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2. (15 points)

(a) **Answer:**

Denote $f(D) = \sum_{i=1}^{7} D^{-l_i}$ For the integers $l_1, \ldots, l_7$ to satisfy Kraft's inequality, we want

$$\begin{aligned} f(D) &= \sum_{i=1}^{7} D^{-l_i} \\ &= D^{-l_1} + D^{-l_2} + D^{-l_3} + D^{-l_4} + D^{-l_5} + D^{-l_6} + D^{-l_7} \\ &= 2D^{-1} + 2D^{-2} + 2D^{-3} + D^{-4} \\ &\leq 1 \end{aligned}$$

Trying values of $D \in \mathbb{Z}_{\geq 2}$ yields

$$\begin{aligned} f(2) &= \frac{29}{16} \not\leq 1 \\ f(3) &= \frac{79}{81} \leq 1. \end{aligned}$$

$\therefore$ the integers $l_1, \ldots, l_7$ satisfy Kraft's inequality in base $\boxed{D = 3}$.

(b) **Answer:**

Claim. $\boxed{l_7^* = 3}$ makes $l_1, \ldots, l_6, l_7^*$ satisfy Kraft's inequality in base 3 with equality.

Proof.

$$\begin{aligned} f(3) &= 3^{-l_1} + 3^{-l_2} + 3^{-l_3} + 3^{-l_4} + 3^{-l_5} + 3^{-l_6} + 3^{-l_7^*} \\ &= 2(3)^{-1} + 2(3)^{-2} + 3(3)^{-3} \\ &= 1 \end{aligned}$$

□

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(c) **Answer:**

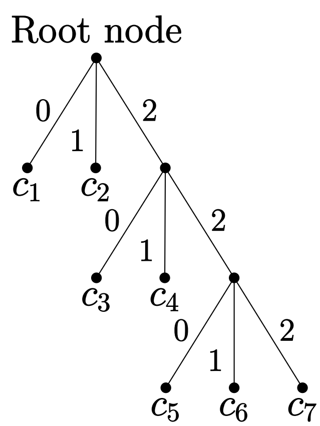

Figure 1: Ternary Huffman tree

From Figure 1, a ternary prefix code with the integers $l_1, \dots, l_6, l_7^*$ as its codeword lengths is $\boxed{\mathcal{C} = \{0, 1, 20, 21, 220, 221, 222\}}$.

(d) **Answer:**

Let $\{X_i\}_{i=1}^{\infty}$ be a DMS with alphabet $\mathcal{X} = \{a_1, \dots, a_7\}$ and PMF $p_X(a_i) = p_i = 3^{-l_i}$ where l_i is the length of codeword c_i for source symbol a_i for $i \in \{1, \dots, 7\}$. Then the Ternary Huffman code is

$$
\begin{aligned}
f : \mathcal{X} &\to \{0, 1, 2\}^* \\
a_1 &\to 0 \\
a_2 &\to 1 \\
a_3 &\to 20 \\
a_4 &\to 21 \\
a_5 &\to 220 \\
a_6 &\to 221 \\
a_7 &\to 222.
\end{aligned}
$$

The source's entropy is

$$
\begin{aligned}
H_3(X) &= -\sum_{i=1}^{7} p_i \log_3 p_i \\
&\boxed{= 1.\overline{4}\,\text{trits}}
\end{aligned}
$$