20053722
Student Number

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5. **Solution:**

Proof. Firstly, we know that $\mathrm{E}[X] = \mathrm{E}[Y] = 0$ and $\mathrm{Var}(X) = \mathrm{Var}(Y) = 1$. Then we have that $\mathrm{E}[X^2] = \mathrm{E}[Y^2] = 1$ and $\mathrm{Cov}(X, Y) = \mathrm{E}[XY] \implies \rho(X, Y) = \mathrm{E}[XY]$.

To prove the inequality, we observe that

$$\begin{aligned}
\mathrm{E}[\max(X^2, Y^2)] &= \mathrm{E}\left[\frac{1}{2}(X^2 + Y^2) + \frac{1}{2}|(X - Y)(X + Y)|\right] \\
&= \mathrm{E}\left[\frac{1}{2}(X^2 + Y^2)\right] + \mathrm{E}\left[\frac{1}{2}|(X - Y)(X + Y)|\right] \\
&= \frac{1}{2}(\mathrm{E}[X^2] + \mathrm{E}[Y^2]) + \frac{1}{2}\,\mathrm{E}[|X - Y| \cdot |X + Y|] \\
&\leq 1 + \frac{1}{2}\sqrt{\mathrm{E}[|X - Y|^2]\,\mathrm{E}[|X + Y|^2]}
\end{aligned}$$

by the Cauchy-Schwartz inequality. Then

$$\begin{aligned}
\mathrm{E}[\max(X^2, Y^2)] &\leq 1 + \frac{1}{2}\sqrt{(\mathrm{E}[X^2] - 2\,\mathrm{E}[XY] + \mathrm{E}[Y^2])(\mathrm{E}[X^2] + 2\,\mathrm{E}[XY] + \mathrm{E}[Y^2])} \\
&= 1 + \sqrt{\frac{1}{4}(2 - 2\,\mathrm{E}[XY])(2 + 2\,\mathrm{E}[XY])} \\
&= 1 + \sqrt{1 - \mathrm{E}[XY]^2} \\
&= 1 + \sqrt{1 - \rho^2(X, Y)}
\end{aligned}$$

$\square$