ELG 7177 Bonus Questions

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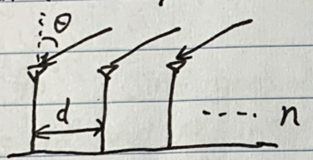
Antenna pattern in free-space propagation

1. $\underline{w}^* = \underline{h}/|\underline{h}| = [e^{j\phi_1}, e^{j\phi_2}, \cdots]^T/\sqrt{n}$

2. Antenna array:



assume we have n receiving antenna, to simplify the question, no noise, and transmitted signal is $x(t)$.

$\underline{w}^*$ means pointing the main beam towards the transmitting direction, so that the amplitude of received signal y_r
$y_r = \underline{w}^{*+} \underline{h}\, x(t)$ is maximized.

The optimal beam former is bringing ~~all the r~~ received signal in every antenna together to the same phase, zero phase

$$y_s = \underline{w}^+ \underline{h}\, x = \sum_{i=1}^{n} a x(t) \exp(j(\phi_i - \phi_{wi}))$$

By assuming $\underline{w} = [1,1,1,\cdots]^T/\sqrt{n}$ and change θ, we have antenna pattern in figure 1

$\underline{w} = [1,1,\cdots]^T/\sqrt{n}$ means the main beam of receiver is pointing to broadside direction, so signal coming from that direction can have maximum amplitude, then maximum SNR among all other directions.

$$y_r(\theta) = \underline{w}^+ \underline{h}\, x = a x(t) \sum_{i=1}^{n} e^{j\Delta\phi(i-1)} = a x(t) e^{j\frac{n-1}{2}\Delta\phi} \frac{\sin(\frac{n}{2}\Delta\phi)}{\sin\frac{1}{2}\Delta\phi}$$

amplitude: $|y_r(\theta)| = \left| a x(t) \frac{\sin(\frac{n}{2}\Delta\phi)}{\sin\frac{1}{2}\Delta\phi} \right|$ by normalizing the amplitude,

we have antenna pattern $F(\theta) = \frac{|y_r(\theta)|}{\max_\theta(y_r(\theta))}$ $\left(\max_\theta(y_r(\theta)) = n\right)$

by observing $|y_r(\theta)|$, we can see amplitude of received signal is effected by factor $\sin(\frac{n}{2}\Delta\phi)/\sin\frac{1}{2}\Delta\phi$

The $F(\theta)$ for $n = 1, 2, 10$, $d = \lambda/2, \lambda, 2\lambda$ and $-90^\circ \le \theta \le 90^\circ$ is given below.

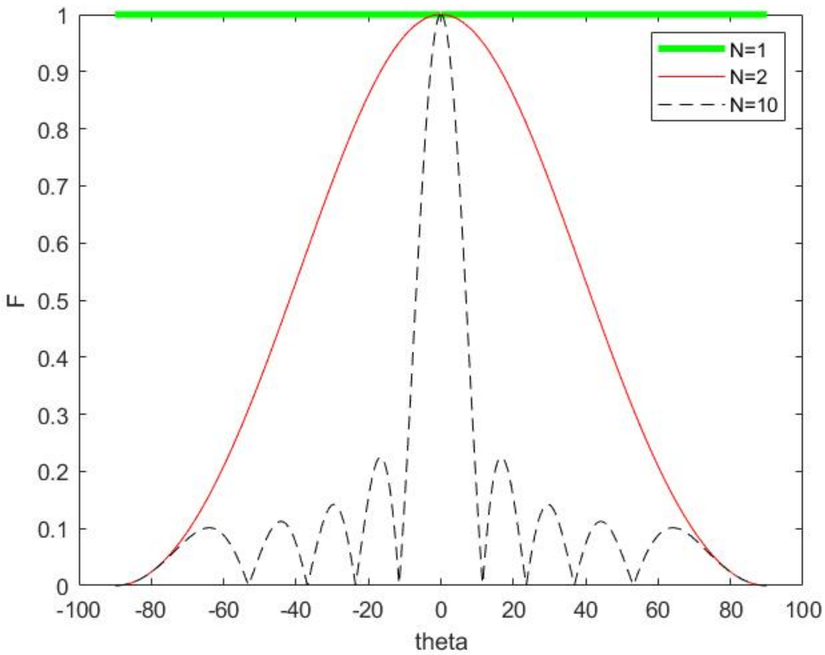


Figura 1. antenna pattern with d = $\frac{\lambda}{2}$

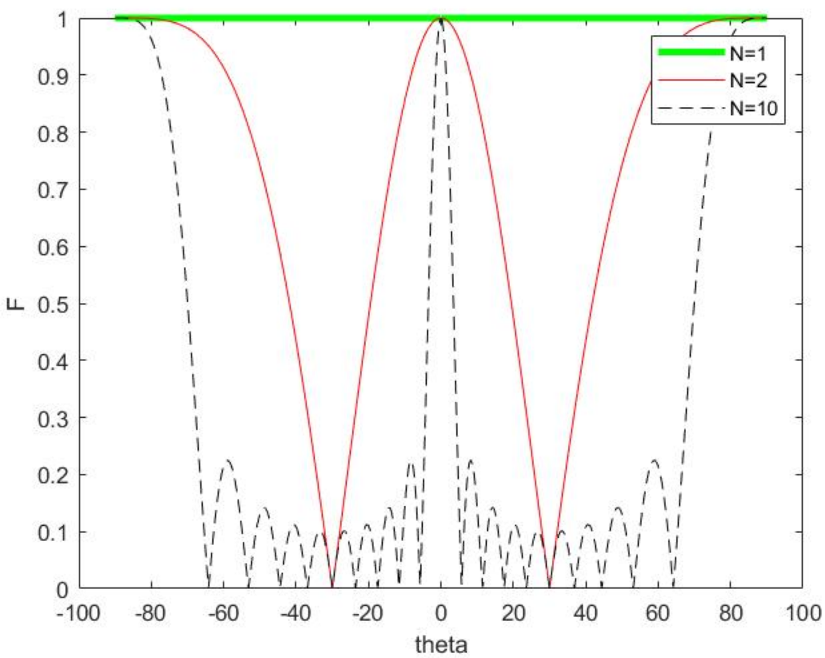


Figura 2. antenna pattern with d = λ

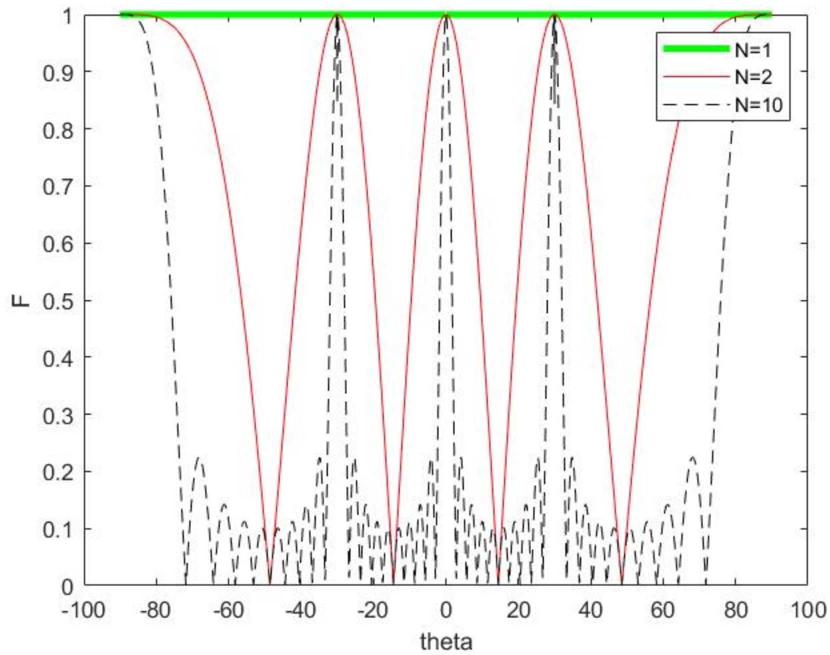


Figura 3. antenna pattern with d = 2λ

As can be seen from the figure, $F(\theta) = \left| \frac{\sin(n \frac{\pi d}{\lambda} \sin\theta)}{n \sin(\frac{\pi d}{\lambda} \sin\theta)} \right|$

when $n=1$, denominter and numerator is same, $F(\theta)=1$,

when $n=2$, $d=\frac{\lambda}{2}$, $F(\theta) = \left| \frac{\sin(\frac{n\pi}{2} \sin\theta)}{n \sin(\frac{\pi}{2} \sin\theta)} \right|$, θ goes from -90° to 90°, $\sin\theta$ goes from -1 to 1, there will be two zeros where $\theta = 90^\circ$ and $\theta = -90^\circ$

when $n=10$, $d=\frac{\lambda}{2}$, in the range $[-1,1]$ of $\sin\theta$, there will be n points that $\frac{n\pi}{2} \cdot \sin\theta = 0$, then it has 10 zeros.

So n determines number of zeros.

When ~~$d=\frac{\lambda}{2}$, λ and 2λ~~, $d=\lambda$, $F(\theta) = \left| \frac{\sin(n\pi \sin\theta)}{n \sin(\pi \sin\theta)} \right|$

$= \left| \frac{\sin(\frac{n\pi}{2}(2\sin\theta))}{n \sin(\frac{\pi}{2}(2\sin\theta))} \right|$ period of $\left| \frac{\sin(\frac{n\pi}{2}\varphi)}{n\sin(\frac{\pi}{2}\varphi)} \right|$ is 2,

and $2\sin\theta \in [-2,2]$ then there will be two periods of $F(\theta)$ in $\theta \in [-90^\circ, 90^\circ]$, if $d=2\lambda$, four periods.

Beam steering:

Q_1 impact of n and d will be same, θ_0 will shift the pattern as in figure 5

Q_2 $\psi = \frac{\Delta\phi - \Delta\phi_w}{2}$, given that $\underline{w} = [e^{j\Delta\phi_w \cdot 0}, e^{j\Delta\phi_w \cdot 1} \ldots e^{j\Delta\phi_w \cdot (n-1)}]/\sqrt{n}$

Q_3 $y_r(\theta) = a \sum_{i=1}^{n} e^{j(i-1)(\Delta\phi - \Delta\phi_w)} = a \sum_{i=1}^{n} e^{j(i-1)(\frac{2\pi}{\lambda} d(\sin\theta - \sin\theta_0))}$

$= a e^{j\frac{n-1}{2}(\frac{2\pi}{\lambda} d(\sin\theta - \sin\theta_0))} \frac{\sin(n\frac{\pi}{\lambda} d(\sin\theta - \sin\theta_0))}{\sin(n\pi\lambda^{-1} d(\sin\theta - \sin\theta_0))}$

Then $F(\theta) = \left| \frac{\sin(n\pi d\lambda^{-1}(\sin\theta - \sin\theta_0))}{n\sin(\pi d\lambda^{-1}(\sin\theta - \sin\theta_0))} \right|$

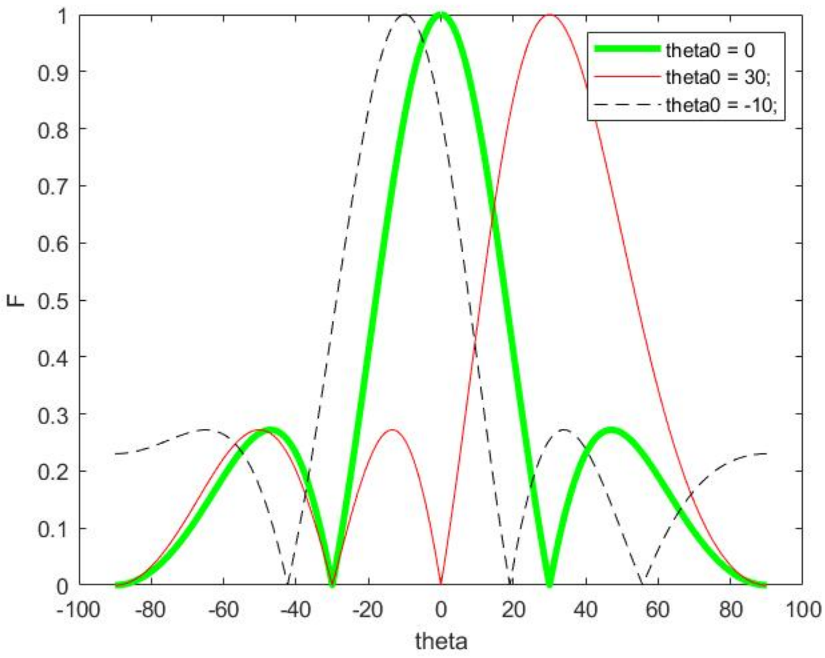


Figura 4. beam steering with $\theta = 30° and - 10°$

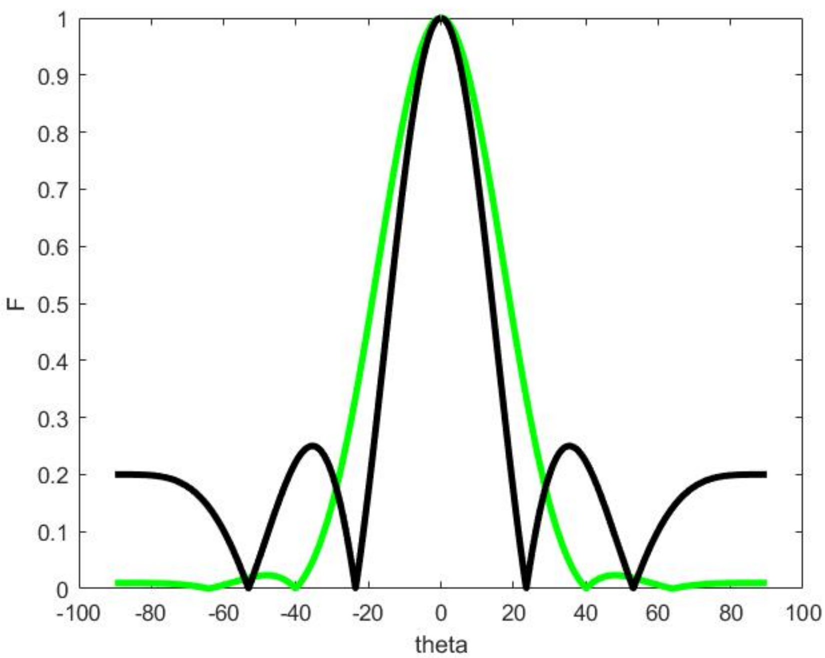
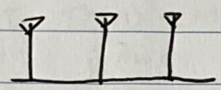


Figura 5. zero steering

Free-Space Propagation MISO Case:

classical phased array:



Assume the receiver is at the direction θ, then

$$\underline{h} = a[1, e^{j\Delta\phi}, e^{j\Delta\phi\cdot 2}, \dots e^{j\Delta\phi(m-1)}]^T$$

where $\Delta\phi = \frac{2\pi}{\lambda} d\sin\theta$ The optimal beamforming is

$$\underline{w}^* = [1, e^{j\Delta\phi}, e^{j\phi_2} \dots e^{j\Delta\phi(m-1)}]^T / \sqrt{m} = \underline{h}/|\underline{h}|$$

It means pointing the main beam towards the receiver and the channel capacity in this scenario:

$$C = \log(1 + \frac{\sigma_x^2}{\sigma_0^2} \underline{h}^+ \underline{w}^* \underline{w}^{*+} \underline{h})$$

$$= \log(1 + \gamma|\underline{h}|^2) = \log(1 + a^2 m^2 \gamma)$$

in this case, the channel capacity equals to the theoretical limit, so optimal Tx beamforming is optimal in the information - theoretic sense as well

Isotropic signaling: Assume $\sigma_0^2 = 1$, then

$$C = \log(1 + \underline{h}^+ R_x \underline{h}) = \log|I + \underline{h}\underline{h}^+ R_x| = \log|I + W R_x|$$

$$= \log|I + U_w \Lambda_w U_w^+ R_x| = \log|I + U_w \Lambda_w U_w^+ R_x U_w|$$

$$= \log|I + \Lambda_w \tilde{R}_x| \qquad \tilde{R}_x = \frac{P}{m} I$$

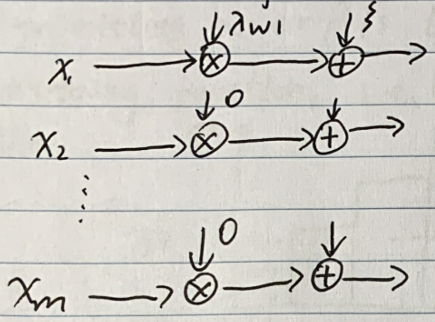
$C = \log|I + \Lambda_w(\frac{P}{m} I)|$ according to Hadamard inequality, this is already optimal, so channel capacity is

$$C = \sum_i \log(1 + \lambda_{wi} d_i) = \log(1 + \lambda_{w_1} d_1) \qquad \lambda_{w_1} = |\underline{h}|^2,\ d_1 = \frac{P}{m}$$

$$= \log(1 + \frac{P}{m}|\underline{h}|^2) = \log(1 + \gamma|\underline{h}|^2/m)$$

γ is the SNR when we have only one antenna (when using optimal strategy)

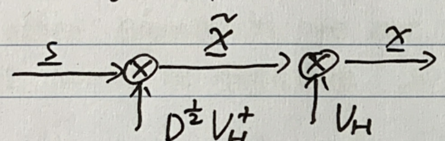
And it transforms to this parallel channel:



power is spreaded out to all eigenmodes, and channel ~~of~~ rank is just 1, then it concentrate all power to one eigenmode (optimal strategy) & transmission rate will be optimized.

Water Filling examples Optimal Tx structure

1. For general case, the Tx structure is designed as follows:

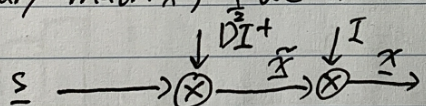


where $\underline{s} \sim CN(\underline{0}, I)$ s_i's are i.i.d., and optimal covariance matrix $R_x^* = V_H D V_H^+$

$H = U_H \Sigma_H V_H^+$ and we have $R_{\tilde{x}} = D^{\frac{1}{2}} V_H^+ \overline{\underline{s}\,\underline{s}^+} V_H D^{\frac{1}{2}} = D$, $\underline{x} = V_H D^{\frac{1}{2}} V_H^+ \underline{s} = R_x^{\frac{1}{2}} \underline{s}$

in this case, $W = V_H \Lambda_W V_H^+ = V_H \lambda_W I V_H^+$ our V_H can be any unitary matrix, we choose $V_H = I$,

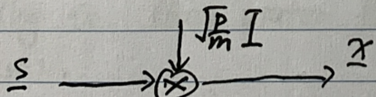
then



where $R_x^* = \frac{P}{m} I$, then $D = \frac{P}{m} I$ and $D^{\frac{1}{2}} = \sqrt{\frac{P}{m}}\, I$

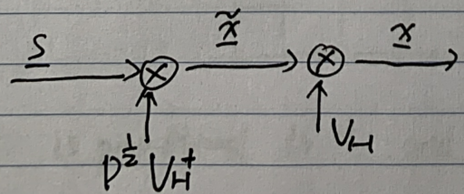
it canbe simplified to:

Tx:



2. We have $R_x^* = V_H D V_H^+$ where $D = \begin{bmatrix} P & & 0 \\ & 0 & \\ 0 & & \ddots\, 0 \end{bmatrix}$ and $\underline{v}_1$ is the eigenvector of R_x^* corresponding to P

so Tx:

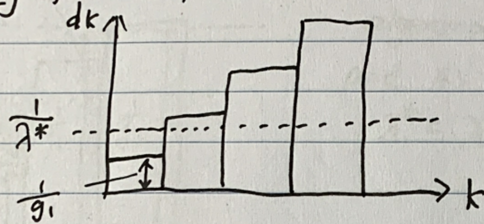


where $D^{\frac{1}{2}} = \begin{bmatrix} \sqrt{P} & & 0 \\ & 0 & \\ 0 & & \ddots\, 0 \end{bmatrix}$

Water Filling Properties

Q1 According to the water filling algorithm, when the SNR is low, and power of noise is normalized $\sigma^2=1$, it shows transmitting power P is low, and $F(\lambda)$ is a monotically decreasing function, we have $1/\lambda^*$ is low, where $F(\lambda^*)=P$

then



in this case, we see that only eigen mode 1 is activated, $d_1>0$ ~~and~~ g_1 is biggest among all g_i's, and because $\sum_{i=1}^{m} d_i = P$, only $d_1>0$ ~~d_i $i\neq 1$~~ $d_i=0, i\neq 1$ then $d_1=P$, all the power is concentrated on eigenmode 1

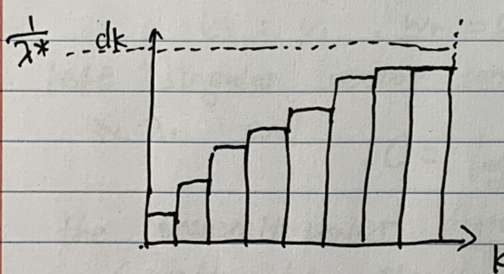
all other eigenmodes are not activated because $\frac{1}{\lambda^*}-\frac{1}{g_i}<0, \forall i\neq 1$

d_i cannot be negative so they're zero.

low SNR: $\frac{1}{g_1}<\frac{1}{\lambda^*}\leq\frac{1}{g_2} \rightarrow g_2\leq\lambda^*<g_1$

then $0<P\leq F(g_2)$

in High SNR case, all eigen modes are activated,



it means $\frac{1}{\lambda^*}>\frac{1}{g_m}$,

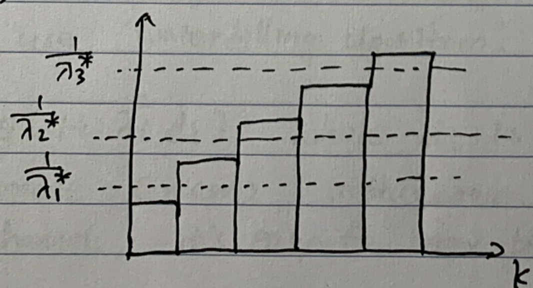
then all the d_i's are >0,

in this case $\lambda^*<g_m$

$\rightarrow P>F(g_m)$

Q2 According to the water filling algorithm figure,

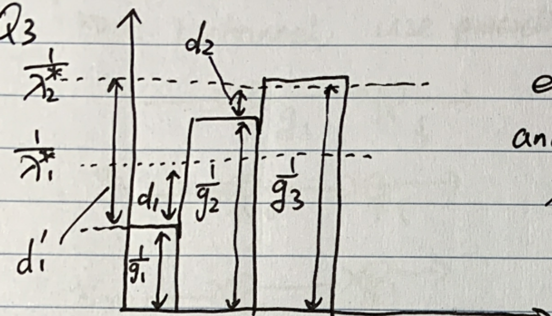
as $\frac{1}{\lambda^*}$ increases,



the number of d_i's that >0 also increased, ~~so~~ then number of active eigenmodes increases with $\frac{1}{\lambda^*}$,

and SNR is proportional to P, and P is propotional to $\frac{1}{\lambda^*}$, so we have number of active eigenmodes increases with SNR

Q3



at low SNR, only strongest eigenmode 1 is activated, $\lambda^* = \lambda^*_1$, and as ~~x~~ SNR increases, $\lambda^* = \lambda_2^*$, we have

$$d_1 = \frac{1}{\lambda_2^*} - \frac{1}{g_1}$$

$$d_2 = \frac{1}{\lambda_2^*} - \frac{1}{g_2}$$

$d_1 + d_2 = P$, but we also noticed that the difference between d_1 and d_2 is fixed as $\frac{1}{\lambda^*}$ increases, it's $\frac{1}{g_2} - \frac{1}{g_1}$, it means as SNR increases, the difference between power allocated to activated eigenmodes ~~are~~ is ~~s~~ fixed, and stronger eigenmode gets more power.

Q4 The capacity of Tx/Rx beamforming is maximized by $\underline{w_t} = \underline{v_1}$, $\underline{w_r} = \underline{u_1}$, where $\underline{v_1}$ and $\underline{u_1}$ are right and left singular vector corresponding to largest singular value of H, ~~λ~~ λ_1 and $C = \log(1 + \lambda_1^2 \cancel{\sigma_x^2} \gamma_t)$ where γ is $\frac{\sigma_x^2}{\sigma_0^2}$, σ_x^2 is the transmit power. After this we normalize noise, $\sigma_0^2 = 1$

Consider doing this in the MIMO case, rank of channel is r, $H = U \Sigma V^+$, $\Sigma = diag(\lambda_1, \cdots \lambda_r)$

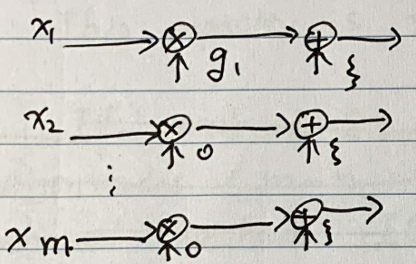
$C = \cancel{\log(1 + \sigma_1^2 \gamma)}$ $\log(1 + \lambda_1^2 \gamma) = \log(1 + \lambda_1^2 \sigma_x^2)$

but if we use 'waterfilling algorithm, we can achieve

$C' = \cancel{x} \sum_{i=1}^{r} \log(1 + \lambda_i^2 d_i)$ where d_i is the power distributed to ith eigenmode $(r < m)$ in this case we have r independent channels, it's easy to know the rate is better

But when the rank of channel is 1, or we have low SNR, they will be equal,

rank 1 channel, use ~~precoding strategy~~ optimal strategy:



in this case $W = V_H \Lambda_W V_H^+$

$= \lambda_1^2 v_1 v_1^+$

and $C' = \log(1 + \lambda_1^2 d_1)$

only $d_1 > 0$, $d_i = 0$ $\forall i \neq 1$,

so $d_1 = P$ as $\sum_i d_i = P$, noise is normalized, $\sigma_0^2 = 1$,

Then we find that capacity is the same as beamforming capacity.

And when we have low SNR, we still have situation that $d_1 = P$ and $d_i = 0$, $\forall i \neq 1$,

$$C' = \log(1 + \lambda_1^2 d_1) = C = \log(1 + \lambda_1^2 \sigma_x^2)$$

The rate of ~~et~~ beamforming and waterfilling is the same.

Q5. Referred to "Channel Capacity Estimation for MIMO Systems with Correlated Noise", by Krusevac, S., et al.

Assume $R_x = \frac{P}{m} I$, we have capacity

$$C = \log \frac{|R_\xi + H R_x H^+|}{|R_\xi|} = \log |I + R_\xi^{-1} H R_x H^+|$$

$$= \log |I + \frac{P}{m} R_\xi^{-1} H H^+|$$

use singular value decomposition and EVD, $R_\xi = U_n \Lambda_n U_n^+$, then $R_\xi^{-1} = U_n \Lambda_n^{-1} U_n^+$ and $H = U_H \Sigma_H V_H^+$,

then we have $C = \log |I + \frac{P}{m} U_n \Lambda_n^{-1} U_n^+ U_H \Sigma_H \Sigma_H^+ U_H^+|$

$$= \log |I + \frac{P}{m} \Lambda_n^{-1} U_n^+ U_H \Sigma_H \Sigma_H^+ U_H^+ U_n| \leq \log |I + \frac{P}{m} \Lambda_n^{-1} \Sigma_H \Sigma_H^+|$$

the equality holds iff $U_n = U_H$

~~I~~ $C = \sum_{i=1}^{m} \log\left(1 + \frac{P}{m} \frac{|\lambda_{Hi}|^2}{\lambda_{ni}}\right)$ where λ_{Hi} is ith singular value of H, and λ_{ni} is ith ~~eigenvec~~ eigenvalue of R_ξ

This capacity is the limit since R_ξ and H is fixed.

~~Q6 If rank of R_ξ is m_r, then it means some eigen modes $(m - m_r)$ don't have noise, for those independent channel we have $I(X;Y) = H(Y)$~~

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