20053722
Student Number

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2.

$$\begin{aligned}
E[X_1^2 X_2^2 \dots X_n^2] &= E[X_1^2] \dots E[X_n^2] && \because X_1, \dots, X_n \text{ are independent} \\
&= \int_{-\infty}^{\infty} x_1^2 f(x_1)\,\mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} x_n^2 f(x_n)\,\mathrm{d}x_n && \text{by LOTUS} \\
&= \int_0^{\infty} \frac{3x_1^2}{(x_1+1)^4}\,\mathrm{d}x_1 \cdots \int_0^{\infty} \frac{3x_n^2}{(x_n+1)^4}\,\mathrm{d}x_n
\end{aligned}$$

Let $I_i = \int_0^\infty \frac{3x_i^2}{(x_i+1)^4}\,\mathrm{d}x_i$. To evaluate I_i, let's apply partial fraction decomposition to its integrand.

$$\begin{aligned}
\frac{3x_i^2}{(x_i+1)^4} &= \frac{A}{x_i+1} + \frac{B}{(x_i+1)^2} + \frac{C}{(x_i+1)^3} + \frac{D}{(x_i+1)^4} \\
&= \frac{A(x_i+1)^3 + B(x_i+1)^2 + C(x_i+1)^1 + D}{(x_i+1)^4} \\
3x_i^2 &= A(x_i+1)^3 + B(x_i+1)^2 + C(x_i+1)^1 + D \\
&= Ax_i^3 + (3A+B)x_i^2 + (3A+2B+C)x_i + A + B + C + D \\
&\Rightarrow \begin{cases} A+B+C+D = 0 \\ 3A+2B+C = 0 \\ 3A+B = 3 \\ A = 0 \end{cases} \\
&\Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right) \\
&\Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right) \\
&\Rightarrow \begin{cases} A = 0 \\ B = 3 \\ C = -6 \\ D = 3 \end{cases}
\end{aligned}$$

$$\Rightarrow \frac{3x_i^2}{(x_i+1)^4} = \frac{3}{(x_i+1)^2} - \frac{6}{(x_i+1)^3} + \frac{3}{(x_i+1)^4}$$

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Therefore,

$$\begin{aligned}
I_i &= \int_0^\infty \frac{3}{(x_i+1)^2} - \frac{6}{(x_i+1)^3} + \frac{3}{(x_i+1)^4}\,\mathrm{d}x_i \\
&= \left[-\frac{3}{(x_i+1)} + \frac{3}{(x_i+1)^2} - \frac{1}{(x_i+1)^3}\right]_{x_i=0}^{x_i=\infty} \\
&= \frac{3}{(0+1)} - \frac{3}{(0+1)^2} + \frac{1}{(0+1)^3} \\
&= 1, \quad \forall i \in \{1, \ldots, n\}
\end{aligned}$$

Hence,

$$\begin{aligned}
E[X_1^2 X_2^2 \ldots X_n^2] &= 1 \times \cdots \times 1 \\
&\boxed{= 1}
\end{aligned}$$