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5. (15 points)

(a) **Answer:**

The transition probability matrix of the stationary Markov source is given by

$$Q = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{bmatrix}$$

Since the source is stationary, $P_{X_i} = \pi$, where $\pi = (\pi_1, \pi_2)$ is the MC's stationary distribution. Then π satisfies

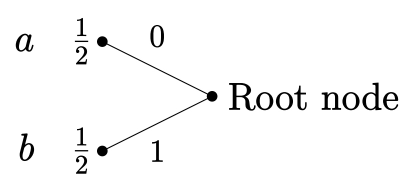
$$\begin{aligned} \pi &= \pi Q \\ (\pi_1, \pi_2) &= (\pi_1, \pi_2) \begin{bmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{bmatrix} \\ &\Rightarrow \begin{cases} \pi_1 = \pi_1\alpha + \pi_2(1-\alpha) \\ \pi_2 = \pi_1(1-\alpha) + \pi_1\alpha \end{cases} \end{aligned}$$

which upon solving it (with $\pi_1 + \pi_2 = 1$), yields

$$\begin{aligned} \pi &= \left(\frac{1}{2}, \frac{1}{2}\right) \\ \therefore P_{X_i}(x) = p_i &= \frac{1}{2}, \quad \forall x \in \mathcal{X}, i \in \mathbb{Z}_{\geq 1} \end{aligned}$$

Lets use Huffman codes in the design, which are always optimal.

Part 1. First-order Huffman code $\mathcal{C}_1$ $(n = 1)$:



$$\begin{aligned} f : \mathcal{X} &\to \{0, 1\}^* \\ a &\to 0 \\ b &\to 1 \end{aligned}$$

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Then the average code rate for $\mathcal{C}_1$, $\overline{R}_1$, is

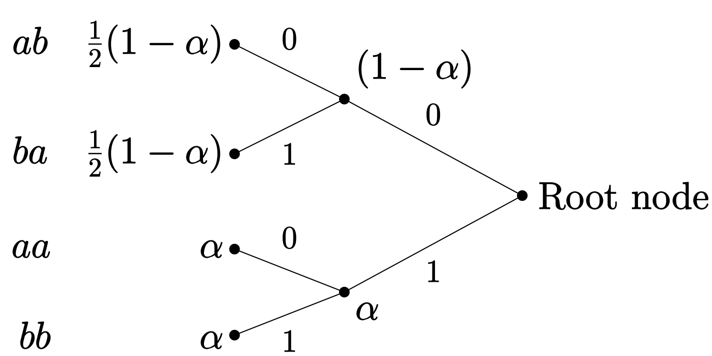
$$\begin{aligned}
\overline{R}_1 &= \frac{\bar{l}_1}{1} \\
&= \sum_{i=1}^{2} \overbrace{p_i}^{\pi_i} l_i \\
&= \frac{1}{2}(1+1) \\
&\boxed{= 1 \, \frac{\text{bits}}{\text{source symbol}}}.
\end{aligned}$$

Part 2. Second-order Huffman code $\mathcal{C}_2$ $(n = 2)$:

Let $\widetilde{\mathcal{X}} = \mathcal{X}^2 = \{aa, ab, ba, bb\}$ with probability distribution

$$\begin{aligned}
p(aa) &= p_{X_1}(a) p_{X_2|X_1}(a|a) = \frac{1}{2} \cdot \alpha = \frac{\alpha}{2} \\
p(ab) &= p_{X_1}(a) p_{X_2|X_1}(b|a) = \frac{1}{2} \cdot (1-\alpha) \\
p(ba) &= p_{X_1}(b) p_{X_2|X_1}(a|b) = \frac{1}{2} \cdot (1-\alpha) \\
p(bb) &= p_{X_1}(b) p_{X_2|X_1}(b|b) = \frac{1}{2} \cdot \alpha = \frac{\alpha}{2}.
\end{aligned}$$

Note. $\alpha \in (\frac{1}{3}, \frac{1}{2}) \Rightarrow 1 - \alpha > \alpha$ and that $\frac{1}{2}(1-\alpha) < \alpha$.



$$\begin{aligned}
f : \mathcal{X} &\to \{0,1\}^* \\
ab &\to 00 \\
ba &\to 01 \\
aa &\to 10 \\
bb &\to 11
\end{aligned}$$

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Then the average code rate for $\mathcal{C}_2$, $\overline{R}_2$, is

$$\begin{aligned}
\overline{R}_2 &= \frac{\bar{l}_2}{2} \\
&= \frac{1}{2}\sum_{i=1}^{4} p_i l_i \\
&= \frac{1}{2}\left(2 \cdot \frac{1}{2}(1-\alpha)\cdot 2 + 2 \cdot \frac{\alpha}{2} \cdot 2\right) \\
&\boxed{= 1 \frac{\text{bits}}{\text{source symbol}}}.
\end{aligned}$$

We have that $\boxed{\overline{R}_1 = \overline{R}_2}$.

(b) **Answer:**

The limit of the average coding rate of an n-th order optimal binary variable-length code for the source as $n \to \infty$ is the source's entropy rate. i.e.,

$$\lim_{n\to\infty} \overline{R} = H(\mathcal{X}).$$

The source entropy can be calculated as

$$\begin{aligned}
H(\mathcal{X}) &= H(X_2|X_1) \\
&= -\sum_i \sum_j \overbrace{P_{X_1}(i)}^{\pi_i} \overbrace{P_{X_2|X_1}(j|i)}^{p_{ij}} \log_2 P_{X_2|X_1}(j|i) \\
&= -\sum_{i=1}^{2}\sum_{j=1}^{2} \pi_i p_{ij} \log_2 p_{ij} \\
&= -\frac{1}{2}\left(2\alpha \log_2 \alpha + 2(1-\alpha)\log_2(1-\alpha)\right) \\
&= -\alpha\log_2\alpha - (1-\alpha)\log_2(1-\alpha) \\
&\boxed{= h_b(\alpha)}
\end{aligned}$$

where h_b is the binary entropy function. This makes sense given part (a) since we have that

$$H(\mathcal{X}) = h_b(\alpha) < \overline{R} = 1 < h_b(\alpha) + \frac{1}{n}, \quad \forall\, \alpha \in \left(\frac{1}{3}, \frac{1}{2}\right), n = 1, 2$$

The fact that average code rate from part (a) didn't decrease as the order of the code increased can be attributed to the fact that $\alpha \in (\frac{1}{3}, \frac{1}{2})$. If it weren't the case, the lengths of the codewords in code $\mathcal{C}_2$ could have been 1,2,3,3, which would have made its average code rate better compared to the first order code.