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4. (15 points)

(a) **Answer:**

Proof. Assuming that the integrals are definite over $\mathbb{R}^n$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n} a(x^n) \ln \frac{a(x^n)}{b(x^n)} \,\mathrm{d}x^n - a \ln \frac{a}{b} \\
&= \int_{\mathbb{R}^n} a(x^n) \ln \frac{a(x^n)}{b(x^n)} \,\mathrm{d}x^n - \int_{\mathbb{R}^n} a(x^n) \,\mathrm{d}x^n \ln \frac{a}{b} \\
&= \int_{\mathbb{R}^n} a(x^n) \ln \frac{a(x^n) b}{b(x^n) a} \,\mathrm{d}x^n \\
&\geq \int_{\mathbb{R}^n} a(x^n) \left(1 - \frac{b(x^n) a}{a(x^n) b}\right) \mathrm{d}x^n && \text{by the Fundamental Inequality for the Logarithm} \\
&= \int_{\mathbb{R}^n} a(x^n) \,\mathrm{d}x^n - \frac{a}{b} \int_{\mathbb{R}^n} b(x^n) \,\mathrm{d}x^n \\
&= a - \frac{a}{b} \cdot b \\
&= 0.
\end{aligned}
$$

$\therefore \int_{\mathbb{R}^n} a(x^n) \ln \frac{a(x^n)}{b(x^n)} \,\mathrm{d}x^n \geq a \ln \frac{a}{b}$. □

(b) **Answer:**

Proof. Let $f_{Y_i}(y) = \int_{\mathbb{R}} f_{Y_i|X_i}(y|x) f_{X_i}(x) \,\mathrm{d}x$ be the pdf of channel output Y_i, $\forall\, i = 1, 2, y \in \mathbb{R}$. Then

$$
\begin{aligned}
&D_{\mathrm{KL}}(Y_1 \,||\, Y_2) \\
&= \int_{\mathbb{R}} f_{Y_1}(y) \ln \frac{f_{Y_1(y)}}{f_{Y_2}(y)} \,\mathrm{d}y \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} f_{Y_1|X_1}(y|x) f_{X_1}(x) \ln \frac{f_{Y_1|X_1}(y|x) f_{X_1}(x)}{f_{Y_2|X_2}(y|x) f_{X_2}(x)} \,\mathrm{d}x \,\mathrm{d}y && \text{by part a} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f_{Y_1|X_1}(y|x) \,\mathrm{d}y \, f_{X_1}(x) \ln \frac{f_{X_1}(x)}{f_{X_2}(x)} \,\mathrm{d}x && \because \text{both outputs are for the } \textit{same} \text{ channel} \\
&= \int_{\mathbb{R}} f_{X_1}(x) \ln \frac{f_{X_1}(x)}{f_{X_2}(x)} \,\mathrm{d}x \\
&= D_{\mathrm{KL}}(X_1 \,||\, X_2)
\end{aligned}
$$

$\therefore D_{\mathrm{KL}}(X_1 \,||\, X_2) \geq D_{\mathrm{KL}}(Y_1 \,||\, Y_2)$. □