20053722
Student Number

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3. A fair coin is tossed successively. Let K_n be the number of tosses until n consecutive heads occur. Find $\mathrm{E}[K_n]$ for $n \in \mathbb{Z}_{\geq}1$. Hint: Condition on K_{n-1} and use the boundary condition $\mathrm{E}[K_1] = 2$.

Solution:

Looking at $\mathrm{E}[K_n|K_{n-1} = k]$, for $k \in \mathbb{Z}_{\geq 1}$, we have two equally likely cases on the kth flip:

- If the $k+1$ flip is heads, we are done. So the expected number is $k+1$
- If the $k+1$ flip is tails, we 'start over'. So the expected number is $k+1+\mathrm{E}[K_n]$

Thus, $\mathrm{E}[K_n|K_{n-1} = k] = \frac{1}{2}(k+1) + \frac{1}{2}(k+1+\mathrm{E}[K_n])$.

It follows that

$$\begin{aligned}
\mathrm{E}[K_n|K_{n-1}] &= \frac{1}{2}(K_{n-1}+1) + \frac{1}{2}(K_{n-1}+1+\mathrm{E}[K_n]) \\
&= K_{n-1} + 1 + \frac{1}{2}\,\mathrm{E}[K_n]
\end{aligned}$$

By the law of total expectation, we have

$$\begin{aligned}
\mathrm{E}[K_n] &= \mathrm{E}[\mathrm{E}[K_n|K_{n-1}]] \\
&= \mathrm{E}[K_{n-1} + 1 + \frac{1}{2}\,\mathrm{E}[K_n]] \\
&= \mathrm{E}[K_{n-1}] + 1 + \frac{1}{2}\,\mathrm{E}[K_n]
\end{aligned}$$

which implies the recurrence relation $\mathrm{E}[K_n] = 2\,\mathrm{E}[K_{n-1}] + 2$.

Solving the recurrence relation:

$$\begin{aligned}
\mathrm{E}[K_n] &= 2^1\,\mathrm{E}[K_{n-1}] + 2(2^0) \\
&= 2^1(2\,\mathrm{E}[K_{n-2}] + 2) + 2(2^0) \\
&= 2^2\,\mathrm{E}[K_{n-2}] + 2(2^0 + 2^1) \\
&= 2^2(2\,\mathrm{E}[K_{n-3}] + 2) + 2(2^0 + 2^1) \\
&= 2^3\,\mathrm{E}[K_{n-3}] + 2(2^0 + 2^1 + 2^2) \\
&\;\;\vdots \\
&= 2^m\,\mathrm{E}[K_{n-m}] + 2\sum_{i=0}^{m-1} 2^i \\
&= 2^m\,\mathrm{E}[K_{n-m}] + 2\left(\frac{2^m - 1}{2-1}\right) \\
&= 2^m\,\mathrm{E}[K_{n-m}] + 2^{m+1} - 2 \\
&\;\;\vdots \\
&= 2^{n-1}\,\mathrm{E}[K_{n-(n-1)}] + 2^{(n-1)+1} - 2 \\
&= 2^{n-1}\,\mathrm{E}[K_1] + 2^n - 2
\end{aligned}$$

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$$= 2^{n-1}(2) + 2^n - 2$$
$$= 2^{n-1}(2) + 2^n - 2$$
$$\boxed{= 2^{n+1} - 2, \quad \forall n \in \mathbb{Z}_{\geq 1}}$$