Atom–photon interaction

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1 Two-level system

We will consider here a formal description of a two-level system illuminated with a coherent radiation that can drive an atomic transition with a certain interaction time that can be set experimentally. The two levels will be labelled $|0\rangle$ and $|1\rangle$ and the energy difference between these two states is $\hbar\omega_a = \mathcal{E}_1 - \mathcal{E}_0$. Since we are in a 2×2 vectorial space, we need four linear operators to contruct a base with which we can write any operator acting on the kets: $\mathbb{1}$, σ_z, σ_+, σ_-. Writing those operators in the $|0\rangle$, $|1\rangle$ representation:

$$\begin{cases} \hat{\mathbb{1}} = |1\rangle\langle 1| + |0\rangle\langle 0| \\ \hat{\sigma_z} = |0\rangle\langle 0| - |1\rangle\langle 1| \\ \hat{\sigma_+} = |1\rangle\langle 0| \\ \hat{\sigma_-} = |0\rangle\langle 1| \end{cases} \tag{1}$$

Following this notation, we can now write the Hamiltonian of the free atom:

$$\hat{\mathcal{H}}_{free} = -\frac{\hbar\omega_a}{2}\hat{\sigma_z} \tag{2}$$

We add now the interaction between the atom and the radiation (radio frequency, laser) and we consider the electric field:

$$\vec{\mathcal{E}} = \hat{\epsilon}\mathcal{E}_0 cos(\omega t) = \frac{1}{2}\mathcal{E}_0(\hat{\epsilon}e^{i\omega_L t} + \hat{\epsilon}^* e^{-i\omega_L t}), \tag{3}$$

where $\hat{\epsilon}$ is the complex polarization. The interaction energy is given by $\mathcal{H}_{int} = -\vec{\mu} \cdot \vec{\mathcal{E}}$, $\vec{\mu}$ being the electric dipole moment of the atom expressed by $\vec{\mu} = -e\vec{r}$. By parity arguments, $\langle 0| \vec{r} |0\rangle = \langle 1| \vec{r} |1\rangle = 0$. Considering the case of linear (real) polarization, we define the operator $\hat{\mu}_\epsilon = \mu_\epsilon \hat{\sigma}_x = e \langle 1| \vec{r} \cdot \hat{\epsilon} |0\rangle \hat{\sigma}_x$ allowing us to write the interaction hamiltonian as:

$$\begin{aligned} \hat{\mathcal{H}}_{int} &= \frac{\mathcal{E}_0 \mu_\epsilon}{2}(e^{i\omega_L t} + e^{-i\omega_L t})\hat{\sigma}_x \\ &= \hbar\Omega_R(e^{i\omega_L t} + e^{-i\omega_L t})\hat{\sigma}_x \end{aligned} \tag{4}$$

The above Hamiltonian has been written with the Rabi frequency defined as $\Omega_R = \mathcal{E}_0\mu_\epsilon/2\hbar$. The full Hamiltonian is:

$$\boxed{\hat{\mathcal{H}} = -\frac{\hbar\omega_a}{2}\hat{\sigma}_z + \hbar\Omega_R(e^{i\omega_L t} + e^{-i\omega_L t})\hat{\sigma}_x} \tag{5}$$

To further proceed on the solutions of the hamiltonian above, we will work on the interaction picture, which will be described in the next section.

1.1 Interaction picture

The interaction picture is a unitary transformation acting on vectors and operators. Let's start by writing a Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1(t)$, where we have decoupled the static and the time dependet part of the Hamiltonian. The central idea of the problem here is that $\mathcal{H}_0$ is the Hamiltonian of an uperturbed system and $\mathcal{H}_1$ is a time dependent perturbation on this system. We work on a basis of well known eigenvectors of $\mathcal{H}_0$ and we are interested in studying the dynamics of the system when the perturbation is on and how the states will evolve with time.

In quantum mechanics, we write an operator that propagates the wavefuntion from one point to another in time: $|\psi(t)\rangle = \mathcal{U} |\psi(0)\rangle$. This time evolution operator is unitary, i.e. $\mathcal{U}^\dagger\mathcal{U} = \mathbb{1}$ which conserves the proper normalization of the states, as required by quantum mechanics:

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)| \mathcal{U}^\dagger\mathcal{U} |\psi(0)\rangle = 1 \tag{6}$$

It is interesting to note that $\mathcal{U}^\dagger$, when acting on wavefunctions propagated in time, brings the states back to $t = 0$: $|\psi(0)\rangle = \mathcal{U}^\dagger |\psi(t)\rangle$. Now, what about how to write those time evolution operators? Well, if the Hamiltonian of the system has a complicated time dependence this problem can be complicated. However, if $[\mathcal{U}, \partial\mathcal{U}/\partial t] = 0$, then:

$$\mathcal{U}(t) = e^{-\frac{i}{\hbar}\int_0^t \mathcal{H} d\tau} \tag{7}$$

In particular, for time independent Hamiltonians, $\mathcal{U} = e^{-i\mathcal{H}t/\hbar}$.

In the interaction picture, we are mainly interested in removing the time evolution contribution due to the well known $\mathcal{H}_0$, which usually contributes only with a phase oscillation on the probability amplitudes. The way to do this is to apply the reverse time evolution operator with respect to $\mathcal{H}_0$. Hence, in the interaction picture, the states transform as:

$$|\psi(t)\rangle_I = e^{i\mathcal{H}_0 t/\hbar} |\psi(t)\rangle_S\,, \tag{8}$$

where S and I stand for Schrodinger and Interation picture respectively. Let's insert the transformed states in the time dependent Schrodinger equation (TDSE):

$$\begin{aligned} i\hbar\partial_t(e^{-i\mathcal{H}_0 t/\hbar}\left|\psi(t)\right\rangle_I) &= (\mathcal{H}_0+\mathcal{H}_1)e^{-i\mathcal{H}_0 t/\hbar}\left|\psi(t)\right\rangle_I \\ i\hbar\partial_t\left|\psi(t)\right\rangle_I &= e^{i\mathcal{H}_0 t/\hbar}\mathcal{H}_1 e^{-i\mathcal{H}_0 t/\hbar}\left|\psi(t)\right\rangle_I \end{aligned} \tag{9}$$

Now we have the motivation to define how operators transform in the interaction picture: $\mathcal{H}_{1I} = e^{i\mathcal{H}_0 t/\hbar}\mathcal{H}_1 e^{-i\mathcal{H}_0 t/\hbar}$, and we recover the beautiful Schrodinger equation–like form in the interaction picture:

$$i\hbar\partial_t\left|\psi(t)\right\rangle_I = \mathcal{H}_{1I}\left|\psi(t)\right\rangle_I \tag{10}$$

That is indeed beautiful, but once we are there, how do we handle the solutions? Well, eventually we will want to come back to the Schrodinger equation to write the complete solution of the problem. Let's start by writing states in a general form: $\left|\psi(t)\right\rangle_I = \sum_j c_j(t)\left|j\right\rangle$. We map the state back to the Schrodinger picture by propagating the wavefunction in time with respect to $\mathcal{H}_0$:

$$\begin{aligned} \left|\psi(t)\right\rangle &= e^{-i\mathcal{H}_0 t/\hbar}\sum_j c_j(t)\left|j\right\rangle \\ &= \sum_j c_j(t)e^{-i\omega_j t}\left|j\right\rangle, \text{with} \omega_j = \mathcal{E}_j/\hbar \end{aligned} \tag{11}$$

We have dropped the S index. Mapping the vector states back to Schrodinger equation adds then the usual phase oscillation to the solution. The job here is now to calculate the various coefficients $c_j(t)$. For this purpose, we can work on the frame of the interaction picture and insert $\left|\psi(t)\right\rangle_I$ in equation 10. The coefficients can then be determined:

$$\dot{c}_k(t) = -\frac{i}{\hbar}\sum_j c_j(t)\left\langle k\right|\mathcal{H}_{1I}\left|j\right\rangle \tag{12}$$

Writing the interaction operator mapped on the Schrodinger picture:

$$\boxed{\dot{c}_k(t) = -\frac{i}{\hbar}\sum_j c_j(t)e^{i\omega_{kj}t}\left\langle k\right|\mathcal{H}_1\left|j\right\rangle} \tag{13}$$

1.2 Rotating Wave Approximation and Rabi oscillations

We turn our attention now back to the full Hamiltonian of the problem. Let's re-write this equation in a more convinient way:

$$\hat{\mathcal{H}} = -\frac{\hbar\omega_L}{2}\hat{\sigma}_z + \hbar\Delta\hat{\Phi} + \hbar\Omega_R(e^{i\omega_L t}+e^{-i\omega_L t})\hat{\sigma}_x, \tag{14}$$

where $\Delta = \omega_a - \omega_L$ and $\hat{\Phi} = \left|1\right\rangle\left\langle 1\right|$. we also shifted the whole energy spectrum by $\hbar\Delta\hat{\mathbb{1}}/2$. Now we write down the transformation to the interaction picture with respect to the first term of expression 14, i.e. $\mathcal{H}_0 = -\frac{\hbar\omega_L}{2}$.

Performing the transformation term by term, given that $[\sigma_z, \hat{\phi}] = 0$, the first term is symmetrical under the interaction picture transformation. We calculate then how $\hat{\sigma_x}$ transforms. Let's write $\hat{\sigma_x} = \hat{\sigma^+} + \hat{\sigma^-}$ and using the baker-Hausdorff lemma:

$$e^B A e^{-B} = B + [B, A] + \frac{1}{2!}[B, [B, A]] + ..., \tag{15}$$

we find:

$$\begin{cases} \sigma_I^+ = e^{i\omega_L t}\sigma^+ \\ \sigma_I^- = e^{-i\omega_L t}\sigma^- \end{cases} \tag{16}$$

Writing the interaction Hamiltonian in the interaction picture:

$$\mathcal{H}_{int}^I = \hbar\Delta\hat{\phi} + \hbar\Omega_R(e^{i\omega_L t} + c.c.)(e^{i\omega_L t}\hat{\sigma}^+ + e^{-i\omega_L t}\hat{\sigma}^-) \tag{17}$$

In this expression some terms do not oscillate and other oscillate with $2\omega_L$. The rotating wave approximation consists in neglecting those fast oscillating terms. The interaction then is finally written:

$$\mathcal{H}_{int}^I = \hbar\Delta\hat{\phi} + \hbar\Omega_R\hat{\sigma}_x \tag{18}$$

In the matrix form:

$$\mathcal{H}_{int}^I = \hbar \begin{bmatrix} 0 & \Omega_R \\ \Omega_R & \Delta \end{bmatrix}$$

1.3 Density matrix formalism

1.4 Bloch sphere representation

2 Ensemble of two-level atoms: mixed states