**MTHE 353 Midterm** Cheat Sheet
by Bryan Hoang, page 1 of 2

## 1 Special Distributions

**Binomial,** Binomial$(n,p)$

$$p(k)=\begin{cases}\binom{n}{k}p^k(1-p)^{n-k}, & \text{if } k\in\{0,1,\dots,n\}\\ 0, & \text{otherwise}\end{cases}$$

where $\binom{n}{k}=\frac{n!}{k!(n-k)!}$
$E[X]=np$
$\mathrm{Var}(X)=np(1-p)$

**Geometric,** Geometric$(p)$

$$p(n)=\begin{cases}p(1-p)^{n-1}, & \text{if } n\in\mathbb{Z}_{>0}\\ 0, & \text{otherwise}\end{cases}$$

$E[X]=\frac{1}{p}$
$\mathrm{Var}(X)=np(1-p)$

**Negative Binomial,** Negative Binomial$(r,p)$

$$p(n)=\begin{cases}\binom{n-1}{r-1}p^r(1-p)^{n-r}, & \text{if } n\in\mathbb{Z}_{\geq r}\\ 0, \text{otherwise}\end{cases}$$

$E[X]=\frac{r}{p}$
$\mathrm{Var}(X)=\frac{r(1-p)}{p^2}$

**Poisson,** Poisson$(\theta)$

$$p(k)=\begin{cases}\frac{\theta^k e^{-\theta}}{k!}, & \text{if } k\in\mathbb{Z}_{\geq 0}\\ 0, & \text{otherwise}\end{cases}$$

$E[X]=\lambda$
$\mathrm{Var}(X)=\lambda$

**Poisson process with rate $\lambda$**
$N(t)\sim\text{Poisson}(\lambda t)$

**Multivariate Hypergeometric**
Usually occurs in situations **without replacement**.
Parameters:

- $N$ objects
- $n_i$ of type $i$, for $i\in\{1,\dots,r\}$
- Pick $k$ objects w/o replacement
- $X_i$ = # of objects of type i picked.

pdfs:
$\frac{\binom{n_1}{x_1}\cdots\binom{n_r}{x_r}}{\binom{N}{k}}$, $\frac{\binom{n_1}{x_1}\cdots\binom{n_{r-1}}{x_{r-1}}\binom{n_r}{k-x_1-\cdots-x_{r-1}}}{\binom{N}{k}}$
$\frac{\binom{n_{i_1}}{x_{i_1}}\cdots\binom{n_{i_d}}{x_{i_d}}\binom{N-n_{i_1}-\cdots-n_{i_d}}{k-x_{i_1}-\cdots-x_{i_d}}}{\binom{N}{k}}$

Support:

- $x$'s $\in\{\max(0,k-(N-n\text{'s})),\dots,n\text{'s}\}$
- sum of $x$'s $\in\{0,\dots,k\}$

**Order Statistics**
Assumptions: iid r.v.'s
pdfs:

$$f_{1,\dots,n}(x_1,\dots,x_n)=\begin{cases}n!f(x_1)\cdots f(x_n), & \text{if } x_1<\cdots<x_n\\ 0, & \text{otherwise}\end{cases}$$

$f_{k,\dots,n}(x_k,\dots,x_n)=n!\frac{F(x_k)^{k-1}}{(k-1)!}f(x_k)\cdots f(x_n)$
$f_{1,\dots,r}(x_1,\dots,x_r)=n!f(x_1)\cdots f(x_r)\frac{(1-F(x_r))^{n-r}}{(n-r)!}$
$f_{k,\dots,r}(x_k,\dots,x_r)=n!\frac{F(x_k)^{k-1}}{(k-1)!}f(x_k)\cdots$
$\quad f(x_r)\frac{(1-F(x_r))^{n-r}}{(n-r)!}$
$f_k(x)=\frac{n!}{(k-1)!(n-k)!}f(x)F(x)^{k-1}(1-F(x))^{n-k}$
$f_{k,r}(x_k,x_r)=n!\frac{f(x_k)F(x_k)^{k-1}}{(k-1)!}\cdot\frac{(F(x_r)-F(x_k))^{r-k-1}}{(r-k-1)!}\cdot\frac{f(x_r)(1-F(x_r))^{n-r}}{(n-r)!}$
$f_1(x)=nf(x)(1-F(x))^{n-1}$
$f_n(x)=nf(x)F(x)^{n-1}$

**Multinomial,** Multinomial$(n,p\text{'s})$

Usually occurs in situations **with replacement**.
Parameters:

- $n$ number of trials (in $\mathbb{Z}$)
- $p_1,\dots,p_r$ event probabilities ($\sum p_i=1$)

pdf:

$$p(x)=\begin{cases}\binom{n}{x_1,\dots,x_r}p_1^{x_1}\cdots p_r^{x_r}, & \text{if } x\text{'s}\in\mathbb{Z}_{\geq 0}\ \&\sum_{i=1}^r x_i=n\\ 0, & \text{otherwise}\end{cases}$$

where $\binom{n}{x_1,\dots,x_r}=\frac{n!}{x_1!\cdots x_n!}$

**Exponential,** Exp$(\lambda)$

$$f(x)=\begin{cases}\lambda e^{-\lambda x}, & \text{if } x\in\mathbb{R}_{\geq 0}\\ 0, & \text{otherwise}\end{cases}$$

$E[X]=\frac{1}{\lambda}$
$\mathrm{Var}(X)=\frac{1}{\lambda^2}$

**Uniform,** Uniform$(a,b)$

$$f(x)=\begin{cases}\frac{1}{b-a}, & \text{if } x\in[a,b]\\ 0, & \text{otherwise}\end{cases}$$

$E[X]=\frac{a+b}{2}$
$\mathrm{Var}(X)=\frac{(b-a)^2}{12}$

**Standard Uniform,** Uniform$(0,1)$

$$f(x)=\begin{cases}1, & \text{if } x\in[0,1]\\ 0, & \text{otherwise}\end{cases}$$

**Normal,** $N(\mu,\sigma^2)$

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2},\quad \text{for } x\in\mathbb{R}$$

**Standard Normal,** $N(0,1)$

$$f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},\quad \text{for } x\in\mathbb{R}$$

**Gamma,** Gamma$(r,\lambda)$

$$f(x)=\begin{cases}\frac{\lambda^r}{\Gamma(r)}x^{r-1}e^{-\lambda x}, & \text{if } x\in\mathbb{R}_{>0}\\ 0, & \text{otherwise}\end{cases},$$

where
$\Gamma(r)=\int_0^\infty y^{r-1}e^{-y}\,\mathrm{d}y,\quad$ for $r>0$,
$\Gamma(0)=1$,
$\Gamma(\frac{1}{2})=\sqrt{\pi}$,
$\Gamma(r)=(r-1)\Gamma(r-1),\quad r\in\mathbb{R}>0$,
$\Gamma(r)=(r-1)!,\quad r\in\mathbb{Z}_{>0}$,
$E[X^k]=\frac{(r+k-1)\cdots r}{\lambda^k}$,
$E[X]=\frac{r}{\lambda}$,
$\mathrm{Var}(X)=\frac{r}{\lambda^2}$

**Chi-squared,** $\chi_n^2\sim\text{Gamma}\left(\frac{n}{2},\frac{1}{2}\right)$

$$f(x)=\begin{cases}\frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}x^{\frac{n}{2}-1}e^{-\frac{x}{2}}, & \text{if } x\in\mathbb{R}_{>0}\\ 0, & \text{otherwise}\end{cases}$$

**Beta,** Beta$(\alpha,\beta)$

$$f(x)=\begin{cases}\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, & \text{if } x\in(0,1)\\ 0, & \text{otherwise}\end{cases}$$

where
$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}=\int_0^1 v^{\alpha-1}(1-v)^{\beta-1}\,\mathrm{d}v$

**Relationships between distributions**

- $X\sim N(0,1)\implies X^2\sim\text{Gamma}(\frac{1}{2},\frac{1}{2})\sim\chi_1^2$
- $X_j\sim\text{Gamma}(r_j,\lambda)\implies\sum X_j\sim\text{Gamma}(\sum r_j,\lambda)$
- The above two points imply that $\sum X_j^2\sim\chi_n^2$
- $X_i\sim\text{Exp}(\lambda)\implies\sum_{i=1}^n X_i\sim\text{Gamma}(n,\lambda)$

## 2 Independence

If $X_1,\dots,X_n$ are mutually independent, then

- so are $g_1(X_1),\dots,g_n(X_n)$
- the joint pdfs and cdfs are the product of the marginals
- with LOTUS, expectation of product is product of expectations

## 3 Transformation of Multiple Random Variables

Let $h:\mathbb{R}^n\to\mathbb{R}^n$ be cont. diff. and **one-to-one (injective/invertible)** with a cont. diff. inverse.

We want to find $\mathbf{Y}=(h\text{'s})^T$.
$f_Y(y_1,\dots,y_n)=f(g_1(y_1,\dots,y_n),\dots,g_n(y_x,\dots,y_n))\left|J_g(y_1,\dots,y_n)\right|$
where

$$J_g(y_1,\dots,y_n)=\begin{vmatrix}\frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_n}\\ \vdots & \ddots & \vdots\\ \frac{\partial g_n}{\partial y_1} & \cdots & \frac{\partial g_n}{\partial y_n}\end{vmatrix}$$

Note the support of its joint pdf.

## 4 Expectation

- The expectation of a random vector is the random vector of expectations.
- **L O T U S**

## 5 Sample Mean and Sample Variance

Let $Z$'s be i.i.d. r.v.'s \w common mean $\mu$ and variance $\sigma^2$. Define the *sample mean* and *sample variance* by
$\overline{Z}=\frac{1}{n}\sum_{i=1}^n Z_i$
$S^2=\frac{1}{n-1}\sum_{i=1}^n(Z_i=Z)^2$
Then the expectation of these …things are what you would **expect**.

## 6 Other Useful Formulas

$\sum_{k=0}^\infty x^k=\frac{1}{1-x},\quad$ if $|x|<1$
$\sum_{k=0}^n\binom{n}{k}x^k y^{n-k}=(x+y)^n$
$\sum_{k=0}^\infty\binom{k+r-1}{r-1}x^k=\frac{1}{(1-x)^r}$
$e^x=\sum_{k=0}^\infty\frac{x^k}{k!}$
$\frac{4}{3}\pi r^3$