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1. *Proof.* Let $X_1, \dots, X_n$ be random variables. Suppose that for all real valued functions $g_1, \dots, g_n$,
$$E[g_1(X_1) \dots g_n(X_n)] = E[g_1(X_1)] \dots E[g_n(X_n)] \tag{1}$$
There are two cases where the expectations will exist.
Case 1. If $X_1, \dots, X_n$ are all jointly discrete random variables, then let $p_X(x_1, \dots, x_n)$ be their joint pmf and for $i \in \{1, \dots, n\}$, let $p_i(x_i)$, be the marginal pmf for X_i. Without loss of generality, define $g_1, \dots, g_n$ by
$$g_i(x_i) = \begin{cases} 1, & \text{if } x_i = a_i \\ 0, & \text{otherwise} \end{cases} \quad \text{for some } a_i \in \mathbb{R},\ \forall i \in \{1, \dots, n\}$$
Then the LHS of (1) becomes
$$\begin{aligned} E[g_1(X_1) \dots g_n(X_n)] &= \sum_x g_1(x_1) \dots g_n(x_n) p_X(x_1, \dots, x_n) && \text{by LOTUS} \\ &= P(X_1 = a_1, \dots, X_n = a_n) \end{aligned}$$
$\because$ of how $g_i,\ i \in \{1, \dots, n\}$, is defined as a pseudo indicator function.
The RHS of (1) is
$$\begin{aligned} E[g_1(X_1)] \dots E[g_n(X_n)] &= \sum_{x_1} g_1(x_1) p_1(x_1) \cdots \sum_{x_n} g_n(x_n) p_n(x_n) && \text{by LOTUS} \\ &= P(X_1 = a_1) \dots P(X_n = a_n) \end{aligned}$$
$\because$ of how $g_i,\ i \in \{1, \dots, n\}$, is defined as a pseudo indicator function.

We now have that
$$P(X_1 = a_1, \dots, X_n = a_n) = P(X_1 = a_1) \dots P(X_n = a_n), \quad \forall a_1, \dots, a_n \in \mathbb{R}$$
Thus, $X_1, \dots, X_n$ are independent in the discrete case.
Case 2. If $X_1, \dots, X_n$ are all jointly continuous random variables, then let $f_X(x_1, \dots, x_n)$ be their joint pdf and for $i \in \{1, \dots, n\}$, let $f_{X_i}(x_i)$, be the marginal pdf for X_i. Without loss of generality, define $g_1, \dots, g_n$ to be the indicator functions for arbitrary sets $A_1, \dots, A_n \subset \mathbb{R}$, respectively. That is, for $i \in \{1, \dots, n\}$,
$$g_i(x_i) = \begin{cases} 1, & \text{if } x_i \in A_i \\ 0, & \text{otherwise} \end{cases}$$
Then the LHS of (1) becomes
$$\begin{aligned} E[g_1(X_1) \dots g_n(X_n)] &= \int_{\mathbb{R}^n} g_1(x_1) \dots g_n(x_n) f_X(x_1, \dots, x_n) \,\mathrm{d}x_1 \dots \mathrm{d}x_n && \text{by LOTUS} \\ &= \int_{A_1, \dots, A_n} f_X(x_1, \dots, x_n) \,\mathrm{d}x_1 \dots \mathrm{d}x_n \\ &= P(X_1 \in A_1, \dots, X_n \in A_n) \end{aligned}$$

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The RHS of (1) is

$$\begin{aligned} E[g_1(X_1)]\dots E[g_n(X_n)] &= \int_{\mathbb{R}} g_1(x_1) f_{X_1}(x_1)\,\mathrm{d}x_1 \cdots \int_{\mathbb{R}} g_n(x_n) f_{X_n}(x_n)\,\mathrm{d}x_n \quad \text{by LOTUS} \\ &= \int_{A_1} f_{X_1}(x_1)\,\mathrm{d}x_1 \cdots \int_{A_n} f_{X_n}(x_n)\,\mathrm{d}x_n \\ &= P(X_1 \in A_1)\dots P(X_n \in A_n) \end{aligned}$$

We have that

$$P(X_1 \in A_1, \dots, X_n \in A_n) = P(X_1 \in A_1)\dots P(X_n \in A_n), \quad \forall A_1, \dots, A_n \subset \mathbb{R}$$

Thus, $X_1, \dots, X_n$ are independent in both cases. $\square$