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3. (15 points)

(a) **Answer:**

$$\begin{aligned}
h_\alpha(f) &= \frac{1}{1-\alpha}\ln\left(\int_{\mathbb{R}}\left(\frac{1}{\sqrt{2\pi\sigma_1^2}}e^{-\frac{x^2}{2\sigma_1^2}}\right)^\alpha \mathrm{d}x\right)\\
&= \frac{1}{1-\alpha}\ln\left(\int_{\mathbb{R}}\left(\frac{1}{\sqrt{2\pi\sigma_1^2}}\right)^\alpha e^{-\frac{x^2\alpha}{2\sigma_1^2}} \mathrm{d}x\right)
\end{aligned}$$

Let $\beta = \frac{\sigma_1}{\sqrt{\alpha}}$. Then,

$$\begin{aligned}
h_\alpha(f) &= \frac{1}{1-\alpha}\ln\left(\left(\frac{1}{\sqrt{2\pi\sigma_1^2}}\right)^\alpha \beta\sqrt{2\pi}\underbrace{\int_{\mathbb{R}}\frac{1}{\beta\sqrt{2\pi}}e^{-\frac{x^2}{2\beta^2}}\mathrm{d}x}_{\text{pdf of a } \mathcal{N}(0,\beta^2)\text{ RV}}\right)\\
&= \frac{1}{1-\alpha}\ln\left(\left(\frac{1}{\sqrt{2\pi\sigma_1^2}}\right)^\alpha \beta\sqrt{2\pi}\right)\\
&= \frac{1}{1-\alpha}\ln\left(-\alpha\ln(\sqrt{2\pi\sigma_1^2}) + \ln(\sqrt{2\pi\sigma_1^2}) - \frac{1}{2}\ln\alpha\right)\\
&= \ln(\sqrt{2\pi\sigma_1^2}) - \frac{1}{2}\cdot\frac{\ln\alpha}{1-\alpha}\\
&\boxed{= \frac{1}{2}\left(\ln(2\pi\sigma_1^2) - \frac{\ln\alpha}{1-\alpha}\right)}
\end{aligned}$$

$$\begin{aligned}
D_{\mathrm{KL}\alpha}(f \parallel g) &= \frac{1}{1-\alpha}\ln\left(\int_{\mathbb{R}}\left(\frac{1}{\sqrt{2\pi\sigma_1^2}}e^{-\frac{x^2}{2\sigma_1^2}}\right)^\alpha\left(\frac{1}{\sqrt{2\pi\sigma_2^2}}e^{-\frac{x^2}{2\sigma_2^2}}\right)^{1-\alpha}\mathrm{d}x\right)\\
&= \frac{1}{1-\alpha}\ln\left(\int_{\mathbb{R}}\frac{1}{\sigma_1^\alpha\sigma_2^{1-\alpha}\sqrt{2\pi}}e^{-\frac{x^2}{2}\left(\frac{\alpha}{\sigma_1^2}+\frac{1-\alpha}{\sigma_2^2}\right)}\mathrm{d}x\right)
\end{aligned}$$

Let $\beta^2 = \frac{\sigma_1^2\sigma_2^2}{\sigma_2^2\alpha+\sigma_1^2(1-\alpha)}$. Then,

$$\begin{aligned}
D_{\mathrm{KL}\alpha}(f \parallel g) &= \frac{1}{1-\alpha}\ln\left(\frac{1}{\sigma_1^\alpha\sigma_2^{1-\alpha}}\beta\underbrace{\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi\beta^2}}e^{-\frac{x^2}{2\beta^2}}}_{\text{pdf of a } \mathcal{N}(0,\beta^2)\text{ RV}}\right)\\
&= \frac{1}{1-\alpha}\ln\left(\frac{1}{\sigma_1^\alpha\sigma_2^{1-\alpha}}\beta\right)\\
&= \frac{1}{1-\alpha}\ln\left(\frac{1}{2}\ln(\beta^2) - \alpha\ln(\sigma_1 - (1-\alpha)\ln(\sigma_2))\right)\\
&= \frac{1}{1-\alpha}\ln\left(\ln(\sigma_1) + \frac{1}{2}\ln\left(\frac{\sigma_2^2}{\alpha\sigma_2^2+(1-\alpha)\sigma_1^2}\right) - \ln(\sigma_1 - (1-\alpha)\ln(\sigma_2))\right)\\
&\boxed{= \ln\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{1}{2(\alpha-1)}\ln\left(\frac{\sigma_2^2}{\alpha\sigma_2^2+(1-\alpha)\sigma_1^2}\right)}
\end{aligned}$$

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(b) **Answer:**

$$\begin{aligned}
\lim_{\alpha\to 1} h_{\alpha(f)} &= \lim_{\alpha\to 1} \frac{1}{2}\left(\ln(2\pi\sigma_1^2) - \frac{\ln\alpha}{1-\alpha}\right) \\
&= \frac{1}{2}\ln(2\pi\sigma_1^2) - \frac{1}{2}\lim_{\alpha\to 1}\frac{\ln\alpha}{1-\alpha} \\
&= \frac{1}{2}\ln(2\pi\sigma_1^2) - \frac{1}{2}\lim_{\alpha\to 1}\frac{\frac{1}{\alpha}}{-1} && \text{by L'Hopital's rule} \\
&= \frac{1}{2}\ln(2\pi\sigma_1^2) - \frac{1}{2}\ln e \\
&= \frac{1}{2}\ln(2\pi e\sigma_1^2) \\
&= h(f)
\end{aligned}$$

$\therefore$ Qualitatively, the limit converges to the regular differential entropy.

$$\begin{aligned}
\lim_{\alpha\to 1} D_{\mathrm{KL}\alpha}(f \,||\, g) &= \lim_{\alpha\to 1}\ln\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{1}{2(\alpha-1)}\ln\left(\frac{\sigma_2^2}{\alpha\sigma_2^2 + (1-\alpha)\sigma_1^2}\right) \\
&= \ln\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{1}{2}\lim_{\alpha\to 1}\frac{1}{\alpha-1}\ln\left(\frac{\sigma_2^2}{\alpha\sigma_2^2 + (1-\alpha)\sigma_1^2}\right) \\
&= \ln\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{1}{2}\lim_{\alpha\to 1}\left(\frac{\alpha\sigma_2^2 + (1-\alpha)\sigma_1^2}{\sigma_2^2}\right)(\sigma_2^2 - \sigma_1^2) && \text{by L'Hopital's rule} \\
&= \ln\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{1}{2\sigma_2^2}(\sigma_1^2 - \sigma_2^2) \\
&= D_{\mathrm{KL}}(f \,||\, g)
\end{aligned}$$

$\therefore$ Qualitatively, the limit converges to the regular Kullback-Leibler divergence.