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1. (10 points) From the question, we immediately have

$$\mathbb{Q} = \begin{bmatrix} \epsilon & ? & ? & ? \\ ? & \epsilon & ? & ? \end{bmatrix}.$$

(a) **Answer:**

$$\begin{aligned} H(Y|X) &= -\sum_{x \in \mathcal{X}} p_X(x) \sum_{y \in \mathcal{Y}} p_{Y|X}(y|x) \log_2 p_{Y|X}(y|x) \\ &= -(\epsilon \log_2 \epsilon) \cancelto{1}{\sum_{x=0}^{1} p_X(x)} - \sum_{x=0}^{1} p_X(x) \left[\sum_{\substack{y=0 \\ y \neq x}}^{3} \underbrace{p_{Y|X}(y|x)}_{a_i} \log_2 \underbrace{p_{Y|X}(y|x)}_{\substack{a_i \\ b_i = 1}} \right] \end{aligned}$$

By the Log sum inequality, we have that

$$\begin{aligned} H(Y|X) &\leq -\epsilon \log_2(\epsilon) - \sum_{x=0}^{1} p_X(x) \left[\underbrace{\left(\sum_{\substack{y=0 \\ y \neq x}}^{3} p_{Y|X}(y|x) \right)}_{=1-\epsilon} \log_2 \frac{\overbrace{\sum_{\substack{y=0 \\ y \neq x}}^{3} p_{Y|X}(y|x)}^{=1-\epsilon}}{\underbrace{\sum_{\substack{y=0 \\ y \neq x}}^{3} 1}_{3}} \right] \\ &= -\epsilon \log_2(\epsilon) - \left(\cancelto{1}{\sum_{x=0}^{1} p_X(x)} \right) (1-\epsilon) \log_2 \frac{1-\epsilon}{3} \\ &= -\epsilon \log_2(\epsilon) - (1-\epsilon) \log_2 \frac{1-\epsilon}{3}. \end{aligned}$$

$$\begin{aligned} \therefore H(Y|X) &\underbrace{\leq}_{\text{w/ equality iff}} -\epsilon \log_2(\epsilon) - (1-\epsilon) \log_2 \frac{1-\epsilon}{3} \\ \frac{\sum_i a_i}{\sum_i b_i} &= \frac{a_i}{b_i} && \forall\, i \\ \iff \frac{1-\epsilon}{3} &= \frac{p_{Y|X}(y|x)}{1} && \forall\, y \neq x \\ \iff p_{Y|X}(y|x) &= \frac{1-\epsilon}{3} && \forall\, y \neq x. \end{aligned}$$

$$\therefore H(Y|X) \text{ is maximized} \iff \boxed{\mathbb{Q} = \begin{bmatrix} \epsilon & \frac{1-\epsilon}{3} & \frac{1-\epsilon}{3} & \frac{1-\epsilon}{3} \\ \frac{1-\epsilon}{3} & \epsilon & \frac{1-\epsilon}{3} & \frac{1-\epsilon}{3} \end{bmatrix}}.$$

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(b) **Answer:**

The channel is quasi-symmetric. The weakly symmetric submatrices are

$$\mathbb{Q} = \begin{bmatrix} \underbrace{\begin{matrix} \epsilon & \frac{1-\epsilon}{3} \\ \frac{1-\epsilon}{3} & \epsilon \end{matrix}}_{\mathbb{Q}_1} & \underbrace{\begin{matrix} \frac{1-\epsilon}{3} & \frac{1-\epsilon}{3} \\ \frac{1-\epsilon}{3} & \frac{1-\epsilon}{3} \end{matrix}}_{\mathbb{Q}_2} \end{bmatrix}$$

Hence, the channel's capacity is

$$C = \sum_{i=1}^{2} a_i C_i$$

where for $i = 1, 2$,

$$\begin{aligned} a_i &= \text{sum of any row in } \mathbb{Q}_i \\ C_i &= \log_2 |\mathcal{Y}_i| - H(\text{any row in matrix } \frac{1}{a_i}\mathbb{Q}_i). \end{aligned}$$

Hence,

$$a_1 = \frac{1 + 2\epsilon}{3}$$

$$\begin{aligned} C_1 &= \log_2(2) - H\left(\epsilon \cdot \frac{3}{1+2\epsilon}, \frac{3}{1+2\epsilon} \cdot \frac{1-\epsilon}{3}\right) \\ &= 1 - H\left(\frac{3\epsilon}{1+2\epsilon}, \frac{1-\epsilon}{1+2\epsilon}\right) \\ &= 1 - \left(-\left(\frac{3\epsilon}{1+2\epsilon}\right)\log_2\left(\frac{3\epsilon}{1+2\epsilon}\right) - \left(\frac{1-\epsilon}{1+2\epsilon}\right)\log_2\left(\frac{1-\epsilon}{1+2\epsilon}\right)\right) \\ &= 1 - \left(-\left(1 - \frac{1-\epsilon}{1+2\epsilon}\right)\log_2\left(1 - \frac{1-\epsilon}{1+2\epsilon}\right) - \left(\frac{1-\epsilon}{1+2\epsilon}\right)\log_2\left(\frac{1-\epsilon}{1+2\epsilon}\right)\right) \\ &= 1 - h_b\left(\frac{1-\epsilon}{1+2\epsilon}\right) \end{aligned}$$

where h_b is the binary entropy function.

$$a_2 = \frac{2(1-\epsilon)}{3}$$

$$\begin{aligned} C_2 &= \log_2(2) - H\left(\frac{3}{2(1-\epsilon)} \cdot \frac{1-\epsilon}{3}, \frac{3}{2(1-\epsilon)} \cdot \frac{1-\epsilon}{3}\right) \\ &= 1 - H\left(\frac{1}{2}, \frac{1}{2}\right) \\ &= 1 - h_b\left(\frac{1}{2}\right) \end{aligned}$$

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$$
\begin{aligned}
\therefore C &= \sum_{i=1}^{2} a_i C_i \\
&= \frac{1+2\epsilon}{3} - \frac{1+2\epsilon}{3} h_b\left(\frac{1-\epsilon}{1+2\epsilon}\right) + \frac{2(1-\epsilon)}{3} - \frac{2(1-\epsilon)}{3} h_b\left(\frac{1}{2}\right) \\
&\boxed{= 1 - h_b\left(\frac{1-\epsilon}{1+2\epsilon}\right) - h_b\left(\frac{1}{2}\right)}
\end{aligned}
$$