20053722
Student Number

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Name

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\begin{aligned}
p_{X_1}(x_1) &= \sum_{x_3=1}^{\infty}\sum_{x_2=1}^{\infty} p_{X_1,\ldots,X_n}(x_1,\ldots,x_n)\\
&= \sum_{x_3=1}^{\infty}\sum_{x_2=1}^{\infty} \left(\frac{1}{2}\right)^{x_3}\left(1-e^{-x_3}\right)^2 e^{-x_3(x_1+x_2-2)}\\
&= \sum_{x_3=1}^{\infty}\sum_{x_2=1}^{\infty} \left(\frac{1}{2}\right)^{x_3}\left(1-e^{-x_3}\right)^2 e^{-x_3(x_1-2)}\left(e^{-x_3}\right)^{x_2}\\
&= \sum_{x_3=1}^{\infty} \left(\frac{1}{2}\right)^{x_3}\left(1-e^{-x_3}\right)^2 e^{-x_3(x_1-2)}\sum_{x_2=1}^{\infty}\left(e^{-x_3}\right)^{x_2}\\
&= \sum_{x_3=1}^{\infty} \left(\frac{1}{2}\right)^{x_3}\left(1-e^{-x_3}\right)^2 e^{-x_3(x_1-2)}\left(\frac{e^{-x_3}}{1-e^{-x_3}}\right) && \because \left|e^{-x_2}\right|<1,\ \forall x_2\in\mathbb{Z}_{>0}\\
&= \sum_{x_3=1}^{\infty} \left(\frac{1}{2}\right)^{x_3}\left(1-e^{-x_3}\right) e^{-x_3x_1}e^{x_3}\\
&= \sum_{x_3=1}^{\infty} \left(\frac{1}{2}\right)^{x_3}\left(e^{-x_1}\right)^{x_3} e^{x_3} - \sum_{x_3=1}^{\infty}\left(\frac{1}{2}\right)^{x_3}\left(e^{-x_1}\right)^{x_3}\\
&= \sum_{x_3=1}^{\infty}\left(\frac{e}{2e^{x_1}}\right)^{x_3} - \sum_{x_3=1}^{\infty}\left(\frac{1}{2e^{x_1}}\right)^{x_3}\\
&= \frac{\frac{e}{2e^{x_1}}}{1-\frac{e}{2e^{x_1}}} - \frac{\frac{1}{2e^{x_1}}}{1-\frac{1}{2e^{x_1}}} && \because \text{both } \left|\frac{e}{2e^{x_1}}\right|<1 \text{ and}\\
& && \left|\frac{1}{2e^{x_1}}\right|<1,\ \forall x_1\in\mathbb{Z}_{>0}\\
&= \frac{e}{2e^{x_1}-e} - \frac{1}{2e^{x_1}-1}
\end{aligned}
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\Rightarrow \boxed{p_{X_1}(x_1) = \begin{cases} \frac{e}{2e^{x_1}-e} - \frac{1}{2e^{x_1}-1}, & \forall x_1\in\mathbb{Z}_{>0}\\ 0, & \text{otherwise}\end{cases}}
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$\square$