Student Number: [redacted] Name: Bryan Hoang

1. (10 points)

(a) **Answer:**

Proof.
Part 1. Proving that $\phi(e_G) = e_H$.
Beginning with the LHS of the equation,

$$\begin{aligned}
\phi(e_G) &= \phi(e_G * e_G) \\
&= \phi(e_G) * \phi(e_G) && \because \phi \text{ is a group homomorphism} \\
\Rightarrow \left(\phi(e_G) * \phi(e_G)\right) * \phi(e_G)^{-1} &= \phi(e_G) * \phi(e_G)^{-1} \\
\phi(e_G) * \left(\phi(e_G) * \phi(e_G)^{-1}\right) &= e_H \\
\phi(e_G) * e_H &= e_H \\
\phi(e_G) &= e_H.
\end{aligned}$$

Part 2. Proving that $\phi(g^{-1}) = \phi(g)^{-1}$.
Let $g \in \boldsymbol{G}$. Then

$$\begin{aligned}
\phi(g) * \phi(g^{-1}) &= \phi(g * g^{-1}) && \because \phi \text{ is a group homomorphism} \\
&= \phi(e_G) \\
&= e_H && \text{by the first part of the proof,}
\end{aligned}$$

and

$$\begin{aligned}
\phi(g^{-1}) * \phi(g) &= \phi(g^{-1} * g) && \because \phi \text{ is a group homomorphism} \\
&= \phi(e_G) \\
&= e_H && \text{by the first part of the proof.}
\end{aligned}$$

Since $\phi(g) * \phi(g^{-1}) = \phi(g^{-1}) * \phi(g) = e_H$, we have that $\phi(g^{-1})$ is the unique inverse of $\phi(g)$ in $\boldsymbol{H}$. Thus, we have shown that $\phi(g^{-1}) = \phi(g)^{-1}$. □

(b) **Answer:**

Proof. Let $g_1, g_2 \in \boldsymbol{G}$. We have

$$\begin{aligned}
\phi(g_1 * g_2) &= (g_1 * g_2)^2 \\
&= (g_1 * g_2) * (g_1 * g_2) \\
&= (g_1 * g_1) * (g_2 * g_2) && \because \text{the group is commutative} \\
&= g_1^2 * g_2^2 \\
&= \phi(g_1) * \phi(g_2).
\end{aligned}$$

Thus, the map ϕ is a homomorphism. □

Example. Consider the group defined by the set of all 2-by-2 real matrices $\boldsymbol{G} = \mathscr{M}_2(\mathbb{R})$ and the operation of

Student Number: [redacted] Name: Bryan Hoang

matrix multiplication $*$. Let $g_1 = \left(\begin{smallmatrix} 3 & 4 \\ 1 & 2 \end{smallmatrix}\right)$ and $g_2 = \left(\begin{smallmatrix} 6 & 2 \\ 3 & 2 \end{smallmatrix}\right)$. Then

$$\begin{aligned}
\phi(g_1 * g_2) &= (g_1 * g_2)^2 \\
&= (g_1 * g_2) * (g_1 * g_2) \\
&= \left(\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} * \begin{bmatrix} 6 & 2 \\ 3 & 2 \end{bmatrix}\right) * \left(\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} * \begin{bmatrix} 6 & 2 \\ 3 & 2 \end{bmatrix}\right) \\
&= \begin{bmatrix} 30 & 14 \\ 12 & 6 \end{bmatrix} * \begin{bmatrix} 30 & 14 \\ 12 & 6 \end{bmatrix} \\
&= \begin{bmatrix} 1068 & 504 \\ 432 & 204 \end{bmatrix} \\
&\neq \begin{bmatrix} 1026 & 408 \\ 402 & 160 \end{bmatrix} \\
&= \begin{bmatrix} 13 & 20 \\ 5 & 8 \end{bmatrix} * \begin{bmatrix} 42 & 16 \\ 24 & 10 \end{bmatrix} \\
&= \left(\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} * \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}\right) * \left(\begin{bmatrix} 6 & 2 \\ 3 & 2 \end{bmatrix} * \begin{bmatrix} 6 & 2 \\ 3 & 2 \end{bmatrix}\right) && = (g_1 * g_1) * (g_2 * g_2) \\
&= g_1^2 * g_2^2 \\
&= \phi(g_1) * \phi(g_2).
\end{aligned}$$

(c) **Answer:**

Proof. Let $g_1, g_2 \in \boldsymbol{G}$. We have

$$\begin{aligned}
\phi(g_1 * g_2) &= (g_1 * g_2)^{-1} \\
&= g_2^{-1} * g_1^{-1} \\
&= g_1^{-1} * g_2^{-1} && \because \boldsymbol{G} \text{ is a commutative group} \\
&= \phi(g_1) * \phi(g_2).
\end{aligned}$$

□

Example. Consider the group defined by the set of all 2-by-2 real matrices $\boldsymbol{G} = \mathscr{M}_2(\mathbb{R})$ and the operation of matrix multiplication $*$. Let $g_1 = \left(\begin{smallmatrix} 3 & 4 \\ 1 & 2 \end{smallmatrix}\right)$ and $g_2 = \left(\begin{smallmatrix} 6 & 2 \\ 3 & 2 \end{smallmatrix}\right)$. Then

$$\begin{aligned}
\phi(g_1 * g_2) &= (g_1 * g_2)^{-1} \\
&= g_2^{-1} * g_1^{-1} \\
&= \begin{bmatrix} 6 & 2 \\ 3 & 2 \end{bmatrix}^{-1} * \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}^{-1} \\
&= \frac{1}{6}\begin{bmatrix} 3 & -7 \\ -6 & 15 \end{bmatrix} \\
&\neq \frac{1}{12}\begin{bmatrix} 16 & -28 \\ -11 & 20 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}^{-1} * \begin{bmatrix} 6 & 2 \\ 3 & 2 \end{bmatrix}^{-1} \\
&= g_1^{-1} * g_2^{-1} \\
&= \phi(g_1) * \phi(g_2).
\end{aligned}$$