20053722
Student Number

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5. For $n = 0, 1, 2, 3, \ldots$, show that

$$\Gamma(n + \frac{1}{2}) = \frac{\sqrt{\pi}(2n)!}{4^n n!} \tag{1}$$

Solution:

Proof. (Induction)
Base case: For $n = 0$, we have that

$$\begin{aligned}\Gamma(0 + \frac{1}{2}) &= \int_0^\infty y^{\frac{1}{2}-1} e^{-y} \, \mathrm{d}y \\ &= \int_0^\infty \frac{e^{-y}}{\sqrt{y}} \, \mathrm{d}y\end{aligned}$$

Let $u = \sqrt{y}$. We then have

$$\Gamma(\frac{1}{2}) = 2 \int_0^\infty \underbrace{e^{-u^2}}_{I} \mathrm{d}u$$

where I is related to the pdf of a Normal(0, $\frac{1}{2}$) random variable. It follows that

$$\begin{aligned}\Gamma(\frac{1}{2}) &= 2\left(\frac{1}{2}\right)\left(\sqrt{\frac{1}{2}}\sqrt{2\pi}\right) \\ &= \sqrt{\pi} \\ &= \frac{\sqrt{\pi}(2 \cdot 0)!}{4^0 (0)!}\end{aligned}$$

which means (1) is true for $n = 0$.

Inductive step: Suppose (1) holds for $k \in \mathbb{Z}_{\geq 0}$. Then we have that

$$\begin{aligned}\Gamma((k+1) + \frac{1}{2}) &= \int_0^\infty y^{(k+1)+\frac{1}{2}-1} e^{-y} \, \mathrm{d}y \\ &= \int_0^\infty y^{k+\frac{1}{2}} e^{-y} \, \mathrm{d}y\end{aligned}$$

Let $u = y^{k+\frac{1}{2}}$ and $\mathrm{d}(v) = e^{-y}$. Then integration by parts gives us

$$\begin{aligned}\Gamma((k+1) + \frac{1}{2}) &= \left[\left(k + \frac{1}{2}\right) y^{k-\frac{1}{2}} e^{-y}\right]_0^\infty - \int_0^\infty \left(k + \frac{1}{2}\right)(-1) y^{k-\frac{1}{2}} e^{-y} \, \mathrm{d}y \\ &= 0 + \left(k + \frac{1}{2}\right) \int_0^\infty y^{k-\frac{1}{2}} e^{-y} \, \mathrm{d}y\end{aligned}$$

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$$= \left(k + \frac{1}{2}\right)\Gamma(k + \frac{1}{2})$$
$$= \left(k + \frac{1}{2}\right)\left(\frac{\sqrt{\pi}(2k)!}{4^k k!}\right)$$

by the inductive hypothesis. Then

$$\Gamma((k+1) + \frac{1}{2}) = \left(k + \frac{1}{2}\right)\left(\frac{\sqrt{\pi}(2k)!}{4^k k!}\right)\left(\frac{4}{4}\right)\left(\frac{k+1}{k+1}\right)$$

by the twice repeated application of **MULTIPLYING BY 1**! Hence, we have that

$$\Gamma((k+1) + \frac{1}{2}) = \frac{\sqrt{\pi}(2k)!(2k+2)(2k+1)}{4^{k+1}(k+1)!}$$
$$= \frac{\sqrt{\pi}(2k+2)!}{4^{k+1}(k+1)!}$$
$$= \frac{\sqrt{\pi}(2(k+1))!}{4^{k+1}(k+1)!}$$

Thus, (1) holds for $k+1$.

Since both the base case and inductive step have been performed, then by mathematical induction, the statement (1) holds for all $n \in \mathbb{Z}_{\geq 0}$. $\square$