20053722
Student Number

Bryan Hoang
Name

2. **Solution:**

Proof. Since $X_1, \dots, X_n$ are independent exponential random variables with parameter λ, then their joint pdf is

$$\begin{aligned} f_X(x_1, \dots, x_n) &= f(x_1)\dots f(x_N) \\ &= \lambda e^{-\lambda x_1} \dots \lambda e^{-\lambda x_n} \\ &= \begin{cases} \lambda^n e^{-\lambda \sum_{i=1}^n x_i}, & \text{if } (x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Also, the joint pdf of $X_{(1)}, \dots, X_{(n)}$ is

$$\begin{aligned} f_{1,\dots,n}(x_1, \dots, x_n) &= n! f(x_1)\dots f(x_n) \\ &= \begin{cases} n! \lambda^n e^{-\lambda \sum_{i=1}^n x_i}, & \text{if } 0 < x_1 < \dots < x_n. \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Given the above facts, let's find the joint pdf of $Y_1, \dots, Y_n$. For $i \in \{1, \dots, n\}$, define the following set of functions:

$$g_i(y_1, \dots, y_n) = \sum_{k=1}^{i} \frac{y_i}{n+1-k}$$

We then have

$$\begin{aligned} |J_g(y_1, \dots, y_n)| &= \begin{vmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial y_1} & \cdots & \frac{\partial g_n}{\partial y_n} \end{vmatrix} \\ &= \begin{vmatrix} \frac{1}{n} & & & & \\ \frac{1}{n} & \frac{1}{n-1} & & \text{\huge 0} & \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & & \\ \vdots & & \ddots & \frac{1}{2} & \\ \frac{1}{n} & \cdots & \cdots & \frac{1}{2} & \frac{1}{1} \end{vmatrix} \\ &= \prod_{i=1}^{n} \frac{1}{i} && \because J_g \text{ is a lower triangular matrix.} \\ &= \frac{1}{n!} \end{aligned}$$

20053722
Student Number

Bryan Hoang
Name

For the support of $Y = (Y_1, \dots, Y_n)$,

$$0 < \frac{y_1}{n} < \frac{y_1}{n} + \frac{y_2}{n-1} < \dots < \frac{y_1}{n} + \dots + y_n$$
$$\Rightarrow (y_1, \dots, y_n) \in \mathbb{R}^n_{\geq 0}$$

It follows that the joint pdf of $Y_1, \dots, Y_n$ is

$$\begin{aligned} f_Y(y_1, \dots, y_n) &= f_{1,\dots,n}(g_1(y_1, \dots, y_n), \dots, g_n(y_1, \dots, y_n))|J_g(y_1, \dots, y_n)| \\ &= n!\lambda^n \left(e^{-\lambda \sum_{i=1}^n \sum_{k=1}^i \frac{y_i}{n+1-k}}\right) \frac{1}{n!} \\ &= \begin{cases} \lambda^n e^{-\lambda \sum_{i=1}^n y_i}, & \text{if } (y_1, \dots, y_n) \in \mathbb{R}^n_{\geq 0} \\ 0, & \text{otherwise} \end{cases} \\ &= f_X(y_1, \dots, y_n) \end{aligned}$$

Therefore, $Y_1, \dots, Y_n$ has the same joint distribution as $X_1, \dots, X_n$.
For $i \in \{1, \dots, n\}$ and $\{j_1, \dots, j_{n-1}\} = \{1, \dots, n\} \setminus \{i\}$, the marginal pdf for Y_i is

$$\begin{aligned} f_{Y_i}(y_i) &= \underset{\mathbb{R}^{n-1}}{\int \cdots \int} f_Y(y_1, \dots, y_n) \, \mathrm{d}y_{j_1} \dots \mathrm{d}y_{j_{n-1}} \\ &= \int_0^\infty \cdots \int_0^\infty \lambda^n e^{-\lambda \sum_{k=1}^n y_k} \, \mathrm{d}y_{j_1} \dots \mathrm{d}y_{j_{n-1}} \\ &= \begin{cases} \lambda e^{-\lambda y_i}, & \text{if } y_i \in \mathbb{R}_{\geq 0} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

It can be seen that $f_Y(y_1, \dots, y_n) = f_{Y_1}(y_1) \dots f_{Y_n}(y_n)$.
Hence, $Y_1, \dots, Y_n$ are also independent. $\square$