137438691328
Student Number

Bryan Hoang
Name

2.

$$\begin{aligned}
f_{X_1,X_2}(x_1,x_2) &= \int_{-\infty}^{\infty} f_X(x_1,x_2,x_3)\,\mathrm{d}x_3 \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_1-x_3)^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_2-x_3)^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_3^2}{2}}\,\mathrm{d}x_3 \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} e^{\frac{-x_1^2+2x_1x_3-x_3^2}{2}} e^{\frac{-x_2^2+2x_2x_3-x_3^2}{2}} e^{-\frac{x_3^2}{2}}\,\mathrm{d}x_3 \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} e^{\frac{-3x_3^2+2x_1x_3+2x_2x_3-x_1^2-x_2^2}{2}}\,\mathrm{d}x_3 \qquad (1)
\end{aligned}$$

Let's try and convert the integrand of the integral in (1) into the form $e^{\frac{-(x_3-\mu)^2}{2\sigma^2}}$ so that the integrand can be treated as a form of a probability distribution function for a Normal(μ,σ^2) random variable. Mainly, let's rearrange the integrand's exponent, **by adding 0**:

$$\begin{aligned}
&\frac{(-3x_3^2+2x_1x_3+2x_2x_3-x_1^2-x_2^2+\frac{2}{3}x_1^2+\frac{2}{3}x_2^2-\frac{2}{3}x_1x_2+)-\frac{2}{3}x_1^2-\frac{2}{3}x_2^2+\frac{2}{3}x_1x_2}{2} \\
&= \frac{-3\left(x_3^2-2x_3(\frac{1}{3}x_1+\frac{1}{3}x_2)+(\frac{1}{9}x_1^2+2(\frac{1}{9}x_1x_2)+\frac{1}{9}x_2^2)\right)}{2} - \frac{\frac{2}{3}x_1^2+\frac{2}{3}x_2^2-\frac{2}{3}x_1x_2}{2} \\
&= \frac{-\left(x_3-(\frac{1}{3}x_1+\frac{1}{3}x_2)\right)^2}{2(\frac{1}{\sqrt{3}})^2} - \frac{2x_2^2-2x_1x_2+2x_1^2}{2(3)} \qquad (2)
\end{aligned}$$

$$(2) \Rightarrow f_{X_1,X_2}(x_1,x_2) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{2x_2^2-2x_1x_2+2x_1^2}{2(3)}} \underbrace{\int_{-\infty}^{\infty} e^{\frac{-\left(x_3-(\frac{1}{3}x_1+\frac{1}{3}x_2)\right)^2}{2(\frac{1}{\sqrt{3}})^2}}\,\mathrm{d}x_3}_{I}$$

$\Rightarrow$ The integrand of I is related to the p.d.f. of a Normal$(\frac{1}{3}x_1+\frac{1}{3}x_2,\frac{1}{3})$ r.v.

$$\Rightarrow I = \sqrt{2\pi}\frac{1}{\sqrt{3}}$$

$$\Rightarrow \boxed{f_{X_1,X_2}(x_1,x_2) = \frac{1}{\sqrt{2\pi}^2\sqrt{3}}\, e^{-\frac{2x_2^2-2x_1x_2+2x_1^2}{2(3)}}, \quad \forall x_1,x_2 \in (-\infty,\infty)}$$

Then for the marginal pdf of X_1,

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\,\mathrm{d}x_2$$