A Specification for Dependent Types in Haskell (Technical Appendix)

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A COMPLETE SYSTEM SPECIFICATION

The complete type system appears in here including the actual rules that we used, automatically generated by Ott. For presentation purposes, we have removed some redundant hypotheses from these rules in the main body of the paper when they were implied via regularity. We have proven (in Coq) that these additional premises are admissible, so their removal does not change the type system.[1][2][3][4][5][6]These redundant hypotheses are marked via square brackets in the complete system below.

We include these redundant hypotheses in our rules for two reasons. First, sometimes these hypotheses simplify the reasoning and allow us to prove properties more independently of one another. For example, in the rule E-Beta rule, we require a_2 to have the same type as a_1. However, this type system supports the preservation lemma so this typing premise will always be derivable. But, it is convenient to prove the regularity property early, so we include that hypothesis.

Another source of redundancy comes from our use of the Coq proof assistant. Some of our proofs require the use of induction on judgments that are not direct premises, but are derived from other premises via regularity. These derivations are always the same height or shorter than the original, so this use of induction is justified. However, while Coq natively supports proofs by induction on derivations, it does not natively support induction on the *heights* of derivations. Therefore, to make these induction hypotheses available for reasoning, we include them as additional premises.

One other minor difference is that this specification also allows the toplevel signature to include type constants T, which must have kind $\star$. These type constants have little interact with the rest of the language.

B TOPLEVEL SIGNATURES

Our results are proven with respect to the following toplevel signatures:

$$\Sigma_1 = \emptyset \cup \{\mathbf{Fix} \sim \lambda^- x{:}\star.\lambda^+ y{:}x.(y\,(\mathbf{Fix}[x]\,y)) : \Pi^- x{:}\star \to (x \to x) \to x\}$$

$$\Sigma_0 = |\Sigma_1|$$

However, our Coq proofs use these signature definitions opaquely. As a result, any pair of toplevel signatures are compatible with the definition of the languages as long as they satisfy the following properties.

(1) $\vDash \Sigma_0$
(2) $\vdash \Sigma_1$
(3) $\Sigma_0 = |\Sigma_1|$

C REDUCTION RELATIONS

C.1 Primitive reduction

$\boxed{\vDash a > b}$ *(primitive reductions on erased terms)*

Beta-AppAbs
$$\frac{[\mathsf{Value}\ (\lambda^\rho x.v)]}{\vDash (\lambda^\rho x.v)\ b^\rho > v\{b/x\}}$$

Beta-CAppCAbs
$$\frac{}{\vDash (\Lambda c.a')[\bullet] > a'\{\bullet/c\}}$$

Beta-Axiom
$$\frac{F \sim a : A \in \Sigma_0}{\vDash F > a}$$

[1] ext_invert.v:E_Pi2,E_Abs2,E_CPi2,E_CAbs2,E_Fam2 [2] ext_invert.v:E_Wff2,E_PiCong2,E_AbsCong2,E_CPiCong2,E_CAbsCong2
[3] ext_red.v:E_Beta2 [4] fc_invert.v:An_Pi_exists2,An_Abs_exists2,An_CPi_exists2,An_CAbs_exists2,An_Fam2
[5] fc_invert.v:An_Sym2,An_Trans2,An_AbsCong_exists2 [6] fc_invert.v:An_AppCong2,An_CPiCong_exists2,An_CAppCong2

C.2 Implicit language one-step reduction

$\boxed{\vDash a \leadsto b}$ *(single-step head reduction for implicit language)*

$$\textsc{E-AbsTerm}\ \frac{\vDash a \leadsto a'}{\vDash \lambda^{-} x.a \leadsto \lambda^{-} x.a'} \qquad \textsc{E-AppLeft}\ \frac{\vDash a \leadsto a'}{\vDash a\ b^{\rho} \leadsto a'\ b^{\rho}} \qquad \textsc{E-CAppLeft}\ \frac{\vDash a \leadsto a'}{\vDash a[\bullet] \leadsto a'[\bullet]} \qquad \textsc{E-AppAbs}\ \frac{[\text{Value } (\lambda^{\rho} x.v)]}{\vDash (\lambda^{\rho} x.v)\ a^{\rho} \leadsto v\{a/x\}}$$

$$\textsc{E-CAppCAbs}\ \frac{}{\vDash (\Lambda c.b)[\bullet] \leadsto b\{\bullet/c\}} \qquad \textsc{E-Axiom}\ \frac{F \sim a : A \in \Sigma_0}{\vDash F \leadsto a}$$

C.3 Parallel reduction

$\boxed{\vDash a \Rightarrow b}$ *(parallel reduction (implicit language))*

$$\textsc{Par-Refl}\ \frac{}{\vDash a \Rightarrow a} \qquad \textsc{Par-Beta}\ \frac{\begin{array}{c}\vDash a \Rightarrow (\lambda^{\rho} x.a') \\ \vDash b \Rightarrow b'\end{array}}{\vDash a\ b^{\rho} \Rightarrow a'\{b'/x\}} \qquad \textsc{Par-App}\ \frac{\vDash a \Rightarrow a' \qquad \vDash b \Rightarrow b'}{\vDash a\ b^{\rho} \Rightarrow a'\ b'^{\rho}} \qquad \textsc{Par-CBeta}\ \frac{\vDash a \Rightarrow (\Lambda c.a')}{\vDash a[\bullet] \Rightarrow a'\{\bullet/c\}}$$

$$\textsc{Par-CApp}\ \frac{\vDash a \Rightarrow a'}{\vDash a[\bullet] \Rightarrow a'[\bullet]} \qquad \textsc{Par-Abs}\ \frac{\vDash a \Rightarrow a'}{\vDash \lambda^{\rho} x.a \Rightarrow \lambda^{\rho} x.a'} \qquad \textsc{Par-Pi}\ \frac{\vDash A \Rightarrow A' \qquad \vDash B \Rightarrow B'}{\vDash \Pi^{\rho} x{:}A \to B \Rightarrow \Pi^{\rho} x{:}A' \to B'}$$

$$\textsc{Par-CAbs}\ \frac{\vDash a \Rightarrow a'}{\vDash \Lambda c.a \Rightarrow \Lambda c.a'} \qquad \textsc{Par-CPi}\ \frac{\begin{array}{c}\vDash A \Rightarrow A' \qquad \vDash B \Rightarrow B' \\ \vDash a \Rightarrow a' \qquad \vDash A_1 \Rightarrow A_1'\end{array}}{\vDash \forall c{:}A \sim_{A_1} B.a \Rightarrow \forall c{:}A' \sim_{A_1'} B'.a'} \qquad \textsc{Par-Axiom}\ \frac{F \sim a : A \in \Sigma_0}{\vDash F \Rightarrow a}$$

C.4 Explicit language one-step reduction

$\boxed{\Gamma \vdash a \leadsto b}$ *(single-step, weak head reduction to values for annotated language)*

$$\textsc{An-AppLeft}\ \frac{\Gamma \vdash a \leadsto a'}{\Gamma \vdash a\ b^{\rho} \leadsto a'\ b^{\rho}} \qquad \textsc{An-AppAbs}\ \frac{[\text{Value } (\lambda^{\rho} x{:}A.w)]}{\Gamma \vdash (\lambda^{\rho} x{:}A.w)\ a^{\rho} \leadsto w\{a/x\}} \qquad \textsc{An-CAppLeft}\ \frac{\Gamma \vdash a \leadsto a'}{\Gamma \vdash a[\gamma] \leadsto a'[\gamma]}$$

$$\textsc{An-CAppCAbs}\ \frac{}{\Gamma \vdash (\Lambda c{:}\phi.b)[\gamma] \leadsto b\{\gamma/c\}} \qquad \textsc{An-AbsTerm}\ \frac{\begin{array}{c}\Gamma \vdash A : \star \\ \Gamma, x : A \vdash b \leadsto b'\end{array}}{\Gamma \vdash (\lambda^{-} x{:}A.b) \leadsto (\lambda^{-} x{:}A.b')} \qquad \textsc{An-Axiom}\ \frac{F \sim a : A \in \Sigma_1}{\Gamma \vdash F \leadsto a}$$

An-ConvTerm
$$\frac{\Gamma \vdash a \rightsquigarrow a'}{\Gamma \vdash a \triangleright \gamma \rightsquigarrow a' \triangleright \gamma}$$

An-Combine
$$\frac{[\text{Value } v]}{\Gamma \vdash (v \triangleright \gamma_1) \triangleright \gamma_2 \rightsquigarrow v \triangleright (\gamma_1 ; \gamma_2)}$$

An-Push
$$\frac{\begin{array}{c}[\text{Value } v] \\ \Gamma ; \widetilde{\Gamma} \vdash \gamma : \Pi^{\rho} x_1 \!:\! A_1 \to B_1 \sim \Pi^{\rho} x_2 \!:\! A_2 \to B_2 \\ b' = b \triangleright \mathbf{sym}\,(\mathbf{piFst}\,\gamma) \\ \gamma' = \gamma @ (b' \mid=\mid_{(\mathbf{piFst}\,\gamma)} b)\end{array}}{\Gamma \vdash (v \triangleright \gamma)\; b^{\rho} \rightsquigarrow (v\; b'^{\rho}) \triangleright \gamma'}$$

An-CPush
$$\frac{\begin{array}{c}[\text{Value } v] \\ \Gamma ; \widetilde{\Gamma} \vdash \gamma : \forall c_1 \!:\! \phi_1 . A_1 \sim \forall c_2 \!:\! \phi_2 . A_2 \\ \gamma_1' = \gamma_1 \triangleright \mathbf{sym}\,(\mathbf{cpiFst}\,\gamma) \\ \gamma' = \gamma @ (\gamma_1' \sim \gamma_1)\end{array}}{\Gamma \vdash (v \triangleright \gamma)[\gamma_1] \rightsquigarrow (v[\gamma_1']) \triangleright \gamma'}$$

D FULL SYSTEM SPECIFICATION: IMPLICIT LANGUAGE TYPE SYSTEM

$\boxed{\Gamma \vDash a : A}$ *(typing)*

E-Star
$$\frac{\vDash \Gamma}{\Gamma \vDash \star : \star}$$

E-Var
$$\frac{\vDash \Gamma \qquad x : A \in \Gamma}{\Gamma \vDash x : A}$$

E-Pi
$$\frac{\begin{array}{c}\Gamma, x : A \vDash B : \star \\ [\Gamma \vDash A : \star]\end{array}}{\Gamma \vDash \Pi^{\rho} x \!:\! A \to B : \star}$$

E-Abs
$$\frac{\begin{array}{c}\Gamma, x : A \vDash a : B \\ [\Gamma \vDash A : \star] \\ (\rho = +) \vee (x \notin \mathsf{fv}\ a)\end{array}}{\Gamma \vDash \lambda^{\rho} x . a : \Pi^{\rho} x \!:\! A \to B}$$

E-App
$$\frac{\begin{array}{c}\Gamma \vDash b : \Pi^{+} x \!:\! A \to B \\ \Gamma \vDash a : A\end{array}}{\Gamma \vDash b\; a^{+} : B\{a/x\}}$$

E-IApp
$$\frac{\begin{array}{c}\Gamma \vDash b : \Pi^{-} x \!:\! A \to B \\ \Gamma \vDash a : A\end{array}}{\Gamma \vDash b\; \square^{-} : B\{a/x\}}$$

E-Conv
$$\frac{\begin{array}{c}\Gamma \vDash a : A \\ \Gamma ; \widetilde{\Gamma} \vDash A \equiv B : \star \\ [\Gamma \vDash B : \star]\end{array}}{\Gamma \vDash a : B}$$

E-CPi
$$\frac{\begin{array}{c}\Gamma, c : \phi \vDash B : \star \\ [\Gamma \vDash \phi \ \mathsf{ok}]\end{array}}{\Gamma \vDash \forall c \!:\! \phi . B : \star}$$

E-CAbs
$$\frac{\begin{array}{c}\Gamma, c : \phi \vDash a : B \\ [\Gamma \vDash \phi \ \mathsf{ok}]\end{array}}{\Gamma \vDash \Lambda c . a : \forall c \!:\! \phi . B}$$

E-CApp
$$\frac{\begin{array}{c}\Gamma \vDash a_1 : \forall c \!:\! (a \sim_A b) . B_1 \\ \Gamma ; \widetilde{\Gamma} \vDash a \equiv b : A\end{array}}{\Gamma \vDash a_1[\bullet] : B_1\{\bullet/c\}}$$

E-Fam
$$\frac{\begin{array}{c}\vDash \Gamma \qquad F \sim a : A \in \Sigma_0 \\ [\varnothing \vDash A : \star]\end{array}}{\Gamma \vDash F : A}$$

$\boxed{\Gamma \vDash \phi \ \mathsf{ok}}$ *(Prop wellformedness)*

E-Wff
$$\frac{\begin{array}{c}\Gamma \vDash a : A \\ \Gamma \vDash b : A \qquad [\Gamma \vDash A : \star]\end{array}}{\Gamma \vDash a \sim_A b \ \mathsf{ok}}$$

$\boxed{\Gamma ; \Delta \vDash \phi_1 \equiv \phi_2}$ *(prop equality)*

E-PropCong
$$\frac{\begin{array}{c}\Gamma ; \Delta \vDash A_1 \equiv A_2 : A \\ \Gamma ; \Delta \vDash B_1 \equiv B_2 : A\end{array}}{\Gamma ; \Delta \vDash A_1 \sim_A B_1 \equiv A_2 \sim_A B_2}$$

E-IsoConv
$$\frac{\begin{array}{c}\Gamma ; \Delta \vDash A \equiv B : \star \\ \Gamma \vDash A_1 \sim_A A_2 \ \mathsf{ok} \\ \Gamma \vDash A_1 \sim_B A_2 \ \mathsf{ok}\end{array}}{\Gamma ; \Delta \vDash A_1 \sim_A A_2 \equiv A_1 \sim_B A_2}$$

E-CPiFst
$$\frac{\Gamma ; \Delta \vDash \forall c \!:\! \phi_1 . B_1 \equiv \forall c \!:\! \phi_2 . B_2 : \star}{\Gamma ; \Delta \vDash \phi_1 \equiv \phi_2}$$

$\boxed{\Gamma; \Delta \vDash a \equiv b : A}$ *(definitional equality)*

E-Assn
$$\frac{\vDash \Gamma \qquad c : (a \sim_A b) \in \Gamma \qquad c \in \Delta}{\Gamma; \Delta \vDash a \equiv b : A}$$

E-Refl
$$\frac{\Gamma \vDash a : A}{\Gamma; \Delta \vDash a \equiv a : A}$$

E-Sym
$$\frac{\Gamma; \Delta \vDash b \equiv a : A}{\Gamma; \Delta \vDash a \equiv b : A}$$

E-Trans
$$\frac{\Gamma; \Delta \vDash a \equiv a_1 : A \qquad \Gamma; \Delta \vDash a_1 \equiv b : A}{\Gamma; \Delta \vDash a \equiv b : A}$$

E-Beta
$$\frac{\Gamma \vDash a_1 : B \qquad [\Gamma \vDash a_2 : B] \qquad \vDash a_1 > a_2}{\Gamma; \Delta \vDash a_1 \equiv a_2 : B}$$

E-PiCong
$$\frac{\Gamma; \Delta \vDash A_1 \equiv A_2 : \star \qquad \Gamma, x : A_1; \Delta \vDash B_1 \equiv B_2 : \star \qquad [\Gamma \vDash A_1 : \star] \qquad [\Gamma \vDash \Pi^\rho x\!:\!A_1 \to B_1 : \star] \qquad [\Gamma \vDash \Pi^\rho x\!:\!A_2 \to B_2 : \star]}{\Gamma; \Delta \vDash (\Pi^\rho x\!:\!A_1 \to B_1) \equiv (\Pi^\rho x\!:\!A_2 \to B_2) : \star}$$

E-AbsCong
$$\frac{\Gamma, x : A_1; \Delta \vDash b_1 \equiv b_2 : B \qquad [\Gamma \vDash A_1 : \star] \qquad (\rho = +) \vee (x \notin \mathsf{fv}\ b_1) \qquad (\rho = +) \vee (x \notin \mathsf{fv}\ b_2)}{\Gamma; \Delta \vDash (\lambda^\rho x.b_1) \equiv (\lambda^\rho x.b_2) : \Pi^\rho x\!:\!A_1 \to B}$$

E-AppCong
$$\frac{\Gamma; \Delta \vDash a_1 \equiv b_1 : \Pi^+ x\!:\!A \to B \qquad \Gamma; \Delta \vDash a_2 \equiv b_2 : A}{\Gamma; \Delta \vDash a_1\ {a_2}^+ \equiv b_1\ {b_2}^+ : B\{a_2/x\}}$$

E-IAppCong
$$\frac{\Gamma; \Delta \vDash a_1 \equiv b_1 : \Pi^- x\!:\!A \to B \qquad \Gamma \vDash a : A}{\Gamma; \Delta \vDash a_1\ \square^- \equiv b_1\ \square^- : B\{a/x\}}$$

E-PiFst
$$\frac{\Gamma; \Delta \vDash \Pi^\rho x\!:\!A_1 \to B_1 \equiv \Pi^\rho x\!:\!A_2 \to B_2 : \star}{\Gamma; \Delta \vDash A_1 \equiv A_2 : \star}$$

E-PiSnd
$$\frac{\Gamma; \Delta \vDash \Pi^\rho x\!:\!A_1 \to B_1 \equiv \Pi^\rho x\!:\!A_2 \to B_2 : \star \qquad \Gamma; \Delta \vDash a_1 \equiv a_2 : A_1}{\Gamma; \Delta \vDash B_1\{a_1/x\} \equiv B_2\{a_2/x\} : \star}$$

E-CPiCong
$$\frac{\Gamma; \Delta \vDash \phi_1 \equiv \phi_2 \qquad \Gamma, c : \phi_1; \Delta \vDash A \equiv B : \star \qquad [\Gamma \vDash \phi_1\ \mathsf{ok}] \qquad [\Gamma \vDash \forall c\!:\!\phi_1.A : \star] \qquad [\Gamma \vDash \forall c\!:\!\phi_2.B : \star]}{\Gamma; \Delta \vDash \forall c\!:\!\phi_1.A \equiv \forall c\!:\!\phi_2.B : \star}$$

E-CAbsCong
$$\frac{\Gamma, c : \phi_1; \Delta \vDash a \equiv b : B \qquad [\Gamma \vDash \phi_1\ \mathsf{ok}]}{\Gamma; \Delta \vDash (\Lambda c.a) \equiv (\Lambda c.b) : \forall c\!:\!\phi_1.B}$$

E-CAppCong
$$\frac{\Gamma; \Delta \vDash a_1 \equiv b_1 : \forall c\!:\!(a \sim_A b).B \qquad \Gamma; \widetilde{\Gamma} \vDash a \equiv b : A}{\Gamma; \Delta \vDash a_1[\bullet] \equiv b_1[\bullet] : B\{\bullet/c\}}$$

$$\frac{\Gamma;\Delta \vDash \forall c{:}(a_1 \sim_A a_2).B_1 \equiv \forall c{:}(a_1' \sim_{A'} a_2').B_2 : \star \quad \Gamma;\widetilde{\Gamma} \vDash a_1 \equiv a_2 : A \quad \Gamma;\widetilde{\Gamma} \vDash a_1' \equiv a_2' : A'}{\Gamma;\Delta \vDash B_1\{\bullet/c\} \equiv B_2\{\bullet/c\} : \star}\ \textsc{E-CPiSnd}$$

$$\frac{\Gamma;\Delta \vDash a \equiv b : A \quad \Gamma;\Delta \vDash a \sim_A b \equiv a' \sim_{A'} b'}{\Gamma;\Delta \vDash a' \equiv b' : A'}\ \textsc{E-Cast}$$

$$\frac{\Gamma;\Delta \vDash a \equiv b : A \quad \Gamma;\widetilde{\Gamma} \vDash A \equiv B : \star}{\Gamma;\Delta \vDash a \equiv b : B}\ \textsc{E-EqConv}$$

$$\frac{\Gamma;\Delta \vDash a \sim_A b \equiv a' \sim_{A'} b'}{\Gamma;\Delta \vDash A \equiv A' : \star}\ \textsc{E-IsoSnd}$$

$\boxed{\vDash \Gamma}$ *(context wellformedness)*

$$\frac{}{\vDash \varnothing}\ \textsc{E-Empty}$$

$$\frac{\vDash \Gamma \quad \Gamma \vDash A : \star \quad x \notin \mathsf{dom}\,\Gamma}{\vDash \Gamma, x : A}\ \textsc{E-ConsTm}$$

$$\frac{\vDash \Gamma \quad \Gamma \vDash \phi\ \mathsf{ok} \quad c \notin \mathsf{dom}\,\Gamma}{\vDash \Gamma, c : \phi}\ \textsc{E-ConsCo}$$

$\boxed{\vDash \Sigma}$ *(signature wellformedness)*

$$\frac{}{\vDash \varnothing}\ \textsc{Sig-Empty}$$

$$\frac{\vDash \Sigma \quad \varnothing \vDash A : \star \quad \varnothing \vDash a : A \quad F \notin \mathsf{dom}\,\Sigma}{\vDash \Sigma \cup \{F \sim a : A\}}\ \textsc{Sig-ConsAx}$$

E FULL SYSTEM SPECIFICATION: EXPLICIT LANGUAGE TYPE SYSTEM

$\boxed{\Gamma \vdash a : A}$ *(typing)*

$$\frac{\vdash \Gamma}{\Gamma \vdash \star : \star}\ \textsc{An-Star}$$

$$\frac{\vdash \Gamma \quad x : A \in \Gamma}{\Gamma \vdash x : A}\ \textsc{An-Var}$$

$$\frac{\Gamma, x : A \vdash B : \star \quad [\Gamma \vdash A : \star]}{\Gamma \vdash \Pi^\rho x{:}A \to B : \star}\ \textsc{An-Pi}$$

$$\frac{[\Gamma \vdash A : \star] \quad \Gamma, x : A \vdash a : B \quad (\rho = +) \lor (x \notin \mathsf{fv}\ |a|)}{\Gamma \vdash \lambda^\rho x{:}A.a : \Pi^\rho x{:}A \to B}\ \textsc{An-Abs}$$

$$\frac{\Gamma \vdash b : \Pi^\rho x{:}A \to B \quad \Gamma \vdash a : A}{\Gamma \vdash b\ a^\rho : B\{a/x\}}\ \textsc{An-App}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma;\widetilde{\Gamma} \vdash \gamma : A \sim B \quad \Gamma \vdash B : \star}{\Gamma \vdash a \triangleright \gamma : B}\ \textsc{An-Conv}$$

$$\frac{[\Gamma \vdash \phi\ \mathsf{ok}] \quad \Gamma, c : \phi \vdash B : \star}{\Gamma \vdash \forall c{:}\phi.B : \star}\ \textsc{An-CPi}$$

$$\frac{[\Gamma \vdash \phi\ \mathsf{ok}] \quad \Gamma, c : \phi \vdash a : B}{\Gamma \vdash \Lambda c{:}\phi.a : \forall c{:}\phi.B}\ \textsc{An-CAbs}$$

$$\frac{\Gamma \vdash a_1 : \forall c{:}a \sim_{A_1} b.B \quad \Gamma;\widetilde{\Gamma} \vdash \gamma : a \sim b}{\Gamma \vdash a_1[\gamma] : B\{\gamma/c\}}\ \textsc{An-CApp}$$

$$\frac{\vdash \Gamma \quad F \sim a : A \in \Sigma_1 \quad [\varnothing \vdash A : \star]}{\Gamma \vdash F : A}\ \textsc{An-Fam}$$

$\boxed{\Gamma \vdash \phi \ \mathsf{ok}}$ *(prop wellformedness)*

$$\textsc{An-Wff}\quad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B \qquad |A| = |B|}{\Gamma \vdash a \sim_A b \ \mathsf{ok}}$$

$\boxed{\Gamma; \Delta \vdash \gamma : \phi_1 \sim \phi_2}$ *(coercion between props)*

$$\textsc{An-PropCong}\quad \frac{\Gamma; \Delta \vdash \gamma_1 : A_1 \sim A_2 \qquad \Gamma; \Delta \vdash \gamma_2 : B_1 \sim B_2 \qquad \Gamma \vdash A_1 \sim_A B_1 \ \mathsf{ok} \qquad \Gamma \vdash A_2 \sim_A B_2 \ \mathsf{ok}}{\Gamma; \Delta \vdash (\gamma_1 \sim_A \gamma_2) : (A_1 \sim_A B_1) \sim (A_2 \sim_A B_2)}$$

$$\textsc{An-CPiFst}\quad \frac{\Gamma; \Delta \vdash \gamma : \forall c{:}\phi_1.A_2 \sim \forall c{:}\phi_2.B_2}{\Gamma; \Delta \vdash \mathbf{cpiFst}\,\gamma : \phi_1 \sim \phi_2}$$

$$\textsc{An-IsoSym}\quad \frac{\Gamma; \Delta \vdash \gamma : \phi_1 \sim \phi_2}{\Gamma; \Delta \vdash \mathbf{sym}\,\gamma : \phi_2 \sim \phi_1}$$

$$\textsc{An-IsoConv}\quad \frac{\Gamma; \Delta \vdash \gamma : A \sim B \qquad \Gamma \vdash a_1 \sim_A a_2 \ \mathsf{ok} \qquad \Gamma \vdash a_1' \sim_B a_2' \ \mathsf{ok} \qquad |a_1| = |a_1'| \qquad |a_2| = |a_2'|}{\Gamma; \Delta \vdash \mathbf{conv}\ (a_1 \sim_A a_2) \sim_\gamma (a_1' \sim_B a_2') : (a_1 \sim_A a_2) \sim (a_1' \sim_B a_2')}$$

$\boxed{\Gamma; \Delta \vdash \gamma : A \sim B}$ *(coercion between types)*

$$\textsc{An-Assn}\quad \frac{\vdash \Gamma \qquad c : a \sim_A b \in \Gamma \qquad c \in \Delta}{\Gamma; \Delta \vdash c : a \sim b}$$

$$\textsc{An-Refl}\quad \frac{\Gamma \vdash a : A}{\Gamma; \Delta \vdash \mathbf{refl}\, a : a \sim a}$$

$$\textsc{An-EraseEq}\quad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B \qquad |a| = |b| \qquad \Gamma; \widetilde{\Gamma} \vdash \gamma : A \sim B}{\Gamma; \Delta \vdash (a \mathrel{|{=}|_\gamma} b) : a \sim b}$$

$$\textsc{An-Sym}\quad \frac{\Gamma \vdash b : B \qquad \Gamma \vdash a : A \qquad [\Gamma; \widetilde{\Gamma} \vdash \gamma_1 : B \sim A] \qquad \Gamma; \Delta \vdash \gamma : b \sim a}{\Gamma; \Delta \vdash \mathbf{sym}\,\gamma : a \sim b}$$

$$\textsc{An-Trans}\quad \frac{\Gamma; \Delta \vdash \gamma_1 : a \sim a_1 \qquad \Gamma; \Delta \vdash \gamma_2 : a_1 \sim b \qquad [\Gamma \vdash a : A] \qquad [\Gamma \vdash a_1 : A_1] \qquad [\Gamma; \widetilde{\Gamma} \vdash \gamma_3 : A \sim A_1]}{\Gamma; \Delta \vdash (\gamma_1; \gamma_2) : a \sim b}$$

$$\textsc{An-Beta}\quad \frac{\Gamma \vdash a_1 : B_0 \qquad \Gamma \vdash a_2 : B_1 \qquad |B_0| = |B_1| \qquad \models |a_1| > |a_2|}{\Gamma; \Delta \vdash \mathbf{red}\, a_1\, a_2 : a_1 \sim a_2}$$

$$\textsc{An-PiCong}\quad \frac{\begin{array}{c}\Gamma; \Delta \vdash \gamma_1 : A_1 \sim A_2 \\ \Gamma, x : A_1; \Delta \vdash \gamma_2 : B_1 \sim B_2 \\ B_3 = B_2\{x \triangleright \mathbf{sym}\,\gamma_1 / x\} \\ \Gamma \vdash \Pi^\rho x{:}A_1 \to B_1 : \star \\ \Gamma \vdash \Pi^\rho x{:}A_1 \to B_2 : \star \\ \Gamma \vdash \Pi^\rho x{:}A_2 \to B_3 : \star\end{array}}{\Gamma; \Delta \vdash (\Pi^\rho x{:}\gamma_1.\gamma_2) : (\Pi^\rho x{:}A_1 \to B_1) \sim (\Pi^\rho x{:}A_2 \to B_3)}$$

AN-ABSCONG
$$\frac{\begin{array}{c}\Gamma;\Delta \vdash \gamma_1 : A_1 \sim A_2 \\ \Gamma, x : A_1;\Delta \vdash \gamma_2 : b_1 \sim b_2 \\ b_3 = b_2\{x \triangleright \mathbf{sym}\,\gamma_1/x\} \\ [\Gamma \vdash A_1 : \star] \qquad \Gamma \vdash A_2 : \star \\ (\rho = +) \vee (x \notin \mathsf{fv}\,|b_1|) \\ (\rho = +) \vee (x \notin \mathsf{fv}\,|b_3|) \\ [\Gamma \vdash (\lambda^\rho x{:}A_1.b_2) : B]\end{array}}{\Gamma;\Delta \vdash (\lambda^\rho x{:}\gamma_1.\gamma_2) : (\lambda^\rho x{:}A_1.b_1) \sim (\lambda^\rho x{:}A_2.b_3)}$$

AN-APPCONG
$$\frac{\begin{array}{c}\Gamma;\Delta \vdash \gamma_1 : a_1 \sim b_1 \\ \Gamma;\Delta \vdash \gamma_2 : a_2 \sim b_2 \\ \Gamma \vdash a_1\ {a_2}^\rho : A \\ \Gamma \vdash b_1\ {b_2}^\rho : B \\ [\Gamma;\widetilde{\Gamma} \vdash \gamma_3 : A \sim B]\end{array}}{\Gamma;\Delta \vdash (\gamma_1\ \gamma_2^\rho) : (a_1\ {a_2}^\rho) \sim (b_1\ {b_2}^\rho)}$$

AN-PIFST
$$\frac{\Gamma;\Delta \vdash \gamma : \Pi^\rho x{:}A_1 \to B_1 \sim \Pi^\rho x{:}A_2 \to B_2}{\Gamma;\Delta \vdash \mathbf{piFst}\,\gamma : A_1 \sim A_2}$$

AN-PISND
$$\frac{\begin{array}{c}\Gamma;\Delta \vdash \gamma_1 : (\Pi^\rho x{:}A_1 \to B_1) \sim (\Pi^\rho x{:}A_2 \to B_2) \\ \Gamma;\Delta \vdash \gamma_2 : a_1 \sim a_2 \\ \Gamma \vdash a_1 : A_1 \qquad \Gamma \vdash a_2 : A_2\end{array}}{\Gamma;\Delta \vdash \gamma_1@\gamma_2 : B_1\{a_1/x\} \sim B_2\{a_2/x\}}$$

AN-CPICONG
$$\frac{\begin{array}{c}\Gamma;\Delta \vdash \gamma_1 : \phi_1 \sim \phi_2 \\ \Gamma, c : \phi_1;\Delta \vdash \gamma_3 : B_1 \sim B_2 \\ B_3 = B_2\{c \triangleright \mathbf{sym}\,\gamma_1/c\} \\ \Gamma \vdash \forall c{:}\phi_1.B_1 : \star \\ [\Gamma \vdash \forall c{:}\phi_2.B_3 : \star] \\ \Gamma \vdash \forall c{:}\phi_1.B_2 : \star\end{array}}{\Gamma;\Delta \vdash (\forall c{:}\gamma_1.\gamma_3) : (\forall c{:}\phi_1.B_1) \sim (\forall c{:}\phi_2.B_3)}$$

AN-CABSCONG
$$\frac{\begin{array}{c}\Gamma;\Delta \vdash \gamma_1 : \phi_1 \sim \phi_2 \\ \Gamma, c : \phi_1;\Delta \vdash \gamma_3 : a_1 \sim a_2 \\ a_3 = a_2\{c \triangleright \mathbf{sym}\,\gamma_1/c\} \\ \Gamma \vdash (\Lambda c{:}\phi_1.a_1) : \forall c{:}\phi_1.B_1 \\ \Gamma \vdash (\Lambda c{:}\phi_1.a_2) : B \\ \Gamma \vdash (\Lambda c{:}\phi_2.a_3) : \forall c{:}\phi_2.B_2 \\ \Gamma;\widetilde{\Gamma} \vdash \gamma_4 : \forall c{:}\phi_1.B_1 \sim \forall c{:}\phi_2.B_2\end{array}}{\Gamma;\Delta \vdash (\lambda c{:}\gamma_1.\gamma_3@\gamma_4) : (\Lambda c{:}\phi_1.a_1) \sim (\Lambda c{:}\phi_2.a_3)}$$

AN-CAPPCONG
$$\frac{\begin{array}{c}\Gamma;\Delta \vdash \gamma_1 : a_1 \sim b_1 \\ \Gamma;\widetilde{\Gamma} \vdash \gamma_2 : a_2 \sim b_2 \\ \Gamma;\widetilde{\Gamma} \vdash \gamma_3 : a_3 \sim b_3 \\ \Gamma \vdash a_1[\gamma_2] : A \\ \Gamma \vdash b_1[\gamma_3] : B \\ [\Gamma;\widetilde{\Gamma} \vdash \gamma_4 : A \sim B]\end{array}}{\Gamma;\Delta \vdash \gamma_1[\gamma_2,\gamma_3] : a_1[\gamma_2] \sim b_1[\gamma_3]}$$

AN-CPISND
$$\frac{\begin{array}{c}\Gamma;\Delta \vdash \gamma_1 : (\forall c_1{:}a \sim_A a'.B_1) \sim (\forall c_2{:}b \sim_B b'.B_2) \\ \Gamma;\widetilde{\Gamma} \vdash \gamma_2 : a \sim a' \\ \Gamma;\widetilde{\Gamma} \vdash \gamma_3 : b \sim b'\end{array}}{\Gamma;\Delta \vdash \gamma_1@(\gamma_2 \sim \gamma_3) : B_1\{\gamma_2/c_1\} \sim B_2\{\gamma_3/c_2\}}$$

AN-CAST
$$\frac{\begin{array}{c}\Gamma;\Delta \vdash \gamma_1 : a \sim a' \\ \Gamma;\Delta \vdash \gamma_2 : (a \sim_A a') \sim (b \sim_B b')\end{array}}{\Gamma;\Delta \vdash \gamma_1 \triangleright \gamma_2 : b \sim b'}$$

AN-ISOSND
$$\frac{\Gamma;\Delta \vdash \gamma : (a \sim_A a') \sim (b \sim_B b')}{\Gamma;\Delta \vdash \mathbf{isoSnd}\,\gamma : A \sim B}$$

$\boxed{\vdash \Gamma}$ *(context wellformedness)*

AN-EMPTY
$$\frac{}{\vdash \varnothing}$$

AN-CONSTM
$$\frac{\begin{array}{c}\vdash \Gamma \\ \Gamma \vdash A : \star \qquad x \notin \mathsf{dom}\,\Gamma\end{array}}{\vdash \Gamma, x : A}$$

AN-CONSCO
$$\frac{\begin{array}{c}\vdash \Gamma \\ \Gamma \vdash \phi\ \mathsf{ok} \qquad c \notin \mathsf{dom}\,\Gamma\end{array}}{\vdash \Gamma, c : \phi}$$

$\boxed{\vdash \Sigma}$ *(signature wellformedness)*

$$\frac{}{\vdash \varnothing}\ \textsc{An-Sig-Empty}$$

$$\frac{\vdash \Sigma \qquad \varnothing \vdash A : \star \qquad \varnothing \vdash a : A \qquad F \notin \mathsf{dom}\,\Sigma}{\vdash \Sigma \cup \{F \sim a : A\}}\ \textsc{An-Sig-ConsAx}$$