1 Introduction

Consider the following system of partial differential equations (PDE).

$$A_t = D_A A_{xx} - \kappa A - \nu A^2 \tag{1}$$

$$G_t = D_G G_{xx} - \mu G + \frac{1}{2}\nu A^2 \tag{2}$$

$$\dot{H}(t) = \begin{cases} -\sigma_N H G(0,t) & \text{, if apoptosis} \\ -\sigma_V H^{2/3} G(0,t) & \text{, if volume} \end{cases} \tag{3}$$

where

$$-D_A A_x(0,t) = J_A(t)$$

and

$$G_x(0,t) = 0$$

2 Non-dimensionalization

Let,

$$\begin{aligned} t = \bar{t}\tau, \quad x &= \bar{x}y \\ A(x,t) &= \bar{A}a(y,\tau) \\ G(x,t) &= \bar{G}g(y,\tau) \\ \kappa(t) &= \bar{\kappa}(\tau) \\ \nu(t) &= \bar{\nu}\hat{\nu}(\tau) \\ \mu(t) &= \bar{\mu}\hat{\mu}(\tau) \\ J_A(t) &= \bar{J}_A j(\tau) \end{aligned}$$

Now, if we were to pick

$$\begin{aligned} \bar{t} &= \bar{\kappa}^{-1}, \\ \bar{x} &= \sqrt{\bar{\kappa}^{-1} D_A}, \\ \bar{A} &= \bar{J}_A(\sqrt{D_A \bar{\kappa}})^{-1}, \end{aligned}$$

and

$$\bar{G} = \bar{\nu}^{-1}\bar{\kappa}^{-1}\bar{A}^2,$$

then we will system of PDEs can represented by the following:

$$a_\tau = a_{yy} - \hat{\kappa}a - \theta_1\hat{\nu}a^2, \quad a_y(0,\tau) = -j(\tau) \tag{4}$$

$$g_\tau = \theta_2 g_{yy} + \frac{1}{2}\hat{\nu}a^2 - \theta_3\hat{\mu}g, \quad g_y(y,\tau) = 0 \tag{5}$$

$$\dot{H}(\tau) = \begin{cases} -\delta_N g(0,\tau)H & \text{, if apoptosis} \\ -\delta_V g(0,\tau)H^{2/3} & \text{, if volume} \end{cases} \tag{6}$$

where

$$\begin{gathered} \theta_1 = \bar{\nu}\kappa^{-1}\bar{A} \qquad \theta_2 = D_A^{-1}D_G \qquad \theta_3 = \bar{\mu}\bar{\kappa}^{-1} \\ \delta_N = \sigma_N\bar{\kappa}^{-1}\bar{G} \qquad \delta_V = \sigma_V\bar{\kappa}^{-1}\bar{G} \end{gathered}$$

are dimensionless constants.

3 Analytical Steady State Solutions

The steady state solution for equation (4) will be the solution to the following equation

$$0 = a_{yy} - \hat{\kappa} a - \theta_1 \hat{\nu} a^2, \quad a_y(0, \tau) = -j(\tau).$$

We will be solving this equation as if it were an ordinary differential equation (ODE) with respect to y. The rate constants $hat\nu$ and $\hat{\kappa}$ are assumed to be constant here. To make deriving the solution even easier, we will change the notation a bit. Let,

$$y = \lambda z,$$
$$\text{and } a(y) = \bar{a} b(z)$$

and take

$$\lambda^2 = \frac{1}{\hat{\kappa}}, \qquad \bar{a} = \frac{\hat{\kappa}}{\theta_1 \nu}$$

so then the equation (4) can be re-written as

$$b_{zz} = b + b^2 \quad \text{with } \frac{\mathrm{d}b}{\mathrm{d}z}(0) = -j\lambda \bar{a}^{-1}.$$

The solution to the above is given by

$$b(z) = -\frac{3}{2} + \frac{3}{2}\left(\frac{1 + c\mathrm{e}^{-z}}{1 - c\mathrm{e}^{-z}}\right)^2$$

where c is determined by finding the positive root of $0 = 1 - 9c - 3c^2 - c^3$. This is based a flux condition of $\frac{\mathrm{d}b}{\mathrm{d}z}(0) = -1$.

The steady state solution for the health equation is as follows:

$$H(\tau) = \begin{cases} e^{-\delta_N g(0,\tau)\tau} & \text{, if apoptosis} \\ \dfrac{1}{(2\delta_V g(0,\tau) + 1)^2} & \text{, if volume} \end{cases}$$

4 Numerics

From here on out, we will assume that the rate constants and outward flux $|j(\tau)|$ are all equal to one. $\hat{\kappa}(\tau) = 1$ however will be compared with $\hat{\kappa} = e^{-\tau}$. The effects of this is seen below.

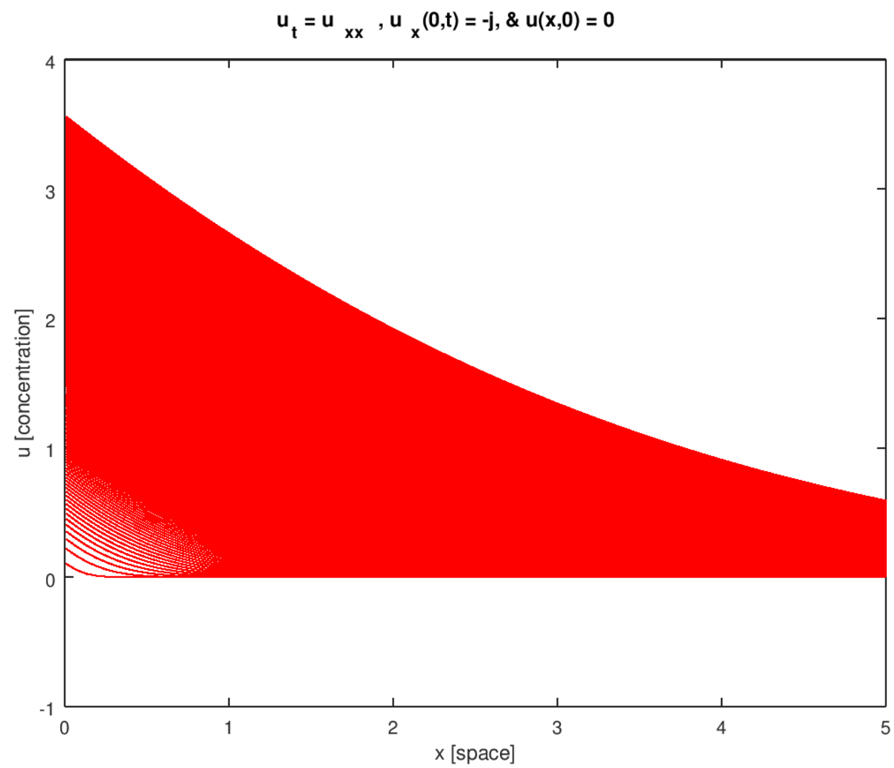

Figure 1: The heat equation with the flux condition.

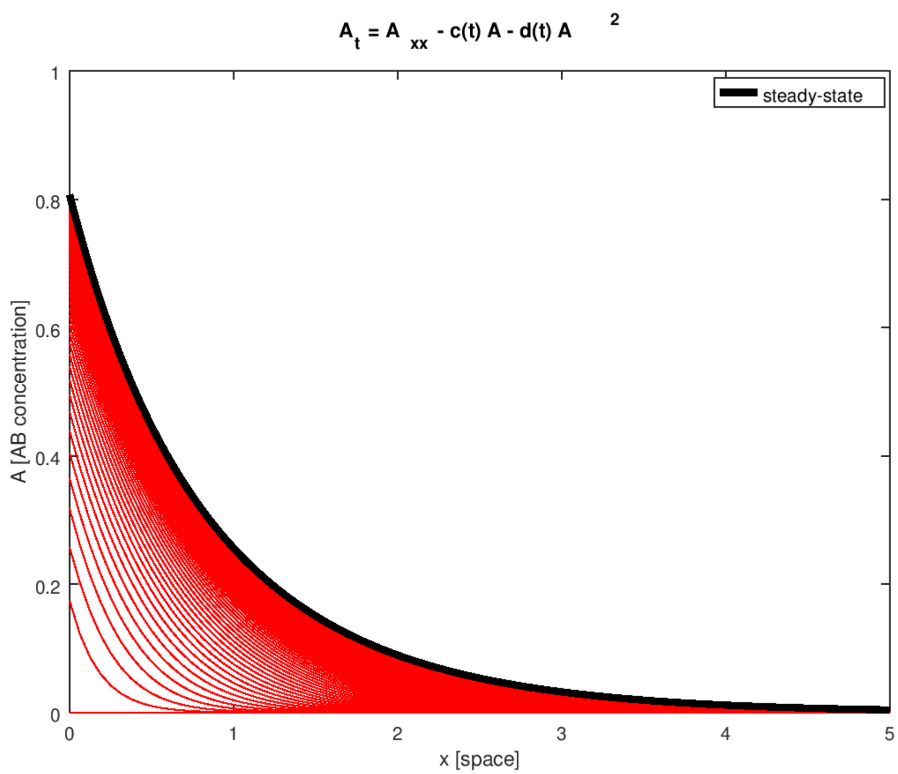

Figure 2: Concentration of normal AB over time along with the steady state solution.

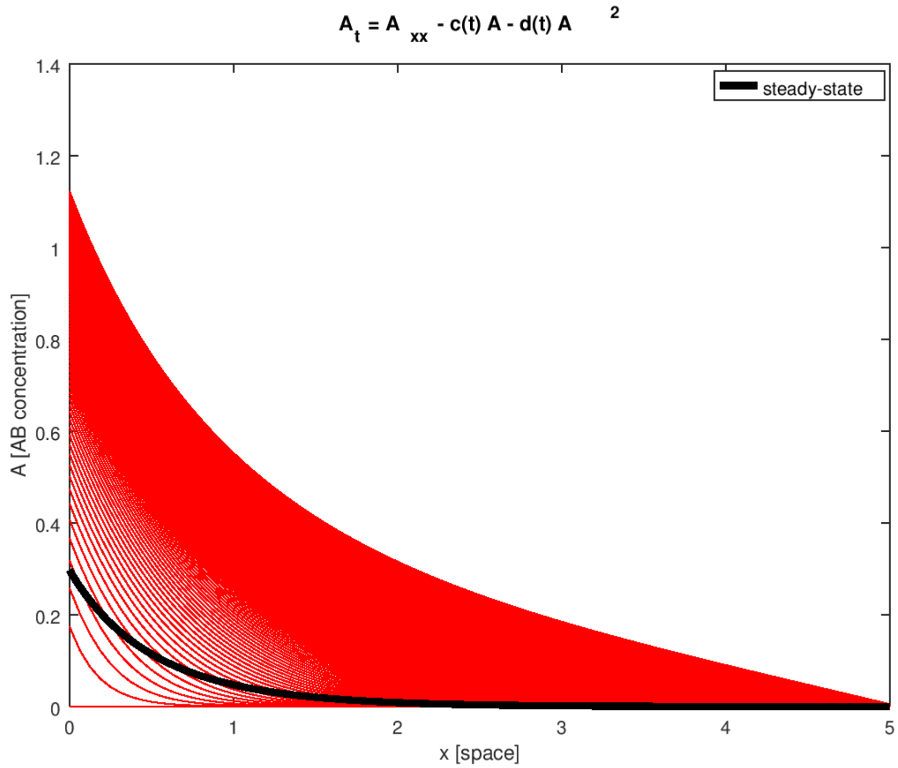

Figure 3: Concentration of normal AB over time with $\hat{\kappa} = e^{-\tau}$ along with the steady state solution.

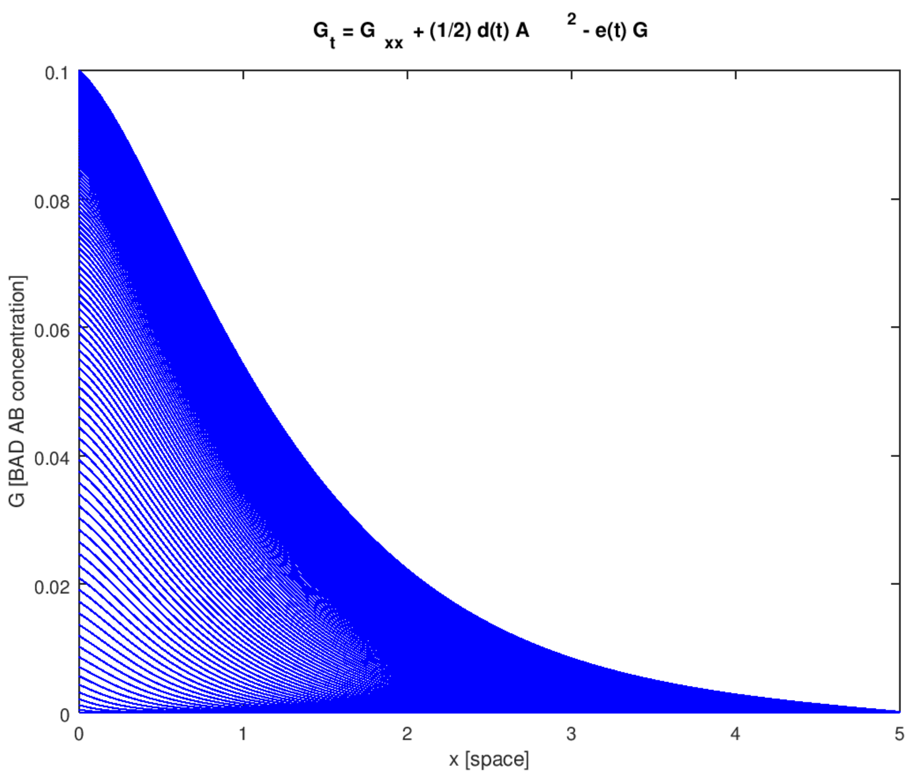

Figure 4: Concentration of bad AB over time.

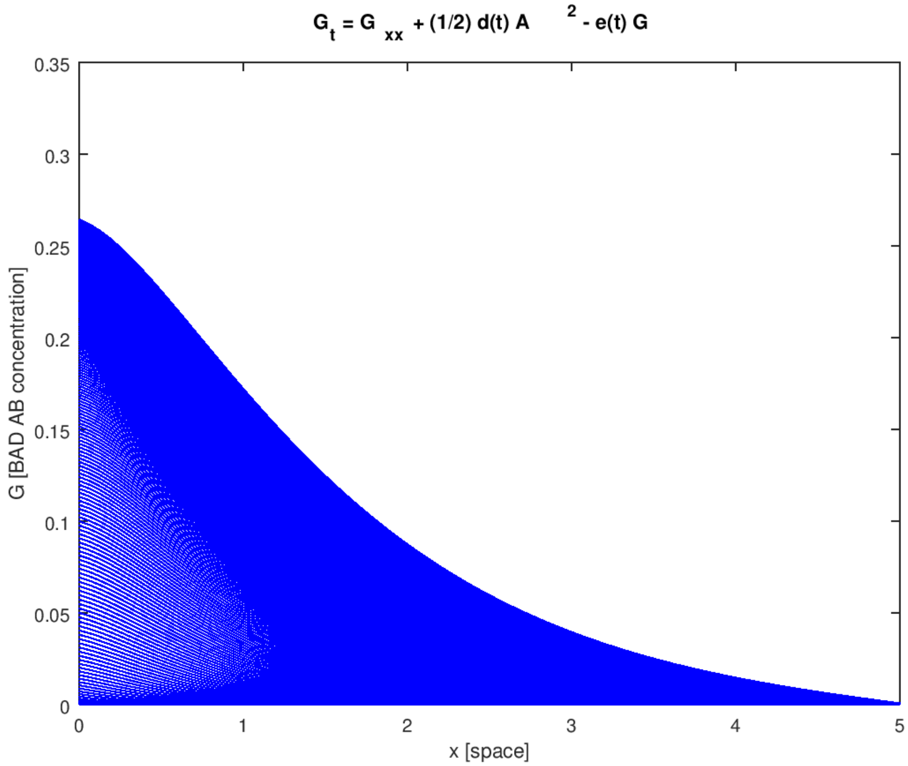

Figure 5: Concentration of bad AB over time with $\hat{\kappa} = e^{-\tau}$.

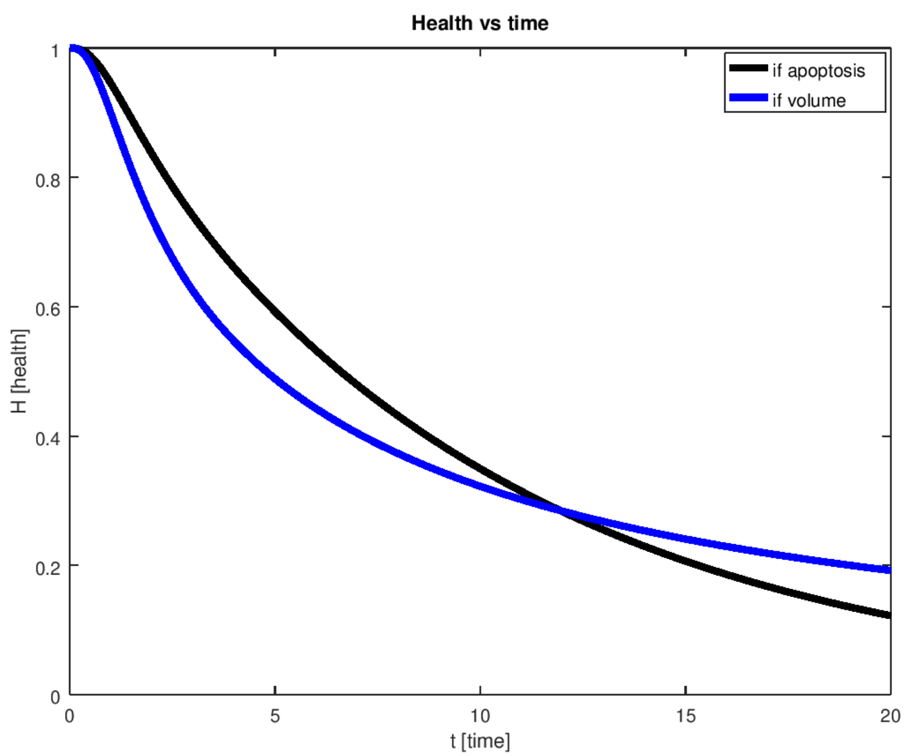

Figure 6: The health of a cell over time.

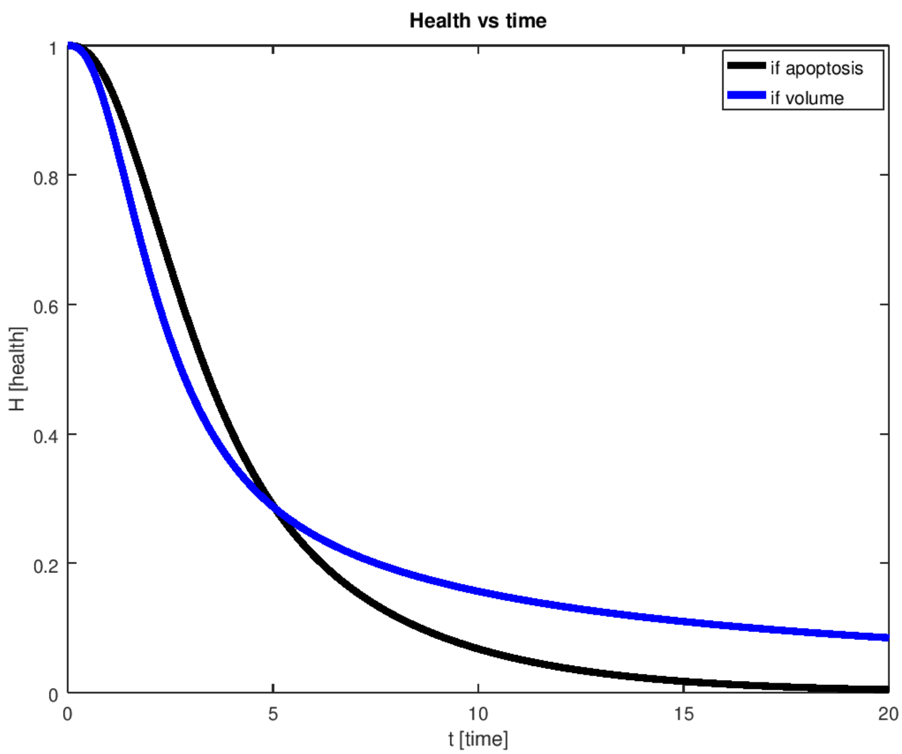

Figure 7: The health of a cell over time with $\hat{\kappa} = e^{-\tau}$.