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Student Number

Bryan Hoang
Name

1. A coin, having probability p of coming up heads, is successively flipped until at least one head and one tail have been flipped.

(a) Find the expected number of flips needed.

Solution:

Let

$$
\begin{aligned}
W &= \#\text{ of flips until one heads and one tails come up} \\
X &= \begin{cases} 1, & \text{if the first flip is heads} \\ 0, & \text{if the first flip is tails} \end{cases} \\
Y &= \#\text{ of flips until one head comes up starting from the second flip} \\
\implies Y &\sim \text{Geometric}(p) \\
Z &= \#\text{ of flips until one tails comes up starting from the second flip} \\
\implies Z &\sim \text{Geometric}(1-p)
\end{aligned}
$$

Then by the law of total expectation,

$$
\begin{aligned}
\mathrm{E}[W] &= \mathrm{E}[\mathrm{E}[W|X]] \\
&= \mathrm{E}[W|X=0]\,\mathrm{P}(X=0) + \mathrm{E}[W|X=1]\,\mathrm{P}(X=1) \\
&= (1+\mathrm{E}[Y])(1-p) + (1+\mathrm{E}[Z])(p) \\
&= \left(1+\frac{1}{p}\right)(1-p) + \left(1+\frac{1}{1-p}\right)p \\
&= 1 + \frac{1}{p} - p - 1 + p + \frac{p}{1-p} \\
&= \frac{1}{p} + \frac{p}{1-p} \\
&\boxed{= \frac{p^2-p+1}{p(1-p)}}
\end{aligned}
$$

(b) Find the expected number of flips that land on heads.

Solution:

Let X and Z be defined as above and let V be the number of heads flipped. Then by the law of total expectation, we have

$$
\begin{aligned}
\mathrm{E}[V] &= \mathrm{E}[\mathrm{E}[V|X]] \\
&= \mathrm{E}[V|X=0]\,\mathrm{P}(X=0) + \mathrm{E}[V|X=1]\,\mathrm{P}(X=1) \\
&= (1)(1-p) + (\mathrm{E}[Z])(p) \\
&= (1-p) + \frac{p}{1-p} \\
&\boxed{= \frac{p^2-p+1}{1-p}}
\end{aligned}
$$