Student Number: 20053722 Name: Bryan Hoang

7.

(a) **Answer:**

Claim: $H_\alpha(p;q) > 0$

Proof. For the case of $\alpha \in (0,1)$, we have that

$$\frac{1}{1-\alpha} > 0 \tag{1}$$

and

$$\begin{aligned}
\alpha - 1 &< 0 \\
\Rightarrow q_i^{\alpha-1} &> 1 && \because q_i \in (0,1),\ \forall\, i \in \{1,\ldots,n\} \\
\Rightarrow \sum_{i=1}^{n} p_i q_i^{\alpha-1} &> \sum_{i=1}^{n} p_i = 1 \\
\Rightarrow \log_2\left(\sum_{i=1}^{n} p_i q_i^{\alpha-1}\right) &> 0 && (2)
\end{aligned}$$

Then by (1) and (2), we have that $H_\alpha(p;q) > 0$.

For the case of $\alpha \in (1,\infty)$, we have that

$$\frac{1}{1-\alpha} < 0 \tag{3}$$

and

$$\begin{aligned}
\alpha - 1 &> 0 \\
\Rightarrow q_i^{\alpha-1} &< 1 && \because q_i \in (0,1),\ \forall\, i \in \{1,\ldots,n\} \\
\Rightarrow \sum_{i=1}^{n} p_i q_i^{\alpha-1} &< \sum_{i=1}^{n} p_i = 1 \\
\Rightarrow \log_2\left(\sum_{i=1}^{n} p_i q_i^{\alpha-1}\right) &< 0 && (4)
\end{aligned}$$

Then by (3) and (4), we have that $H_\alpha(p;q) > 0$.

Therefore, $\forall\, \alpha \in (0,1) \cup (1,\infty)$, $H_\alpha(p;q) > 0$. □

(b) **Answer:**

Claim: $H_\alpha(p;q)$ is non-increasing in α.

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Proof. Let $p' = (p'_1, \ldots, p'_n)$ be a probability distribution with $p'_i = \frac{p_i q_i^{\alpha-1}}{\sum_{i=1}^n p_i q_i^{\alpha-1}}, \forall\, i \in \{1, \ldots, n\}$. It can be seen that $p'_i \in (0,1)\ \ \forall\, i \in \{1, \ldots, n\}$ and that $\sum_{i=1}^n p'_i = 1$. Then

$$
\begin{aligned}
&D(p' || p) \\
&= \sum_{i=1}^n p'_i \log_2 \frac{p'_i}{p_i} \\
&= \sum_{i=1}^n \frac{p_i q_i^{\alpha-1}}{\sum_{i=1}^n p_i q_i^{\alpha-1}} \log_2 \frac{\frac{\cancel{p_i} q_i^{\alpha-1}}{\sum_{i=1}^n p_i q_i^{\alpha-1}}}{\cancel{p_i}} \\
&= \sum_{i=1}^n \frac{p_i q_i^{\alpha-1}}{\sum_{i=1}^n p_i q_i^{\alpha-1}} \log_2 \frac{1}{\sum_{i=1}^n p_i q_i^{\alpha-1}} + \sum_{i=1}^n \frac{p_i q_i^{\alpha-1}}{\sum_{i=1}^n p_i q_i^{\alpha-1}} \log_2 q_i^{\alpha} + \sum_{i=1}^n \frac{p_i q_i^{\alpha-1}}{\sum_{i=1}^n p_i q_i^{\alpha-1}} \log_2 \frac{1}{q_i} \\
&= \log_2 \frac{1}{\sum_{i=1}^n p_i q_i^{\alpha-1}} + \frac{\alpha}{\sum_{i=1}^n p_i q_i^{\alpha-1}} \sum_{i=1}^n p_i q_i^{\alpha-1} \log_2 q_i - \frac{1}{\sum_{i=1}^n p_i q_i^{\alpha-1}} \sum_{i=1}^n p_i q_i^{\alpha-1} \log_2 q_i \\
&= -\log_2 \sum_{i=1}^n p_i q_i^{\alpha-1} + \frac{\alpha - 1}{\sum_{i=1}^n p_i q_i^{\alpha-1}} \sum_{i=1}^n p_i q_i^{\alpha-1} \log_2 q_i \qquad (5)
\end{aligned}
$$

Taking the derivative of $H_\alpha(p; q)$ with respect to α gives

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\alpha} H_\alpha(p; q) \\
&= \frac{1}{(1-\alpha)^2} \log_2 \sum_{i=1}^n p_i q_i^{\alpha-1} + \frac{1}{1-\alpha} \cdot \frac{1}{\ln 2 \sum_{i=1}^n p_i q_i^{\alpha-1}} \cdot \sum_{i=1}^n p_i q_i^{\alpha-1} \ln q_i \\
&= \frac{1}{(1-\alpha)^2} \left(\log_2 \sum_{i=1}^n p_i q_i^{\alpha-1} + \frac{1-\alpha}{\sum_{i=1}^n p_i q_i^{\alpha-1}} \cdot \sum_{i=1}^n p_i q_i^{\alpha-1} \frac{\ln q_i}{\ln 2} \right) \\
&= \frac{1}{(1-\alpha)^2} \left(\log_2 \sum_{i=1}^n p_i q_i^{\alpha-1} - \frac{\alpha-1}{\sum_{i=1}^n p_i q_i^{\alpha-1}} \cdot \sum_{i=1}^n p_i q_i^{\alpha-1} \log_2 q_i \right) \\
&= \frac{1}{(1-\alpha)^2} \left(-D(p' || p) \right) \qquad \text{by (5)} \\
&\leq 0
\end{aligned}
$$

since $\frac{1}{(1-\alpha)^2} > 0$ and by the nonnegativity of divergence. Thus, we have shown that $H_\alpha(p; q)$ is non-increasing in α. $\square$