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4. **Solution:**

To convert the product into a sum for the CTL, let's take the natural logarithm of both sides of the inequality. A transformation of random variables will be necessary, and so let $h(x) = \ln(x)$ and let $Y_i = h(X_i)$, for $i \in \{1, \dots, 100\}$. Then $g(y) = h^{-1}(y) = e^y$. With $f_X(x)$ as the common pdf of the X_i's, it follows that the common pdf of each Y_i is

$$\begin{aligned} f_Y(y) &= f_X(g(y))|J_g(y)| \\ &= f_X(e^y)e^y \\ &= \frac{1}{e}1_{\{y\le 1\}}e^y \\ &= e^{y-1}1_{\{y\le 1\}} \end{aligned}$$

The common expectation, μ, of each Y_i is

$$\begin{aligned} \mathrm{E}[Y] &= \int_{-\infty}^{1} y f_Y(y)\,\mathrm{d}y \\ &= \int_{-\infty}^{1} y e^{y-1}\,\mathrm{d}y \\ &= (y-1)e^{y-1}\Big|_{y=-\infty}^{y=1} \\ &= 0 \end{aligned}$$

and the common second moment of each Y_i is

$$\begin{aligned} \mathrm{E}[Y^2] &= \int_{-\infty}^{1} y^2 e^{y-1}\,\mathrm{d}y \\ &= (y^2 - 2y + 2)e^{y-1}\Big|_{y=-\infty}^{y=1} \\ &= 1 \end{aligned}$$

Thus the common variance, σ^2, of each Y_i is

$$\begin{aligned} \mathrm{Var}(Y) &= \mathrm{E}[Y^2] - \mathrm{E}[Y]^2 \\ &= 1 \end{aligned}$$

Evaluating the probability yields

$$\begin{aligned} \mathrm{P}\left(\prod_{i=1}^{100} X_i \le c\right) &= \mathrm{P}\left(\ln\left(\prod_{i=1}^{100} X_i\right) \le \ln(c)\right) \\ &= \mathrm{P}\left(\sum_{i=1}^{100} Y_i \le \ln(c)\right) \end{aligned}$$

Let $S_n = \sum_{i=1}^{100} Y_i$ and $Z \sim \mathrm{N}(0,1)$. Then by the central limit theorem, we have that

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} Z$$

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$$\implies \frac{S_{100}}{10} \dot{\sim} Z$$

Hence, $\mathrm{P}\big(\prod_{i=1}^{100} X_i \leq c\big) = \mathrm{P}(\frac{S_n}{10} \leq \frac{1}{10}\ln(c)) = \frac{1}{2}$ when $\frac{1}{10}\ln(c) = 0$ by the symmetry of the normal distribution.

Therefore, $\boxed{c = 1}$.