20053722
Student Number

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3.

i. Finding the joint pmf of (X_1, X_2, X_3, X_4), $p(x_1, x_2, x_3, x_4)$.
Note. The majority of the number crunching discussed here will be done in part ii below.

Now, the first thing to observe is that the total number of possible arrangements for the placement of the balls in the boxes is $4! = 24$. We can then think about the situation as a combinatorial problem, where the probability that a given configuration of fixed points occur is equal to the number of permutations which satisfy the fixed point condition divided by the total number of permutations possible. The notion of a **derangement** appears during some of the calculations in part ii.

The support of the joint pmf can be characterized as

$$\operatorname{supp}(p) = \{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^n \mid x_1 + x_2 + x_3 + x_4 = k,\ k \in \{0, 1, 2, 4\}\}$$

which contains 12 points.

The calculations for the joint pmf at $k = 0$ and $k = 4$ are seen in part ii, while the values for that pmf at $k = 1$ and $k = 2$ were obtained by dividing the pmf of X, p_X, at 1 and 2 by the corresponding number of points in the support of p. It then follows that the joint pmf is given by

$$\boxed{p(x_1, x_2, x_3, x_4) = \begin{cases} \dfrac{9}{24}, & \text{if } x_1 + x_2 + x_3 + x_4 = 0 \qquad \text{(1 point)} \\ \dfrac{1}{12}, & \text{if } x_1 + x_2 + x_3 + x_4 = 1 \qquad \text{(4 points)} \\ \dfrac{1}{24}, & \text{if } x_1 + x_2 + x_3 + x_4 = 2 \qquad \text{(6 points)} \\ \dfrac{1}{24}, & \text{if } x_1 + x_2 + x_3 + x_4 = 4 \qquad \text{(1 point)} \\ 0, & \text{otherwise} \qquad \text{(12 points total)} \end{cases}}$$

ii. Finding the pmf of X, $p_X(x)$, where X can be interpreted as the number of balls placed into their corresponding box.

The support of p_X is the set $\{0, 1, 2, 3, 4\}$. Let's calculate $p_X(x)$ for $x \in \{0, 1, 2, 4\}$. 3 is not a part of the support since it it is impossible to have 3 of the balls belong to their corresponding box and for the fourth one to not belong to its box.

For $x = 0$,

$$\begin{aligned} p_X(0) &= \frac{\binom{4}{0} !4}{4!} && \because \text{ it is the derangement of 4 objects} \\ &= \frac{\lfloor \frac{24}{e} + \frac{1}{2} \rfloor}{24} \\ &= \frac{9}{24} \end{aligned}$$

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For $x = 1$,

$$\begin{aligned} p_X(1) &= \frac{\binom{4}{1}!3}{24} && \because \text{ one ball is fixed, while the other 3 are om a derangement} \\ &= \frac{4\lfloor \frac{3!}{e} + \frac{1}{2} \rfloor}{24} \\ &= \frac{(4)(2)}{24} \\ &= \frac{1}{3} \end{aligned}$$

For $x = 2$

$$\begin{aligned} p_X(2) &= \frac{\binom{4}{2}!2}{24} \\ &= \frac{(6)\lfloor \frac{2!}{e} + \frac{1}{2} \rfloor}{24} \\ &= \frac{(6)(1)}{24} \\ &= \frac{1}{4} \end{aligned}$$

For $x = 4$,

$$\begin{aligned} p_X(4) &= \frac{\binom{4}{4}}{24} \\ &= \frac{1}{24} \end{aligned}$$

Therefore,

$$\boxed{p_X(x) = \begin{cases} \frac{9}{24}, & \text{if } x = 0 \\ \frac{1}{3}, & \text{if } x = 1 \\ \frac{1}{4}, & \text{if } x = 2 \\ \frac{1}{24}, & \text{if } x = 4 \\ 0, & \text{otherwise} \end{cases}}$$

iii. Finding $E[X]$.

$$\begin{aligned} E[X] &= \sum_{x \in \{0,1,2,4\}} x p_X(x) \\ &= (0)(\frac{9}{24}) + (1)(\frac{1}{3}) + (2)(\frac{1}{4}) + (4)(\frac{1}{24}) \\ &= \frac{1}{3} + \frac{1}{2} + \frac{1}{6} \\ &\boxed{= 1} \end{aligned}$$