Student Number: 20053722 Name: Bryan Hoang

6. **Answer:**

(i) (a). *Proof.* Suppose $X_1, X_2, \dots, X_n$ are independent. Then

$$
\begin{aligned}
& I(X_1, X_2, \dots, X_n; Y_1, Y_2, \dots, Y_n) \\
&= H(X_1, \dots, X_n) - H(X_1, \dots, X_n | Y_1, \dots, Y_n) \\
&= \sum_{i=1}^{n} H(X_i) - H(X_1, \dots, X_n | Y_1, \dots, Y_n) && \text{by the independence of the } X_i\text{'s} \\
&= \sum_{i=1}^{n} H(X_i) - \sum_{i=1}^{n} H(X_i | X_1, \dots, X_{i-1}, Y_1, \dots, Y_n) && \text{by the chain rule} \\
&\geq \sum_{i=1}^{n} H(X_i) - \sum_{i=1}^{n} H(X_i | Y_i) && \because \text{conditioning reduces entropy} \\
&= \sum_{i=1}^{n} H(X_i) - H(X_i | Y_i) \\
&= \sum_{i=1}^{n} I(X_i; Y_i)
\end{aligned}
$$

□

(b). *Proof.* Suppose that for a given X_i, the random variable Y_i is conditionally independent of the remaining random variables, for $i \in \{1, \dots, n\}$. Then from the independence bound on entropy, we have

$$
H(Y^n) \leq \sum_{i=1}^{n} H(Y_i) \tag{1}
$$

By the conditional independence assumption, we have

$$
\begin{aligned}
H(Y^n | X^n) &= \mathrm{E}[-\log_2 P_{Y^n|X^n}(Y^n \mid X^n)] \\
&= \mathrm{E}\left[-\sum_{i=1}^{n} \log_2 P_{Y_i|X_i}(Y_i \mid X_i)\right] \\
&= \sum_{i=1}^{n} H(Y_i | X_i)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
I(X^n; Y^n) &= H(Y^n) - H(Y^n | X^n) \\
&\leq \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i | X_i) \\
&= \sum_{i=1}^{n} I(X_i; Y_i)
\end{aligned}
$$

□