20053722
Student Number

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3.

(a) **Solution:**

Proof. Suppose that $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$. Then

$$\begin{aligned}
&\mathrm{P}\Big(\Big\{\omega \in \Omega : \lim_{n\to\infty} (X_n + Y_n)(\omega) = (X+Y)(\omega)\Big\}\Big) \\
&= \mathrm{P}\Big(\Big\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) + Y_n(\omega) = X(\omega) + Y(\omega)\Big\}\Big) \\
&= \mathrm{P}\Big(\Big\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) + \lim_{n\to\infty} Y_n(\omega) = X(\omega) + Y(\omega)\Big\}\Big) \\
&= \mathrm{P}\Big(\Big\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) \text{ and } \lim_{n\to\infty} Y_n(\omega) = Y(\omega)\Big\}\Big) \\
&= 1
\end{aligned}$$

since

$$\mathrm{P}\Big(\Big\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\Big\}\Big) = 1$$

and

$$\mathrm{P}\Big(\Big\{\omega \in \Omega : \lim_{n\to\infty} Y_n(\omega) = Y(\omega)\Big\}\Big) = 1$$

Therefore, $X_n + Y_n \xrightarrow{a.s.} X + Y$. □

(b) **Solution:**

Proof. We have

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 &= \frac{1}{n}\sum_{i=1}^{n}\Big(X_i^2 - 2X_i\overline{X}_n + \overline{X}_n^2\Big) \\
&= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} 2X_i\overline{X}_n + \frac{1}{n}\sum_{i=1}^{n}\overline{X}_n^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}(X_i^2) - 2\overline{X}_n^2 + \overline{X}_n^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}(X_i^2) - \overline{X}_n^2 \qquad (1)
\end{aligned}$$

The first term is essentially a sample mean of the squared X_i's. For instance, define $Y_i = X_i^2$ and let $\overline{Y}_n$ be the sample mean of the $Y_i's$. Then since each Y_i are i.i.d. with finite mean and variance (since that X_i's are i.i.d. with finite mean and variance), then by the strong law of large numbers, we have that

$$\frac{1}{n}\sum_{i=1}^{n}(X_i^2) = \overline{Y}_n \xrightarrow{a.s.} \mathrm{E}[Y_i] = \mathrm{E}[X_i^2]$$

But we also know that

$$\mathrm{Var}(X_i) = \mathrm{E}[X_i^2] - \mathrm{E}[X_i]^2$$

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$$\sigma^2 = \mathrm{E}[X_i^2] - \mu^2$$
$$\implies \mathrm{E}[X_i^2] = \sigma^2 + \mu^2$$

Hence, $\frac{1}{n}\sum_{i=1}^{n}(X_i^2) \xrightarrow{a.s.} \sigma^2 + \mu^2$.

For the second term in (1), since $\overline{X}_n \xrightarrow{a.s.} \mu$ and $f(x) = x^2$ is a continuous function, then it follows that $\overline{X}_n^2 \xrightarrow{a.s.} \mu^2$.

By part a, we have that (1) $\xrightarrow{a.s.} \sigma^2 + \mu^2 - \mu^2 = \sigma^2$

Therefore, $\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 \xrightarrow{a.s.} \sigma^2$. $\square$