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5.

(a) **Solution:**

Proof. Suppose that $g(\cdot)$ is a function for which $\mathrm{E}[\mathrm{E}[Y|X]g(X)]$ and $\mathrm{E}[Yg(X)]$ exist. Then by LOTUS, we have

$$\begin{aligned}
\mathrm{E}[\mathrm{E}[Y|X]g(X)] &= \sum_x \mathrm{E}[Y|X=x]\cdot g(x)\cdot p_X(x) \\
&= \sum_x \sum_y y\cdot p_{Y|X}(y|x)\cdot g(x)\cdot p_X(x) \\
&= \sum_x \sum_y y\cdot \frac{p_{X,Y}(x,y)}{p_X(x)}\cdot g(x)\cdot p_X(x) \\
&= \sum_x \sum_y y\cdot g(x)\cdot p_{X,Y}(x,y) \\
&= \mathrm{E}[Yg(X)]
\end{aligned}$$

by LOTUS. □

(b) **Solution:**

Proof. Suppose that $\phi(\cdot)$ is a function satisfying $\mathrm{E}[\phi(X)g(X)] = \mathrm{E}[Yg(X)]$ for all functions $g(\cdot)$ for which the expectations exist. Then WLOG, consider the function

$$g_x(X) = \begin{cases} 1, & \text{if } X = x \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\mathrm{E}[\phi(X)g_x(X)] = \phi(x)$$

and

$$\mathrm{E}[Yg_x(X)] = \mathrm{E}[Y|X=x]$$

Therefore, we have that

$$\begin{aligned}
&\phi(x) = \mathrm{E}[Y|X=x] \\
\implies &\phi(X) = \mathrm{E}[Y|X]
\end{aligned}$$

with probability 1. □