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Student Number

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4. For the random matrix to be singular, we want

$$\begin{vmatrix} X_1 & 0 & 0 \\ 0 & X_2 & X_2 \\ 0 & X_3 & X_2 \end{vmatrix} = 0$$

$$X_1(X_2^2 - X_2X_3) = 0$$

$$X_1X_2(X_2 - X_3) = 0$$

\Rightarrow We want to find $P(\{X_1 = 0\} \cup \{X_2 = 0\} \cup \{X_2 = X_3\})$. Then

$$\begin{aligned} & P(\{X_1 = 0\} \cup \{X_2 = 0\} \cup \{X_2 = X_3\}) \\ &= P(X_1 = 0) + P(X_2 = 0) + P(X_2 = X_3) - P(\{X_1 = 0\} \cap \{X_2 = 0\}) \\ &\quad - P(\{X_1 = 0\} \cap \{X_2 = X_3\}) - P(\{X_2 = 0\} \cap \{X_2 = X_3\}) \\ &\quad + P(\{X_1 = 0\} \cap \{X_2 = 0\} \cap \{X_2 = X_3\}) \\ &= P(X_1 = 0) + P(X_2 = 0) + P(X_2 = X_3) \\ &\quad - P(X_1 = 0)P(X_2 = 0) - P(X_1 = 0)P(X_2 = X_3) - 0 - 0 \\ &\quad \because \{X_2 = 0\} \cap \{X_2 = X_3\} = \emptyset \text{ as } X_3 \text{ is a geometric r.v.} \end{aligned}$$

Finding each of the individual probabilities:

$$P(X_1 = 0) = P(X_2 = 0) = \frac{\theta^0 e^{-\theta}}{0!} = e^{-\theta} \quad \because X_1 \text{ and } X_2 \text{ are both distributed as Poisson}(\theta)$$

$$\begin{aligned} P(X_2 = X_3) &= \sum_{n=0}^{\infty} P(X_2 = n, X_3 = n) \\ &= \sum_{n=0}^{\infty} P(X_2 = n)P(X_3 = n) && \because X_2 \text{ and } X_3 \text{ are independent} \\ &= \sum_{n=1}^{\infty} P(X_2 = n)P(X_3 = n) && \because P(X_3 = 0) = 0 \\ &= \sum_{n=1}^{\infty} \frac{\theta^n e^{-\theta}}{n!} = e^{-\theta} \left(\frac{1}{2}\right)^n \\ &= e^{-\theta} \sum_{n=1}^{\infty} \frac{\left(\frac{\theta}{2}\right)^n}{n!} \\ &= e^{-\theta} \left(e^{\frac{\theta}{2}} - 1\right) \\ &= e^{-\frac{\theta}{2}} - e^{-\theta} \end{aligned}$$

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Then the probability that the random matrix is singular is

$$\begin{aligned} &= P(X_1 = 0) + P(X_2 = 0) + P(X_2 = X_3) - P(X_1 = 0)P(X_2 = 0) \\ &\quad - P(X_1 = 0)P(X_2 = X_3) \\ &= e^{-\theta} + e^{-\theta} + e^{-\frac{\theta}{2}} - e^{-\theta} - e^{-2\theta} - e^{-\frac{3\theta}{2}} + e^{-2\theta} \\ &\quad \boxed{= e^{-\frac{\theta}{2}} + e^{-\theta} - e^{-\frac{3\theta}{2}}} \end{aligned}$$

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