MATH/MTHE 477/877 – Winter 2022

Homework 1

due Monday, Jan. 31

1. Let $\mathcal{X}$ be a finite alphabet, assume $\hat{\mathcal{X}} = \mathcal{X}$, and that $d(x, \hat{x})$ satisfies

$$d(x, \hat{x}) = 0 \quad \text{if and only if } x = \hat{x}.$$

Define

$$D^* = \min_{\hat{x} \in \hat{\mathcal{X}}} Ed(X, \hat{x})$$

Assume that $H(X) > 0$ and prove the following:

   (a) $R(0) = H(X)$,

   (b) $R(D) = 0$ if $D \geq D^*$,

   (c) $R(D) > 0$ if $0 \leq D < D^*$.

2. Let $X \sim \text{Bernoulli}(1/2)$ with alphabet $\mathcal{X} = \{0, 1\}$ and let the reproduction alphabet and distortion measure be given by $\hat{\mathcal{X}} = \{0, e, 1\}$ and

$$d(x, \hat{x}) = \begin{cases} 0 & \text{if } x = \hat{x}, \\ 1 & \text{if } \hat{x} = e, \\ \infty & \text{if } x = 0 \text{ and } \hat{x} = 1, \text{ or } x = 1 \text{ and } \hat{x} = 0. \end{cases}$$

Determine $R(D)$.

Hint: We use the convention $0 \cdot \infty = 0$. You can *assume* that in the definition of $R(D)$ it is enough to consider *symmetric* conditional distributions $p(\hat{x}|x)$ (i.e., such that $p(0|0) = p(1|1)$). Then $I(X; \hat{X}) = H(X) - H(X|\hat{X})$ and $Ed(X, \hat{X})$ are is easy to evaluate.

3. Let $X_1, X_2, \ldots, X_n, \ldots$ be a Bernoulli(1/2) source with alphabet $\{0, 1\}$ and let $d(x, \hat{x})$ be the Hamming distortion. The sequence X^n is to be encoded so that its reconstruction satisfies $Ed(X^n, \hat{X}^n) \leq D$ for some $D \in [0, 1]$. Two coding schemes are proposed.

Scheme 1. Whenever the source produces a '0', it is flipped to a '1' with probability ρ (it stays as '0' with probability $1 - \rho$), where the value of ρ is chosen in accordance with the maximum allowed expected distortion D. (These flips are independent of each other.) Whenever the source produces a '1', the bit remains unchanged. In this manner, the original symmetric source is converted into an asymmetric source. Next, this asymmetric source is *losslessly* encoded, using R_1 bits/source symbol. The reconstruction (or decoding) is performed by reconstructing (losslessly) the asymmetric source. Assume n is large and let $R_1(D)$ denote the minimum rate R_1 needed by this scheme to yield an expected distortion at most D.

Scheme 2. Given a source sequence of length n, only the first nR_2 bits are *losslessly* encoded. The decoding consists of reconstructing (losslessly) these first nR_2 bits and padding them with 0s to obtain a sequence of total length n. For large n let $R_2(D)$ denote the minimum number of bits/source symbol needed by this scheme to yield an expected distortion not exceeding D.

   (a) Determine $R_1(D)$ as a function of D for $0 \leq D \leq 1$.

   (b) Determine $R_2(D)$ as a function of D for $0 \leq D \leq 1$.

(c) Compare Schemes 1 and 2 by plotting $R_1(D)$ and $R_2(D)$ in the same figure. Also include the plot of the rate-distortion function $R(D)$ of the given source.

4. Problem 6.17 of the MTHE 477 textbook (F. Alajaji and P.-N. Chen, An Introduction to Single-User Information Theory, Springer, 2018). Note that in the book, the source and reproduction alphabets are denoted by $\mathcal{Z}$ and $\hat{\mathcal{Z}}$, respectively.

   Hint: The capacity of the q-ary symmetric DMC is given in Example 4.19 of the book and the rate-distortion function of the q-ary symmetric DMS is given in Observation 6.25.