

Student Number: XXXXXXXXXXName: Bryan Hoang

7. (15 points)

(i)

(a) **Answer:***Proof.* To prove both inequalities, first observe that

$$\begin{aligned}
& \frac{1}{n}H(X^n) + \frac{1}{n}D(P_{X^n}||P_{\hat{X}_n}) \\
&= \frac{1}{n} \left(- \sum_{x^n \in X^n} P_{X^n}(x^n) \log_2 P_{X^n}(x^n) - \sum_{x^n \in X^n} P_{X^n}(x^n) \log_2 \frac{P_{\hat{X}_n}(x^n)}{P_{X^n}(x^n)} \right) \\
&= -\frac{1}{n} \sum_{x^n \in X^n} P_{X^n}(x^n) \log_2 \frac{P_{\hat{X}_n}(x^n)}{P_{X^n}(x^n)} \\
&= \frac{1}{n} \sum_{x^n \in X^n} P_{X^n}(x^n) (-\log_2 P_{\hat{X}_n}(x^n)). \tag{1}
\end{aligned}$$

We also know that by the definition of $l(c_{x^n})$,

$$-\log_2 P_{\hat{X}_n}(x^n) \leq l(c_{x^n}) < -\log_2 P_{\hat{X}_n}(x^n) + 1, \quad \forall x^n \in \mathcal{X}^n. \tag{2}$$

Part 1. $\left(\frac{1}{n}H(X^n) + \frac{1}{n}D(P_{X^n}||P_{\hat{X}_n}) \leq \bar{R}_n \right)$

$$\begin{aligned}
& \frac{1}{n}H(X^n) + \frac{1}{n}D(P_{X^n}||P_{\hat{X}_n}) \\
&= \frac{1}{n} \sum_{x^n \in X^n} P_{X^n}(x^n) (-\log_2 P_{\hat{X}_n}(x^n)) && \text{by (1)} \\
&\leq \frac{1}{n} \sum_{x^n \in X^n} P_{X^n}(x^n) l(c_{x^n}) && \text{by (2)} \\
&= \bar{R}_n && \text{by the definition of } \bar{R}_n.
\end{aligned}$$

Part 2. $\left(\bar{R}_n < \frac{1}{n}H(X^n) + \frac{1}{n}D(P_{X^n}||P_{\hat{X}_n}) + \frac{1}{n} \right)$

$$\begin{aligned}
& \bar{R}_n \\
&= \frac{1}{n} \sum_{x^n \in X^n} P_{X^n}(x^n) l(c_{x^n}) && \text{by the definition of } \bar{R}_n. \\
&< \frac{1}{n} \sum_{x^n \in X^n} P_{X^n}(x^n) (-\log_2 P_{\hat{X}_n}(x^n)) + \frac{1}{n} \sum_{x^n \in X^n} P_{X^n}(x^n) && \text{by (2)} \\
&= \frac{1}{n}H(X^n) + \frac{1}{n}D(P_{X^n}||P_{\hat{X}_n}) + \frac{1}{n} && \text{by (1)}.
\end{aligned}$$

□

Student Number: XXXXXXXXXXName: Bryan Hoang(b) **Answer:**

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \hat{R}_n \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} H(X^n) + \frac{1}{n} D(P_{X^n} \| P_{\hat{X}_n}) \\
&= H(X_2 | X_1) + \lim_{n \rightarrow \infty} \frac{1}{n} D(P_{X^n} \| P_{\hat{X}_n}) \\
&= H(X_2 | X_1) + \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n D(P_{X_i | X^{i-1}} \| P_{\hat{X}_n}) \\
&\vdots \\
&= H(P_{X_2, X_1}; P_{\hat{X}_2, \hat{X}_1}) - H(P_{X_1}; P_{\hat{X}_1}).
\end{aligned}$$