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4. (30 points)

(a) **Answer:**
From (ii) and (iii), we get

$$\begin{aligned} r_i &= H(Q_{\Delta_i}(Y_i)) \\ &= h(Y_i) - \log_2 \Delta_i \\ \Rightarrow \Delta_i &= 2^{h(Y_i)-r_i} \\ \rightarrow \text{(i)} \Rightarrow D_i &= \frac{1}{12} 2^{2h(Y_i)} 2^{-2r_i}. \end{aligned} \tag{1}$$

Therefore, the optimization problem is a CMP to minimize

$$D(\mathbf{r}) \triangleq \sum_{i=1}^{k} D_i = \frac{1}{12} \sum_{i=1}^{k} 2^{2h(Y_i)} 2^{-2r_i} \tag{2}$$

subject to $\sum_{i=1}^{k} r_i \leq R$. Since D_i is strictly decreasing in r_i, it follows that $\sum_{i=1}^{k} r_i = R$ for the optimal $\mathbf{r} = (r_1, \dots, r_k)$.
To use the Lagrange multiplier method, let

$$\begin{aligned} f(\mathbf{r}) &\triangleq D(\mathbf{r}) = \frac{1}{12} \sum_{i=1}^{k} 2^{2h(Y_i)} 2^{-2r_i}, \\ g(\mathbf{r}) &\triangleq \sum_{i=1}^{k} r_i - R, \\ J(\mathbf{r}, \lambda) &\triangleq f(\mathbf{r}) + \lambda g(\mathbf{r}). \end{aligned}$$

Then solving

$$\frac{\partial}{\partial r_j} J(\mathbf{r}, \lambda) = \frac{1}{12} 2^{2h(Y_i)} (-2 \ln 2) 2^{-2r_i} + \lambda = 0$$

yields

$$r_i = h(Y_i) + \underbrace{\log_2 \frac{\ln 2}{12\lambda}}_{\hat{\lambda}}.$$

From the constraint $\sum_{j=1}^{k} r_j = R$, $\hat{\lambda}$ must satisfy

$$\begin{aligned} R &= \sum_{j=1}^{k} h(Y_j) + k\hat{\lambda} \\ \Rightarrow \hat{\lambda} &= \frac{R}{k} - \frac{1}{k} \sum_{j=1}^{k} h(Y_j). \end{aligned}$$

Substituting into

$$r_i = h(Y_i) + \hat{\lambda} s$$

yields

$$r_i = h(Y_i) + \frac{R}{k} - \frac{1}{k} \sum_{j=1}^{k} h(Y_j). \tag{3}$$

Then by (1) we get that $\forall i = 1, \dots, k$,

$$\boxed{\Delta_i = 2^{-\frac{R}{k} + \frac{1}{k} \sum_{j=1}^{k} h(Y_j)}}.$$

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Hence, substituting (3) into (2) gives the the minimum total distortion as

$$\begin{aligned} D(\mathbf{r}) &= \frac{1}{12}\sum_{i=1}^{k} 2^{2h(Y_i)} 2^{-2\left(\frac{R}{k}+h(Y_i)-\frac{1}{k}\sum_{j=1}^{k} h(Y_j)\right)} \\ &\boxed{= \frac{k}{12} 2^{\frac{2}{k}\sum_{j=1}^{k} h(Y_j)} 2^{-\left(\frac{R}{k}\right)}}. \end{aligned}$$

The above minimum distortion is the same as if k copies of a random variable with differential entropy $\frac{1}{k}\sum_{j=1}^{k} h(Y_j)$ were quantized using the uniform rate allocation $r_i = \frac{R}{k}$.

(b) **Answer:**
Since we have that $\forall i = 1, \dots, k$, $Y_i \sim N(\mu, \sigma_i^2)$, it follows that

$$\begin{aligned} h(Y_i) &= \frac{1}{2}\log_2(2\pi e \sigma_i^2) \\ \Rightarrow \frac{1}{k}\sum_{j=1}^{k} h(Y_j) &= \frac{1}{k}\sum_{j=1}^{k} \frac{1}{2}\log_2(2\pi e \sigma_i^2) \\ &= \frac{1}{2}\log_2(2\pi e) + \frac{1}{2}\log_2\left(\left(\prod_{j=1}^{k} \sigma_j^2\right)^{\frac{1}{k}}\right). \end{aligned}$$

Then by part (a),

$$\begin{aligned} D(\mathbf{r}) &= \frac{k}{12} 2^{\frac{2}{k}\sum_{j=1}^{k} h(Y_j)} 2^{-\left(\frac{R}{k}\right)} \\ &\boxed{= \frac{k\pi e}{6}\left(\prod_{j=1}^{k} \sigma_j^2\right)^{\frac{1}{k}} 2^{-2\left(\frac{R}{k}\right)}}. \end{aligned}$$

(c) **Answer:**

Proof. □