Rank aggregation and Kemeny compatible functions

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Abstract

Rank aggregation is a method by which a consensus ranking (output) is computed from a (multi)set of rankings (input). The consensus ranking must be as representative as possible of the input multiset of rankings. The main goal is to best reflect the common points between the input rankings without overstating the elements considered as "relevant" by a few rankings. The problem of aggregating rankings has been investigated in many communities : databases, algorithmics, artificial intelligence, social science. According to the context and the elements to be ranked, the multiset of rankings may have different features. The input rankings may be complete or not (whether or not each input ranking contains the exact same set of elements) and may have tied elements (i.e., elements placed at the same rank in a given ranking). Several scoring functions have been introduced in the literature, each has been defined to fit with the kind of input rankings to be aggregated and the context. We present in this report a new family of functions covering all the traditionally used measures in the context of rank aggregation.

1 Introduction

Aggregating several rankings to get a consensus of the input rankings is an important problem investigated since the eighteenth century in the context of elections (the voters had to rank a set of candidates). Nowadays, rank aggregation has applications in many contexts : *metasearch* for the Web ([DKNS01]), databases ([DKNS01], [FKM$^+$04], [SAYBD$^+$13]]), artificial intelligence ([PHG00]), social choice theory ([CFR10])... As the problem of aggregating rankings is known to be NP-Hard ([DKNS01], [BBG$^+$17]), many approximations and heuristics have been introduced so far([Ail10], [ACN08], [FKM$^+$04]). Most communities have exclusively investigated the context in which the input rankings are permutations of the same elements (that is, there are no tied elements in any ranking and all rankings have the exact same set of elements). Some others have started to consider rankings with ties ([FKM$^+$04], [BYB$^+$15], in which several elements can have the same rank in the same ranking.

However, real datasets are rarely complete. Indeed, it is very often the case that all the rankings to be aggregated do not contain the same set of elements. This point must get our attention as we must wonder how missing elements must be interpreted, according to the context. Different contexts can lead to different interpretations of missing elements; this requires the use of appropriated measures to aggregate the rankings. For each interpretation, a measure (*scoring function*) has been designed to aggregate rankings ([Bra15]). This lead to a bunch of different measures, one for each interpretation. We present in this report a new family of functions covering all the generalizations of the Kendall-τ distance (traditionally used in the context of rank aggregation) to fit with as many contexts as possible. Gathering all the current available scoring functions into one family of functions enables a generalization of several existing algorithms (including an exact algorithm). The generalization for any kind of dataset and currently available scoring function.

We will introduce first some definitions about rankings, and then about rank aggregation. We will review the currently used scoring functions and the context in which each of them must be used. Finally we will present the family of functions covering all the functions previously introduced.

2 Definitions related to rankings

Let us call U (for *Universe*) a set of elements to rank. The elements of U may be candidates for election, URL (web search engines and *metasearch*, [DKNS01]), genes (biological context, [BRDCB14]), athletes (sports, [BBN13]), movies ... According to the context, the rankings may have tied elements ([FKM$^+$04], [BYB$^+$15]). A ranking can also be complete or incomplete towards U, whether each element of U is in the ranking or not.

2.1 Rankings : several features

A ranking on U is a list of subsets of U such that any two subsets are disjoint.
A complete ranking on U is a ranking on U such that the union of its subsets is equal to U.
A ranking without ties is a ranking such that all the subsets have only one element.
For the sake of readability, we will sometimes call *permutation of* U a complete ranking without ties on U.

2.2 Rank of an element

Let $r = [P_1, P_2, ..., P_k]$ be a ranking . The *rank of x in r*, denoted $r[x]$, is :

$$r[x] = \begin{cases} 0 & \text{if } \forall\, 1 \leq i \leq k, x \notin P_i \\ 1 + \sum_{j=1}^{\rho-1} |P_j| & \text{, where } \rho \text{ is such that } x \in P_\rho \end{cases}$$

For example, in the following ranking $r = [\{A\}, \{B, C\}, \{D\}]$, we have $r[A] = 1$, $r[B] = 2$, $r[C] = 2$, $r[D] = 4$ and $r[E] = 0$.

Finally, if r is a ranking, we set *el(r)* = $\{x \mid r[x] \neq 0\}$.

2.3 Unification process

Let r be an incomplete ranking on U. r' is the *unified ranking of r* if r' is a complete ranking and $\forall x \in U$,

$$r'[x] = \begin{cases} r[x] & \text{if } x \in c \\ 1 + \sum_{y \in U} \mathbf{1}_{y \in r} & \text{if } x \notin c \end{cases}$$

The ranking r' is the ranking r in which an additional set, called *unification bucket*, has been added at the end. The unification bucket contains all the elements of r' which are not in r. For example, if $U = \{A, B, C, D, E\}$ and $r = \{A\}, \{D\}$, then the unified ranking of r is $r' = [\{A\}, \{D\}, \{B, C, E\}]$.

3 Rank aggregation

As already noticed in the introduction, rank aggregation has mainly been investigated when the rankings to aggregate are permutations of a same set ([DKNS01], [Cop51], [dB81], [BBG$^+$17]). In this situation, to quantify the dissimilarity between two permutations, the *Kendall-τ distance* is used. Several generalizations ([FKM$^+$04], [Bra15]) enable the aggregation of rankings with ties and incomplete rankings. We will first introduce the case when the rankings to aggregate are permutations of a same set and then we will introduce a more general framework.

Notations If U is a set,

- $P(U)$ is the set of all the permutations of U.
- $C(U)$ is the set of all the complete rankings on U (with or without ties).
- $A(U)$ is the set of all the rankings on U (complete or incomplete, with or without ties).

3.1 Aggregation of permutations

Definition 3.1 (Kendall-τ distance) *The Kendall-τ distance D is defined as follows: $\forall r \in P(U)$ and $s \in P(U)$,*

$$D(r, s) = |\{(x, y) \in U^2 : x < y \wedge (r[x] < r[y] \wedge s[x] > s[y] \vee r[x] > r[y] \wedge s[x] < s[y])\}| \quad (1)$$

Definition 3.2 (Kemeny score) *Let $\mathscr{R}$ be a multiset of permutations of U. The Kemeny score between $\pi \in P(U)$ and $\mathscr{R}$ is defined as follows :*

$$K(\pi, \mathscr{R}) = \sum_{\sigma \in \mathscr{R}} D(\pi, \sigma) \quad (2)$$

Intuitively, the Kemeny score is the scoring function which evaluates how much the consensus ranking (which must be a permutation of U) represents well the input rankings.

Definition 3.3 (Median or optimal consensus) *Let $\mathscr{R}$ be a multiset of permutations of U. A median of $\mathscr{R}$, also called optimal consensus, denoted m^*, is a ranking $c \in P(U)$ such that*

$$\forall \pi \in \mathscr{P},\ K(m^*, \mathscr{R}) \leq K(\pi, \mathscr{R}) \quad (3)$$

Complexity of the problem Finding a median is NP-hard when $|U| \geq 4$ and $|\mathscr{R}| \geq 7$ ([BBD09], [DKNS01], [BBG$^+$17]). Several points are still open, especially the case when $|\mathscr{R}| = 3$.

3.2 Rankings with ties

The rankings to aggregate may have tied elements. A generalization of the Kendall-τ distance and Kemeny score called respectively *Generalized Kendall-τ distance* and *Generalized Kemeny score* have been designed ([FKM$^+$04]) to handle ties (for complete rankings). This generalization brings in a parameter $p \in]0; 1]$, called *Kemeny parameter*. p is the cost to pay for (un)tying a pair of elements.

Definition 3.4 (Generalized Kendall-τ distance) *The Generalized Kendall-τ distance, noted G^p, is defined as follows: $\forall (r,s) \in C(U)^2$,*

$$\begin{aligned} G^p(r,s) = &|\{(x,y) \in U^2 : x < y \wedge (r[x] < r[y] \wedge s[x] > s[y] \vee r[x] > r[y] \wedge s[x] < s[y])\}| \\ &+ p * |\{(x,y) \in U^2 : x < y \wedge (r[x] = r[y] \wedge s[x] \neq s[y] \vee r[x] \neq r[y] \wedge s[x] = s[y])\}| \end{aligned} \tag{4}$$

Definition 3.5 (Generalized Kemeny score) *Let $\mathscr{R}$ be a multiset of complete rankings. The Generalized Kemeny score between $c \in C(U)$ and $\mathscr{R}$ is defined as follows:*

$$K^p(c, \mathscr{R}) = \sum_{r \in \mathscr{R}} G^p(c, r) \tag{5}$$

Definition 3.6 (Median or optimal consensus) *Let $\mathscr{R}$ be a multiset of complete rankings. A median of $\mathscr{R}$, also called optimal consensus, denoted m^*, is a ranking $c \in C(U)$ such that:*

$$\forall s \in C(U),\ K^p(m^*, \mathscr{R}) \leq K^p(s, \mathscr{R}) \tag{6}$$

Note: If U is a set of n elements, the number of possible consensus for $\mathscr{R} \subseteq C(U)$ (candidates to be a median) is the cardinality of $C(U)$. This number is called the *n-th ordered Bell number* ([Goo75]). This number grows exponentially with n. For example, if U contains 2 elements A and B, the cardinal of $C(U)$ is $3 : \{[\{A\},\{B\}],[\{B\},\{A\}],[\{A,B\}]\}$. If U contains 20 elements, the cardinal of $C(U)$ is 2.67×10^{23}.

As noted previously, the rankings that have to be aggregated may be incomplete. For example, if several Web search engines return their top-10 for a search, there is absolutely no reason for the set of the ten websites to be the same for each web search engine. This is why handling incomplete rankings is particularly important. We are now going to present two scoring functions used to aggregate incomplete rankings.

3.3 Handling incomplete rankings

There are several ways to handle incomplete rankings ([Bra15]). To choose an adapted scoring function, we must wonder if, in a ranking r, a missing element must be considered as less important to a present element regarding r.

Case 1: When missing elements are less important than the present ones In several situations, the missing elements must be considered as less important. For example, a website which has not been returned by a web search engine S must be considered as less important (at least from the point of view of S) than the websites that S has returned. In this situation, the unification process can be applied (Section 2.3) to complete the rankings. The Generalized Kendall-τ distance (definition 3.4) and the Generalized Kemeny score (definition 3.5) can be used on the resulting unified rankings.

Remove bias due to unification process Applying the unification process can lead to bias results. Indeed, in a ranking, adding all the missing elements in a same unification bucket arbitrary creates ties. To remove the bias, [BRDCB14] has introduced a pseudometric so that the missing elements can be considered as less important than the present elements without tying all the missing elements within a same unification bucket. In a same ranking, two missing elements are now considered as incomparable. The formal definition of the *Generalized Kendall-τ pseudometric* is the following one.

Definition 3.7 (Generalized Kendall-τ pseudometric) *Let p be the Kemeny parameter. The Generalized Kendall-τ pseudometric, noted M^p, is defined as follows: $\forall (r,s) \in A(U)^2$,*

$$\begin{aligned} M^p(r,s) = |\{(x,y) \in U^2 : x < y \wedge (x \in el(r) \vee y \in el(r)) \wedge (x \in el(s) \vee y \in el(s)) \\ \wedge (r[y] \wedge s[x] > s[y] \vee r[x] > r[y] \wedge s[x] < s[y])\}| \\ + p * |\{(x,y) \in U^2 : x < y \wedge (x \in el(r) \vee y \in el(r)) \wedge (x \in el(s) \vee y \in el(s)) \\ \wedge (r[x] = r[y] \wedge s[x] \neq s[y] \vee r[x] \neq r[y] \wedge s[x] = s[y])\}| \end{aligned} \tag{7}$$

Definition 3.8 (Genralized Kemeny score related to pseudometric) *Let $\mathscr{R}$ be a multiset of complete rankings. We define the Generalized Kemeny score related to pseudometric between $c \in C(U)$ and $\mathscr{R}$ is defined as follows:*

$$K^p_M(c, \mathscr{R}) = \sum_{r \in \mathscr{R}} M^p(c, r) \tag{8}$$

Definition 3.9 (Median or optimal consensus) *Let $\mathscr{R}$ be a multiset of complete rankings. A median or optimal consensus for $\mathscr{R}$, denoted m^*, is a ranking $c \in C(U)$ such that:*

$$\forall s \in C(U), K^p_M(m^*, \mathscr{R}) \leq K^p_M(s, \mathscr{R}) \tag{9}$$

Case 2: When missing elements and present elements cannot be compared In some other situations, it may not make sense to compare present and missing elements in a ranking. For example, a movie that has not been seen by a user should not been rated by this user. It does not mean that the user rated this movie negatively. Let us consider another example: a Formula 1 racing driver who could not take part of a race should not be ranked as if he did not arrive after the present drivers of this race. When it makes no sense to compare present and missing elements, we will consider the Generalized induced Kendall-τ measure.

Definition 3.10 (Generalized induced Kendall-τ measure) *Let p be the Kemeny parameter. The Generalized induced Kendall-τ measure, noted I^p, is defined as follows : $\forall (r,s) \in A(U)^2$,*

$$\begin{aligned} I^p(r,s) = |\{(x,y) \in (el(r) \cap el(s))^2 : x < y \wedge (r[x] < r[y] \wedge s[x] > s[y] \vee r[x] > r[y] \wedge s[x] < s[y])\}| \\ + p * |\{(x,y) \in (el(r) \cap el(s))^2 : x < y \wedge (r[x] = r[y] \wedge s[x] \neq s[y] \vee r[x] \neq r[y] \wedge s[x] = s[y])\}| \end{aligned} \tag{10}$$

Definition 3.11 (Generalized induced Kemeny score) *Let $\mathscr{R}$ be a multiset of complete rankings. We define the Generalized induced Kemeny score between $c \in C(U)$ and $\mathscr{R}$ as follows :*

$$K^p_I(c, \mathscr{R}) = \sum_{r \in \mathscr{R}} I^p(c, r) \tag{11}$$

Definition 3.12 (Median or optimal consensus) *Once again, a median or optimal consensus for $\mathscr{R}$ a multiset of rankings, noted m^*, is the ranking $c \in C(U)$ such that :*

$$\forall s \in C(U), K_I^p(m^*, \mathscr{R}) \leq K_I^p(s, \mathscr{R}) \tag{12}$$

We have presented the different scoring functions to minimize according to the features of the input rankings, including the interpretation of the missing elements in the input rankings. These scoring functions are very similar and we gathered them into a general family of functions.

4 Generalizations

In the previous section, we highlighted the necessity of considering several scoring functions. Each of them is relevant in a specific context. For example, the Generalized induced Kendall-τ measure is relevant when at least one of the input rankings is incomplete and the missing elements can not be compared to the present elements (in terms of importance regarding the incomplete ranking). All the scoring functions are generalizations of the Kemeny score defined in definition 3.2. We gathered all these scoring functions into a family of functions called *Kemeny-compatible functions*. We will detail the formalism of the Kemeny compatible functions and demonstrate that all the scoring functions mentioned in the previous section are in this family of functions.

4.1 Notations

Let $\mathscr{R}$ be a multiset of rankings and U the set of all the elements present in at least one ranking in $\mathscr{R}$. Let's set :

- $\Omega^{\mathscr{R}}(x \prec y)$ the number of rankings in $\mathscr{R}$ in which x and y are present and x is before y.
- $\Omega^{\mathscr{R}}(x \succ y)$ the number of rankings in $\mathscr{R}$ in which x and y are present and x is after y.
- $\Omega^{\mathscr{R}}(x \equiv y)$ the number of rankings in $\mathscr{R}$ in which x and y are present and x is tied with y.
- $\Omega^{\mathscr{R}}(x\neg y)$ the number of rankings in $\mathscr{R}$ in which x is present and y is missing.
- $\Omega^{\mathscr{R}}(\neg xy)$ the number of rankings in $\mathscr{R}$ in which x is missing y is present.
- $\Omega^{\mathscr{R}}(\neg x\neg y)$ the number of rankings in which both x and y are missing.
- $u^{\mathscr{R}}_{x,y}$ the vector $(\Omega^{\mathscr{R}}(x \prec y), \Omega^{\mathscr{R}}(x \succ y), \Omega^{\mathscr{R}}(x \equiv y), \Omega^{\mathscr{R}}(x\neg y), \Omega^{\mathscr{R}}(\neg xy), \Omega^{\mathscr{R}}(\neg x\neg y))$

For a set U, we set MS(U) the set of all the multisets of U.

4.2 Kemeny compatibility

Definition 4.1 (Kemeny-compatible function) *Let $f\colon Df \mapsto \mathbb{R}$ be a function where $Df \subseteq C(U) \times MS(A(U))$. The function f is Kemeny-compatible if for any totally ordered set U, $\exists\ v = (v_1, ..., v_6) \in \mathbb{R}^6$ and $v' = (v'_1, ..., v'_6) \in \mathbb{R}^6$ such that $\forall\ (c, \mathscr{R}) \in Df$,*

$$f(c, \mathscr{R}) = \sum_{\substack{(x,y)\in U^2 \\ c[x] < c[y]}} |< v, u^{\mathscr{R}}_{x,y} >| + \sum_{\substack{(x,y)\in U^2 \\ c[x]=c[y] \\ x<y}} |< v', u^{\mathscr{R}}_{x,y} >|$$

For a Kemeny-compatible function f , v and v' are called the *cost vectors of f*.

Intuitively, a Kemeny-compatible function is a function with two parameters: a complete ranking (according to Df, it may be a permutation or a ranking with ties) and a multiset of rankings (according to Df, the rankings may be complete or incomplete, with or without ties). The Kemeny compatible function computes a score, which is necessary a positive real value. For $\mathscr{R}$ a (multi)set of rankings, c_1 and c_2 two consensuses and f a Kemeny-compatible function, if $f(c_1, \mathscr{R}) < f(c_2, \mathscr{R})$ we can deduce that regarding f, c_1 represents $\mathscr{R}$ better than c_2.

The score can be computed from the pair of elements of the consensus (so, the pair of elements of U, as the consensus is a complete ranking). For each pair of elements (x, y) such that x is before y (resp. tied) in the consensus, the cost of placing x before y (resp. tied) is computed: this value is the dot product $< v, u^{\mathscr{R}}_{x,y} >$ (resp $< v', u^{\mathscr{R}}_{x,y} >$). This score depends on the number of rankings placing x before y, x after y, x and y tied, and, if the rankings are incomplete, the number of rankings with x and not y, y and not x and neither x nor y. The cost vectors v and v' are therefore weighting coefficients.

We are now going to demonstrate that all the different scoring functions introduced in the previous (Kemeny score and the generalizations) are Kemeny-compatible functions.

4.3 Link with the classical scoring functions

Property 1 *The Kemeny score, (definition 3.2) related to the Kendall-τ distance, is Kemeny-compatible.*

Demonstration *Let U be a totally ordered set. The Kemeny score is defined $\forall(\pi, \mathscr{R}) \in P(U) \times MS(P(U))$. According to the definition 3.1, we know that $\forall(\pi, \mathscr{R}) \in P(U) \times MS(P(U))$,*

$$\begin{aligned}
S(\pi, \mathscr{R}) &= \sum_{\sigma \in \mathscr{R}} D(\pi, \sigma) \\
&= \sum_{\sigma \in \mathscr{R}} |\{(x, y) \in U^2 : x < y \wedge (\pi[x] < \pi[y] \wedge \sigma[x] > \sigma[y] \vee \pi[x] > \pi[y] \wedge \sigma[x] < \sigma[y])\}| \\
&= \sum_{\sigma \in \mathscr{R}} |\{(x, y) \in U^2 : \pi[x] < \pi[y] \wedge \sigma[x] > \sigma[y]\}| \\
&= \sum_{\sigma \in \mathscr{R}} \sum_{\substack{(x,y) \in U^2 \\ \pi[x] < \pi[y]}} \mathbf{1}_{\sigma[x] > \sigma[y]} = \sum_{\substack{(x,y) \in U^2 \\ \pi[x] < \pi[y]}} \sum_{\sigma \in \mathscr{R}} \mathbf{1}_{\sigma[x] > \sigma[y]} = \sum_{\substack{(x,y) \in U^2 \\ \pi[x] < \pi[y]}} \Omega^{\mathscr{R}}_{x>y} \\
&= \sum_{\substack{(x,y) \in U^2 \\ \pi[x] < \pi[y]}} |< v, u^{\mathscr{R}}_{x,y} >| + \sum_{\substack{(x,y) \in U^2 \\ \pi[x] = \pi[y] \\ x<y}} |< v', u^{\mathscr{R}}_{x,y} >|
\end{aligned} \tag{13}$$

with $v = (0, 1, 0, 0, 0, 0)$ and $v' = (0, 0, 0, 0, 0, 0)$

Property 2 *The Generalized Kemeny score (definition 3.5), related to the Generalized Kendall-τ distance, is Kemeny-compatible.*

Demonstration *Let U be a totally ordered set. The Generalized Kemeny score is defined $\forall(c,\mathscr{R}) \in (C(U) \times MS(C(U)))$. According to the definition 3.4, we know that $\forall(c,\mathscr{R}) \in C(U) \times MS(C(U))$,*

$$
\begin{aligned}
K(c,\mathscr{R}) &= \sum_{r\in\mathscr{R}} G(c,r) \\
&= \sum_{r\in\mathscr{R}} |\{(x,y) \in U^2 : x < y \wedge (c[x] < c[y] \wedge r[x] > r[y] \vee c[x] > c[y] \wedge r[x] < r[y])\}| \\
&+ p * \sum_{r\in\mathscr{R}} |\{(x,y) \in U^2 : x < y \wedge (c[x] = c[y] \wedge r[x] \neq r[y] \vee c[x] \neq c[y] \wedge r[x] = r[y])\}| \\
&= \sum_{r\in\mathscr{R}} |\{(x,y) \in U^2 : c[x] < c[y] \wedge r[x] > r[y]\}| + p * \sum_{r\in\mathscr{R}} |\{(x,y) \in U^2 : c[x] < c[y] \wedge r[x] = r[y]\}| \\
&+ p * \sum_{r\in\mathscr{R}} |\{(x,y) \in U^2 : x < y \wedge c[x] = c[y] \wedge r[x] \neq r[y]\}| \\
&= (\sum_{r\in\mathscr{R}} \sum_{\substack{(x,y)\in U^2 \\ c[x] < c[y]}} \mathbf{1}_{r[x]>r[y]} + p * \mathbf{1}_{r[x]=r[y]}) + (p * \sum_{r\in\mathscr{R}} \sum_{\substack{(x,y)\in U^2 \\ c[x] = c[y] \\ x<y}} \mathbf{1}_{r[x]\neq r[y]}) \\
&= (\sum_{\substack{(x,y)\in U^2 \\ c[x] < c[y]}} \sum_{r\in\mathscr{R}} \mathbf{1}_{r[x]>r[y]} + p * \mathbf{1}_{r[x]=r[y]}) + (p * \sum_{\substack{(x,y)\in U^2 \\ c[x] = c[y] \\ x<y}} \sum_{r\in\mathscr{R}} \mathbf{1}_{r[x]\neq r[y]}) \\
&= (\sum_{\substack{(x,y)\in U^2 \\ c[x] < c[y]}} card^{\mathscr{R}}_{x>y} + p * card^{\mathscr{R}}_{x=y}) + (p * \sum_{\substack{(x,y)\in U^2 \\ c[x] = c[y] \\ x<y}} card^{\mathscr{R}}_{x>y} + card^{\mathscr{R}}_{x<y}) \\
&= \sum_{\substack{(x,y)\in U^2 \\ c[x] < c[y]}} |< v, u^{\mathscr{R}}_{x,y} >| + \sum_{\substack{(x,y)\in U^2 \\ c[x]=c[y] \\ x<y}} |< v', u^{\mathscr{R}}_{x,y} >|
\end{aligned}
\tag{14}
$$

with $v = (0, 1, p, 0, 0, 0)$ and $v' = (p, p, 0, 0, 0, 0)$

Property 3 *The Generalized induced Kemeny score (definition 3.11), related to the Generalized induced Kendall-τ measure, is Kemeny-compatible.*

Demonstration *Let U be a totally ordered set. The Generalized induced Kemeny score is defined $\forall(c,\mathscr{R}) \in C(U) \times MS(T(U))$. According to the definition 3.11, we know that $\forall(c,\mathscr{R}) \in C(U) \times MS(T(U))$,*

$$
\begin{aligned}
M(c,\mathscr{R}) &= \sum_{r\in\mathscr{R}} I(c,r) \\
&= \sum_{r\in\mathscr{R}} |\{(x,y)\in(el(c)\cap el(r))^2 : x<y \wedge (c[x]<c[y]\wedge r[x]>r[y] \vee c[x]>c[y]\wedge r[x]<r[y])\}| \\
&+ p*\sum_{r\in\mathscr{R}} |\{(x,y)\in(el(c)\cap el(r))^2 : x<y \wedge (c[x]=c[y]\wedge r[x]\neq r[y] \vee c[x]\neq c[y]\wedge r[x]=r[y])\}| \\
&= \sum_{r\in\mathscr{R}} |\{(x,y)\in(el(c)\cap el(r))^2 : c[x]<c[y]\wedge r[x]>r[y]\}| \\
&+ p*\sum_{r\in\mathscr{R}} |\{(x,y)\in(el(c)\cap el(r))^2 : c[x]<c[y]\wedge r[x]=r[y]\}| \\
&+ p*\sum_{r\in\mathscr{R}} |\{(x,y)\in(el(c)\cap el(r))^2 : x<y\wedge c[x]=c[y]\wedge r[x]\neq r[y]\}| \\
&= \sum_{r\in\mathscr{R}} \sum_{\substack{(x,y)\in U^2\\ c[x]<c[y]}} \mathbf{1}_{x\in el(r)} * \mathbf{1}_{y\in el(r)} * (\mathbf{1}_{r[x]>r[y]} + p*\mathbf{1}_{r[x]=r[y]}) \\
&+ p*\sum_{r\in\mathscr{R}} \sum_{\substack{(x,y)\in U^2\\ c[x]=c[y]\\ x<y}} \mathbf{1}_{x\in el(r)} * \mathbf{1}_{y\in el(r)} * (\mathbf{1}_{r[x]\neq r[y]}) \\
&= \sum_{\substack{(x,y)\in U^2\\ c[x]<c[y]}} \sum_{r\in\mathscr{R}} \mathbf{1}_{x\in el(r)} * \mathbf{1}_{y\in el(r)} * (\mathbf{1}_{r[x]>r[y]} + p*\mathbf{1}_{r[x]=r[y]}) \\
&+ p*\sum_{\substack{(x,y)\in U^2\\ c[x]=c[y]\\ x<y}} \sum_{r\in\mathscr{R}} \mathbf{1}_{x\in el(r)} * \mathbf{1}_{y\in el(r)} * (\mathbf{1}_{r[x]\neq r[y]}) \\
&= (\sum_{\substack{(x,y)\in U^2\\ c[x]<c[y]}} \Omega^{\mathscr{R}}_{x>y} + p*\Omega^{\mathscr{R}}_{x=y}) + (p*\sum_{\substack{(x,y)\in U^2\\ c[x]=c[y]\\ x<y}} \Omega^{\mathscr{R}}_{x>y} + \Omega^{\mathscr{R}}_{x<y}) \\
&= \sum_{\substack{(x,y)\in U^2\\ c[x]<c[y]}} |< v, u^{\mathscr{R}}_{x,y} >| + \sum_{\substack{(x,y)\in U^2\\ c[x]=c[y]\\ x<y}} |< v', u^{\mathscr{R}}_{x,y} >|
\end{aligned}
\tag{15}
$$

with $v=(0,1,p,0,0,0)$ *and* $v'=(p,p,0,0,0,0)$

Property 4 *The Generalized Kemeny score related to pseudometric (definition 3.8, related to the Generalized Kendall-τ pseudometric, is Kemeny-compatible.*

Demonstration *Same as the previous demonstrations.*

5 Conclusion

Aggregating rankings is an important issue as this problem has applications in many communities. Few studies have so far considered incomplete rankings, although in real contexts, the rankings to aggregate are rarely complete. Providing tools to aggregate incomplete rankings is challenging in so far as there not one single interpretation of the missing elements. As there were several scoring functions to aggregate rankings, we gathered them into one family of functions called Kemeny-compatible functions. Considering the whole family and not only one single scoring functions enabled a generalization of several existing algorithms to make them compatible with any Kemeny-compatible function.

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