MATH/MTHE 477/877 – Winter 2022

Homework 3

due Monday, March 7

1. Suppose the source random variable X has the two-sided exponential (Laplace) pdf $f(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$. Let $b > a > 0$ and define the three-level quantizer Q by

$$Q(x) = \begin{cases} b & \text{if } x > a \\ 0 & \text{if } -a \le x \le a \\ -b & \text{if } x < -a. \end{cases}$$

   (a) Considering the mean squared distortion, find an expression for b as a function of a so that the centroid condition is met.

   (b) Find a such that the resulting quantizer Q (with b as above) is a Lloyd-Max quantizer. Calculate the resulting mean squared error.

2. Consider the rth power distortion measure $d(x, y) = |x - y|^r$, where $r \ge 2$ is an integer. Let the pdf $f(x)$ be positive and *constant* on the bounded interval (a, b). Show that the centroid of the cell (a, b) with respect to f and d is the midpoint $(a + b)/2$. (Hint: make use the fact that the function $|x|^r$ is differentiable everywhere.)

3. Write a MATLAB program that implements the training-set version of the Lloyd design algorithm for MSE distortion. First your program should generate (say) $M = 5,000$ random samples from the standard normal distribution. Then the program uses this training set to design MSE-optimized quantizers of rates $R = 1$, 2, and 3 bits/sample. You can set the stopping threshold as $\epsilon = 0.001$. Report on the resulting three quantizers and their distortions. For the $R = 1$, how close is your quantizer to the Lloyd-Max quantizer obtained on slides 24-25?

4. Let X be a random variable with finite variance, Q an MSE optimal N-level quantizer for X, let $Y = aX + b$, where $a \ne 0$ and b are real constants, and $\hat{Q}$ an MSE optimal N-level quantizer for Y.

   (a) Let f denote the pdf of X and g the pdf of Y. Assume that N is large enough so that the distortion of Q is well approximated by the high-rate formula $D_N = (1/12)N^{-2}\|f\|_{1/3}$. By finding $\|g\|_{1/3}$ in terms of $\|f\|_{1/3}$, express the high-rate approximation for the distortion $E[(Y - \hat{Q}(Y))^2]$ in terms of $E[(X - Q(X))^2]$.

   (b) Here we don't assume that N is large and X has a pdf. Assuming Q is unique, describe $\hat{Q}$ in terms of the code points and decision levels of Q. What is the relation between $E[(X - Q(X))^2]$ and $E[(Y - \hat{Q}(Y))^2]$? Does this reinforce the answer to part (a)?

5. Let X be a random variable with "smooth" pdf $f(x)$ which vanishes outside a finite interval $[a, b]$. Let Q_N be the N-level uniform quantizer over $[a, b]$. Assume that N is large and find (approximately) the distortion of Q_N if the rth power distortion measure $d(x, y) = |x - y|^r$ $(r > 0)$ is used.