20053722
Student Number

Bryan Hoang
Name

4.

i. Finding the joint pdf of $(Y_1, \dots, Y_n)$, $f_Y(y_1, \dots, y_n)$.
Since $X_1, \dots, X_N$ are independent, their joint pdf is then

$$\begin{aligned} f_X(x_1, \dots, x_n) &= f_{X_1}(x_1) \dots f_{X_n}(x_n) \\ &= \begin{cases} e^{-\sum_{i=1}^n x_i}, & \text{if } (x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Let $g_1(y_1, \dots, y_n) = y_1$ and $g_k(y_1, \dots, y_n) = y_k - y_{k-1}$, $\forall k \in \{2, \dots, n\}$. We then have

$$\begin{aligned} J_g &= \begin{vmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial y_1} & \cdots & \frac{\partial g_n}{\partial y_n} \end{vmatrix} \\ &= \begin{vmatrix} 1 & & & & \\ -1 & 1 & & \huge{0} & \\ & \ddots & \ddots & & \\ & \huge{0} & & 1 & \\ & & & -1 & 1 \end{vmatrix} \end{aligned}$$

Hopfully the illustration above gets the point across.

$$\begin{aligned} &= \prod_{i=1}^n 1 && \because \text{ the matrix is lower triangular} \\ &= 1 \end{aligned}$$

Then the joint pdf of $(Y_1, \dots, Y_n)$ is

$$\begin{aligned} f_Y(y_1, \dots, y_n) &= f_X(y_1, y_2 - y_1, \dots, y_n - y_{n-1})|J_g| \\ &= e^{-y_1} e^{-(y_2 - y_1)} \dots e^{-(y_n - y_{n-1})} |1| \\ &\boxed{= \begin{cases} e^{-y_n}, & \text{if } 0 \leq y_1 \leq y_2 \leq \dots \leq y_n < \infty \\ 0, & \text{otherwise} \end{cases}} \end{aligned}$$

20053722
Student Number

Bryan Hoang
Name

ii. Finding the marginal pdf of Y_n, $f_{Y_n}(y_n)$.

$$\begin{aligned}
f_{Y_n}(y_n) &= \int_0^{y_n}\int_0^{y_{n-1}}\cdots\int_0^{y_2} f_Y(y_1,\ldots,y_n)\,\mathrm{d}y_1\ldots\,\mathrm{d}y_{n-2}\,\mathrm{d}y_{n-1}\\
&= \int_0^{y_n}\int_0^{y_{n-1}}\cdots\int_0^{y_2} e^{-y_n}\,\mathrm{d}y_1\ldots\,\mathrm{d}y_{n-2}\,\mathrm{d}y_{n-1}\\
&= e^{-y_n}\int_0^{y_n}\int_0^{y_{n-1}}\cdots\int_0^{y_2} \mathrm{d}y_1\ldots\,\mathrm{d}y_{n-2}\,\mathrm{d}y_{n-1}\\
&\boxed{= \begin{cases} e^{-y_n}\frac{y_n^{n-1}}{(n-1)!}, & \text{if } y_n \in [0,\infty)\\ 0, & \text{otherwise}\end{cases}}
\end{aligned}$$

iii. Determining if $Y_1,\ldots,Y_n$ are mutually independent.

Claim. $Y_1,\ldots,Y_n$ are not mutually independent.

Proof. (Contradiction) Suppose $Y_1,\ldots,Y_n$ are mutually independent. Then we can write

$$\begin{aligned}
f_Y(y_1,\ldots,y_n) &= f_{Y_n}(y_n)f_{Y_1,\ldots,Y_{n-1}}(y_1,\ldots,y_{n-1})\\
\Rightarrow e^{-y_n} &= e^{-y_n}\frac{y_n^{n-1}}{(n-1)!}f_{Y_1,\ldots,Y_{n-1}}(y_1,\ldots,y_{n-1})\\
\Rightarrow f_{Y_1,\ldots,Y_{n-1}}(y_1,\ldots,y_{n-1}) &= \frac{(n-1)!}{y_n^{n-1}}
\end{aligned}$$

which is a contradiction since $f_{Y_1,\ldots,Y_{n-1}}(y_1,\ldots,y_{n-1})$ cannot depend on y_n.

Hence, it follows that $\boxed{Y_1,\ldots,Y_n \text{ are } \textbf{not} \text{ mutually independent.}}$ $\square$