

Student Number: 20053722Name: Bryan Hoang3. (a) **Answer:**

Since conditioning decreases entropy, we have that

$$\begin{aligned} H(X|Y) &\geq H(X|Y, Z) \\ -H(X|Z) &\leq -H(X|Y, Z) \end{aligned} \quad (1)$$

We also have the following identity for the conditional mutual information:

$$I(X; Y|Z) = H(X|Z) - H(X|Y, Z) \quad (2)$$

Thus,

$$\begin{aligned} H(X|Z) - H(X|Y) &\leq H(X|Z) - H(X|Y, Z) && \text{by (1)} \\ \boxed{H(X|Z) - H(X|Y) \leq I(X; Y|Z)} &&& \text{by (2)} \end{aligned}$$

(b) **Answer:**

If  $Z = g(Y)$ , then  $X \rightarrow Y \rightarrow Z$ . Thus,

$$\begin{aligned} I(X; Y) &\geq I(X; g(Y)) \\ \Rightarrow -I(X; g(Y)) &\geq -I(X; Y) \end{aligned} \quad (3)$$

By the properties of mutual information, we also have that

$$H(X|g(Y)) = H(X) - I(X; g(Y)) \quad \text{and} \quad H(X|Y) = H(X) - I(X; Y)$$

Then by (3), it follows that

$$\begin{aligned} H(X) - I(X; g(Y)) &\geq H(X) - I(X; Y) \\ \Rightarrow \boxed{H(X|g(Y)) \geq H(X|Y)} \end{aligned}$$

(c) **Answer:**

$$\begin{aligned} \frac{1}{2}H(X_1, X_2) &= \frac{1}{2}(H(X_2|X_1) + H(X_1)) && \text{by the chain rule} \\ &= \frac{1}{2}(H(X_2|X_1) + H(X_1)) && \because X_1 \text{ and } X_2 \text{ are identically distributed} \\ &\geq \frac{1}{2}(2 \cdot H(X_2|X_1)) && \because \text{conditioning reduces entropy} \\ &= H(X_2|X_1) \end{aligned}$$

Therefore, we have  $\boxed{H(X_2|X_1) \leq \frac{1}{2}H(X_1, X_2)}.$