20053722
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3.

(a) Suppose X is a non-negative random variable with $\mathrm{P}(X \geq 10) = 0.1$. What is the smallest possible value of $\mathrm{E}[X]$? Give a distribution for X under which the mean of X achieves this smallest value.

Solution:

By Markov's inequality, we have that

$$\begin{aligned} \mathrm{P}(X \geq 10) &\leq \frac{\mathrm{E}[X]}{10} \\ 0.1 &\leq \frac{\mathrm{E}[X]}{10} \\ &\boxed{\implies \mathrm{E}[X] \geq 1} \end{aligned}$$

Let X have a distribution such that takes that value 0 with probability $\frac{9}{10}$ and the value 10 with probability $\frac{1}{10}$. Then $\mathrm{P}(X \geq 10) = \mathrm{P}(X = 10) = 0.1$. By Markov's inequality, we have that $\mathrm{E}[X] \geq 1$. But we also have $\mathrm{E}[X] = 0 \cdot \frac{9}{10} + 10 \cdot \frac{1}{10} = 1$ as desired.

(b) We have shown that if X is a random variable with mean μ and variance σ^2, where $\sigma^2 < \infty$, then $\mathrm{P}(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$ for any positive integer k, using Chebyshev's inequality. For any positive integer k, give a distribution for X that satisfies $\mathrm{P}(|X - \mu| \geq k\sigma) = \frac{1}{k^2}$.

Solution:

Let

$$X = \begin{cases} -1, & \text{with probability } \frac{1}{2k^2} \\ 0, & \text{with probability } 1 - \frac{1}{k^2} \\ 1, & \text{with probability } \frac{1}{2k^2} \end{cases}$$

Then the mean of X is

$$\begin{aligned} \mu &= \mathrm{E}[X] \\ &= -1 \cdot \frac{1}{2k^2} + 0 \cdot \left(1 - \frac{1}{k^2}\right) + 1 \cdot \frac{1}{2k^2} \\ &= 0 \end{aligned}$$

The second moment of X is

$$\begin{aligned} \mathrm{E}[X^2] &= (-1)^2 \cdot \frac{1}{2k^2} + 0^2 \cdot \left(1 - \frac{1}{k^2}\right) + 1^2 \cdot \frac{1}{2k^2} \\ &= \frac{1}{k^2} \end{aligned}$$

Then the variance of X is

$$\sigma^2 = \mathrm{Var}(X)$$

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$$= \mathrm{E}[X^2] - \mathrm{E}[X]^2$$
$$= \frac{1}{k^2}$$

Thus, we have that

$$\begin{aligned}
\mathrm{P}(|X - \mu| \geq k\sigma) &= \mathrm{P}(|X - 0| \geq k \cdot \frac{1}{k}) \\
&= \mathrm{P}(|X| \geq 1) \\
&= \mathrm{P}(|X| = 1) \\
&= \mathrm{P}(X = -1) + \mathrm{P}(X = 1) \\
&= \frac{1}{2k^2} + \frac{1}{2k^2} \\
&= \frac{1}{k^2}
\end{aligned}$$

as desired.