Student Number: [redacted] Name: Bryan Hoang

2. (10 points)

(a) **Answer:**

$$
\begin{aligned}
&h(X) \\
&= -\int_0^{\infty} f_X(x) \ln f_X(x) \, \mathrm{d}x \\
&= -\int_0^{\infty} \frac{1}{\sigma x\sqrt{2\pi}} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} \ln \frac{1}{\sigma x\sqrt{2\pi}} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} \, \mathrm{d}x \\
&= -\int_0^{\infty} \frac{1}{\sigma x\sqrt{2\pi}} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} \left(-\frac{(\ln x-\mu)^2}{2\sigma^2} - \ln \sigma x\sqrt{2\pi}\right) \mathrm{d}x
\end{aligned}
$$

Let $u = \ln x \Rightarrow \mathrm{d}u = \frac{1}{x}$ & $x = e^u$ & $u \in \mathbb{R}$. Then

$$
\begin{aligned}
&h(X) \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(u-\mu)^2}{2\sigma^2}} \left(\frac{(u-\mu)^2}{2\sigma^2} + \ln e^u \sigma\sqrt{2\pi}\right) \mathrm{d}u \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u-\mu)^2}{2\sigma^2}} \frac{(u-\mu)^2}{2\sigma^2} \, \mathrm{d}u + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u-\mu)^2}{2\sigma^2}} u \, \mathrm{d}u + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u-\mu)^2}{2\sigma^2}} \ln \sqrt{2\pi\sigma^2} \, \mathrm{d}u \\
&= \frac{1}{2\sigma^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u-\mu)^2}{2\sigma^2}} (u-\mu)^2 \, \mathrm{d}u}_{\sigma^2} + \underbrace{\int_{-\infty}^{\infty} u \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u-\mu)^2}{2\sigma^2}} \, \mathrm{d}u}_{\mu} + \frac{1}{2} \ln 2\pi\sigma^2 \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(u-\mu)^2}{2\sigma^2}} \, \mathrm{d}u}_{1} \\
&= \frac{1}{2} \ln e + \mu + \ln \frac{1}{2} 2\pi\sigma^2 \\
&\boxed{= \mu + \frac{1}{2} \ln 2\pi e\sigma^2 \text{ (nats)}}
\end{aligned}
$$

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(b) **Answer:**

Proof. Let g_X be an arbitrary pdf for the random variable X and f_X be the pdf of the log-normal distribution as from part (a). Then calculating the divergence between g_X and f_x yields

$$
\begin{aligned}
0 &\leq D_{\mathrm{KL}}(g_X \,||\, f_X) \\
&= \int_0^\infty g_X(x) \ln \frac{g_X(x)}{f_X(x)} \,\mathrm{d}x \\
&= \int_0^\infty g_X(x) \ln g_X(x) \,\mathrm{d}x - \int_0^\infty g_X(x) \ln f_X(x) \,\mathrm{d}x \\
&= -h(g_X) - \int_0^\infty g_X(x) \left[\ln\left(\frac{1}{\sigma x \sqrt{2\pi}} \right) - \frac{(\ln x - \mu)^2}{2\sigma^2} \right] \mathrm{d}x \\
&= -h(g_X) + \int_0^\infty g_X(x) \ln x\sqrt{2\pi\sigma^2} \,\mathrm{d}x + \int_0^\infty g_X(x) \frac{(\ln x - \mu)^2}{2\sigma^2} \,\mathrm{d}x \\
&= -h(g_X) + \int_0^\infty g_X(x) \ln x \,\mathrm{d}x + \ln(\sqrt{2\pi\sigma^2}) \int_0^\infty g_X(x) \,\mathrm{d}x + \frac{1}{2\sigma^2} \int_0^\infty g_X(x)(\ln(x) - \mu)^2 \,\mathrm{d}x \\
&= -h(g_X) + \mathrm{E}_{g_X}[\ln x] + \ln(\sqrt{2\pi\sigma^2}) + \frac{1}{2\sigma^2} \mathrm{E}_{g_X}[(\ln(x) - \mu)^2] \\
&= -h(g_X) + \mu + \frac{1}{2}\ln(2\pi\sigma^2) + \frac{1}{2} \\
&= -h(g_X) + \mu + \frac{1}{2}\ln(2\pi\sigma^2) + \frac{1}{2}\ln e \\
&= -h(g_X) + \mu + \frac{1}{2}\ln 2\pi e\sigma^2 \\
&= -h(g_X) + h(f_X) \quad \text{by part (a).}
\end{aligned}
$$

$\therefore h(g_X) \leq h(f_X)$, with equality iff $h(g_X) = h(f_X)$ almost everywhere. $\square$