MATH - MTHE 474/874 - Information Theory

Fall 2021

Homework # 1

Due Date: Friday October 1, 2021

Material: *Information measures.*

Readings: Chapter 2 of the textbook.

The referred problems are from the textbook.

(1) Problem 2.9.

(2) A single unbiased die is tossed once. If the face of the die is 1, 2, 3 or 4, an unbiased coin is tossed once. If the face of the die is 5 or 6, the coin is tossed twice. If we let X denote the face of the die and Y denote the number of heads obtained, determine $H(X,Y)$ and $I(X;Y)$ in bits.

(3) Problem 2.11.

(4) Answer the following questions.

(a) Let $f(y)$ be an arbitrary function defined for $y \geq 1$. Let X be a discrete random variable with alphabet $\mathcal{X} = \{a_1, a_2, \ldots, a_N\}$ and probability distribution $\{p_1, p_2, \ldots, p_N\}$, where $p_i = Pr\{X = a_i\}$, $i = 1, 2, \ldots, N$. Define the *f-entropy* of X by

$$H_f(X) = \sum_{i=1}^{N} p_i \; f\left(\frac{1}{p_i}\right).$$

If $f(\cdot)$ is concave, show that the following inequality is always satisfied:

$$H_f(X) \leq f(N).$$

(b) Let $p(y|x)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$, be a conditional distribution defined on the finite alphabets $\mathcal{X}$ and $\mathcal{Y}$. Let $p_1(\cdot)$ and $p_2(\cdot)$ be two distributions on $\mathcal{X}$, and $q_1(\cdot)$ and $q_2(\cdot)$ be two corresponding distributions on $\mathcal{Y}$ defined by

$$q_i(y) = \sum_{x \in \mathcal{X}} p(y|x) p_i(x),$$

$\forall y \in \mathcal{Y}$ and $i = 1, 2$. Show that

$$D(p_1 || p_2) \geq D(q_1 || q_2).$$

(5) *Information theoretic metric.* A real-valued function $d(\cdot, \cdot)$ defined on $\mathcal{X} \times \mathcal{X}$ is said to be a metric if for all $x, y \in \mathcal{X}$,

- $d(x, y) \geq 0$,
- $d(x, y) = d(y, x)$,
- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) + d(y, z) \geq d(x, z)$ for all $z \in \mathcal{X}$.

(a) Define the information-theoretic function $\Delta(\cdot, \cdot)$ of discrete random variables by

$$\Delta(X, Y) := H(X|Y) + H(Y|X).$$

Show that $\Delta(\cdot, \cdot)$ satisfies all the properties required of a metric, except that $X = Y$ is not a necessary condition for $\Delta(X, Y) = 0$. What is the necessary and sufficient condition for $\Delta(X, Y) = 0$?

(b) Show that

$$|H(X) - H(Y)| \leq \Delta(X, Y),$$

and

$$|H(X_1|X_2) - H(Y_1|Y_2)| \leq \Delta(X_1, Y_1) + \Delta(X_2, Y_2).$$

(6) Answer the following problems.

(i) For the sequence of (jointly distributed) random variables $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n$, show the following:

(a) If $X_1, X_2, \ldots, X_n$ are independent, then

$$I(X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots, Y_n) \geq \sum_{i=1}^{n} I(X_i; Y_i).$$

(b) If given X_i, the random variable Y_i is conditionally independent of the remaining random variables, for $i = 1, \ldots, n$, then

$$I(X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots, Y_n) \le \sum_{i=1}^{n} I(X_i; Y_i).$$

(ii) **[MATH 874 only]** Problem 2.28.

(7) Given a fixed positive integer $n > 1$, consider two probability distributions

$$p = (p_1, p_2, \ldots, p_n) \qquad \text{and} \qquad q = (q_1, q_2, \ldots, q_n)$$

with support $\mathcal{X} = \{1, 2, \ldots, n\}$; i.e. $p_i, q_i > 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$. Given $\alpha > 0$ and $\alpha \neq 1$, the *Rényi cross-entropy* between p and q of order α is given by

$$H_\alpha(p; q) := \frac{1}{1-\alpha} \log_2 \left(\sum_{i=1}^{n} p_i q_i^{\alpha-1} \right).$$

(a) Prove whether or not $H_\alpha(p; q) > 0$.

(b) Prove whether or not $H_\alpha(p; q)$ is non-increasing in α.

(c) **[MATH 874 only]** Determine $\lim_{\alpha \to 1} H_\alpha(p; q)$.

(d) **[MATH 874 only]** Compare $\lim_{\alpha \to 1} H_\alpha(p; q)$ to both $H(p)$ and $D(p\|q)$.