20053722
Student Number

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2. **Solution:**

Firstly, we know that $X_{(1)} \sim \text{Beta}(1, n)$ and $X_{(n)} \sim \text{Beta}(n, 1)$. Then the marginal and joint pdfs of $X_{(1)}$ and $X_{(n)}$ are

$$
\begin{aligned}
f_1(x) &= \begin{cases} \frac{1}{B(1,n)} x^{1-1}(1-x)^{n-1}, & \text{if } x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} n(1-x)^{n-1}, & \text{if } x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \\
f_n(x) &= \begin{cases} \frac{1}{B(n,1)} x^{n-1}(1-x)^{1-1}, & \text{if } x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} nx^{n-1}, & \text{if } x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \\
f_{1,n}(x_1, x_n) &= n! \frac{f(x_1)F(x_1)^{1-1}}{(1-1)!} \cdot \frac{(F(x_n) - F(x_1))^{n-1-1}}{(n-1-1)!} \cdot \frac{f(x_n)(1-F(x_n))^{n-n}}{(n-n)!} \\
&= \begin{cases} n(n-1)(x_n - x_1)^{n-2}, & \text{if } x \in (0,1) \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

The expectations and variances of $X_{(1)}$ and $X_{(n)}$ are

$$
\begin{aligned}
\mathrm{E}[X_{(1)}] &= \frac{1}{n+1} \\
\mathrm{E}[X_{(n)}] &= \frac{n}{n+1} \\
\mathrm{Var}(X_{(1)}) &= \frac{n}{(n+2)(n+1)^2} \\
\mathrm{Var}(X_{(n)}) &= \frac{n}{(n+2)(n+1)^2}
\end{aligned}
$$

We also have

$$
\begin{aligned}
\mathrm{E}[X_{(1)}X_{(n)}] &= \int_0^1 \int_0^{x_n} x_1 x_n f_{1,n}(x_1, x_n) \, \mathrm{d}x_1 \, \mathrm{d}x_n \\
&= \int_0^1 \int_0^{x_n} x_1 x_n n(n-1)(x_n - x_1)^{n-2} \, \mathrm{d}x_1 \, \mathrm{d}x_n
\end{aligned}
$$

Let $u = \frac{x_1}{x_n} \implies \mathrm{d}u = \frac{1}{x_n} \, \mathrm{d}x_1$. Then

$$
\begin{aligned}
\mathrm{E}[X_{(1)}X_{(n)}] &= \int_0^1 \int_0^1 u x_n x_n n(n-1) x_n^{n-2} (1-u)^{n-2} x_n \, \mathrm{d}u \, \mathrm{d}x_n \\
&= n(n-1) \int_0^1 x_n^{n+1} \int_0^1 \underbrace{u(1-u)^{n-2}}_{I} \, \mathrm{d}u \, \mathrm{d}x_n
\end{aligned}
$$

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where $I = B(2, n-1)f(x)$, with $f(x)$ being tha pdf of a $\text{Beta}(2, n-1)$ random variable.

$$\begin{aligned}
\implies \mathrm{E}[X_{(1)}X_{(n)}] &= n(n-1)B(2,n-1)\int_0^1 x_n^{n+1}\,\mathrm{d}x_n \\
&= n(n-1)\frac{\Gamma(2)\Gamma(n-1)}{\Gamma(n+1)} \cdot \frac{1}{n+2} \\
&= \frac{n(n-1)}{n+2} \cdot \frac{1!(n-2)!}{n!} \\
&= \frac{1}{n+2}
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{Cov}(X_{(1)}X_{(n)}) &= \mathrm{E}[X_{(1)}X_{(n)}] - \mathrm{E}[X_{(1)}]\,\mathrm{E}[X_{(n)}] \\
&= \frac{1}{n+2} - \frac{1}{n+1} \cdot \frac{n}{n+1} \\
&= \frac{1}{(n+2)(n+1)^2}
\end{aligned}$$

Therefore, the correlation coefficient between $X_{(1)}$ and $X_{(n)}$ is

$$\begin{aligned}
\rho(X_{(1)}X_{(n)}) &= \frac{\mathrm{Cov}(X_{(1)}X_{(n)})}{\sqrt{\mathrm{Var}(X_{(1)})\,\mathrm{Var}(X_{(n)})}} \\
&= \frac{\frac{1}{(n+2)(n+1)^2}}{\left(\frac{n}{(n+2)(n+1)^2}\right)} \\
&\boxed{= \frac{1}{n}}
\end{aligned}$$