MATH/MTHE 477/877 – Winter 2022

Homework 4

due Monday, March 28

1. *(Linear prediction)* Let $Y, X_1, X_2, \ldots, X_m$ be random variables with finite second moments and and let $\hat{Y} = \sum_{i=1}^m a_i X_i$ be an *optimal* linear predictor for Y minimizing the mean square prediction error $E[(Y - \hat{Y})^2]$. Prove that
$$E[(Y - \hat{Y})^2] = E(Y^2) - E(\hat{Y}^2).$$

2. *(Linear Prediction)* Suppose X and Y are jointly distributed random variables with finite mean and variance. We want to estimate the value of Y by an *affine* function of X as $\hat{Y} = aX + b$ so that the mean squared error
$$E[(Y - \hat{Y})^2] = E[(Y - (aX + b))^2]$$
is *minimized* over all choices of $a, b \in \mathbb{R}$.

   (a) Use the orthogonality principle to show that the optimal choice for a and b is
$$a = \frac{\mathrm{Cov}(X, Y)}{\mathrm{Var}(X)}, \qquad b = E(Y) - aE(X).$$
(Hint: $\hat{Y} = aX + b$ is a linear predictor of Y from the random variables X and $X_0 = 1$ (constant 1 random variable) with prediction coefficients a and b.)

   (b) Show also that this optimal $\hat{Y}$ can be expressed as
$$\hat{Y} = m_Y + \frac{\rho\sigma_Y}{\sigma_X}(X - m_X)$$
where $m_X = E(X)$, $m_Y = E(Y)$, $\sigma_X = \sqrt{\mathrm{Var}(X)}$, $\sigma_Y = \sqrt{\mathrm{Var}(Y)}$, and $\rho = \rho(X, Y) = \dfrac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}$ is the correlation of X and Y.

   (c) Show that the resulting minimum mean squared error is
$$E[(Y - (aX + b))^2] = \sigma_Y^2(1 - \rho^2).$$

3. *(Predictive quantization)* A discrete-time wide sense stationary process $\{X_n\}$ has zero mean and autocorrelation values $r_0 = 4$, $r_1 = 2$, $r_2 = 1$, and $r_k = 0$ if $k \geq 3$.

   Two DPCM coders are designed for the input process $\{X_n\}$. In both coders, the predictor is designed to be optimal for $\{X_n\}$. Coder 1 uses a first-order predictor and a 64 level quantizer, Coder 2 uses a second-order predictor and a 256 level quantizer.

   (a) Use the orthogonality principle to determine the predictor coefficients for both DPCM coders.

   (b) In both coders, the quantizer design assumes approximately zero mean and uniform distribution for the prediction error over the interval $[-B, B]$ where B is a sufficiently large number so that the overload distortion is negligible. What is approximately the SNR improvement achieved by switching from Coder 1 to Coder 2?

4. (*Transform coder with uniform quantization and entropy coding*) Let $\boldsymbol{X} = (X_1, \ldots, X_k)^t$ be a zero-mean random vector and $\boldsymbol{A}$ a $k \times k$ orthogonal matrix. Consider the transform coder in which the the ith component of $(Y_1, \ldots, Y_k)^t = \boldsymbol{Y} = \boldsymbol{AX}$ is uniformly quantized with an infinite-level uniform quantizer Q_{Δ_i} having step size $\Delta_i > 0$. Since Q_{Δ_i} has infinitely many levels, in general its output cannot be losslessly encoded using finite binary strings of equal length. Suppose instead that we encode $\hat{Y}_i = Q_{\Delta_i}(Y_i)$ using an optimal variable-length binary lossless code. Then the rate of the quantizer r_i (in bits) is defined to be the expected code length of this binary code. At the decoder, the components $\hat{Y}_i = Q_{\Delta_i}(Y_i)$, $i = 1, \ldots, k$ are reconstructed and the system's output is $\hat{\boldsymbol{X}} = \boldsymbol{A}^{-1}\hat{\boldsymbol{Y}}$.

   (a) Assume the ith transform coefficient Y_i has pdf f_i. We make the following approximations:

      (i) The MSE distortion of Q_{Δ_i} is given by

$$D_i = E[(Y_i - Q_{\Delta_i}(Y_i))^2] = \frac{\Delta_i^2}{12}.$$

      (ii) The average code length r_i is equal to its theoretical lower bound, the entropy of $Q_{\Delta_i}(Y_i)$:

$$r_i = H(Q_{\Delta_i}(Y_i)).$$

      (iii) The entropy $H(Q_{\Delta_i}(Y_i))$ can be calculated as

$$H(Q_{\Delta_i}(Y_i)) = h(Y_i) - \log \Delta_i$$

      where $h(Y_i) = -\int f_i(x) \log f_i(x)\, dx$ is the differential entropy of Y_i, which is assumed to be finite.

   (*Note*: We know from the lossless source coding theorem that (ii) is accurate within 1 bit. It can be shown that with some regularity assumption on f_i, the approximations (i) and (iii) become accurate as $\Delta_i \to 0$, i.e., for small distortions/large rates.)

   Given an overall bit quota $R > 0$, apply one of the techniques we used in class to prove the optimal bit allocation formula to find $\Delta_1, \Delta_2, \ldots, \Delta_k$ which minimize the overall distortion $\sum_{i=1}^k D_i$ subject to the rate constraint $\sum_{i=1}^k r_i \le R$. Also, express the resulting minimum distortion as a function of R and the $h(Y_i)$'s and interpret the result.

   (b) Assume each Y_i is a zero-mean Gaussian random variable with positive variance σ_i^2. Calculate $h(Y_i)$ and use the result to calculate the minimum overall distortion $\sum_{i=1}^k D_i$ in part (a).

   (c) Assume $\boldsymbol{X} = (X_1, \ldots, X_k)^t$ is a zero-mean Gaussian random vector and suppose the uniform quantizers are chosen to minimize $\sum_{i=1}^k E[(Y_i - Q_{\Delta_i}(Y_i))^2]$ for a given bit quota R as in part (a). Combine your answer to (b) with the proof of optimality of the K-L transform (learned in class) to show that under the assumptions we have made the K-L transform is the optimal transform for this system in the sense of minimizing the total end-to-end distortion $E[\|\boldsymbol{X} - \hat{\boldsymbol{X}}\|^2]$. Justify each step of your proof.

   (Hint: Write out the distortion $E[\|\boldsymbol{X} - \hat{\boldsymbol{X}}\|^2] = D_{\boldsymbol{A}}$ of the system for an arbitrary orthogonal $\boldsymbol{A}$ and show that $D_{\boldsymbol{A}} \ge D_{\boldsymbol{T}}$, where $\boldsymbol{T}$ is the K-L transform matrix for $\boldsymbol{X}$.)