Antenna pattern in free-space propagation

1. $\underline{w}^* = \underline{h}/|\underline{h}| = [e^{j\phi_1}, e^{j\phi_2}, \cdots]^T/\sqrt{n}$

2. Antenna array:

assume we have n receiving antenna, to simplify the question, no noise, and transmitted signal is $x(t)$.

$\underline{w}^*$ means pointing the main beam towards the transmitting direction, so that the amplitude of received signal y_r
$y_r = \underline{w}^{*+}\underline{h}\,x(t)$ is maximized.

The optimal beam former is bringing ~~all the r~~ received signal in every antenna together to the same phase, zero phase

$$y_s = \underline{w}^+\underline{h}\,x = \sum_{i=1}^{n} a x(t) \exp(j(\phi_i - \phi_{wi}))$$

By assuming $\underline{w} = [1, 1, 1, \cdots]^T/\sqrt{n}$ and change θ, we have antenna pattern in figure 1

$\underline{w} = [1, 1, \cdots]^T/\sqrt{n}$ means the main beam of receiver is pointing to broadside direction, so signal coming from that direction can have maximum amplitude, then maximum SNR among all other directions.

$$y_r(\theta) = \underline{w}^+\underline{h}\,x = a x(t) \sum_{i=1}^{n} e^{j\Delta\phi(i-1)} = a x(t) e^{j\frac{n-1}{2}\Delta\phi} \frac{\sin(\frac{n}{2}\Delta\phi)}{\sin\frac{1}{2}\Delta\phi}$$

amplitude: $|y_r(\theta)| = \left| a x(t) \frac{\sin(\frac{n}{2}\Delta\phi)}{\sin\frac{1}{2}\Delta\phi} \right|$ by normalizing the amplitude,

we have antenna pattern $F(\theta) = \dfrac{|y_r(\theta)|}{\max_\theta(y_r(\theta))}$ $\left(\max_\theta(|y_r(\theta)|) = n\right)$

by observing $|y_r(\theta)|$, we can see amplitude of received signal is effected by factor $\sin(\frac{n}{2}\Delta\phi)/\sin\frac{1}{2}\Delta\phi$

The $F(\theta)$ for $n = 1, 2, 10$, $d = \lambda/2, \lambda, 2\lambda$ and $-90° \le \theta \le 90°$ is given below.

As can be seen from the figure, $F(\theta)=\left|\frac{\sin(n\frac{\pi d}{\lambda}\sin\theta)}{n\sin(\frac{\pi d}{\lambda}\sin\theta)}\right|$

when $n=1$, denominter and numerator is same, $F(\theta)=1$,
when $n=2$, $d=\frac{\lambda}{2}$, $F(\theta)=\left|\frac{\sin(\frac{n\pi}{2}\sin\theta)}{n\sin(\frac{\pi}{2}\sin\theta)}\right|$, θ goes from -90° to 90°, $\sin\theta$ goes from -1 to 1, there will be two zeros where $\theta=90^\circ$ and $\theta=-90^\circ$
when $n=10$, $d=\frac{\lambda}{2}$, in the range $[-1,1]$ of $\sin\theta$, there will be n points that $\frac{n\pi}{2}\cdot\sin\theta=0$, then it has 10 zeros.
So n determines number of zeros.

When ~~$d=\frac{\lambda}{2}$, λ and 2λ~~, $d=\lambda$, $F(\theta)=\left|\frac{\sin(n\pi\sin\theta)}{n\sin(\pi\sin\theta)}\right|$
$=\left|\frac{\sin(\frac{n\pi}{2}(2\sin\theta))}{n\sin(\frac{\pi}{2}(2\sin\theta))}\right|$ period of $\left|\frac{\sin(\frac{n\pi}{2}\varphi)}{n\sin(\frac{\pi}{2}\varphi)}\right|$ is 2, and $2\sin\theta\in[-2,2]$ then there will be two periods of $F(\theta)$ in $\theta\in[-90^\circ,90^\circ]$, if $d=2\lambda$, four periods.

Beam steering:

Q1 impact of n and d will be same, θ_0 will shift the pattern as in figure 5

Q2 $\varphi=\frac{\Delta\phi-\Delta\phi_w}{2}$, given that $\underline{w}=[e^{j\Delta\phi_w\cdot 0}, e^{j\Delta\phi_w\cdot 1}\dots e^{j\Delta\phi_w\cdot(n-1)}]/\sqrt{n}$

Q3 $y_r(\theta)=a\sum_{i=1}^{n}e^{j(i-1)(\Delta\phi-\Delta\phi_w)}=a\sum_{i=1}^{n}e^{j(i-1)(\frac{2\pi}{\lambda}d(\sin\theta-\sin\theta_0))}$

$=a e^{j\frac{n-1}{2}(\frac{2\pi}{\lambda}d(\sin\theta-\sin\theta_0))}\frac{\sin(n\frac{\pi}{\lambda}d(\sin\theta-\sin\theta_0))}{\sin(n\pi\lambda^{-1}d(\sin\theta-\sin\theta_0))}$

Then $F(\theta)=\left|\frac{\sin(n\pi d\lambda^{-1}(\sin\theta-\sin\theta_0))}{n\sin(\pi d\lambda^{-1}(\sin\theta-\sin\theta_0))}\right|$

Free-Space Propagation MISO Case:

classical phased array:

Assume the receiver is at the direction θ, then

$$\underline{h} = a[1, e^{j\Delta\phi}, e^{j\Delta\phi\cdot 2}, \dots e^{j\Delta\phi(m-1)}]^T$$

where $\Delta\phi = \frac{2\pi}{\lambda} d\sin\theta$ The optimal beamforming is

$$\underline{w}^* = [1, e^{j\Delta\phi}, e^{j\Delta\phi_2} \dots e^{j\Delta\phi(m-1)}]^T/\sqrt{m} = \underline{h}/|\underline{h}|$$

It means pointing the main beam towards the receiver and the channel capacity in this scenario:

$$C = \log\left(1 + \frac{\sigma_x^2}{\sigma_0^2} \underline{h}^+ \underline{w}^* \underline{w}^{*+} \underline{h}\right)$$

$$= \log(1 + \gamma|\underline{h}|^2) = \log(1 + a^2 m^2 \gamma)$$

in this case, the channel capacity equals to the theoretical limit, so optimal Tx beamforming is opstimal in the information-theoretic sense as well

Isotropic signaling: Assume $\sigma_0^2 = 1$, then

$$C = \log(1 + \underline{h}^+ R_x \underline{h}) = \log|I + \underline{h}\underline{h}^+ R_x| = \log|I + W R_x|$$

$$= \log|I + U_w \Lambda_w U_w^+ R_x| = \log|I + U_w \Lambda_w U_w^+ R_x U_w|$$

$$= \log|I + \Lambda_w \tilde{R_x}| \qquad \tilde{R_x} = \frac{P}{m} I$$

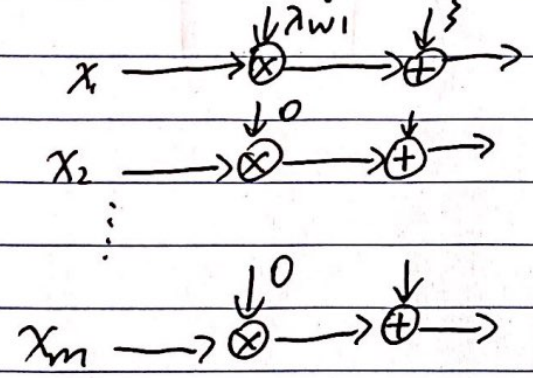
$C = \log|I + \Lambda_w(\frac{P}{m} I)|$ according to Hadamard inequality, this is already optimal, so channel capacity is

$$C = \sum_i \log(1 + \lambda_{w_i} d_i) = \log(1 + \lambda_{w_1} d_1) \qquad \lambda_{w_1} = |\underline{h}|^2,\ d_1 = \frac{P}{m}$$

$$= \log\left(1 + \frac{P}{m}|\underline{h}|^2\right) = \log(1 + \gamma|\underline{h}|^2/m)$$

γ is the SNR when we have only one antenna (when using optimal strategy)

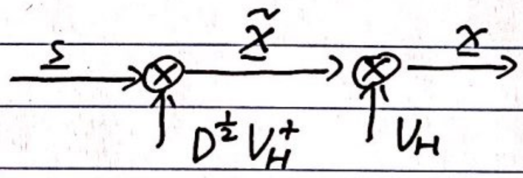
And it transforms to this parallel channel :



power is spreaded out to all eigenmodes, and channel ~~of~~ rank is just 1, then if concentrate all power to one eigenmode (optimal strategy) & transmission rate will be optimized.

Water Filling examples Optimal T_x structure

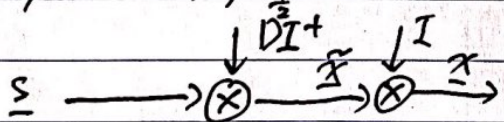
1. For general case, the T_x structure is designed as follows:



where $\underline{s} \sim CN(\underline{0}, I)$ s_i's are i.i.d., and optimal covariance matrix $R_x^* = V_H D V_H^+$

$H = U_H \Sigma_H V_H^+$ and we have $R_{\tilde{x}} = D^{\frac{1}{2}} V_H^+ \underline{s}\,\underline{s}^+ V_H D^{\frac{1}{2}} = D$, $\underline{x} = V_H D^{\frac{1}{2}} V_H^+ \underline{s} = R_x^{\frac{1}{2}} \underline{s}$

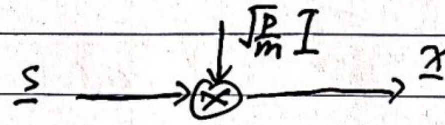
In this case, $W = V_H \Lambda_W V_H^+ = V_H \lambda_W I V_H^+$ our V_H can be any unitary matrix, we choose $V_H = I$, then



where $R_x^* = \frac{P}{m} I$, then $D = \frac{P}{m} I$ and $D^{\frac{1}{2}} = \sqrt{\frac{P}{m}}\, I$

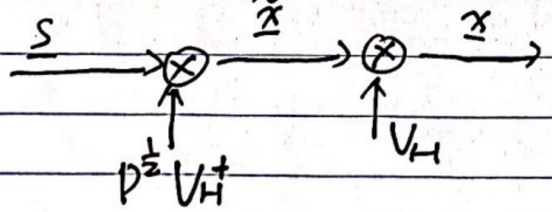
it canbe simplified to :

T_x :



2. We have $R_x^* = V_H D V_H^+$ where $D = \begin{bmatrix} P & & 0 \\ & 0 & \\ 0 & & \ddots\, 0 \end{bmatrix}$ and $\underline{v_1}$ is the eigenvector of R_x^* corresponding to P

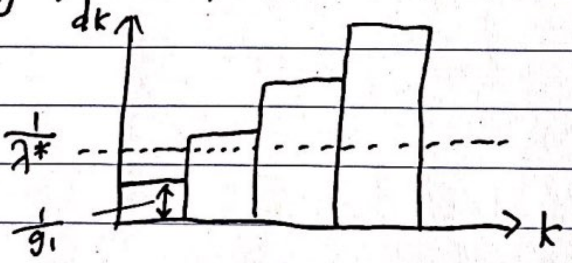
so T_x:



where $D^{\frac{1}{2}} = \begin{bmatrix} \sqrt{P} & & 0 \\ & 0 & \\ 0 & & \ddots\, 0 \end{bmatrix}$

Water Filling Properties

Q1 According to the water filling algorithm, when the SNR is low, and power of noise is normalized $\sigma^2 = 1$, it shows transmitting power P is low, and $F(\lambda)$ is a monotically decreasing function, we have $1/\lambda^*$ is low, where $F(\lambda^*) = P$ then



in this case, we see that only eigen mode 1 is activated, $d_1 > 0$ ~~in th~~ g_1 is biggest among all g_i's, and because $\sum_{i=1}^{m} d_i = P$, only $d_1 > 0$ ~~$d_i : i \neq 1$~~ $d_i = 0, i \neq 1$ then $d_1 = P$, all the power is concentrated on eigenmode 1

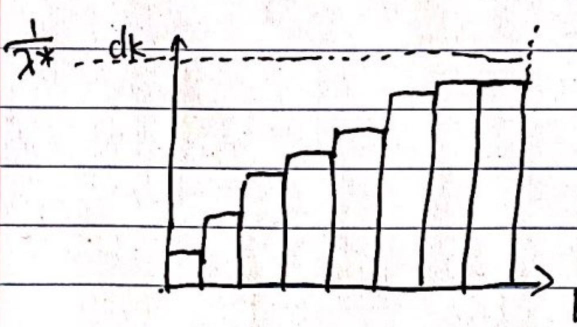
all other eigenmodes are not activated because $\frac{1}{\lambda^*} - \frac{1}{g_i} < 0, \forall i \neq 1$

d_i cannot be negative so they're zero.

low SNR: $\frac{1}{g_1} < \frac{1}{\lambda^*} \leq \frac{1}{g_2} \rightarrow g_2 \leq \lambda^* < g_1$

then $0 < P \leq F(g_2)$

in High SNR case, all eigen modes are activated,



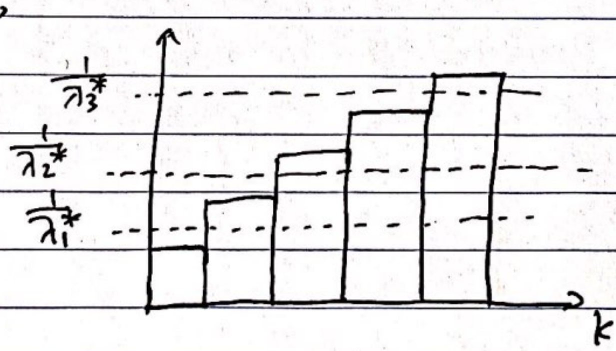
it means $\frac{1}{\lambda^*} > \frac{1}{g_m}$,

then all the d_i's are > 0,

in this case $\lambda^* < g_m$

$\rightarrow P > F(g_m)$

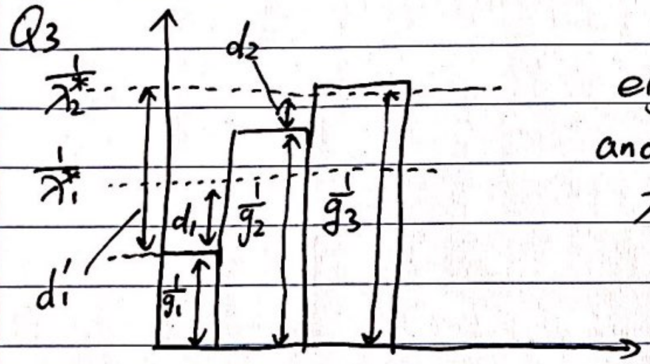
Q2 According to the water filling algorithm figure,
as $\frac{1}{\lambda^*}$ increases,



the number of d_i's that > 0 also increased, ~~so~~ then number of active eigenmodes increases with $\frac{1}{\lambda^*}$,

and SNR is proportional to P, and P is propotional to $\frac{1}{\lambda^*}$, so we have number of active eigenmodes increases with SNR

Q3



at low SNR, only strongest eigenmode 1 is activated, $\lambda^* = \lambda^*_1$, and as ~~p~~ SNR increases, $\lambda^* = \lambda^*_2$, we have

$$d_1 = \frac{1}{\lambda_2^*} - \frac{1}{g_1}$$

$$d_2 = \frac{1}{\lambda_2^*} - \frac{1}{g_2}$$

$d_1 + d_2 = P$, but we also noticed that the difference between d_1 and d_2 is fixed as $\frac{1}{\lambda^*}$ increases, it's $\frac{1}{g_2} - \frac{1}{g_1}$, it means as SNR increases, the difference between power allocated to activated eigenmodes ~~are~~ is fixed, and stronger eigenmode gets more power.

Q4 The capacity of Tx/Rx beamforming is maximized by $\underline{w}_t = \underline{v}_1$, $\underline{w}_r = \underline{u}_1$, where $\underline{v}_1$ and $\underline{u}_1$ are right and left singular vector corresponding to largest singular value of H, $\cancel{\sigma_1}$ λ_1 and

$$C = \log(1 + \lambda_1^2 \cancel{\sigma_1^2} \gamma_t)$$

where γ is $\frac{\sigma_x^2}{\sigma_0^2}$, σ_x^2 is the transmit power. After this we normalize noise, $\sigma_0^2 = 1$

Consider doing this in the MIMO case, rank of channel is r, $H = U \Sigma V^+$, $\Sigma = diag(\lambda_1, \cdots \lambda_r)$

$$C = \cancel{\log(1+\sigma_1^2\gamma)}\ \log(1 + \lambda_1^2 \gamma) = \log(1 + \lambda_1^2 \sigma_x^2)$$

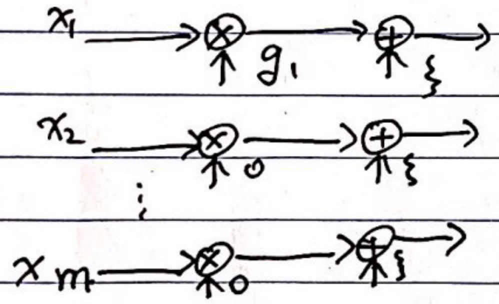
but if we use 'waterfilling algorithm, we can achieve

$$C' = \sum_{i=1}^{r} \log(1 + \lambda_i^2 d_i)$$

where d_i is the power distributed to ith eigenmode $(r<m)$ in this case we have r independent channels, it's easy to know the rate is better

But when the rank of channel is 1, or we have low SNR, they will be equal,

rank 1 channel, use ~~precoding strategy~~ optimal strategy:



in this case $W = V_H \Lambda_W U_H^+$

$= \lambda_1^2 \underline{v_1} \underline{v_1}^+$

and $C' = \log(1 + \lambda_1^2 d_1)$

only $d_1 > 0$, $d_i = 0 \ \forall i \neq 1$,

so $d_1 = P$ as $\sum_i d_i = P$, noise is normalized, $\sigma_0^2 = 1$,

Then we find that capacity is the same as beamforming capacity

And when we have low SNR, we still have situation that $d_1 = P$ and $d_i = 0$, $\forall i \neq 1$,

$$C' = \log(1 + \lambda_1^2 d_1) = C = \log(1 + \lambda_1^2 \sigma_x^2)$$

The rate of ~~bt~~ beamforming and waterfilling is the same.

Q5. Referred to "Channel Capacity Estimation for MIMO Systems with Correlated Noise", by Krusevac, S., et al.

Assume $R_x = \frac{P}{m} I$, we have capacity

$$C = \log \frac{|R_\xi + H R_x H^+|}{|R_\xi|} = \log |I + R_\xi^{-1} H R_x H^+|$$

$$= \log |I + \frac{P}{m} R_\xi^{-1} H H^+|$$ use singular value decomposition and EVD, $R_\xi = U_n \Lambda_n U_n^+$, then $R_\xi^{-1} = U_n \Lambda_n^{-1} U_n^+$ and $H = U_H \Sigma_H V_H^+$,

then we have $C = \log |I + \frac{P}{m} U_n \Lambda_n^{-1} U_n^+ U_H \Sigma_H \Sigma_H^+ U_H^+|$

$$= \log |I + \frac{P}{m} \Lambda_n^{-1} U_n^+ U_H \Sigma_H \Sigma_H^+ U_H^+ U_n| \leq \log |I + \frac{P}{m} \Lambda_n^{-1} \Sigma_H \Sigma_H^+|$$

the equality holds iff $U_n = U_H$

$$\cancel{I=}\ C=\sum_{i=1}^{m}\log\left(1+\frac{P}{m}\frac{|\lambda_{Hi}|^2}{\lambda_{ni}}\right)$$ where λ_{Hi} is ith singular value of H, and λ_{ni} is ith ~~eigenvec~~ eigenvalue of R_ξ

This capacity is the limit since R_ξ and H is fixed.

~~Q6 If rank of R_ξ is r_n, then it means some eigenmodes $(m-r_n)$ don't have noise, for those independent channel we have $I(X;Y)=H(Y)$~~

Q6. If R_ξ is singular, it means some entries of ξ has variance 0, and this shows ~~not~~ some receiving antennas may not be working.