20053722
Student Number

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4. **Solution:**

Proof. Let $Y = W$. Then define $g_1(x) = x\sqrt{\frac{y}{n}}$ and $g_2(y) = y$. Then Jacobian determinant is

$$\begin{aligned}
|J_g(x,y)| &= \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix} \\
&= \begin{vmatrix} \sqrt{\frac{y}{n}} & x\frac{1}{2\sqrt{ny}} \\ 0 & 1 \end{vmatrix} \\
&= \sqrt{\frac{y}{n}}
\end{aligned}$$

Since Z and W are independent, then the joint pdf of (Z, W) is

$$f_{Z,W}(z,w) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cdot \frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} w^{\frac{n}{2}-1} e^{-\frac{w}{2}}, & \text{if } z \in (-\infty,\infty) \text{ and } w \in (0,\infty) \\ 0, & \text{otherwise} \end{cases}$$

Hence, the joint pmf of (X, Y) is

$$\begin{aligned}
f_{X,Y}(x,y) &= f_{Z,W}(g_1(x,y), g_2(x,y))\,|J_g(x,y)| \\
&= f_{Z,W}\left(x\sqrt{\frac{y}{n}}, y\right) \left|\sqrt{\frac{y}{n}}\right| \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(x\sqrt{\frac{y}{n}}\right)^2}{2}} \cdot \frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} y^{\frac{n}{2}-1} e^{-\frac{y}{2}} \cdot \sqrt{\frac{y}{n}} \\
&= \frac{1}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \cdot \frac{1}{\sqrt{2}} \left(\frac{1}{2}\right)^{\frac{n}{2}} \cdot y^{\frac{n}{2}+\frac{1}{2}-1} \cdot e^{-\frac{y}{2}-\frac{x^2 y}{2n}} \\
&= \begin{cases} \frac{1}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \left(\frac{1}{2}\right)^{\frac{n+1}{2}} y^{\left(\frac{n+1}{2}\right)-1} e^{-\left(\frac{1+\frac{x^2}{n}}{2}\right)y}, & \text{if } x \in (-\infty,\infty) \text{ and } y \in (0,\infty) \\ 0, & \text{otherwise} \end{cases} \qquad (1)
\end{aligned}$$

Now, we can compute the pdf of X using (1).

$$\begin{aligned}
f(x) &= \int_{\mathbb{R}} f_{X,Y}(x,y)\,\mathrm{d}y \\
&= \int_0^\infty \frac{1}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \left(\frac{1}{2}\right)^{\frac{n+1}{2}} y^{\left(\frac{n+1}{2}\right)-1} e^{-\left(\frac{1+\frac{x^2}{n}}{2}\right)y}\,\mathrm{d}y \\
&= \frac{1}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \left(\frac{1}{2}\right)^{\frac{n+1}{2}} \int_0^\infty y^{\left(\frac{n+1}{2}\right)-1} e^{-\left(\frac{1+\frac{x^2}{n}}{2}\right)y}\,\mathrm{d}y \qquad (2)
\end{aligned}$$

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The integrand of the integral in (2) looks like the pdf of a Gamma$\left(\frac{n+1}{2}, \frac{\left(1+\frac{x^2}{n}\right)}{2}\right)$ random variable. Let's do some ***magic*** and **multiply by 1 TWICE** (i.e. $\mathbf{1} = \mathbf{1} \cdot \mathbf{1}$). (2) then becomes

$$f(x) = \frac{1}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \left(\frac{1}{2}\right)^{\frac{n+1}{2}} \left(\frac{\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{2}}}{\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{2}}}\right) \left(\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+1}{2})}\right) \int_0^\infty y^{\left(\frac{n+1}{2}\right)-1} e^{-\left(\frac{1+\frac{x^2}{n}}{2}\right)y}\, \mathrm{d}y$$

$$= \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \left(1+\frac{x^2}{n}\right)^{-\left(\frac{n+1}{2}\right)} \int_0^\infty \frac{\left(\frac{1+\frac{x^2}{n}}{2}\right)^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} y^{\left(\frac{n+1}{2}\right)-1} e^{-\left(\frac{1+\frac{x^2}{n}}{2}\right)y}\, \mathrm{d}y$$

$$= \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \left(1+\frac{x^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}, \quad x \in (-\infty, \infty) \qquad \square$$