Advanced Practice Problems

MATH 145 / MATH 147

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Problem 1 ([).

1] We say that a set X of real numbers is bounded above *if there exists a real number α such that $x \leq \alpha$ for all $x \in X$. Similarly, X is* bounded below *if there exists a real number β such that $x \geq \beta$ for all $x \in X$.*

Let S be the set of all real numbers satisfying

$$x^3 + 2x < 4.$$

Is the set bounded above? Is it bounded below? Prove your answers.

Solution.

We can define S as

$$S = \{x \in \mathbb{R} \mid x^3 + 2x - 4 < 0\}.$$

Let

$$g(x) = x^3 + 2x - 4.$$

Since $g(x)$ is an odd polynomial with a positive leading coefficient, it has end behaviour from $QIII$ to QI. Thus, there exists some r such that $g(r) = 0$.

We check if g is strictly increasing by considering

$$g(x) - g(y) = (x^3 + 2x) - (y^3 + 2y) = (x - y)(x^2 + xy + y^2) + 2(x - y) = (x - y)(x^2 + xy + y^2 + 2).$$

Because for any real x, y, the term $x^2 + xy + y^2 + 2$ is always positive, and if $x > y$ then $(x - y) > 0$, it follows that

$$g(x) - g(y) > 0 \quad \text{whenever } x > y.$$

This shows g is strictly increasing.

Since g is strictly increasing and $g(r) = 0$, it follows that

$$g(x) < 0 \quad \text{for } x < r, \quad \text{and} \quad g(x) > 0 \quad \text{for } x > r.$$

Thus,

$$S = \{x \in \mathbb{R} : x < r\}.$$

Therefore, S is bounded above by r, but since it contains arbitrarily large negative numbers, it is not bounded below. $\square$

Problem 2 ([).

2]

The complex numbers $\mathbb{C}$ are a number system which, as a set, is the set of all ordered pairs of real numbers $\mathbb{C} = \{(a,b) \mid a,b \in \mathbb{R}\}$. The ordered pair (a,b) is usually written $a+bi$. Complex numbers can be added, subtracted, multiplied, and divided as follows.

- *Addition:*
$$(a+bi)+(c+di) = (a+c)+(b+d)i.$$
- *Subtraction:*
$$(a+bi)-(c+di) = (a-c)+(b-d)i.$$
- *Multiplication:*
$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i.$$
- *Division (for $c+di \neq 0$):*
$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}.$$

Prove that there is no order relation "$<$" on $\mathbb{C}$ that satisfies the five order axioms:

1. *For any $a,b \in \mathbb{C}$, exactly one of the following holds: $a<b$, $a=b$, or $b<a$.*
2. *For any $a,b,c \in \mathbb{C}$, if $a<b$ and $b<c$, then $a<c$.*
3. *For any $a,b,c \in \mathbb{C}$, if $a<b$, then $a+c<b+c$.*
4. *For any $a,b,c \in \mathbb{C}$, if $a<b$ and $0<c$, then $ac<bc$.*
5. *For any $a,b,c \in \mathbb{C}$, if $a<b$ and $c<0$, then $bc<ac$.*

Solution.

We proceed by contradiction.

Suppose there exists an order relation "$<$" on $\mathbb{C}$ satisfying the five order axioms above, making $(\mathbb{C},+,\cdot,<)$ an ordered field.

Consider the imaginary unit $i \in \mathbb{C}$ where $i^2=-1$.

By axiom (1), exactly one of the following must hold:

$$i>0, \quad i=0, \quad \text{or} \quad i<0.$$

We know $i \neq 0$, so either $i>0$ or $i<0$.

Case 1: Suppose $i>0$.

Then by axiom (4), let $a=0$, $b=i$, $c=i$

$$0 \cdot i < i \cdot i.$$

This implies $0<-1$.

Case 2: Suppose $i < 0$.
Then $-i > 0$ and by the same reasoning

$$0 < (-i)^2 = (-1)^2(i)^2 = -1$$

In both cases, we conclude

$$0 < -1.$$

However, in any ordered field, $1 > 0$ (because $1 = 1$ and is the multiplicative identity). Adding inequalities preserves order by axiom (3), so

$$-1 > 0 \implies (-1) + 1 > 0 + 1 \implies 0 > 1,$$

which contradicts the fact that $1 > 0$.

Therefore, our assumption that such an order relation exists on $\mathbb{C}$ leads to a contradiction.

Conclusion: There is no order relation "<" on the complex numbers $\mathbb{C}$ satisfying all five axioms of an ordered field. □

Problem 3 ([).

*3] **Q3.** Given two sets, A and B, the Cartesian product $A \times B$ is defined as the set of all ordered tuples (a, b), where $a \in A$ and $b \in B$. Using the set notation, this can be written as*

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Let A and B be sets. A function from A to B, denoted by $f : A \to B$, is a subset f of $A \times B$ such that for all $a \in A$ there exists a unique $b \in B$ such that $(a, b) \in f$. We write $b = f(a)$ and call the b the image of a under f. We say $f : A \to B$ is injective if for all $b \in B$ there exists at most one $a \in A$ such that $f(a) = b$. We say $f : A \to B$ is surjective if $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$. Finally, $f : A \to B$ is bijective if it is both injective and surjective. Determine if the following functions are injective and/or surjective.

1. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$, $f(x, y) = \dfrac{x + 1}{x^2 + y^2 + 2}$
2. $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(x, y) = xy$
3. $f : \mathbb{R} \to (0, 1)$, $f(x) = \dfrac{x^2}{1 + x^2}$

Solution.

1.

(a) Injective

We want to check if the function is injective. The function f is injective if whenever $f(x_1, y_1) = f(x_2, y_2)$, it implies that $(x_1, y_1) = (x_2, y_2)$.

Counterexample: Let $f(0, 0) = \frac{1}{2}$ and $f(1, 1) = \frac{1}{2}$, but $(0, 0) \neq (1, 1)$.

Thus, the function is **not injective**.

Surjective

We need to check if every rational number is attainable.

$$f(x,y)=\frac{x+1}{x^2+y^2+2}, x,y\in\mathbb{Z}$$

Let $f(x,y)=-1$. -1 can be written as $-k/k$ or $k/-k$ for all k
Case 1:

$$\frac{x+1}{x^2+y^2+2}=k/-k$$

But, x^2+y^2+2 is strictly positive for all $x,y\in\mathbb{Z}$
Case 2:

$$\frac{x+1}{x^2+y^2+2}=-k/k$$

We get the following system of equations.

$$x+1=-k \text{ and } x^2+y^2+2=k$$

From $x+1=-k$, we get $x=-(k+1)$. Substituting into the second equation:

$$x^2+y^2+2=k \quad\Rightarrow\quad (-(k+1))^2+y^2+2=k$$

$$(k+1)^2+y^2+2=k \quad\Rightarrow\quad k^2+2k+1+y^2+2=k$$

$$k^2+k+3+y^2=0$$

$$k^2+y^2+3=-k$$

Since the left side, k^2+y^2+3 is always positive, and the right side is negative
we reach a contradiction. That is, a positive number cannot equal a negative number.
Hence, no such integers x,y exist for which $f(x,y)=-1$. Therefore, $-1\notin \mathrm{Im}(f)$, so f is not surjective.

Conclusion: The function $f:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Q}$, defined by $f(x,y)=\dfrac{x+1}{x^2+y^2+2}$, is neither injective nor surjective.

2.

Consider the function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ given by $f(x,y)=xy$.

Injective:

Counterexample: Let $(x_1,y_1)=(2,3)$ and $(x_2,y_2)=(1,6)$. Then

$$f(2,3)=6=f(1,6),$$

but $(2,3)\neq(1,6)$. So the function is **not injective**.

Surjective:

Let $r\in\mathbb{R}$. We want to find $(x,y)\in\mathbb{R}\times\mathbb{R}$ such that $xy=r$.

Case 1: If $r = 0$, then $f(0, y) = 0$ for any y, so 0 is in the image.
Case 2: If $r \neq 0$, let $x = r$, $y = 1$. Then $f(x, y) = r \cdot 1 = r$.
Hence, every real number has a preimage, so the function is **surjective**.

Conclusion: The function $f(x, y) = xy$ is **not injective** but **surjective**.

3.
Consider the function $f : \mathbb{R} \to (0, 1)$ defined by

$$f(x) = \frac{x^2}{1 + x^2}.$$

Injective:
Counterexample:

$$f(1) = \frac{1}{2} = f(-1),$$

but $1 \neq -1$. Therefore, the function is **not injective**.
Surjective:
Let $y \in (0, 1)$. We want to find $x \in \mathbb{R}$ such that

$$f(x) = \frac{x^2}{1 + x^2} = y.$$

Evaluate $f(\pm\sqrt{\frac{y}{1-y}})$

$$f(\pm\sqrt{\frac{y}{1-y}}) = \frac{(\pm\sqrt{\frac{y}{1-y}})^2}{1 + (\pm\sqrt{\frac{y}{1-y}})^2}$$

$$f\left(\pm\sqrt{\frac{y}{1-y}}\right) = \frac{\left(\frac{y}{1-y}\right)}{1 + \left(\frac{y}{1-y}\right)}.$$

Simplify the denominator:

$$1 + \frac{y}{1-y} = \frac{(1-y) + y}{1-y} = \frac{1}{1-y}.$$

So the entire expression becomes:

$$\frac{\frac{y}{1-y}}{\frac{1}{1-y}} = y.$$

Therefore,

$$f\left(\pm\sqrt{\frac{y}{1-y}}\right) = y.$$

surjective.

Conclusion: The function $f(x) = \frac{x^2}{1+x^2}$ is **not injective** but **surjective**.

□

Problem 4 ([]).

*4] **Q4.** Let X be a set and let $P(X)$ denote the set of all subsets of X. Prove that there does not exist a surjective function $f : X \to P(X)$. (Hint: $A = \{x \in X : x \notin f(x)\}$.)*

Solution.

Assume for contradiction that:
$f : X \to P(X)$ is surjective.
For every subset $A \subseteq X$, $\exists a \in X$ such that $f(a) = A$.
Let

$$A = \{x \in X \mid x \notin f(x)\}.$$

This is a well-defined subset of X, so $A \in P(X)$.
Since f is surjective, $\exists a \in X$ such that $f(a) = A$.
Suppose $a \notin A$. By definition of A,

$$a \notin A \Rightarrow a \in f(a).$$

But $f(a) = A$, so

$$a \in A,$$

a contradiction.
Suppose $a \in A$. Then by definition of A,

$$a \in A \Rightarrow a \notin f(a).$$

Since $f(a) = A$, this means

$$a \notin A,$$

again a contradiction.
Thus, in both cases we get a contradiction. Therefore f is not surjective. □

Problem 5 ([]).

5] A nonempty subset X of $\mathbb{Z}$ is said to be sticky if for all $x, y \in X$ and $n \in \mathbb{Z}$, $x + y \in X$ and $xn \in X$.

***(a)** Fix $n \in \mathbb{Z}$. Find the smallest sticky subset of $\mathbb{Z}$ which contains n.*

***(b)** Fix $n \in \mathbb{Z}$, different from ± 1. Find all sticky subsets $X \subseteq \mathbb{Z}$ such that (1) $X \neq \mathbb{Z}$; (2) $n \in X$; (3) whenever $X \subseteq Y \subsetneq \mathbb{Z}$, then $X = Y$. How many sticky subsets can you find?*

Solution.

(a)
Given $n \in \mathbb{Z}$, define

$$X_n = \{kn \mid k \in \mathbb{Z}\} = n\mathbb{Z}.$$

We claim that X_n is the smallest sticky subset of $\mathbb{Z}$ containing n.
Proof that X_n is sticky:

Let $x = an$ and $y = bn$ be arbitrary elements of X_n with $a, b \in \mathbb{Z}$.

Then:

$$x + y = an + bn = (a + b)n \in X_n$$

and for any $m \in \mathbb{Z}$:

$$xm = anm = (am)n \in X_n.$$

Thus, X_n is closed under addition and multiplication by any integer, so X_n is sticky.

Minimality of X_n:

Suppose Y is any sticky subset of $\mathbb{Z}$ containing n. Then:

- Since $n \in Y$ and Y is sticky, for any integer k, by repeatedly adding n to itself (closure under addition), we get $kn \in Y$.
- Also, closure under multiplication by any integer implies all integer multiples of n are in Y.

Hence $n\mathbb{Z} \subseteq Y$.

Therefore, the smallest sticky subset of $\mathbb{Z}$ containing n is $n\mathbb{Z}$.

(b) Fix $n \in \mathbb{Z}$ with $n \neq \pm 1$.

We are asked to find all proper sticky subsets $X \subset \mathbb{Z}$ satisfying:

(i) $X \neq \mathbb{Z}$,

(ii) $n \in X$,

(iii) X is maximal (under inclusion) among all proper sticky subsets of $\mathbb{Z}$ containing n.

Recall: A subset $X \subset \mathbb{Z}$ is **sticky** if:

- $x, y \in X \Rightarrow x + y \in X$ (closed under addition),
- $x \in X, m \in \mathbb{Z} \Rightarrow mx \in X$ (closed under integer multiplication).

It is known that the sticky subsets of $\mathbb{Z}$ are exactly the additive subgroups of $\mathbb{Z}$, which are all of the form

$$d\mathbb{Z} = \{dk \mid k \in \mathbb{Z}\}$$

for some $d \in \mathbb{N}$.

We seek all such $d\mathbb{Z} \subsetneq \mathbb{Z}$ such that:

- $n \in d\mathbb{Z} \iff d \mid n$, and
- $d\mathbb{Z}$ is maximal (under inclusion) among all proper sticky subsets containing n.

Since $d_1\mathbb{Z} \subset d_2\mathbb{Z} \iff d_2 \mid d_1$, maximality of $d\mathbb{Z}$ among proper sticky subsets means there is no smaller $d' > 1$ with $d' \mid n$ and $d' \mid d$.

In other words, d must be **minimal** among all positive divisors of n greater than 1.

These minimal positive divisors of n are exactly the **distinct prime divisors** of n.

Conclusion: Let $p_1, \ldots, p_k$ be the distinct prime divisors of n. Then the sticky subsets $X \subsetneq \mathbb{Z}$ satisfying the given conditions are:

$$X = p_i\mathbb{Z} \quad \text{for each prime } p_i \mid n.$$

There are exactly k such subsets, one for each distinct prime divisor of n.

□

Problem 6 ([).

*6] **Q6.** Suppose that for each natural number n, we make a statement $P(n)$ (which can either be true or false). The Principle of Mathematical Induction tells us that:*

1. *If $P(1)$ is true, and*
2. *If for any natural number k, $P(k)$ being true implies that $P(k+1)$ is also true,*

then $P(n)$ is true for all $n \in \mathbb{N}$.

Find the flaw in the following proof by mathematical induction, which seems to suggest that all real numbers are equal.

Let $P(n)$ be the statement that in any set of real numbers $\{x_1, x_2, \ldots, x_n\}$, all of the numbers have the same value.

It is clear that $P(1)$ is true. Now assume that $P(k)$ is true for some natural number k. Consider any set of $(k+1)$ real numbers $\{x_1, x_2, \ldots, x_{k+1}\}$.

If we remove x_1, then we have a set of k numbers, so by $P(k)$, we have that

$$x_2 = x_3 = \cdots = x_{k+1}.$$

If, instead, we remove x_k, then we again have a set of k numbers, so

$$x_1 = x_2 = \cdots = x_{k-1} = x_k.$$

Combining these two results, we see that

$$x_1 = x_2 = x_3 = \cdots = x_{k+1},$$

proving that $P(k+1)$ is true. By the Principle of Mathematical Induction, $P(n)$ is true for all natural numbers n.

Solution.

We are asked to identify the flaw in a proof that falsely claims all real numbers are equal, using mathematical induction on the following statement:

$$P(n)\text{: "In any set of } n \text{ real numbers, all the numbers are equal."}$$

Base Case ($n = 1$): The statement $P(1)$ is true. Any set with a single real number trivially has all elements equal.

Inductive Step: The proof assumes $P(k)$ is true for some $k \in \mathbb{N}$ and attempts to prove $P(k+1)$. The idea is to take a set of $k+1$ real numbers $\{x_1, x_2, \ldots, x_{k+1}\}$ and consider two overlapping subsets of size k:

- Removing x_1 gives the set $\{x_2, x_3, \ldots, x_{k+1}\}$. By the inductive hypothesis, all these values are equal:
$$x_2 = x_3 = \cdots = x_{k+1}.$$
- Removing x_{k+1} gives the set $\{x_1, x_2, \ldots, x_k\}$. Again by the inductive hypothesis,
$$x_1 = x_2 = \cdots = x_k.$$

Since both subsets include x_2, the proof claims this common value links the two equalities and implies
$$x_1 = x_2 = \cdots = x_{k+1},$$
and thus concludes that $P(k+1)$ is true.

Flaw in the Argument:

This reasoning appears valid for $k \geq 2$, but fails when $k = 1$, which is exactly the first step of the induction.

To illustrate the failure:

- Let $k = 1$. Then we try to prove $P(2)$: "Any set of two real numbers are equal."
- The set is $\{x_1, x_2\}$.
- Removing x_1 leaves $\{x_2\}$, which trivially satisfies $P(1)$, so no new information is obtained.
- Removing x_2 leaves $\{x_1\}$, which again trivially satisfies $P(1)$.

However, since these two subsets have no elements in common (they are disjoint), we cannot compare or "link" their values. There's no overlapping element (like x_2 in the general case) to ensure consistency between the values.

As a result, we cannot conclude that $x_1 = x_2$. This breaks the inductive step at $k = 1$, and therefore the entire induction argument collapses.

Conclusion: The flaw lies in the assumption that the two overlapping subsets used in the inductive step always share a common element. This is not true when $k = 1$, making the inductive step invalid and the proof incorrect. Hence, the conclusion that all real numbers are equal is false. $\square$

Problem 7 ([]).

*7] **Q7.** Let $a_n > 0$ be a positive real number for each natural number n and define $S_n = a_1 + a_2 + \cdots + a_n$. It is clear that $0 \leq S_n \leq S$, for some $S \in \mathbb{R}$.*

Find examples of a_n satisfying the following conditions:

***(a)** The set $\{S_n\}$ is not bounded above, i.e., for any real number M, there exists $n \in \mathbb{N}$ such that $S_n > M$.*

***(b)** The set $\{S_n\}$ is bounded above, i.e., there exists a fixed number S such that $S_n \leq S$ for all $n \in \mathbb{N}$.*

Solution.

(a) We need to show that the sequence of partial sums S_n diverges to infinity.

Let $a_n = 1$ for all n. Then, the partial sum is given by

$$S_n = \sum_{k=1}^{n} 1 = n.$$

This sequence grows without bound:

$$S_1 = 1, \quad S_2 = 2, \quad S_3 = 3, \quad \ldots, \quad S_n = n \quad \text{as} \quad n \to \infty.$$

Thus, this sequence is not bounded above.

(b) The set $\{S_n\}$ is bounded above, i.e., there exists a fixed number S such that $S_n \leq S$ for all $n \in \mathbb{N}$.

Solution: We need the infinite series to converge.

Let $a_n = \frac{1}{2^n}$ for all n. Then, the partial sum is:

$$S_n = \sum_{k=1}^{n} \frac{1}{2^k}.$$

The infinite series is given by

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

This series converges to 1, so $S_n \leq 1$ for all n, and S_n converges to 1. $\square$