Implicit Differentiation and the Chain Rule

The chain rule tells us that:

$$\frac{d}{dx}(f \circ g) = \frac{df}{dg}\frac{dg}{dx}.$$

While implicitly differentiating an expression like $x + y^2$ we use the chain rule as follows:

$$\frac{d}{dx}(y^2) = \frac{d(y^2)}{dy}\frac{dy}{dx} = 2yy'.$$

Why can we treat y as a function of x in this way?

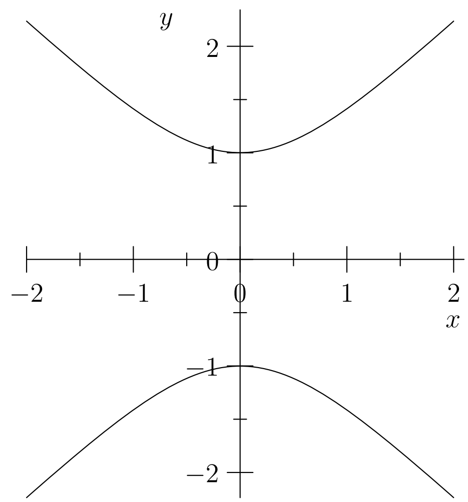

Figure 1: The hyperbola $y^2 - x^2 = 1$.

Consider the equation $y^2 - x^2 = 1$, which describes the hyperbola shown in Figure 1. We cannot write y as a function of x, but if we start with a point (x, y) on the graph and then change its x coordinate by sliding the point along the graph its y coordinate will be constrained to change as well. The change in y is *implied* by the change in x and the constraint $y^2 - x^2 = 1$. Thus, it makes sense to think about $y' = \frac{dy}{dx}$, the rate of change of y with respect to x.

Given that $y^2 - x^2 = 1$:

a) Use implicit differentiation to find y'.

b) Check your work by using Figure 1 to estimate the slope of the tangent line to the hyperbola when $y = -1$ and when $x = 1$.

c) Check your work for $y > 0$ by solving for y and using the direct method to take the derivative.

Activity Total: 4/5

Given that $y^2 - x^2 = 1$

a. Use implicit differentiation to find y'.

$$\frac{d}{dx}(y^2 - x^2 = 1)$$

$$2yy' - 2x = 0$$

$$y' = \frac{2x}{2y}$$

$$y' = \frac{x}{y} = \frac{x}{\pm\sqrt{x^2+1}}$$

b. Check your work by using Figure 1 to estimate the slope of the tangent line to the hyperbola when $y = -1$ and when $x = 1$

m at $y = -1$ is 0.

m at $x = 1$ is $\approx \frac{-1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$.

c. Check your work for $y > 0$ by solving for y and using the direct method to take the derivative.

$$y^2 - x^2 = 1$$

$$y^2 = x^2 + 1$$

$$y = \pm\sqrt{x^2+1}$$

$$y = +\sqrt{x^2+1}$$

$$y' = \frac{x}{y}$$

$$y' = \frac{x}{\sqrt{x+1}}$$

$$= \frac{1}{\sqrt{1+1}}$$

$$y' = \frac{1}{\sqrt{2}} \text{ at } x = 1$$

$$y' = \frac{x}{y}$$

$$= \frac{x}{-1}$$

$$y' = -x$$

$$y' = 0 \text{ at } y = -1$$

$$\frac{d}{dx}\left[(x^2+1)^{\frac{1}{2}}\right]$$

$$= \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)$$

$$= x(x^2+1)^{-\frac{1}{2}}$$

$$y' = \frac{x}{\sqrt{x^2+1}} = \frac{x}{y}$$

check whether explicit differentiation yields similar result with implicit differentiation.