## 1C. Slope and derivative

**1C-1** a) Use the difference quotient definition of derivative to calculate the rate of change of the area of a disk with respect to its radius. (Your answer should be the circumference of the disk.)

b) Use the difference quotient definition of derivative to calculate the rate of change of the volume of a ball with respect to the radius. (Your answer should be the surface area of the ball.)

**1C-2** Let $f(x) = (x-a)g(x)$. Use the definition of the derivative to calculate that $f'(a) = g(a)$, assuming that $g$ is continuous.

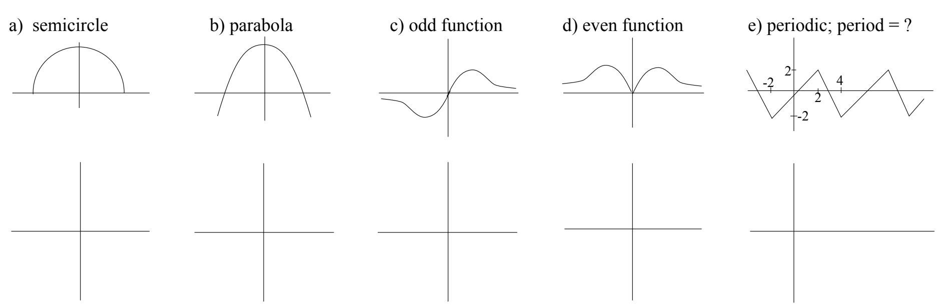
**1C-3** Calculate the derivative of each of these functions directly from the definition.

a) $f(x) = 1/(2x+1)$ b) $f(x) = 2x^2 + 5x + 4$
c) $f(x) = 1/(x^2+1)$ d) $f(x) = 1/\sqrt{x}$
e) For part (a) and (b) find points where the slope is $+1$, $-1$, $0$.

**1C-4** Write an equation for the tangent line for the following functions
a) $f(x) = 1/(2x+1)$ at $x = 1$ b) $f(x) = 2x^2 + 5x + 4$ at $x = a$
c) $f(x) = 1/(x^2+1)$ at $x = 0$ d) $f(x) = 1/\sqrt{x}$ at $x = a$

**1C-5** Find all tangent lines through the origin to the graph of $y = 1 + (x-1)^2$.

**1C-6** Graph the derivative of the following functions directly below the graph of the function. It is very helpful to know that the derivative of an odd function is even and the derivative of an even function is odd (see **1F-6**).




## 1D. Limits and continuity

# 1C-1

(a) Use the difference quotient definition of derivative to calculate the rate of change of the area of a disk with respect to its radius. (Your answer should be the circumference of the disk.)

$A_c = \pi r^2$

$$\frac{\Delta A}{\Delta r} = \frac{\pi(r+\Delta r)^2 - \pi r^2}{\Delta r} = \frac{\pi(r^2 + 2r\Delta r + \Delta r^2) - \pi r^2}{\Delta r}$$

$$= \frac{\pi r^2 + \pi(2r\Delta r + \Delta r^2) - \pi r^2}{\Delta r}$$

$$= \frac{\pi(2r\Delta r + \Delta r^2)}{\Delta r}$$

$$= \pi(2r + \Delta r)$$

$$= 2\pi r \text{ as } \Delta r \to 0$$

(b) Use the difference quotient definition of derivative to calculate the rate of change of the volume of the ball with respect to the radius. (Your answer should be the surface area of the ball.)

$V_c = \frac{4}{3}\pi r^3$

$$\frac{\Delta V}{\Delta r} = \frac{4\pi(r+\Delta r)^3}{3\Delta r} - \frac{4\pi r^3}{3\Delta r}$$

$$= \frac{4\pi(r^3 + 3r^2\Delta r + 3r\Delta r^2 + \Delta r^3)}{3\Delta r} - \frac{4\pi r^3}{3\Delta r}$$

$$= \frac{4\pi(3r^2\Delta r + 3r\Delta r^2 + \Delta r^3)}{3\Delta r}$$

$$= 4\pi r^2 + r\Delta r + \frac{\Delta r^2}{3}$$

$$= 4\pi r^2 \text{ as } \Delta r \to 0$$

# 1C-2

Let $f(x) = (x-a)g(x)$. Use definition of the derivative to calculate that $f'(a) = g(a)$, assuming that $g$ is continuous.

- Derivative is defined as

$$f'(a) = \lim_{x\to a} \frac{f(x)}{x-a}$$

$$= \lim_{x\to a} g(x)$$

$f(x) = (x-a)g(x)$

$f(a) = (a-a)g(a)$

$f(a) = 0 \cdot g(a)$

$f(a) = 0$

$f(x) = (x-a)g(x)$

$g(x) = \frac{f(x)}{x-a}$

$g(a) = \lim_{x\to a} \frac{f(x)}{x-a}$

$f'(a) = g(a)$

# 1C-3

Calculate the derivative from the definition

a. $f(x) = 1/(2x+1)$

$$= \frac{(2(x+h)+1)^{-1} - (2x+1)^{-1}}{h}$$

$$= \frac{(2x+2h+1)^{-1} - (2x+1)^{-1}}{h} \cdot \frac{(2x+2h+1)(2x+1)}{(2x+2h+1)(2x+1)}$$

$$= \frac{2x+1-(2x+2h+1)}{h(2x+2h+1)(2x+1)}$$

$$= \frac{2x+1-2x-2h-1}{h(2x+2h+1)(2x+1)}$$

$$= \frac{-2h}{h(2x+2h+1)(2x+1)}$$

$$f'(x) = \lim_{h\to 0} \frac{-2}{(2x+2h+1)(2x+1)}$$

$$f'(x) = \frac{-2}{(2x+1)(2x+1)}$$

$$f'(x) = \frac{-2}{(2x+1)^2}$$

b. $f(x) = 2x^2 + 5x + 4$

$$\frac{\Delta f(x)}{h} = \frac{2(x+h)^2 + 5(x+h) + 4 - (2x^2+5x+4)}{h}$$

$$= \frac{2(x^2+2hx+h^2) + 5x + 5h + 4 - 2x^2 - 5x - 4}{h}$$

$$= \frac{2x^2 + 4hx + 2h^2 + 5x + 5h + 4 - 2x^2 - 5x - 4}{h}$$

$$= \frac{4hx + 2h^2 + 5h}{h} = \frac{h(4x+2h+5)}{h}$$

$$\frac{\Delta f(x)}{h} = 4x + 2h + 5$$

$$f'(a) = \lim_{h\to 0} 4x + 2h + 5$$

$$f'(a) = 4x + 5$$

(c) $f(x) = 1/(x^2+1)$

$$\frac{\Delta f}{h} = \frac{((x+h)^2+1)^{-1} - (x^2+1)^{-1}}{h}$$

$$= \frac{x^2+1-((x+h)^2+1)}{h((x+h)^2+1)(x^2+1)}$$

$$= \frac{x^2+1-[x^2+2hx+h^2+1]}{h((x+h)^2+1)(x^2+1)}$$

$$= \frac{x^2+1-x^2-2hx-h^2-1}{h((x+h)^2+1)(x^2+1)}$$

$$= \frac{h(-2x-h)}{h((x+h)^2+1)(x^2+1)}$$

$$\frac{\Delta f(x)}{h} = \frac{-2x-h}{[(x+h)^2+1](x^2+1)}$$

$$f'(x) = \lim_{h\to 0} \frac{-2x-h}{[(x+h)^2+1](x^2+1)}$$

$$f'(x) = \frac{-2x}{(x^2+1)^2}$$

(d) $f(x) = 1/\sqrt{x}$

$$\frac{\Delta f(x)}{h} = \frac{(\sqrt{x+h})^{-1} - (\sqrt{x})^{-1}}{h}$$

$$= \frac{\sqrt{x} - \sqrt{x+h}}{h(\sqrt{x+h})(\sqrt{x})} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}$$

$$= \frac{x + \sqrt{x}\sqrt{x+h} - \sqrt{x}\sqrt{x+h} - (x+h)}{h(\sqrt{x+h})(\sqrt{x})(\sqrt{x}+\sqrt{x+h})}$$

$$\frac{\Delta f(x)}{h} = \frac{-1}{(\sqrt{x+h})(\sqrt{x})(\sqrt{x}+\sqrt{x+h})}$$

$$f'(x) = \lim_{h\to 0} \frac{-1}{(\sqrt{x+h})(\sqrt{x})(\sqrt{x}+\sqrt{x+h})}$$

$$= \frac{-1}{(\sqrt{x})(\sqrt{x})(\sqrt{x}+\sqrt{x})} = \frac{-1}{x(2\sqrt{x})}$$

$$= \frac{-1}{2x\sqrt{x}} \quad \text{or } -\frac{1}{2}x^{-3/2}$$

# 1C-3 Cont...

e. For part (a) and (b) find points where the slope is $+1, -1, 0$.

Part a

$$f'(x) = \frac{-2}{(2x+1)^2}$$

(a)

| | slope | | |
|---|---|---|---|
| | +1 | -1 | 0 |
| x | - | $\frac{-1\pm\sqrt{2}}{2}$ | - |

$f'(x) < 0$

$$-1 = \frac{-2}{(2x+1)^2}$$

$$-2 = -(2x+1)^2$$

$$\frac{2x}{2} = \frac{\sqrt{2}-1}{2}$$

$$x = \frac{-1\pm\sqrt{2}}{2}$$

Part b

$f'(a) = 4x + 5$

| | slope | | |
|---|---|---|---|
| | +1 | -1 | 0 |
| x | -1 | -3/2 | -5/4 |

$1 = 4x + 5$

$4x = 1 - 5$

$\frac{4x}{4} = \frac{-4}{4}$

$x = -1$

$-1 = 4x + 5$

$4x = -1 - 5$

$\frac{4x}{4} = \frac{-6}{4}$

$x = -\frac{3}{2}$

$0 = 4x + 5$

$\frac{4x}{4} = \frac{-5}{4}$

$x = -\frac{5}{4}$

# 1C-5

Write an eq. for the tangent line for the following functions

a. $f(x) = \frac{1}{(2x+1)}$ at $x = 1$

$y = \frac{1}{2(1)+1}$ $\quad x, y = (1, \frac{1}{3})$

$m = -2/9$

$y = \frac{1}{3}$

$$\frac{\Delta f(x)}{h} = \frac{[2(x+h)+1]^{-1} - (2x+1)^{-1}}{h}$$

$$= \frac{(2x+2h+1)^{-1} - (2x+1)^{-1}}{h}$$

$$= \frac{2x+1-2x-2h-1}{h(2x+2h+1)(2x+1)}$$

$$= \frac{-2h}{h(2x+2h+1)(2x+1)}$$

$$f'(x) = \lim_{h\to 0} \frac{-2}{(2x+2h+1)(2x+1)}$$

$$f'(x) = \frac{-2}{(2x+1)^2}$$

$y - y_1 = m(x - x_1)$

$y - \frac{1}{3} = -\frac{2}{9}(x-1)$

$y - \frac{1}{3} = \frac{-2x+2}{9}$

$y = \frac{-2x+2+3}{9}$

$y = \frac{-2x+5}{9}$

$m = \frac{-2}{(2(1)+1)^2}$

$m = \frac{-2}{9}$

(b) $f(x) = 2x^2 + 5x + 4$ at $x = a$

$f'(x) = 4x + 5$

$y = 2a^2 + 5a + 4$

$m = 4x + 5$

$m = 4a + 5$

$y - y_1 = m(x - x_1)$

$y - 2a^2 + 5a + 4 = (4a+5)(-a+x)$

$y = -4a^2 + 4ax - 5a + 5x + 2a^2 + 5a + 4$

$= -2a^2 + 4ax + 5x + 4$

$y = -2a^2 + (4a+5)x + 4$

(c) $f(x) = 1/(x^2+1)$ at $x = 0$

$y = 1/((0)^2+1)$

$= 1/1$

$y = 1$

$y - y_1 = m(x - x_1)$

$y = 1$

$$f'(x) = \frac{-2x}{(x^2+1)^2}$$

$$m = \frac{-2(0)}{[(0)^2+1]^2}$$

$m = 0$

(d) $f(x) = 1/\sqrt{x}$ at $x = a$

$y = 1/\sqrt{a}$

$$f'(x) = \frac{-1}{2x\sqrt{x}}$$

$$m = \frac{-1}{2a\sqrt{a}}$$

$y - y_1 = m(x - x_1)$

$$y - \frac{1}{\sqrt{a}} = \frac{-1}{2a\sqrt{a}}(x - a)$$

$$y = \frac{1}{\sqrt{a}} - \frac{x}{2a\sqrt{a}} + \frac{a}{2a\sqrt{a}}$$

$$y = \frac{2a - x + a}{2a\sqrt{a}}$$

$$y = \frac{3a - x}{2a\sqrt{a}}$$

$$= \frac{3}{2\sqrt{a}} - \frac{x}{2a\sqrt{a}}$$

Answer

$$\frac{3}{2\sqrt{a}} - \frac{x}{a\sqrt{a}}$$

I don't understand why (2) got cancelled?

Total Score (Page 3) - 11/14

# 1C-5

Find all tangent lines through the origin to the graph $y = 1 + (x-1)^2$.

$f'(x) = 2(x-1)$

tangent line on point $(a, 1+(a-1)^2)$,

$y = mx + b$

$y = 2(a-1)(x-a) + 1 + (a-1)^2$

origin $(0,0)$

$0 = 2(a-1)(0-a) + 1 + (a-1)^2$

$= -2a^2 + 2a + 1 + a^2 - 2a + 1$

$0 = -a^2 + 2$

$a^2 = 2$

$a = \pm\sqrt{2}$

Answer

$y = 2(\sqrt{2}-1)x$

$y = -2(\sqrt{2}+1)x$

Find the $f'(x)$ of ~~$f(x) = 1+(x-1)^2$~~

$$\frac{\Delta f(x)}{h} = \frac{1+((x+h)-1)^2 - (1+(x-1)^2)}{h}$$

$$= \frac{1+(x+h)^2-(x+h)-(x+h)+1-(1+x^2-x-x+1)}{h}$$

$$= \frac{(x+h)^2-2(x+h)-x^2+2x}{h}$$

$$= \frac{x^2+2hx+h^2-2x-2h-x^2+2x}{h}$$

$$= \frac{2hx+h^2-2h}{h} = \frac{h[2x+h-2]}{h}$$

$$= 2x + h - 2$$

$$f'(x) = \lim_{h\to 0} 2x + h - 2 = 2(x-1)$$

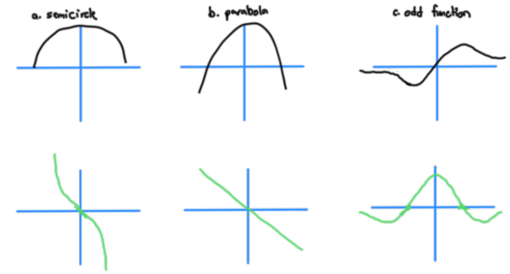
# 1C-6

Graph the derivative of the function

a. semicircle

b. parabola

c. odd function



d. even function

e. periodic function, period = 6