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1. (25 points)

(a) **Answer:**

Proof. Let's start on the LHS.

$$
\begin{aligned}
R(0) &= \min_{p(\hat{x}|x):Ed(X,\hat{X})\leq 0} I(X;\hat{X}) && \text{by definition of } R(D) \\
&= \min_{p(\hat{x}|x):Ed(X,\hat{X})\leq 0} (H(X) - H(X \mid \hat{X})). && (1)
\end{aligned}
$$

Claim. $\forall\, p(\hat{x}|x) : Ed(X,\hat{X}) \leq 0,\ H(X \mid \hat{X}) = 0$

Proof. $\forall\, p(\hat{x}|x)$,

$$
\begin{aligned}
Ed(X,\hat{X}) &\leq 0 \\
\sum_{x\in\mathcal{X}} \sum_{\hat{x}\in\hat{\mathcal{X}}} p(x)p(\hat{x}|x)d(x,\hat{x}) &\leq 0 \\
\Rightarrow d(x,\hat{x}) &= 0 && \forall\, x \in \mathcal{X},\ \hat{x} \in \hat{\mathcal{X}} \\
\Rightarrow x &= \hat{x} && \forall\, x \in \mathcal{X},\ \hat{x} \in \hat{\mathcal{X}} \\
\Rightarrow H(X|\hat{X}) &= 0
\end{aligned}
$$

□

With the claim, (1) becomes

$$
\begin{aligned}
R(0) &= \min_{p(\hat{x}|x):Ed(X,\hat{X})\leq 0} H(X) \\
&= H(X).
\end{aligned}
$$

since $H(X)$ doesn't depend on $p(\hat{x}|x)$. □

(b) **Answer:**

Proof. By the non-increasing property of $R(D)$, it is sufficient to show that $R(D^*) = 0$ to prove the result. First, let $\hat{x}^* \in \hat{X}$ be the element that achieve D^*. Then let

$$
p(\hat{x}|x) = \begin{cases} 1, & \hat{x} = \hat{x}^* \\ 0, & \text{otherwise.} \end{cases} \tag{2}
$$

It follows that

$$
\begin{aligned}
Ed(X,\hat{X}) &= \sum_{x\in\mathcal{X}} \sum_{\hat{x}\in\hat{\mathcal{X}}} p(x)p(\hat{x}|x)d(x,\hat{x}) \\
&= \sum_{x\in\mathcal{X}} p(x)d(x,\hat{x}^*) \\
&= Ed(X,\hat{x}^*) \\
&= D^*
\end{aligned}
$$

which satisfies the constraint for minimizing $I(X;\hat{X})$. By (2), we also have that $X \perp\!\!\!\perp \hat{X}$ which means that $I(X;\hat{X}) = 0$. By the nonnegativity of mutual information,

$$
\begin{aligned}
R(D^*) &= \min_{p(\hat{x}|x):Ed(X,\hat{X})\leq D^*} I(X;\hat{X}) \\
&= 0.
\end{aligned}
$$

$\therefore \forall\, D \geq D^*,\ R(D^*) = 0.$ □

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(c) **Answer:**

Proof. (by contraction)
Suppose that for $0 \leq D \leq D^*,\ R(D) = 0$. Then $\exists\, p(\hat{x}|x) : Ed(X, \hat{X}) \leq D$ and that $I(X; \hat{X}) = 0$, which implies that $X \perp\!\!\!\perp \hat{X}$. But we also have that

$$\begin{aligned}
Ed(X, \hat{X}) &= \sum_{x \in \mathcal{X}} \sum_{\hat{x} \in \hat{\mathcal{X}}} p(x) p(\hat{x}|x) d(x, \hat{x}) \\
&= \sum_{\hat{x} \in \hat{\mathcal{X}}} p(\hat{x}) \sum_{x \in \mathcal{X}} p(x) d(x, \hat{x}) && \because X \perp\!\!\!\perp \hat{X} \\
&= \sum_{\hat{x} \in \hat{\mathcal{X}}} p(\hat{x}) Ed(X, \hat{x}) \\
&\geq \left(\sum_{\hat{x} \in \hat{\mathcal{X}}} p(\hat{x}) \right) D^* && \text{by the definition of } D^* \\
&= D^*
\end{aligned}$$

So we have $Ed(X, \hat{X}) \geq D^*$, but that contradicts the assumption of $Ed(X, \hat{X}) \leq D \leq D^*$, leading to a contradiction. Therefore, for $0 \leq D \leq D^*,\ R(D) > 0$. □