# Advanced Practice Problems

MATH 145 / MATH 147

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**Problem 1** ([).

*1] We say that a set $X$ of real numbers is* bounded above *if there exists a real number $\alpha$ such that $x \leq \alpha$ for all $x \in X$. Similarly, $X$ is* bounded below *if there exists a real number $\beta$ such that $x \geq \beta$ for all $x \in X$.*

*Let $S$ be the set of all real numbers satisfying*

$$x^3 + 2x < 4.$$

*Is the set bounded above? Is it bounded below? Prove your answers.*

**Solution.**

We can define $S$ as

$$S = \{x \in \mathbb{R} \mid x^3 + 2x - 4 < 0\}.$$

Let

$$g(x) = x^3 + 2x - 4.$$

Since $g(x)$ is an odd polynomial with a positive leading coefficient, it has end behaviour from $QIII$ to $QI$. Thus, there exists some $r$ such that $g(r) = 0$.

We check if $g$ is strictly increasing by considering

$$g(x) - g(y) = (x^3 + 2x) - (y^3 + 2y) = (x-y)(x^2 + xy + y^2) + 2(x-y) = (x-y)(x^2 + xy + y^2 + 2).$$

Because for any real $x, y$, the term $x^2 + xy + y^2 + 2$ is always positive, and if $x > y$ then $(x - y) > 0$, it follows that

$$g(x) - g(y) > 0 \quad \text{whenever } x > y.$$

This shows $g$ is strictly increasing.

Since $g$ is strictly increasing and $g(r) = 0$, it follows that

$$g(x) < 0 \quad \text{for } x < r, \quad \text{and} \quad g(x) > 0 \quad \text{for } x > r.$$

Thus,

$$S = \{x \in \mathbb{R} : x < r\}.$$

Therefore, $S$ is bounded above by $r$, but since it contains arbitrarily large negative numbers, it is not bounded below. $\square$

**Problem 2** ([).

*2]*

*The complex numbers $\mathbb{C}$ are a number system which, as a set, is the set of all ordered pairs of real numbers $\mathbb{C} = \{(a, b) \mid a, b \in \mathbb{R}\}$. The ordered pair $(a, b)$ is usually written $a+bi$. Complex numbers can be added, subtracted, multiplied, and divided as follows.*

- *Addition:*
$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$
- *Subtraction:*
$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$
- *Multiplication:*
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$
- *Division (for $c + di \neq 0$):*
$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.$$

*Prove that there is no order relation "$<$" on $\mathbb{C}$ that satisfies the five order axioms:*

1. *For any $a, b \in \mathbb{C}$, exactly one of the following holds: $a < b$, $a = b$, or $b < a$.*
2. *For any $a, b, c \in \mathbb{C}$, if $a < b$ and $b < c$, then $a < c$.*
3. *For any $a, b, c \in \mathbb{C}$, if $a < b$, then $a + c < b + c$.*
4. *For any $a, b, c \in \mathbb{C}$, if $a < b$ and $0 < c$, then $ac < bc$.*
5. *For any $a, b, c \in \mathbb{C}$, if $a < b$ and $c < 0$, then $bc < ac$.*

**Solution.**

We proceed by contradiction.

Suppose there exists an order relation "$<$" on $\mathbb{C}$ satisfying the five order axioms above, making $(\mathbb{C}, +, \cdot, <)$ an ordered field.

Consider the imaginary unit $i \in \mathbb{C}$ where $i^2 = -1$.

By axiom (1), exactly one of the following must hold:

$$i > 0, \quad i = 0, \quad \text{or} \quad i < 0.$$

We know $i \neq 0$, so either $i > 0$ or $i < 0$.

**Case 1:** Suppose $i > 0$.

Then by axiom (4), let $a = 0$, $b = i$, $c = i$

$$0 \cdot i < i \cdot i.$$

This implies $0 < -1$.

**Case 2:** Suppose $i < 0$.
Then $-i > 0$ and by the same reasoning

$$0 < (-i)^2 = (-1)^2(i)^2 = -1$$

In both cases, we conclude

$$0 < -1.$$

However, in any ordered field, $1 > 0$ (because $1 = 1$ and is the multiplicative identity). Adding inequalities preserves order by axiom (3), so

$$-1 > 0 \implies (-1) + 1 > 0 + 1 \implies 0 > 1,$$

which contradicts the fact that $1 > 0$.
Therefore, our assumption that such an order relation exists on $\mathbb{C}$ leads to a contradiction.

**Conclusion:** There is no order relation "$<$" on the complex numbers $\mathbb{C}$ satisfying all five axioms of an ordered field. □

**Problem 3** ([).

*3] **Q3.** Given two sets, $A$ and $B$, the Cartesian product $A \times B$ is defined as the set of all ordered tuples $(a, b)$, where $a \in A$ and $b \in B$. Using the set notation, this can be written as*

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

*Let $A$ and $B$ be sets. A function from $A$ to $B$, denoted by $f : A \to B$, is a subset $f$ of $A \times B$ such that for all $a \in A$ there exists a unique $b \in B$ such that $(a, b) \in f$. We write $b = f(a)$ and call the $b$ the image of $a$ under $f$. We say $f : A \to B$ is injective if for all $b \in B$ there exists at most one $a \in A$ such that $f(a) = b$. We say $f : A \to B$ is surjective if $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$. Finally, $f : A \to B$ is bijective if it is both injective and surjective. Determine if the following functions are injective and/or surjective.*

1. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$, $f(x, y) = \dfrac{x + 1}{x^2 + y^2 + 2}$
2. $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(x, y) = xy$
3. $f : \mathbb{R} \to (0, 1)$, $f(x) = \dfrac{x^2}{1 + x^2}$

**Solution.**
**1.**
**(a) Injective**
We want to check if the function is injective. The function $f$ is injective if whenever $f(x_1, y_1) = f(x_2, y_2)$, it implies that $(x_1, y_1) = (x_2, y_2)$.
*Counterexample:* Let $f(0, 0) = \frac{1}{2}$ and $f(1, 1) = \frac{1}{2}$, but $(0, 0) \neq (1, 1)$.
Thus, the function is **not injective**.
**Surjective**

We need to check if every rational number is attainable.

$$f(x,y) = \frac{x+1}{x^2+y^2+2}, x, y \in \mathbb{Z}$$

Let $f(x,y) = -1$. $-1$ can be written as $-k/k$ or $k/-k$ for all $k$

**Case 1:**

$$\frac{x+1}{x^2+y^2+2} = k/-k$$

But, $x^2+y^2+2$ is strictly positive for all $x, y \in \mathbb{Z}$

**Case 2:**

$$\frac{x+1}{x^2+y^2+2} = -k/k$$

We get the following system of equations.

$$x+1 = -k \text{ and } x^2+y^2+2 = k$$

From $x+1 = -k$, we get $x = -(k+1)$. Substituting into the second equation:

$$x^2+y^2+2 = k \quad \Rightarrow \quad (-(k+1))^2+y^2+2 = k$$

$$(k+1)^2+y^2+2 = k \quad \Rightarrow \quad k^2+2k+1+y^2+2 = k$$

$$k^2+k+3+y^2 = 0$$

$$k^2+y^2+3 = -k$$

Since the left side, $k^2+y^2+3$ is always positive, and the right side is negative
we reach a contradiction. That is, a positive number cannot equal a negative number.
Hence, no such integers $x, y$ exist for which $f(x,y) = -1$. Therefore, $-1 \notin \text{Im}(f)$, so $f$ is not surjective.

**Conclusion:** The function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$, defined by $f(x,y) = \dfrac{x+1}{x^2+y^2+2}$, is neither injective nor surjective.

**2.**

Consider the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x,y) = xy$.

**Injective:**

*Counterexample:* Let $(x_1, y_1) = (2,3)$ and $(x_2, y_2) = (1,6)$. Then

$$f(2,3) = 6 = f(1,6),$$

but $(2,3) \neq (1,6)$. So the function is **not injective**.

**Surjective:**

Let $r \in \mathbb{R}$. We want to find $(x,y) \in \mathbb{R} \times \mathbb{R}$ such that $xy = r$.

*Case 1:* If $r = 0$, then $f(0, y) = 0$ for any $y$, so 0 is in the image.
*Case 2:* If $r \neq 0$, let $x = r$, $y = 1$. Then $f(x, y) = r \cdot 1 = r$.
Hence, every real number has a preimage, so the function is **surjective**.

**Conclusion:** The function $f(x, y) = xy$ is **not injective** but **surjective**.

**3.**
Consider the function $f : \mathbb{R} \to (0, 1)$ defined by

$$f(x) = \frac{x^2}{1 + x^2}.$$

**Injective:**
*Counterexample:*

$$f(1) = \frac{1}{2} = f(-1),$$

but $1 \neq -1$. Therefore, the function is **not injective**.
**Surjective:**
Let $y \in (0, 1)$. We want to find $x \in \mathbb{R}$ such that

$$f(x) = \frac{x^2}{1 + x^2} = y.$$

Evaluate $f(\pm\sqrt{\frac{y}{1-y}})$

$$f(\pm\sqrt{\frac{y}{1-y}}) = \frac{(\pm\sqrt{\frac{y}{1-y}})^2}{1 + (\pm\sqrt{\frac{y}{1-y}})^2}$$

$$f\left(\pm\sqrt{\frac{y}{1-y}}\right) = \frac{\left(\frac{y}{1-y}\right)}{1 + \left(\frac{y}{1-y}\right)}.$$

Simplify the denominator:

$$1 + \frac{y}{1-y} = \frac{(1-y) + y}{1-y} = \frac{1}{1-y}.$$

So the entire expression becomes:

$$\frac{\frac{y}{1-y}}{\frac{1}{1-y}} = y.$$

**Therefore,**

$$f\left(\pm\sqrt{\frac{y}{1-y}}\right) = y.$$

Thus, for any $y \in [0, 1)$, there exists $x \in \mathbb{R}$ (namely $x = \pm\sqrt{\frac{y}{1-y}}$) such that $f(x) = y$.

Moreover, note that $\frac{y}{1-y} > 0$ for all $y \in [0,1)$, so the square root is real. $\pm\sqrt{\frac{y}{1-y}} \in \mathbb{R}$, since the argument of the square root greater than 0.

**Therefore, the function is surjective.**

**Conclusion:** The function $f(x) = \frac{x^2}{1+x^2}$ is **not injective** but **surjective**.

$\square$

**Problem 4** ([).

*4] **Q4.** Let $X$ be a set and let $P(X)$ denote the set of all subsets of $X$. Prove that there does not exist a surjective function $f : X \to P(X)$. (Hint: $A = \{x \in X : x \notin f(x)\}$.)*

**Solution.**

Assume for contradiction that:
$f : X \to P(X)$ is surjective.
For every subset $A \subseteq X$, $\exists a \in X$ such that $f(a) = A$.
Let

$$A = \{x \in X \mid x \notin f(x)\}.$$

This is a well-defined subset of $X$, so $A \in P(X)$.
Since $f$ is surjective, $\exists a \in X$ such that $f(a) = A$.
Suppose $a \notin A$. By definition of $A$,

$$a \notin A \Rightarrow a \in f(a).$$

But $f(a) = A$, so

$$a \in A,$$

a contradiction.
Suppose $a \in A$. Then by definition of $A$,

$$a \in A \Rightarrow a \notin f(a).$$

Since $f(a) = A$, this means

$$a \notin A,$$

again a contradiction.
Thus, in both cases we get a contradiction. Therefore $f$ is not surjective. $\square$

**Problem 5** ([).

*5] A nonempty subset $X$ of $\mathbb{Z}$ is said to be sticky if for all $x, y \in X$ and $n \in \mathbb{Z}$, $x + y \in X$ and $xn \in X$.*

***(a)** Fix $n \in \mathbb{Z}$. Find the smallest sticky subset of $\mathbb{Z}$ which contains $n$.*

***(b)** Fix $n \in \mathbb{Z}$, different from $\pm 1$. Find all sticky subsets $X \subseteq \mathbb{Z}$ such that (1) $X \neq \mathbb{Z}$; (2) $n \in X$; (3) whenever $X \subseteq Y \subsetneq \mathbb{Z}$, then $X = Y$. How many sticky subsets can you find?*

**Solution.**

**(a)**

Given $n \in \mathbb{Z}$, define
$$X_n = \{kn \mid k \in \mathbb{Z}\} = n\mathbb{Z}.$$

We claim that $X_n$ is the smallest sticky subset of $\mathbb{Z}$ containing $n$.

*Proof that $X_n$ is sticky:*

Let $x = an$ and $y = bn$ be arbitrary elements of $X_n$ with $a, b \in \mathbb{Z}$.

Then:
$$x + y = an + bn = (a + b)n \in X_n$$
and for any $m \in \mathbb{Z}$:
$$xm = anm = (am)n \in X_n.$$

Thus, $X_n$ is closed under addition and multiplication by any integer, so $X_n$ is sticky.

*Minimality of $X_n$:*

Suppose $Y$ is any sticky subset of $\mathbb{Z}$ containing $n$. Then:

- Since $n \in Y$ and $Y$ is sticky, for any integer $k$, by repeatedly adding $n$ to itself (closure under addition), we get $kn \in Y$.
- Also, closure under multiplication by any integer implies all integer multiples of $n$ are in $Y$.

Hence $n\mathbb{Z} \subseteq Y$.

Therefore, the smallest sticky subset of $\mathbb{Z}$ containing $n$ is $n\mathbb{Z}$.

**(b)** Fix $n \in \mathbb{Z}$ with $n \neq \pm 1$.

We are asked to find all proper sticky subsets $X \subset \mathbb{Z}$ satisfying:

(i) $X \neq \mathbb{Z}$,

(ii) $n \in X$,

(iii) $X$ is maximal (under inclusion) among all proper sticky subsets of $\mathbb{Z}$ containing $n$.

*Recall:* A subset $X \subset \mathbb{Z}$ is **sticky** if:

- $x, y \in X \Rightarrow x + y \in X$ (closed under addition),
- $x \in X, m \in \mathbb{Z} \Rightarrow mx \in X$ (closed under integer multiplication).

It is known that the sticky subsets of $\mathbb{Z}$ are exactly the additive subgroups of $\mathbb{Z}$, which are all of the form
$$d\mathbb{Z} = \{dk \mid k \in \mathbb{Z}\}$$
for some $d \in \mathbb{N}$.

We seek all such $d\mathbb{Z} \subsetneq \mathbb{Z}$ such that:

- $n \in d\mathbb{Z} \iff d \mid n$, and
- $d\mathbb{Z}$ is maximal (under inclusion) among all proper sticky subsets containing $n$.

Since $d_1\mathbb{Z} \subset d_2\mathbb{Z} \iff d_2 \mid d_1$, maximality of $d\mathbb{Z}$ among proper sticky subsets means there is no smaller $d' > 1$ with $d' \mid n$ and $d' \mid d$.

In other words, $d$ must be **minimal** among all positive divisors of $n$ greater than 1.

These minimal positive divisors of $n$ are exactly the **distinct prime divisors** of $n$.

**Conclusion:** Let $p_1, \ldots, p_k$ be the distinct prime divisors of $n$. Then the sticky subsets $X \subsetneq \mathbb{Z}$ satisfying the given conditions are:

$$X = p_i\mathbb{Z} \quad \text{for each prime } p_i \mid n.$$

There are exactly $k$ such subsets, one for each distinct prime divisor of $n$.

□

**Problem 6** ([).

*6] **Q6.** Suppose that for each natural number $n$, we make a statement $P(n)$ (which can either be true or false). The Principle of Mathematical Induction tells us that:*

1. *If $P(1)$ is true, and*
2. *If for any natural number $k$, $P(k)$ being true implies that $P(k+1)$ is also true,*

*then $P(n)$ is true for all $n \in \mathbb{N}$.*

*Find the flaw in the following proof by mathematical induction, which seems to suggest that all real numbers are equal.*

*Let $P(n)$ be the statement that in any set of real numbers $\{x_1, x_2, \ldots, x_n\}$, all of the numbers have the same value.*

*It is clear that $P(1)$ is true. Now assume that $P(k)$ is true for some natural number $k$. Consider any set of $(k+1)$ real numbers $\{x_1, x_2, \ldots, x_{k+1}\}$.*

*If we remove $x_1$, then we have a set of $k$ numbers, so by $P(k)$, we have that*

$$x_2 = x_3 = \cdots = x_{k+1}.$$

*If, instead, we remove $x_k$, then we again have a set of $k$ numbers, so*

$$x_1 = x_2 = \cdots = x_{k-1} = x_k.$$

*Combining these two results, we see that*

$$x_1 = x_2 = x_3 = \cdots = x_{k+1},$$

*proving that $P(k+1)$ is true. By the Principle of Mathematical Induction, $P(n)$ is true for all natural numbers $n$.*

**Solution.**

We are asked to identify the flaw in a proof that falsely claims all real numbers are equal, using mathematical induction on the following statement:

$$P(n)\text{: "In any set of } n \text{ real numbers, all the numbers are equal."}$$

**Base Case ($n = 1$):** The statement $P(1)$ is true. Any set with a single real number trivially has all elements equal.

**Inductive Step:** The proof assumes $P(k)$ is true for some $k \in \mathbb{N}$ and attempts to prove $P(k+1)$. The idea is to take a set of $k+1$ real numbers $\{x_1, x_2, \ldots, x_{k+1}\}$ and consider two overlapping subsets of size $k$:

- Removing $x_1$ gives the set $\{x_2, x_3, \ldots, x_{k+1}\}$. By the inductive hypothesis, all these values are equal:
$$x_2 = x_3 = \cdots = x_{k+1}.$$
- Removing $x_{k+1}$ gives the set $\{x_1, x_2, \ldots, x_k\}$. Again by the inductive hypothesis,
$$x_1 = x_2 = \cdots = x_k.$$

Since both subsets include $x_2$, the proof claims this common value links the two equalities and implies
$$x_1 = x_2 = \cdots = x_{k+1},$$
and thus concludes that $P(k+1)$ is true.

**Flaw in the Argument:**

This reasoning appears valid for $k \geq 2$, but fails when $k = 1$, which is exactly the first step of the induction.

To illustrate the failure:

- Let $k = 1$. Then we try to prove $P(2)$: "Any set of two real numbers are equal."
- The set is $\{x_1, x_2\}$.
- Removing $x_1$ leaves $\{x_2\}$, which trivially satisfies $P(1)$, so no new information is obtained.
- Removing $x_2$ leaves $\{x_1\}$, which again trivially satisfies $P(1)$.

However, since these two subsets have no elements in common (they are disjoint), we cannot compare or "link" their values. There's no overlapping element (like $x_2$ in the general case) to ensure consistency between the values.

As a result, we cannot conclude that $x_1 = x_2$. This breaks the inductive step at $k = 1$, and therefore the entire induction argument collapses.

**Conclusion:** The flaw lies in the assumption that the two overlapping subsets used in the inductive step always share a common element. This is not true when $k = 1$, making the inductive step invalid and the proof incorrect. Hence, the conclusion that all real numbers are equal is false. □

**Problem 7** **([).**

*7] **Q7.** Let $a_n > 0$ be a positive real number for each natural number $n$ and define $S_n = a_1 + a_2 + \cdots + a_n$. It is clear that $0 \leq S_n \leq S$, for some $S \in \mathbb{R}$.*

*Find examples of $a_n$ satisfying the following conditions:*

***(a)*** *The set $\{S_n\}$ is not bounded above, i.e., for any real number $M$, there exists $n \in \mathbb{N}$ such that $S_n > M$.*

***(b)*** *The set $\{S_n\}$ is bounded above, i.e., there exists a fixed number $S$ such that $S_n \leq S$ for all $n \in \mathbb{N}$.*

**Solution.**

**(a)** We need to show that the sequence of partial sums $S_n$ diverges to infinity.

Let $a_n = 1$ for all $n$. Then, the partial sum is given by

$$S_n = \sum_{k=1}^{n} 1 = n.$$

This sequence grows without bound:

$$S_1 = 1, \quad S_2 = 2, \quad S_3 = 3, \quad \dots, \quad S_n = n \quad \text{as} \quad n \to \infty.$$

Thus, this sequence is not bounded above.

**(b)** The set $\{S_n\}$ is bounded above, i.e., there exists a fixed number $S$ such that $S_n \leq S$ for all $n \in \mathbb{N}$.

*Solution:* We need the infinite series to converge.

Let $a_n = \frac{1}{2^n}$ for all $n$. Then, the partial sum is:

$$S_n = \sum_{k=1}^{n} \frac{1}{2^k}.$$

The infinite series is given by

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

This series converges to 1, so $S_n \leq 1$ for all $n$, and $S_n$ converges to 1. $\square$