Bonus question: lecture 6:

Q: What's the R^* when channel isn't full rank channel (R* will not be proportional to I anymore)

Assume the rank of channel is r, then the optimal covariance matrix is R^*,

$$R^* = U_W \Lambda U_W^+ \quad \text{where} \quad \Lambda = \frac{P}{r} I_r ,$$

$$I_r = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{bmatrix}$$ the first r diagonal elements are 1.

to generate this signal with covariance matrix R^*, assume channel matrix H, $n \times m$ matrix, $H = U_H \Sigma V_H^+$ $\quad V_H = U_W$, and $W = H^+ H$

generate $m \times 1$ ~~independent~~ signal $\underline{s}$ with i.i.d. Gaussian entries with variance 1 and independent to each other, namely, $\mathbb{E}[\underline{s}\,\underline{s}^+] = I$

$$\underline{s} \to \bigotimes \xrightarrow{\tilde{x}} \bigotimes \xrightarrow{x} \bigotimes \to \bigoplus \xrightarrow{y} \bigotimes \to \tilde{y}$$

$$\uparrow \Lambda^{\frac{1}{2}} V_H^+ \qquad \uparrow V_H \qquad \uparrow H \qquad \uparrow \underline{z} \qquad \uparrow U_H^+$$

$$\Lambda^{\frac{1}{2}} = \sqrt{\tfrac{P}{r}}\, I_r$$

This is how to generate R^* and do processing, we form r eigenmodes, but is it true that we allocate no power to eigenmode j? ($j > r$)

assume $V_H = [\underline{v}_1 \; \underline{v}_2 \cdots \underline{v}_m]$ and because

$\tilde{\underline{x}} = V^+ \underline{x}$, then $\tilde{x}_j = \underline{v}_j^+ \underline{x}$,

$$\mathbb{E}[\tilde{x}_j \tilde{x}_j^*] = \mathbb{E}(\underline{v}_j^+ \underline{x})(\underline{v}_j^+ \underline{x})^* = \mathbb{E}\Big(\sum_{l=1}^{m} v_{jl}^* x_l\Big)\Big(\sum_{k=1}^{m} v_{jk} x_k^*\Big)$$

$$= \sum_{l=1}^{m}\sum_{k=1}^{m} v_{jl}^* v_{jk} R_{lk} = (V_H^+ R^* V_H)_{jj}$$ $\quad ( )_{jj}$ means jj th element of a matrix,

$$(V_H^+ R^* V_H) = (V_H^+ V_H \Lambda V_H^+ V_H)_{jj} = (\Lambda)_{jj} = 0,$$

so $\mathbb{E}[\tilde{x}_j \tilde{x}_j^*] = 0$, we allocate no power to jth eigen mode.

Q: MIMO capacity: $C \approx m\log\frac{P}{m} + \log|W|$

when $m=1$, equivalently: $C_1 = \log P + \log|h_{11}|^2$

Which one is better?

Because we are in high-SNR regime, we only consider the dominant factor $m\log\frac{P}{m}$ or $\log P$, notice $m\log\frac{P}{m}$, $P/m < P$, but ~~it's~~ it's inside logarism, attenuation is small, and the factor in the front, m, is dominant, the influence of P/m is small, so we approximately say that $m\log\frac{P}{m}$ is linearly growing with m, we can know $m \neq 1$ is better.

Q_1 Consider free-space propagation at far field: $h_{ij} = 1$ for all i, j, compare C, C_1, C_b

Isotropic signaling: $R_x = \frac{P}{m} I$ then

$$C = \log(1 + \lambda_{w_1} \lambda_1) = \log(1 + \frac{P}{m} \cdot mn) = \log(1 + nP)$$

SISO: $C_1 = \log(1+P)$

beamforming: $C_b = \log(1 + mnP)$

so $C_b > C > C_1$

Q_2 assume $H = I$ and compare 3 capacities.

$C = m\log(1 + \frac{P}{m})$ $\quad$ $C_1 = \log(1+P)$

$C_b = \log(1+P)$

so $C > C_1 = C_b$

Q3 high SNR regime is defined when all eigenmodes are active, derive a condition on SNR for this to be the case

We assume all eigenvalues of $W = H^{\dagger}H$ is arranged in an order $\lambda_{w_1} \geq \lambda_{w_2} \geq \cdots \lambda_{w_m}$

then define function $F(\mu) = \sum_{i=1}^{m} \left(\frac{1}{\mu} - \frac{1}{\lambda_{w_i}}\right)_+$

if we want all eigenmode to be active, we need

$\frac{1}{\mu} > \frac{1}{\lambda_{w_m}}$ when $F(\mu_1) = P$, $\mu_1 = F^{-1}(P)$

then we need $\frac{1}{F^{-1}(P)} > \frac{1}{\lambda_{w_m}}$, $F^{-1}(P) < \lambda_{w_m}$

(we normalize noise so $P = \gamma$)

※: closed-form solution: $P > \frac{m}{\lambda_{w_m}} - \sum_{i=1}^{m} \lambda_{w_i}^{-1}$

Q4. when W is rank deficient, find an equivalent of (4)

answered at the beginning

Q1 : Using the WF solution, show that $r = \text{rank}(H) = \text{rank}(W)$

~~B~~ $\text{rank}(H) = \text{rank}(W)$, we assume the rank to be r, then ~~accordi~~ assume we we waterfilling algorithm, and only have r eigenmodes, this means

$R^* = U_W \Lambda U_W^{\dagger}$ and $\Lambda = \text{diag}\{\lambda_1, \cdots \lambda_r, 0, \cdots 0\}$

for each λ_i, $i \in 1, \cdots r$, we can use α_i ~~to~~ and γ to denote it, $\lambda_i = \alpha_i \gamma$, capacity :

$$C = \sum_{i=1}^{r} \log(1 + \lambda_{w_i} \lambda_i) = \sum_{i=1}^{r} \log(1 + \lambda_{w_i} \alpha_i \gamma)$$

multiplexing gain $g = \lim_{\gamma \to \infty} \frac{C}{\log \gamma} = \lim_{\gamma \to \infty} \frac{\sum_{i=1}^{r} \log(1 + \lambda_{w_i} \alpha_i \gamma)}{\log \gamma}$

$= \sum_{i=1}^{r} 1 = r$, because $\lim_{\gamma \to \infty} \frac{\log(1 + \lambda_{w_i} \alpha_i \gamma)}{\log \gamma} \to 1$

then $g = \text{rank}(H) = \text{rank}(W)$

Q2 when is beamforming optimal at high SNR ?

when rank of channel is 1

Q: Low SNR Regime : compare $\lambda_{w1} P$ and $|h_{11}|^2 P$:

We know $\underline{x}^\dagger W \underline{x} \leq \lambda_1(W) |\underline{x}|^2$, let $\underline{x} = [1, 0, 0, \dots 0]^T$

then we have $W_{11} \leq \lambda_{w1}$

and $W_{11} = \sum_{i=1}^{n} |h_{i1}|^2 \geq |h_{11}|^2$ so we have

$\lambda_{w1} \geq |h_{11}|^2$,

then $C = \lambda_{w1} P \geq C_1 = |h_{11}|^2 P$

Q1 : consider free-space propagation at far field :

$h_{ij} = 1$ for all i, j. Compare C, C_1, C_b

MIMO capacity : $C = \lambda_{w1} P = mnP$

beam forming capacity : $C_b = \lambda_{w1} P = mnP$

which makes sense because in low-SNR regime MIMO. we are doing beamforming.

SISO : ~~C_b~~ $C_1 = |h_{11}|^2 P = P$

so $C = C_b > C_1$ $(m > 1, n > 1)$

Q2 : Assume $H = I$, compare 3 capacities

$C = \lambda_{w1} P = P = C_b$, $C_1 = P$

so $C = C_b = C_1$

Hilroy

Q3. Derive a condition on the SNR that only one eigenmode is active, which eigenmode is active?

Assume eigenvalues of W and R^* are ranked in a descendent order, $\lambda_{w_1} > \lambda_{w_2}$

then when we only have one ~~activate~~ active eigenmode, the first eigenmode is active,

$$P = \lambda_1 = \frac{1}{\mu} - \frac{1}{\lambda_{w_1}}$$

$$\frac{1}{\lambda_{w_1}} < \frac{1}{\mu} < \frac{1}{\lambda_{w_2}} \Rightarrow 0 < P < \frac{1}{\lambda_{w_2}} - \frac{1}{\lambda_{w_1}}$$

Q4: Is it possible for low and high SNR regimes to overlap i.e. the SNR is high but only 1 eigenmode is active?

When we are doing beam forming, nomatter how high is the SNR, only 1 eigenmode is active.

Q5, Justify the definition "wideband regime"

In SISO channel, $C = \Delta f \log(1 + P_x / N_0 \Delta f)$ (bit/s)

when $\Delta f \to \infty$, $C \approx \Delta f \cdot (\frac{P_x}{N_0 \Delta f}) = P_x / N_0$ (bit/s)

then spectral efficiency $C = \frac{P_x}{N_0 \Delta f} = \gamma$ (bit/s/Hz)

this is exactly the form of low-SNR regime with ~~antenna~~ channel gain = 1

Optimal covariance matrix R^* for key hole channel:

$$H = \underline{h}_r \underline{h}_t^+, \qquad W = H^+H = |\underline{h}_r|^2 \underline{h}_t \underline{h}_t^+$$

$$C = \max_{\mathrm{tr}R \le P} \log|I + WR| = \max_{\mathrm{tr}R \le P} \log\left| 1 + |\underline{h}_r|^2 \underline{h}_t^+ R \underline{h}_t \right|$$

$$\underline{h}_t^+ R \underline{h}_t \le \lambda_1(R) |\underline{h}_t|^2, \text{ and}$$

$$R^* = P \frac{\underline{h}_t}{|\underline{h}_t|} \cdot \frac{\underline{h}_t^+}{|\underline{h}_t|}$$

Q_1 Show that 1. Isotropic signaling is optimal:

$$C = \max_R \min_{H \in S_H} \log|I + HRH^+|$$

where $S_H := \{H : \mathrm{tr} H^+H \ge G\}$

so we assume $W = H^+H = U_W \Lambda_W U_W^+$, and

$\Lambda_W = \mathrm{diag}\{\lambda_{w1}, \cdots \lambda_{wm}\}$ all of them are variables.

$R = U_R \Lambda U_R$ $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots \lambda_m\}$

① Assume: $W = \Lambda_W$, and $R = \Lambda$, we have

$$C_H = \min_{H \in S_H} \log|I + \Lambda_W \Lambda| = \min_{H \in S_H} \sum_{i=1}^m \log(1 + \lambda_{wi} \lambda_i)$$

we want to minimize C_H, use KKT:

$$L = \sum_{i=1}^m \log(1 + \lambda_{wi}\lambda_i) - \mu\left(\sum_i \lambda_{wi} - G\right) - \sum_i \lambda_{wi}\mu_i$$

with
$$\begin{cases} \frac{\partial L}{\partial \lambda_{wi}} = 0 = \frac{\lambda_i}{1 + \lambda_{wi}\lambda_i} - \mu - \mu_i = 0 \\ \mu\left(\sum_i \lambda_{wi} - G\right) = 0, \quad \lambda_{wi}\mu_i = 0 \ \forall i, \\ \lambda_{wi} \ge 0, \quad \sum_i \lambda_{wi} \ge G \\ \mu \ge 0, \quad \mu_i \ge 0 \end{cases}$$

Assume $\lambda_{wk} > 0$, we have $\mu_k = 0$,

$$\lambda_{wk} = (\mu^{-1} - \lambda_k^{-1})_+$$, assume $\sum_i \lambda_{wi} > G_T$

then $\mu = 0$, $\lambda_{wk} = \infty$, doesn't make sense,

so $\sum_i \lambda_{wi} = G_T$,

Thus, $C_H = \sum_i \log(\mu^{-1}\lambda_i) \geq \log(\mu^{-1}\lambda_m)$

λ_m is the smallest eigenvalue of R, equality is achieved when only $\lambda_{wm} \neq 0$, others are zero, then we maximize C_H, we need to maximize the smallest eigenvalue of R, the best we can do is to let $R = P/m\, I$ so the smallest eigenvalue is maximized.

※ The solution for general case isn't found

Q2 the capacity in this scenario is:

$$C = \log(1 + \lambda_{wm}\lambda_m) = \log(1 + G_T \frac{P}{m})$$

because $\sum_i \lambda_{wi} = G_T$ and only $\lambda_{wm} \neq 0$.

Q3. Compared with high SNR regime, we are using isotropic signaling but capacity is still SISO capacity.

Q: Show that for any SNR, $C_2(\gamma) \leq C_1(\gamma) \leq C_2(m\gamma)$

We normalize ~~SNR~~ noise, then use P to denote SNR γ, low SNR scheme is proved, in high SNR case, full CSI: $C_1 = \sum_{i=1}^{m} \log(1+\lambda_{wi}\lambda_i) \approx m\log\frac{P}{m} + \log|W|$

no CSI: $C_2 = \log|I + \frac{P}{m}\Sigma W| \approx \log|\frac{P}{m}W| =$
$m\log\frac{P}{m} + \log|W|$

$$\frac{C_1}{C_2} = 1,$$

in rank deficiency case, $\text{rank}(W) = r$,

$$C_1 = r\log\frac{P}{r} + \sum_{i=1}^{r}\log\lambda_{wr}$$

$$C_2 = r\log\frac{P}{m} + \sum_{i=1}^{r}\log\lambda_{wr}$$

then $\quad C_2(mP) = r\log P + \sum_{i=1}^{r}\log\lambda_{wr}$

$C_1 \le C_2(mP)$, equality is achieved when $r=1$

general case: general SNR,

full CSI: $C_1 = \sum_{i=1}^{m}\log(1+\lambda_i\lambda_{wi})$

no CSI: $C_2 = \log|I + \frac{P}{m}W| = \sum_{i=1}^{m}\log(1+\frac{P}{m}\lambda_{wi})$

because C_1 is ~~minimized~~ maximized form of C_2,
$C_2 \le C_1$, the equality achieves when all λ_{wi}'s are same.

$$C_2(mP) = \sum_i \log(1+P\lambda_{wi})$$

because $\lambda_i \le P \ \forall i$, then $\log(1+\lambda_{wi}\lambda_i)$
$\le \log(1+P\lambda_{wi})$ $\quad C_1 = \sum_i \log(1+\lambda_{wi}\lambda_i)$
$\le C_2(mP) = \sum_i \log(1+P\lambda_{wi})$

the equality holds when $\text{rank}(W) = 1$

So $\quad 1 \le G_{CSI} \le m$

max-SNR / MMSE Beamformer

Q2 : Consider the special case of $R_z = \sigma_0^2 I$

When $R_{\underline{z}} = \sigma_0^2 I$, we can write $\underline{z}$ as ξ, where $\mathbb{E}[\xi\xi^+] = \sigma_0^2 I$, and

$$\underline{y} = \underline{h}x + \xi$$

this is SIMO channel we already known, and best beamformer i

$$\underline{w}^* = \alpha R_{\underline{z}}^{-1}\underline{h} = \alpha \underline{h}$$

and this result is exactly the best beamformer in SIMO case

SIC Rx (V-BLAST) Q1 : show optimal Rx cancels interference from all yet-to-be detected signals

proof : $R_{\xi'} = \left(\sum_{j>k} P_j \underline{h}_j \underline{h}_j^+ + \sigma_0^2 I\right)$ we let $\underline{e}_j$ to be the unit ~~unitary~~ vector of $\underline{h}_j$ and $|e_j| = 1$,

then we form a matrix $U = [\underline{e}_1, \cdots \underline{e}_m]$ notice that $\underline{e}_{k+1} \cdots \underline{e}_m$ are direction vectors for $\underline{h}_{k+1} \cdots \underline{h}_m$ and we complete other vectors, ~~and~~ to form a unitary matrix, because $\underline{h}_k^+\underline{h}_j = 0 \ \forall j>k$, the direction vector $\underline{e}$ is also in U.

$$\underline{w}_k^* = R_{\xi'}^{-1}\underline{h}_k$$

write $R_{\xi'}^{-1}$ as $U(\Lambda + \sigma_0^2 I)U^+$ ~~$U\Lambda U^+$~~,

$\Lambda = \mathrm{diag}[0, 0, \cdots, P_{k+1}|\underline{h}_{k+1}|^2, \cdots, P_m|\underline{h}_m|^2]$

$$\underline{w}_k^+\underline{h}_j = \underline{h}_k^+ R_{\xi'}^{-1}\underline{h}_j = \underline{h}_k^+ U(\Lambda + \sigma_0^2 I)^{-1} U^+ \underline{h}_j$$

$\underline{h}_k^+ U = [0, 0, \cdots, |\underline{h}_k|, 0 \cdots 0]$ $\quad U^+\underline{h}_j = [0, 0, \cdots, |\underline{h}_j|, 0, \cdots 0]$

with non-zero entry on the kth and jth entry respectively.

then $\underline{w}_k^+\underline{h}_j = ((\Lambda + \sigma_0^2 I)^{-1})_{kj}|\underline{h}_k||\underline{h}_j|$ where $(\ )_{kj}$ is kjth element of a matrix, so if $k \neq j$,

then $\underline{w}_k^+\underline{h}_j = 0$

~~per antenna~~ stream SNR $\gamma_k = \dfrac{|y_{sk}|^2}{|y_{nk}|^2}$

$$= \frac{P_k |\underline{w}_k^+ \underline{h}_k|^2}{\underline{w}_k^+ R_\xi' \underline{w}_k} = \frac{P_k (\underline{h}_k^+ R_\xi'^{-1} \underline{h}_k)(\underline{h}_k^+ R_\xi'^{-1} \underline{h}_k)}{\underline{h}_k^+ R_\xi'^{-1} \underline{h}_k}$$

$= P_k (\underline{h}_k^+ R_\xi'^{-1} \underline{h}_k)$ $\quad$ $\underline{h}_k^+ R_\xi'^{-1} \underline{h}_k$ is the kkth element of $\Lambda + \sigma_0^2 I$ times $|\underline{h}_k|^2$,

then $\gamma_k = \frac{P_k}{\sigma_0^2} |\underline{h}_k|^2$

$$C_k = \log\left(1 + \frac{|y_{sk}|^2}{|y_{nk}|^2}\right) = \log\left(1 + \frac{P_k |\underline{h}_k|^2}{\sigma_0^2}\right)$$

because after apply beamforming, $y_r = \underline{w}_k^+ \underline{h}_k x_k + \underline{w}_k^+ \underline{\xi}$
it is same as ~~$y_r = h$~~ $y = \underline{h}_k x_k + \underline{\xi}$
this means equivalently, it is a SIMO chanel with single user.

Q2: Show that the same result hold for general full-rank channel at high-SNR, $\sigma_0 \to 0$

Write H as $U \Sigma V^+$ where H is $n \times m$ matrix,
$\Sigma = \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix}$ $\quad$ Σ_1 is $m \times m$ square matrix $(n > m)$

then $\underline{h}_k^+ R_\xi^{-1} \underline{h}_j$ is the kjth entry of matrix
$$H^+ R_\xi^{-1} H$$
Let $\underline{x}_j = [0, 0, \dots 0, x_{k+1}, \dots x_m]^T$ then
$$R_\xi^{-1} = (H \overline{\underline{x}_j \underline{x}_j^+} H^+ + \sigma_0^2 I)^{-1}$$
$$= (H R_{xj} H^+ + \sigma_0^2 I)^{-1} = (U \Sigma V^+ R_{xj} V \Sigma^+ U^+ + \sigma_0^2 I)^{-1}$$
$$\Sigma V^+ R_{xj} V \Sigma^+ = \begin{bmatrix} \Sigma_1 V^+ \\ 0 \end{bmatrix} R_{xj} \begin{bmatrix} V \Sigma_1 & 0 \end{bmatrix} = \begin{bmatrix} \Sigma_1 V^+ R_{xj} V \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$$

then, $H^+ R_j^{-1} H = V\Sigma^+ U^+ U \left(\begin{bmatrix} \Sigma_1 V^+ R_{xj} V \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} + \sigma_0^2 I \right)^{-1} U^+ U \Sigma V^+$

$$= V\Sigma^+ \left(\begin{bmatrix} \Sigma_1 V^+ R_{xj} V \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} + \sigma_0^2 I \right)^{-1} \Sigma V^+$$

$$\left(\begin{bmatrix} \Sigma_1 V^+ R_{xj} V \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} + \sigma_0^2 I \right)^{-1} = \begin{bmatrix} A & 0 \\ 0 & \frac{1}{\sigma_0^2} I_2 \end{bmatrix}$$

We let $I_1 = \mathrm{diag}\{1, 1, \cdots 1\}$ an $m \times m$ unit matrix and I_2 $(n-m) \times (n-m)$ identity matrix,

$$A = (\Sigma_1 V^+ R_{xj} V \Sigma_1 + \sigma_0^2 I)^{-1}$$

then

$$V\Sigma^+ \begin{bmatrix} A & 0 \\ 0 & \frac{1}{\sigma_0^2} I_2 \end{bmatrix} \Sigma V^+ =$$

$$[V\Sigma_1 \;\; 0] \begin{bmatrix} A & 0 \\ 0 & \frac{1}{\sigma_0^2} I_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 V^+ \\ 0 \end{bmatrix} = V\Sigma_1 A \Sigma_1 V^+$$

$$= (\Sigma_1^{-1} V^+)^{-1} A (V\Sigma_1^{-1})^{-1}$$

$$= (V\Sigma_1^{-1} \Sigma_1 V^+ R_{xj} V^+ \Sigma_1 \Sigma_1^{-1} V^+ + \sigma_0^2 V \Sigma_1^{-2} V^+)^{-1}$$

$$= (\sigma_0^2 V \Sigma_1^{-2} V^+ + R_{xj})^{-1}$$

recall the ~~low~~ high SNR condition, we have

$$\lim_{\sigma_0^2 \to 0} (\sigma_0^2 V \Sigma_1^{-2} V^+ + R_{xj})^{-1}_{kj} \to 0$$

because R_{xj} is diagonal, the off-diagonal entries of the inverse matrix will go to zero.

Q_3 : What about low SNR regime? $\sigma_0 \to \infty$? optimal R_x?

$$SNR = \frac{P_k |\underline{w}_k \underline{h}_k|^2}{\underline{w}_k^+ R_3 \underline{w}_k} \approx \frac{P_k |\underline{w}_k^+ \underline{h}_k|^2}{\sigma_0^2 |\underline{w}_k|^2} \leq \frac{P_k |\underline{h}_k|^2}{\sigma_0^2}$$

when $\underline{w}_k = \alpha \underline{h}_k$, equality holds, so
optimal R_x : $\underline{w}_k = \alpha \underline{h}_k$

This means if SNR is low, we should point the beam to user k if we want to maximize SNR for user k.

SDMA vs FDMA:

Q_2 evaluate C_{1a}, C_{sa} for this example
compare to C_i' C_s', comment on the difference

$C_{1a} = \delta f \log(1 + 10 \cdot 10) \approx 6.6$ Mb/s
$C_{sa} = N C_{1a} = 66$ Mb/s

we can see it doubles the C_1 and C_s, but improvement is not much, because log increases slowly, and for C_i' ~~linear~~ it is propotional, it increases much faster.

SDMA via Null forming

Q2 : find the system capacity of SDMA via null forming

When we are we SDMA, capacity for user k :

$$Y_{rk} = \underline{w}_k^+ \underline{Y}_k = \underline{w}_k^+ \underline{h}_k x_k + \underline{w}_k^+ \underline{\xi},$$

$$C_k = \log\left(1 + \frac{|\underline{w}_k^+ \underline{h}_k|^2 P_k}{|\underline{w}_k|^2 \sigma_0^2}\right)$$

The system capacity $C_s = \sum_{k=1}^{m} \log\left(1 + \frac{|\underline{w}_k^+ \underline{h}_k|^2 P_k}{|\underline{w}_k|^2 \sigma_0^2}\right)$

when channel is orthogonal, $\underline{h}_i^+ \underline{h}_j = 0 \quad \forall i \neq j$, the system capacity is optimized,

$$C_s^* = \sum_{k=1}^{m} \log(1 + \frac{P_k |\underline{h}_k|^2}{\sigma^2})$$ in this case, this is the favorable propagation in V-BLAST

Free-space MAC: 1. Consider free-space MAC, m users

$$C = \log|I + \sum_{k}^{m} P_k \underline{h}_k \underline{h}_k^+| = \log|I + mP\underline{h}\underline{h}^+|$$

$$= \log(1 + mNP)$$ where $\underline{h}$ is $N \times 1$ vector, $\underline{h} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

In this case, we can't use ZF-SDMA

2. Do the same for an orthogonal MAC, $H = I$

$$C = \log|I + WP| = \log|I + IP| = \sum_{k=1}^{m} \log(1 + P_k) =$$

$m\log(1+P)$

This is equivalent to m independent SIMO channels

~~and m~~ channel gain is 1,

K-user MAC ~~Q4: assuming~~ Q3: how does sum capacity

$$\sum_{k=1}^{K} R_k < C_{sum} = \log|I + \frac{WP}{\sigma^2}| = \log|I + \frac{1}{\sigma^2} \sum_{k=1}^{K} P_k \underline{h}_k \underline{h}_k^+|$$

can the two be equal?

We can see the K-user MAC capacity is a special case of full MIMO capacity ~~$C = \log|I + WR_x|$~~

$$C = \max_{R_x} \log|I + WR_x|$$ if we set R_x to P

then we can know the full MIMO capacity is greater equal than MAC capacity, but when transmitting antennas of full MIMO do not cooperate with each other, it is equivalent with K independent user scenario.

so, when MIMO transmitter antennas transmit independent streams, the two capacities are equal.

Q4, Assume all per-user channels are orthogonal to each other,

First we consider 2 user case, we have Base station equipped with N antennas, rate of user 1 and 2 are R_1 and R_2, capacity denoted by C_1 and C_2

$$R_1 < C_1 = \log\left(1+\frac{P_1|\underline{h}_1|^2}{\sigma_0^2}\right)$$

$$R_2 < C_2 = \log\left(1+\frac{P_2|\underline{h}_2|^2}{\sigma_0^2}\right)$$

$$R_1+R_2 < \log\left|I + \sigma_0^{-2}\sum_{k=1}^{2} P_k \underline{h}_k \underline{h}_k^{+}\right| = \log|I + WP|$$

where $W = H^{+}H = [\underline{h}_1, \underline{h}_2]^{+}[\underline{h}_1, \underline{h}_2] = \begin{bmatrix} |\underline{h}_1|^2 & 0 \\ 0 & |\underline{h}_2|^2 \end{bmatrix}$

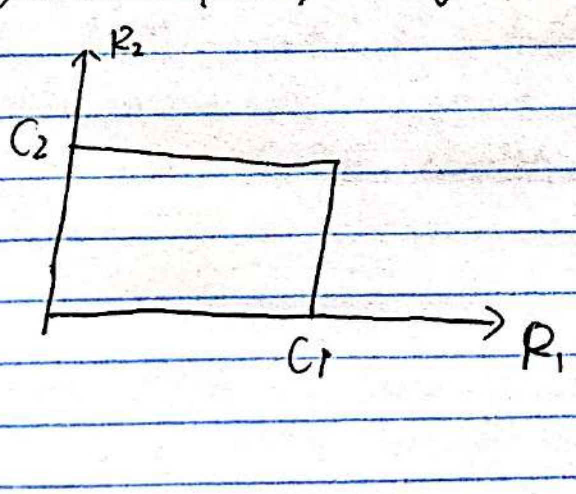
~~then~~ and $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$ then $\log|I+WP|$

$$= \log(1+P_1|\underline{h}_1|^2/\sigma_0^2) + \log(1+P_2|\underline{h}_2|^2/\sigma_0^2)$$

we find the intersections of line $R_1 = \log(1+P_1|\underline{h}_1|^2/\sigma_0^2)$ $R_2 = \log(1+P_2|\underline{h}_2|^2/\sigma_0^2)$ and $R_1+R_2 = \log(1+P_1|\underline{h}_1|^2/\sigma_0^2)+\log(1+P_2|\underline{h}_2|^2/\sigma_0^2)$

clearly, it's $(R_1, R_2) = (C_1, C_2)$

then capacity region:



We extend the result to ~~k~~ m users, then the hyperplane $\sum_{k=1}^{m} R_k = C_{sum}$ intersects planes $R_k = C_k$ all at one single point: $(R_1, \cdots R_m) = (C_1, \cdots C_m)$

to achieve the intersection, we multiply $\underline{w}_k$ to received signal $\underline{y}$, according to ~~quest~~ the derivation for questions in V-BLAST, when channels are orthogonal, optimal beamformer $\underline{w}_k$ has property

$$\underline{w}_k^+ \underline{h}_j = 0 \quad \text{for } \forall j \neq k,$$

then $\underline{w}^+ \underline{y} = \underline{w}^+ \sum_{k=1}^{m} \underline{h}_k x_k + \underline{w}_k^+ \underline{z}$

$$= \cancel{\underline{w}^+ |h_k| x_k} \; \underline{w}_k^+ \underline{h}_k x_k + \underline{w}_k^+ \underline{z}$$

we apply this to every k, then get every signal x_k

To achieve other points, just reduce the corresponding user's rate.

Thus, we can know the optimal Rx is just performing SDMA, previously we found rate of SDMA is smaller than rate of MAC, because MAC rate is optimized, how ever in this case they are equal, because in this case, every signal of MAC are independent, we can no longer derive information about one signal from others, it is the same condition with SDMA.