Exam Score - 5/10

1. Compute the following derivatives. (Simplify your answers when possible.)

(a) $f'(x)$ where $f(x) = \dfrac{x}{1-x^2}$

$$f'(x) = \frac{(1-x^2) - x(-2x)}{(1-x^2)^2}$$

$$= \frac{1-x^2+2x^2}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2}$$

(b) $f'(x)$ where $f(x) = \ln(\cos x) - \dfrac{1}{2}\sin^2(x)$

$$f'(x) = \frac{1}{\cos x}(-\sin x) - \sin x \cos x$$

$$= -\tan x - \sin x \cos x$$

This answer is acceptable, I think :)

$$= -\frac{\sin x}{\cos x} - \frac{\sin x \cos^2 x}{\cos x}$$

$$= -\sin x\left(\frac{1+\cos^2 x}{\cos x}\right)$$

(c) $f^{(5)}(x)$, the fifth derivative of f, where $f(x) = xe^x$

$f'(x) = e^x + xe^x$

$f^{(2)}x = e^x + e^x + xe^x$

...

$f^{(5)}x = 5e^x + xe^x$

2. Find the equation of the tangent line to the "astroid" curve defined implicitly by the equation

$$x^{2/3} + y^{2/3} = 4$$

at the point $(-\sqrt{27}, 1)$.

$\Rightarrow \frac{d}{dx}\left(x^{2/3} + y^{2/3} = 4\right)$

$\Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$

$-x^{-1/3} = y^{-1/3}y'$

$y' = \frac{-y^{1/3}}{x^{1/3}} = \frac{-1}{27^{1/6}}$

X

not simplified enough and didn't notice the sign.

$y' = \frac{-y^{1/3}}{x^{1/3}}$

$= \frac{-(1)^{1/3}}{(-3^{3/2})^{1/3}}$

$= \frac{-1}{-3^{1/2}} = \frac{1}{\sqrt{3}}$

3. A particle is moving along a vertical axis so that its position y (in meters) at time t (in seconds) is given by the equation

$$y(t) = t^3 - 3t + 3, \quad t \geq 0.$$

Determine the total distance traveled by the particle in the first three seconds.

$$\begin{aligned} y'(t) &= 3t^2 - 3 \\ &= 3(3)^2 - 3 \\ &= 27 - 3 \\ &= 24 \text{ meters} \end{aligned}$$

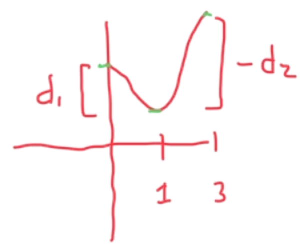
Here, I am calculating $\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$ which would be the net distance. The problem is asking for TOTAL distance!

Look at min and max,

$$y'(t) = 3t^2 - 3$$

At $t < 1$, $y'(t) < 0$ which means $y(t)$ is decreasing.

At $t > 1$, $y'(t) > 0$ which means $y(t)$ is increasing.



$y(0) = 3$

$y(1) = 1$

$$\begin{aligned} y(3) &= 3(3)^3 - 3(3) + 3 \\ &= 27 - 9 + 3 \\ &= 21 \end{aligned}$$

$$\begin{aligned} \text{total dist} &= d_1 + d_2 \\ &= [y(0) - y(1)] + [y(3) - y(1)] \\ &= (3 - 1) + (21 - 1) \\ &= 2 + 20 = 22 \text{ meters} \end{aligned}$$

Note: I could have answered this myself if I just thought about the problem.

4. State the product rule for the derivative of a pair of differentiable functions f and g using your favorite notation. Then use the DEFINITION of the derivative to prove the product rule. Briefly justify your reasoning at each step.

Let $F(x) = f(x)g(x)$ and $F(x+h) = f(x+h)g(x+h)$;

The product rule states that,

$$F'(x) = f'(x)g(x) + f(x)g'(x)$$

We can then use the definition of a derivative to prove the equation above;

This one was fun.
I could never have thought about this manipulation though.
But I am going to give myself the credit because I did the problem even if I had some help from the net.

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)(g(x+h) - g(x)) + (f(x+h) - f(x))g(x)}{h}$$

$$= \lim_{h\to 0} f(x+h)\frac{g(x+h) + g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h}$$

$$= f(x)\lim_{h\to 0}\frac{g(x+h) + g(x)}{h} + g(x)\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= f(x)g'(x) + g(x)f'(x) = f'(x)g(x) + f(x)g'(x)$$

5. Does there exist a set of real numbers a, b and c for which the function

$$f(x) = \begin{cases} \tan^{-1}(x) & x \leq 0 \\ ax^2 + bx + c, & 0 < x < 2 \\ x^3 - \frac{1}{4}x^2 + 5, & x \geq 2 \end{cases}$$

is differentiable (i.e. everywhere differentiable)? Explain why or why not. (Here $\tan^{-1}(x)$ denotes the inverse of the tangent function.)

For $f(x)$ to be differentiable, it must satisfy the following conditions;

a. continous

b. equal two-sided limits of $f'(x)$

$y = \tan^{-1} x$

$\tan y = x$

$\sec^2 y \, y' = 1$

$y' = \frac{1}{\sec^2 y} = \frac{1}{1 + (\tan y)^2} = \frac{1}{1+x^2}$

$\sin^2 y + \cos^2 y = 1$

$\tan^2 y + 1 = \sec^2 y$

Checking for condition (a), we need to evaluate the value of the function at each junction;

$\tan^{-1}(0) = a(0)^2 + b(0) + c$

$\tan^{-1}(0) = c$

$0 = c$

$a(2)^2 + b(2) + c = (2)^3 - \frac{1}{4}(2)^2 + 5$

$4a + 2b + c = 8 - 1 + 5$

$4a + 2b + c = 12$

Thus, $f(x)$ is continous if $4a + 2b = 12$.

Checking for condition (b);

$\frac{1}{1+x^2} = 2ax + b$

$1 = 2a(0) + b$

$1 = b$

$2ax + b = 3x^2 - \frac{1}{2}x$

$2a(2) + 1 = 3(2)^2 - \frac{1}{2}(2)$

$4a = 12 - 1 - 1$

$4a = 10$ literally $\frac{5}{2}$

Plugging in these values on our previous equation;

$4a + 2b = 12$

$10 + 2(1) = 12$

$12 = 12$

$\therefore f(x)$ is differentiable if $a = \frac{2}{5}$, $b = 1$ and $c = 0$.

$\frac{5}{2}$

OMG, my algebra sucks

6. Suppose that f satisfies the equation $f(x+y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers x and y. Suppose further that

$$\lim_{x \to 0} \frac{f(x)}{x} = 1.$$

(a) Find $f(0)$.

$$f(x+y) = f(x) + f(y) + x^2y + xy^2$$
$$f(y) = f(0) + f(y)$$
$$f(0) = 0$$

(b) Find $f'(0)$.

$$f(y) = f(0) + f(y)$$
$$f'(y) = f'(0) + f'(y)$$
$$f'(0) = 0 \quad X$$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{f(h) - 0}{h} = \lim_{h \to 0} \frac{f(h)}{h} = 1$$

(c) Find $f'(x)$.

$$\lim_{y \to 0} \frac{f(x+y) - f(x)}{y} = \lim_{y \to 0} \frac{f(x) + f(y) + x^2y + xy^2 - f(x)}{y}$$

$$f'(x) = \lim_{y \to 0} \frac{f_y}{y}$$

$$f'(x) = 1 \quad X$$

I was almost right here.
You can't cancel x^2y because
y is not supposed to be zero!!!

$$\lim_{y \to 0} \frac{f(y)}{y} + x^2 + xy$$
$$= 1 + x^2 + 0 \qquad \text{Almost !!!!}$$
$$= x^2 + 1$$