20053722
Student Number

Bryan Hoang
Name

4. **Solution:**

The first jump from the jth best vertex has an equal probability of $\frac{1}{j-1}$ to land on any of the other $j-1$ better vertices where those vertices have their expected number of jumps to reach B. Then we have the following recursive relationship:

$$M_j = 1 + \frac{1}{j-1}(M_{j-1} + \cdots + M_2 + M_1)$$

where $M_1 = 0$ and $M_2 = 1$

$$= 1 + \frac{1}{j-1}\sum_{i=2}^{j-1} M_i \tag{1}$$

Let $P(j)$ be the statement that $M_j = \sum_{i=1}^{j-1} \frac{1}{i}$ for $j \in \mathbb{Z}_{\geq 2}$. We will prove $P(j)$ by strong induction.

Proof. (Strong Induction)
Base case: For $j = 2$, it is clear that

$$M_2 = 1 = \frac{1}{1}$$

Thus, $P(j)$ is true for $j = 2$.
Inductive step: Suppose $P(j)$ is true $\forall j \in \{2, \ldots, k\}$ for some $k \in \mathbb{Z}_{\geq 2}$. Then

$$\begin{aligned}
M_{k+1} &= 1 + \frac{1}{k}\sum_{i=2}^{k} M_i && \text{by (1)} \\
&= 1 + \frac{1}{k}\sum_{i=2}^{k}\left(\sum_{m=1}^{i-1}\frac{1}{m}\right) && \text{by the inductive hypothesis} \\
&= 1 + \frac{1}{k}\left((k-1)\frac{1}{1} + (k-2)\frac{1}{2} + \cdots + \frac{1}{k-1}\right) \\
&= 1 + \frac{1}{k}\left(\sum_{i=1}^{k-1}\frac{k-i}{i}\right) \\
&= 1 + \frac{1}{k}\left(\left(k\sum_{i=1}^{k-1}\frac{1}{i}\right) - (k-1)\right) \\
&= 1 + \sum_{i=1}^{k-1}\frac{1}{i} - 1 + \frac{1}{k} \\
&= \sum_{i=1}^{k}\frac{1}{i}
\end{aligned}$$

Since both the base case and inductive step have been performed, then by mathematical induction, the statement $P(j)$ holds for all $j \in \mathbb{Z}_{\geq 2}$. □