20053722
Student Number

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1.

(a) Finding $P(X_n = X_{(k)})$ for $k = 1, \ldots, n$.

**Solution:**

As noted by the given hint, since all orderings of $X_1, \ldots, X_n$ are equally likely,

$$P(X_n = X_{(1)}) = \cdots = P(X_n = X_{(n)}) \tag{1}$$

Given the sample space, we also have

$$\sum_{i=1}^{n} P(X_n = X_{(i)}) = 1 \tag{2}$$

Given (1) and (2), it follows that

$$\begin{aligned}
\sum_{i=1}^{n} P(X_n = X_{(i)}) &= 1 \\
\sum_{i=1}^{n} P(X_n = X_{(k)}) &= 1, && \text{for } k \in \{1, \ldots, n\}, \text{ by (1)} \\
nP(X_n = X_{(k)}) &= 1, && \text{for } k \in \{1, \ldots, n\}
\end{aligned}$$

$$\Rightarrow \boxed{P(X_n = X_{(k)}) = \frac{1}{n}, \text{ for } k \in \{1, \ldots, n\}}$$

(b) Showing that $(X_n, X_{(n)})$ does not have a joint pdf.

**Solution:**

*Proof.* (Contradiction)
Suppose that $(X_n, X_{(n)})$ does have a joint pdf, $f(x_1, x_2)$. From (a), we know that

$$P(X_n = X_{(n)}) = \frac{1}{n} \tag{3}$$

But since we know that the joint pdf of $(X_n, X_{(n)})$ exists, we can compute the LHS of (3) as follows:

$$\begin{aligned}
P(X_n = X_{(n)}) &= \iint_{\{(x,x) \mid x \in \mathbb{R}\}} f(x_1, x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&= \int_{\mathbb{R}} \int_{x_2}^{x_2} f(x_1, x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2
\end{aligned}$$

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\begin{aligned}
&= \int_{\mathbb{R}} 0 \, \mathrm{d}x_2 \\
&= 0 \\
&\neq \frac{1}{n}
\end{aligned}
$$

which contradicts (3) $\Rightarrow\!\Leftarrow$. Therefore, $(X_n, X_{(n)})$ does not have a joint pdf. $\square$