20053722
Student Number

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1. **Solution:**

Proof. To prove the equality, we shall start with the RHS.

$$\begin{aligned}
\text{RHS} &= \mathrm{E}[\mathrm{Cov}(X_1, X_2)|Y] + \mathrm{Cov}(\mathrm{E}[X_1|Y], \mathrm{E}[X_2|Y]) \\
&= \mathrm{E}[\mathrm{E}[X_1X_1|Y] - \mathrm{E}[X_1|Y]\,\mathrm{E}[X_2|Y]] + \mathrm{E}[\mathrm{E}[X_1|Y]\,\mathrm{E}[X_2|Y]] \\
&\quad - \mathrm{E}[\mathrm{E}[X_1|Y]]\,\mathrm{E}[\mathrm{E}[X_2|Y]] \\
&= \mathrm{E}[\mathrm{E}[X_1X_1|Y]] - \mathrm{E}[\mathrm{E}[X_1|Y]\,\mathrm{E}[X_2|Y]] + \mathrm{E}[\mathrm{E}[X_1|Y]\,\mathrm{E}[X_2|Y]] - \mathrm{E}[X_1]\,\mathrm{E}[X_2]
\end{aligned}$$

by the linearity of expectation and the law of total expectation

$$= \mathrm{E}[X_1X_1] - \mathrm{E}[X_1]\,\mathrm{E}[X_2]$$

by the law of total expectation

$$\begin{aligned}
&= \mathrm{Cov}(X_1X_2) \\
&= \text{LHS}
\end{aligned}$$

□