Product of Linear Approximations

Suppose we have two complicated functions and we need an estimate of the value of their product. We could multiply the functions out and then approximate the result, or we could approximate each function separately and then find the product of the two (simple) approximations. Does it matter which we do? Not very much.

Prove that the linear approximation of $f(x) \cdot g(x)$ equals the (linear part of the) product of the linear approximations of $f(x)$ and $g(x)$.

$$f(x) \approx f(x_0) + f'(x_0)\Delta x$$

$$g(x) \approx g(x_0) + g'(x_0)\Delta x$$

$$u(x) = f(x) \cdot g(x)$$

$$u(x) \approx u(x_0) + u'(x_0)\Delta x$$

$$\approx f(x_0)g(x_0) + \frac{d}{dx}\left[f(x_0)g(x_0)\right]\Delta x$$

$$\approx f(x_0)g(x_0) + \left[f'(x_0)g(x_0) + f(x_0)g'(x_0)\right]\Delta x$$

$$= \left[f(x_0) + f'(x_0)\Delta x\right]\left[g(x_0) + g'(x_0)\Delta x\right]$$

$$= f(x_0)g(x_0) + f(x_0)g'(x_0)\Delta x + g(x_0)f'(x_0)\Delta x + f'(x_0)g'(x_0)(\Delta x)^2$$

$$= f(x_0)g(x_0) + \left[f(x_0)g'(x_0) + g(x_0)f'(x_0)\right]\Delta x + f'(x_0)g'(x_0)(\Delta x)^2$$

linear approximation of $u(x)$ – $f(x)g(x)$ – is equal with, at least the linear part, of the product of the linear approximations of $f(x)$ and $g(x)$.