20053722
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2. Finding $E[X_iX_j]$, for $i, j \in \{1, \ldots, k\}$.

Solution:

Let $X_i = X_{i,1} + \cdots + X_{i,n}$ where

$$X_{i,k} = \begin{cases} 1, & \text{if the } k\text{th multinomial experiment has outcome i} \\ 0, & \text{otherwise} \end{cases}$$

and let $X_j = X_{j,1} + \cdots + X_{j,n}$ where

$$X_{j,k} = \begin{cases} 1, & \text{if the } k\text{th multinomial experiment has outcome j} \\ 0, & \text{otherwise} \end{cases}$$

We then have

$$\begin{aligned} E[X_iX_j] &= E\left[\left(\sum_{l=1}^{n} X_{i,l}\right)\left(\sum_{m=1}^{n} X_{j,m}\right)\right] \\ &= E\left[\sum_{l=1}^{n}\sum_{m=1}^{n} X_{i,l}X_{j,m}\right] && \text{by the distributivity of summation} \\ &= \sum_{l=1}^{n}\sum_{m=1}^{n} E\left[X_{i,l}X_{j,m}\right] && \text{by the linearity of expectation} \end{aligned}$$

For each $l, m \in \{1, \ldots, n\}$,

$$E\left[X_{i,l}X_{j,m}\right] = \begin{cases} p_ip_j, & \text{if } l \neq m \\ 0, & \text{if } l = m \end{cases}$$

Since $l = m$ only n times in out of the total n^2 terms in the summation, we have that

$$\boxed{E[X_iX_j] = (n^2 - n)p_ip_j}$$