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5. **Solution:**

Proof. Let $M \in \mathbb{R}_{>0}$ be such that $\sigma_i^2 < M$ for all $i \geq 1$. First,

$$\begin{aligned}
\mathrm{E}[\overline{X}_n] &= \mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[X_i] && \text{by the linearity of expectation} \\
&= \frac{1}{n}\sum_{i=1}^{n} \mu && \text{since } \mathrm{E}[X_i] = \mu \text{ by the premises} \\
&= \frac{1}{n} \cdot n \cdot \mu \\
&= \mu && (1)
\end{aligned}$$

We also have

$$\begin{aligned}
\mathrm{Var}(\overline{X}_n) &= \mathrm{Var}(\frac{1}{n}\sum_{i=1}^{n} X_i) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(X_i) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \sigma_i^2 \\
&\leq \frac{1}{n^2}\sum_{i=1}^{n} M && \text{by assumption} \\
&= \frac{1}{n^2} \cdot n \cdot M \\
&= \frac{M}{n} && (2)
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{P}(|\overline{X}_n - \mu| > \varepsilon) &= \mathrm{P}(|\overline{X}_n - \mathrm{E}[X_n]| > \varepsilon) && \text{by (1)} \\
&\leq \frac{\mathrm{Var}(\overline{X}_n)}{\varepsilon^2} && \text{by Chebyshev's inequality} \\
&\leq \frac{1}{\varepsilon^2} \cdot \frac{M}{n} && \text{by (2)}
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{1}{\varepsilon^2} \cdot \frac{M}{n} = 0$, it follows that $\lim_{n\to\infty} \mathrm{P}(|\overline{X}_n - \mu| > \varepsilon) = 0$. Therefore, $\overline{X}_n \xrightarrow{p} \mu$. $\square$