Activity Score: 2/3
Recitation Score: 2/2

Finding a Formula for the Best Degree n Approximation

Linear approximation uses the values of a function and its first derivative at x_0 — or equivalently the slope of the tangent line and the point of tangency — to find the equation of a degree 1 polynomial approximating the function. The formula for the linear approximation of a function $f(x)$ near the value $x = x_0$ is:

$$f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0).$$

The formula for quadratic approximation:

$$f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

describes the best degree 2 approximation of the function $f(x)$ for $x \approx x_0$. We could also describe a degree 0 approximation: $f(x) \approx f(x_0)$ when x is sufficiently close to x_0. As the degree of the approximation increases we add new terms, but the lower degree terms stay the same.

Can we define higher degree approximations, and if so, what terms should we add to do so? We'll focus on finding approximations near the value $x_0 = 0$; the calculations for the general case are very similar.

a) Find the best third degree approximation of a function $f(x)$ near $x = 0$:

$$A(x) \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

In other words, find values for the a_i so that the first, second, and third derivatives of $f(x)$ are the same as the first, second and third derivatives of $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ when $x = 0$. (You may wish to refer to your notes on the derivation of the formula for the quadratic approximation.)

b) Use the fact that the n^{th} derivative of x^m is 0 for $m < n$ to find the n^{th} derivative of:

$$A(x) = a_0 + a_1 x^1 + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n.$$

c) Use your answer to (b) to give a formula for an n^{th} degree polynomial approximation of a function $f(x)$ near $x = 0$.

a. Find the best degree approximation of a function $f(x)$ near $x=0$:

$$A(x) \approx a_0 + a_1x + a_2x^2 + a_3x^3$$

In other words, find values for a_i so that the first, second, and third derivatives of $f(x)$ are the same as the first, second and third derivatives of $a_0 + a_1x + a_2x^2 + a_3x^3$ when $x=0$

Approximations	at $x=0$
$A(x) \approx a_0 + a_1x + a_2x^2 + a_3x^3$	$A(x) \approx a_0$
$A'(x) \approx a_1 + 2a_2x + 3a_3x^2$	$A'(x) \approx a_1$
$A''(x) \approx 2a_2 + 6a_3x$	$\frac{1}{2}A''(x) \approx a_2$
$A'''(x) \approx 6a_3$	$\frac{1}{6}A'''(x) \approx a_3$

$f(x) \approx f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3$, when $x \approx 0$

b. Use the fact that the n^{th} derivative of x^m is 0 for $m<n$ to find the n^{th} derivative of:

$$A(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n$$

$$A^{(n)}(x) = n!\,a_n$$

this was mentioned in part A of

c. Use your answer in (b) to give a formula for the nth polynomial approximation of a function $f(x)$ near $x=0$.

$A^{(n)} = n!\,a_n$

$a_n = \frac{1}{n!}A^{(n)}$

Got several aspects of the notation wrong but

$$f(x) \approx f(0) + f^{(1)}(0)x + \frac{1}{2}f^{(2)}(0)x^2 + \ldots + \frac{1}{(n-1)!}f^{(n-1)}(0)x^{(n-1)} + \frac{1}{n!}f^{(n)}(0)x^n$$

answered correctly. I just got the notation wrong.

Recitation:

Let $Q(f) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2$

Goal: Show $Q(fg) = Q(Q(f)Q(g))$

or

Show that the quadratic approximation of a product is equal to the quadratic approximation of the quadratic approximations of the components.

$Q(fg) = Q(Q(f)Q(g))$

$Q(fg) = fg + [f'g + fg']x + \frac{1}{2}[f''g + 2f'g' + fg'']x^2$

$Q(Q(f)Q(g)) = (f + f'x + \frac{1}{2}f''x^2)(g + g'x + \frac{1}{2}g''x^2)$

$= fg + fg'x + \frac{1}{2}fg''x^2 + f'gx + f'g'x^2 + \frac{1}{2}f'g''x^3$
$+ \frac{1}{2}f''gx^2 + \frac{1}{2}f''g'x^3 + \frac{1}{4}f''g''x^4$

$= fg + [f'g + fg']x + \frac{1}{2}[f''g + 2f'g' + fg'']x^2$

$\therefore Q(fg) = Q(Q(f)Q(g))$