20053722
Student Number

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4. **Solution:**

Proof. By the Cauchy-Schwartz inequality, we have that

$$|\operatorname{Cov}(I_A, I_B)| \leq \sqrt{\operatorname{Var}(I_A)\operatorname{Var}(I_B)}$$

Let's find the supremums of $\operatorname{Var}(I_A)$ and $\operatorname{Var}(I_B)$, which will be the same since they are both indicator functions of arbitrary events A and B. WLOG, examining $\operatorname{Var}(I_A)$ gives us

$$\begin{aligned}\operatorname{Var}(I_A) &= \operatorname{E}[I_A^2] - \operatorname{E}[I_A]^2 \\ &= \operatorname{P}(A) - \operatorname{P}(A)^2\end{aligned}$$

since I_A is an indicator function, so $I_A^2 = I_A$ and $\operatorname{E}[I_A] = \operatorname{P}(A)$. We then have

$$\begin{aligned}\sup_{\operatorname{P}(A)\in[0,1]} \{\operatorname{Var}(I_A)\} &= \sup_{x\in[0,1]} \{x(1-x)\} \\ &= \frac{1}{4} \\ &= \sup_{\operatorname{P}(B)\in[0,1]} \{\operatorname{Var}(I_B)\}\end{aligned}$$

Thus, it follows that

$$\begin{aligned}|\operatorname{Cov}(I_A, I_B)| &\leq \sqrt{\operatorname{Var}(I_A)\operatorname{Var}(I_B)} \\ &\leq \sqrt{\sup_{\operatorname{P}(A)\in[0,1]} \{\operatorname{Var}(I_A)\} \cdot \sup_{\operatorname{P}(B)\in[0,1]} \{\operatorname{Var}(I_B)\}} \\ &= \sqrt{\frac{1}{4}\cdot\frac{1}{4}} \\ &= \frac{1}{4} \\ \implies \operatorname{Cov}(I_A, I_B) &\in \left[-\frac{1}{4}, \frac{1}{4}\right]\end{aligned}$$

□

First, note that

$$\begin{aligned}\operatorname{Cov}(I_A, I_B) &= \operatorname{E}[I_A I_B] - \operatorname{E}[I_A]\operatorname{E}[I_B] \\ &= \operatorname{P}(A \cap B) - \operatorname{P}(A)\operatorname{P}(B)\end{aligned}$$

Now, consider rolling a fair six sided die. Let $A = B =$ "an odd number was rolled". Then

$$\begin{aligned}\operatorname{Cov}(I_A, I_B) &= \operatorname{P}(A \cap B) - \operatorname{P}(A)\operatorname{P}(B) \\ &= \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2} \\ &= +\frac{1}{4}\end{aligned}$$

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If we let A be as above and $B =$ "an even number was rolled", then

$$\begin{aligned}
\operatorname{Cov}(I_A, I_B) &= \mathrm{P}(A \cap B) - \mathrm{P}(A)\,\mathrm{P}(B) \\
&= 0 - \frac{1}{2} \cdot \frac{1}{2} \\
&= -\frac{1}{4}
\end{aligned}$$