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2. (10 points) **Answer:**

Proof. Multiplying out the right-hand side gives

$$X^3 + AX + B = X^3 - (e_1 + e_2 + e_3)X^2 + (e_1e_2 + e_1e_3 + e_2e_3)X - e_1e_2e_3.$$

Comparing the coefficients yields

$$\begin{cases} 0 = e_1 + e_2 + e_3 & (1) \\ A = e_1e_2 + e_1e_3 + e_2e_3 & (2) \\ B = e_1e_2e_3. & (3) \end{cases}$$

Part 1. ($\Rightarrow$) First, suppose that

$$4A^3 + 27B^2 = 0 \tag{4}$$

Then substituting (2) and (3) into (4) yields

$$0 = (4e_2^3 + 12e_3e_2^2 + 12e_3^2e_2 + 4e_3^3)e_1^3 + (12e_3e_2^3 + 51e_3^2e_2^2 + 12e_3^3e_2)e_1^2 + (12e_3^2e_2^3 + 12e_3^3e_2^2)e_1 + 4e_3^3e_2^3.$$

Next, substituting in $e_1 = -e_2 - e_3$ from (1) lets us obtain

$$\begin{aligned} 0 &= -4e_2^6 - 12e_3e_2^5 + 3e_3^2e_2^4 + 26e_3^3e_2^3 + 3e_3^4e_2^2 - 12e_3^5e_2 - 4e_3^6 \\ &= -(e_2 - e_3)^2(e_2 + 2e_3)^2(e_3 + 2e_2)^2 \\ &= (e_2 - e_3)^2(e_1 - e_3)^2(e_1 - e_2)^2, \end{aligned}$$

which implies that at least two of the e_i are the same.

Part 2. ($\Leftarrow$) Next, suppose that two of the e_i are the same, say $e_2 = e_3$. Then

$$\begin{cases} 0 = e_1 + 2e_2 & (5) \\ A = 2e_1e_2 + e_2^2 & (6) \\ B = e_1e_2^2. & (7) \end{cases}$$

Substituting (5) into (6) and (7) gives

$$\begin{cases} A = -3e_2^2 \\ B = -2e_2^3. \end{cases}$$

Therefore,

$$\begin{aligned} 4A^3 + 27B^2 &= 4(-3e_2^2)^3 + 27(-2e_2^3)^2 \\ &= 0. \end{aligned}$$

$\square$