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3.

Answer:

Since conditioning decreases entropy, we have that

$$\begin{aligned} H(X|Y) &\geq H(X|Y, Z) \\ -H(X|Z) &\leq -H(X|Y, Z) \end{aligned} \quad (1)$$

We also have the following identity for the conditional mutual information:

$$I(X; Y|Z) = H(X|Z) - H(X|Y, Z) \quad (2)$$

Thus,

$$H(X|Z) - H(X|Y) \leq H(X|Z) - H(X|Y, Z) \quad \text{by (1)}$$

$$\boxed{H(X|Z) - H(X|Y) \leq I(X; Y|Z)} \quad \text{by (2)}$$

(b) Answer:If $Z = g(Y)$, then $X \rightarrow Y \rightarrow Z$. Thus,

$$\begin{aligned} I(X; Y) &\geq I(X; g(Y)) \\ \Rightarrow -I(X; g(Y)) &\geq -I(X; Y) \end{aligned} \quad (3)$$

By the properties of mutual information, we also have that

$$H(X|g(Y)) = H(X) - I(X; g(Y)) \quad \text{and} \quad H(X|Y) = H(X) - I(X; Y)$$

Then by (3), it follows that

$$\begin{aligned} H(X) - I(X; g(Y)) &\geq H(X) - I(X; Y) \\ \Rightarrow \boxed{H(X|g(Y))} &\geq \boxed{H(X|Y)} \end{aligned}$$

(c) Answer:

$$\begin{aligned} \frac{1}{2}H(X_1, X_2) &= \frac{1}{2}(H(X_2|X_1) + H(X_1)) && \text{by the chain rule} \\ &= \frac{1}{2}(H(X_2|X_1) + H(X_2)) && \because X_1 \text{ and } X_2 \text{ are identically distributed} \\ &\geq \frac{1}{2}(2 \cdot H(X_2|X_1)) && \because \text{conditioning reduces entropy} \\ &= H(X_2|X_1) \end{aligned}$$

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Therefore, we have
$$H(X_2|X_1) \leq \frac{1}{2}H(X_1, X_2).$$