

Advanced Practice Problems

MATH 145 / MATH 147

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Problem 1 ([]).

1] We say that a set X of real numbers is bounded above if there exists a real number α such that $x \leq \alpha$ for all $x \in X$. Similarly, X is bounded below if there exists a real number β such that $x \geq \beta$ for all $x \in X$.

Let S be the set of all real numbers satisfying

$$x^3 + 2x < 4.$$

Is the set bounded above? Is it bounded below? Prove your answers.

Solution.

We can define S as

$$S = \{x \in \mathbb{R} \mid x^3 + 2x - 4 < 0\}.$$

Let

$$g(x) = x^3 + 2x - 4.$$

Suppose there exists some r such that $g(r) = 0$.

We check if g is strictly increasing by considering

$$g(x) - g(y) = (x^3 + 2x) - (y^3 + 2y) = (x - y)(x^2 + xy + y^2) + 2(x - y) = (x - y)(x^2 + xy + y^2 + 2).$$

Because for any real x, y , the term $x^2 + xy + y^2 + 2$ is always positive, and if $x > y$ then $(x - y) > 0$, it follows that

$$g(x) - g(y) > 0 \quad \text{whenever } x > y.$$

This shows g is strictly increasing.

Since g is strictly increasing and $g(r) = 0$, it follows that

$$g(x) < 0 \quad \text{for } x < r, \quad \text{and} \quad g(x) > 0 \quad \text{for } x > r.$$

Thus,

$$S = \{x \in \mathbb{R} : x < r\}.$$

Therefore, S is bounded above by r , but since it contains arbitrarily large negative numbers, it is not bounded below. \square

Problem 2 ([]).

2]

The complex numbers \mathbb{C} are a number system which, as a set, is the set of all ordered pairs of real numbers $\mathbb{C} = \{(a, b) \mid a, b \in \mathbb{R}\}$. The ordered pair (a, b) is usually written $a + bi$. Complex numbers can be added, subtracted, multiplied, and divided as follows.

- *Addition:*

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

- *Subtraction:*

$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$

- *Multiplication:*

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

- *Division (for $c + di \neq 0$):*

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.$$

Prove that there is no order relation " $<$ " on \mathbb{C} that satisfies the five order axioms:

1. For any $a, b \in \mathbb{C}$, exactly one of the following holds: $a < b$, $a = b$, or $b < a$.
2. For any $a, b, c \in \mathbb{C}$, if $a < b$ and $b < c$, then $a < c$.
3. For any $a, b, c \in \mathbb{C}$, if $a < b$, then $a + c < b + c$.
4. For any $a, b, c \in \mathbb{C}$, if $a < b$ and $0 < c$, then $ac < bc$.
5. For any $a, b, c \in \mathbb{C}$, if $a < b$ and $c < 0$, then $bc < ac$.

Solution.

We proceed by contradiction.

Suppose there exists an order relation " $<$ " on \mathbb{C} satisfying the five order axioms above, making $(\mathbb{C}, +, \cdot, <)$ an ordered field.

Consider the imaginary unit $i \in \mathbb{C}$ where $i^2 = -1$.

By axiom (1), exactly one of the following must hold:

$$i > 0, \quad i = 0, \quad \text{or} \quad i < 0.$$

We know $i \neq 0$, so either $i > 0$ or $i < 0$.

Case 1: Suppose $i > 0$.

Then by axiom (4), let $a = 0$, $b = i$, $c = i$

$$0 \cdot i < i \cdot i.$$

This implies $0 < -1$, which is a contradiction.

Case 2: Suppose $i < 0$.

Then $-i > 0$ and by the same reasoning

$$0 < (-i)^2 = (-1)^2(i)^2 = -1$$

In both cases, we conclude

$$0 < -1.$$

However, in any ordered field, $1 > 0$ (because $1 = 1$ and is the multiplicative identity).

Adding inequalities preserves order by axiom (3), so

$$-1 > 0 \implies (-1) + 1 > 0 + 1 \implies 0 > 1,$$

which contradicts the fact that $1 > 0$.

Therefore, our assumption that such an order relation exists on \mathbb{C} leads to a contradiction.

Conclusion: There is no order relation " $<$ " on the complex numbers \mathbb{C} satisfying all five axioms of an ordered field. \square

Problem 3 ([]).

3] **Q3.** Given two sets, A and B , the Cartesian product $A \times B$ is defined as the set of all ordered tuples (a, b) , where $a \in A$ and $b \in B$. Using the set notation, this can be written as

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Let A and B be sets. A function from A to B , denoted by $f : A \rightarrow B$, is a subset f of $A \times B$ such that for all $a \in A$ there exists a unique $b \in B$ such that $(a, b) \in f$. We write $b = f(a)$ and call the b the image of a under f . We say $f : A \rightarrow B$ is injective if for all $b \in B$ there exists at most one $a \in A$ such that $f(a) = b$. We say $f : A \rightarrow B$ is surjective if $\forall b \in B, \exists a \in A$ such that $f(a) = b$. Finally, $f : A \rightarrow B$ is bijective if it is both injective and surjective. Determine if the following functions are injective and/or surjective.

$$1. f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Q}, f(x, y) = \frac{x + 1}{x^2 + y^2 + 2}$$

$$2. f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, f(x, y) = xy$$

$$3. f : \mathbb{R} \rightarrow (0, 1), f(x) = \frac{x^2}{1 + x^2}$$

Solution.

1.

(a) **Injective**

We want to check if the function is injective. The function f is injective if whenever $f(x_1, y_1) = f(x_2, y_2)$, it implies that $(x_1, y_1) = (x_2, y_2)$.

$$\text{Suppose } f(x_1, y_1) = f(x_2, y_2)$$

Then, we have:

$$\frac{x_1 + 1}{x_1^2 + y_1^2 + 2} = \frac{x_2 + 1}{x_2^2 + y_2^2 + 2}$$

This does not necessarily imply that $(x_1, y_1) = (x_2, y_2)$.

Counterexample: Let $f(0, 0) = \frac{1}{2}$ and $f(1, 1) = \frac{1}{2}$, but $(0, 0) \neq (1, 1)$.

Thus, the function is **not injective**.

Surjective

We need to check if every rational number is attainable.

$$f(x, y) = \frac{x + 1}{x^2 + y^2 + 2}, x, y \in \mathbb{Z}$$

Let $f(x, y) = -1$. -1 can be written as $-k/k$ or $k/-k$ for all k

Case 1:

$$\frac{x + 1}{x^2 + y^2 + 2} = k/-k$$

But, $x^2 + y^2 + 2$ is strictly positive for all $x, y \in \mathbb{Z}$

Case 2:

$$\frac{x + 1}{x^2 + y^2 + 2} = -k/k$$

We get the following system of equations.

$$x + 1 = -k \text{ and } x^2 + y^2 + 2 = k$$

From $x + 1 = -k$, we get $x = -(k + 1)$. Substituting into the second equation:

$$x^2 + y^2 + 2 = k \Rightarrow (-(k + 1))^2 + y^2 + 2 = k$$

$$(k + 1)^2 + y^2 + 2 = k \Rightarrow k^2 + 2k + 1 + y^2 + 2 = k$$

$$k^2 + k + 3 + y^2 = 0$$

$$k^2 + y^2 + 3 = -k$$

Since the left side, $k^2 + y^2 + 3$ is always positive, and the right side is negative we reach a contradiction. That is, a positive number cannot equal a negative number.

Hence, no such integers x, y exist for which $f(x, y) = -1$. Therefore, $-1 \notin \text{Im}(f)$, so f is not surjective.

Conclusion: The function $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Q}$, defined by $f(x, y) = \frac{x + 1}{x^2 + y^2 + 2}$, is neither injective nor surjective.

2.

Consider the function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x, y) = xy$.

Injective:

Suppose $f(x_1, y_1) = f(x_2, y_2)$, i.e.,

$$x_1 y_1 = x_2 y_2.$$

This does not imply $(x_1, y_1) = (x_2, y_2)$.

Counterexample: Let $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (1, 6)$. Then

$$f(2, 3) = 6 = f(1, 6),$$

but $(2, 3) \neq (1, 6)$. So the function is **not injective**.

Surjective:

Let $r \in \mathbb{R}$. We want to find $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $xy = r$.

Case 1: If $r = 0$, then $f(0, y) = 0$ for any y , so 0 is in the image.

Case 2: If $r \neq 0$, let $x = r$, $y = 1$. Then $f(x, y) = r \cdot 1 = r$.

Hence, every real number has a preimage, so the function is **surjective**.

Conclusion: The function $f(x, y) = xy$ is **not injective** but **surjective**.

3.

Consider the function $f : \mathbb{R} \rightarrow (0, 1)$ defined by

$$f(x) = \frac{x^2}{1 + x^2}.$$

Injective:

Suppose $f(x_1) = f(x_2)$. Then

$$\frac{x_1^2}{1 + x_1^2} = \frac{x_2^2}{1 + x_2^2}.$$

Cross-multiplying:

$$x_1^2(1 + x_2^2) = x_2^2(1 + x_1^2).$$

Expanding both sides:

$$x_1^2 + x_1^2 x_2^2 = x_2^2 + x_1^2 x_2^2.$$

Subtracting $x_1^2 x_2^2$ from both sides:

$$x_1^2 = x_2^2 \quad \Rightarrow \quad x_1 = \pm x_2.$$

So the function maps both x and $-x$ to the same value. For example:

$$f(1) = \frac{1}{2} = f(-1),$$

but $1 \neq -1$. Therefore, the function is **not injective**.

Surjective:

Let $y \in (0, 1)$. We want to find $x \in \mathbb{R}$ such that

$$\frac{x^2}{1 + x^2} = y.$$

Multiply both sides by $1 + x^2$:

$$x^2 = y(1 + x^2) \Rightarrow x^2 = y + yx^2.$$

Rewriting:

$$x^2 - yx^2 = y \Rightarrow x^2(1 - y) = y \Rightarrow x^2 = \frac{y}{1 - y}.$$

Since $y \in (0, 1)$, the right-hand side is positive. So such an x exists, namely $x = \pm \sqrt{\frac{y}{1-y}} \in \mathbb{R}$.

Hence, the function is **surjective**.

Conclusion: The function $f(x) = \frac{x^2}{1+x^2}$ is **not injective** but **surjective**.

□

Problem 4 ([]).

4] **Q4.** Let X be a set and let $P(X)$ denote the set of all subsets of X . Prove that there does not exist a surjective function $f : X \rightarrow P(X)$. (Hint: $A = \{x \in X : x \notin f(x)\}$.)

Solution.

Assume for contradiction that:

$f : X \rightarrow P(X)$ is surjective.

For every subset $A \subseteq X$, $\exists a \in X$ such that $f(a) = A$.

Let

$$A = \{x \in X \mid x \notin f(x)\}.$$

This is a well-defined subset of X , so $A \in P(X)$.

Since f is surjective, $\exists a \in X$ such that $f(a) = A$.

Suppose $a \notin A$. By definition of A ,

$$a \notin A \Rightarrow a \in f(a).$$

But $f(a) = A$, so

$$a \in A,$$

a contradiction.

Suppose $a \in A$. Then by definition of A ,

$$a \in A \Rightarrow a \notin f(a).$$

Since $f(a) = A$, this means

$$a \notin A,$$

again a contradiction.

Thus, in both cases we get a contradiction. Therefore f is not surjective.

□

Problem 5 ([]).

5] A nonempty subset X of \mathbb{Z} is said to be sticky if for all $x, y \in X$ and $n \in \mathbb{Z}$, $x + y \in X$ and $xn \in X$.

(a) Fix $n \in \mathbb{Z}$. Find the smallest sticky subset of \mathbb{Z} which contains n .

(b) Fix $n \in \mathbb{Z}$, different from ± 1 . Find all sticky subsets $X \subset \mathbb{Z}$ such that (1) $X \neq \mathbb{Z}$; (2) $n \in X$; (3) whenever $X \subset Y$ which is a sticky subset of \mathbb{Z} , then $X = Y$. How many sticky subsets can you find?

Solution.

(a)

Given $n \in \mathbb{Z}$, define

$$X_n = \{kn \mid k \in \mathbb{Z}\} = n\mathbb{Z}.$$

We claim that X_n is the smallest sticky subset of \mathbb{Z} containing n .

Proof that X_n is sticky:

Let $x = an$ and $y = bn$ be arbitrary elements of X_n with $a, b \in \mathbb{Z}$.

Then:

$$x + y = an + bn = (a + b)n \in X_n$$

and for any $m \in \mathbb{Z}$:

$$xm = anm = (am)n \in X_n.$$

Thus, X_n is closed under addition and multiplication by any integer, so X_n is sticky.

Minimality of X_n :

Suppose Y is any sticky subset of \mathbb{Z} containing n . Then:

- Since $n \in Y$ and Y is sticky, for any integer k , by repeatedly adding n to itself (closure under addition), we get $kn \in Y$.
- Also, closure under multiplication by any integer implies all integer multiples of n are in Y .

Hence $n\mathbb{Z} \subseteq Y$.

Therefore, the smallest sticky subset of \mathbb{Z} containing n is $n\mathbb{Z}$.

(b)

Fix $n \in \mathbb{Z}$ with $n \neq \pm 1$.

We are asked to find all sticky subsets $X \subset \mathbb{Z}$ satisfying:

- (i) $X \neq \mathbb{Z}$,
- (ii) $n \in X$,
- (iii) X is maximal among all proper sticky subsets of \mathbb{Z} containing n .

Recall: A subset $X \subset \mathbb{Z}$ is sticky if it is closed under:

- addition: if $x, y \in X$ then $x + y \in X$, and

- integer multiplication: if $x \in X$, $m \in \mathbb{Z}$ then $mx \in X$.

As shown in part (a), the smallest sticky subset of \mathbb{Z} containing n is:

$$X_n = n\mathbb{Z} = \{kn \mid k \in \mathbb{Z}\}.$$

More generally, any sticky subset of \mathbb{Z} is of the form $d\mathbb{Z}$ for some $d \in \mathbb{N}$.

We want all proper sticky subsets $X = d\mathbb{Z} \subsetneq \mathbb{Z}$ such that:

- $n \in d\mathbb{Z} \iff d \mid n$,
- and $d\mathbb{Z}$ is maximal among such proper sticky sets.

So we want all divisors $d > 1$ of n such that $d\mathbb{Z}$ is not properly contained in any other proper sticky subset that contains n .

But since $d_1\mathbb{Z} \subset d_2\mathbb{Z} \iff d_2 \mid d_1$, the *maximal* proper sticky subsets containing n are the ones where d is the smallest possible positive divisor of n greater than 1—but in fact, every $d > 1$ dividing n gives a distinct such sticky set $d\mathbb{Z}$, and they are all maximal under inclusion among proper sticky sets containing n .

Conclusion: The sticky subsets $X \subsetneq \mathbb{Z}$ satisfying the conditions are precisely the sets:

$$X = d\mathbb{Z} \quad \text{where } d > 1 \text{ and } d \mid n.$$

The number of such subsets is equal to the number of positive divisors of n greater than 1. □

Problem 6 ([]).

6] **Q6.** Suppose that for each natural number n , we make a statement $P(n)$ (which can either be true or false). The Principle of Mathematical Induction tells us that:

1. If $P(1)$ is true, and
2. If for any natural number k , $P(k)$ being true implies that $P(k+1)$ is also true,

then $P(n)$ is true for all $n \in \mathbb{N}$.

Find the flaw in the following proof by mathematical induction, which seems to suggest that all real numbers are equal.

Let $P(n)$ be the statement that in any set of real numbers $\{x_1, x_2, \dots, x_n\}$, all of the numbers have the same value.

It is clear that $P(1)$ is true. Now assume that $P(k)$ is true for some natural number k . Consider any set of $(k+1)$ real numbers $\{x_1, x_2, \dots, x_{k+1}\}$.

If we remove x_1 , then we have a set of k numbers, so by $P(k)$, we have that

$$x_2 = x_3 = \dots = x_{k+1}.$$

If, instead, we remove x_k , then we again have a set of k numbers, so

$$x_1 = x_2 = \dots = x_{k-1} = x_k.$$

Combining these two results, we see that

$$x_1 = x_2 = x_3 = \cdots = x_{k+1},$$

proving that $P(k+1)$ is true. By the Principle of Mathematical Induction, $P(n)$ is true for all natural numbers n .

Solution.

We are asked to identify the flaw in a proof that falsely claims all real numbers are equal, using mathematical induction on the following statement:

$P(n)$: "In any set of n real numbers, all the numbers are equal."

Base Case ($n = 1$): The statement $P(1)$ is true. Any set with a single real number trivially has all elements equal.

Inductive Step: The proof assumes $P(k)$ is true for some $k \in \mathbb{N}$ and attempts to prove $P(k+1)$. The idea is to take a set of $k+1$ real numbers $\{x_1, x_2, \dots, x_{k+1}\}$ and consider two overlapping subsets of size k :

- Removing x_1 gives the set $\{x_2, x_3, \dots, x_{k+1}\}$. By the inductive hypothesis, all these values are equal:

$$x_2 = x_3 = \cdots = x_{k+1}.$$

- Removing x_{k+1} gives the set $\{x_1, x_2, \dots, x_k\}$. Again by the inductive hypothesis,

$$x_1 = x_2 = \cdots = x_k.$$

Since both subsets include x_2 , the proof claims this common value links the two equalities and implies

$$x_1 = x_2 = \cdots = x_{k+1},$$

and thus concludes that $P(k+1)$ is true.

Flaw in the Argument:

This reasoning appears valid for $k \geq 2$, but fails when $k = 1$, which is exactly the first step of the induction.

To illustrate the failure:

- Let $k = 1$. Then we try to prove $P(2)$: "Any set of two real numbers are equal."
- The set is $\{x_1, x_2\}$.
- Removing x_1 leaves $\{x_2\}$, which trivially satisfies $P(1)$, so no new information is obtained.
- Removing x_2 leaves $\{x_1\}$, which again trivially satisfies $P(1)$.

However, since these two subsets have no elements in common (they are disjoint), we cannot compare or "link" their values. There's no overlapping element (like x_2 in the general case) to ensure consistency between the values.

As a result, we cannot conclude that $x_1 = x_2$. This breaks the inductive step at $k = 1$, and therefore the entire induction argument collapses.

Conclusion: The flaw lies in the assumption that the two overlapping subsets used in the inductive step always share a common element. This is not true when $k = 1$, making the inductive step invalid and the proof incorrect. Hence, the conclusion that all real numbers are equal is false. \square

Problem 7 ([]).

7] **Q7.** Let $a_n > 0$ be a positive real number for each natural number n and define $S_n = a_1 + a_2 + \cdots + a_n$. It is clear that $0 \leq S_n \leq S$, for some $S \in \mathbb{R}$.

Find examples of a_n satisfying the following conditions:

(a) The set $\{S_n\}$ is not bounded above, i.e., for any real number M , there exists $n \in \mathbb{N}$ such that $S_n > M$.

(b) The set $\{S_n\}$ is bounded above, i.e., there exists a fixed number S such that $S_n \leq S$ for all $n \in \mathbb{N}$.

Solution.

(a) We need to show that the sequence of partial sums S_n diverges to infinity.

Let $a_n = 1$ for all n . Then, the partial sum is given by

$$S_n = \sum_{k=1}^n 1 = n.$$

This sequence grows without bound:

$$S_1 = 1, \quad S_2 = 2, \quad S_3 = 3, \quad \dots, \quad S_n = n \quad \text{as } n \rightarrow \infty.$$

Thus, this sequence is not bounded above.

(b) The set $\{S_n\}$ is bounded above, i.e., there exists a fixed number S such that $S_n \leq S$ for all $n \in \mathbb{N}$.

Solution: We need the infinite series to converge.

Let $a_n = \frac{1}{2^n}$ for all n . Then, the partial sum is:

$$S_n = \sum_{k=1}^n \frac{1}{2^k}.$$

The infinite series is given by

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

This series converges to 1, so $S_n \leq 1$ for all n , and S_n converges to 1. \square