MTHE - MATH 474/874 - Information Theory
Fall 2021

Homework # 5

Due Date: Monday December 6, 2021

Material: *Joint source-channel coding, differential entropy and Gaussian channels.*

Readings: Section 4.6 and Chapter 5 of the textbook.

The referred problems are from the textbook.

(1) Consider the binary Polya contagion Markov source treated in Example 3.17 in the textbook with memory $M = 2$. We are interested in sending this source over the memoryless binary symmetric erasure channel (BSEC) with crossover probability ε and erasure probability α using rate-R_{sc} block source-channel codes.

(a) Write down the sufficient condition for reliable transmissibility of the source over the BSEC via rate-R_{sc} source-channel codes in terms of ε, α, R_{sc} and the source parameters $\rho := R/T$ and $\delta := \Delta/T$.

(b) If $\rho = \delta = 1/2$ and $\varepsilon = \alpha = 0.1$, determine the permissible range of rates R_{sc} for reliably communicating the source over the channel.

(2) Answer the following questions.

(a) Let X be a log-normal random variable with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ and pdf given by

$$f_X(x) = \frac{1}{\sigma x \sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}, \quad x > 0.$$

Determine its differential entropy (in nats).

(b) Show that among all continuous random variables X admitting a pdf with support $(0, \infty)$ and finite differential entropy and satisfying $E[\ln(X)] = \mu$ and $E\big[(\ln(X) - \mu)^2\big] = \sigma^2$, where $\mu \in \mathbb{R}$ and $\sigma > 0$ are fixed parameters, the log-normal random variable with parameters μ and σ maximizes differential entropy.

(3) Rényi information measures: Given two pdfs f and g with common support $\mathcal{S} \subseteq \mathbb{R}$, consider the following Rényi information measures (in nats) of *order* α, for $\alpha > 0$, $\alpha \neq 1$.

Rényi differential entropy of f: $h_\alpha(f) := \frac{1}{1-\alpha} \ln \left(\int_{\mathcal{S}} f(x)^\alpha \, dx \right)$.

Rényi divergence between f *and* g: $D_\alpha(f \| g) := \frac{1}{\alpha-1} \ln \left(\int_{\mathcal{S}} f(x)^\alpha g(x)^{1-\alpha} \, dx \right)$.

(a) Determine each of the above quantities for the Gaussian densities $f \sim \mathcal{N}(0, \sigma_1^2)$ and $g \sim \mathcal{N}(0, \sigma_2^2)$.

(b) Find the limits as $\alpha \to 1$ of the measures obtained in (a) and comment qualitatively.

(4) Answer the following questions.

(a) *Integral analogue of the log-sum inequality*: Given non-negative functions $a(\cdot)$ and $b(\cdot)$ on $\mathbb{R}^n$, show that

$$\int a(x^n) \ln \frac{a(x^n)}{b(x^n)} \, dx^n \geq a \ln \frac{a}{b}$$

where $a := \int a(x^n) \, dx^n$ and $b := \int b(x^n) \, dx^n$ and all integrals are assumed to exist.

(b) *Data processing inequality for the divergence:* Consider a memoryless continuous channel with real input and output alphabets $\mathcal{X} = \mathcal{Y} = \mathbb{R}$ and transition pdf $f_{Y|X}$ with support $\mathbb{R}^2$. Let random variables X_1 and X_2, having common support $\mathbb{R}$ and respective pdfs f_{X_1} and f_{X_2}, be two possible inputs to the channel, with corresponding channel outputs Y_1 and Y_2, respectively. Show that

$$D(X_1 \| X_2) \geq D(Y_1 \| Y_2).$$

(5) Determine (with justification) whether each of the following statements is *True* or *False*:

(a) If X and Y are real-valued independent random variables, then

$$h(-4X - 3Y - 10) \geq h(X) + 2 \quad \text{(in bits)}.$$

(b) Suppose that random variables X, Y and Z are jointly Gaussian, each with mean 0 and variance 1. Assume that $X \to Y \to Z$ and that $E[XY] = \rho$, where $0 < \rho < 1$. Then

$$I(X; Z) > \frac{1}{2} \log_2 \left[\frac{1}{1 - \rho^2} \right].$$

(c) Consider a network of two parallel memoryless Gaussian channels:

$$\begin{cases} Y_1 = X_1 + Z_1 \\ Y_2 = X_2 + Z_2 \end{cases}$$

under an overall power constraint $E[X_1^2] + E[X_2^2] \leq P$, where (X_1, X_2) and (Z_1, Z_2) are independent and the noise variables Z_1 and Z_2 are zero-mean Gaussian with covariance matrix

$$K = \begin{pmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^2 \end{pmatrix}, \qquad \sigma > 0.$$

If $P > 3\sigma^2$, then the optimal input power allocation that maximizes the overall channel capacity is

$$(P_1, P_2) = \left(\frac{P + \sigma^2}{2}, \frac{P - \sigma^2}{2} \right).$$

(6) Consider two zero-mean Gaussian random variables X_1 and X_2 with (positive) variances P_1 and P_2, respectively: $X_1 \sim \mathcal{N}(0, P_1)$ and $X_2 \sim \mathcal{N}(0, P_2)$. Let

$$Y_1 = X_1 + Z$$

and

$$Y_2 = X_2 + Z$$

where $Z \sim \mathcal{N}(0, \sigma^2)$ is a zero-mean Gaussian random variable with (positive) variance σ^2 and is independent from both X_1 and X_2.

(a) Determine in nats $D(X_1 \| X_2)$ and $D(Y_1 \| Y_2)$ in terms of P_1, P_2 and σ^2.

(b) Compare $D(X_1 \| X_2)$ to $D(Y_1 \| Y_2)$ and comment qualitatively.

(7) Answer the following questions.

(i) Problem 5.19

(ii) [**MATH 874 only**] Problem 5.18.
