MTHE - MATH 474/874 - Information Theory
Fall 2021

Homework # 4

Due Date: Friday November 19, 2021

Material: *Channel capacity and joint source-channel coding.*

Readings: Chapter 4 of the textbook (except Section 4.4).

The referred problems are from the textbook.

(1) A DMC has input alphabet $\mathcal{X} = \{0, 1\}$ and output alphabet $\mathcal{Y} = \{0, 1, 2, 3\}$. Let X be the channel input, and Y be the channel output. Suppose that

$$P_{Y|X}(0|0) = P_{Y|X}(1|1) = \epsilon$$

where $0 \leq \epsilon \leq 1$.

(a) Determine (in terms of ϵ) the channel transition matrix that maximizes $H(Y|X)$.

(b) Using the transition matrix obtained in (a), find (in terms of ϵ) the channel capacity.

(2) Problem 4.6.

(3) For each of the following discrete memoryless channels (DMCs) find the capacity (in bits) and determine in each case the capacity achieving input distribution.

(a) *DMC 1*: $\big(\mathcal{X}_1 = \{a_1, a_2\}, \mathcal{Y}_1 = \{b_1, b_2\}, Q_1 = [P_{Y_1|X_1}]\big)$ with

$$Q_1 = \begin{bmatrix} 1 & 0 \\ \beta & 1-\beta \end{bmatrix}$$

where $0 < \beta < 1$.

(b) *DMC 2*: $\big(\mathcal{X}_2 = \{a_3, a_4, a_5\}, \mathcal{Y}_1 = \{b_3, b_4, b_5\}, Q_2 = [P_{Y_2|X_2}]\big)$ with

$$Q_2 = \begin{bmatrix} 1-\epsilon & \gamma\epsilon & (1-\gamma)\epsilon \\ (1-\gamma)\epsilon & 1-\epsilon & \gamma\epsilon \\ \gamma\epsilon & (1-\gamma)\epsilon & 1-\epsilon \end{bmatrix}$$

where $0 < \epsilon, \gamma < 1$.

(c) *DMC 3*: $\big(\mathcal{X}_3 = \{a_1, a_2, a_3, a_4, a_5\}, \mathcal{Y}_3 = \{b_1, b_2, b_3, b_4, b_5\}, Q_3 = [P_{Y_3|X_3}]\big)$ with

$$P_{Y_3|X_3}(b|a) = \begin{cases} P_{Y_1|X_1}(b|a) & \text{if } (a,b) \in \mathcal{X}_1 \times \mathcal{Y}_1 \\ P_{Y_2|X_2}(b|a) & \text{if } (a,b) \in \mathcal{X}_2 \times \mathcal{Y}_2 \\ 0 & \text{otherwise} \end{cases}$$

where $P_{Y_i|X_i}$, $\mathcal{X}_i$, $\mathcal{Y}_i$, $i = 1, 2$, are defined in parts (a) and (b) above.

(4) Determine (with justification) whether each of the following statements is *True* or *False*:

(a) Let X and Y be the input and output, respectively, of a DMC $(\mathcal{X}, \mathcal{Y}, Q_1 = [P_{Y|X}])$. Let Y be also fed to the input of another DMC $(\mathcal{Y}, \mathcal{Z}, Q_2 = [P_{Z|Y}])$ resulting in output Z. Assume that $|\mathcal{X}| = 8$, $|\mathcal{Y}| = 4$ and $|\mathcal{Z}| = 8$. Then the capacity of the channel with input X, output Z and transition matrix $Q_3 = Q_1 Q_2$ (i.e., the channel formed by concatenating Q_1 with Q_2) satisfies

$$C = \max_{P_X} I(X;Z) \le 2 \quad \text{(in bits)}.$$

(b) Let $X_1, X_2, \ldots, X_n$ be inputs to a DMC $(\mathcal{X}, \mathcal{Y}, P_{Y|X})$, with finite input alphabet $\mathcal{X}$ and finite output alphabet $\mathcal{Y}$, and let $Y_1, Y_2, \ldots, Y_n$ be the corresponding outputs. Then it is possible to construct the channel's transition distribution $P_{Y|X}$ and an input distribution P_{X^n} such that the inputs $X_1, X_2, \ldots, X_n$ are dependent while the corresponding outputs $Y_1, Y_2, \ldots, Y_n$ are independent.

(c) It is not possible to transmit a stationary binary Markov source $\{U_i\}_{i=1}^{\infty}$ with $P_{U_i|U_{i-1}}(0|0) = P_{U_i|U_{i-1}}(1|1) = 0.9$ via a rate-one block source-channel code over the memoryless binary erasure channel (BEC) with erasure probability $\alpha = 0.5$ and losslessly recover it for sufficiently long coding blocklengths.

(5) Given a channel with finite input and output alphabets $\mathcal{X}$ and $\mathcal{Y}$, respectively, and given codebook $\mathcal{C} = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_M\}$ of size M and blocklength n with $\boldsymbol{c}_i = (c_{i,1}, \ldots, c_{i,n}) \in \mathcal{X}^n$, if an n-tuple $y^n \in \mathcal{Y}^n$ is received at the channel output, then under *maximum a posteriori* (MAP) decoding, y^n is decoded into the codeword $\boldsymbol{c}^* \in \mathcal{C}$ that maximizes

$P(X^n = \boldsymbol{c}|Y^n = y^n)$ among all codewords $\boldsymbol{c} \in \mathcal{C}$. Assume that the channel input n-tuple is governed by the following distribution

$$P_{X^n}(x^n) = \begin{cases} Q(\boldsymbol{c}_i) & \text{if } x^n \in \mathcal{C} \\ 0 & \text{if } x^n \notin \mathcal{C} \end{cases}$$

where $Q(\cdot)$ is an assigned distribution on the codewords in $\mathcal{C}$ (such that $Q(\boldsymbol{c}_i) \geq 0$, $i = 1, \ldots, M$, and $\sum_{i=1}^{M} Q(\boldsymbol{c}_i) = 1$).

(a) Show that MAP decoding minimizes the probability of error.

(b) Assume that the channel is the memoryless binary symmetric channel (BSC) with crossover probability $\varepsilon = 0.1$. Assume also that the used code is $\mathcal{C} = \{\boldsymbol{c}_1, \boldsymbol{c}_2\}$, where the codewords are given by $\boldsymbol{c}_1 = 00$ and $\boldsymbol{c}_2 = 11$ with assigned probabilities $Q(\boldsymbol{c}_1) = p = 1 - Q(\boldsymbol{c}_2)$, where $0.1 \leq p < 1/2$.

   (i) Determine the code's MAP decoding rule and calculate its resulting probability of error in terms of p.

   (ii) Find the value of $p \in [0.1, 1/2)$ that minimizes the probability of error and calculate the resulting (minimal) probability of error.

(6) Answer the following problems.

(i) Consider a discrete memoryless channel with input alphabet

$$\mathcal{X} = \{0, 1, \ldots, q, q+1, q+2, q+3\},$$

output alphabet

$$\mathcal{Y} = \{0, 1, \ldots, q, q+1, q+2, q+3, q+4\},$$

where $q \geq 2$ is a fixed integer, and transition distribution $P_{Y|X}$ given by

$$P_{Y|X}(y|x) = \begin{cases} 1-\alpha & \text{if } y = x \text{ and } x \in \{0, 1, \ldots, q-1\} \\ \alpha & \text{if } y = q \text{ and } x \in \{0, 1, \ldots, q-1\} \\ 1-\epsilon & \text{if } y = x+1 \text{ and } x \in \{q, q+1, q+2, q+3\} \\ \epsilon & \text{if } y = q+4 \text{ and } x = q \\ \epsilon & \text{if } y = x \text{ and } x \in \{q+1, q+2, q+3\} \\ 0 & \text{otherwise} \end{cases}$$

where $0 \leq \alpha, \epsilon \leq 1$.

(a) Find the capacity C of the channel (in bits) in terms of α, ϵ and q.

(b) Determine the values of α and ϵ that yield the largest possible value of C.

(c) We wish to transmit an arbitrary source $\{U_i\}_{i=1}^{\infty}$ with memory via a rate-one source-channel code over this channel with parameters α and ϵ obtained in part (b). What is the largest possible size of the source alphabet $\mathcal{U}$ for which the source can be reliably sent of the channel? Describe the rate-one source-channel code which can achieve such communication.

(ii) [**MATH 874 only**] Problem 4.8.

(7) *Feedback Capacity.*

(i) Problem 4.28, Part (a).

Hint: Since $C_{op,FB} \geq C_{op}$, it suffices to show that $C_{op,FB} \leq C_{op} = \max_{P_X} I(X;Y)$ to conclude that $C_{op,FB} = C_{op}$. In other words, show (invoking the data processing and Fano inequalities) that for any achievable rate with feedback R,

$$R \leq C_{op} = \max_{P_X} I(X;Y),$$

and hence $C_{op,FB} = \sup\{R\colon R \text{ is achievable with feedback}\} \leq C_{op}$.

(ii) [**MATH 874 only**] Problem 4.28, Part (b).