

20053722
Student Number

Bryan Hoang
Name

4. **Solution:**

Note.

$$(\min(X_1, \dots, X_n), \max(X_1, \dots, X_n)) = (X_{(1)}, X_{(n)})$$

Since X_1, \dots, X_n are mutually independent Uniform(0,1) random variables, the joint pdf of $X_{(1)}$ and $X_{(n)}$ is

$$\begin{aligned} f_{1,n}(x_1, x_n) &= n! f(x_1) f(x_n) \frac{(F(x_n) - F(x_1))^{n-2}}{(n-2)!} \\ &= n(n-1)(1)(1)(x_n - x_1)^{n-2} \\ &= \begin{cases} n(n-1)(x_n - x_1)^{n-2}, & \text{if } 0 < x_1 < x_n < 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Given the above observations, the probability that the interval $(X_{(1)}, X_{(n)})$ contains the value $\frac{1}{2}$ is

$$\begin{aligned} P(X_{(1)} < \frac{1}{2} < X_{(n)}) &= \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} f_{1,n}(x_1, x_n) dx_1 dx_n \\ &= \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} n(n-1)(x_n - x_1)^{n-2} dx_1 dx_n \\ &= \int_{\frac{1}{2}}^1 n \left[-(x_n - x_1)^{n-1} \right]_{x_1=0}^{x_1=\frac{1}{2}} dx_n \\ &= \int_{\frac{1}{2}}^1 n \left(- \left(x_n - \frac{1}{2} \right)^{n-1} + x_n^{n-1} \right) dx_n \\ &= \left[- \left(x_n - \frac{1}{2} \right)^n + x_n^n \right]_{x_n=\frac{1}{2}}^{x_n=1} \\ &= - \left(\frac{1}{2} \right)^n + 1 + 0 - \left(\frac{1}{2} \right)^n \\ &= 1 - \left(\frac{1}{2} \right)^{n-1} \end{aligned}$$

The smallest n such that this probability is at least 0.95 is

$$\begin{aligned} 1 - \left(\frac{1}{2} \right)^{n-1} &> 0.95 \\ \left(\frac{1}{2} \right)^{n-1} &< 0.05 \end{aligned}$$

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$$(n-1)(-\ln(2)) < \ln\left(\frac{1}{20}\right)$$
$$n > \frac{\ln(20)}{\ln(2)} + 1$$

Then

$$\frac{\ln(20)}{\ln(2)} + 1 \approx 5.3$$
$$\Rightarrow \left\lceil \frac{\ln(20)}{\ln(2)} + 1 \right\rceil = 6$$
$$\Rightarrow \boxed{n > 6}$$