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2. (10 points) **Answer:**

Proof. Let P, Q, and R be three distributions defined on a common finite alphabet $\mathcal{X}$. Then starting from the RHS of the equation and applying logarithm identities yields

$$
\begin{aligned}
& D\left(P \,\middle\|\, \frac{P+Q}{2}\right) + D\left(Q \,\middle\|\, \frac{P+Q}{2}\right) + 2D\left(\frac{P+Q}{2} \,\middle\|\, R\right) \\
&= \sum_{x \in \mathcal{X}} P(x) \log \frac{P(x)}{\left(\frac{P(x)+Q(x)}{2}\right)} + \sum_{x \in \mathcal{X}} Q(x) \log \frac{P(x)}{\left(\frac{P(x)+Q(x)}{2}\right)} + \cancel{2} \sum_{x \in \mathcal{X}} \frac{P(x)+Q(x)}{\cancel{2}} \log \frac{\left(\frac{P(x)+Q(x)}{2}\right)}{R(x)} \\
&= \cancel{\sum_{x \in \mathcal{X}} P(x) \log 2} + \cancel{\sum_{x \in \mathcal{X}} P(x) \log P(x)} - \cancel{\sum_{x \in \mathcal{X}} P(x) \log P(x)} - \cancel{\sum_{x \in \mathcal{X}} P(x) \log Q(x)} \\
&\quad + \cancel{\sum_{x \in \mathcal{X}} Q(x) \log 2} + \cancel{\sum_{x \in \mathcal{X}} Q(x) \log Q(x)} - \cancel{\sum_{x \in \mathcal{X}} Q(x) \log P(x)} - \cancel{\sum_{x \in \mathcal{X}} Q(x) \log Q(x)} \\
&\quad + \sum_{x \in \mathcal{X}} P(x) \log P(x) + \cancel{\sum_{x \in \mathcal{X}} P(x) \log Q(x)} - \cancel{\sum_{x \in \mathcal{X}} P(x) \log 2} - \sum_{x \in \mathcal{X}} P(x) \log R(x) \\
&\quad + \cancel{\sum_{x \in \mathcal{X}} Q(x) \log P(x)} + \sum_{x \in \mathcal{X}} Q(x) \log Q(x) - \cancel{\sum_{x \in \mathcal{X}} Q(x) \log 2} - \sum_{x \in \mathcal{X}} Q(x) \log R(x) \\
&= \sum_{x \in \mathcal{X}} P(x) \log P(x) - \sum_{x \in \mathcal{X}} P(x) \log R(x) + \sum_{x \in \mathcal{X}} Q(x) \log Q(x) - \sum_{x \in \mathcal{X}} Q(x) \log R(x) \\
&= \sum_{x \in \mathcal{X}} P(x) \log \frac{P(x)}{R(x)} + \sum_{x \in \mathcal{X}} Q(x) \log \frac{Q(x)}{R(x)} \\
&= D(P \parallel R) + D(Q \parallel R)
\end{aligned}
$$

□