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6. (20 points)

(i)

(a) **Answer:**

Proof. To prove the result, it is sufficient to show that $\forall\, n \in \mathbb{Z}_{\geq 0}$,

$$\frac{1}{n-1}H(X_{(n-1)+1}^{2(n-1)}|X^{n-1}) \geq \frac{1}{n}H(X_{n+1}^{2n}|X^n)$$

$$\frac{1}{n-1}H(X_n^{2n-2}|X^{n-1}) - \frac{1}{n}H(X_{n+1}^{2n}|X^n) \geq 0$$

Thus, we have

$$\begin{aligned}
&\frac{1}{n-1}H(X_n^{2n-2}|X^{n-1}) - \frac{1}{n}H(X_{n+1}^{2n}|X^n) \\
&= \frac{1}{n-1}H(X_n^{2n-2}|X^{n-1}) - \frac{1}{n}H(X_{n+1}^{2n}|X^n) \\
&= \frac{1}{n-1}\left[H(X_1^{2n-2}) - H(X^{n-1})\right] - \frac{1}{n}\left[H(X_1^{2n}) - H(X^n)\right] \\
&= \frac{1}{n-1}\left[\sum_{i=1}^{2n-2} H(X_i|X^{i-1}) - \sum_{i=1}^{n-1} H(X_i|X^{i-1})\right] - \frac{1}{n}\left[\sum_{i=1}^{2n} H(X_i|X^{i-1}) - \sum_{i=1}^{n} H(X_i|X^{i-1})\right] \\
&= \frac{1}{n-1}\left[\sum_{i=n}^{2n-2} H(X_i|X^{i-1})\right] - \frac{1}{n}\left[\sum_{i=n+1}^{2n} H(X_i|X^{i-1})\right]
\end{aligned}$$

By the stationarity of the source, we can re-index the first summation.

$$\begin{aligned}
&= \frac{1}{n-1}\left[\sum_{i=n+1}^{2n-1} H(X_i|X_2^{i-1})\right] - \frac{1}{n}\left[\sum_{i=n+1}^{2n} H(X_i|X^{i-1})\right] \\
&= \frac{1}{n(n-1)}\left[\sum_{i=n+1}^{2n-1} nH(X_i|X_2^{i-1}) - (n-1)H(X_i|X_1^{i-1})\right] - \frac{1}{n}H(X_{2n}|X^{2n-1})
\end{aligned}$$

Since conditioning reduces entropy, we then have that

$$\begin{aligned}
&\geq \frac{1}{n(n-1)}\left[\sum_{i=n+1}^{2n-1} nH(X_i|X_1^{i-1}) - (n-1)H(X_i|X_1^{i-1})\right] - \frac{1}{n}H(X_{2n}|X^{2n-1}) \\
&= \frac{1}{n(n-1)}\left[\sum_{i=n+1}^{2n-1} H(X_i|X_1^{i-1})\right] - \frac{1}{n(n-1)}(n-1)H(X_{2n}|X^{2n-1}) \\
&= \frac{1}{n(n-1)}\left[\sum_{i=n+1}^{2n-1} \underbrace{H(X_i|X_1^{i-1}) - H(X_{2n}|X^{2n-1})}_{D_i}\right] \\
&\geq 0
\end{aligned}$$

since the sequence of conditional entropies of a stationary source is nonincreasing, we have that $D_i \geq 0,\ \forall\, i \in \{n+1, \ldots, 2n-1\}$ □

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(b) **Answer:**

Proof.

Part 1. Showing that $H(X_{2n}|X^{2n-1}) \leq \frac{1}{n}H(X_{n+1}^{2n}|X^n)$.

$$
\begin{aligned}
&\frac{1}{n}H(X_{n+1}^{2n}|X^n) - H(X_{2n}|X^{2n-1}) \\
&= \frac{1}{n}\Big[H(X^{2n}) - H(X^n)\Big] - H(X_{2n}|X^{2n-1}) \\
&= \frac{1}{n}\left[\sum_{i=1}^{2n} H(X_i|X^{i-1}) - \sum_{i=1}^{n} H(X_i|X^{i-1})\right] - H(X_{2n}|X^{2n-1}) \\
&= \frac{1}{n}\left[\sum_{i=n+1}^{2n} H(X_i|X^{i-1})\right] - \frac{1}{n}\cdot nH(X_{2n}|X^{2n-1}) \\
&= \frac{1}{n}\left[\sum_{i=n+1}^{2n} \underbrace{H(X_i|X^{i-1}) - H(X_{2n}|X^{2n-1})}_{D_i}\right] \\
&\geq 0
\end{aligned}
$$

since the sequence of conditional entropies of a stationary source is nonincreasing, we have that $\forall\, i \in \{n+1, \ldots, 2n\}$,

$$
\begin{cases} D_i \geq 0 & \text{if } i \neq 2n, \\ D_i = 0 & \text{if } i = 2n. \end{cases}
$$

Part 2. Showing that $\frac{1}{n}H(X_{n+1}^{2n}|X^n) \leq \frac{1}{n}H(X^n)$.

$$
\begin{aligned}
&\frac{1}{n}H(X^n) - \frac{1}{n}H(X_{n+1}^{2n}|X^n) \\
&= \frac{1}{n}\Big[H(X^n) - \Big(H(X^{2n}) - H(X^n)\Big)\Big] \\
&= \frac{1}{n}\Big[H(X^n) - \Big(H(X^n|X_{n+1}^{2n}) + H(X_{n+1}^{2n}) - H(X^n)\Big)\Big] \\
&= \frac{1}{n}\Big[H(X^n) - \Big(H(X^n|X_{n+1}^{2n}) + H(X^n) - H(X^n)\Big)\Big] && \text{by stationarity} \\
&= \frac{1}{n}\Big[H(X^n) - H(X^n|X_{n+1}^{2n})\Big] \\
&\geq 0 && \because \text{ conditioning reduces entropy}
\end{aligned}
$$

$\square$