20053722
Student Number

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2.

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \int_{-\infty}^{\infty} f_X(x_1,x_2,x_3)\,\mathrm{d}x_3 \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_1-x_3)^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_2-x_3)^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_3^2}{2}}\,\mathrm{d}x_3 \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} e^{\frac{-x_1^2+2x_1x_3-x_3^2}{2}} e^{\frac{-x_2^2+2x_2x_3-x_3^2}{2}} e^{-\frac{x_3^2}{2}}\,\mathrm{d}x_3 \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} e^{\frac{-3x_3^2+2x_1x_3+2x_2x_3-x_1^2-x_2^2}{2}}\,\mathrm{d}x_3 \end{aligned} \tag{1}$$

Let's try and convert the integrand of the integral in (1) into the form $e^{\frac{-(x_3-\mu)^2}{2\sigma^2}}$ so that the integrand can be treated as a form of a p.fd.f. for a Normal(μ,σ^2) random variable. Mainly, let's rearrange the integrand's exponent, **by adding 0**:

$$\begin{aligned} &\frac{(-3x_3^2+2x_1x_3+2x_2x_3-x_1^2-x_2^2+\frac{2}{3}x_1^2+\frac{2}{3}x_2^2-\frac{2}{3}x_1x_2+)-\frac{2}{3}x_1^2-\frac{2}{3}x_2^2+\frac{2}{3}x_1x_2}{2} \\ =&\frac{-3\left(x_3^2-2x_3(\frac{1}{3}x_1+\frac{1}{3}x_2)+(\frac{1}{9}x_1^2+2(\frac{1}{9}x_1x_2)+\frac{1}{9}x_2^2)\right)}{2}-\frac{\frac{2}{3}x_1^2+\frac{2}{3}x_2^2-\frac{2}{3}x_1x_2}{2} \\ =&\frac{-\left(x_3-(\frac{1}{3}x_1+\frac{1}{3}x_2)\right)^2}{2(\frac{1}{\sqrt{3}})^2}-\frac{2x_2^2-2x_1x_2+2x_1^2}{2(3)} \end{aligned} \tag{2}$$

$$(2) \Rightarrow f_{X_1,X_2}(x_1,x_2) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{2x_2^2-2x_1x_2+2x_1^2}{2(3)}} \underbrace{\int_{-\infty}^{\infty} e^{\frac{-\left(x_3-(\frac{1}{3}x_1+\frac{1}{3}x_2)\right)^2}{2(\frac{1}{\sqrt{3}})^2}}\,\mathrm{d}x_3}_{I_1}$$

$$\Rightarrow \text{The integrand of } I_1 \text{ is related to the p.d.f. of a Normal}(\frac{1}{3}x_1+\frac{1}{3}x_2,\frac{1}{3}) \text{ r.v.}$$

$$\Rightarrow I_1 = \sqrt{2\pi}\frac{1}{\sqrt{3}}$$

$$\Rightarrow \boxed{f_{X_1,X_2}(x_1,x_2) = \frac{1}{\sqrt{2\pi}^2\sqrt{3}}\, e^{-\frac{2x_2^2-2x_1x_2+2x_1^2}{2(3)}}, \quad \forall x_1,x_2\in(-\infty,\infty)}$$

Then for the marginal pdf of X_1,

$$\begin{aligned} f_{X_1}(x_1) &= \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\,\mathrm{d}x_2 \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}^2\sqrt{3}}\, e^{-\frac{2x_2^2-2x_1x_2+2x_1^2}{2(3)}}\,\mathrm{d}x_2 \\ &= \frac{1}{\sqrt{2\pi}^2\sqrt{3}} \int_{-\infty}^{\infty} e^{-\frac{2x_2^2-2x_1x_2+2x_1^2}{2(3)}}\,\mathrm{d}x_2 \end{aligned} \tag{3}$$

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Let's do the same thing we did for (1) to (3). i.e. **add 0.**

$$\begin{aligned}
&\frac{(-2x_2^2 + 2x_1x_2 - 2x_1^2 + \frac{3}{2}x_1^2) - \frac{3}{2}x_1^2}{2(3)} \\
=&\frac{-2(x_2^2 + 2x_2(\frac{1}{2}x_1) + \frac{1}{4}x_1^2)}{2(3)} - \frac{\frac{3}{2}x_1^2}{2(3)} \\
=&\frac{-(x_2 - \frac{1}{2}x_1)^2}{2\sqrt{\frac{3}{2}}^2} - \frac{1}{4}x_1^2 \qquad (4)
\end{aligned}$$

$$(4) \Rightarrow f_{X_1}(x_1) = \frac{1}{\sqrt{2\pi}^2\sqrt{3}} e^{-\frac{1}{4}x_1^2} \underbrace{\int_{-\infty}^{\infty} e^{\frac{-(x_2 - \frac{1}{2}x_1)^2}{2\sqrt{\frac{3}{2}}^2}} \, \mathrm{d}x_2}_{I_2}$$

$$\Rightarrow \text{The integrand of } I_2 \text{ is related to the p.d.f. of a Normal}(\frac{1}{2}x_1, \frac{3}{2}) \text{ r.v.}$$

$$\Rightarrow I_2 = \sqrt{2\pi}\sqrt{\frac{3}{2}}$$

$$\Rightarrow \boxed{f_{X_1}(x_1) = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{x_1^2}{2(2)}}, \quad \forall x_1 \in (-\infty, \infty)}$$

□