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1. (10 points)

(a) **Answer:**

Example. Let the sample space be $\mathcal{X} = \{0, 1\}$ and consider the following three probability distributions

$$p(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0, \\ \frac{1}{2} & \text{if } x = 1, \end{cases}$$
$$r(x) = \begin{cases} \frac{1}{4} & \text{if } x = 0, \\ \frac{3}{4} & \text{if } x = 1, \end{cases}$$
$$q(x) = \begin{cases} \frac{1}{8} & \text{if } x = 0, \\ \frac{7}{8} & \text{if } x = 1, \end{cases}$$

Then the triangle inequality for divergence implies that

$$\begin{aligned} D(p||q) &\leq D(p||r) + D(r||q) \\ \sum_{x\in\{0,1\}} p(x) \log \frac{p(x)}{q(x)} &\leq \sum_{x\in\{0,1\}} p(x) \log \frac{p(x)}{r(x)} + \sum_{x\in\{0,1\}} r(x) \log \frac{r(x)}{q(x)} \\ \frac{1}{2} \log \frac{\frac{1}{2}}{\frac{1}{8}} + \frac{1}{2} \log \frac{\frac{1}{2}}{\frac{7}{8}} &\leq \frac{1}{2} \log \frac{\frac{1}{2}}{\frac{1}{4}} + \frac{1}{2} \log \frac{\frac{1}{2}}{\frac{3}{4}} + \frac{1}{4} \log \frac{\frac{1}{4}}{\frac{1}{8}} + \frac{3}{4} \log \frac{\frac{3}{4}}{\frac{7}{8}} \\ 0.41 &\leq 0.20 \end{aligned}$$

which is false, thus making Divergence violate the triangle inequality.

(b) **Answer:**

Proof. Let p and q be two PMFs on the same alphabet $\mathcal{X}$.

To apply Jensen's Inequality, let

$$f(x) = -\log x$$

be a convex function and define the random variable Y to have the alphabet

$$\mathcal{Y} = \left\{ \frac{q(x)}{p(x)} : x \in \mathcal{X} \right\}$$

with the PMF

$$P\left(Y = \frac{q(x)}{p(x)}\right) = p(x), \ \forall\, x \in \mathcal{X}$$

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Then

$$
\begin{aligned}
D(p||q) &= \mathrm{E}_p\left[\log \frac{p(X)}{q(X)}\right] \\
&= \mathrm{E}_p\left[-\log \frac{q(X)}{p(X)}\right] \\
&= \mathrm{E}_p\left[-\log \frac{q(X)}{p(X)}\right] \\
&= \mathrm{E}_p[f(Y)] \\
&\geq f(\mathrm{E}[Y]) && \text{by Jensen's Inequality} \\
&= -\log \sum_{x \in \mathcal{X}} \frac{q(x)}{\cancel{p(x)}} \cdot \cancel{p(x)} \\
&= -\log 1 \\
&= 0
\end{aligned}
$$

Therefore, divergence is non-negative. □