

$$\begin{array}{lll} \text{a) } V = \frac{1}{3}\pi r^2 h, & \frac{dr}{dh} = ? & \text{b) } PV^c = nRT, & \frac{dP}{dV} = ? \\ \text{c) } c^2 = a^2 + b^2 - 2ab \cos \theta, & \frac{da}{db} = ? \end{array}$$

1G. Higher derivatives**1G-1** Calculate y'' for the following functions.

$$\begin{array}{ll} \text{a) } 3x^2 + 2x + 4\sqrt{x} & \text{b) } \frac{x}{x+5} \\ \text{c) } \frac{-5}{x+5} & \text{d) } \frac{x^2+5x}{x+5} \end{array}$$

1G-2 Find all functions $f(x)$ whose third derivative $f'''(x)$ is identically zero. (“Identically” is math jargon for “always” or “for every value of x ”.)**1G-3** Calculate y'' using implicit differentiation and simplify as much as possible.

$$x^2a^2 + y^2b^2 = 1$$

1G-4 Find the formula for the n th derivative $y^{(n)}$ of $y = 1/(x+1)$.**1G-5** Let $y = u(x)v(x)$.a) Find y' , y'' , and y''' .b) The general formula for $y^{(n)}$, the n -th derivative, is called *Leibniz' formula*: it uses the same coefficients as the binomial theorem , and looks like

$$y^{(n)} = u^{(n)}v + \binom{n}{1}u^{(n-1)}v^{(1)} + \binom{n}{2}u^{(n-2)}v^{(2)} + \dots + uv^{(n)}$$

Use this to check your answers in part (a), and use it to calculate $y^{(p+q)}$, if $y = x^p(1+x)^q$.**1H. Exponentials and Logarithms: Algebra****1H-1** The *half-life* λ of a radioactive substance decaying according to the law $y = y_0e^{-kt}$ is defined to be the time it takes the amount to decrease to 1/2 of the initial amount y_0 .a) Express the half-life λ in terms of k . (Do this from scratch — don't just plug into formulas given here or elsewhere.)

IG-1 Calculate y'' of the following functions

$$a. 3x^2 + 2x + 4\sqrt{x}$$

$$y' = \frac{d}{dx} \left[\frac{d}{dx} (3x^2 + 2x + 4\sqrt{x}) \right]$$

$$= \frac{d}{dx} (6x + 2 + 2x^{-1/2})$$

$$y'' = 6 - x^{-3/2}$$

$$b. \frac{x}{x+5}$$

$$y' = \frac{d}{dx} \left[\frac{d}{dx} (x(x+5)^{-1}) \right]$$

$$= \frac{d}{dx} [(1)(x+5)^{-1} + (x)(-1)(x+5)^{-2}(1)]$$

$$= \frac{d}{dx} [(x+5) - x](x+5)^{-2} = \frac{d}{dx} (5)(x+5)^{-2}$$

$$= 5(-2)(x+5)^{-3}(1)$$

$$y'' = -10(x+5)^{-3}$$

$$c. \frac{-5}{x+5}$$

$$y' = \frac{d}{dx} \left[\frac{d}{dx} (-5)(x+5)^{-1} \right]$$

$$= \frac{d}{dx} [(-5)(-1)(x+5)^{-2}(1)] = \frac{d}{dx} 5(x+5)^{-2}$$

$$= 5(-2)(x+5)^{-3}$$

$$y'' = -10(x+5)^{-3}$$

$$d. \frac{x^2 + 5x}{x+5}$$

$$y' = \frac{d}{dx} \left[\frac{d}{dx} (x^2 + 5x)(x+5)^{-1} \right]$$

$$= \frac{d}{dx} (2x+5)(x+5)^{-1} + (x^2 + 5x)(-1)(x+5)^{-2}$$

$$= (2)(x+5)^{-1} + (2x+5)(-1)(x+5)^{-2} - [(2x+5)(x+5)^{-2} + (x^2 + 5x)(-2)(x+5)^{-3}]$$

$$= 2(x+5)^{-1} - (2x+5)(x+5)^{-2} - (2x+5)(x+5)^{-1} + 2(x^2 + 5x)(x+5)^{-3}$$

$$= [2(x+5)^{-1} - (2x+5)(x+5)^{-2} - (2x+5)(x+5)^{-1} + 2x(x+5)](x+5)^{-3}$$

$$= [(x+5) - (2x+5) - (2x+5) + 2x](x+5)^{-3}$$

$$= [(x+5)(0)](x+5)^{-3}$$

$$y'' = 0$$

IG-2 Find all functions $f(x)$ whose third derivative is identically zero.

Will consult answer key **F**

IG-3 Calculate y'' using implicit differentiation and simplify as much as possible

$$x^2a^2 + y^2b^2 = 1$$

$$\frac{d}{dx} (x^2a^2 + y^2b^2 = 1)$$

$$2a^2x + 2b^2yy' = 0$$

$$y' = \frac{-2a^2x}{2b^2y}$$

$$y' = \frac{-a^2x}{b^2y}$$

$$\frac{d}{dx} (-a^2x)(b^2y)^{-1}$$

$$-a^2(-1)(b^2y)^{-2}(b^2)(y')$$

$$y'' = \frac{a^2b^2y'}{(b^2y)^2}$$

$$= \frac{a^2b^2(-a^2x)(b^2y)^{-1}}{(b^2y)^2}$$

$$y'' = \frac{-a^4b^2x}{(b^2y)^3}$$

IG-4 Find the formula for the n th derivative

$$y^{(n)} \text{ of } y = 1/(x+1).$$

$$y^{(1)} = (x+1)^{-2}$$

$$y^{(2)} = 2(x+1)^{-3}$$

$$y^{(3)} = -6(x+1)^{-4}$$

$$y^{(4)} = 24(x+1)^{-5}$$

b. The general formula for $y^{(n)}$, the n th derivative, is called the Leibniz' formula. It uses the same coefficients as the binomial theorem.

$$y^{(n)} = u^{(n)} v + \binom{n}{1} u^{(n-1)} v^{(1)} + \binom{n}{2} u^{(n-2)} v^{(2)} \dots u v^{(n)}$$

Use this to check your answer in part (a), and calculate $y^{(p+q)}$, if $y = x^p(1+x)^q$

$$y^{(p+q)} \Big|_{p+q=n} = x^{(p+q)}(1+x) + \binom{n}{1} x^{(p+q-1)}(1+x)^{(1)} + \binom{n}{2} x^{(p+q-2)}(1+x)^{(2)} \dots x(1+x)^{(p+q-1)}$$

, because $(1+x)^{(n)} = 0 \Leftrightarrow n > 1$

$$= x^{(p+q)}(1+x) + \binom{n}{1} x^{(p+q-1)}$$

IG-1 The half-life λ of a radioactive substance decaying according to the law $y = y_0 e^{-kt}$ is defined to be the time it takes the amount to decrease to $1/2$ of the initial amount y_0 .

a. Express the half-life λ in terms of k . (Do this from scratch - don't just plug into formulas given here or anywhere.)

$$\frac{y}{y_0} = e^{-kt}$$

$$-kt = \ln \left(\frac{y}{y_0} \right)$$

$$-k\lambda = \ln \left(\frac{1}{2} \right)$$

$$-k = \frac{\ln 1 - \ln 2}{\lambda}$$

$$k = \frac{\ln 2 - \ln 1}{\lambda} = \frac{\ln 2}{\lambda} \Rightarrow \lambda = \frac{\ln 2}{k}$$

b. Show using your expression for λ that if at time t , the amount is y , then at time $t_1 + \lambda$ it will be $y/2$ no matter what t_1 is.

$$\lambda = \frac{\ln y_0 - \ln y}{k}$$

$$t_1 + \lambda = \frac{\ln y_0 - \ln y_1}{k}$$

$$k(t_1 + \lambda) = \ln y_0 - \ln y_1 + \ln 2$$

$$-k(t_1 + \lambda) = \ln y_1 - \ln y_0 - \ln 2$$

$$-k(t_1 + \lambda) = \ln \left(\frac{y_1}{2y_0} \right)$$

$$e^{-k(t_1 + \lambda)} = y_1 / 2y_0$$

$$= \underline{y_1}$$