Some results about cocycles

Bruno Seefeld

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Abstract

Some interesting exercises/resutls about cocycles

1 Non-continuity of Lyapunov exponents

Given two matrices $A_1, A_2 \in \mathrm{GL}_2(\mathbb{R})$ and numbers p_1, p_2 with $p_1 + p_2 = 1, p_1, p_2 > 0$, we can consider the random matrix $A^(n) = A_{i_n} \dots A_{i_1}$ with $i_k \in \{1, 2\}$ with a Bernoulli $B(p_1)$ distribution. We can consider the assymptotic exponential rate of growth of the norm of $A^{(n)}$ given by

$$\lambda^+(A_1, A_2, p_1, p_2) = \lim_{n\to\infty} \frac{1}{n} \log \|A^{(n)}\|. \tag{1}$$

The following theorem [1] says that the quantity λ^+ is continuous on $\mathrm{GL}_2(\mathbb{R})$ and $(0, 1)^2$:

Theorem 1.1 (Bocker-Viana) *The function $\lambda^+ : GL_d(\mathbb{R}) \times GL_d(\mathbb{R}) \times (0, 1)^2 \to \mathbb{R}$ is continuous.*

What happens at $p_1 = 1$? We'll show an example where we don't have continuity of λ^+ at this point.

Let's take

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

For $p_1 = 1$ we have $A^{(n)} = A_1^n$ almost surely, therefore $\lambda^+(A_1, A_2, 1, 0) = \log 2$. In order to simplify notation denote $p_2 = p$ and $X_n = \frac{1}{n} \log \|A^{(n)}\|$, we'll show that if $p > 0$ then $\lambda^+(A_1, A_2, (1 - p), p) = 0$, and then there is no continuity at $(A_1, A_2, 1, 0)$. It's enough to show that the sequence of random

[1] This is true for dimension d and d matrices

variables X_n converge in probability to 0, therefore it cannot converge almost surely to something else.

By observing that $A_1A_2A_1 = A_2$ and $A_2^2 = -Id$ we can write the norm of the products $\|A^{(n)}\|$ as $\|A_1^{n_1}A_2^{n_2}\|$, with the number n_1 being given by a random walk $\{S_n\}$

for example:

$$\begin{aligned} A^{10} =& A_1^4A_2A_1A_2A_1^3 = && (2)\\ A_1^4A_2(A_1A_2A_1)A_1^2 =& A_1^4A_2A_2A_1^2 = && (3)\\ A_1^4(A_2^2)A_1^2 =& A_1^6A_2^2 && (4)\\ \{S_n\}_{n=1}^{10} =& \{1, 2, 3, -3, -2, 2, 3, 4, 5, 6\} && (5) \end{aligned}$$

We have then $\frac{1}{n}\log\|A^{(n)}\| \leq \frac{n_1}{n}\log 2$. Notice that the random walk S_n is bounded by a classical $\{1, -1\}$ random walk W_n by $|S_n| \leq |W_n| + 1$, therefore $\mathbb{E}[S_n^2] \leq Cn$ for some $C > 0$. We have

$$\mathbb{P}(\frac{n_1}{n} > \epsilon) = \mathbb{P}((\frac{n_1}{n})^2 > \epsilon^2) \leq \frac{Cn}{n^2} \to 0 \tag{6}$$

so we have the desired convergence.