Student Number: [redacted] Name: Bryan Hoang

2. (25 points) **Answer:**

Claim.

$$R(D) = \begin{cases} 1 - D, & 0 \leq D \leq 1, \\ 0, & D \geq 1. \end{cases}$$

Proof. Let $0 \leq D \leq 1$. First, we will show that $1 - D$ is the minimum achievable mutual information given the distortion constraint under the previous assumption. Assume that $p(\hat{x}|x)$ satisfies the distortion constraint. For that to occur, we have

$$\begin{aligned} Ed(X, \hat{X}) &= \sum_{x \in \mathcal{X}} \sum_{\hat{x} \in \hat{\mathcal{X}}} p(x) p(\hat{x}|x) d(x, \hat{x}) \\ &= \sum_{\hat{x} \in \hat{\mathcal{X}}} \left(\frac{1}{2} p(\hat{x}|0) d(0, \hat{x}) + \frac{1}{2} p(\hat{x}|1) d(1, \hat{x}) \right) \\ &= \frac{1}{2} (p(e|0) + \infty \cdot p(1|0) + p(e|1) + \infty \cdot p(0|1)) \\ &\leq D. \end{aligned} \tag{1}$$

That implies that $p(1|0) = p(0|1) = 0$ for the distortion constraint to be satisfied.

With that in mind, let $p \in \mathbb{R} : p \in [0, 1]$ and define $p(\hat{x}|x)$ by

$$\begin{aligned} p_{\hat{X}|X}(0|0) = p_{\hat{X}|X}(1|1) = p \\ p_{\hat{X}|X}(e|0) = p_{\hat{X}|X}(e|1) = 1 - p \end{aligned}$$

which implies that for $\hat{x} \in \hat{X}$,

$$p_{\hat{X}}(\hat{x}) = \begin{cases} \frac{1}{2} p, & \hat{x} = 0, \\ (1 - p), & \hat{x} = e, \\ \frac{1}{2} p, & \hat{x} = 1. \end{cases}$$

Then by Bayes' theorem,

$$\begin{aligned} p_{X|\hat{X}}(0|0) = p_{X|\hat{X}}(1|1) &= \frac{\frac{1}{2} p}{\frac{1}{2} p + \frac{1}{2}(0)} = 1 \\ p_{X|\hat{X}}(0|e) = p_{X|\hat{X}}(1|e) &= \frac{\frac{1}{2}(1 - p)}{\frac{1}{2}(1 - p) + \frac{1}{2}(1 - p)} = \frac{1}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} H(X|\hat{X}) &= \sum_{x \in \mathcal{X}} \sum_{\hat{x} \in \hat{\mathcal{X}}} p(\hat{x}) p(x|\hat{x}) \log p(x|\hat{x}) \\ &= 0 + \frac{1}{2}(1 - p) \log_2(\frac{1}{2}) + 0 + 0 + \frac{1}{2}(1 - p) \log_2(\frac{1}{2}) + 0 \\ &= 1 - p. \end{aligned}$$

From (1) and the definition of $p_{\hat{X}|X}$, we also have

$$\begin{aligned} Ed(X, \hat{X}) = \frac{1}{2}((1 - p) + (1 - p)) \\ &= 1 - p \\ &\leq D \end{aligned}$$

Student Number: [redacted] Name: Bryan Hoang

which implies that $H(X|\hat{X}) = 1 - p \leq D$. Thus,

$$\begin{aligned} I(X;\hat{X}) &= H(X) - H(X|\hat{X}) \\ &\geq 1 - D && \because H(X) = 1 \text{ for Bernoulli}(\frac{1}{2}). \end{aligned}$$

Now that we have shown that $1 - D$ is the minimum achievable mutual information given the distortion constraint, let's show an achievable distribution that makes $I(X;\hat{X}) = 1 - D$.

If we set $p = 1 - D$, then the previously defined distribution becomes

$$p(\hat{x}|x) = \begin{cases} 1 - D, & \hat{x} = x, \\ D, & \hat{x} = e, \\ 0, & \hat{x} \neq x. \end{cases}$$

Clearly, the distortion constraint is satisfied,

$$Ed(X,\hat{X}) = \frac{1}{2}(D + D) = D$$

and $I(X,\hat{X}) = 1 - (1 - p) = 1 - D$. By the nonnegativity and nonincreasing properties of $R(D)$, we have

$$R(D) = \begin{cases} 1 - D, & 0 \leq D \leq 1, \\ 0, & D \geq 1. \end{cases}$$

$\square$