Exercise about cocycles

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January 3, 2025

Abstract

showing that random products of a hyperbolic matrix and a rotation has zero Lyapunov exponent.

1 Non-continuity of Lyapunov exponents

Given two matrices $A_1, A_2 \in \mathrm{GL}_2(\mathbb{R})$ and numbers p_1, p_2 with $p_1 + p_2 = 1, p_1, p_2 > 0$, we can consider the random matrix $A^(n) = A_{i_n} \dots A_{i_1}$ with $i_k \in \{1, 2\}$ with a Bernoulli $B(p_1)$ distribution. We can consider the assymptotic exponential rate of growth of the norm of $A^{(n)}$ given by

$$\lambda^+(A_1, A_2, p_1, p_2) = \lim_{n\to\infty} \frac{1}{n} \log \|A^{(n)}\|. \tag{1}$$

The following theorem [1] says that the quantity λ^+ is continuous on $\mathrm{GL}_2(\mathbb{R})$ and $(0, 1)^2$:

Theorem 1.1 (Bocker-Viana)**.** *The function $\lambda^+ : GL_d(\mathbb{R}) \times GL_d(\mathbb{R}) \times (0, 1)^2 \to \mathbb{R}$ is continuous.*

What happens at $p_1 = 1$? We'll show an example where we don't have continuity of λ^+ at this point.

Let's take

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

For $p_1 = 1$ we have $A^{(n)} = A_1^n$ almost surely, therefore $\lambda^+(A_1, A_2, 1, 0) = \log 2$. In order to simplify notation denote $p_2 = p$ and $X_n = \frac{1}{n} \log \|A^{(n)}\|$, we'll show that if $p > 0$ then $\lambda^+(A_1, A_2, (1-p), p) = 0$, and then there is no continuity at $(A_1, A_2, 1, 0)$.

[1]This is true for dimension d and d matrices

1.1 Probabilistic approach

It's enough to show that the sequence of random variables X_n converge in probability to 0, therefore it cannot converge almost surely to something else.

By observing that $A_1A_2A_1 = A_2$ and $A_2^2 = -Id$ we can write the norm of the products $\|A^{(n)}\|$ as $\|A_1^{n_1}A_2^{n_2}\|$, with the number n_1 being given by a random walk

$$S_{n+1} = \begin{cases} S_n + 1 & \text{if} \quad X_{n+1} = A_1 \\ -S_n & if \quad X_{n+1} = A_2 \end{cases}$$

for example:

$$A^{10} = A_1^4A_2A_1A_2A_1^3 = \tag{2}$$

$$A_1^4A_2(A_1A_2A_1)A_1^2 = A_1^4A_2A_2A_1^2 = \tag{3}$$

$$A_1^4(A_2^2)A_1^2 = A_1^6A_2^2 \tag{4}$$

$$\{S_n\}_{n=1}^{10} = \{1, 2, 3, -3, -2, 2, 3, 4, 5, 6\} \tag{5}$$

We have then $\frac{1}{n}\log\|A^{(n)}\| \leq \frac{n_1}{n}\log 2$. The convergence in probability to 0 of X_n will be estabilished then by showing $\frac{|S_n|}{n} \to 0$ in probability.

Let ξ_i be 1 if $X_i = A_2$ and 0 otherwise. For every product $A^{(n)}$ we can assositate a sequence of runs $\xi_1, \ldots, \xi_n$. For each of these runs we'll consider the random variables ν_k that count the number of zeros between 1's. More formally denote $\tau_0 = 0$ $\tau_k = \inf\{k > \tau_{k-1} | \xi_k = 1\}$ the time that a one appears after the last apperance of a 1 (in position τ_{k-1}). We have then $\nu_k = \tau_k - \tau_{k-1} - 1$.

For example: the run associated with the product 2 is $\{0, 0, 0, 1, 0, 1, 0, 0, 0, 0\}$ so that $\tau_1 = 3, \tau_2 = 5$ and $\nu_1 = 0, \nu_2 = 1$.

Notice that $\mathbb{P}[\nu_k \geq l] = (1-p)^l$, so that the independent random variables ν_k are i.i.d geometrically distributed. We consider the process $T_m = S_{\tau_m - 1}$. Depending on the parity of m we have:

$$T_m = \nu_1 - \nu_2 + \ldots \pm \nu_m. \tag{6}$$

Therefore

$$\mathbb{E}[T_m] = \begin{cases} 0, & \text{if} \quad \text{m} \quad \text{is even} \\ \frac{1-p}{p} & \text{if} \quad \text{m} \quad \text{is odd} \end{cases}$$

and the variance is given by $\frac{1-p}{p^2}m$.
Applying the Chernoff bound we get, for any $\delta > 0$:

$$\mathbb{P}(|T_m| > m^{1/2+\delta}) \leq e^{-\frac{p^2}{2^{2\epsilon}(1-p)}m^{2\delta}}$$

for some ϵ depending only on δ.

Writing $\nu_1 + \nu_2 + \ldots + \nu_k = \tau_k = \tau_0 - 1$ and applying the strong law of large number we get $\tau_k = (\frac{1-p}{p} + o(1))k$. So $T_m = S_{\tau_m - 1} = S_{\Omega(m)}$ and the large deviation argument also holds for S_m.

Combining all this to show $\frac{|S_m|}{m}$ converges in probability to 0: let $\epsilon > 0$, choose m large enough so that

$$\frac{1}{m^{1/2+\delta}} < \epsilon$$

. we have $\mathbb{P}\{\frac{|S_m|}{m} > \epsilon\} \leq e^{-cm^{2\delta}}$ for some positive constant c depending only on p, δ.

1.2 Dynamical approach

[2]

We'll interpret the problem as a linear cocycle over a Bernoulli shift and show that an induced cocycle has 0 Lyapinov exponent, therefore the original one also has this property by

Proposition 1.1. *For a linear cocycle $F : M \times \mathbb{R}^d \to M \times \mathbb{R}^d$ with invariant measure μ and $G : Z \times \mathbb{R}^d \to Z \times \mathbb{R}^d$ an induced cocycle with invariant measure ν, there is a function $c : M \to R, c \geq 1$ such that the Lyapunov exponent $\lambda^+(x, G) = c(c)\lambda^+(x, F)$ μ almost everywhere.*

Let $M = \{0, 1\}^{\mathbb{Z}}$, $A : M \to \mathrm{GL}_2(\mathbb{R})$ be given by

$$A(x_n)_n = \begin{cases} A_1 & \text{if} \quad x_0 = 0 \\ A_2 & \text{if} \quad x_0 = 1 \end{cases}$$

and $Z = \{x \in M | x_0 = 1\}$, the measure μ being Bernoulli with parameter $1 - p$.

The first return map to Z, $g : Z \to Z$, is a shift on infinitely many symbols. More formally, consider $\mathcal{C} : Z \to \mathbb{N}^{\mathbb{Z}}$ given by $\mathcal{C}(x_n)_n = (\nu_n)_n$ with $\nu_n = \tau_n - \tau_{n-1} - 1$, with τ_n being as in the probabilistic approach: the position of the n-th 1, in this case the n-th return time. We have

Proposition 1.2. *With the above $\sigma \circ \mathcal{C} = \mathcal{C} \circ g$*

Proof. Take $x \in Z$, write $\mathcal{C}(x) = (\nu_n)_n$, we have $g(x) = y$, with $y_k = x_{\tau_1 + k}$, so $\mathcal{C}y = (\nu_{n+1})_n = \sigma\mathcal{C}(x)$. □

Notice also that the induced invariant measure ν_I on $\mathbb{N}^{\mathbb{Z}}$is Bernoulli. To see this let's calculate the value of ν_I on cylinders. It's enough to consider the cases

[2]This solution was informed to me by Jairo Bochi

of a one coordinate cylinder, since

$$\begin{aligned}\nu_I([j_1 = a_1; \dots, j_k = a_k]) = \\ \nu(\mathcal{C}^{-1}[j_1 = a_1; \dots, j_k = a_k]) = \\ \prod_l \nu(\mathcal{C}^{-1}[j_l = a_l]).\end{aligned}$$

By invariance it's enough to consider that the coordinate is the 0 one, so

$$\begin{aligned}\nu(\mathcal{C}^{-1})[k] = & \frac{\mu([1 \quad 0_k \quad 1])}{p} = \\ \frac{p^2}{p}(1-p)^k = & p(1-p)^k\end{aligned}$$

summing over k we get that it's in fact a probability measure.

Consider the induced cocycle $B : Z \to \mathrm{GL}_2(\mathbb{R})$ given by $B(x) = A^{r(x)}(x)$, with $r(x) = \tau_1$ being the first return time. We have $B(x) = A_1^{\nu_1} A_2$ and by induction $B^n(x) = A_1^{\nu_n} A_2 A_1^{\nu_{n-1}} A_2 \dots A_1{}_1^{\nu} A_2$. Therefore

$$\begin{aligned}\frac{1}{n} \log \|B^n(x)\| \leq \\ & \frac{1}{n}(\nu_1 - \nu_2 + \nu_3 - \dots \pm \nu_n) = \\ & \frac{1}{n} \sum_{k=1} (-1)^k r(g^k(x))\end{aligned}$$

.

In this approach we see the sum in the inequality above as an alternate Birkhoff sum of an integrable function on $(\mathbb{N}^{\mathbb{Z}}, \nu_I, \sigma)$.

Lemma 1.2. *Let (X, μ, T) be a measure preserving dynamical system and $f \in L_1(X, \mu)$, then $\frac{1}{n} \sum_{k=0}^{n-1} (-1)^k f(T^k(x)) \to 0$ μ a.e.*

Proof. Let $AS_n(X) = \sum_{k=0}^{n-1} (-1)^k f(T^k(x))$ be the n-th alternating sum and S_n be the Birkhoff sum of T^2, we have $AS_{2n-1}(x) = f(x) + S_n(Tx) - S_n(x)$. Applying Birkhoff ergodic theorem we get that the result holds along the odd integers. Analogously it hold for the even integers. So it holds for any subsequence. $\square$

2 Remark

The first approach is slightly more elaborate but we get exponential convergence, at least in probability. By using a "soft analysis" result like Birkhoff ergodic theorem we get a simpler proof, but don't have any infromation about the speed of convergence to 0.