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5. Since $X_1, \dots, X_n$ are independent random variables, then the joint probability distribution function of the random variables is

$$\begin{aligned} f_{X_1,\dots,X_n}(x_1,\dots,x_n) =& f_1(x_1)\dots f_n(x_n) \\ =& \begin{cases} \prod_{k=1}^{n} ke^{-kx_k}, & \text{if } (x_1,\dots,x_n) \in \mathbb{R}^n_{\geq 0} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

To compute $P(\min(X_1,\dots,X_n) = X_n)$, we need to integrate the joint pdf over the "region" where $X_1 \geq X_n$, and $X_2 \geq X_n$, ..., and $X_{n-1} \geq X_n$. Hence

$$\begin{aligned} & P(\min(X_1,\dots,X_n) = X_n) \\ &= \int_0^\infty \int_{x_n}^\infty \dots \int_{x_n}^\infty f_{X_1,\dots,X_n}(x_1,\dots,x_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_{n-1}\,\mathrm{d}x_n \\ &= \int_0^\infty \int_{x_n}^\infty \dots \int_{x_n}^\infty \prod_{k=1}^{n} ke^{-kx_k}\,\mathrm{d}x_1 \dots \mathrm{d}x_{n-1}\,\mathrm{d}x_n \\ &= \int_0^\infty ne^{-nx_n} \int_{x_n}^\infty (n-1)e^{-(n-1)x_{n-1}}\,\mathrm{d}x_{n-1} \dots \int_{x_n}^\infty e^{-x_1}\,\mathrm{d}x_1\,\mathrm{d}x_n \\ &= \int_0^\infty n\left(e^{-nx_n}\right)\left(e^{-(n-1)x_n}\right)\dots\left(e^{-x_n}\right)\,\mathrm{d}x_n \\ &= \int_0^\infty n\prod_{k=1}^{n} e^{-kx_n}\,\mathrm{d}x_n \\ &= \int_0^\infty ne^{-x_n\sum_{k=1}^{n} k}\,\mathrm{d}x_n \\ &= \int_0^\infty ne^{-\frac{n(n+1)}{2}x_n}\,\mathrm{d}x_n \end{aligned}$$

$\because$ the sum of n natural numbers is a triangular number. We then have

$$\begin{aligned} &= -\frac{2}{n+1}\left[e^{-\frac{n(n+1)}{2}x_n}\right]_{x_n=0}^{x_n=\infty} \\ \Rightarrow & \boxed{P(\min(X_1,\dots,X_n) = X_n) = \frac{2}{n+1}, \quad \forall n \in \mathbb{Z}_{>0}} \end{aligned}$$

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Computing $P(X_n < X_{n-1} < \cdots < X_2 < X_1)$ is done as follows:

$$\begin{aligned}
&P(X_n < X_{n-1} < \cdots < X_2 < X_1) \\
&= \int_0^\infty \int_{x_n}^\infty \int_{x_{n-1}}^\infty \cdots \int_{x_3}^\infty \int_{x_2}^\infty f_{X_1,\ldots,X_n}(x_1,\ldots,x_n)\,\mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_{n-2}\,\mathrm{d}x_{n-1}\,\mathrm{d}x_n \\
&= \int_0^\infty \int_{x_n}^\infty \int_{x_{n-1}}^\infty \cdots \int_{x_3}^\infty \int_{x_2}^\infty \prod_{k=1}^{n} ke^{-kx_k}\;\mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_{n-2}\,\mathrm{d}x_{n-1}\,\mathrm{d}x_n \\
&= \int_0^\infty ne^{-nx_n} \int_{x_n}^\infty (n-1)e^{-(n-1)x_{n-1}} \int_{x_{n-1}}^\infty (n-2)e^{-(n-2)x_{n-2}} \\
&\quad \cdots \int_{x_3}^\infty 2e^{-2x_2} \underbrace{\int_{x_2}^\infty e^{-x_1}\,\mathrm{d}x_1}_{\phi_1} \mathrm{d}x_2 \ldots \mathrm{d}x_{n-2}\,\mathrm{d}x_{n-1}\,\mathrm{d}x_n \qquad (1)
\end{aligned}$$

To evaluate this **beautiful integral**, let ϕ_1 be defined as above, and let

$$\phi_k = \int_{x_{k+1}}^\infty f_k(x_k)\phi_{k-1}\,\mathrm{d}x_k, \quad \forall k \in \mathbb{Z}_{>1}$$

Then $(\phi_k)_{k\in\mathbb{Z}_{>0}}$ defines a sequence of the inner integrals of (1). Let's try and find an explicit formula for each element in the sequence and use that to calculate the integral. Firstly let's find a general form for each element. For the first element, We have

$$\phi_1 = \int_{x_2}^\infty e^{-x_1}\,\mathrm{d}x_1 = e^{-x_2} \qquad (2)$$

Claim. $\forall k \in \mathbb{Z}_{>1}$, ϕ_k is of the form $\alpha_k e^{-\beta_k x_{k+1}}$ for some constants α_k and β_k.

Proof. (Induction)
Base case:
For $k = 1$, from (2), we see that $\alpha_1 = \beta_1 = 1$.
Inductive step:
For $k \geq 1$, assume the claim holds. Then

$$\begin{aligned}
\phi_{k+1} &= \int_{x_{k+2}}^\infty f_{k+1}(x_{k+1})\phi_k\,\mathrm{d}x_{k+1} \\
&= \int_{x_{k+2}}^\infty (k+1)e^{-(k+1)x_{k+1}}\alpha_k e^{-\beta_k x_{k+1}}\,\mathrm{d}x_{x+1} \\
&= \alpha_k(k+1)\int_{x_{k+2}}^\infty e^{-(\beta_k+k+1)x_{k+1}}\,\mathrm{d}x_{x+1} \\
&= \alpha_k(k+1)\frac{1}{\beta_k+k+1}e^{-(\beta_k+k+1)x_{k+2}}
\end{aligned}$$

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Let $\alpha_{k+1} = \alpha_k(k+1)\frac{1}{\beta_k+k+1}$ and $\beta_{k+1} = \beta_k + k + 1$. Then ϕ_k is of the form $\alpha_{k+1}e^{-\beta_{k+1}x_{(k+1)+1}}$. Since the base case and the inductive step both hold, the claim must be true for all positive integers. $\square$

The proof above provides recursive formulas for α_k and β_k (where $\alpha_1 = \beta_1 = 1$), that is $\forall k \in \mathbb{Z}_{>1}$,

$$\beta_k = \beta_{k-1} + k \qquad \begin{aligned} \alpha_k &= \alpha_{k-1}\frac{1}{\beta_{k-1}+k} \\ &= \alpha_{k-1}\frac{1}{\beta_k} \end{aligned}$$

Now we can solve for the for the explicit formula's of the coefficients. For β_k,

$$\begin{aligned} \beta_k &= \left(\sum_{i=1}^{k-1} i\right) + k \\ &= \sum_{i=1}^{k} i \\ &= \frac{k(k+1)}{2} \end{aligned}$$

For α_k,

$$\begin{aligned} \alpha_k &= \alpha_{k-1}\frac{k}{\frac{k(k+1)}{2}} \\ &= \alpha_{k-1}\frac{2}{k+1} \\ &= \left(\prod_{i=1}^{k-1}\frac{2}{i+1}\right)\frac{2}{k+1} \\ &= \prod_{i=1}^{k}\frac{2}{i+1} \\ &= \frac{2^k}{(k+1)!} \end{aligned}$$

Now that we know what each term in the sequence $(\phi_k)_{k\in\mathbb{Z}_{>0}}$ looks like, we can evaluate (1) using the $k = n$ term of the sequence, and letting $x_n = 0$, since the lower bound of the

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outermost integral is 0. Then

$$\begin{aligned} P(X_n < X_{n-1} < \cdots < X_2 < X_1) &= \phi_k\big|_{x_n=0}\,\mathrm{d}x_n \\ &= \frac{2^n}{(n+1)!}e^{-\frac{n(n+1)}{2}(0)} \\ &\boxed{= \frac{2^n}{(n+1)!}, \quad \forall n \in \mathbb{Z}_{>1}} \end{aligned}$$