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7. (15 points)

(i)

(a) **Answer:**

Proof. To prove both inequalities, first observe that

$$\begin{aligned}
&\frac{1}{n}H(X^n) + \frac{1}{n}D(P_{X^n}||P_{\hat{X}_n}) \\
&= \frac{1}{n}\left(-\sum_{x^n \in X^n} P_{X^n}(x^n)\log_2 P_{X^n}(x^n) - \sum_{x^n \in X^n} P_{X^n}(x^n)\log_2 \frac{P_{\hat{X}^n}(x^n)}{P_{X^n}(x^n)}\right) \\
&= -\frac{1}{n}\sum_{x^n \in X^n} P_{X^n}(x^n)\log_2 \cancel{P_{X^n}(x^n)}\frac{P_{\hat{X}^n}(x^n)}{\cancel{P_{X^n}(x^n)}} \\
&= \frac{1}{n}\sum_{x^n \in X^n} P_{X^n}(x^n)(-\log_2 P_{\hat{X}^n}(x^n)). \qquad (1)
\end{aligned}$$

We also know that by the definition of $l(c_{x^n})$,

$$-\log_2 P_{\hat{X}_n}(x^n) \le l(c_{x^n}) < -\log_2 P_{\hat{X}_n}(x^n) + 1, \quad \forall x^n \in \mathcal{X}^n. \qquad (2)$$

Part 1. $\left(\frac{1}{n}H(X^n) + \frac{1}{n}D(P_{X^n}||P_{\hat{X}_n}) \le \bar{R}_n\right)$

$$\begin{aligned}
&\frac{1}{n}H(X^n) + \frac{1}{n}D(P_{X^n}||P_{\hat{X}_n}) \\
&= \frac{1}{n}\sum_{x^n \in X^n} P_{X^n}(x^n)(-\log_2 P_{\hat{X}^n}(x^n)) && \text{by (1)} \\
&\le \frac{1}{n}\sum_{x^n \in X^n} P_{X^n}(x^n) l(c_{x^n}) && \text{by (2)} \\
&= \bar{R}_n && \text{by the definition of } \bar{R}_n.
\end{aligned}$$

Part 2. $\left(\bar{R}_n < \frac{1}{n}H(X^n) + \frac{1}{n}D(P_{X^n}||P_{\hat{X}_n}) + \frac{1}{n}\right)$

$$\begin{aligned}
&\bar{R}_n \\
&= \frac{1}{n}\sum_{x^n \in X^n} P_{X^n}(x^n) l(c_{x^n}) && \text{by the definition of } \bar{R}_n. \\
&< \frac{1}{n}\sum_{x^n \in X^n} P_{X^n}(x^n)(-\log_2 P_{\hat{X}^n}(x^n)) + \frac{1}{n}\sum_{x^n \in X^n} P_{X^n}(x^n) && \text{by (2)} \\
&= \frac{1}{n}H(X^n) + \frac{1}{n}D(P_{X^n}||P_{\hat{X}_n}) + \frac{1}{n} && \text{by (1)}.
\end{aligned}$$

□

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(b) **Answer:**

$$
\begin{aligned}
&\lim_{n\to\infty} \hat{R}_n \\
&= \lim_{n\to\infty} \frac{1}{n} H(X^n) + \frac{1}{n} D(P_{X^n} || P_{\hat{X}_n}) \\
&= H(X_2|X_1) + \lim_{n\to\infty} \frac{1}{n} D(P_{X^n} || P_{\hat{X}_n}) \\
&= H(X_2|X_1) + \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} D(P_{X_i|X^{i-1}} || P_{\hat{X}_n}) \\
&\vdots \\
&\boxed{= H(P_{X_2,X_1}; P_{\hat{X}_2,\hat{X}_1}) - H(P_{X_1}; P_{\hat{X}_1})}.
\end{aligned}
$$