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3. (20 points)

(a) **Answer:**

Proof. Define

$$c = \left(\sum_{x^n \in \mathcal{X}^n} 2^{-l(x^n)} \right)^{-1}.$$

Then $q(x^n) = c2^{-l(x^n)}$ is a valid pmf on $\mathcal{X}^n$ and since $c \geq 1$ by Kraft's inequality,

$$l(x^n) \geq -\log q(x^n). \tag{1}$$

Then the corresponding Shannon-Fano code has codeword lengths $l_q(x^n) = \lceil -\log q(x^n) \rceil$.
Thus,

$$\begin{aligned}
\frac{1}{n} E[l_q(X^n)] &= \frac{1}{n} \sum_{x^n \in \mathcal{X}^n} p(x^n) \lceil -\log q(x^n) \rceil \\
&\leq \frac{1}{n} \sum_{x^n \in \mathcal{X}^n} p(x^n)(-\log q(x^n) + 1) \\
&\leq \frac{1}{n} \sum_{x^n \in \mathcal{X}^n} p(x^n) l(x^n) + \frac{1}{n} && \text{by (1)} \\
&= \frac{1}{n} E[l(X^n)] + \frac{1}{n}.
\end{aligned}$$

□

(b) **Answer:**

Proof.
Part 1. ($\Rightarrow$)
Assume that there exists a sequence $\{C_n\}$ of prefix codes $C_n : \mathcal{X}^n \to \{0,1\}^*$ which is universal with respect to $\mathcal{P}$. Then $\forall\, p \in \mathcal{P}$,

$$\lim_{n\to\infty} R(C_n, p) = 0.$$

By part (a), $\exists\, q \in \mathcal{P} : \frac{1}{n} E[l_q(X^n)] \leq \frac{1}{n} E[l(X^n)] + \frac{1}{n}$, where $l(X^n) = |C_n|$.
Thus, $\lim_{n\to\infty} R(S_n, p) = 0$ for some sequence $\{S_n\}$ of Shannon-Fano codes, which means that $\{S_n\}$ is universal with respect to $\mathcal{P}$.
Therefore, by Corollary 6 on slide 49, we have that $\forall\, p \in \mathcal{P}$,

$$\lim_{n\to\infty} \frac{1}{n} D(p||q_n) = 0.$$

Part 2. ($\Leftarrow$)
Assume that $\exists \{q_n\}$ a sequence of probability distributions such that $\forall\, p \in \mathcal{P}$,

$$\lim_{n\to\infty} \frac{1}{n} D(p||q_n) = 0.$$

Then by Corollary 6 on slide 49, we can get a sequence of Shannon-Fano codes $\{C_n\}$ obtained from $\{q_n\}$ which is universal with respect to $\mathcal{P}$. Since Shannon-Fano codes are also prefix codes, the proof is complete.

□