20053722
Student Number

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Name

3.

i. Finding $g_n(x)$.

Solution:

Let $Y_n = g_n(x) = nX_k$. To find the pdf of Y_n, let's compute the cdf of Y_n. With F_k as the cdf of the order statistic $X_{(k)}$, then

$$\begin{aligned}
F_{Y_n}(y) &= P(Y_n \leq y) \\
&= P(nX_{(k)} \leq y) \\
&= P\left(X_{(k)} \leq \frac{y}{n}\right) \\
&= F_k\left(\frac{y}{n}\right)
\end{aligned}$$

With f_k as the marginal pdf of the order statistic $X_{(k)}$, and f and F as the common pdf and cdf of the X_i's, respectively, it follows that the pdf of Y_n (and equivalently of $g_n(x)$) is

$$\begin{aligned}
f_{g_n}(y) &= f_{Y_n}(y) \\
&= \frac{\partial}{\partial y} F_k\left(\frac{y}{n}\right) \\
&= \frac{1}{n} f_k\left(\frac{y}{n}\right) \\
&= \frac{1}{n}\left(\frac{n!}{(k-1)!(n-k)!} \overbrace{f\left(\frac{y}{n}\right)}^{\neq 0 \text{ if } y \in (0,n)} F\left(\frac{y}{n}\right)^{k-1}\left(1 - F\left(\frac{y}{n}\right)\right)^{n-k}\right) \\
&\boxed{= \begin{cases} \frac{(n-1)!}{(k-1)!(n-k)!}\left(\frac{y}{n}\right)^{k-1}\left(1 - \frac{y}{n}\right)^{n-k}, & \text{if } y \in (0,n) \\ 0, & \text{otherwise} \end{cases}}
\end{aligned}$$

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ii. Showing that $g_n(x)$ converges pointwise to the pdf of a Gamma$(k, 1)$ distribution.

Solution:

Proof. Let $x \in \mathbb{R}$. Then

$$\begin{aligned}
&\lim_{n\to\infty} f_{g_n}(x) \\
&= \lim_{n\to\infty} \begin{cases} \frac{(n-1)!}{(k-1)!(n-k)!} \left(\frac{x}{n}\right)^{k-1} \left(1-\frac{x}{n}\right)^{n-k}, & \text{if } x \in (0,n) \\ 0, & \text{otherwise} \end{cases} \\
&= \lim_{n\to\infty} \begin{cases} \frac{1}{\Gamma(k)}\, x^{k-1} \frac{(n-1)!}{(n-k)!n^{k-1}} \left(1+\frac{(-x)}{n}\right)^{n} \left(1-\frac{x}{n}\right)^{-k}, & \text{if } x \in (0,n) \\ 0, & \text{otherwise} \end{cases} \\
&= \lim_{n\to\infty} \begin{cases} \frac{1}{\Gamma(k)}\, x^{k-1} \frac{(n-1)\ldots(n-k+1)(n-k)!}{(n-k)!n^{k-1}} \left(1+\frac{(-x)}{n}\right)^{n} \left(1-\frac{x}{n}\right)^{-k}, & \text{if } x \in (0,n) \\ 0, & \text{otherwise} \end{cases} \\
&= \lim_{n\to\infty} \begin{cases} \frac{1}{\Gamma(k)}\, x^{k-1} \frac{(n-1)\ldots(n-k+1)}{n^{k-1}} \left(1+\frac{(-x)}{n}\right)^{n} \left(1-\frac{x}{n}\right)^{-k}, & \text{if } x \in (0,n) \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

where $(n-1)\ldots(n-k+1)$ is a polynomial is degree $k-1$. We then have

$$\begin{aligned}
&= \begin{cases} \frac{1}{\Gamma(k)}\, x^{k-1} \lim\limits_{n\to\infty} \frac{(n-1)\ldots(n-k+1)}{n^{k-1}} \left(1+\frac{(-x)}{n}\right)^{n} \left(1-\frac{x}{n}\right)^{-k}, & \text{if } x \in (0,n) \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{\Gamma(k)}\, x^{k-1} \lim\limits_{n\to\infty} \frac{(n-1)\ldots(n-k+1)}{n^{k-1}} \lim\limits_{n\to\infty} \left(1+\frac{(-x)}{n}\right)^{n} \lim\limits_{n\to\infty} \left(1-\frac{x}{n}\right)^{-k}, & \text{if } x \in \lim\limits_{n\to\infty} (0,n) \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

Please excuse my notation for the support of the function.

$$\begin{aligned}
&= \begin{cases} \frac{1}{\Gamma(k)}\, x^{k-1} (1)\left(e^{-x}\right)(1), & \text{if } x \in (0,\infty) \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1^k}{\Gamma(k)}\, x^{k-1} \left(e^{-(1)x}\right), & \text{if } x \in (0,\infty) \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

which is the pdf of a Gamma$(k, 1)$ distribution. □