Student Number: [redacted] Name: Bryan Hoang

1. (15 points)

(a) **Answer:**

The binary Polya contagion Markov source treated in Example 3.17 in the textbook with memory $M = 2$, $\{Z_n\}_{n=1}^{\infty}$, is also stationary with alphabet $\mathcal{Z} = \{0, 1\}$. Then for $a, b \in \mathcal{Z}$, its transition probabilities are

$$\begin{aligned}
\mathbb{P}(Z_n = 1 \mid Z_{n-1} = a, Z_{n-2} = b) &= \frac{R + (a+b)\Delta}{T + 2\Delta} \\
&= \frac{\rho + (a+b)\delta}{1 + 2\delta} \\
\mathbb{P}(Z_n = 0 \mid Z_{n-1} = a, Z_{n-2} = b) &= 1 - \mathbb{P}(Z_n = 1 \mid Z_{n-1} = a, Z_{n-2} = b) \\
&= 1 - \frac{\rho + (a+b)\delta}{1 + 2\delta}.
\end{aligned}$$

Let's define a new Markov source $\{X_n\}_{n=1}^{\infty}$ with $X_n = (Z_n, Z_{n+1})\ \forall\, n \in Z_{n\geq 1}$ and state space $\mathcal{X} = \mathcal{Z}^2$.

Claim. $\{X_n\}_{n=1}^{\infty}$ is a Markov source with memory 1.

Proof. $\forall\, i \in \mathbb{Z}_{>1}, x^i \in \mathcal{X}^i$,

$$\begin{aligned}
&\mathbb{P}(X_i = x_i \mid X^{i-1} = x^{i-1}) \\
&= \mathbb{P}(Z_i = z_i, Z_{i+1} = z_{i+1} \mid Z^{i-1} = z^{i-1}, Z_2^i = z_2^i) \\
&= \mathbb{P}(Z_i = z_i, Z_{i+1} = z_{i+1} \mid Z^{i} = z^{i}) \\
&= \mathbb{P}(Z_i = z_i, Z_{i+1} = z_{i+1} \mid Z_{i-1} = z_{i-1}, Z_i = z_i) && \because \{Z_n\}_{n=1}^{\infty} \text{ has memory 2} \\
&= \mathbb{P}(X_i = x_i \mid X_{i-1} = x_{i-1})
\end{aligned}$$

□

Claim. $\{X_n\}_{n=1}^{\infty}$ is a stationary Markov source with stationary distribution

$$\pi = \left(\frac{(1-\rho)(1-\rho+\delta)}{1+\delta}, \frac{\rho(1-\rho)}{1+\delta}, \frac{\rho(1-\rho)}{1+\delta}, \frac{\rho(\rho+\delta)}{1+\delta}\right)$$

Proof. The transition probability matrix of $\{X_n\}_{n=1}^{\infty}$ is

$$\mathcal{Q} = \begin{bmatrix}
1 - \frac{\rho}{1+2\delta} & \frac{\rho}{1+2\delta} & 0 & 0 \\
0 & 0 & 1 - \frac{\rho+\delta}{1+2\delta} & \frac{\rho+\delta}{1+2\delta} \\
1 - \frac{\rho+\delta}{1+2\delta} & \frac{\rho+\delta}{1+2\delta} & 0 & 0 \\
0 & 0 & 1 - \frac{\rho+2\delta}{1+2\delta} & \frac{\rho+2\delta}{1+2\delta}
\end{bmatrix}$$

Let $\pi = (\pi_{0,0}, \pi_{0,1}, \pi_{1,0}, \pi_{1,1})$. Then solving $\pi = \pi\mathcal{Q}$ and $\sum_{i,j\in\mathcal{Z}} \pi_{i,j} = 1$ yields

$$\pi = \left(\frac{(1-\rho)(1-\rho+\delta)}{1+\delta}, \frac{\rho(1-\rho)}{1+\delta}, \frac{\rho(1-\rho)}{1+\delta}, \frac{\rho(\rho+\delta)}{1+\delta}\right).$$

$\therefore \{X_n\}_{n=1}^{\infty}$ is a stationary process. □

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The last quantity required to find the sufficient condition is the Entropy rate of $\{Z_n\}_{n=1}^{\infty}$, $H(\mathcal{Z})$.

$$
\begin{aligned}
&\mathrm{H}(\mathcal{Z}) \\
&= \mathrm{H}(Z_3 \mid Z_2, Z_1) && \because \{Z_n\}_{n=1}^{\infty}\text{'s stationary} \\
&= \mathrm{H}(Z_2, Z_3 \mid Z_1, Z_2) \\
&= \mathrm{H}(X_2 \mid X_1) \\
&= -\sum_{x_1 \in \mathcal{X}} p_{X_1}(x_1) \sum_{x_2 \in \mathcal{X}} p_{X_2|X_1}(x_2|x_1) \log_2 p_{X_2|X_1}(x_2|x_1) \\
&= -\sum_{x_1 \in \mathcal{X}} \pi(x_1) \sum_{x_2 \in \mathcal{X}} p_{X_2|X_1}(x_2|x_1) \log_2 p_{X_2|X_1}(x_2|x_1) && \because \{X_n\}_{n=1}^{\infty}\text{'s stationary} \\
&= \frac{(1-\rho)(1-\rho+\delta)}{1+\delta} h_b\left(\frac{\rho}{1+2\delta}\right) + \frac{2\rho(1-\rho+\delta)}{1+\delta} h_b\left(\frac{\rho+\delta}{1+2\delta}\right) + \frac{\rho(\rho+\delta)}{1+\delta} h_b\left(\frac{\rho+2\delta}{1+2\delta}\right) && \blacksquare
\end{aligned}
$$

where h_b is the binary entropy function. Then by the forward-part of lossless joint source-channel coding theorem, the Markov source $\{Z_n\}_{n=1}^{\infty}$ can be reliably transmitted over the BSEC via an m-to-n_m source-channel block code if

$$R_{sc}\,\mathrm{H}(\mathcal{X}) < C.$$

Therefore, the sufficient condition for reliable transmissibility of the source over the BSEC is

$$
\begin{aligned}
&R_{sc} < \frac{C}{\mathrm{H}(\mathcal{X})} \\
\Rightarrow &\boxed{R_{sc} < \frac{(1-\alpha)\left(1 - h_b\left(\frac{\varepsilon}{1-\alpha}\right)\right)}{\frac{(1-\rho)(1-\rho+\delta)}{1+\delta} h_b\left(\frac{\rho}{1+2\delta}\right) + \frac{2\rho(1-\rho+\delta)}{1+\delta} h_b\left(\frac{\rho+\delta}{1+2\delta}\right) + \frac{\rho(\rho+\delta)}{1+\delta} h_b\left(\frac{\rho+2\delta}{1+2\delta}\right)}}
\end{aligned}
$$

(b) **Answer:**
For $\rho = \delta = \frac{1}{2}$ and $\varepsilon = \alpha = 0.1$, then

$$
\begin{aligned}
&R_{sc} \in \left(0, \frac{C}{\mathrm{H}(\mathcal{X})}\right) \\
\Rightarrow &R_{sc} \in \left(0, \frac{0.447}{\frac{1}{3} h_b\left(\frac{1}{4}\right) + \frac{1}{3} h_b\left(\frac{1}{2}\right) + \frac{1}{3} h_b\left(\frac{3}{4}\right)}\right) \\
\Rightarrow &\boxed{R_{sc} \in (0, 0.511)}
\end{aligned}
$$