)x = x$ therefore $f(x) = 1$, and $f(ix)(ix) = -ix$ and $f(ix) = -1$, taking $x \to 0$ we see that f is not continuous at 0, a contradiction. Therefore the ideal is not self adjoint.

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Suppose p, q are Murray-von Neumann equivalent, with $p = vv^*$, $q = v^*v$. Let $v = u|v|$ be the polar decomposition of v, with u unitary and $|v|$ self adjoint. We have $p = p^2 = u|v|^2u^*$ and $q = |v|u * u|v| = |v|^2$ clearly $p = uqu^*$.

Now suppose p, q, $1-p, 1-q$ are unitarlly equivalent. Say $1-p = u(1-q)u^* = uu^* - uqu^*$, we have $p = uqu^*$ so we can suppose these pairs are conjugated by the same unitary transformation. Let's calculate $p = p^2 = (uqu^*)(uqu^*) = (uq)(qu^*) = (uq)(uq)^*$, take $v = uq$. we have $p = vv^*$ and $v^*v = qu^*uq = q$, therefore they are Murray-von Neumann equivalent.

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Suppose p, q are Murray-von Neumann equivalent, we have $p = uqu^*$ for u unitary. Take $x \in \text{Range}(p)$, then $px = x$, we have $u^*(x) = qu^*(x)$ so $u^*(x) \in \text{Range}(q)$, now take $y \in \text{Range}(q)$ $uy = uqy = puy$ having $y = u^*(puy)$, with all this we get $\text{Range}(q) = u^*\text{Range}p$. By reversing the roles of q and p and u and u^* we have $\text{Range}p = u\text{Range}q$. By using that $(1-p) = u(1-q)u^*$ we also have that $\text{Null}(p) = q\text{Null}(q)$ and $\text{Null}(q) = u^*\text{Null}(p)$.

Now suppose $\text{Range}(p) = u\text{Range}(q)$ and $\text{Range}(q) = u^*\text{Range}(p)$ and $\text{Null}(p) = u\text{Null}(q)$, $\text{Null}(q) = u^*\text{Null}p$. Take $x \in H$, we know from the above that $u^*px = qy$ for some y. We also know $qu^*(1-p)x = 0$, then $qu^*x = qu^*px = qqy = qy$ therefore $u^*px = qu^*x$ and they are Murray-von Neumann equivalent.

So we have the following characterization of Murray-von Neumann equivalence: p, q are Murray-von Neumann equivalent if and only if $\text{Range}(p) = u\text{Range}(q)$ and $\text{Range}(q) = u^*\text{Range}(p)$ and $\text{Null}(p) = u\text{Null}(q)$, $\text{Null}(q) = u^*\text{Null}(p)$.