20053722
Student Number

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2.

(a) By conditioning on N, show that the moment generating function of Y is given by

$$m_Y(t) = m_N(\ln(m_X(t))) \tag{1}$$

Solution:

Proof. We begin with the LHS of (1).

$$\begin{aligned}
\text{LHS} &= m_Y(t) \\
&= \mathrm{E}\big[e^{tY}\big] && \text{by definition of } m_Y(t) \\
&= \mathrm{E}\big[\mathrm{E}\big[e^{tY}\big|N\big]\big] && \text{by the law of total expectation} \\
&= \sum_{n=0}^{\infty} \mathrm{E}\big[e^{tY}\big|N=n\big]p_N(n) && \text{where } p_N \text{ is the pmf of } N \\
&= 1 + \sum_{n=1}^{\infty} \mathrm{E}\Big[e^{t\sum_{i=1}^{n} X_i}\Big]p_N(n) && \text{by the definition of } Y \\
&= 1 + \sum_{n=1}^{\infty} \mathrm{E}\left[\prod_{i=1}^{n} e^{tX_i}\right]p_N(n) \\
&= 1 + \sum_{n=1}^{\infty} \mathrm{E}\big[e^{tX}\big]^n p_N(n) && \text{since the } X_i\text{'s are iid w.r.t. } X \\
&= \sum_{n=0}^{\infty} m_X(t)^n p_N(n) \\
&= \mathrm{E}\big[m_X(t)^N\big] && \text{by the law of the unconscious statistician} \\
&= \mathrm{E}\big[e^{\ln(m_X)\cdot N}\big] \\
&= m_N(\ln(m_X(t))) \\
&= \text{RHS} && \square
\end{aligned}$$

(b) Let N have a Poisson(λ) distribution and suppose N independent Bernoulli trials are conducted, where the probability of success in each trial is p. Let Y denote the total number of successes in the conducted trials. Compute the moment generating function of Y and use this to determine the distribution of Y.

Solution:

Let X_i be the indicator function for the success of the ith experiment, $i \in \{1, \dots, N\}$, which implies that the X_i's are iid to $X \sim \text{Bernoulli}(p)$. Then

$$Y = \begin{cases} \sum_{i=1}^{N} X_i & \text{if } N \in \mathbb{Z}_{\geq 1} \\ 0, & \text{if } N = 0 \end{cases}$$

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From the lecture material, we know that

$$m_N(t) = e^{\lambda(e^t - 1)}, \quad \forall t \in \mathbb{R}$$

Computing the mgf of X gives us

$$\begin{aligned} m_X(t) &= \mathrm{E}\big[e^{tX}\big] \\ &= \sum_{x=0}^{1} e^{tx} p_X(x) && \text{where } p_X \text{ is the pmf of } X \\ &= 1 - p + pe^t, \quad \forall t \in \mathbb{R} \end{aligned}$$

By part a, the mgf of Y is

$$\begin{aligned} m_Y(t) &= m_N(\ln(m_X(t))) \\ &= m_N(\ln\big(1 - p + pe^t\big)) \\ &= e^{\lambda\left(e^{\ln\left(1-p+pe^t\right)} - 1\right)} \\ &= e^{\lambda\left(1-p+pe^t-1\right)} \\ &= e^{(\lambda p)(e^t - 1)} \end{aligned}$$

which is the mgf of a random variable with a Poisson(λp) distribution. Thus, $\boxed{Y \sim \text{Poisson}(\lambda p)}$.