Student Number: [redacted] Name: Bryan Hoang

7. (20 points)

(i)

(a) **Answer:**

To determine the entropy rate of the stationary Markov source, we need to determine its stationary (and hence initial) distribution.

First, note that the transition matrix $\mathbb{Q}$ of the Markov source is

$$\mathbb{Q} = \begin{bmatrix} 1-\gamma & \gamma & 0 \\ 0 & 1-\beta & \beta \\ \alpha & 0 & 1-\alpha \end{bmatrix}$$

Let $\pi = (\pi_0, \pi_1, \pi_2)$ be stationary distribution of the Markov source. Then

$$\begin{aligned} \pi &= \pi\mathbb{Q} \\ (\pi_0, \pi_1, \pi_2) &= (\pi_0, \pi_1, \pi_2) \begin{bmatrix} 1-\gamma & \gamma & 0 \\ 0 & 1-\beta & \beta \\ \alpha & 0 & 1-\alpha \end{bmatrix} \\ \Rightarrow &\begin{cases} \pi_0 = (1-\gamma)\pi_0 + \alpha\pi_2 \\ \pi_1 = \gamma\pi_0 + (1-\beta)\pi_1 \\ \pi_2 = \beta\pi_1 + (1-\alpha)\pi_2 \end{cases} \end{aligned} \tag{1}$$

Solving (1) in addition to $\pi_0 + \pi_1 + \pi_2 = 1$ yields

$$\pi = \left(\frac{\alpha\beta}{\alpha\beta + \alpha\gamma + \beta\gamma}, \frac{\alpha\gamma}{\alpha\beta + \alpha\gamma + \beta\gamma}, \frac{\beta\gamma}{\alpha\beta + \alpha\gamma + \beta\gamma} \right).$$

Thus, the entropy rate is

$$\begin{aligned} H(\mathcal{X}) &= H(X_2|X_1) \\ &= -\sum_{x_1\in\mathcal{X}} \sum_{x_2\in\mathcal{X}} \pi(x_1) P_{X_2|X_1}(x_2|x_1) \log_2 P_{X_2|X_1}(x_2|x_1) \\ &= \frac{\alpha\beta}{\alpha\beta + \alpha\gamma + \beta\gamma} \Big[(1-\gamma)\log_2(1-\gamma) + \gamma\log_2\gamma\Big] \\ &\quad + \frac{\alpha\gamma}{\alpha\beta + \alpha\gamma + \beta\gamma} \Big[(1-\beta)\log_2(1-\beta) + \beta\log_2\beta\Big] \\ &\quad + \frac{\beta\gamma}{\alpha\beta + \alpha\gamma + \beta\gamma} \Big[(1-\alpha)\log_2(1-\alpha) + \alpha\log_2\alpha\Big] \\ &\boxed{= \frac{\alpha\beta}{\alpha\beta + \alpha\gamma + \beta\gamma} h_b(\gamma) + \frac{\alpha\gamma}{\alpha\beta + \alpha\gamma + \beta\gamma} h_b(\beta) + \frac{\beta\gamma}{\alpha\beta + \alpha\gamma + \beta\gamma} h_b(\alpha),} \end{aligned}$$

where $h_b(\alpha) \coloneqq (1-\alpha)\log_2(1-\alpha) + \alpha\log_2\alpha$ is the binary entropy function.

(b) **Answer:**

If $\alpha = 1$, then the transition matrix $\mathbb{Q}$ becomes

$$\mathbb{Q} = \begin{bmatrix} 1-\gamma & \gamma & 0 \\ 0 & 1-\beta & \beta \\ 1 & 0 & 0 \end{bmatrix},$$

and the stationary distribution π becomes

$$\pi = \left(\frac{\beta}{\beta+\gamma+\beta\gamma}, \frac{\gamma}{\beta+\gamma+\beta\gamma}, \frac{\beta\gamma}{\beta+\gamma+\beta\gamma} \right).$$

Since a unique stationary distribution still exists, **the Markov source $\{X_n\}$ is irreducible**. Thus, the entropy rate in this case is

$$\begin{aligned} H(\mathcal{X}) &= H(X_2|X_1) \\ &= -\sum_{x_1\in\mathcal{X}} \sum_{x_2\in\mathcal{X}} \pi(x_1) P_{X_2|X_1}(x_2|x_1) \log_2 P_{X_2|X_1}(x_2|x_1) \\ &= \frac{\beta}{\beta+\gamma+\beta\gamma}\Big[(1-\gamma)\log_2(1-\gamma) + \gamma\log_2\gamma\Big] \\ &\quad + \frac{\gamma}{\beta+\gamma+\beta\gamma}\Big[(1-\beta)\log_2(1-\beta) + \beta\log_2\beta\Big] + \cancel{\frac{\beta\gamma}{\beta+\gamma+\beta\gamma}\Big[0\Big]} \\ &\boxed{= \frac{\beta}{\beta+\gamma+\beta\gamma} h_b(\gamma) + \frac{\gamma}{\beta+\gamma+\beta\gamma} h_b(\beta),} \end{aligned}$$

(c) **Answer:**

If $\alpha = 1$ and $\beta = \gamma = \frac{1}{3}$, then the transition matrix $\mathbb{Q}$ becomes

$$\mathbb{Q} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 1 & 0 & 0 \end{bmatrix}$$

and the stationary distribution π becomes

$$\pi = \left(\frac{3}{7}, \frac{3}{7}, \frac{1}{7}\right).$$

Student Number: [redacted] Name: Bryan Hoang

Then computing the source redundancies gives

$$\begin{aligned}
\rho_D &= \log_2|\mathcal{X}| - H(X_1) \\
&= \log_2 3 - \left[-\frac{3}{7}\log_2\frac{3}{7} - \frac{3}{7}\log_2\frac{3}{7} - \frac{1}{7}\log_2\frac{1}{7}\right] \\
&\boxed{\approx 0.136\,\text{bits}} \\
\rho_M &= H(X_1) - H(\mathcal{X}) \\
&= \log_2 3 - \left[\frac{(\frac{1}{3})}{(\frac{1}{3}) + (\frac{1}{3}) + (\frac{1}{3})(\frac{1}{3})} h_b\left(\frac{1}{3}\right) + \frac{(\frac{1}{3})}{(\frac{1}{3}) + (\frac{1}{3}) + (\frac{1}{3})(\frac{1}{3})} h_b\left(\frac{1}{3}\right)\right] \\
&\boxed{\approx 0.662\,\text{bits}} \\
\rho_T &= \rho_D + \rho_M \\
&\approx 0.136\,\text{bits} + 0.662\,\text{bits} \\
&\boxed{\approx 0.798\,\text{bits}}
\end{aligned}$$

(d) **Answer:**

Given that the the Markov source is stationary and irreducible, it follows that it is also ergodic. Hence, its sample average converges to a constnt given by the expected value. i.e., the WLLN holds for this Markov source. Thus, we have that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i \\
&= \mathrm{E}[X_1] \\
&= \sum_{x\in\mathcal{X}} x\pi(x) \\
&= 0 \cdot \frac{3}{7} + 1 \cdot \frac{3}{7} + 2 \cdot \frac{1}{7} \\
&\boxed{= \frac{5}{7}}
\end{aligned}$$