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5. Showing that (1) is true.

$$\lim_{n\to\infty} P(X_{(n)} - \ln n \le x) = \exp\left(-e^{-x}\right) \tag{1}$$

Solution:

Proof. Since the mean of each $X_i,\ i \in \{1, \ldots, n\}$, is 1, that means $\lambda = 1$. Given the premise, the pdf of $X_(n)$ is

$$f_n(x) = nf(x)F(x)^{n-1}$$

Then the cdf of $X_(n)$ is

$$\begin{aligned} F_n(x) &= \int_{\infty}^{x} nf(x)F(x)^{n-1}\,\mathrm{d}x_n \\ &= F(x)^n \end{aligned}$$

where $F(x)$ is the common cdf of the X_i

Hence, the LHS of (1) is

$$\begin{aligned} \lim_{n\to\infty} P(X_{(n)} - \ln n \le x) &= \lim_{n\to\infty} P(X_{(n)} \le x + \ln n) \\ &= \lim_{n\to\infty} F_n(x + \ln n) \\ &= \lim_{n\to\infty} F(x + \ln n)^n \\ &= \lim_{n\to\infty} (1 - e^{-(x+\ln n)})^n \\ &= \lim_{n\to\infty} \left(1 + \frac{(-e^{-x})}{n}\right)^n \\ &= \exp\left(-e^{-x}\right) \\ &= \text{RHS of (1)} \end{aligned}$$

□