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4. **Solution:**

Proof. Let $a \in [0, M)$ and $A = \{X \geq a\}$. Then we claim that

$$X \leq M I_A + a I_{A^C} \tag{1}$$

The inequality in (1) is true since if $I_A = 1$, then the inequality becomes

$$X \leq M = M(1) + a(0)$$

which is true by the hypothesis about X. If $I_{A^C} = 1$, then the inequality becomes

$$X < a = M(0) + a(1)$$

which is true when $I_{A^C} = 1$.

By taking the expectations of both sides of (1), we see that

$$\begin{aligned} \mathrm{E}[X] &\leq \mathrm{E}[M I_A + a I_{A^C}] \\ \mathrm{E}[X] &\leq M\,\mathrm{E}[I_A] + a\,\mathrm{E}[I_{A^C}] \end{aligned}$$

by the linearity of expectation. Then

$$\begin{aligned} \mathrm{E}[X] &\leq M\,\mathrm{P}(X \geq a) + a\,\mathrm{P}(X < a) \\ \mathrm{E}[X] &\leq M\,\mathrm{P}(X \geq a) + a - a\,\mathrm{P}(X \geq a) \\ \implies \mathrm{P}(X \geq a)(M - a) &\geq \mathrm{E}[X] - a \\ \mathrm{P}(X \geq a) &\geq \frac{\mathrm{E}[X] - a}{M - a} \end{aligned}$$

□