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Student Number

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3.

Note. Since X_1, X_2, X_3 are iid r.v.'s, their joint pdf will be

$$f_X(x_1, x_2, x_3) = \begin{cases} f(x_1)f(x_2)f(x_3), & \text{if } x_1, x_2, x_3 \in S_X \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where S_X is the support of the random variables.

(a) The common distribution of X_i is the Uniform(0, 1) distribution.

Solution:

By (1), the joint pdf is then

$$f_X(x_1, x_2, x_3) = \begin{cases} 1, & \text{if } x_1, x_2, x_3 \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

That means the joint pdf of the order statistics $X_{(1)}, X_{(2)}, X_{(3)}$ is

$$\begin{aligned} f_{1,2,3}(x_1, x_2, x_3) &= \begin{cases} 3!(1), & \text{if } 0 < x_1 < x_2 < x_3 < \infty \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 6, & \text{if } x_1, x_2, x_3 \in [0, 1] \text{ and } x_1 < x_2 < x_3 \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{2}$$

Since X_1, X_2, X_3 are uniformly distributed, the probability that the second-largest value, $X_{(2)}$ (i.e., the median), is closer to the smallest value, $X_{(1)}$, rather than than to the largest value, $X_{(3)}$, is the same in either case. It can be seen in (2) that computing the probability will be symmetric in either case due to the lack of dependence on the values of x_1, x_2, x_3. Since the cardinality of the sample space in question is 2, the probability is then equal to $\boxed{\frac{1}{2}}$.

(b) The common distribution of X_i is the Exponential(λ) distribution.

Solution:

By (1), the joint pdf is then

$$f_X(x_1, x_2, x_3) = \begin{cases} \lambda^3 e^{-\lambda(x_1+x_2+x_3)}, & \text{if } x_1, x_2, x_3 \in [0, \infty) \\ 0, & \text{otherwise} \end{cases}$$

That means the joint pdf of the order statistics $X_{(1)}, X_{(2)}, X_{(3)}$ is

$$f_{1,2,3}(x_1, x_2, x_3) = \begin{cases} 6\lambda^3 e^{-\lambda(x_1+x_2+x_3)}, & \text{if } 0 < x_1 < x_2 < x_3 < \infty \\ 0, & \text{otherwise} \end{cases}$$

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The probability we want to compute is

$$
\begin{aligned}
& P(X_{(2)} - X_{(1)} < X_{(3)} - X_{(2)}) \\
&= P(2X_{(2)} < X_{(1)} + X_{(3)}) \\
&= P(X_{(2)} < \frac{X_{(1)} + X_{(3)}}{2}) \\
&= \iiint_{\mathbb{R}} f_{1,2,3}(x_1, x_2, x_3)\, \mathrm{d}x_2\, \mathrm{d}x_1\, \mathrm{d}x_3 \\
&= \int_0^\infty \int_0^{x_3} \int_{x_1}^{\frac{x_1+x_3}{2}} 6\lambda^3 e^{-\lambda(x_1+x_2+x_3)}\, \mathrm{d}x_2\, \mathrm{d}x_1\, \mathrm{d}x_3 \\
&= \int_0^\infty \int_0^{x_3} 6\lambda^2 \left[-e^{-\lambda(x_1+x_2+x_3)}\right]_{x_2=x_1}^{x_2=\frac{x_1+x_3}{2}} \mathrm{d}x_1\, \mathrm{d}x_3 \\
&= \int_0^\infty \int_0^{x_3} 6\lambda^2 \left(-e^{-\frac{3}{2}\lambda(x_1+x_3)} + e^{-\lambda(2x_1+x_3)}\right) \mathrm{d}x_1\, \mathrm{d}x_3 \\
&= \int_0^\infty \lambda \left(\left[4e^{-\frac{3}{2}\lambda(x_1+x_3)}\right]_{x_1=0}^{x_1=x_3} + \left[-3e^{-\lambda(2x_1+x_3)}\right]_{x_1=0}^{x_1=x_3}\right) \mathrm{d}x_3 \\
&= \int_0^\infty \lambda \left(4\left(e^{-3\lambda x_3} - e^{-\frac{3}{2}\lambda x_3}\right) - 3\left(e^{-3\lambda x_3} - e^{-\lambda x_3}\right)\right) \mathrm{d}x_3 \\
&= \left[-\frac{4}{3}e^{-3\lambda x_3} + \frac{8}{3}e^{-\frac{3}{2}\lambda x_3} + e^{-3\lambda x_3} - 3e^{-\lambda x_3}\right]_{x_3=0}^{x_3=\infty} \\
&= \frac{4}{3} - \frac{8}{3} - 1 + 3 \\
& \boxed{= \frac{2}{3}}
\end{aligned}
$$