20053722
Student Number

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4. **Solution:**

Note.

$$(\min(X_1, \ldots, X_n), \max(X_1, \ldots, X_n)) = (X_{(1)}, X_{(n)})$$

Since $X_1, \ldots, X_n$ are mutually independent Uniform(0,1) random variables, the joint pdf of $X_{(1)}$ and $X_{(n)}$ is

$$\begin{aligned} f_{1,n}(x_1, x_n) &= n! f(x_1) f(x_n) \frac{(F(x_n) - F(x_1))^{n-2}}{(n-2)!} \\ &= n(n-1)(1)(1)(x_n - x_1)^{n-2} \\ &= \begin{cases} n(n-1)(x_n - x_1)^{n-2}, & \text{if } 0 < x_1 < x_n < 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Given the above observations, the probability that the interval $(X_{(1)}, X_{(n)})$ contains the value $\frac{1}{2}$ is

$$\begin{aligned} P(X_{(1)} < \frac{1}{2} < X_{(n)}) &= \int_{\frac{1}{2}}^{1} \int_{0}^{\frac{1}{2}} f_{1,n}(x_1, x_n) \, \mathrm{d}x_1 \, \mathrm{d}x_n \\ &= \int_{\frac{1}{2}}^{1} \int_{0}^{\frac{1}{2}} n(n-1)(x_n - x_1)^{n-2} \, \mathrm{d}x_1 \, \mathrm{d}x_n \\ &= \int_{\frac{1}{2}}^{1} n \left[-(x_n - x_1)^{n-1} \right]_{x_1=0}^{x_1=\frac{1}{2}} \mathrm{d}x_n \\ &= \int_{\frac{1}{2}}^{1} n \left(-\left(x_n - \frac{1}{2} \right)^{n-1} + x_n^{n-1} \right) \mathrm{d}x_n \\ &= \left[-\left(x_n - \frac{1}{2} \right)^{n} + x_n^{n} \right]_{x_n=\frac{1}{2}}^{x_n=1} \\ &= -\left(\frac{1}{2} \right)^{n} + 1 + 0 - \left(\frac{1}{2} \right)^{n} \\ &\boxed{= 1 - \left(\frac{1}{2} \right)^{n-1}} \end{aligned}$$

The smallest n such that this probability is at least 0.95 is

$$\begin{aligned} 1 - \left(\frac{1}{2} \right)^{n-1} &> 0.95 \\ \left(\frac{1}{2} \right)^{n-1} &< 0.05 \end{aligned}$$

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$$
\begin{aligned}
(n-1)(-\ln(2)) &< \ln\left(\frac{1}{20}\right) \\
n &> \frac{\ln(20)}{\ln(2)} + 1
\end{aligned}
$$

Then

$$
\begin{aligned}
\left\lfloor \frac{\ln(20)}{\ln(2)} + 1 \right\rfloor &= \lfloor 5.3\ldots \rfloor \\
&= 5
\end{aligned}
$$

$$
\Rightarrow \boxed{n > 5}
$$