20053722
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(a) **Solution:**

Proof. Let $\varepsilon \in \mathbb{R}_{>0}$. Then

$$\begin{aligned} \mathrm{P}(X > M + \varepsilon) &= \mathrm{P}(X - X_n > M - X_n + \varepsilon) \\ &= \mathrm{P}(X - X_n > M - X_n + \varepsilon | M - X_n \geq 0)\,\mathrm{P}(M - X_n \geq 0) \\ &\quad + \mathrm{P}(X - X_n > M - X_n + \varepsilon | M - X_n < 0)\,\mathrm{P}(M - X_n < 0) \end{aligned}$$

by the law of total probability. Then

$$\mathrm{P}(X > M + \varepsilon) = \mathrm{P}(X - X_n > M - X_n + \varepsilon | M - X_n \geq 0)$$

since $\mathrm{P}(|X_n| \leq M = 1)$ by assumption. It follows that

$$\begin{aligned} \mathrm{P}(X > M + \varepsilon) &\leq \mathrm{P}(X - X_n > \varepsilon) \\ &\leq \mathrm{P}(|X - X_n| > \varepsilon) \to 0 \end{aligned}$$

as $n \to \infty$ since $X_n \to X$ by assumption. Hence, $\mathrm{P}(X > M + \varepsilon) = 0\ \forall \varepsilon \in \mathbb{R}_{>0}$. This implies that $\mathrm{P}(X > M) = 0$, and so $\mathrm{P}(X \leq M) = 1$. A similar argument can be used to show that $\mathrm{P}(-X < -M - \varepsilon) = 0$, and so $\mathrm{P}(-X \geq M) = 1$.

Therefore, $\mathrm{P}(|X| \leq M) = 1$. □

(b) **Solution:**

Proof. Let $r \in \mathbb{Z}_{>0}$ and let $B = \{\omega : |X(\omega)| \leq M\}$.

If $\omega \in A_n(\varepsilon)^c$, then $|X_n(\omega) - X(\omega)| \leq \varepsilon \implies |X_n(\omega) - X(\omega)|^r \leq \varepsilon^r$.

If $\omega \in A_n(\varepsilon) \cap B$, then

$$\begin{aligned} |X_n(\omega) - X(\omega)| &\leq |X_n(\omega)| + |X(\omega)| && \text{by the triangle inequality} \\ &\leq 2M && \text{since } \mathrm{P}(X_n \leq M) = 1 \text{ and } \omega \in B \\ \implies |X_n(\omega) - X(\omega)|^r &\leq (2M)^r \end{aligned}$$

Hence, $|X_n - X|^r \leq \varepsilon^r I_{A_n(\varepsilon)^c} + (2M)^r I_{A_n(\varepsilon)}\ \forall \omega \in B$. Since by part a, $\mathrm{P}((B) = 1)$, then

$$|X_n - X|^r \leq \varepsilon^r I_{A_n(\varepsilon)^c} + (2M)^r I_{A_n(\varepsilon)}$$

with probability 1.

By taking the expectations of both sides, we get that

$$\begin{aligned} \mathrm{E}[|X_n - X|^r] &\leq \mathrm{E}[\varepsilon^r I_{A_n(\varepsilon)^c} + (2M)^r I_{A_n(\varepsilon)}] \\ &= \varepsilon^r\,\mathrm{P}(A_n(\varepsilon)^c) + (2M)^r\,\mathrm{P}(A_n(\varepsilon)) \\ &\leq \varepsilon^r + (2M)^r\,\mathrm{P}(A_n(\varepsilon)) \end{aligned}$$

$\forall n \in \mathbb{Z}_{>0}$ because $\mathrm{P}(A_n(\varepsilon)^c) \leq 1$. Since $X_n \xrightarrow{p} X$ by assumption, it follows that $\lim_{n\to\infty} \mathrm{P}(A_n(\varepsilon)) = 0$.

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We then have

$$\lim_{n\to\infty} \mathrm{E}[|X_n - X|^r] \leq \varepsilon^r$$

$\forall \varepsilon \in \mathbb{R}_{>0}$, which implies that $\lim_{n\to\infty} \mathrm{E}[|X_n - X|^r] = 0$. Therefore, $X_n \xrightarrow{r.m.} X$. $\square$