Syndecity of the set ot return times for skew-products over ergodic systems

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Abstract

1 Introduction

Let $(X, \mathcal{X}, \mu)$ be a probability space and T a measure preserving transformation. For an integrable function $f \in L_1(X, \mu)$ we define for each $n \in \mathbb{N}$ the cocycle $a_f(n, x) = \sum_{i=0}^{n-1} f(T^j x)$. This cocycle is important when studying the skew-product over X, i.e, the measurable transformation $S_f : X \times \mathbb{R} \to X \times \mathbb{R}$ given by:

$$S_f(x, t) = (T(x), t + f(x)) \tag{1}$$

the iterates of a point (x, t) will be $S_f^n(x, t) = (T^n(x), t + a_f(n, x))$.

We say that the cocycle a_f is recurrent when for all measurable sets of positive μ measure A we have: for $\epsilon > 0$ we can find $n \in \mathbb{N}$ with $\mu(T^{-n}(A) \cap A \cap \{|a_f(n,)| < \epsilon\}\}) > 0$. There is to say, not only we have recurrence for the dynamics T but the points get back with small sum.

If the dynamical system (X, T) is ergodic and f has zero average, then we can expect that the orbits visit the positive and negative regions in a balanced way so that we can comeback with small sum (this is more intuitive when we imagine f bounded). In fact this is true by the following theorem of Atkinson:

Theorem 1.1 (Atkinson) *For an atomless probability space $(X, \mathcal{X}, \mu)$, an ergodic translation T, and f integrable, a_f is recurrent iff $\int f d\mu = 0$.*

Let ν be the product measure $\mu \times$ Leb. We're interested in the recurrence times for a set $A \times B \subset X \times \mathbb{R}$,i.e, the set $R_\epsilon(A \times B) = \{n \in \mathbb{N} | \nu(S_f^{-n}(A \times B) \cap (A \times B)) > \nu(A \times B)^2 - \epsilon\}$. This are the times that the set comes back to itself but with large measure. For the non skew-product case, with only a measure preserving transformation on a finite measure space Kintchine theorem says that $R_\epsilon(A)$ is syndetic, the proof is an aplication of von-Neumann ergodic theorem. (ref petersen).

One may ask if in the infinite invariant measure we have an analogous Khintchine theorem. In the skew-product setting:

Problem: Given $(X \times \mathbb{R}, \nu, S_f)$ with a_f recurrent, $\epsilon > 0$ and $A \subset X \times \mathbb{R}$ of positive measure, is the set $R_\epsilon(A) = \{n \in \mathbb{N} | \nu(S_f^{-n} A \cap A) > \nu(A)^2 - \epsilon\}$ syndetic?

In the next section we show that an example by Aaronson (discrepancy skew product over badly approximable irrational numbers) gives a negative answer to that question when we consider the skew-product defined on $X \times \mathbb{Z}$.

2 Discrepancy skew products over irrational rotations

Consider $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ a badly approximable irrational, there is:

$$\inf\{q^2 |\alpha - \frac{p}{q}| : \quad q \in \mathbb{N}^+, p \in \mathbb{Z}\} > 0.$$

we'll consider the example from (ref Aaronson), called *discrepancy skew product over* α:

$$\begin{aligned} T : \mathbb{T} &\to \mathbb{T} \\ x &\mapsto x + \alpha \\ f : \mathbb{T} &\to \mathbb{Z} \\ x &\mapsto 2\mathbb{1}_{[0,1/2)} x - \mathbb{1}_{[1/2,0)} x. \end{aligned}$$

Clearly this satisfyes the hypothesis of Atkinson theorem 1.1 considering ν the product of Lebesgue and the counting measure on $\mathbb{Z}$.

Let $A = \mathbb{T} \times \{0\}$, $\epsilon > 0$. Notice that $(x, 0) \in S_f^{-n}(A) \cap A$ iff $x \in \{y \in \mathbb{T} : a_f(n, y) = 0\}$, therefore $\nu(S_f^{-n} A \cap A) = \mu(\{a_f(n, ;) = 0\})$.

The main result of the paper is that the dynamical system $(\mathbb{T} \times \mathbb{Z}, S_f)$ is *boundedly rationally ergodic*. Denote $\Psi_n : \mathbb{T} \to \mathbb{N}$ the function:

$$\begin{aligned} \psi(x) = \sum_{k=1}^{n-1} \mathbb{1}_{\mathbb{T} \times 0} S_f(x, 0) = & \quad (2) \\ \#\{1 \leq k \leq n-1 : a_f(k, x) = 0\} & \quad (3) \end{aligned}$$

a corollary of the main result of (ref paper) is:

Corollary 2.0.1 *There exists $M > 1$ with $\int_{\mathbb{T}} \Psi_n = M^{\pm} \frac{n}{\log^{\frac{1}{2}} n}$.*

with $x = M^{\pm} y$ meaning $\frac{1}{M} \leq \frac{x}{y} \leq M$.

When α is quadratic we have, from the paper (ref old paper by Aaronson), that:

$$\sum_{k=1}^{n} \nu(S_f^{-k} A \cap A) = \int_{\mathbb{T}} \Psi_n \tag{4}$$

Proposition 2.1 *For α a quadratic irrational and $(\mathbb{T} \times \mathbb{Z}, S_f)$ the discrepancy skew product over α, for $\epsilon > 0$ small enough, $A = \mathbb{T} \times \{0\}$ the set $R_\epsilon(A)$ is not syndetic.*

Proof: Assume that the $R_\epsilon(A)$ is syndetic. Then there exists $\tau > 1$ such that for all $j \in \mathbb{N}$, $R_\epsilon(A) \cap \{j, j+1, \ldots, j+\tau\} \neq \emptyset$. This means that for N large, $\# R_\epsilon \cap \{1, \ldots, N\} \geq \frac{N}{\tau}$.

Notice that

$$\sum_{k=1}^{n} \nu(S_f^{-k} A \cap A) \geq (1-\epsilon) \# R_\epsilon \cap \{1, \ldots, n\} \geq \tag{5}$$

$$(1-\epsilon)\frac{n}{\tau} > M \frac{n}{\log^{\frac{1}{2}} n} \quad \text{for} \quad n \quad \text{large enough} \tag{6}$$

combining the last inequality with formula 4 and the corollary 2.0.1 we get a contradiction.

3 Further questions

1. What happens when α is not quadratic?
2. What happens if it is Liouville?
3. What are some necessary conditions on the dynamics and on the integrable function that makes the return times of the skew-product syndetic or not?
4. In the case of the discrepancy skew-product, is the set of small returns syndetic? There is: is $\{n | \mu(S_f^{-n}(A) \cap A \cap \{a_f(n:) = 0\}) > 0\}$ syndetic?