Syndecity of the set ot return times for skew-products over ergodic systems

Bruno Seefeld

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Abstract

1 Introduction

Let $(X, \mathcal{X}, \mu)$ be a probability space and T a mpt. For an integrable function $f \in L_1(X, \mu)$ we define for each $n \in \mathbb{N}$ the cocycle $a_f(n, x) = \sum_{i=0}^{n-1} f(T^j x)$. This cocycle is important when studying the skew-product over X, i.e, the measurable transformation $S_f : X \times \mathbb{R} \to X \times \mathbb{R}$ given by:

$$S_f(x, t) = (T(x), t + a(1, x)) \tag{1}$$

the iterates of a point (x, t) will be $S_f^n(x, t) = (T^n(x), a_f(n, x))$.

We say that the cocycle a_f is recurrent when for all measurable sets of positive μ measure A we have: for $\epsilon > 0$ we can find $n \in \mathbb{N}$ with $\{T^{-n}(A) \cap A \cap \{|a_f(n,)| < \epsilon\}\}$. There is to say, not only we have recurrence for the dynamics T but the points get back with small sum.

If the dynamical system (X, T) is ergodic and f has zero average, then we can expect that the orbits visit the positive and negative regions in a balanced way so that we can comeback with small sum (this is more intuitive when we imagine f bounded). In fact this is true by the following theorem of Atkinson:

Theorem 1.1 (Atkinson) *For an atomless probability space $(X, \mathcal{X}, \mu)$, an ergodic translation T, and f integrable, a_f is recurrent iff $\int f d\mu = 0$.*

Let ν be the product measure $\mu \times$ Leb. We're interested in the recurrence times for a set $A \times B \subset X \times \mathbb{R}$,i.e, the set $R_\lambda(A \times B) = \{n \in \mathbb{N} | \nu(S_f^{-n}(A \times B) \cap (A \times B)) > \lambda \nu(A \times B)^2\}$. This are the times that the sets comes back to itself bu with positive measure. For the non skew-product case, with only a a measure preserving transformation on a finite measure space Kintchine theorem says that $R_\lambda(A)$ is syndetic, the proof is an aplication of von-Neumann ergodic theorem.