Student Number: 20053722 Name: Bryan Hoang

4. **Answer:**

(a) *Proof.* Let $Y = p_i$. Then $P(Y = p_i) = p_i$, which implies that

$$H_f(X) = E\left[f\left(\frac{1}{Y}\right)\right] \tag{1}$$

Since f is concave, then by Jensen's inequality, we have

$$H_f(X) \leq f\left(E\left[\frac{1}{Y}\right]\right) \qquad \text{by (1)} \tag{2}$$

Computing $E\left[\frac{1}{Y}\right]$ yields

$$\begin{aligned} E\left[\frac{1}{Y}\right] &= \sum_{i=1}^{N} p_i \cdot \frac{1}{p_i} \\ &= \sum_{i=1}^{N} 1 \\ &= N \end{aligned} \tag{3}$$

Therefore, by (2) and (3), we have $H_f(X) \leq f(N)$. □

(b) *Proof.* By the definition of Divergence, we have

$$\begin{aligned} D(q_1||q_2) &= \sum_{b\in\mathcal{Y}} q_1(b) \log_2 \frac{q_1(b)}{q_2(b)} \\ &= \sum_{b\in\mathcal{Y}} \left[\left(\sum_{a\in\mathcal{X}} p(b|a)p_1(a)\right) \log_2 \frac{\sum_{a\in\mathcal{X}} p(b|a)p_1(a)}{\sum_{a\in\mathcal{X}} p(b|a)p_2(a)}\right] \\ &\leq \sum_{b\in\mathcal{Y}} \sum_{a\in\mathcal{X}} p(b|a)p_1(a) \log_2 \frac{\cancel{p(b|a)}p_1(a)}{\cancel{p(b|a)}p_2(a)} && \text{by the Log-Sum Inequality} \\ &= \sum_{a\in\mathcal{X}} p_1(a) \log_2\left(\frac{p_1(a)}{p_2(a)}\right) \cancelto{1}{\sum_{b\in\mathcal{Y}} p(b|a)} \\ &= D(p_1||p_2) \end{aligned}$$

□