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2. (20 points)

(a) **Answer:**
With the given transition matrix, we can determine the stationary distribution of the Markov source to be $\pi = (\frac{1}{2}, \frac{1}{2})$. To calculate $\bar{T}(x^n)$

$$\bar{T}(x^n) = \hat{F}(x^n) + \frac{1}{2}p(x^n),$$

we will first calculate $\hat{F}(x^n)$ using

$$\hat{F}(x^k) = \hat{F}(x^{k-1}) + p(x^{k-1}) \sum_{y^k < x^k} p(y_k|x^{k-1}),$$

for $k \in \mathbb{Z}_{\geq 2}$. Thus,

$$\begin{aligned}
\hat{F}(1) &= p(1) = \pi(1) = \frac{1}{2} \\
\hat{F}(10) &= \hat{F}(1) = \frac{1}{2} \\
\hat{F}(100) &= \hat{F}(10) = \frac{1}{2} \\
\hat{F}(1001) &= \hat{F}(100) + p(100)p(0|100) = \frac{1}{2} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{29}{54} \\
\hat{F}(10011) &= \hat{F}(1001) + p(1001)p(0|1001) = \frac{29}{54} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{95}{162} \\
\hat{F}(100110) &= \hat{F}(10011) = \frac{95}{162}
\end{aligned}$$

Therefore,

$$\bar{T}(100110) = \frac{95}{162} + \frac{1}{2}p(100110) = \boxed{\frac{289}{486} = 0.5946502058}.$$

(b) **Answer:**
Given the codeword 00001110000, we compute

$$z = \sum_{i=1}^{11} b_i 2^{-i} = 2^{-5} + 2^{-6} + 2^{-7} = \frac{7}{128} = 0.0546875.$$

Then starting from $k = 1$, we find sequentially

$$\max_{y \in \mathcal{X} : \hat{F}(x^{k-1}y) < z} y$$

and let $x_k = y$. We then repeat this for $k = 2, \ldots, n$. The resulting x^n will satisfy $\hat{F}(x^n) < z < F(x^n)$, so we decode x^n.
Note that $p(a) + p(b) = 0.2 + 0.3 = 0.5$.
For $k = 1$:

$$\begin{aligned}
\hat{F}(a) &= 0 < z \\
\hat{F}(b) &= 0.2 > z \\
&\Rightarrow x_1 = a.
\end{aligned}$$

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For $k = 2$:

$$\hat{F}(aa) = \hat{F}(a) = 0 < z$$
$$\hat{F}(ab) = \hat{F}(a) + p(a)p(a) = 0 + 0.2 \cdot 0.2 = 0.04 < z$$
$$\hat{F}(ac) = \hat{F}(a) + p(a)(p(a) + p(b)) = 0 + 0.2(0.2 + 0.3) = 0.1 > z$$
$$\Rightarrow x_2 = b.$$

For $k = 3$:

$$p(ab) = 0.2 \cdot 0.3 = 0.06$$
$$\hat{F}(aba) = \hat{F}(ab) = 0.04 < z$$
$$\hat{F}(abb) = \hat{F}(ab) + p(ab)p(a) = 0.04 + 0.06 \cdot 0.2 = 0.052 < z$$
$$\hat{F}(abc) = \hat{F}(ab) + p(ab)(p(a) + p(b)) = 0.04 + 0.06 \cdot 0.5 = 0.07 > z$$
$$\Rightarrow x_3 = b.$$

For $k = 4$:

$$p(abb) = 0.2 \cdot 0.3 \cdot 0.3 = 0.018$$
$$\hat{F}(abba) = \hat{F}(abb) = 0.052 < z$$
$$\hat{F}(abbb) = \hat{F}(abb) + p(abb)p(a) = 0.052 + 0.018 \cdot 0.2 = 0.0556 > z$$
$$\Rightarrow x_4 = a.$$

For $k = 5$:

$$p(abba) = 0.2 \cdot 0.3 \cdot 0.3 \cdot 0.2 = 0.0036$$
$$\hat{F}(abbaa) = \hat{F}(abba) = 0.052 < z$$
$$\hat{F}(abbab) = \hat{F}(abba) + p(abba)p(a) = 0.052 + 0.0036 \cdot 0.2 = 0.05272 < z$$
$$\hat{F}(abbac) = \hat{F}(abba) + p(abba)(p(a) + p(b)) = 0.052 + 0.0036 \cdot 0.5 = 0.0538 < z$$
$$\Rightarrow x_5 = c.$$

$\therefore \boxed{x^5 = x_1 x_2 x_3 x_4 x_5 = abbac}$.

(c) **Answer:**