18.01 EXERCISES

Unit 1. Differentiation

1A. Graphing

1A-1 By completing the square, use translation and change of scale to sketch

a) $y = x^2 - 2x - 1$ b) $y = 3x^2 + 6x + 2$

1A-2 Sketch, using translation and change of scale

a) $y = 1 + |x + 2|$ b) $y = \dfrac{2}{(x-1)^2}$

1A-3 Identify each of the following as even, odd , or neither

a) $\dfrac{x^3 + 3x}{1 - x^4}$ b) $\sin^2 x$

c) $\dfrac{\tan x}{1 + x^2}$ d) $(1 + x)^4$

e) $J_0(x^2)$, where $J_0(x)$ is a function you never heard of

1A-4 a) Show that every polynomial is the sum of an even and an odd function.

b) Generalize part (a) to an arbitrary function $f(x)$ by writing

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) + f(-x)}{2}$$

Verify this equation, and then show that the two functions on the right are respectively even and odd.

c) How would you write $\dfrac{1}{x + a}$ as the sum of an even and an odd function?

1A-5. Find the inverse to each of the following, and sketch both $f(x)$ and the inverse function $g(x)$. Restrict the domain if necessary. (Write $y = f(x)$ and solve for y; then interchange x and y.)

a) $\dfrac{x - 1}{2x + 3}$ b) $x^2 + 2x$

1A-6 Express in the form $A \sin(x + c)$

a) $\sin x + \sqrt{3} \cos x$ b) $\sin x - \cos x$

1A-7 Find the period , amplitude , and phase angle, and use these to sketch

a) $3 \sin(2x - \pi)$ b) $-4 \cos(x + \pi/2)$

Total Score (Page 1) = $\frac{13}{21}$

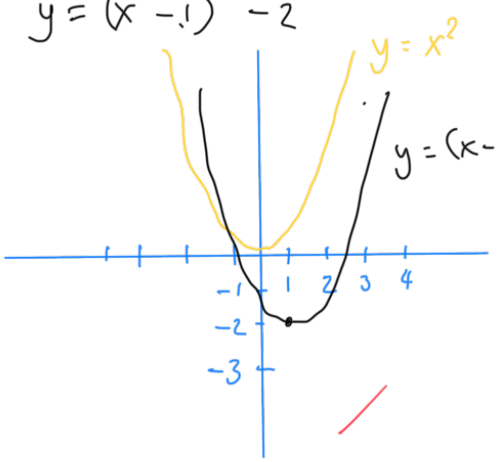
Answers

1A-1 – Complete the square, translate and sketch the functions

a. $y = x^2 - 2x - 1$

$= x^2 - 2x + 1 - 1 - 1$

$y = (x-1)^2 - 2$



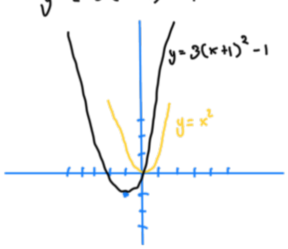
b $y = 3x^2 + 6x + 2$

$\frac{y}{3} = x^2 + 2x + \frac{2}{3}$

$= x^2 + 2x + 1 + \frac{2}{3} - 1$

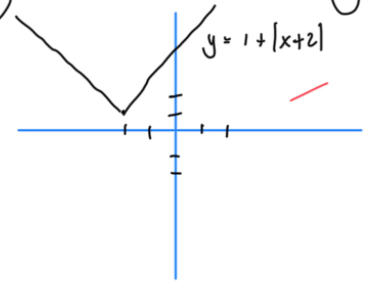
$\frac{y}{3} = (x+1)^2 - \frac{1}{3}$

$y = 3(x+1)^2 - 1$

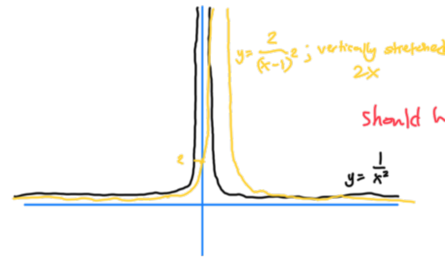


1A-2 Sketch using translation and change of scale

a. $y = 1 + |x+2|$



b. $y = \frac{2}{(x-1)^2}$



should have changed the scale

1A-3 Identify whether odd, even or neither.

a. $\frac{x^3 + 3x}{1 - x^4}$

even if:

$f(x) = f(-x)$

$\frac{x^3+3x}{1-x^4} = \frac{(-x)^3 + 3(-x)}{1-(-x)^4}$

$= \frac{-x^3 - 3x}{1 - x^4}$

$\frac{x^3+3x}{1-x^4} \overset{\times}{=} \frac{-(x^3+3x)}{1-x^4}$

$f(x) = -f(-x)$

$f(x) = -\left(\frac{-(x^3+3x)}{1-x^4}\right)$

$f(x) \overset{\times}{=} \frac{x^3+3x}{-1+x^4}$

Answer - Neither

b. $\sin^2 x$

Answer - Even

$f(x) = f(-x)$

$(\sin x)^2 = (\sin -x)^2$

$\sin^2 x \overset{\checkmark}{=} \sin^2 x$

$f(x) = -f(-x)$

$\sin^2 x = -(\sin -x)^2$

$\sin^2 x \overset{\times}{=} -\sin^2 x$

c. $\frac{\tan x}{1 + x^2}$

Answer - Neither

$f(x) = f(-x)$

$= \frac{\tan -x}{1 + (-x)^2}$

$f(x) \overset{\times}{=} \frac{-\tan x}{1 + x^2}$

$f(x) = -f(-x)$

$= -\left(\frac{-\tan x}{1 + x^2}\right)$

$f(x) \overset{\times}{=} \frac{\tan x}{-x^2 - 1}$

d. $(1+x)^4$

Answer - Even

$f(x) = f(-x)$

$= 1 + (-x)^4$

$f(x) \overset{\checkmark}{=} 1 + x^4$

e. $J_0(x^2)$, where J_0 is a function you've never heard of.

Answer - Even

$f(x) = f(-x)$

$= J_0((-x)^2)$

$f(x) = J_0(x^2)$

1A - 4

ⓐ Show that every polynomial is a sum of odd and even functions

E(x) - even function
O(x) - odd function

$f(x) = E(x) + O(x)$ ✓

$f(-x) = E(-x) + O(-x) = E(x) - O(x)$

$f(x) + f(-x) = E(x) + \cancel{O(x)} + E(x) - \cancel{O(x)}$

$f(x) + f(-x) = 2E(x)$

Eq. 1 - $E(x) = \dfrac{f(x) + f(-x)}{2}$

$f(x) - f(-x) = \cancel{E(x)} + O(x) - \cancel{E(x)} + O(x)$

$f(x) - f(-x) = 2O(x)$

Eq. 2 - $O(x) = \dfrac{f(x) - f(-x)}{2}$

ⓑ Generalize part a to an arbitrary function f(x)

$f(x) = O(x) + E(x)$

$f(x) = \dfrac{f(x) + f(-x)}{2} + \dfrac{f(x) - f(-x)}{2}$ ✓

ⓒ Write $f(x) = \dfrac{1}{x+a}$ as sum of even and odd functions

$f(x) = \dfrac{f(x)+f(-x)}{2} + \dfrac{f(x) - f(-x)}{2}$

$= \frac{1}{2}\left[\frac{1}{x+a} + \frac{-1}{x-a} + \frac{1}{x+a} - \frac{-1}{x-a}\right]$

$= \frac{1}{2}\left[\frac{x-a-x-a}{(x+a)(x-a)} + \frac{x-a-(-x-a)}{(x+a)(x-a)}\right]$

$= \dfrac{-2a}{2(x+a)(x-a)} + \dfrac{2x+2a}{2(x+a)(x-a)}$ ✗

$= \dfrac{-a}{(x+a)(x-a)} + \dfrac{1}{x-a}$

1A - 5 Find the inverse function & sketch

ⓐ $\dfrac{x-1}{2x+3}$

$g(x) = f^{-1}(x)$

$g(x) = \dfrac{3x+1}{1-2x}$

$y = \dfrac{x-1}{2x+3}$

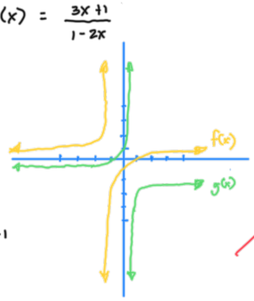
$x = \dfrac{y-1}{2y+3}$

$y - 1 = 2xy + 3x$

$y - 2xy = 3x + 1$

$y(1-2x) = 3x+1$

$y = \dfrac{3x+1}{1-2x}$



✓

ⓑ $x^2 + 2x$

$g(x) = f^{-1}(x)$

$g(x) = \sqrt{x+1} - 1$

$y = x^2 + 2x$

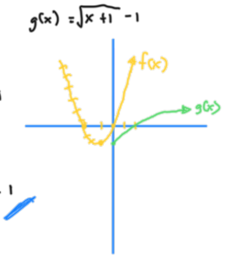
$x = y^2 + 2y$

$= y^2 + 2y + 1 - 1$

$x + 1 = (y+1)^2$

$y + 1 = \sqrt{x+1}$

$y = \sqrt{x+1} - 1$



✗

1A - 6 Express in form $A\sin(x+c)$

a. $\sin x + \sqrt{3}\cos x$

Eq 1 $A\sin(x+c) = A\sin x\cos c + A\sin c\cos x$

Eq 2 $(A\sin x)^2 + (A\cos x)^2 = A^2$

$A\sin x\cos c + A\sin c\cos x = \sin x + \sqrt{3}\cos x$

$(A\cos c)\sin x + (A\sin c)\cos x = \sin x + \sqrt{3}\cos x$

$A\cos c = 1 \qquad A\sin c = \sqrt{3}$

$A\sin^2 x + A\cos^2 x = A^2$

$1 + 3 = A^2$

$2 = A$

$A\cos c = 1$

$\cos c = \frac{1}{2}$

$c = \frac{\pi}{3}$

$\therefore \sin x + \sqrt{3}\cos x = 2\sin(x + \pi/3)$ ✓

b. $\sin x - \cos x$

$A\sin x\cos c + A\sin c\cos x = \sin x - \cos x$

$(A\cos c)\sin x + (A\sin c)\cos x = \sin x - \cos x$

$A\cos c = 1 \qquad A\sin c = -1$

$A^2 = (A\sin x)^2 + (A\cos x)^2$

$= (-1)^2 + (1)^2$

$A^2 = 2$

$A = \pm\sqrt{2}$

Solution $A = \sqrt{2}$

$A\cos c = 1 \qquad A\sin c = -1$

$\cos c = \frac{1}{\sqrt{2}} \qquad \sin c = \frac{-1}{\sqrt{2}}$

$c = \frac{\pi}{4}$ or $-\frac{\pi}{4} \qquad c = -\frac{\pi}{4}$

$\therefore \sin x - \cos x = \sqrt{2}\sin(x - \pi/4)$ ✓

1A - 7 Find the period, amplitude, and phase angle to sketch

a. $3\sin(2x - \pi)$

Amplitude = 3 ✓

Period = π ✓

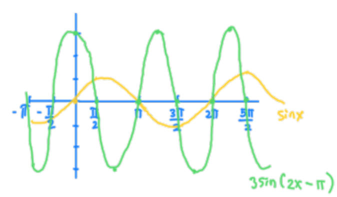
Phase angle = $-\pi$ ✗ $\frac{\pi}{2}$

$A = 3$

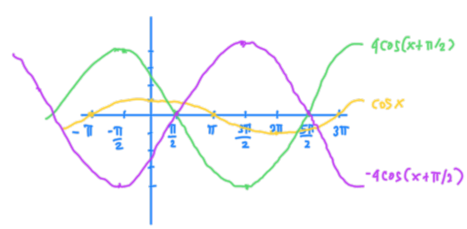
$B = 2$

$C = -\pi$

$D = 0$



b. $-4\cos(x + \pi/2)$



Amplitude = 4 ✓

Period = 2π ✓

Phase Angle = $\frac{\pi}{2}$ ✗ 0