Student Number: 20053722 Name: Bryan Hoang

(a) **Answer:**

Since conditioning decreases entropy, we have that

$$\begin{aligned} H(X|Y) &\geq H(X|Y,Z) \\ -H(X|Z) &\leq -H(X|Y,Z) \end{aligned} \tag{1}$$

We also have the following identity for the conditional mutual information:

$$I(X;Y|Z) = H(X|Z) - H(X|Y,Z) \tag{2}$$

Thus,

$$\begin{aligned} H(X|Z) - H(X|Y) &\leq H(X|Z) - H(X|Y,Z) && \text{by (1)} \\ \boxed{H(X|Z) - H(X|Y) \leq I(X;Y|Z)} & && \text{by (2)} \end{aligned}$$

(b) **Answer:**

If $Z = g(Y)$, then $X \to Y \to Z$. Thus,

$$\begin{aligned} I(X;Y) &\geq I(X;g(Y)) \\ \Rightarrow -I(X;g(Y)) &\geq -I(X;Y) \end{aligned} \tag{3}$$

By the properties of mutual information, we also have that

$$H(X|g(Y)) = H(X) - I(X;g(Y)) \qquad \text{and} \qquad H(X|Y) = H(X) - I(X;Y)$$

Then by (3), it follows that

$$\begin{aligned} H(X) - I(X;g(Y)) &\geq H(X) - I(X;Y) \\ \Rightarrow & \boxed{H(X|g(Y)) \geq H(X|Y)} \end{aligned}$$

(c) **Answer:**

$$\begin{aligned} \frac{1}{2}H(X_1,X_2) &= \frac{1}{2}(H(X_2|X_1) + H(X_1)) && \text{by the chain rule} \\ &= \frac{1}{2}(H(X_2|X_1) + H(X_2)) && \because X_1 \text{ and } X_2 \text{ are identically distributed} \\ &\geq \frac{1}{2}(2 \cdot H(X_2|X_1)) && \because \text{conditioning reduces entropy} \\ &= H(X_2|X_1) \end{aligned}$$

Therefore, we have $\boxed{H(X_2|X_1) \leq \frac{1}{2}H(X_1,X_2)}$.