$restart;$
$with(LinearAlgebra):$
$unprotect(\mathrm{D});$

Opgave 6.1

Delopgave 6.1.(a)

Nedenfor ses matricen P der beskriver musens mulige bevægelser

$$P := Matrix\left(\left[\left[0, \frac{1}{2}, 0, \frac{1}{2}, 0\right], \left[\frac{1}{2}, 0, \frac{1}{3}, 0, 0\right], \left[0, \frac{1}{2}, 0, \frac{1}{2}, 0\right], \left[\frac{1}{2}, 0, \frac{1}{3}, 0, 0\right], \left[0, 0, \frac{1}{3}, 0, 1\right]\right]\right);$$

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 1 \end{bmatrix} \quad \textbf{(1)}$$

Delopgave 6.1.(b)

Nedenfor ses sandsyndligheden for - *alt efter hvilket rum musen står i* - at musen finder frem til osten på højst 5 minutter. Jeg har opløftet til 4 potens, da det er min opfattelse at musen kun bevæger sig efter hvert minut. På 5 minutter når den således at bevæge sig 4 gange, hvis vi går ud fra at den bliver, i det rum den startede, i et minut.

$seq(Matrix(P^4)[5, i], i = 1..5)$

$$\frac{1}{6}, \frac{11}{36}, \frac{4}{9}, \frac{11}{36}, 1 \quad \textbf{(2)}$$

Delopgave 6.1.(c)

Fra den første delopgave har vi overgangsmatricen. Hvis vi ikke benyttede maple skulle til at opskrive og finde røderne i det karakteristisk polynomium for at bestemme egenværdier og igennem disses egenrum bestemme egenvektorerne. Med Maple ved hånden går det dog væsentlig hurtigere.

$(L, T) := Eigenvectors(P);$

$$\begin{bmatrix} \frac{1}{6}\sqrt{30} \\ -\frac{1}{6}\sqrt{30} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2}\sqrt{30}-3 & -\frac{1}{2}\sqrt{30}-3 & 0 & 2 & 0 \\ \frac{1}{2}\left(\frac{1}{6}\sqrt{30}-1\right)\sqrt{30} & -\frac{1}{2}\left(-\frac{1}{6}\sqrt{30}-1\right)\sqrt{30} & 0 & 0 & -1 \\ \frac{1}{2}\sqrt{30}-3 & -\frac{1}{2}\sqrt{30}-3 & 0 & -3 & 0 \\ \frac{1}{2}\left(\frac{1}{6}\sqrt{30}-1\right)\sqrt{30} & -\frac{1}{2}\left(-\frac{1}{6}\sqrt{30}-1\right)\sqrt{30} & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \tag{3}$$

Om koodinattransformationsmatricen T gælder at $S := T^{(-1)}$:

I maple findes diagonalmatricen på følgende måde.

$DiagonalMatrix(L);$

$$\begin{bmatrix} \frac{1}{6}\sqrt{30} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{6}\sqrt{30} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

Vi ved fra NVP s 138 (6.4 Potensopløftning af matricer) at følgende gælder $P^k = S^{-1} D^k S$

$simplify(S\text{^}(-1).(DiagonalMatrix(L)\text{^}k).S);$

$$\begin{bmatrix} -\frac{1}{2}\frac{\sqrt{30}}{5+\sqrt{30}} & -\frac{1}{2}\sqrt{30}-3 & 0 & 2 & 0 \\ \frac{5}{2}-\frac{1}{2}\sqrt{30} & \frac{5}{2}+\frac{1}{2}\sqrt{30} & 0 & 0 & -1 \\ -\frac{1}{2}\frac{\sqrt{30}}{5+\sqrt{30}} & -\frac{1}{2}\sqrt{30}-3 & 0 & -3 & 0 \\ \frac{5}{2}-\frac{1}{2}\sqrt{30} & \frac{5}{2}+\frac{1}{2}\sqrt{30} & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} . \begin{bmatrix} \frac{1}{6}\sqrt{30} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{6}\sqrt{30} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}^k . [\tag{5}$$

$$\left[\frac{3}{5\left(\sqrt{30}-6\right)}, \frac{1}{2}\frac{\sqrt{30}+6}{\left(5+\sqrt{30}\right)\left(\sqrt{30}-6\right)}, \frac{2}{5\left(\sqrt{30}-6\right)}, \frac{1}{2}\frac{\sqrt{30}+6}{\left(5+\sqrt{30}\right)\left(\sqrt{30}-6\right)}, 0\right],$$

$$\left[-\frac{1}{10}\frac{\sqrt{30}}{5+\sqrt{30}}, \frac{1}{2\left(5+\sqrt{30}\right)}, -\frac{1}{15}\frac{\sqrt{30}}{5+\sqrt{30}}, \frac{1}{2\left(5+\sqrt{30}\right)}, 0\right],$$

$$\left[1, 1, 1, 1, 1\right],$$

$$\left[\frac{1}{5}, 0, -\frac{1}{5}, 0, 0\right],$$

$$\left[0, -\frac{1}{2}, 0, \frac{1}{2}, 0\right]\right]$$

Vi er dog interesseret i at få k ind i matricen og bestemme produktet af matricerne. Dette gøres ved følgende udregning som godt kan være en smule svær at overskue.

simplify(*S*^(-1).*DiagonalMatrix*([*L*[1]^*k*, *L*[2]^*k*, *L*[3]^*k*, *L*[4]^*k*, *L*[5]^*k*]).*S*);

$$\left[\left[-\frac{3}{10}\frac{\sqrt{30}\,6^{-k}30^{\frac{1}{2}k}\left(1+(-1)^{k}\right)}{\left(5+\sqrt{30}\right)\left(\sqrt{30}-6\right)}, \frac{1}{4}\frac{\left(\sqrt{30}+6\right)\sqrt{30}\,6^{-k}30^{\frac{1}{2}k}\left(-1+(-1)^{k}\right)}{\left(5+\sqrt{30}\right)^{2}\left(\sqrt{30}-6\right)},\right.\right. \tag{6}$$

$$\left.-\frac{1}{5}\frac{\sqrt{30}\,6^{-k}30^{\frac{1}{2}k}\left(1+(-1)^{k}\right)}{\left(5+\sqrt{30}\right)\left(\sqrt{30}-6\right)}, \frac{1}{4}\frac{\left(\sqrt{30}+6\right)\sqrt{30}\,6^{-k}30^{\frac{1}{2}k}\left(-1+(-1)^{k}\right)}{\left(5+\sqrt{30}\right)^{2}\left(\sqrt{30}-6\right)}, 0\right],$$

$$\left[\frac{3}{10}\frac{6^{-k}\left(5\,30^{\frac{1}{2}k}-30^{\frac{1}{2}k+\frac{1}{2}}+5\,(-1)^{k+1}30^{\frac{1}{2}k}+(-1)^{k}30^{\frac{1}{2}k+\frac{1}{2}}\right)}{\sqrt{30}-6},\right.$$

$$-\frac{1}{4}\frac{\sqrt{30}\,6^{-k}30^{\frac{1}{2}k}\left(1+(-1)^{k}\right)}{\left(5+\sqrt{30}\right)\left(\sqrt{30}-6\right)},$$

$$\frac{1}{5}\frac{6^{-k}\left(5\,30^{\frac{1}{2}k}-30^{\frac{1}{2}k+\frac{1}{2}}+5\,(-1)^{k+1}30^{\frac{1}{2}k}+(-1)^{k}30^{\frac{1}{2}k+\frac{1}{2}}\right)}{\sqrt{30}-6},$$

$$\left.-\frac{1}{4}\frac{\sqrt{30}\,6^{-k}30^{\frac{1}{2}k}\left(1+(-1)^{k}\right)}{\left(5+\sqrt{30}\right)\left(\sqrt{30}-6\right)}, 0\right],$$

$$\left[-\frac{3}{10}\frac{\sqrt{30}\,6^{-k}30^{\frac{1}{2}k}\left(1+(-1)^{k}\right)}{\left(5+\sqrt{30}\right)\left(\sqrt{30}-6\right)}, \frac{1}{4}\frac{\left(\sqrt{30}+6\right)\sqrt{30}\,6^{-k}30^{\frac{1}{2}k}\left(-1+(-1)^{k}\right)}{\left(5+\sqrt{30}\right)^{2}\left(\sqrt{30}-6\right)},\right.$$

$$\left.-\frac{1}{5}\frac{\sqrt{30}\,6^{-k}30^{\frac{1}{2}k}\left(1+(-1)^{k}\right)}{\left(5+\sqrt{30}\right)\left(\sqrt{30}-6\right)}, \frac{1}{4}\frac{\left(\sqrt{30}+6\right)\sqrt{30}\,6^{-k}30^{\frac{1}{2}k}\left(-1+(-1)^{k}\right)}{\left(5+\sqrt{30}\right)^{2}\left(\sqrt{30}-6\right)}, 0\right],$$

$$\left[\frac{3}{10}\frac{6^{-k}\left(5\,30^{\frac{1}{2}k}-30^{\frac{1}{2}k+\frac{1}{2}}+5\,(-1)^{k+1}30^{\frac{1}{2}k}+(-1)^{k}30^{\frac{1}{2}k+\frac{1}{2}}\right)}{\sqrt{30}-6},\right.$$

$$-\frac{1}{4}\,\frac{\sqrt{30}\,6^{-k}\,30^{\frac{1}{2}k}\left(1+(-1)^k\right)}{\left(5+\sqrt{30}\right)\left(\sqrt{30}-6\right)},$$

$$\frac{1}{5}\,\frac{6^{-k}\left(5\,30^{\frac{1}{2}k}-30^{\frac{1}{2}k+\frac{1}{2}}+5\,(-1)^{k+1}\,30^{\frac{1}{2}k}+(-1)^k\,30^{\frac{1}{2}k+\frac{1}{2}}\right)}{\sqrt{30}-6},$$

$$\left.-\frac{1}{4}\,\frac{\sqrt{30}\,6^{-k}\,30^{\frac{1}{2}k}\left(1+(-1)^k\right)}{\left(5+\sqrt{30}\right)\left(\sqrt{30}-6\right)},0\right],$$

$$\left[\frac{1}{5}\,\frac{1}{\left(5+\sqrt{30}\right)\left(\sqrt{30}-6\right)}\left(15\,6^{-k}\,30^{\frac{1}{2}k}+3\,6^{-k}\,30^{\frac{1}{2}k+\frac{1}{2}}+15\,(-1)^k\right.\right.$$

$$\left.{}^{+1}\,30^{\frac{1}{2}k}\,6^{-k}+3\,(-1)^k\,30^{\frac{1}{2}k+\frac{1}{2}}\,6^{-k}-5\sqrt{30}\right),$$

$$\frac{1}{2}\,\frac{6^{-k}\,30^{\frac{1}{2}k+\frac{1}{2}}+6^{1-k}\,30^{\frac{1}{2}k}+(-1)^k\,30^{\frac{1}{2}k+\frac{1}{2}}\,6^{-k}+(-1)^{k+1}\,6^{1-k}\,30^{\frac{1}{2}k}-2\sqrt{30}}{\left(5+\sqrt{30}\right)\left(\sqrt{30}-6\right)},$$

$$\frac{1}{5}\,\frac{1}{\left(5+\sqrt{30}\right)\left(\sqrt{30}-6\right)}\left(10\,6^{-k}\,30^{\frac{1}{2}k}+2\,6^{-k}\,30^{\frac{1}{2}k+\frac{1}{2}}+10\,(-1)^{k+1}\,30^{\frac{1}{2}k}\,6^{-k}\right.$$

$$\left.+2\,(-1)^k\,30^{\frac{1}{2}k+\frac{1}{2}}\,6^{-k}-5\sqrt{30}\right),$$

$$\frac{1}{2}\,\frac{6^{-k}\,30^{\frac{1}{2}k+\frac{1}{2}}+6^{1-k}\,30^{\frac{1}{2}k}+(-1)^k\,30^{\frac{1}{2}k+\frac{1}{2}}\,6^{-k}+(-1)^{k+1}\,6^{1-k}\,30^{\frac{1}{2}k}-2\sqrt{30}}{\left(5+\sqrt{30}\right)\left(\sqrt{30}-6\right)},$$

$$\left.\left.1\right]\right]$$

Det kan være lidt svært at se, men det vi har her ovenfor er en (5 × 5) matrix.

Vi er faktisk kun interreseret i den femte række. Det er fordi den femte rækker, med udgangspunkt i rummene 1 til 5, beskriver sandsynligheden for at musen har fundet osten efter *k* minutter. Derfor har jeg valgt at isolere række fem nedenfor.

R5 := *simplify*(*Row*(%, 5));

$$\left[\frac{1}{5}\,\frac{1}{\left(5+\sqrt{30}\right)\left(\sqrt{30}-6\right)}\left(15\,6^{-k}\,30^{\frac{1}{2}k}+3\,6^{-k}\,30^{\frac{1}{2}k+\frac{1}{2}}+15\,(-1)^{k+1}\,30^{\frac{1}{2}k}\,6^{-k}+3\,(\right.\right. \tag{7}$$

$$-1)^{k} 30^{\frac{1}{2} k+\frac{1}{2}} 6^{-k}-5 \sqrt{30}\Big),$$

$$\frac{1}{2} \frac{6^{-k} 30^{\frac{1}{2} k+\frac{1}{2}}+6^{1-k} 30^{\frac{1}{2} k}+(-1)^{k} 30^{\frac{1}{2} k+\frac{1}{2}} 6^{-k}+(-1)^{k+1} 6^{1-k} 30^{\frac{1}{2} k}-2 \sqrt{30}}{(5+\sqrt{30})(\sqrt{30}-6)},$$

$$\frac{1}{5} \frac{1}{(5+\sqrt{30})(\sqrt{30}-6)}\Big(10\, 6^{-k} 30^{\frac{1}{2} k}+2\, 6^{-k} 30^{\frac{1}{2} k+\frac{1}{2}}+10\,(-1)^{k+1} 30^{\frac{1}{2} k} 6^{-k}$$

$$+2\,(-1)^{k} 30^{\frac{1}{2} k+\frac{1}{2}} 6^{-k}-5 \sqrt{30}\Big),$$

$$\frac{1}{2} \frac{6^{-k} 30^{\frac{1}{2} k+\frac{1}{2}}+6^{1-k} 30^{\frac{1}{2} k}+(-1)^{k} 30^{\frac{1}{2} k+\frac{1}{2}} 6^{-k}+(-1)^{k+1} 6^{1-k} 30^{\frac{1}{2} k}-2 \sqrt{30}}{(5+\sqrt{30})(\sqrt{30}-6)},$$

$$1\Big]$$

Jeg har nu en rækkevektor med 5 udtryk af k . Disse 5 udtryk beskriver som ovenfor nævnt, sandsynligheden for at træffe musen i rum 5 efter k minutter. Jeg tager nu blot og finder grænseværdien for disse 5 udtryk.

seq(limit(R5[i], k = infinity), i = 1 ..5);

$$1, 1, 1, 1, 1 \tag{8}$$

Vi ser her at alle udtrykkene går mod 1, når k går mod uendelig.