36. Solution:

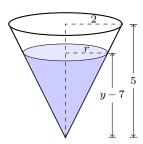


Figure 1: Diagram of the conical portion of the water tank

Let y denote the height of any given "drop" of water in the tank, where y=0 m is the base of the tank and y=5+7=12 m is the top of the tank. Then y-7 for $y\in[7,12]$ represents the height of water in the conical portion of the tank.

Let $W_S(y)$ denote the total amount of work done by letting a slice water of thickness Δy in the tank at height y drain and let W_T be the integral we want to construct. Then

$$\begin{split} W_S(y) &= F \cdot d \\ &= (mg)(y) \\ &= (\rho V)gy \\ &= \rho(\pi r^2 \Delta y)gy \end{split}$$

From Figure 1, we can represent a water slice's radius using similar triangles:

$$\begin{split} \frac{r}{y-7} &= \frac{2}{5} \\ r &= \frac{2}{5}(y-7) \\ \Rightarrow W_S(y) &= \rho \pi \bigg(\frac{2}{5}(y-7)\bigg)^2 gy \Delta y \\ &= \frac{4}{25} \rho g \pi y (y-7)^2 \Delta y \\ &= 1568 \pi y (y-7)^2 \Delta y \end{split}$$

: the total amount of work done can be represented as

$$W_T = \sum W_S(y)$$

$$= 1568\pi \int_9^{12} y(y-7)^2 \,\mathrm{d}y$$

Note. If we defined y differently, then the integrand (d and r as functions of y) and the bounds of the integral would change. But the overall value of the integral would remain unchanged.

37. Solution:

The integrand is a rational function that can undergo partial fraction decomposition to simplify the evaluation of the integral, since $deg(4x-2) < deg(x^2-2x+1)$. i.e., the integrand is a proper rational function.

Let I denote the value of the indefinite integral. Then using partial fraction decomposition yields

$$I = \int \frac{4x - 2}{x^2 - 2x + 1} dx$$

$$= 2 \int \frac{2x - 1}{(x - 1)^2} dx$$

$$= 2 \int \frac{c_1}{x - 1} + \frac{c_2}{(x - 1)^2} dx$$

$$\Rightarrow \frac{2x - 1}{(x - 1)^2} = \frac{c_1}{x - 1} + \frac{c_2}{(x - 1)^2}$$

$$2x - 1 = c_1(x - 1) + c_2$$

$$2x - 1 = c_1x - c_1 + c_2$$

$$\Rightarrow \begin{cases} 2x = c_1x \\ -1 = -c_1 + c_2 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = 2 \\ c_2 = 1 \end{cases}$$

$$(1)$$

Note. The constants can also be solved for by substituting in certain values of x into (1).

$$\Rightarrow I = 2 \int \frac{2}{x-1} + \frac{1}{(x-1)^2} \, \mathrm{d}x$$

$$= 2(2 \ln |x-1| - (x-1)^{-1}) + C.$$