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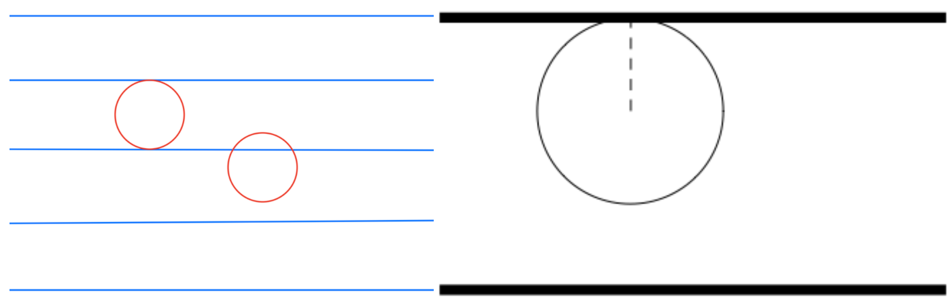
Homework 3

Notes:

1. Do all assigned problems.
2. The set is worth 105 points evenly distributed among problems for the entire set.
3. No late HW is accepted.

Problem 3-1 (35 points): Buffon's Needle refers to a simple Monte Carlo method for the estimation of the value of $\pi = 3.14159265$...among others. Here is in a nutshell of such experiment: Suppose you draw many parallel, and equally distanced (1 inch, for example), lines on a desktop and suppose you have many 1-inch-long needles. Dropping a needle randomly to the table, you see: (1) The needle crosses or touches one of the lines, or (2) the needle crosses no lines. Repeating such droppings N times, you noticed N_C times the needle crosses a line.



For our project, instead of dropping needles, you perform numerical experiments for tossing a disc of diameter d (to be specified below) to parallel lines of distance $w = 1$. Estimate (numerically, of course) the probability when any part of the disc crosses a parallel line for $n = 4{,}444{,}444$ tosses with each of the following disc diameters $d = \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}, \frac{9}{10}, \frac{10}{10}, \frac{15}{10}, \frac{20}{10}, \frac{30}{10}$.

For $d < w$, you may cross no more than 1 line and the crossing probability depends on "d".
For $d > w$, you may cross more than 1 line and, as such, you must specify the probabilities for crossing, at least, 1-, 2-, 3-, 4-... lines. Obviously, for $d = \frac{20}{10}$ (and $w = 1$, as always),

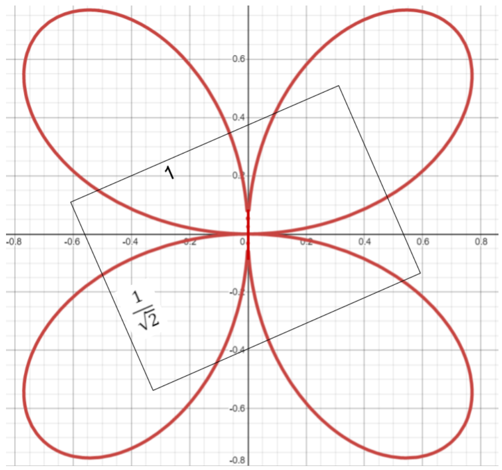
$$P(1 \text{ line}) = 100\%$$
$$P(2 \text{ line}) = 100\%$$
$$P(3 \text{ line}) = 000\%$$

In the case of $P(3 \text{ line})$, we have a tiny exception the disc touches all three lines.

Please estimate, and make a plot for, the probabilities as a function of d.

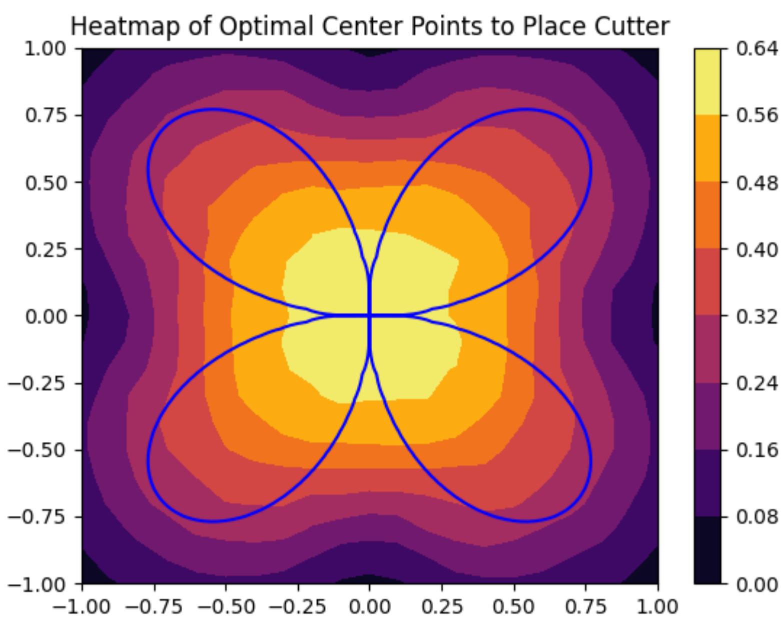
Note: The above $n_{\text{realistic}} = 4{,}444{,}444$ is a realistic number and one can get crazy with $n_{\text{dream}} = 4{,}444{,}444{,}444$ for better accuracy. For me, seven "4"'s are enough.

Problem 3-2 (35 points): The polar equation of a 4-leaf "rose" as graphed is $r = \sin 2\theta$ whose Cartesian coordinate equation is $(x^2 + y^2)^3 = 4x^2y^2$.

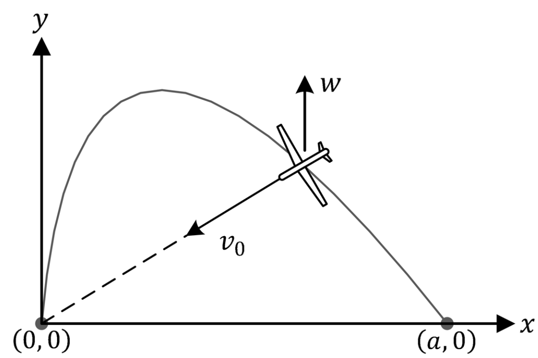


Please write a program to enable you to place a rectangular cutter (as shown) of sides $1 \times \frac{1}{\sqrt{2}}$ to cut the most area of the rose.

Notes: (1) The following heatmap shows the area as a function of the center coordinates of the cutter for a given angle. (2) It's highly desirable to generate a 3D heatmap by varying the angle too. Now, we are quite confident that placing the cutter at the center of the rose will likely cut the most area. This depends, of course, on the size and orientation of the cutter. (3) This problem is much richer than a HW one and elucubrating it will likely result in a nice paper.



Problem 3.3 (35 Points): A flying plane at $(a, 0)$ approaches an airport whose coordinates are $(0, 0)$, which means the plan is a miles east to the airport. A steady south wind of speed w blows from south to north. While approaching the airport, the plane maintains a constant speed v_0 relative to the wind and always maintains its heading directly toward the airport. It is interesting to construct the DE and find its solution as the plane's trajectory.



The plane's velocity components relative to the airport tower are

$$\begin{cases} \dfrac{dx}{dt} = -v_0 \cos\alpha = -v_0 \dfrac{x}{\sqrt{x^2 + y^2}} \\ \dfrac{dy}{dt} = -v_0 \sin\alpha + w = -v_0 \dfrac{y}{\sqrt{x^2 + y^2}} + w \end{cases}$$

which is a system of two DEs with x and y are dependent variables and t independent variable. We are not interested in time so we may eliminate explicit dependence on time and build the following DE between x and y:

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{y}{x} - k\sqrt{1 + \left(\dfrac{y}{x}\right)^2} \\ y(x = a) = 0 \end{cases}$$

where we have defined $k = \frac{w}{v_0}$.

Now, you are given the values $a = 100$, $w = 44$, and $v_0 = 88$. Please use any solution method including the Euler's methods and the Runge-Kutta method to compute the plane's trajectory until it lands at the airport.

Notes: (1) A trajectory is an assembly of points in the xy-plane, or a function $F(x, y) = 0$, or a curve (as shown above). (2) You may express the trajectory in a table or in a more-preferred curve. (3) The problem can be made more realistic and more sophisticated by adjusting the wind's direction, etc. I'm not going to do it for a course.