Return times in infinite ergodic theory

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1 Examples

1.1 Bernoulli shift

Let $X = \{-1, 1\}^{\mathbb{Z}}$ and $\phi : X \to \mathbb{R}$ be given by $\phi(x_n)_n = x_0$. Take $U = \{x \in X | x_0 = 1\}$ which we denote by $[0 : 1]$. We have, for $n > 0$, $\sigma^{-n}(U) \cap U = [0 : 1, n : 1]$. When n is even, $\sigma^{-n}(U) \cap U \cap \{|S_\phi(n,)| < \epsilon\} = \emptyset$; when $n \geq 3$ is odd, $\sigma^{-n}(U) \cap U \cap |\{S_\phi(n,)| < \epsilon\} = [0 : 1, W, n : 1]$ with W being a block of length $n - 1$ summing to -2.

The measure of each block is $\frac{1}{2}^{n+1}$ and there are $\binom{n-1}{\frac{n+1}{2}}$ such blocks, therefore the intersection has measure $B_n = \frac{1}{2}^{n+1}\binom{n-1}{\frac{n+1}{2}}$, so that the set $R(U)$ is syndetic. Since $B_n \to 0$, the set $R_\lambda(U)$ is not syndetic though.

1.2 Some irrational rotations

2 The general case

Let $(X, \mathcal{B}, \mu)$ be an infinite measure σ- finite measure space and T a measure preserving conservative ergodic transformation. We have:

Theorem 2.1 (Hopf ratio ergodic theorem) *For $f, g \in L_1(X, \mu)$, $g \geq 0$,*

$$\frac{S_n(f)}{S_n(g)} \to \frac{\int f d\mu}{\int g f \mu} \quad a.e \ on \quad X$$

and by taking a sequence of sets $A_n \uparrow X$ and $g_N = \mathbb{1}_{A_N}$ we get

Corollary 2.1.1 (Birkhoff ergodic theorem) *For $f \in L_1(X, \mu)$*

$$\frac{1}{n} S_n(f) \to 0 \quad a.e \ on \quad X.$$

So by taking a set $A \in \mathcal{B}$ with $0 < \mu(A) < \infty$ we can apply the Dominated Convergence Theorem and get that $\frac{1}{n}\sum_{k=0}^{n-1} \mu(A \cap T^{-k}A) \to 0$. Therefore the set of λ-large return times of A cannot be syndetic, since if there was a gap τ we would have for all n large $\frac{1}{n}\sum_{k=0}^{n-1} \mu(A \cap T^{-k}A) \geq \frac{1}{\tau}\lambda\mu(A)^2 > 0$, a contradiction.

2.1 Skew-products

For a probability space $(X, \mathcal{B}, \mu)$, an ergodic measure preserving transformation T, and an integrable function $\phi \in L_1(X, \mu)$ we can define the **skew-product** $F_\phi : X \times \mathbb{R} \to X \times \mathbb{R}$ by

$$F_\phi(x, t) = (T(x), t + \phi(x))$$

with the invariant measure being the product of μ and Lebesgue.

The second coordinate of the iterates of F_ϕ is given by $S_n(x)$ for $n > 0$ and $\sum_{j=-1}^{-n} \phi(T^j x)$ for $n < 0$. There is nothing special with the group $\mathbb{R}$, we could change it by any locally compact group and it's Haar measure.

With this last information in mind consider $\phi : X \to \mathbb{Z}$ with zero integral.

=======next writing -Atkinson for $\mathbb{Z}$ -atkinson showed that a certain set is infinite -the results show that it can't be syndetic for a Khitnchine type bound -can it be true if we remove the bound?.