1J. Trigonometric functions

1J-1 Calculate the derivatives of the following functions

a) $\sin(5x^2)$ b) $\sin^2(3x)$ c) $\ln(\cos(2x))$

d) $\ln(2\cos x)$ e) $\dfrac{\sin x}{x}$ f) $\cos(x+y);\ \ y = f(x)$

g) $\cos(x+y);\ \ y$ constant h) $e^{\sin^2 x}$ i) $\ln(x^2 \sin x)$

j) $e^{2x}\sin(10x)$ k) $\tan^2(3x)$ l) $\sec\sqrt{1-x^2}$

m) The following three functions have the same derivative: $\cos(2x)$, $\cos^2 x - \sin^2 x$, and $2\cos^2 x$. Verify this. Are the three functions equal? Explain.

n) $\sec(5x)\tan(5x)$ o) $\sec^2(3x) - \tan^2(3x)$ p) $\sin(\sqrt{x^2+1})$

q) $\cos^2(\sqrt{1-x^2})$ r) $\tan^2(\dfrac{x}{x+1})$

1J-2 Calculate $\displaystyle\lim_{x\to\pi/2} \frac{\cos x}{x-\pi/2}$ by relating it to a value of $(\cos x)'$.

1J-3 a) Let $a > 0$ be a given constant. Find in terms of a the value of $k > 0$ for which $y = \sin(kx)$ and $y = \cos(kx)$ both satisfy the equation

$$y'' + ay = 0.$$

Use this value of k in each of the following parts.

b) Show that $y = c_1\sin(kx) + c_2\cos(kx)$ is also a solution to the equation in (a), for any constants c_1 and c_2.

c) Show that the function $y = \sin(kx + \phi)$ (whose graph is a sine wave with phase shift ϕ) also satisfies the equation in (a), for any constant ϕ.

d) Show that the function in (c) is already included among the functions of part (b), by using the trigonometric addition formula for the sine function. In other words, given k and ϕ, find values of c_1 and c_2 for which

$$\sin(kx+\phi) = c_1\sin(kx) + c_2\cos(kx)$$

1J-4 a) Show that a chord of the unit circle with angle θ has length $\sqrt{2-2\cos\theta}$. Deduce from the half-angle formula

$$\sin(\theta/2) = \sqrt{\frac{1-\cos\theta}{2}}$$

that the length of the chord is

$$2\sin(\theta/2)$$

1J-1 Calculate the derivatives of the following functions

a. $\sin(5x^2)$

$$\frac{d}{dx}\sin(5x^2) = \cos(5x^2)\,10x$$
$$= 10x\cos(5x^2)$$

b. $\sin^2(3x)$

$$\frac{d}{dx}(\sin(3x))^2 = 2\sin(3x)\cos(3x)(3)$$
$$= 6\sin(3x)\cos(3x)$$

c. $\ln(\cos(2x))$

$$\frac{d}{dx}\ln(\cos(2x)) = \frac{1}{\cos(2x)}\cdot -\sin(2x)\cdot 2$$
$$= \frac{-2\sin(2x)}{\cos(2x)}$$
$$= -2\tan 2x$$

not sure though

d. $\ln(2\cos x)$

$$\frac{d}{dx}\ln(2\cos x) = \frac{1}{2\cos x}\cdot -2\sin x$$
$$= \frac{-2\sin x}{2\cos x} = -\tan x$$

e. $\frac{\sin x}{x}$

$$\frac{d}{dx}\sin x\,(x)^{-1} = \cos x\,(x)^{-1} - \sin x\,(x)^{-2}$$
$$= \frac{x\cos x - \sin x}{x^2}$$

f. $\cos(x+y)$; $y = f(x)$

$\cos x\cos y - \sin x\sin y$

$$\frac{d}{dx}\cos x\cos y - \sin x\sin y$$
$$= -\sin x\cos y - \cos x\sin y\,y'$$
$$+ \cos x\sin y + \sin x\cos y\,y'$$
$$= -\sin(x+y) + \sin(x+y)y'$$
$$= -(1+y')\sin(x+y)$$ from answer key

g. $\cos(x+y)$; y constant

$$\frac{d}{dx}\cos(x+y) = -\sin(x+y)(1)$$
$$= -\sin(x+y)$$

h. $e^{\sin^2 x}$

$$u' = (\ln e^{\sin^2 x})'\,e^{\sin^2 x}$$
$$= (\sin^2 x)'\cdot e^{\sin^2 x}$$
$$= 2\sin x\cos x\cdot e^{\sin^2 x}$$
$$= 2\sin x\cos x\,e^{\sin^2 x}$$

i. $\ln(x^2\sin x)$

$$\frac{d}{dx}\ln(x^2\sin x) = \frac{1}{x^2\sin x}\cdot 2x\sin x + x^2\cos x$$
$$= \frac{2x\sin x + x^2\cos x}{x^2\sin x}$$
$$= \frac{2x\sin x}{x^2\sin x} + \frac{x^2\cos x}{x^2\sin x} = \frac{2}{x} + \cot x$$

from key; failed to simplify

j. $e^{2x}\sin(10x)$

$$\frac{d}{dx}e^{2x}\sin(10x) = 2e^{2x}\sin(10x) + e^{2x}\cos(10x)(10)$$
$$= 2e^{2x}[\sin(10x) + 5\cos(10x)]$$

k. $\tan^2(3x)$

$$\frac{d}{dx}\tan^2(3x) = 2\tan(3x)\cdot\frac{d}{dx}\sin(3x)[\cos(3x)]^{-1}$$

Ans. $\frac{6\sin x}{\cos^3 x}$

l. $\sec\sqrt{1-x^2}$

$$\frac{d}{dx}\sec\sqrt{1-x^2}$$
$$= \cot\sqrt{1-x^2}\cdot\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$$
$$= \frac{x\cot\sqrt{1-x^2}}{\sqrt{1-x^2}} = \frac{x\sqrt{1-x^2}\cot\sqrt{1-x^2}}{1-x^2}$$

m. The following three functions have the same derivative: $\cos(2x)$, $\cos^2 x - \sin^2 x$, and $2\cos^2 x$. Verify this. Are the three functions equal? Explain.

sin double-angle formula
$\sin 2x = 2\sin x\cos x$

$$\frac{d}{dx}\cos(2x) = -2\sin(2x)$$
$$= -2(2\sin x\cos x) = -4\sin x\cos x$$

$$\frac{d}{dx}\cos^2 x - \sin^2 x = -2\cos x\sin x - 2\sin x\cos x$$
$$= -4\cos x\sin x$$

$$\frac{d}{dx}2\cos^2 x = 2\cdot 2\cos x\cdot(-\sin x)$$
$$= -4\cos x\sin x$$

Indeed, $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x$; using $\sin^2 x = 1 - \cos^2 x$

n. $\sec(5x)\tan(5x)$

$$\frac{d}{dx}\sec(5x)\tan(5x)$$
$$\frac{d}{dx}(1)[\cos(5x)]^{-1}[\sin(5x)(\cos(5x)^{-1}]$$
$$= [(-5)[\cos(5x)]^{-2}(-\sin(5x))\tan(5x)] + \cos(5x)[\cos(5x)]^{-1}$$
$$+ \sin(5x)(-1)[\cos(5x)]^{-2}(-\sin(5x)(5)$$
$$= 5\tan^2(5x)[\cos(5x)]^{-1} + 1 + 5[\cos(5x)]^{-2}$$
$$= 5\tan^2(5x)\sec(5x) + 5\sec^2(5x) + 1$$

o. $\sec^2(3x) - \tan^2(3x)$

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$$\frac{d}{dx}\sec^2(3x) - \tan^2(3x)$$
$$= \tan(3x)\sec(3x)\,2\sec(3x) - [1+\tan^2(3x)]$$
$$= 6\tan(3x)\sec^2(3x) - 6\tan(3x)\sec^2(3x)$$
$$= 0$$

p. $\sin(\sqrt{x^2+1})$

$$\frac{d}{dx}\sin(\sqrt{x^2+1}) = \cos(\sqrt{x^2+1})\left[\tfrac{1}{2}(x^2+1)^{-\frac{1}{2}}\right]2x$$
$$= \frac{x\cos(\sqrt{x^2+1})}{\sqrt{x^2+1}} \text{ or } \frac{x}{\sqrt{x^2+1}}\cos(\sqrt{x^2+1})$$

q. $\cos^2(\sqrt{1-x^2})$

$$\frac{d}{dx}\cos^2(\sqrt{1-x^2}) = 2\cos(\sqrt{1-x^2})[-\sin(\sqrt{1-x^2})]\left[-\tfrac{1}{2}(1-x^2)^{-\frac{1}{2}}\right]2x$$
$$= \frac{+2x\cos(\sqrt{1-x^2})\sin(\sqrt{1-x^2})}{\sqrt{1-x^2}} \text{ or } \frac{2}{\sqrt{1-x^2}}\sin(2\sqrt{1-x^2})$$

r. $\tan^2\left(\frac{x}{x+1}\right)$

$$2\tan\left(\frac{x}{x+1}\right)\sec^2\left(\frac{x}{x+1}\right)\left[(1)(x+1)^{-1} + (x)(-1)(x+1)^{-2}(1)\right]$$
$$2\tan\left(\frac{x}{x+1}\right)\sec^2\left(\frac{x}{x+1}\right)\left[\frac{(x+1)-x}{(x+1)^2}\right] =$$

1J-2 Calculate $\lim_{x\to\pi/2}\frac{\cos x}{x-\pi/2}$ by relating it to a value of $(\cos x)'$.

$$\lim_{h\to 0}\frac{\cos(\pi/2+h) - \cos\pi/2}{h} = \frac{d}{dx}\cos x\Big|_{x=\frac{\pi}{2}} = -\sin\left(\frac{\pi}{2}\right)$$
$$= -1$$

1J-3 a. Let $a>0$ be a given constant. Find in terms of a the value of $k>0$ for which $y=\sin(kx)$ and $y=\cos(kx)$ both satisfy the equation

$$y'' + ay = 0$$

$y=\sin(kx)$ | $y=\cos(kx)$
$y'=k\cos(kx)$ | $y'=-k\sin(kx)$
$y''=-k^2\sin(kx)$ | $y''=-k^2\cos(kx)$

$$-k^2\sin(kx)+a\sin(kx) = -k^2\cos(kx)+a\cos(kx)$$
$$0 = \cos(kx)[-k^2+a]$$
$$k^2 = a$$
$$k = \sqrt{a}$$

Use this value of k in each of the following parts

b. Show that $y=c_1\sin(kx)+c_2\cos(kx)$ is also a solution to the equation in (a), for any constants c_1 and c_2.

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c. Show that the function $y=\sin(kx+\phi)$ (whose graph is a sine wave with phase shift ϕ) also satisfies the eq. in (a), for any constant ϕ.

d. Show that the function in (c) is already included among the functions of part (b), by using the trigonometric addition formula for the sine function. In other words, given k and ϕ, find values c_1 and c_2 for which

$$\sin(kx+\phi) = c_1\sin(kx)+c_2\cos(kx)$$