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5. (10 points)

(a) **Answer:**

Let P_X and P_Y be the common marginal PMFs of X and Y respectively. Then we have

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \log_2 \frac{[P_{X^n,Y^n}(X^n,Y^n)]^{1-\alpha}}{[P_{X^n}(X^n)]^{1-\alpha}[P_{Y^n}(Y^n)]^{\alpha}} \\
&= \lim_{n\to\infty} \frac{1}{n} \left(\log_2 \left[\frac{P_{X^n,Y^n}(X^n,Y^n)}{P_{X^n}(X^n)} \right]^{1-\alpha} - \log_2 [P_{Y^n}(Y^n)]^{\alpha} \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \left((1-\alpha) \log_2 \frac{P_{X^n,Y^n}(X^n,Y^n)}{P_{X^n}(X^n)} - \alpha \log_2 P_{Y^n}(Y^n) \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \left((1-\alpha) \log_2 \frac{\prod_{i=1}^n P_{X,Y}(X_i,Y_i)}{\prod_{i=1}^n P_X(X_i)} - \alpha \log_2 \prod_{i=1}^n P_Y(Y_i) \right) && \because \text{the RVs are iid} \\
&= \lim_{n\to\infty} \frac{1}{n} \left((1-\alpha) \sum_{i=1}^n \log_2 \frac{P_{X,Y}(X_i,Y_i)}{P_X(X_i)} - \alpha \sum_{i=1}^n \log_2 P_Y(Y_i) \right) \\
&= (1-\alpha) \lim_{n\to\infty} \left(\frac{1}{n} \sum_{i=1}^n \log_2 \frac{P_{X,Y}(X_i,Y_i)}{P_X(X_i)} \right) - \alpha \lim_{n\to\infty} \left(\frac{1}{n} \sum_{i=1}^n \log_2 P_Y(Y_i) \right) \\
&= (1-\alpha)\,\mathrm{E}\left[\log_2 \frac{P_{X,Y}(X,Y)}{P_X(X)} \right] + \alpha\,\mathrm{E}[-\log_2 P_Y(Y)] && \text{by the WLLN} \\
&= (-1)(-1)(1-\alpha)\,\mathrm{E}\left[\log_2 P_{Y|X}(Y \mid X) \right] + \alpha H(Y) \\
&\boxed{= -(1-\alpha)H(Y|X) + \alpha H(Y)} && (1)
\end{aligned}
$$

(b) **Answer:**

Table 1: The joint distribution defined by $P_{X,Y}$

X \ Y	0	1	2	$P_X(x)$
0	$\frac{1-\epsilon}{2}$	0	$\frac{\epsilon}{2}$	$\frac{1}{2}$
1	0	$\frac{1-\epsilon}{2}$	$\frac{\epsilon}{2}$	$\frac{1}{2}$
$P_Y(y)$	$\frac{1-\epsilon}{2}$	$\frac{1-\epsilon}{2}$	ϵ	

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Based on Table 1, the necessary conditional PMF to calculate $H(Y|X)$ can be computed using

$$P_{Y|X} = \frac{P_{X,Y}}{P_X}$$

Then evaluating (1) from part (a) gives

$$\begin{aligned}
&-(1-\alpha)H(Y|X) + \alpha H(Y) \\
&= \frac{1}{2}\left[\cancel{2}\left(\frac{1-\epsilon}{\cancel{2}}\right)\log_2 \frac{\frac{1-\epsilon}{\cancel{2}}}{\frac{1}{\cancel{2}}} + \cancel{2}\left(\frac{\epsilon}{\cancel{2}}\right)\log_2 \frac{\frac{\epsilon}{\cancel{2}}}{\frac{1}{\cancel{2}}}\right] - \frac{1}{2}\left[\cancel{2}\left(\frac{1-\epsilon}{\cancel{2}}\right)\log_2 \frac{1-\epsilon}{2} + \epsilon \log_2 \epsilon\right] \\
&= \frac{1}{2}\left[(1-\epsilon)\log_2(1-\epsilon) + \epsilon\log_2\epsilon\right] - \frac{1}{2}\left[(1-\epsilon)\log_2(1-\epsilon) - (1-\epsilon)\cancelto{1}{\log_2 2} + \epsilon\log_2\epsilon\right] \\
&\boxed{= \frac{1}{2}(1-\epsilon)}
\end{aligned}$$