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Student Number

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4. For the random matrix to be singular, we want

$$\begin{vmatrix} X_1 & 0 & 0 \\ 0 & X_2 & X_2 \\ 0 & X_3 & X_2 \end{vmatrix} = 0$$
$$X_1(X_2^2 - X_2X_3) = 0$$
$$X_1X_2(X_2 - X_3) = 0$$

$\Rightarrow$We want to find $P(\{X_1 = 0\} \cup \{X_2 = 0\} \cup \{X_2 = X_3\})$. Then

$$\begin{aligned}
&P(\{X_1 = 0\} \cup \{X_2 = 0\} \cup \{X_2 = X_3\}) \\
&= P(X_1 = 0) + P(X_2 = 0) + P(X_2 = X_3) - P(\{X_1 = 0\} \cap \{X_2 = 0\}) \\
&\quad - P(\{X_1 = 0\} \cap \{X_2 = X_3\}) - P(\{X_2 = 0\} \cap \{X_2 = X_3\}) \\
&\quad + P(\{X_1 = 0\} \cap \{X_2 = 0\} \cap \{X_2 = X_3\}) \\
&= P(X_1 = 0) + P(X_2 = 0) + P(X_2 = X_3) \\
&\quad - P(X_1 = 0)P(X_2 = 0) - P(X_1 = 0)P(X_2 = X_3) - 0 - 0 && \because\ P(X_3) = 0 \text{ for a geometric r.v.}
\end{aligned}$$

Finding each of the individual probabilities:

$$P(X_1 = 0) = P(X_2 = 0) = \frac{\theta^0 e^{-\theta}}{0!} = e^{-\theta} \quad \because X_1 \text{ and } X_2 \text{ are both distributed as Poisson}(\theta)$$

$$\begin{aligned}
P(X_2 = X_3) &= \sum_{n=0}^{\infty} P(X_2 = n, X_3 = n) \\
&= \sum_{n=0}^{\infty} P(X_2 = n)P(X_3 = n) && \because X_2 \text{ and } X_3 \text{ are independent} \\
&= \sum_{n=1}^{\infty} P(X_2 = n)P(X_3 = n) && \because P(X_3 = 0) = 0 \\
&= \sum_{n=1}^{\infty} \frac{\theta^n e^{-\theta}}{n!} = e^{-\theta}\left(\frac{1}{2}\right)^n \\
&= e^{-\theta} \sum_{n=1}^{\infty} \frac{\left(\frac{\theta}{2}\right)^n}{n!} \\
&= e^{-\theta}\left(e^{\frac{\theta}{2}} - 1\right) \\
&= e^{-\frac{\theta}{2}} - e^{-\theta}
\end{aligned}$$

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Then the probability that the random matrix is singular is

$$
\begin{aligned}
&= P(X_1 = 0) + P(X_2 = 0) + P(X_2 = X_3) - P(X_1 = 0)P(X_2 = 0) \\
&\quad - P(X_1 = 0)P(X_2 = X_3) \\
&= e^{-\theta} + e^{-\theta} + e^{-\frac{\theta}{2}} - e^{-\theta} - e^{-2\theta} - e^{-\frac{3\theta}{2}} + e^{-2\theta} \\
&\boxed{= e^{-\frac{\theta}{2}} + e^{-\theta} - e^{-\frac{3\theta}{2}}}
\end{aligned}
$$

$\square$