MODULATION GROUPS

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ABSTRACT. Conjectures of Braverman and Kazhdan, Ngô and Sakellaridis have motivated the development of Schwartz spaces for certain spherical varieties. We prove that under suitable assumptions these Schwartz spaces are naturally a representation of a group that we christen the modulation group. This provides a broad generalization of the defining representation of the metaplectic group. The example of a vector space and the zero locus of a quadric cone in an even number of variables are discussed in detail. In both of these cases the modulation group is closely related to algebraic groups, and we propose a conjectural method of linking modulation groups to ind-algebraic groups in general. At the end of the paper we discuss adelization and the relationship between representations of modulation groups and the Poisson summation conjecture.

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1. Introduction

Let F be a characteristic zero local field, let H be an affine algebraic group over F, and let X be an affine H-scheme such that $X^{\text{sm}}(F) \neq \emptyset$. We then have a unitary representation

$$L^{2}(X^{sm}(F), \mathcal{L}^{1/2})$$

of H(F), where $\mathcal{L}^{1/2}$ is the sheaf of half-densities. If $X^{\text{sm}}(F)$ admits an H(F) eigenmeasure then one can alternately form the Hilbert space using the eigenmeasure (see Remark 3.1).

In various settings there are additional automorphisms of $L^2(X^{sm}(F), \mathcal{L}^{1/2})$ that allow us to extend the action of H(F) to an action of a larger abstract group. These larger groups have been studied in special cases, but never systematically. Our aim in this paper is to unify these special cases into a broader framework, and conjecturally relate them to the F-points of ind-algebraic groups.

Our ultimate motive, however, is to use this study to add structure to certain Schwartz spaces that appear in the Poisson summation conjecture. We give more details in §1.6 below. Ultimately the constructions in this paper upgrade any suitable Poisson summation formula to representations generalizing the global Weil representation.

1.1. Small modulation groups. Suppose that we are given a (right) representation V of H and an H-equivariant map

$$\omega: X \longrightarrow V$$
.

We define a left action of $V^{\vee}(F) \rtimes H(F)$ on $L^{2}(X^{\mathrm{sm}}(F), \mathcal{L}^{1/2})$ by

$$\mathcal{R}_{\omega}: V^{\vee}(F) \rtimes H(F) \times L^{2}(X^{\mathrm{sm}}(F), \mathcal{L}^{1/2}) \longrightarrow L^{2}(X^{\mathrm{sm}}(F), \mathcal{L}^{1/2})$$

$$((\lambda, h), f) \longmapsto \psi(\lambda \circ \omega(x)) f(xh),$$

where ψ is a non-trivial additive character of F.

Definition 1.1. The (spectral) small modulation group $\Psi_{\omega}^{s}\{F\}$ is the image of $V^{\vee}(F) \rtimes H(F)$ in $\operatorname{Aut}(L^{2}(X^{\operatorname{sm}}(F), \mathcal{L}^{1/2}))$.

The odd notation $\Psi^s_{\omega}\{F\}$ in the theorem is due to the fact that, a priori, $\Psi^s_{\omega}\{F\}$ is not the F-points of an algebraic group. Nevertheless, the definition furnishes a surjection

$$\mathcal{R}_{\omega}: V^{\vee}(F) \rtimes H(F) \to \Psi_{\omega}^{\mathrm{s}}\{F\}$$

of abstract groups. Under natural assumptions on H and X, we check that in fact $\Psi_{\omega}^{s}\{F\}$ is the F-points of an affine algebraic group Ψ_{ω}^{s} over F in Proposition 3.8. We refer to Ψ_{ω}^{s} as the algebraic small modulation group.

1.2. The modulation group. In many cases there exists an isometry

$$\mathcal{F}_{X,\psi}: L^2(X^{\mathrm{sm}}(F),\mathcal{L}^{1/2}) \longrightarrow L^2(X^{\mathrm{sm}}(F),\mathcal{L}^{1/2})$$

and isomorphisms $\iota: H \to H$ such that

$$\mathcal{F}_{X,\psi} \circ \mathcal{R}_{\omega}(h) = \mathcal{R}_{\omega}(\iota(h)) \circ \mathcal{F}_{X,\psi}$$

for $h \in H(F)$. Given a character $\chi : \Psi_{\omega}^{s}(F) \to \mathbb{C}^{\times}$ we define the **modulation group**

$$(1.2.1) \qquad \Psi_{\omega}\{F\} \coloneqq \Psi_{\omega,\chi,\mathcal{F}_{X,\psi}}\{F\} = \langle \mathcal{F}_{X,\psi}, (\mathcal{R}_{\omega} \otimes \chi)(\Psi_{\omega}^{s}\{F\}) \rangle < \operatorname{Aut}(L^{2}(X^{sm}(F), \mathcal{L}^{1/2}))$$

The odd notation is a hint that $\Psi_{\omega}\{F\}$ is not the F-points of a group scheme in general.

We compute the modulation group in a family of test cases in §6 and §7. The reader can refer to these sections for more details on the theorems we are about to state. In §6, we consider the classic setting in which $X = V \cong \mathbb{G}_a^n$ equipped with the canonical action of GL_V . We consider the two equivariant maps $id: X \to V$ and $Sym^2: X \to Sym^2 V^{\vee}$. Let

$$W \coloneqq V \oplus V^{\vee},$$

viewed as a symplectic space in the canonical manner, let H_W be the Heisenberg group, and let $\widetilde{J}_W(F) = H_W(F) \rtimes \mathrm{Mp}_W(F)$ be the 8-fold cover of the corresponding Jacobi group considered in [GKT25].

Theorem 1.2. The modulation group $\Psi_{id}\{F\}$ is the image of

$$H_W(F) \rtimes \langle w, GL_V(F) \rangle \leq \widetilde{J}_W(F)$$

under the Heisenberg-Weil representation. Here $w \in \mathrm{Mp}_W(F)$ is defined as in (6.1.8). The modulation group $\Psi_{\mathrm{Sym}^2}\{F\}$ is the image of a certain subgroup of $\mathrm{Mp}_W(F)$ under the Heisenberg-Weil representation.

This theorem shows that Ψ_{ω} need not be the points of an algebraic group, although in both cases it is closely related to algebraic groups.

In §7, we consider the case where X is the zero locus of a split nondegenerate quadratic form on a vector space $V \cong \mathbb{G}_a^{2n}$ equipped with the natural action of the similitude group GO_{2n} . For technical reasons we extend this to an action of $\mathbb{G}_m \times GO_{2n}$ with \mathbb{G}_m acting by scaling. In this setting we consider the inclusion $\omega: X \to V$.

Theorem 1.3. In the setting above, $\Psi_{\omega}\{F\}$ is the image of $GO_{2n+2}(F)$ under the minimal representation.

Theorems 1.2 and 1.3 indicate that the representations

$$\Psi_{\omega}\{F\} \longrightarrow \operatorname{Aut}(L^2(X^{\operatorname{sm}}(F), \mathcal{L}^{1/2}))$$

provide an interesting generalization of minimal representations. One can think of them as a group-theoretic enrichment of Fourier transforms in the same manner as the metaplectic group is a group-theoretic enrichment of the Fourier transform on a vector space. 1.3. Schwartz spaces. In all of the examples above there is a Schwartz space

$$\mathcal{S}(X(F),\mathcal{L}^{1/2}) < L^2(X^{\mathrm{sm}}(F),\mathcal{L}^{1/2})$$

that is preserved by $\Psi_{\omega}\{F\}$. Based on conjectures of Braverman-Kahzdan [BK00], Ngô [Ngô14], and Sakellaridis [Sak12] we expect this is true in some degree of generality.

Using the examples discussed previously, we highlight properties of Schwartz spaces that to our knowledge, have not been discussed in the literature. In most cases we do not have conceptual explanations for these properties. In fact, we do not even know in what generality they will hold.

Therefore, to explain our work, we will proceed as follows. We will label several statements as "Ansatz." What we mean by this is statements that should be true in some unknown level of generality. We will then isolate two broad families of examples and make the formal conjecture that the ansatz are true in these two families. Briefly, these are certain horospherical varieties and certain reductive monoids. Finally, we will prove the conjectures in the simplest working examples in each family. Organizing the paper in this manner highlights the structural phenomenon that we believe will provide an important guide to future research.

1.4. Boundary terms. Assume that H acts with dense open orbit X° on X. The boundary of the Schwartz space is the quotient

$$\mathcal{S}(X(F),\mathcal{L}^{1/2})/\mathcal{S}(X^{\circ}(F),\mathcal{L}^{1/2})$$

We can regard $S(X^{\circ}(F), \mathcal{L}^{1/2})$ as understood, so the boundary isolates exactly the exotic structure of the full Schwartz space.

Motivated by the orbit method, we introduce an action of $\Psi^{\rm s}_{\omega}$ on $T^*X^{\rm sm}$ in §3.3. In the setting of Theorem 1.2 and Theorem 1.3 we exhibit parallel $\Psi^{\rm s}_{\omega}\{F\}$ -invariant filtrations of $\mathcal{S}(X(F),\mathcal{L}^{1/2})$ and $\Psi^{\rm s}_{\omega}$ -invariant filtrations of $\overline{T^*X^{\rm sm}}$. We refer the reader to §9 for precise statements.

- Remark 1.4. The paper [Hsu21] gives a systematic description of the boundary of the Schwartz space for certain horospherical varieties (the boundary is referred to as the asymptotics in loc. cit.). This work will likely be crucial for a conceptual understanding of the phenomena we observe qualitatively in §9.
- 1.5. Algebraicity. In the settings of both Theorem 1.2 and Theorem 1.3 the modulation group is closely related to an algebraic group. Moreover, as just explained, there is an empirical link between action of the spectral small modulation group on $S(X(F), \mathcal{L}^{1/2})$ and the algebraic small modulation group on $\overline{T^*X^{\text{sm}}}$.

Thus, though the modulation group is not necessarily the F-points of an algebraic group, we expect that it is closely related to an ind-algebraic group. To investigate this, we restrict to the Archimedean setting. Without loss of generality we may assume $F = \mathbb{R}$. In §8 we

isolate assumptions under which the action of $\Psi_{\omega}(\mathbb{R})$ on $L^2(X^{\mathrm{sm}}(\mathbb{R}), \mathcal{L}^{1/2})$ induces an action of $\Psi_{\omega}\{\mathbb{R}\}$ on \mathcal{D}_X , the algebra of algebraic differential operators on X.

In Ansatz 8.14 we give a precise mathematical formulation of the following slogan: The action of $\Psi_{\omega}\{\mathbb{R}\}$ admits a semiclassical limit, which is an action of an ind-algebraic group Ψ_{ω} on $\overline{T^*X^{\mathrm{sm}}}$. We prove this conjecture in the settings of Theorem 1.2 and 1.3 in §8.4.

Remark 1.5. Ben-Zvi, Sakellaridis, and Venkatesh have suggested that Fourier transforms on $L^2(X^{\text{sm}}(F), \mathcal{L}^{1/2})$ should correspond to choices of Lagrangian subvarieties of $\overline{T^*X^{\text{sm}}}$. In cases that can be reduced to the vector spaces setting this is discussed in [BSV24, §10.9]. One consequence of Ansatz 8.14 is that, roughly, Fourier transforms pass to automorphisms of the cotangent bundle.

1.6. Global analogues of the metaplectic representation. One expects that for a broad class of spaces there exists Poisson summation formulae for the Schwartz spaces considered above [BK00, Ngô14, Ngô20, Sak13]. In the literature these conjectural Poisson summation formulae are usually stated for a limited class of test functions (see Ansatz 10.1). The problem is that in general one needs to introduce "boundary terms" that remain mysterious in general. We make this more precise in §9 and §10.

We came to the definition of modulation groups after studying these boundary terms. In §11, we show that the existence of the Poisson summation formula in a strong form (i.e. including boundary terms) is more or less equivalent to the existence of a space

$$\{\Theta_f: \Psi_\omega(F) \backslash \Psi_\omega\{\mathbb{A}_F\} \to \mathbb{C}: f \in \mathcal{S}\left(X(\mathbb{A}_F), \mathcal{L}^{1/2}\right)\},\,$$

that provides a representation of $\Psi_{\omega}\{\mathbb{A}_F\}$. The space (1.6.1) is a generalization of the metaplectic representation.

We mentioned above that in the local setting modulation groups provide a group-theoretic enrichment of the Fourier transform. The theory discussed above implies that adelic modulation groups provide a group-theoretic enrichment of Poisson summation formulae. This generalizes the manner in which the adelic metaplectic representation provides a group theoretic enrichment of the Poisson summation formula for vector spaces.

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2. Preliminaries

2.1. Function spaces. For a scheme X smooth and of finite type over F with $X(F) \neq \emptyset$, we have spaces

$$C_c^{\infty}(X(F)) \leq \mathcal{S}(X(F)) \leq C^{\infty}(X(F)).$$

When F is non-Archimedean we define

$$\mathcal{S}_{\mathrm{ES}}(X(F))$$

to be $C_c^{\infty}(X(F))$ when F is non-Archimedean. When F is Archimedean, view $X(F) = \operatorname{Res}_{F/\mathbb{R}}X(\mathbb{R})$ as a real algebraic variety and let $\mathcal{S}_{ES}(X(F))$ be the Schwartz space defined in [ES18]; this is again a nuclear Fréchet space. When X is smooth we set $\mathcal{S}(X(F)) = \mathcal{S}_{ES}(X(F))$.

Let $V \cong \mathbb{G}_a^n$ be a vector space over F. We recall that when F is Archimedean the space of **tempered functions** on V(F) is the space $\mathcal{T}(V(F))$ of all $f \in C^{\infty}(V(F))$ that are of at worst polynomial growth in the following sense: for all $\alpha, \alpha' \in \mathbb{Z}_{\geq 0}^n$ there exists $p_{\alpha,\alpha'} \in F[x_1,\ldots,x_n]$ such that $\left|\frac{\partial^{|\alpha|}}{\partial \alpha_x}\frac{\partial^{|\alpha'|}}{\partial \alpha' x}f(x)\right| \leq |p_{\alpha,\alpha'}(x)|$ for all $x \in V(F)$. Here we omit α' and the corresponding terms if F is real. If F is non-Archimedean, $\mathcal{T}(V(F)) := C^{\infty}(V(F))$.

Let X be an affine scheme over F with $X^{sm}(F) \neq \emptyset$.

Definition 2.1. A subspace $S \leq C^{\infty}(X^{\text{sm}}(F))$ is **local** with respect to X if for any closed immersion $X \to \mathbb{G}_a^n$, S is closed under multiplication by the restriction of functions in $\mathcal{T}(F^n)$.

3. Modulation groups

3.1. **Modulations.** Let F be a local field of characteristic zero and let $L^2(F^n)$ be defined with respect to a choice of additive Haar measure. Let $\langle , \rangle : F^n \times F^n \to F$ be a perfect pairing and let $\psi : F \to \mathbb{C}^\times$ be a nontrivial additive character. This gives rise to a Fourier transform $\mathcal{F}_{\psi} : L^2(F^n) \to L^2(F^n)$. The Fourier transform \mathcal{F}_{ψ} intertwines translation $f \mapsto f(\cdot + v)$ with a modulation, namely multiplication by $\overline{\psi}(\langle v, \cdot \rangle)$. If we replace F^n by a nonlinear space X then translations do not exist. However, we can define an analogue of modulation in great generality.

To explain this, suppose we are given the following data:

- (A1) an affine algebraic group H over F,
- (A2) an action $X \times H \to X$ of H on an affine scheme X of finite type over F,
- (A3) a (right) linear representation $V \times H \to V$ with $V \cong \mathbb{G}_a^n$ for some n, and
- (A4) an *H*-equivariant morphism $\omega: X \to V$.

In the following we denote by R an F-algebra, used to define the points of schemes. Let V^{\vee} be the dual of V, so that $V^{\vee}(R) = \operatorname{Hom}_{\mathbf{Mod}_R}(V(R), R)$. We form the semi-direct product $V^{\vee} \rtimes H$. The group law is given at the level of points by

$$(3.1.1) \qquad (\lambda, h)(\lambda', h') = (\lambda + \lambda' \circ h, hh').$$

To proceed we will use the bundle of 1/2 densities $\mathcal{L}^{1/2}$ on $X^{\mathrm{sm}}(F)$ and the corresponding canonical Hilbert space $L^2(X^{\mathrm{sm}}(F), \mathcal{L}^{1/2})$ and let

$$\mathcal{R}: H(F) \times L^{2}(X^{\mathrm{sm}}(F), \mathcal{L}^{1/2}) \longrightarrow L^{2}(X^{\mathrm{sm}}(F), \mathcal{L}^{1/2})$$
$$(h, f) \longmapsto (x \mapsto f(xh))$$

denote the induced unitary representation of H(F). A convenient reference is [Li18, §3.1].

Remark 3.1. For readers unfamiliar with this language, we point out that in many cases one can avoid considering half-densities as follows. Assume that X^{sm} admits an open dense H-orbit O such that O(F) is nonempty. Then O(F) is open [Con12, Proposition 3.1] in $X^{\text{sm}}(F)$, and its complement is cut off by polynomials so that it has (Borel) measure zero. Assume moreover that O(F) admits an H(F)-eigenmeasure dx; thus $d(xh) = \chi(h)dx$ for some quasi-character $\chi: H(F) \to \mathbb{R}_{>0}$. One has an isometry

$$\phi_{dx}: L^2(O(F), dx) \xrightarrow{\sim} L^2(X^{\text{sm}}(F), \mathcal{L}^{1/2})$$

$$f \longmapsto f dx^{1/2}.$$

and
$$\phi_{dx}(\mathcal{R}(h)f) = \chi^{-1/2}(h)\mathcal{R}(h)\phi_{dx}(f)$$
.

The representation \mathcal{R} extends to a unitary representation

(3.1.2)
$$\mathcal{R}_{\omega}: V^{\vee}(F) \rtimes H(F) \times L^{2}(X^{\mathrm{sm}}(F), \mathcal{L}^{1/2}) \longrightarrow L^{2}(X^{\mathrm{sm}}(F), \mathcal{L}^{1/2})$$
$$((\lambda, h), f) \longmapsto \psi(\lambda \circ \omega(x)) f(xh)$$

of the semi-direct product $V^{\vee}(F) \rtimes H(F)$. By abuse of notation we let

$$\mathcal{R}_{\omega}: V^{\vee}(F) \rtimes H(F) \longrightarrow \mathrm{U}(L^{2}(X^{\mathrm{sm}}(F), \mathcal{L}^{1/2}))$$

denote the corresponding homomorphism. Here for Hilbert spaces W we write U(W) for the group of unitary operators on W. The (spectral) small modulation group $\Psi^s_{\omega}\{F\}$ is defined as the image of $V^{\vee}(F) \rtimes H(F)$ in $U(L^2(X^{sm}(F), \mathcal{L}^{1/2}))$. In particular we have a surjection

$$(3.1.3) \mathcal{R}_{\omega}: V^{\vee}(F) \rtimes H(F) \longrightarrow \Psi_{\omega}^{s}\{F\}.$$

Under fairly weak assumptions the action of $\Psi^{\rm s}_{\omega}\{F\}$ on $L^2(X^{\rm sm}(F),\mathcal{L}^{1/2})$ is irreducible. To prove this we first prove the following:

Lemma 3.2. Let Z be a separated scheme of finite type over F equipped with an action of the smooth algebraic group H. Then every H(F) orbit in Z(F) is locally closed.

Proof. Every H orbit in Z is locally closed in the Zariski topology by [Mil17, Proposition 7.17]. Thus we may assume that Z is a single H-orbit, and that Z(F) is nonempty. Choosing $z \in Z(F)$, we have an isomorphism of H-schemes $H_z \setminus H \cong Z$. Thus by [GH24, Lemma 17.4.3], every H(F)-orbit in Z(F) is closed in Z(F).

Choose $x_0 \in X^{\mathrm{sm}}(F)$. Then $\omega(x_0) \in V(F) = (V^{\vee})^{\vee}(F)$.

Theorem 3.3. Assume the orbit $O(x_0)$ of x_0 under H is open and dense in X and that H(F) acts transitively on $O(x_0)(F)$. Assume moreover that $H_{x_0} = H_{\omega(x_0)}$. Then the representation of $\Psi^s_{\omega}\{F\}$ on $L^2(X^{\mathrm{sm}}(F), \mathcal{L}^{1/2})$ is irreducible.

Proof. We employ the Mackey machine. The continuous linear dual of $V^{\vee}(F)$ is V(F). We point out that V(F) is second countable and $V^{\vee}(F) \rtimes H(F)/V^{\vee}(F) = H(F)$ is σ -compact. Moreover H(F)-orbits in V(F) are locally closed by Lemma 3.2. Thus $V^{\vee}(F)$ is Mackey compatible in the terminology of [KT13, Remark 4.26(2)].

We have a character

$$\psi_{x_0}: V^{\vee}(F) \longrightarrow F$$

$$\lambda \longmapsto \psi(\lambda(\omega(x_0))).$$

The stabilizer in H(F) of this character is $H_{\omega(x_0)}(F)$, and the character extends to $V^{\vee}(F) \times H(F)_{\omega(x_0)}$. By [KT13, Theorem 4.29] we deduce that the unnormalized induction

(3.1.4)
$$\operatorname{ind}_{V^{\vee}(F) \rtimes H_{\omega(x_0)}(F)}^{\Psi_{\omega}^{s}(F)} \left((\lambda, h) \mapsto (\delta_H^{-1/2} \delta_{H_{\omega(x_0)}}^{1/2})(h) \psi_{x_0}(\lambda) \right)$$

is irreducible. Since we assumed $H_{\omega(x_0)} = H_{x_0}$ the representation (3.1.4) is isomorphic to $L^2(X^{\text{sm}}(F), \mathcal{L}^{1/2})$. We point out that the modular quasi-characters are introduced due to the fact that $L^2(X^{\text{sm}}(F), \mathcal{L}^{1/2})$ is a space of half-densities, see [Li18, (3.2)].

For use in §8 we prove the following lemma:

Lemma 3.4. Suppose that $x_0 \in X(F)$ and the H-orbit $O(x_0)$ of x_0 is Zariski dense in X. If H is connected then $O(x_0)(F)$ is Zariski dense in X.

Proof. For clarity we write |Y| for the underlying topological space of a scheme Y. It suffices to show that $\overline{O(x_0)(F)}$, the closure of $O(x_0)(F)$ in $|O(x_0)|$, is equal to $|O(x_0)|$. Let $p:|H| \to |O(x_0)|$ be the action map induced by the choice of basepoint x_0 . It is surjective. The set $p^{-1}(\overline{O(x_0)(F)}) \subseteq |H|$ is closed and contains H(F). On the other hand since H is connected and F is of characteristic zero, [Bor91, Chapter V, Proposition 18.3] asserts that H(F) is dense in |H|. Hence $p^{-1}(\overline{O(x_0)(F)}) = |H|$ and $\overline{O(x_0)(F)} = |O(x_0)|$.

- 3.2. The algebraic small modulation group. Under mild assumptions $\Psi_{\omega}^{s}\{F\}$ is the F-points of an algebraic group. Let H_X denote the kernel of the H-action on X^{sm} . By [DG80, §II.1.3.6.(c)], it is a closed normal subgroup scheme of H. Moreover, let H_V be the kernel of action H on V. We impose the following assumptions:
 - (M1) The image of $X^{\rm sm}(F)$ under ω spans V(F).
 - (M2) There exists a cocharacter $c: \mathbb{G}_m \to Z_H$ and an integer $k \in \mathbb{Z}_{>0}$ such that

$$\omega(xc(t)) = t^k \omega(x)$$

for all $x \in X(R)$ and $t \in R^{\times}$.

(M3) $H^1(F, H_X) = 0$.

(M4)
$$H_X(F) = \{ h \in H(F) : xh = x \text{ for all } x \in X^{sm}(F) \}.$$

Since V(F) spans V(R) as an R-module, it follows that (M1) is equivalent to

(M1') For any F-algebra R, $\omega(X^{\rm sm}(R))$ spans V(R) as an R-module.

In particular, under (M1) we have

$$(3.2.1) H_X \le H_V,$$

Lemma 3.5. Assume that

(M0) $\omega: X \longrightarrow V$ is an immersion.

Then (M1) implies $H_X = H_V$ and (M4).

Proof. By (3.2.1), the assumption (M1) implies $H_X \leq H_V$. Since immersions are injective on points we deduce $H_X = H_V$. Moreover, the subset V(F) spans V(R) as an R-module, implying

$$H_V(F) = \{ h \in H(F) : vh = v \text{ for all } v \in V(F) \}.$$

By (M1) again we deduce (M4).

Lemma 3.6. The kernel of \mathcal{R}_{ω} in (3.1.3) is

$$\{(0,h)\in V^{\vee}(F)\rtimes H(F): xh=x \text{ for all } x\in X^{\mathrm{sm}}(F)\},$$

which is canonically isomorphic to $H_X(F)$.

Proof. Let $(\lambda, h) \in \ker \mathcal{R}_{\omega}$. Suppose $x \neq xh$ for some $x \in X^{\mathrm{sm}}(F)$. Then there exists $f \in L^2(X^{\mathrm{sm}}(F), \mathcal{L}^{1/2})$ such that $x \in \mathrm{supp}(f)$ while $xh \notin \mathrm{supp}(f)$. But then $\mathcal{R}_{\omega}(\lambda, h)f \neq f$, a contradiction. Hence $h \in H_X(F)$ by (M4).

Consequently, for any $(x, (\lambda, h)) \in X^{\mathrm{sm}}(F) \times \ker \mathcal{R}_{\omega}$

$$\psi(\lambda \circ \omega(x)) = 1.$$

By (M2), for $t \in F^{\times}$ and $x \in X^{\text{sm}}(F)$, we have

$$1 = \psi(\lambda \circ \omega(xc(t))) = \psi(\lambda(t^k \omega(x))) = \psi(t^k \lambda(\omega(x))).$$

On the other hand, we claim that any character $\psi': F \to \mathbb{C}^{\times}$ trivial on $(F^{\times})^k$ is trivial. Assuming the claim we can apply (M1) to see that $\lambda = 0$ and complete the proof.

The claim is easy to check in the Archimedean case. If F is non-Archimedean, let ϖ be a uniformizer of F. By continuity, we can find $N \in \mathbb{Z}_{\geq 1}$ such that $1 + \varpi^N \mathcal{O}_F \leq (1 + \varpi \mathcal{O}_F)^k$. Then for $n \in \mathbb{Z}$, we have $\varpi^{-nk}(1 + \varpi^N \mathcal{O}_F) \leq (\varpi^{-n}(1 + \varpi \mathcal{O}_F))^k \subseteq \ker \psi'$. Since $\varpi^{-nk} \in \ker \psi'$, we see $\varpi^{-nk+N}\mathcal{O}_F \leq \ker \psi'$ for all $n \in \mathbb{Z}$. The claim then follows from the observation that $F = \bigcup_{n \in \mathbb{Z}} \varpi^{-nk+N} \mathcal{O}_F$.

Definition 3.7. The algebraic small modulation group $\Psi^{s,alg}_{\omega}$ of $\omega: X \to V$ is

$$\Psi^{\mathrm{s,alg}}_{\omega} \coloneqq V^{\vee} \rtimes (H/H_X).$$

Proposition 3.8. The map \mathcal{R}_{ω} of (3.1.3) induces an isomorphism $\Psi_{\omega}^{s,alg}(F) \cong \Psi_{\omega}^{s}\{F\}$.

Proof. By Lemma 3.6 the map \mathcal{R}_{ω} induces an isomorphism

$$V^{\vee}(F) \rtimes (H(F)/H_X(F)) \cong \Psi_{\omega}^{\mathrm{s}}\{F\}$$

On other other hand, by (M3) the left hand side is $\Psi^{s,alg}_{\omega}(F)$.

Due to the proposition we omit the superscript alg from $\Psi^{s,alg}_{\omega}$. Thus the algebraic small modulation group is the algebraic group Ψ^{s}_{ω} and the spectral small modulation group is its F-points $\Psi^{s}_{\omega}(F)$. Moreover, in this setting there is no real need to use the adjectives "algebraic" and "spectral."

3.3. Action on the cotangent bundle. We pause to explain how $V^{\vee} \rtimes H$ acts on the cotangent bundle T^*X^{sm} of X^{sm} . We will not use the results of this section until §8 and 9. For this section F is an arbitrary field.

For lack of a reference, we give an explicit description of the functor of points of the cotangent space. This is necessary so the reader can check that the actions we construct below are well-defined. Let X_0 be scheme smooth over a field F. The cotangent bundle of X_0 is defined as

$$T^*X_0\coloneqq \underline{\operatorname{Spec}}_{X_0} \mathrm{Sym}_{\mathcal{O}_{X_0}}(\Omega_{X_0/F}^{\vee}).$$

where $\Omega_{X_0/F} := \Omega_{X_0/\operatorname{Spec} F}$ is the sheaf of Kähler differentials of X_0 . Since X_0 is smooth, $\Omega_{X_0/F}$ is a finite locally free \mathcal{O}_{X_0} -module. Using the description of the functor of points of a relative spectrum in [Sta21, Tag 01LQ] we have

$$T^*X_0(R) = \{(x,\phi) : x \in X_0(R), \phi \in \operatorname{Hom}_{\mathbf{Alg}_{\mathcal{O}_{X_0}}}(\operatorname{Sym}_{\mathcal{O}_{X_0}}\Omega^{\vee}_{X_0/F}, x_*\mathcal{O}_{\operatorname{Spec} R})\}$$

$$= \{(x,\phi) : x \in X_0(R), \phi \in \operatorname{Hom}_{\mathbf{Mod}_{\mathcal{O}_{X_0}}}(\Omega^{\vee}_{X_0/F}, x_*\mathcal{O}_{\operatorname{Spec} R})\}$$

$$= \{(x,\phi) : x \in X_0(R), \phi \in \operatorname{Hom}_{\mathbf{Mod}_{\mathcal{O}_{\operatorname{Spec} R}}}(x^*(\Omega^{\vee}_{X_0/F}), \mathcal{O}_{\operatorname{Spec} R})\}.$$

In the last equation we have used the standard identities in [Sta21, Tag 0094].

Over field-valued points, this description simplifies somewhat. Let $x \in X_0(F)$. We point out that duals commute with pullbacks for finite locally free modules, so $x^*(\Omega_{X_0/F}^{\vee}) \cong (x^*\Omega_{X_0/F})^{\vee}$. Hence

$$\operatorname{Hom}_{\operatorname{\mathbf{Mod}}_{\mathcal{O}_{\operatorname{Spec} F}}}(x^{*}(\Omega_{X_{0}/F}^{\vee}), \mathcal{O}_{\operatorname{Spec} F}) \cong (x^{*}\Omega_{X_{0}/F})(\operatorname{Spec} F) \cong T_{X_{0},x}^{*},$$

where $T_{X_0,x}^*$ is the Zariski cotangent space of X_0 at x. Here we are identifying $x \in X_0(F)$ with a closed point of the underlying topological space of X_0 . In other words,

$$(3.3.1) T^*X_0(F) = \{(x,v) : x \in X_0(F), v \in T^*_{X_0,x}\}.$$

For this reason, we define

$$T_{X_0,x}^* \coloneqq \operatorname{Hom}_{\mathbf{Mod}_{\mathcal{O}_{\operatorname{Spec} R}}}(x^*(\Omega_{X_0/F}^{\vee}), \mathcal{O}_{\operatorname{Spec} R})$$

for any $x \in X_0(R)$. This is naturally an R-module.

Lemma 3.9. Assume X_0 admits a right H-action. Then T^*X_0 admits a canonical H-action. It is given on points in an F-algebra R by

$$T^*X_0(R) \times H(R) \longrightarrow T^*X_0(R)$$

 $((x,\phi),h) \longmapsto (xh,h^*\phi)$

where $h^*: T^*_{X_0,x} \to T^*_{X_0,xh}$ is given by precomposing with the isomorphism $(xh)^*(\Omega^{\vee}_{(X_0)_R/R}) \to x^*(\Omega^{\vee}_{(X_0)_R/R})$.

Lemma 3.10. Let $V \cong \mathbb{G}_a^n$ over F and let $\omega : X_0 \to V$ be a morphism of F-schemes. There is an action of V^{\vee} on T^*X_0 given on points in an F-algebra R by

$$T^*X_0(R) \times V^{\vee}(R) \longrightarrow T^*X_0(R)$$

 $((x,\phi),\lambda) \longmapsto (x,\phi+\omega_x^*\lambda)$

Here $\omega_x^*: V^{\vee}(R) = T_{V,\omega(x)}^* \to T_{X_0,x}^*$ is given by precomposing with $\Omega_{(X_0)_R/R}^{\vee} \to (\omega^* \Omega_{V_R/R}^{\vee})^{\vee} \cong \omega^* \Omega_{V_R/R}^{\vee}$.

We return to the setting of (A1), (A2), (A3) and (A4); the field F can be arbitrary.

Lemma 3.11. We have an action of $V^{\vee} \times H$ on T^*X^{sm} given on the level of points by

(3.3.2)
$$T^*X^{\mathrm{sm}}(R) \times V^{\vee}(R) \rtimes H(R) \longrightarrow T^*X^{\mathrm{sm}}(R)$$
$$((x,\phi),(\lambda,h)) \longmapsto (xh,h^*\phi + \omega_{ab}^*(\lambda \circ h^{-1}))$$

Here we are using notations in Lemma 3.9 and Lemma 3.10.

3.4. Fourier transforms. To proceed we require Fourier transforms. We begin with the abstract setup. Let F be a local field. Let

$$\nu: \mathbb{G}_m \longrightarrow Z_H$$

be a cocharacter. We assume there is an automorphism $\iota: H \to H$ such that $\iota \circ \nu(x) = \nu(x)^{-1}$. We then assume the existence of an isometry

$$(3.4.1) \mathcal{F}_X: L^2(X^{\mathrm{sm}}(F), \mathcal{L}^{1/2}) \xrightarrow{\sim} L^2(X^{\mathrm{sm}}(F), \mathcal{L}^{1/2})$$

such that

(3.4.2)
$$\mathcal{F}_X \circ \mathcal{R}(h) = \mathcal{R}(\iota(h)) \circ \mathcal{F}_X.$$

Finally, let

$$\chi: \Psi^{\mathbf{s}}_{\omega}(F) \longrightarrow \mathbb{C}^{\times}$$

be a character.

Definition 3.12. The modulation group $\Psi_{\omega}\{F\}$ attached to ω , \mathcal{F}_X , and χ is the subgroup of $U(L^2(X^{sm}(F), \mathcal{L}^{1/2}))$ generated by $\mathcal{F}_{X,\psi}$ and $(\mathcal{R}_{\omega} \otimes \chi)(\Psi_{\omega}^{s}(F))$:

$$\Psi_{\omega}\{F\} \coloneqq \Psi_{\omega,\mathcal{F}_X,\chi}\{F\} \coloneqq \langle \mathcal{F}_X, (\mathcal{R}_{\omega} \otimes \chi)(\Psi_{\omega}^{\mathrm{s}}(F)) \rangle.$$

The group $\Psi_{\omega}\{F\}$ is not a priori the F-points of an algebraic group, or even a group scheme. We have adopted the odd notation to emphasize this point.

Remarks.

- (1) We will see in (7.2.11) an example in which it is convenient to twist the action of the small modulation group by a character as in Definition 3.12. It would be desirable to have a conceptual explanation for this phenomenon. In any case one should view Definition 3.12 as a working definition.
- (2) In some situations it is more natural to work with a family of Fourier transforms (see [BK99]). The considerations of this paper adapt in the natural manner to this setting.
- 3.5. Schwartz spaces. For many purposes, the Hilbert space $L^2(X^{sm}(F), \mathcal{L}^{1/2})$ is unwieldy. Instead, one would like to work with an H(F)-invariant Schwartz space

$$S(X(F), \mathcal{L}^{1/2}) \le L^2(X^{sm}(F), \mathcal{L}^{1/2}) \cap C^{\infty}(X^{sm}(F), \mathcal{L}^{1/2})$$

satisfying the following axioms.

- (S1) $S(X(F), \mathcal{L}^{1/2})$ is preserved by \mathcal{F}_X .
- (S_2) $S(X(F), \mathcal{L}^{1/2})$ is local with respect to $X^{sm} \subseteq X$ in the sense of Definition 2.1.
- (S3) $S(X(F), \mathcal{L}^{1/2})$ is dense in $L^2(X^{sm}(F), \mathcal{L}^{1/2})$.
- $(\mathcal{S}4)\ \mathcal{S}(X^{\mathrm{sm}}(F),\mathcal{L}^{1/2}) \leq \mathcal{S}(X(F),\mathcal{L}^{1/2}).$

Clearly (S4) implies (S3). We separate the two because (S3) is easier to prove in practice. There are additional desiderata one could impose, but we prefer to keep things as simple as possible.

Lemma 3.13. Under (S1), (S2), and (S3) the Schwartz space $S(X(F), \mathcal{L}^{1/2})$ is preserved by $\Psi_{\omega}\{F\}$ and the induced map $\Psi_{\omega}\{F\} \to U(S(X(F), \mathcal{L}^{1/2}))$ is injective.

In favorable circumstances the following holds:

Ansatz 3.14. There exists \mathcal{F} satisfying (3.4.2) and a Schwartz space $\mathcal{S}(X(F), \mathcal{L}^{1/2})$ satisfying ($\mathcal{S}1$), ($\mathcal{S}2$), ($\mathcal{S}3$), and ($\mathcal{S}4$).

Question 3.15. When Ansatz 3.14 holds, is there a notion of smooth representation for $\Psi_{\omega}\{F\}$ such that $\mathcal{S}(X(F),\mathcal{L}^{1/2})$ is the space of smooth vectors in $L^2(X^{\mathrm{sm}}(F),\mathcal{L}^{1/2})$?

When $\Psi_{\omega}\{F\}$ is a Lie group or a locally compact totally disconnected group it is reasonable to use the usual notion of smoothness. However, we do not know if $\Psi_{\omega}\{F\}$ is a Lie group or a locally compact totally disconnected group in general.

In the setting of Theorems 1.2 and 1.3 the answer to this question is affirmative if we use the usual notion of smoothness (see Proposition 6.5 and (7.2.8) below).

4. Reductive monoids

We briefly review Ngô's construction of an L-monoid in [Ngô20, §5]. Let F be either a local field or a global field, and let G be a connected reductive group over F. Let $\rho: {}^LG \to \operatorname{GL}_V(\mathbb{C})$ be a finite dimensional representation of LG that factors through ${}^LG \to \widehat{G}(\mathbb{C}) \rtimes \operatorname{Gal}(E/F)$ for some finite Galois extension E/F. This is under the assumption that one is using the Weil form of the L-group; one could alternately work with the Galois form and then the assumption would be automatic. For simplicity we assume that we are given an isomorphism

$$d: G/G^{\operatorname{der}} \xrightarrow{\sim} \mathbb{G}_m,$$

and that $\rho \circ \widehat{d} : \mathbb{G}_m \longrightarrow \mathrm{GL}_V$ is the inclusion of \mathbb{G}_m into the center.

Under this assumption one obtains a $G \times G$ -equivariant immersion

$$G \longrightarrow E_{\omega_{\rho}} := \bigoplus_{\mathfrak{D}} E_{\mathfrak{D}}$$

where each $E_{\mathfrak{D}}$ is an F vector space [Ngô20, §5.1]. Here we are using notation as in [GGTH+25, §6.3]. By loc. cit., each $E_{\mathfrak{D}}$ is an irreducible representation of $G \times G$ and $E_{\mathfrak{D}} \cong E_{\mathfrak{D}'}$ if and only if $\mathfrak{D} = \mathfrak{D}'$.

The closure of the image of G is M_{ρ} , and hence we have a $G \times G$ -equivariant closed immersion

$$(4.0.1) \omega_{\rho}: M_{\rho} \longrightarrow \bigoplus_{\mathfrak{D}} E_{\mathfrak{D}}.$$

- 4.1. Modulation groups for reductive monoids. We now explain how to define modulation groups attached to representations $\rho: {}^LG \to \mathrm{GL}_{V_\rho}(\mathbb{C})$. We take
 - $H = G \times G$.
 - $X = M_{\rho}$ with right $G \times G$ -action.
 - $V = E_{\omega_{\rho}}$, with right $G \times G$ -action.
 - $\omega := \omega_{\rho} : M_{\rho} \to E_{\omega_{\rho}}$, as in (4.0.1).

We henceforth assume that

$$(4.1.1) H^1(F, Z_G) = 0.$$

Lemma 4.1. In the setting above, under assumption (4.1.1), the assumptions (M1), (M2), (M3), and (M4) hold.

Proof. The fact that the $E_{\mathfrak{D}}$ are pairwise nonisomorphic irreducible representations of $G \times G$ implies (M1).

Assumption (M2) is a consequence of our assumptions on d and the construction of M_{ρ} .

To prove (M3) we point out that in the case at hand $H_X = (G \times G)_{M_\rho} = \Delta(Z_G)$, the image of the center Z_G under the diagonal embedding $\Delta : G \to G \times G$. Indeed, we certainly have $\Delta(Z_G) \leq (G \times G)_{M_\rho}$, and it is easy to check that $(G \times G)_G \leq \Delta(Z_G)$. We now apply (4.1.1).

To prove
$$(M4)$$
 we use Lemma 3.5.

We set

$$(4.1.2) \Psi_{\rho}^{s} \coloneqq \Psi_{\omega_{\rho}}^{s}$$

and call it the small modulation group of ρ .

We originally defined the small modulation group using the unitary representation $\Psi_{\rho}^{s}(F) \to \operatorname{Aut}(L^{2}(M_{\rho}^{sm}(F), \mathcal{L}^{1/2}))$ on the space of half-densities. This representation is equivalent to the representation

$$\Psi_{\rho}^{s}(F) \times L^{2}(G(F)) \longrightarrow L^{2}(G(F))$$

$$((\lambda, (g_{1}, g_{2})), f) \longmapsto (x \mapsto \psi(\lambda \circ \omega_{\rho}(x)) f(g_{1}^{-1} x g_{2}))$$

where we use the Haar measure to define $L^2(G(F))$. Indeed, $M_{\rho}^{\text{sm}}(F) - G(F)$ is of measure zero, and the passage between the space of half densities and Haar measures is a special case of Remark 3.1. We use this observation in the following lemma:

Lemma 4.2. The representation of $\Psi_{\rho}^{s}(F)$ on $L^{2}(G(F))$ is irreducible.

Proof. The $G \times G$ orbit of the identity I in $G \subset M_{\rho}$ is just G, and it is open and dense in M_{ρ} . Moreover $G(F) \times G(F)$ certainly acts transitively on G(F), and $(G \times G)_{I} = (G \times G)_{\omega(I)}$. Thus we can apply Theorem 3.3.

Definition 4.3. Assume G is split. The modulation group $\Psi_{\rho}\{F\}$ of ρ over F is the modulation group attached to \mathcal{F}_{ρ} , ω_{ρ} , and the trivial character $\chi: G(F) \to \mathbb{C}^{\times}$:

(4.1.3)
$$\Psi_{\rho}\{F\} := \langle \mathcal{F}_{\rho}, \Psi_{\omega_{\rho}}^{s}(F) \rangle.$$

Here we are using Definition 3.12. We have assumed G is split because it is unclear which character χ to choose in general.

The following is a summary of some of the results of [GGTH+25]:

Theorem 4.4. When X is a reductive monoid as above, there is a Fourier transform \mathcal{F}_{ρ} satisfying (3.4.2) and a Schwartz space satisfying (S1), (S3), (S4).

5. Horospherical varieties

Let $P \leq G$ be a parabolic subgroup of a simple and simply connected group G. Let

$$(5.0.1) X_P^{\circ} := P^{\operatorname{der}} \backslash G, \quad X_P := \overline{P^{\operatorname{der}} \backslash G}.$$

These spaces are both examples of horospherical varieties. Let M be a Levi subgroup of P and let $M^{ab} = M/M^{der}$. The space X_P admits a canonical right action $X_P \times M^{ab} \times G \longrightarrow X_P$.

Fourier transforms and Schwartz spaces for this general class of varieties were first considered in [BK02]. To simplify our discussion, we will focus on the case where P is a self-associate maximal parabolic subgroup. We have a closed immersion

that is a lift of the usual Plücker embedding. We refer to [GHL25, Lemma 3.4] for details. Here V_P is a right representation of G. It is a fundamental representation of highest weight $-\omega_P$ with respect to a choice of maximal torus $T \leq M$. The character ω_P extends to M and factors through $M^{\rm ab}$. The map ${\rm Pl}_P$ intertwines the action of $M^{\rm ab}$ on X_P with the action of $M^{\rm ab}$ on V_P via scaling by the character ω_P .

Lemma 5.1. The actions of $M^{ab} \times G$ on X_P and V_P factor through

$$H := (M^{\mathrm{ab}} \times G)/(M^{\mathrm{ab}} \times G)_{X_P}.$$

With this choice of H, $X = X_P$ and $\omega = \operatorname{Pl}_P$ the assumptions (M1), (M2), (M3), and (M4) are valid. Moreover X_P is normal and $\operatorname{codim}(X - X^{\operatorname{sm}}, X) \geq 2$.

Proof. The fact that V_P is an irreducible representation of G implies (M1). Moreover, since ω_P is an immersion, Lemma 3.5 (with $H = (M^{ab} \times G)$) implies $(M^{ab} \times G)_{X_P} = (M^{ab} \times G)_{V_P}$. The assertion on the actions follows.

Our comments above on the action of M^{ab} imply (M2). Condition (M3) and (M4) are trivially valid.

Since X_P° is smooth and strongly quasi-affine [BG02, Theorem 1.1.2] it is normal [GGHL25, Lemma 3.1]. Since $X_P - X_P^{\circ}$ has dimension 0 [VP73, Theorem 1 and 2] the codimension statement is easy to check.

Theorem 5.2. When X is a horospherical variety as above and G is a classical group or G_2 then Ansatz 3.14 is valid.

Proof. For the construction of the Schwartz space and the Fourier transform see [GHL25], which refines earlier work of [BK02]. These references use an eigenmeasure to trivialize the space of half-densities as in Remark 3.1, so we will omit $\mathcal{L}^{1/2}$ from notation. It is obvious from the definition that functions in the Schwartz space $\mathcal{S}(X_P(F))$ are smooth. The fact that the Schwartz space is contained in $L^2(X^{\circ}(F))$ is [GHL25, Corollary 5.8]. See [GHL25, 5.24] for condition ($\mathcal{S}1$) and the existence of the Fourier transform. None of the result just mention depend on our assumption that G is a classical group or G_2 .

If G is a classical group or G_2 the proof of (S_2) and (S_4) (and hence (S_3)) is contained in [Hsu21].

6. Modulation groups of vector spaces and the Weil representation

Throughout this section, we assume that F is a local field of characteristic zero. Let

- $X = V \cong \mathbb{G}_a^n$ a finite dimensional F-vector space,
- $H = GL_V$, acting on V on the right.

We wish to compute modulation groups in this setting. This requires a Fourier transform. We choose a perfect pairing

$$(6.0.1) \qquad \langle , \rangle : V(F) \times V(F) \longrightarrow F$$

and use it to define a Fourier transform

(6.0.2)
$$\mathcal{F}_{V,\psi} := \mathcal{F}_{\langle , \rangle, \psi} : \mathcal{S}(V(F), \mathcal{L}^{1/2}) \longrightarrow \mathcal{S}(V(F), \mathcal{L}^{1/2}).$$

The notation here is the same as that of [GGHL25, §2.7], which can be consulted for more details. The Fourier transform extends an isometry of $L^2(V(F), \mathcal{L}^{1/2})$.

The modulation groups Ψ_{ω} computed in this section are both defined using the Fourier transform $\mathcal{F}_{V,\psi}$, but with two different choices of equivariant maps ω .

6.1. The identity map. For this subsection we take $\omega = \mathrm{id} : X \to V$, the identity map. Since GL_V acts faithfully on V, one sees $H_X = H_V$ is the trivial group. It is straightforward to check the conditions (M1)-(M4). By Definition 3.7

$$(6.1.1) \Psi_{\rm id}^s = V^{\vee} \rtimes \operatorname{GL}_V.$$

To compute the modulation group $\Psi_{id}\{F\}$, we embed Ψ_{id}^s into a Jacobi group and employ the Heisenberg-Weil representation to describe $\Psi_{id}\{F\}$ explicitly.

Let $W := V \oplus V$. Our choice of perfect pairing (6.0.1) induces a symplectic pairing

(6.1.2)
$$\langle , \rangle_{\wedge} : W(F) \times W(F) \longrightarrow F$$
$$((v_1, \lambda_1), (v_2, \lambda_2)) \longmapsto \langle v_2, \lambda_1 \rangle - \langle v_1, \lambda_2 \rangle.$$

Let Sp_W denote the corresponding symplectic group. We will also make use of the Heisenberg group $\operatorname{H}_W = W \ltimes \mathbb{G}_a$. We recall that the group law is given on points in an F-algebra R by

$$(6.1.3) (w_1, t_1).(w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2 \rangle_{\wedge}), w_1, w_2 \in W(R), t_1, t_2 \in R.$$

There is a representation

$$\rho_{\psi,W}: \mathcal{H}_W(F) \times L^2(V(F), \mathcal{L}^{1/2}) \longrightarrow L^2(V(F), \mathcal{L}^{1/2})$$

given as follows: for $f \in L^2(V(F), \mathcal{L}^{1/2})$ and $x \in V(F)$,

(6.1.4)
$$\rho_{\psi,W}((v,\lambda),t)f(x) = \psi(\langle x,\lambda\rangle + \frac{1}{2}\langle v,\lambda\rangle + t)f(x+v).$$

Let GL_V act on $W = V \oplus V$ via its natural action on the first factor and the dual action with respect to \langle , \rangle on the second factor. This action preserves the symplectic form and induces a closed immersion of groups

$$(6.1.5) m: GL_V \to Sp_W$$

with image a Levi subgroup. For F-algebras R let

We then have a closed immersion of groups

(6.1.7)
$$n: \operatorname{Sym}_{\langle,\rangle}(R) \longrightarrow \operatorname{Sp}_{W}(R) \\ b \longmapsto ((v,\lambda) \mapsto (v,\lambda + bv))$$

Lastly, consider $w \in \operatorname{Sp}_W(F)$ that acts on W by

$$(6.1.8) (v,\lambda)w = (-\lambda,v).$$

Since w represents the longest Weyl element, the collection of elements m(g), n(b), and w generate $\operatorname{Sp}_W(F)$ by the Bruhat decomposition with respect to Siegel parabolic $P = m(\operatorname{GL}_V)n(\operatorname{Sym}_{(\cdot)}(V))$. We let $\widetilde{\operatorname{GL}}_V$ be the algebraic subgroup of Sp_V such that

(6.1.9)
$$\widetilde{\operatorname{GL}}_{V}(k) = \langle w, \operatorname{GL}_{V}(k) \rangle$$

for all fields k/F.

The action of GL_V on W extends to an action on H_W . Consequently, we have the following closed immersion of group schemes:

The group J_W is known as Jacobi group associated with the symplectic space W. The representation $\rho_{\psi,W}$ does not in general extend to $J_W(F)$, but it does lift to a metaplectic cover as we now recall. It is convenient to use [GKT25] as a reference, but warn the reader that in loc. cit. only the non-Archimedean case is treated. The statements we make will be valid for general local fields.

Following [GKT25, §9.2.2], let $\operatorname{Mp}_W(F) := \operatorname{Mp}_{W,\psi,V^{\vee}}^{(8)}(F)$ be the metaplectic 8-fold cover of $\operatorname{Sp}_W(F)$ attached to W and the additive character ψ . In other words we take $Z = \mu_8$ in the notation of loc. cit. Let $(\omega_{\psi,W}, L^2(V(F), \mathcal{L}^{1/2}))$ be the Schrödinger model of the Weil representation of $\operatorname{Mp}_W(F)$. By [GKT25, Theorem 9.9], we can explicitly describe the action on a set of generators as follows:

(6.1.11)
$$\omega_{\psi,W}(m(g),1)f(x) = f(xg),$$

$$\omega_{\psi,W}(n(b),1)f(x) = \overline{\psi}(\frac{1}{2}\langle bx, x \rangle)f(x),$$

$$\omega_{\psi,W}(w,1)f(x) = \mathcal{F}_{V,\psi}(f)(x)$$

$$\omega_{\psi,W}(I,z)f(x) = zf(x)$$

We refer to loc. cit. for the (standard) unexplained notation. We point out that the usual factor of $|\det g|^{1/2}$ does not appear because we are working with half densities. Moreover, we are using the fact that the cover $\operatorname{Mp}_W(F) \to \operatorname{Sp}_W(F)$ splits over P(F), and over the subgroup generated by w [GKT25, Corollary 2.21, Corollary 2.27].

Let $\widetilde{J}_W(F) := H_W(F) \times Mp_W(F)$. Then we have the Heisenberg-Weil representation

(6.1.12)
$$\omega_{\psi}^{\mathbf{J}} := \rho_{\psi,W} \otimes \omega_{\psi,W} : \widetilde{\mathbf{J}}_{W}(F) \longrightarrow \operatorname{Aut}(L^{2}(V(F), \mathcal{L}^{1/2})).$$

We point out that in general the map ω_{ψ}^{J} is not injective. Indeed, when $F \neq \mathbb{R}$ even its restriction to the center of $H_{W}(F)$ is not injective because ψ has a nontrivial kernel.

Theorem 6.1. The modulation group $\Psi_{id}\{F\}$ coincides with the image of

$$H_W(F) \rtimes \langle w, GL_V(F) \rangle < \widetilde{J}_W(F),$$

under ω_{ψ}^{J} .

Proof. We have an injective group homomorphism

$$\Psi_{\mathrm{id}}^{\mathrm{s}}(F) \longrightarrow \mathrm{H}_{W}(F) \rtimes \mathrm{GL}_{V}(F)$$

 $(\lambda, h) \longmapsto (((0, \lambda), 0), m(h))$

that intertwines the actions of $\Psi_{id}^{s}(F)$ and $H_{W}(F)$ on $L^{2}(V(F), \mathcal{L}^{1/2})$.

The formulae (6.1.11) imply that

(6.1.13)
$$\Psi_{\mathrm{id}}\{F\} := \langle \mathcal{F}, V^{\vee}(F) \rtimes \mathrm{GL}_{V}(F) \rangle = \omega_{\psi}^{\mathrm{J}}(\langle w, (\{0\} \times V(F)) \rtimes \mathrm{GL}_{V}(F) \rangle) \\ \leq \omega_{\psi}^{\mathrm{J}}(\mathrm{H}_{W}(F) \rtimes \langle w, \mathrm{GL}_{V}(F) \rangle).$$

On the other hand, $W(F) < \langle w, (\{0\} \times V(F)) \rangle \times GL_V(F) \rangle$, and for any $\lambda, v \in V(F)$

$$(((v,0),0).((0,\lambda),0)).(((v,0),0).((0,-\lambda),0)) = ((0,0),\langle\lambda,v\rangle_{\wedge}),$$

This implies that $\langle w, (\{0\} \times V(F)) \times GL_V(F) \rangle = H_W(F) \times \langle w, GL_V(F) \rangle$ and hence (6.1.13) is an equality.

6.2. Symmetric squares of vector spaces. Let $V_0 \cong \mathbb{G}_a^n$ and let $\langle , \rangle : V_0(F) \times V_0(F) \to F$ be a perfect pairing. We take

$$(6.2.1) X := V_0 \text{ and } H = GL_{V_0}$$

acting on X on the right. We then take $V = \operatorname{Sym}_{\langle , \rangle}^{\vee}$ and let ω be the map given on points in an F-algebra R by

(6.2.2)
$$\omega: X(R) \longrightarrow \operatorname{Sym}_{\langle,\rangle}^{\vee}(R)$$
$$v \longmapsto \left(b \mapsto \frac{1}{2} \langle -b(v), v \rangle\right)$$

The group H_X is trivial and verifying the conditions (M1)-(M4) is straightforward. In this setting

(6.2.3)
$$\Psi_{\omega}^{s} = \operatorname{Sym}_{\langle,\rangle} \rtimes \operatorname{GL}_{V_{0}}.$$

For our use in a moment we recall

Proposition 6.2. Let G be a reductive group over a field k, let $P \leq G$ be a minimal parabolic subgroup, let $M \leq P$ be a Levi subgroup, let $N_P \leq P$ be the unipotent radical, and let $N_{P^{op}}$ be the unipotent radical of the parabolic subgroup P^{op} opposite P with respect to M. Then

$$G(k) = N_P(k)N_{P^{\mathrm{op}}}(k)P(k).$$

Proof. See the comment after [BT65, Corollaire 6.26].

Let $W_0 = V_0 \oplus V_0$ equipped with the symplectic form $\langle , \rangle_{\wedge}$ attached to \langle , \rangle as in (6.1.2). We then have a symplectic group Sp_{W_0} etc. as in §6.1. Let

(6.2.4)
$$\operatorname{Mp}_{W_0}(F)' \coloneqq \langle w, P_{W_0}(F) \rangle$$

be the subgroup generated by w and $P_{W_0}(F)$. It follows from Proposition 6.2 that the natural map $\operatorname{Mp}_{W_0}(F)' \to \operatorname{Sp}_{W_0}(F)$ is surjective.

Theorem 6.3. One has $\Psi_{\operatorname{Sym}^2}\{F\} = \omega_{\psi,W_0}(\operatorname{Mp}_{W_0}(F)')$.

Proof. Using notation from (6.1.5) and (6.1.7) we have an isomorphism

$$\Psi^{\mathrm{s}}_{\omega}(R) \xrightarrow{\tilde{}} P_{W_0}(R)$$

 $(b,g) \longmapsto n(b)m(g).$

The isomorphism intertwines the actions of $\Psi_{\omega}^{s}(F)$ and $P_{W_0}(F)$ on $L^2(V_0(F), \mathcal{L}^{1/2})$. Using (6.1.11) we see that

$$\Psi_{\operatorname{Sym}^2}\{F\} = \langle \mathcal{F}, \omega_{\psi, W_0}(P_{W_0}(F)) \rangle = \omega_{\psi, W_0}(\langle w, P_{W_0}(F) \rangle).$$

6.3. The standard representation. We let

(6.3.1) $X := M_n = M_{\text{st}}$ equipped with the usual action of $H = GL_n \times GL_n$

and

$$\omega \coloneqq \omega_{\operatorname{st}} = \operatorname{id} : X \longrightarrow V \coloneqq M_n.$$

Then $H_X \cong \mathbb{G}_m$,

(6.3.2)
$$H/H_X = P(GL_n^2) := (GL_n \times GL_n)/\operatorname{diag}(\mathbb{G}_m),$$

and conditions (M1)-(M4) are satisfied. By (6.3.2) we have

(6.3.3)
$$\Psi_{\mathrm{id}}^{\mathrm{s}} = V^{\vee} \rtimes \mathrm{P}(\mathrm{GL}_{n}^{2}).$$

Let $W = V \oplus V^{\vee}$, and define H_W , etc. as in §6.1. The action of $GL_n \times GL_n$ on X = V induces a closed immersion

$$P(GL_n^2) \longrightarrow GL_V \xrightarrow{m} P_W$$

and we identify $P(GL_n^2)$ with its image. Following the same argument as for Theorem 6.1, we obtain

Theorem 6.4. One has
$$\Psi_{\mathrm{id}}\{F\} = \omega_{\psi}^{\mathrm{J}}(\mathrm{H}_{W}(F) \rtimes \langle w, \mathrm{P}(\mathrm{GL}_{n}^{2})(F) \rangle).$$

6.4. Smooth vectors.

Proposition 6.5. In the settings of Theorems 6.1 and 6.4, the Schwartz space is the space of smooth vectors under the action of the modulation group.

Proof. Assume first that V is as in Theorem 6.1. We have

$$(6.4.1) S(V(F)) \le L^2(V(F))^{sm}.$$

The Schwartz space S(V(F)) is the space of smooth vectors in $L^2(V(F))$ under the action of $H_W(F)$. Indeed, when $F = \mathbb{R}$ this is [How80, p. 827]; the argument generalizes in a standard manner to the case $F = \mathbb{C}$. When F is non-Archimedean it follows from [GKT25, Theorem 1.13]. This completes the proof in the setting of Theorem 6.1.

The proof in the setting of Theorem 6.4 is analogous to the proof in the setting of Theorem 6.1.

7. Modulation groups of quadric cones

For the moment we take F to be an arbitrary characteristic zero field so that we can develop notation that will also appear in other sections. Let $V_0 = \mathbb{G}_a^{2j}$ for some $j \in \mathbb{Z}_{\geq 0}$ and let $Q_0 : V_0 \to \mathbb{G}_a$ be an anisotropic form. Let $V_i = \mathbb{G}_a^{2i+2j}$ be a space with a nondegenerate quadratic form Q_i in the same Witt class as Q_0 . We choose bases so that the matrix of the quadratic form Q_{i+1} is given inductively by

$$\left(7.0.1\right) \qquad \left(\begin{smallmatrix} & J_i \\ & \end{smallmatrix}\right)$$

where J_i is the matrix of \mathcal{Q}_i . Let \langle , \rangle_i be the pairing defined by \mathcal{Q}_i . We let \mathcal{O}_{V_i} and \mathcal{GO}_{V_i} be the orthogonal group and orthogonal similitude group of V_i and let \mathcal{GSO}_{V_i} be the neutral component of \mathcal{GO}_{V_i} . We denote by

$$\nu: \mathrm{GSO}_{V_i} \longrightarrow \mathbb{G}_m$$

the similitude norm. Let

$$C_i \subset V_i$$

be the vanishing locus of Q_i and let $C_i^{\circ} := C_i - \{0\}$. For F-algebras R we have a group homomorphism

(7.0.2)
$$m: R^{\times} \times GO_{V_{i}}(R) \longrightarrow GO_{V_{i+1}}(R)$$
$$(a,g) \longmapsto \begin{pmatrix} a\nu(g) & g \\ & a^{-1} \end{pmatrix}.$$

Let

(7.0.3)
$$\overline{n}: V_{i}(R) \longrightarrow \mathcal{O}_{V_{i+1}}(R) \\
v \longmapsto \begin{pmatrix} 1 \\ -J_{i}v^{t} & I_{i} \\ -Q_{i}(v) & v & 1 \end{pmatrix}, \\
n: V_{i}(R) \longrightarrow \mathcal{O}_{V_{i+1}}(R) \\
v \longmapsto \begin{pmatrix} 1 & v & -Q_{i}(v) \\ I_{i} & -J_{i}v^{t} \\ 1 \end{pmatrix}.$$

We let

(7.0.4)
$$Q_{i}(R) := \{ m(a,g)n(v) : (a,g,v) \in R^{\times} \times O_{V_{i}}(R) \times V_{i}(R) \},$$
$$Q_{i}^{op}(R) := \{ m(a,g)\overline{n}(v) : (a,g,v) \in R^{\times} \times O_{V_{i}}(R) \times V_{i}(R) \}.$$

These are parabolic subgroups of $O_{V_{i+1}}$ opposite each other with respect to $m(\mathbb{G}_m \times O_{V_i})$. We let N_i (resp. N_i^{op}) be the unipotent radical of Q_i (resp. Q_i^{op}). Thus N_i is the image of n and N_i^{op} is the image of \overline{n} . Similarly, we let

$$(7.0.5) \widetilde{Q}_i(R) := \{ m(a,g)n(v) : (a,g,v) \in R^{\times} \times GO_{V_i}(R) \times V_i(R) \},$$

which is a parabolic subgroup of $GSO_{V_{i+1}}$, and let \widetilde{Q}_i^{op} be its opposite parabolic. Usually in the theory described below one restricts attention to orthogonal groups, but in order to construct a group action satisfying the assumptions in §3.1 and to make contact with reductive monoids in §7.3 we extend the theory to similitudes.

Let

be the minimal nilpotent orbit. By restricting the adjoint action, we obtain an action of Q_n^{op} on \mathbb{O}_{n+2} .

The following is proved in [Tom25] using the description of the minimal nilpotent orbit given in [Jia21].

Lemma 7.1. For dim $V_n > 2$ there is a unique open Q_n^{op} -orbit in \mathbb{O}_n . Its complement has codimension at least 2.

One has a canonical action of O_{V_n} on $T^*C_n^{\circ}$ as in Lemma 3.11; it extends to the affine closure $\overline{T^*C_n^{\circ}}^{\text{aff}}$. In Proposition 8.22 we require the following

Proposition 7.2. For dim $V_n \ge 2$ the affine closure $\overline{T^*C_n^{\circ}}^{\text{aff}}$ is isomorphic as a O_{V_n} -scheme to the closure of \mathbb{O}_{n+1} in $\mathfrak{o}_{V_{n+1}}$.

This is part of the folklore. We give a proof because the morphism will be helpful to us later in §8.4.2.

Proof. For F-algebras R let $R[\epsilon]$ denote the ring of dual numbers. We have

(7.0.7)
$$TC_n^{\circ}(R) = C_n^{\circ}(R[\epsilon]) \\ = \{(c, v) \in C_n^{\circ}(R) \times V_n(R) : \langle c, v \rangle_n = 0\}.$$

Use \langle , \rangle_n to identify V_n with V_n^{\vee} . Then (7.0.7) implies $T^*C_n^{\circ}$ is the quotient of $C_n^{\circ} \times V_n$ by the action of \mathbb{G}_a given by

(7.0.8)
$$C_n^{\circ}(R) \times V_n(R) \times R \longrightarrow C_n^{\circ}(R) \times V_n(R)$$
$$((c, v), a) \longmapsto (c, v + ac).$$

Let $\mathbb{O}_{n+1} \subset \mathfrak{o}_{V_{n+1}}$ be the minimal nilpotent orbit. We have a map

(7.0.9)
$$a: C_n^{\circ}(R) \times V_n(R) \longrightarrow \mathbb{O}_{n+1}(R)$$
$$(c,v) \longmapsto \overline{n}(v) \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & -J_n c^t \\ 0 & 0 & 0 \end{pmatrix} \overline{n}(v)^{-1}.$$

Here we are using notation as in (7.0.3). Thus

$$(7.0.10) a(c,v) = \begin{pmatrix} \langle c,v\rangle_n & c & 0 \\ J_n c^t \mathcal{Q}_n(v) - \langle c,v\rangle_n J_n v^t & ((-J_n v)_i c_j + (J_n c^t)_i v_j)_{ij} & -J_n c^t \\ 0 & -\mathcal{Q}_n(v) c + \langle v,c\rangle_n v & -\langle v,c\rangle_n \end{pmatrix}.$$

One has a(c, v+tc) = a(c, v) for all $t \in R$. Hence a descends to a morphism $\overline{a} : T^*C_n^{\circ} \to \mathbb{O}_{n+1}$. We claim that this map is injective on \overline{F} -points. Indeed, assume a(c, v) = a(c', v') for $(c, v), (c', v') \in T^*C_n^{\circ}(\overline{F})$. Then it is clear that c = c', and then a(c, v - v') = a(c, 0). One checks directly that this implies v - v' = tc for some $t \in \overline{F}$.

One has $\dim \mathbb{O}_{n+1} = 2n - 2$. Thus by comparing dimensions the image of a is the open Q_{n+1}^{op} -orbit in \mathbb{O}_{n+1} mentioned in Lemma 7.1. It is the underlying topological space of a reduced open subscheme $\mathbb{O}'_{n+1} \subset \mathbb{O}_{n+1}$. Since $T^*C_n^{\circ}$ and \mathbb{O}_{n+1} are both smooth they are in particular integral and normal. The map $\overline{a}: T^*C_n^{\circ}(\overline{F}) \to \mathbb{O}'_{n+1}(\overline{F})$ is bijective, and hence the map $\overline{a}: T^*C_n^{\circ} \to \mathbb{O}_{n+1}$ is an isomorphism onto its image by a standard argument recorded in [GH24, Proposition 1.2.9]; the proof therein does not use the assumption that the schemes are affine.

Now the complement of \mathbb{O}'_{n+1} in \mathbb{O}_{n+1} is of codimension at least 2. Moreover $\overline{\mathbb{O}}_{n+1}$ is known to be normal and equal to $\mathbb{O}_{n+1} \sqcup \{0\}$ [Jan04, §8.6]. It follows that the closure $\overline{\mathbb{O}'}_{n+1} = \overline{\mathbb{O}}_{n+1}$ of \mathbb{O}'_{n+1} in $\mathfrak{o}_{V_{n+1}}$ may be identified with the affine closure of \mathbb{O}'_{n+1} [GW10, Theorem 6.45] and hence the isomorphism $\overline{a}: T^*C_n^{\circ} \xrightarrow{\sim} \mathbb{O}'_{n+1}$ extends to an isomorphism $\overline{a}: \overline{T^*C_n^{\circ}} \xrightarrow{\sim} \overline{\mathbb{O}}_{V_{n+1}}$.

7.1. The small modulation group for quadric cones. We now assume F is a local field of characteristic 0. We take $X = C_n$, $V = V_n$, and let

$$\omega: C_n \longrightarrow V_n$$

be the inclusion. We take $H = \mathbb{G}_m \times \mathrm{GO}_{V_n}$ where the action is given by

$$V_n(R) \times H(R) \longrightarrow V_n(R)$$

 $(v, (a, h)) \longmapsto avh.$

Then

$$H_X(R) := \{(\lambda^{-1}, \lambda I_{V_n}) : \lambda \in R^{\times}\} \cong R^{\times}.$$

The hypothesis (M1)-(M4) are valid.

Remark 7.3. The space C_n is a horospherical variety in the sense of §5, but for convenience we have not taken the group H in the current section to be the same as the group H in §5. The two actions encode essentially the same information.

Using the pairing \langle , \rangle_n we identify V with its dual. Then according to Definition 3.7, we have

$$\Psi_{\omega}^{s} = V \rtimes (\mathbb{G}_{m} \times GO_{V_{i}})/H_{X}.$$

Let ω_{C_n} be the top-degree differential form on C_n° such that

(7.1.1)
$$dv_1 \wedge \cdots \wedge dv_{\dim(V_n)} = d(\mathcal{Q}_n(v)) \wedge \omega_{C_n}(v).$$

The corresponding density $|\omega_{C_n}|$ on $C_n^{\circ}(F)$ defines an eigenmeasure under the action of H(F). Explicitly,

(7.1.2)
$$|\omega_{C_n}|(avh) = |a|^{2n-2} |\nu(h)|^{n-1} |\omega_{C_n}|(v).$$

As in Remark 3.1 we therefore have an isomorphism

(7.1.3)
$$L^{2}(C_{n}^{\circ}(F)) \xrightarrow{\tilde{}} L^{2}(C_{n}^{\circ}(F), \mathcal{L}^{1/2})$$
$$f \xrightarrow{\tilde{}} f |\omega_{C_{n}}|^{1/2}$$

where on the left hand side the Hilbert space is defined using the measure $|\omega_{C_n}|$. The action of $\Psi^s_{\omega}(F)$ on the right hand side is intertwined with the action on the left hand side given by

$$(7.1.4) \quad \mathcal{R}_{\omega,\psi}(v \rtimes m(a,h))f(c) = \psi(\langle v, c \rangle)|a|^{n-1}|\nu(h)|^{\frac{n-1}{2}}f(ach) \text{ for } (a,h) \in F^{\times} \times GO_{V_n}(F).$$

To ease comparison with [GK23] we work with $L^2(C_n^{\circ}(F))$ instead of $L^2(C_n^{\circ}(F), \mathcal{L}^{1/2})$.

- 7.2. **Modulation groups for quadric cones.** For the remainder of this section we assume the following:
 - (1) When F is non-Archimedean we assume $n \ge 3$ and that the Witt index of V_n is at least n-1.
 - (2) When F is Archimedean we assume it is real.

The first assumption is made so that we can apply the results of [GK23]. The second is due to the fact that we were unable to locate references for the minimal representation of the orthogonal group over the complex numbers.

When F is non-Archimedean we let χ be the character associated by class field theory to $F(\sqrt{\det J_n})/F$. When F is real and \mathcal{Q}_n has signature (p,q) we let $\chi: F^{\times} \to \mathbb{C}^{\times}$ to be the character that is trivial on $\mathbb{R}_{>0}$ and assigns $(-1)^{(p-q)/2}$ to $\mathbb{R}_{<0}$.

7.2.1. The minimal representation of $O_{V_{n+1}}(F)$. We use the isomorphism $\overline{n}: V_n \to N_n^{\text{op}}$ of (7.0.3) to identify V_n with N_n^{op} . For $(v, a, h) \in V_n(F) \times F^{\times} \times GO_{V_n}(F)$ one has

(7.2.1)
$$m(a,h)^{-1}\overline{n}(v)m(a,h) = \overline{n}(avh).$$

Use the exponential map to identify $N_n(F) \times N_n^{\text{op}}(F)$ with its Lie algebra; it is a subalgebra of $\mathfrak{o}_{V_{n+1}}$. Restricting the Killing form $\langle X, Y \rangle := \operatorname{tr} XY$ to $N_n(F) \times N_n^{\text{op}}(F)$ we see that

$$-\langle n(v), \overline{n}(v') \rangle = \langle v, v' \rangle_n$$

Let $\widetilde{\tau}$ be the representation of $\widetilde{Q}_n(F)$ on the space of smooth functions $C^{\infty}(C_n^{\circ}(F))$ defined by

for
$$(v, a, g, c) \in V_n(F) \times F^{\times} \times GO_{V_n}(F) \times C_n^{\circ}(F)$$
.

Let $\tau = \widetilde{\tau}|_{Q_n(F)}$. Then τ extends to a unitary representation

$$\tau: \mathcal{O}_{V_{n+1}}(F) \times L^2(C_n^{\circ}(F)) \longrightarrow L^2(C_n^{\circ}(F)),$$

namely, the minimal representation (see [KM11, GK23]).

Identify $O_{V_{n+1}}(F)$ with a subgroup of $GO_{V_{n+1}}(F)$ in the evident manner. Let

(7.2.4)
$$w_0 = \binom{1}{1} I_{V_n}^{1} \in \mathcal{O}_{V_{n+1}}(F).$$

Lemma 7.4. The representation $\widetilde{\tau}$ extends to a unitary representation of $GO_{V_{n+1}}(F)$ on $L^2(C_n^{\circ}(F))$, still denoted τ , such that $\widetilde{\tau}|_{O_{V_{n+1}}(F)} = \tau$.

Proof. Any element of $GO_{V_{n+1}}(F)$ is of the form m(a,g)h where $(a,g,h) \in F^{\times} \times GO_{V_n}(F) \times O_{V_{n+1}}(F)$. We claim that the desired extension is given by

$$\widetilde{\tau}(m(a,g)h) \coloneqq \widetilde{\tau}(m(a,g))\tau(h).$$

To prove that this is an extension of our original representation it suffices to show that

(7.2.5)
$$\tau(h)\widetilde{\tau}(m(a,g)) = \widetilde{\tau}(m(a,g))\tau(m(a,g)^{-1}hm(a,g)).$$

It suffices to check (7.2.5) for a set of h generating $O_{V_{n+1}}(F)$ as a group. Thus by Proposition 6.2 the Bruhat decomposition it suffices to check (7.2.5) if $h \in Q_{n+1}(F)$ or if $h = w_0$. If $h \in Q_{n+1}(F)$ then (7.2.5) is valid because $\widetilde{\tau}|_{Q_{n+1}(F)} = \tau|_{Q_{n+1}(F)}$.

To check (7.2.5) when $h = w_0$ we use the formula

(7.2.6)
$$\tau(w_0)(f)(c) = \int_F \Psi(t) \mathcal{R}_t(f)(c) dt.$$

Here $\Psi(t)$ is a certain distribution,

(7.2.7)
$$\mathcal{R}_t(f)(c) \coloneqq \int_{x \in C_n^c(F): \langle c, x \rangle_n = t} f(x) |\omega_{c,t}|(x).$$

is the Radon transform [GK23, (1.5)] [KM11, (5.2.2)], and $|\omega_{c,t}|$ is a suitable family of measures (see [GK23, §3.2]). We point out that in [KM11] a different choice of w_0 and pairing is used, but (7.2.6) is still valid.

Following the proof of [GK23, Proposition 3.16], we then have

$$\tau(w_{0})\widetilde{\tau}(m(a,g))f(c)$$

$$= \chi(a)|a|^{n-1}|\nu(g)|^{(n-1)/2} \int_{F} \Psi(t) \int_{x \in C_{n}^{\circ}(F):\langle c,x\rangle_{n}=t} f(axg)|\omega_{c,t}|(x)dt$$

$$= \chi(a)|a|^{1-n}|\nu(g)|^{(1-n)/2} \int_{F} \Psi(t) \int_{x \in C_{n}^{\circ}(F):\langle c,x\rangle_{n}=t} f(axg)|\omega_{a^{-1}\nu(g)^{-1}cg,t}|(axg)dt$$

$$= \chi(a)|a|^{1-n}|\nu(g)|^{(1-n)/2} \int_{F} \Psi(t) \int_{x \in C_{n}^{\circ}(F):\langle c,axg^{-1}\rangle_{n}=t} f(x)|\omega_{a^{-1}\nu(g)^{-1}cg,t}|(x)dt$$

$$= \chi(a)|a|^{1-n}|\nu(g)|^{(1-n)/2} \int_{F} \Psi(t) \int_{x \in C_{n}^{\circ}(F):\langle a^{-1}\nu(g)^{-1}cg,x\rangle_{n}=t} f(x)|\omega_{a^{-1}\nu(g)^{-1}cg,t}|(x)dt$$

Since $m(a,g)^{-1}w_0m(a,g) = \binom{a^{-2}\nu(g)^{-1}}{a^2\nu(g)}w_0$ this is $\widetilde{\tau}(m(a,g))\tau(m(a,g)^{-1}w_0m(a,g))f(c)$. This completes the proof of (7.2.5) when $h = w_0$ and hence the proof of the lemma.

7.2.2. The modulation group. We define

$$\mathcal{S}(C_n(F)) = L^2(C_n^{\circ}(F))^{\mathrm{sm}}$$

where we take smooth vectors with respect to the action of $GO_{V_{n+1}}(F)$ via $\widetilde{\tau}$, or equivalently with respect to $O_{V_{n+1}}(F)$ via τ .

We define

$$\mathcal{F}_C \coloneqq \widetilde{\tau}(w_0)$$

with w_0 as in (7.2.4). Thus we have an isomorphism

$$\mathcal{F}_C: \mathcal{S}(C_n(F)) \longrightarrow \mathcal{S}(C_n(F)).$$

It is reasonable to view this as a Fourier transform. Indeed, there is a formula for it entirely analogous to the usual Fourier transform; see [GK23, 1.5] or [GHL25, Corollary 6.9].

Consider the action $\mathcal{R}_{\omega,\chi}: \Psi^{\mathbf{s}}_{\omega}(F) \times L^2(C_n^{\circ}(F)) \to L^2(C_n^{\circ}(F))$ given by (7.2.10)

$$\mathcal{R}_{\omega,\chi}((v,m(a,h)))f(c) = \psi(\langle v,c\rangle)\chi(a)|a|^{n-1}|\nu(h)|^{\frac{n-1}{2}}f(ach) \text{ for } (a,h) \in F^{\times} \times GO_{V_n}(F).$$

This is a twist of the action (7.1.4). The **modulation group** of the inclusion $\omega: C_n \to V_n$ is the group

(7.2.11)
$$\Psi_{\omega}\{F\} := \langle \mathcal{F}_C, \mathcal{R}_{\omega, \chi}(\Psi_{\omega}^{\mathrm{s}}(F)) \rangle.$$

Theorem 7.5. One has

$$\Psi_{\omega}\{F\} = \widetilde{\tau}(\mathrm{GO}_{V_{n+1}}(F)).$$

Proof. We have an isomorphism

$$\Psi_{\omega}^{s}(F) \xrightarrow{\sim} (F^{\times} \times \widetilde{Q}_{n}(F)) / H_{X}(F)$$
$$(v, (a, q)) \longmapsto n(v) m(a, q)$$

that intertwines the action (7.2.10) and $\widetilde{\tau}$ by (7.2.3) and (7.2.2). Thus it suffices to check that $GO_{V_{n+1}}(F)$ is generated by $\widetilde{Q}_n(F)$ and w_0 . This is a consequence of Proposition 6.2

7.3. The modulation group for the Rankin-Selberg monoid. Let M_{\otimes} be the reductive monoid whose points in an F-algebra R are given by

$$(7.3.1) M_{\otimes}(R) = \{(X_1, X_2) \in M_2(R) \times M_2(R) : \det(X_1) = \det(X_2)\}.$$

Let $G := M_{\otimes}^{\times}$ be its group of units. We refer to M_{\otimes} as the Rankin-Selberg monoid. It is the reductive monoid attached to the tensor product representation

$$\otimes : {}^{L}G \longrightarrow \mathrm{GL}_{4}(\mathbb{C}).$$

There is a canonical closed immersion

$$(7.3.2) \omega_{\otimes}: M_{\otimes} \to M_2 \times M_2$$

We can choose data so that this is the morphism (4.0.1).

We view M_2 and $M_2 \times M_2$ as quadratic spaces equipped with quadratic forms $X \mapsto \det X$ and $(X,Y) \mapsto \det X - \det Y$. Let

$$W \coloneqq \mathbb{G}_a \oplus M_2 \oplus M_2 \oplus \mathbb{G}_a$$

equipped with the quadratic form $(a, X, Y, b) \mapsto ab + \det X - \det Y$.

We have a Fourier transform

$$\mathcal{F}_{M_{\otimes}}: \mathcal{S}(M_{\otimes}(F)) \longrightarrow \mathcal{S}(M_{\otimes}(F)).$$

When we refer to modulation groups in this section, we will always mean modulation groups with respect to this transform.

In the present case, the small modulation group is equal to the group

$$\Psi_{\otimes}^{\mathrm{s}} = M_2 \times M_2 \rtimes (G \times G) / \Delta(Z_G).$$

Here we have used the quadratic form to identify $M_2 \times M_2$ with its dual. Let

$$(7.3.3) \qquad (\mathrm{GSO}_{M_2} \times \mathrm{GSO}_{M_2})^{\circ}(R) \coloneqq \{(g_1, g_2) \in \mathrm{GSO}_{M_2}(R) \times \mathrm{GSO}_{M_2}(R) : \nu(g_1) = \nu(g_2)\}.$$

Proposition 7.6. We have $\Psi_{\otimes}^s = M_2 \times M_2 \rtimes (GSO_{M_2} \times GSO_{M_2})^{\circ}$.

Proof. The action map

$$M_2 \times \operatorname{GL}_2 \times \operatorname{GL}_2 \longrightarrow M_2$$

induces an isomorphism $\operatorname{GL}_2 \times \operatorname{GL}_2/\Delta(Z_{\operatorname{GL}_2}) \xrightarrow{\sim} \operatorname{GSO}_{M_2}$.

Theorem 7.7. One has $\Psi_{\otimes}\{F\} = \Psi_{\omega}(GSO_W(F))$.

Proof. The quadratic space $M_2 \times M_2$ is isomorphic to V_4 and the quadratic space W is isomorphic to V_5 . We have a closed immersion $(GSO_{M_2} \times GSO_{M_2})^{\circ} \to GSO_{W_8}$ given on points by

$$(7.3.4) (g_1, g_2) \longmapsto \begin{pmatrix} \nu(g_1) & & \\ & g_1 & \\ & & 1 \end{pmatrix}$$

which extends to a homomorphism $M_2 \times M_2 \times (\text{GSO}_{M_2} \times \text{GSO}_{M_2})^{\circ} \to \text{GSO}_{W_8}$ sending $M_2 \times M_2$ to N. By the proof of Theorem 7.5 and Proposition 7.6 we deduce that

$$(7.3.5) \Psi_{\otimes} \{F\} = \widetilde{\tau} \left(\langle w_0, N(F) \rtimes (\mathrm{GSO}_{M_2} \times \mathrm{GSO}_{M_2})^{\circ}(F) \rangle \right).$$

Consider the subgroup

$$(7.3.6) \langle w_0, N(F) \times (GSO_{M_2} \times GSO_{M_2})^{\circ}(F) \rangle \leq GSO_W(F)$$

We claim that this inequality is in fact an equality. Proving the claim will complete the proof of the theorem.

For $a \in F^{\times}$, choose $h \in (GSO_{M_2} \times GSO_{M_2})^{\circ}(F)$ so that $a = \nu(h)$. Then

$$\begin{pmatrix} a & & & & \\ & I_{M_2 \times M_2} & & \\ & & a^{-1} \end{pmatrix} = \begin{pmatrix} \nu(h) & & \\ & h & \\ & & 1 \end{pmatrix} \begin{pmatrix} & I_{M_2 \times M_2} & 1 \\ 1 & & & \\ & & 1 \end{pmatrix} \begin{pmatrix} \nu(h)^{-1} & & \\ & h^{-1} & \\ & & 1 \end{pmatrix} \begin{pmatrix} & I_{M_2 \times M_2} & 1 \\ 1 & & & \\ & & 1 \end{pmatrix}.$$

Using this observation we deduce that the left hand side of (7.3.6) contains the F-points of a maximal torus of $GSO_W(F)$. One can now deduce the theorem from [BT73, Proposition 6.2 and 6.11].

8. Deformation quantization and the semi-classical limit

8.1. **Differential operators.** Let F be a field. For an F-scheme X, we have a sheaf $\underline{\operatorname{End}_F}(\mathcal{O}_X)$ of F-linear endomorphisms of \mathcal{O}_X and and a sheaf $\underline{\operatorname{End}_{\mathcal{O}_X}}(\mathcal{O}_X)$ of \mathcal{O}_X -linear endomorphisms of \mathcal{O}_X . The sheaf $\underline{\operatorname{End}_F}(\mathcal{O}_X)$ is a non-commutative \mathcal{O}_X -algebra so the usual commutator bracket $[\cdot,\cdot]$ is defined, and $\underline{\operatorname{End}_{\mathcal{O}_X}}(\mathcal{O}_X)$ is canonically isomorphic to \mathcal{O}_X .

Set
$$\mathcal{D}_X^{\leq 0} = \operatorname{End}_{\mathcal{O}_X}(\mathcal{O}_X)$$
, and for $n \geq 1$ set

$$\underline{\mathcal{D}_X^{\leq n}} \coloneqq \left\{ T \in \underline{\operatorname{End}_F}(\mathcal{O}_X) \mid [T, f] \in \underline{\mathcal{D}_X^{n-1}} \text{ for all } f \in \underline{\mathcal{D}_X^{\leq 0}} \right\}.$$

The sheaf of differential operators on X is given by the directed union

$$\underline{\mathcal{D}_X} \coloneqq \bigcup_{n>0} \underline{\mathcal{D}_X^{\leq n}}.$$

This is a (non-commutative) \mathcal{O}_X -subalgebra of $\underline{\operatorname{End}_F}(\mathcal{O}_X)$, filtered by $(\mathcal{D}_X^{\leq n})_{n\geq 0}$.

Throughout the rest of this subsection, let X be a scheme of finite type over F. We assume:

(M5) X is normal and $\operatorname{codim}(X - X^{\operatorname{sm}}, X) \ge 2$.

Let $j: X^{\mathrm{sm}} \to X$ denote the inclusion.

Lemma 8.1. The canonical map $\mathcal{O}_X \to j_*\mathcal{O}_{X^{\mathrm{sm}}}$ is an isomorphism of \mathcal{O}_X -modules.

Proof. If U is any open subscheme of X then it again satisfies (M5). Thus the lemma from [GW10, Theorem 6.45].

For any open subscheme U of X, note that

$$j_* \operatorname{End}_F(\mathcal{O}_{X^{\operatorname{sm}}})(U) = \operatorname{End}_F(\mathcal{O}_{U^{\operatorname{sm}}}) = \operatorname{End}_F((j_*\mathcal{O}_{X^{\operatorname{sm}}})|_U)$$

By Lemma 8.1, the canonical isomorphism $\mathcal{O}_X \to j_* \mathcal{O}_{X^{\mathrm{sm}}}$ then induces a canonical isomorphism $\mathrm{End}_F(\mathcal{O}_X|_U) \cong \mathrm{End}_F((j_* \mathcal{O}_{X^{\mathrm{sm}}})|_U)$. Varying U yields the following

Corollary 8.2. The canonical map $\underline{\operatorname{End}_F}(\mathcal{O}_X) \to j_*\underline{\operatorname{End}_F}(\mathcal{O}_{X^{\operatorname{sm}}})$ is an isomorphism. \square

Corollary 8.3. The canonical map $\underline{\mathcal{D}_X} \to j_* \underline{\mathcal{D}_{X^{\mathrm{sm}}}}$ is an isomorphism of filtered \mathcal{O}_X -algebras.

Put

$$(8.1.1) \mathcal{D}_X \coloneqq \mathcal{D}_X(X).$$

and similarly $\mathcal{D}_X^{\leq n} \coloneqq \mathcal{D}_X^{\leq n}(X)$. Then Corollary 8.3 implies the restriction

$$\mathcal{D}_X \longrightarrow \mathcal{D}_{X^{\mathrm{sm}}}$$

is an isomorphism of filtered $\mathcal{O}_X(X)$ -algebras.

For an F-algebras A, define the algebra of differential operators $\mathcal{D}_A := \bigcup_{n \geq 0} \mathcal{D}_A^{\leq n}$ in the usual manner. Then the canonical projection $\operatorname{End}_F(\mathcal{O}_X) \to \operatorname{End}_F(\mathcal{O}_X(X))$ restricts to a map

$$(8.1.2) \mathcal{D}_X \longrightarrow \mathcal{D}_{\mathcal{O}_X(X)}.$$

of filtered F-algebras. The following is [Sta21, Tag 0G44]:

Lemma 8.4. If X is affine, then (8.1.2) is an isomorphism.

Let H be an affine F-group scheme and X is an affine F-scheme. In particular we have a group anti-homomorphism $H(F) \to \operatorname{Aut}_{\mathbf{Sch}_F}(X)$. Since X is affine, the usual anti-equivalence gives a group anti-homomorphism $\operatorname{Aut}_{\mathbf{Sch}_F}(X) \to \operatorname{Aut}_{\mathbf{Alg}_F}(\mathcal{O}_X(X))$. Finally, conjugation defines a group homomorphism $\operatorname{Aut}_{\mathbf{Alg}_F}(\mathcal{O}_X(X)) \to \operatorname{Aut}_{\mathbf{Mod}_F}(\operatorname{End}_F(\mathcal{O}_X(X)))$. In sum, all of these compose to a group homomorphism

$$(8.1.3) H(F) \to \operatorname{Aut}_{\mathbf{Mod}_F}(\operatorname{End}_F(\mathcal{O}_X(X)))$$

defining a left H(F)-action on $\operatorname{End}_F(\mathcal{O}_X(X))$. Notice $H_X(F)$ acts trivially on $\operatorname{End}_F(\mathcal{O}_X(X))$ as it acts trivially on X.

Lemma 8.5. For each $n \ge 0$, $\mathcal{D}_{\mathcal{O}_X(X)}^{\le n}$ is an H(F)-invariant subspace of $\operatorname{End}_F(\mathcal{O}_X(X))$.

Proof. This follows from an induction based on the identity [h.T, h.S] = h.[T, S] for $T, S \in \operatorname{End}_F(\mathcal{O}_X(X))$ and $h \in H(F)$.

Assume X is affine. We equip \mathcal{D}_X with the unique left H(F)-action such that (8.1.2) is H(F)-equivariant. Similarly we equip $\mathcal{D}_{X^{\text{sm}}}$ with the unique left H(F)-action making (8.1.1) an H(F)-equivariant isomorphism.

We conclude this subsection by briefly explaining the passage from algebraic differential operators to analytic ones. Let X_0 be a scheme smooth over F. Recall the sheaf $\underline{\mathrm{Der}_F}(\mathcal{O}_{X_0})$ of derivations and the following result from [Gro67, Théorème 16.11.2].

Lemma 8.6. One has $\underline{\mathrm{Der}_F}(\mathcal{O}_{X_0}) \subseteq \mathcal{D}_{X_0}^{\leq 1}$. The induced map

$$\mathcal{O}_{X_0} \oplus \underline{\mathrm{Der}_F}(\mathcal{O}_{X_0}) \to \underline{\mathcal{D}_{X_0}^{\leq 1}}$$

is an isomorphism of \mathcal{O}_{X_0} -modules. Moreover, $\underline{\mathcal{D}_{X_0}}$ is the \mathcal{O}_{X_0} -subalgebra of $\underline{\operatorname{End}_F}(\mathcal{O}_{X_0})$ generated by $\mathcal{D}_{X_0}^{\leq 1}$.

Note the last assertion does not imply that \mathcal{D}_{X_0} is generated by $\mathcal{D}_{X_0}^{\leq 1}$. Nevertheless this is true when X_0 is affine [Muh88, Theorem 1.15].

By definition, there is a canonical isomorphism of sheaves

$$\underline{\mathrm{Der}_F}(\mathcal{O}_{X_0}) \longrightarrow \Omega^{\vee}_{X_0/F}$$

and $\Omega_{X_0/F}^{\vee}$ is canonically isomorphic to the sheaf of sections of the tangent bundle $TX_0 \to X_0$. Now let F be a local field. Assume $X_0(F)$ is Zariski dense in X_0 . In particular, $U(F) = X_0(F) \cap U$ is Zariski dense in U for every nonempty open subscheme U of X_0 , and U(F) is an F-analytic manifold. By identifying $\mathcal{O}_X(U)$ with $\operatorname{Hom}_{\mathbf{Sch}_F}(U, \mathbb{G}_a)$, taking F-points gives rise to an F-algebra homomorphism

(8.1.4)
$$\mathcal{O}_{X_0}(U) \to C^{\infty}(U(F), F).$$

By assumption U(F) is Zariski dense is U, so this is injective. Similarly, taking F-points gives an injection

(8.1.5)
$$\Omega_{X_0/F}^{\vee}(U) \cong \operatorname{Hom}_{\mathbf{Sch}_{X_0}}(U, TX_0) \to \Gamma(U(F), TX_0(F))$$

where $\Gamma(U(F), TX_0(F))$ denotes the space of sections of $TX_0(F) \to X_0(F)$ over U(F). All these allow us to realize $\underline{\mathrm{Der}_F}(\mathcal{O}_{X_0})$ as vector fields on $X_0(F)$.

8.2. Action of the small modulation group on $\mathcal{D}_{X\mathbb{C}}$. We now impose the notation of §3.1 in the case of an Archimedean local field F. By Weil restriction there is no loss of generality in assuming $F = \mathbb{R}$.

Therefore, we are given an affine algebraic group H over \mathbb{R} , an affine H-scheme X of finite type over \mathbb{R} and a right representation of H on a finite dimensional \mathbb{R} -vector space V together with an H-equivariant map $\omega: X \to V$. We assume the data satisfy (M1), (M2), (M3), (M4) and (M5). Assume further that

(M6) $X^{\text{sm}}(\mathbb{R})$ is (Zariski) dense in X^{sm} .

In particular, $U^{\mathrm{sm}}(\mathbb{R})$ is Zariski dense in U^{sm} for each open subscheme U of X. Hence the assumption (M6) implies the canonical map $\mathcal{O}_X(U) \to C^{\infty}(U^{\mathrm{sm}}(\mathbb{R}))$ of (8.1.4) is injective. Here we put $C^{\infty}(U^{\mathrm{sm}}(\mathbb{R})) := C^{\infty}(U^{\mathrm{sm}}(\mathbb{R}), \mathbb{C})$.

Remark 8.7. We point out that (M6) is often automatic. If H is connected and $x \in X(\mathbb{R})$ is a point such that the orbit $O(x_0) \subset X$ is dense, then (M6) follows from Lemma 3.4.

Let C_X^{∞} denote the Zariski sheaf on X^{sm} given by

$$C_X^{\infty}(U) \coloneqq C^{\infty}(U(\mathbb{R})).$$

For $v^{\vee} \in V^{\vee}(\mathbb{R})$, $\theta \in \mathcal{D}_{X^{\mathrm{sm}}}$ and open subschemes $U \subseteq X^{\mathrm{sm}}$ we define $v^{\vee}(\theta)|_{U} \in \mathrm{End}_{\mathbb{C}}C_{X}^{\infty}(U)$ to be the arrow making the following diagram commute:

(8.2.1)
$$C^{\infty}(U(F)) \xrightarrow{\cdot \psi(v^{\vee} \circ \omega)} C^{\infty}(U(F))$$

$$\downarrow v^{\vee}(\theta)|_{U}$$

$$C^{\infty}(U(F)) \xrightarrow{\cdot \psi(v^{\vee} \circ \omega)} C^{\infty}(U(F)).$$

where the horizontal maps are given by multiplication by the function $x \mapsto \psi(v^{\vee} \circ \omega(x))$. The endomorphisms $v^{\vee}(\theta)|_{U}$ together glue to

$$v^{\vee}(\theta) \in \operatorname{End}_{\mathbb{C}}C_X^{\infty}$$
.

Repeating the construction with X^{sm} replaced by its open subspaces, we obtain an injective morphism of sheaves of \mathbb{R} -algebras

$$(8.2.2) v^{\vee} : \underline{\mathcal{D}_{X^{\mathrm{sm}}}} \longrightarrow \underline{\mathrm{End}_{\mathbb{C}}} C_X^{\infty}$$

The horizontal arrows in (8.2.1) are multiplication by transcendental functions. Nonetheless we have:

Proposition 8.8. For $v^{\vee} \in V^{\vee}(\mathbb{R})$, the map of (8.2.2) has image in $\underline{\mathcal{D}_{X^{\text{sm}}}} \otimes_{\mathbb{R}} \mathbb{C}$. Moreover, $v^{\vee}(\underline{\mathcal{D}_{X^{\text{sm}}}^{\leq n}}) \leq \underline{\mathcal{D}_{X^{\text{sm}}}^{\leq n}} \otimes_{\mathbb{R}} \mathbb{C}$ for each $n \geq 0$.

Proof. The problem is local in nature, so we may assume $X^{\text{sm}} = \text{Spec } A$ is affine with $A = \mathbb{R}[x_1, \dots, x_n]/I$. Since A is smooth, by Lemma 8.6 and Lemma 8.4, it suffices to show $v^{\vee}(\text{Der}_{\mathbb{R}}(A)) \subseteq \mathcal{D}_A^{\leq 1} \otimes_{\mathbb{R}} \mathbb{C}$. Here $\text{Der}_{\mathbb{R}}(A)$ is the set of \mathbb{R} -linear derivations on A.

Let $\theta \in \operatorname{Der}_{\mathbb{R}}(A)$. Let $f := v^{\vee} \circ \omega|_{X^{\operatorname{sm}}} \in A$ and by abuse of notation denote by f a lift to $\mathbb{R}[x_1, \ldots, x_n]$. By definition

$$v^{\vee}(\theta) = e^{-2\pi i r f} \circ \theta \circ e^{2\pi i r f}$$

for some $r \in \mathbb{R}^{\times}$. Say $\theta = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} \mod I$ for some $a_j \in \mathbb{R}[x_1, \dots, x_n]$. Then

$$v^{\vee}(\theta) = \sum_{j=1}^{n} a_j \left(\frac{\partial}{\partial x_j} + 2\pi i r \frac{\partial}{\partial x_j}(f) \right) \mod I = \theta + 2\pi i r \theta(f).$$

Since $\theta(f) \in A$, it follows that $v^{\vee}(\theta) \in \mathcal{D}_A^{\leq 1} \otimes_{\mathbb{R}} \mathbb{C}$.

Let

$$\mathcal{D}_{X\mathbb{C}} \coloneqq \mathcal{D}_X \otimes_{\mathbb{R}} \mathbb{C} \cong \mathcal{D}_{X^{\mathrm{sm}}} \otimes_{\mathbb{R}} \mathbb{C}$$

Then (8.1.3) and (8.2.1) define an action

(8.2.3)
$$\Psi_{\omega}^{s}(\mathbb{R}) \times \mathcal{D}_{X\mathbb{C}} \longrightarrow \mathcal{D}_{X\mathbb{C}}.$$

Ansatz 8.9. The action of \mathcal{D}_X on $C^{\infty}(X^{\mathrm{sm}}(\mathbb{R}))$ preserves $\mathcal{S}(X(\mathbb{R}),\mathcal{L}^{1/2})$.

We assume Ansatz 8.9 moving forward. For $\theta \in \mathcal{D}_{X\mathbb{C}}$ the **Fourier transform** $\mathcal{F}_{X,\psi}(\theta)$ is the unique endomorphism on $\mathcal{S}(X(F), \mathcal{L}^{1/2})$ making the following diagram commute:

(8.2.4)
$$\mathcal{S}(X(\mathbb{R}), \mathcal{L}^{1/2}) \xrightarrow{\mathcal{F}_{X,\psi}} \mathcal{S}(X(\mathbb{R}), \mathcal{L}^{1/2}) \\ \downarrow \psi \qquad \qquad \downarrow \mathcal{F}_{X,\psi}(\theta) \\ \mathcal{S}(X(\mathbb{R}), \mathcal{L}^{1/2}) \xrightarrow{\mathcal{F}_{X,\psi}} \mathcal{S}(X(\mathbb{R}), \mathcal{L}^{1/2})$$

Ansatz 8.10. If $\theta \in \mathcal{D}_X$ then $\mathcal{F}_{X,\psi}(\theta)$ lies in the image of $\mathcal{D}_X \to \operatorname{Aut}(\mathcal{S}(X(\mathbb{R}),\mathcal{L}^{1/2}))$.

Remark 8.11. In a follow up paper to [Hsu21] Hsu plans to prove Ansatz 8.9 and Ansatz 8.10 when X is a horospherical variety as in §5.

Lemma 8.12. Assuming (S3), Ansatz 8.9 and 8.10 there is a unique action

(8.2.5)
$$\Psi_{\omega}\{\mathbb{R}\} \times \mathcal{D}_{X\mathbb{C}} \longrightarrow \mathcal{D}_{X\mathbb{C}}.$$

such that $\Psi^s_{\omega}(\mathbb{R})$ acts via (8.2.3) and $\mathcal{F}_{X,\psi}$ acts via (8.2.4).

Proof. By (S3) $S(X(\mathbb{R}), \mathcal{L}^{1/2})$ is dense in $L^2(X^{sm}(\mathbb{R}), \mathcal{L}^{1/2})$. Thus an automorphism in $\Psi_{\omega}\{\mathbb{R}\}$, which is originally defined as a group of automorphisms of $L^2(X^{sm}(\mathbb{R}), \mathcal{L}^{1/2})$, is uniquely determined by its action on $S(X(\mathbb{R}), \mathcal{L}^{1/2})$. The lemma follows.

8.3. Passage to the semi-classical limit. Consider the associated graded

$$\operatorname{gr} \underline{\mathcal{D}_X} = \bigoplus_{n \ge 0} \underline{\mathcal{D}_X^{\le n}} / \underline{\mathcal{D}_X^{\le n-1}}$$

where we set $\underline{\mathcal{D}_X^{\leq -1}} = 0$. This is a sheaf of commutative \mathcal{O}_X -algebras. Similarly one defines $\operatorname{gr} \mathcal{D}_{X^{\operatorname{sm}}}$. By Corollary 8.3, there is a canonical isomorphism

(8.3.1)
$$\operatorname{gr} \mathcal{D}_X \longrightarrow j_* \left(\operatorname{gr} \mathcal{D}_{X^{\operatorname{sm}}} \right)$$

of commutative \mathcal{O}_X -algebras.

By [HTT08, §1.1] one has an identification

(8.3.2)
$$\operatorname{gr} \underline{\mathcal{D}_{X^{\operatorname{sm}}}} \cong \pi_* \mathcal{O}_{T^*X^{\operatorname{sm}}} \cong \operatorname{Sym}_{\mathcal{O}_{X^{\operatorname{sm}}}} \Omega_{X^{\operatorname{sm}}/\mathbb{R}}^{\vee}.$$

where $\pi: T^*X^{\mathrm{sm}} \to X^{\mathrm{sm}}$ is the bundle projection. Strictly speaking they work over the the complex numbers but the argument is valid over characteristic zero fields. The map can be realized as follows. By definition $\Omega_{X^{\mathrm{sm}}/\mathbb{R}}^{\vee} \cong \underline{\mathrm{Der}_{\mathbb{R}}}(\mathcal{O}_{X^{\mathrm{sm}}})$ canonically. The composition

$$\Omega_{X^{\mathrm{sm}}/\mathbb{R}}^{\vee} \to \mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{X^{\mathrm{sm}}}) \leq \mathrm{gr}\,\mathcal{D}_{X^{\mathrm{sm}}}$$

extends to the $\mathcal{O}_{X^{\mathrm{sm}}}$ -algebra morphism $\mathrm{Sym}_{\mathcal{O}_{X^{\mathrm{sm}}}}\Omega^{\vee}_{X^{\mathrm{sm}}/\mathbb{R}} \to \mathrm{gr}\,\underline{\mathcal{D}_{X^{\mathrm{sm}}}}$ of (8.3.2). Taking global sections of (8.3.2) we have an isomorphism

$$(\operatorname{gr} \mathcal{D}_{X^{\operatorname{sm}}})(X^{\operatorname{sm}}) \cong \mathcal{O}_{T^*X^{\operatorname{sm}}}(T^*X^{\operatorname{sm}})$$

By (8.3.1), one has

$$(\operatorname{gr} \mathcal{D}_X)(X) \cong (\operatorname{gr} \mathcal{D}_{X^{\operatorname{sm}}})(X^{\operatorname{sm}})$$

The action of the small modulation group Ψ^s_{ω} on T^*X^{sm} in Lemma 3.11 induces a left $\Psi^s_{\omega}(\mathbb{R})$ -action on $\mathcal{O}_{T^*X^{\mathrm{sm}}}(T^*X^{\mathrm{sm}})$. Under the identifications above, this yields an action

(8.3.3)
$$\Psi_{\omega}^{s}(\mathbb{R}) \times (\operatorname{gr} \mathcal{D}_{X})(X) \longrightarrow (\operatorname{gr} \mathcal{D}_{X})(X)$$

There is a canonical injective $\mathcal{O}_X(X)$ -algebra homomorphism

(8.3.4)
$$\operatorname{gr} \mathcal{D}_X := \bigoplus_{n>0} \mathcal{D}_X^{\leq n} / \mathcal{D}_X^{\leq n-1} \longrightarrow (\operatorname{gr} \underline{\mathcal{D}_X})(X).$$

The left $H(\mathbb{R})$ -action on \mathcal{D}_X induces an $H(\mathbb{R})$ -action on $\operatorname{gr} \mathcal{D}_X$. With these actions, the map of (8.3.4) is $H(\mathbb{R})$ -equivariant.

Remark 8.13. By Proposition 8.8, the action (8.2.3) induces an $\Psi^{\rm s}_{\omega}(\mathbb{R})$ -action on gr $\mathcal{D}_{X\mathbb{C}}$. However, the $V^{\vee}(\mathbb{R})$ -action is trivial, as opposed opposed to (8.3.3).

We now prepare to explain how a conjugate of the action (8.3.3) may extend to an action of $\Psi_{\omega}\{\mathbb{R}\}$, and formulate how this action "corresponds" to the action on $\mathcal{D}_{X\mathbb{C}}$. Let

$$(8.3.5) \sigma: \mathcal{D}_X \longrightarrow \operatorname{gr} \mathcal{D}_X$$

be the canonical map; it is called the symbol map. It is a multiplicative, but not additive $H(\mathbb{R})$ -equivariant homomorphism. Using the symbol map the associated graded $\operatorname{gr} \mathcal{D}_X$ inherits the structure of a commutative Poisson algebra. The Poisson bracket is determined by $\{f,g\} := \sigma([\widetilde{f},\widetilde{g}])$, where f and g are homogeneous elements of $\operatorname{gr} \mathcal{D}_X$ and \widetilde{f} and \widetilde{g} are (arbitrary) lifts of f and g (respectively) to \mathcal{D}_X .

Assume that $\mathfrak{x} \subset \mathcal{D}_{X\mathbb{C}}$ is a complex Lie subalgebra under the commutator that is $H(\mathbb{R})$ stable. We assume moreover that \mathfrak{x} admits a decomposition

$$\mathfrak{x} = \bigoplus_{i} \mathfrak{x}_{j}$$

into $H(\mathbb{R})$ -invariant subspaces such that

$$\sigma|_{\mathfrak{x}_j}:\mathfrak{x}_j\longrightarrow\operatorname{gr}\mathcal{D}_X$$

is \mathbb{R} -linear. Taking direct sum we obtain an \mathbb{C} -linear map $\Sigma : \mathfrak{x} \to \operatorname{gr} \mathcal{D}_{X\mathbb{C}}$. Let $\mathfrak{x}' \subset \operatorname{gr} \mathcal{D}_{X\mathbb{C}}$ be its image, which is a real Lie algebra under the Poisson bracket. We point out that $\Sigma \neq \sigma|_{\mathfrak{x}}$ because in general σ is not linear.

We observe that if \mathfrak{x} is preserved by $\Psi^{s}_{\omega}(\mathbb{R})$ then $\Psi^{s}_{\omega}(\mathbb{R})$ acts on \mathfrak{x}' by transfer of structure. It preserves the Poisson bracket on \mathfrak{x}' .

Ansatz 8.14. Assume Ansatz 8.9 and Ansatz 8.10. There exists a complex Lie subalgebra $\mathfrak{x} \subset \mathcal{D}_{X\mathbb{C}}$ as above that satisfies the following:

- (1) \mathfrak{x} generates $\mathcal{D}_{X\mathbb{C}}$ as an associative algebra;
- (2) \mathfrak{x} is preserved by the action of $\Psi_{\omega}\{\mathbb{R}\}$;
- (3) \mathfrak{x}' generates gr $\mathcal{D}_{X\mathbb{C}}$ as a commutative algebra;
- (4) There is an ind-group scheme $\Psi^{\text{ia}}_{\omega}$ containing Ψ^{s}_{ω} as a subgroup such that the action of Ψ^{s}_{ω} on T^*X^{sm} extends to an action of $\Psi^{\text{ia}}_{\omega}$ on $\overline{T^*X^{\text{sm}}}^{\text{aff}}$;
- (5) The action of $\Psi^{\text{ia}}_{\omega}(\mathbb{R})$ on $\operatorname{gr} \mathcal{D}_{X\mathbb{C}}$ preserves \mathfrak{x}' . Moreover, the images of the homomorphisms

$$\Psi^{\mathrm{ia}}_{\omega}(\mathbb{R}) \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathfrak{x}')$$

$$\Psi_{\omega}\{\mathbb{R}\} \longrightarrow \operatorname{Aut}_{\mathbb{C}}(\mathfrak{x}')$$

are $\operatorname{Aut}_{\mathbb{C}}(\mathfrak{x}')$ -conjugate.

Remarks.

- (1) In (5) we are using the fact that if H is an ind-algebraic group over a field k, acting on a k-scheme Y, then H(k) acts on $\mathcal{O}_Y(Y)$ by pullback of functions.
- (2) The modulation group $\Psi_{\omega}\{\mathbb{R}\}=\Psi_{\omega,\chi}\{\mathbb{R}\}$ depends on a choice of character χ . Changing χ does not affect the image of the lower automorphism in (5). In particular, Ψ_{ω}^{ia} can be taken to be independent of χ .
- (3) When $X = \overline{P^{\text{der}}\backslash G}^{\text{aff}}$ we suspect that something similar to the procedure in §7.2 may yield the conjecture.
- (4) When $X = M_{\rho}$, one might try to look for the group ind-schemes in (4) using the Kac-Moody groups discussed in [Sha05].

Conjecture 8.15. Ansatz 8.14 is true if X is a reductive monoid as in §4 or if X is a horospherical variety as in §5.

8.4. **Examples.** We prove Ansatz 8.14 in the cases considered in §6 and §7. Let $\alpha \in \mathbb{R}^{\times}$ be chosen so that

$$\psi(t) = e^{2\pi i \alpha t}.$$

8.4.1. Affine Space. In this subsection we take $X = V = \mathbb{G}_a^n$ as in §6 with $H = \operatorname{GL}_n$ acting on the right. We use the notation of loc. cit. In particular, we identify V with its dual V^{\vee} using a perfect pairing as in (6.0.1). For computations we pick the pairing given on points in an \mathbb{R} -algebra R by

$$V(R) \times V(R) \longrightarrow R$$

 $((v_j), (w_j)) \longmapsto \sum_{j=1}^{n} v_j w_j.$

We then have a symplectic space $W = V \oplus V$ with form $\langle \rangle_{\wedge}$ defined as in (6.1.2). Thus we have identifications

$$T^*V = V \oplus V^{\vee} = V \oplus V = W.$$

In this case the algebra $\mathcal{D}_{V\mathbb{C}}$ is the Weyl algebra

(8.4.2)
$$\mathcal{D}_{V\mathbb{C}} = \mathbb{C}\left[x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right]$$

Its action on $C^{\infty}(V(\mathbb{R}))$ preserves the usual Schwartz space $\mathcal{S}(V(\mathbb{R}))$, so Ansatz 8.9 holds. Ansatz 8.10 holds as well. Indeed, standard facts on the Fourier transform imply that for $p(x_i, \frac{\partial}{\partial x_1}) \in \mathcal{D}_{V\mathbb{C}}$ one has

(8.4.3)
$$\mathcal{F}_{X,\psi}(p)\left(x_j, \frac{\partial}{\partial x_j}\right) = p\left(\frac{1}{2\pi i\alpha} \frac{\partial}{\partial x_j}, -2\pi i\alpha x_j\right).$$

Moreover, for $A \in GL_n(\mathbb{R})$

(8.4.4)
$$A.p\left(x_j, \frac{\partial}{\partial x_j}\right) = p\left((x_j)A, \left(\frac{\partial}{\partial x_j}\right)A^{-t}\right).$$

In the notation of Ansatz 8.14 we choose

(8.4.5)
$$\mathfrak{x} := \left(x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) \subset \mathcal{D}_V$$

where the brackets indicate the Lie subalgebra over \mathbb{C} generated by the given elements. We have an isomorphism of Lie algebras

(8.4.6)
$$\text{Lie } H_W \xrightarrow{\sim} \mathfrak{x}$$

$$((v_j), (w_j), t) \longmapsto \sum_{j=1}^n v_j x_j + \sum_{j=1}^n w_j \frac{\partial}{\partial x_j} + t$$

We set

$$\mathfrak{x}_0 := \mathbb{C}, \quad \mathfrak{x}_1 := \bigoplus_j \mathbb{C} x_j, \quad \mathfrak{x}_2 := \bigoplus_j \mathbb{C} \frac{\partial}{\partial x_j}.$$

Then $\mathfrak{x} = \mathfrak{x}_0 \oplus \mathfrak{x}_1 \oplus \mathfrak{x}_2$ and each \mathfrak{x}_i is $H(\mathbb{R})$ -invariant.

Proposition 8.16. When $\omega: X \to V$ is the identity map, Ansatz 8.14 is valid with $\Psi^{\text{ia}}_{\omega} = H_W/Z_{H_W} \rtimes \widetilde{\operatorname{GL}}_V$ and $\mathfrak{x} = \operatorname{Lie} H_W$.

Proof. Assertion (1) is clear. Recall that we computed $\Psi_{\omega}\{\mathbb{R}\}$ in Theorem 6.1. Using (6.1.4), the pullback of the action of $\Psi_{\omega}\{\mathbb{R}\}$ on \mathcal{D}_{V} to $H_{W}(\mathbb{R})$ is given explicitly as follows:

$$(8.4.7) \qquad ((v,\lambda),t) \cdot \left(\sum_{i=1}^n a_i x_i + \sum_{i=1}^n b_i \frac{\partial}{\partial x_i} + c\right) = \sum_{i=1}^n a_i (x_i + v_i) + \sum_{i=1}^n b_i \left(\frac{\partial}{\partial x_i} - 2\pi i \alpha \lambda_i\right) + c.$$

The action of all of $\Psi_{\omega}\{\mathbb{R}\}$ is now determined by (8.4.3) and (8.4.4). Assumption (2) follows. The symbol map has image

$$\mathbb{C}[x_1,\ldots,x_n,\xi_1,\ldots,\xi_n] = \Gamma(V_{\mathbb{C}},\mathcal{O}_{V_{\mathbb{C}}}) \otimes_{\mathbb{C}} \Gamma(V_{\mathbb{C}},\mathcal{O}_{V_{\mathbb{C}}}) = \Gamma(T^*V_{\mathbb{C}},\mathcal{O}_{T^*V_{\mathbb{C}}}).$$

One has $\sigma(x_i) = x_i$ and $\sigma\left(\frac{d}{dx_i}\right) = \xi_i$. Thus $\sigma|_{\mathfrak{t}_i}$ is linear for $1 \le i \le 3$ and (3) is valid. Now consider the action of $\Psi_{\mathrm{Id}}^{\mathrm{ia}} := \mathrm{H}_W/Z_{\mathrm{H}_W} \rtimes \widetilde{\mathrm{GL}}_V$ on $T^*V := V \times V$ given by

$$(a,b)w = (b,a)$$

 $(a,b)((v,\lambda),g) = ((a+v)g,(b+\lambda)g^{-t}).$

for $((a,b),((v,\lambda),g)) \in T^*V(R) \times \Psi_{\mathrm{Id}}^{\mathrm{ia}}(R)$. We have a closed immersion given on points by

$$\Psi_{\mathrm{Id}}^{\mathrm{s}}(R) \longrightarrow \Psi_{\mathrm{Id}}^{\mathrm{ia}}(R)$$
$$(\lambda, g) \longmapsto (((0, \lambda), 0), g).$$

Using this closed immersion to identify Ψ^s_{Id} with a subgroup of Ψ_{Id} we see that the action of $\Psi^{\mathrm{ia}}_{\mathrm{Id}}$ on T^*V does indeed extend the action of Ψ^s_{Id} on T^*V . This proves (4). One can now check (5) directly.

Proposition 8.17. When $\omega: X \to \operatorname{Sym}_{\langle,\rangle}^{\vee}$ is the map of (6.2.2) then Ansatz 8.14 is valid with $\Psi_{\omega}^{\mathrm{ia}} = \operatorname{Sp}_{W}$.

Proof. We have already verified (1) and (3) in the proof of Proposition 8.16. Due to our choice of pairing we may identify $J \in \text{Sym}_{\langle , \rangle}(R)$ with a symmetric $n \times n$ matrix. Then writing $a = (a_1, \ldots, a_n)$ the action of J on \mathfrak{x} is given by

$$(8.4.8) J.\left(\sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c\right) = \left(\sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i \left(\frac{\partial}{\partial x_i} - \pi i \alpha \frac{\partial}{\partial x_i} \left(x^t J x\right)\right) + c\right)$$

This implies (2).

We set $\Psi^{\text{ia}}_{\omega} := \operatorname{Sp}_{W}$ and identify Ψ^{s}_{ω} with the standard Siegel parabolic subgroup of $\Psi^{\text{ia}}_{\omega}$ in the usual manner. Then the standard representation of $\Psi^{\text{ia}}_{\omega}$ on $W = V \oplus V$ extends the action of Ψ^{s}_{ω} , yielding (4). One can now use (8.4.8) to check (5) directly.

Remark 8.18. Kontsevich and Belov-Kanel conjecture that the ind-group of all automorphisms of the Weyl Algebra and the group of Poisson-automorphisms of the commutative algebra $\mathbb{R}[x_1,\ldots,x_n;\xi_1,\ldots,\xi_n]$ are isomorphic [BKK05]. Above, we have only considered the relationship between certain subgroups of these automorphism groups. It is possible that a generalization of the Kontsevich Belov-Kanel conjecture holds for the ind-group of

automorphisms of \mathcal{D}_X and the Poisson-automorphisms of $\mathcal{O}_{T^*X^{\mathrm{sm}}}(T^*X^{\mathrm{sm}})$ for more general schemes (e.g. reductive monoids and horospherical spaces).

8.4.2. Quadric Cones. We shall now verify Ansatz 8.14 for the cones C_n of §7.

For comparison with [KM11] it is helpful to change the quadratic form defining our orthogonal group. We therefore use different notation from §7. Thus we write

$$(8.4.9) J_{p,q} \coloneqq \begin{pmatrix} I_p \\ -I_a \end{pmatrix}$$

and let $O_{p,q}$ be the associated orthogonal group, etc. We write

$$\epsilon_i = \begin{cases} 1 & \text{if } 1 \le i \le p \\ -1 & \text{if } p+1 \le i \le p+q \end{cases}$$

We assume p+q is even. Let $\mathcal{Q}_{p,q}$ be the quadratic form on $V_{p,q} := \mathbb{G}_a^{p+q}$ associated to $J_{p,q}$ and let $C_{p,q} \subset V_{p,q}$ be the vanishing locus of $\mathcal{Q}_{p,q}$.

We define a family of differential operators using the identification

(8.4.10)
$$\Gamma(C_{p,q}, \mathcal{O}_{C_{p,q}}) = \mathbb{R}[x_1, \dots, x_{p+q}]/\mathcal{Q}_{p,q}$$

Let

$$\Box \coloneqq \sum_{i=1}^{n} \epsilon_i \frac{\partial^2}{\partial x_i^2}$$

be the Laplace-Beltrami operator.

Consider the following operators:

- (1) x_i , $1 \le i \le p + q$
- (2) $X_{ij} := \epsilon_i \epsilon_j x_i \frac{\partial}{\partial x_j} x_j \frac{\partial}{\partial x_i}, \ 1 \le i < j \le p + q$
- (3) $E = \sum_{i} x_{i} \frac{\partial}{\partial x_{i}}$
- (4) $P_i := \epsilon_i x_i \Box \dot{-} (2E + p + q 2) \frac{\partial}{\partial x_i}, \ 1 \le i \le p + q$

These all define elements of \mathcal{D}_C [KM11, §1.1].

Let

$$\mathfrak{x} = \mathfrak{x}_2 \oplus \mathfrak{x}_1 \oplus \mathfrak{x}_0$$

where

(8.4.11)
$$\mathfrak{x}_{2} := \langle P_{1}, \dots, P_{p+q} \rangle$$

$$\mathfrak{x}_{1} := \langle E + \frac{p+q-2}{2}, \{ X_{ij} : 1 \le i < j \le p+q \} \rangle$$

$$\mathfrak{x}_{0} := \langle x_{1}, \dots, x_{p+q} \rangle$$

and the brackets denote the \mathbb{C} -span. The commutators of these elements in \mathcal{D}_C are computed in [KM11, §2.4].

Proposition 8.19. The space \mathfrak{x} is a Lie subalgebra of $(\mathcal{D}_C)_{\mathbb{C}}$ under the bracket and is isomorphic to $(\mathfrak{o}_{p+1,q+1})_{\mathbb{C}}$. The induced map $\mathcal{U}((\mathfrak{o}_{p+1,q+1})_{\mathbb{C}}) \to \mathcal{D}_C$ is surjective.

Proof. We observe that all nondegenerate quadratic forms on a complex vector space are equivalent. Thus the embedding $\mathfrak{x} \subset \mathcal{D}_{C\mathbb{C}}$ is equivalent to the embedding constructed in [Gon82, Example]. With this in mind the main result of [LSS88] implies the proposition. \square

As in §7.2.2, $S(C_{p,q}(\mathbb{R})) = L^2(C_{p,q}^{\circ}(\mathbb{R}))^{sm}$, where the superscript indicates vectors smooth under the action of $O_{p+1,q+1}(\mathbb{R})$.

Lemma 8.20. An $O_{p+1,q+1}(\mathbb{C})$ -conjugate of the action of $\mathfrak{o}_{p+1,q+1}$ on $\mathcal{S}(C_{p,q}(\mathbb{R}))$ via \mathcal{D}_C coincides with the infinitesimal action of the minimal representation.

Proof. By [KM11, (2.3.14), (2.3.19)] the action of x_i and P_i can be realized using a $O_{p,q}(\mathbb{C})$ conjugate of the infinitesimal action. Since these operators generate the Lie algebra by
[KM11, Lemma 2.4.8] we deduce the lemma.

Proposition 8.21. Ansatz 8.9 and 8.10 are valid for \mathcal{D}_C . One has

$$\mathcal{F}_{C_{p,q},\psi} \circ E \circ \mathcal{F}_{C_{p,q},\psi} = -(E+p+q-2) \qquad \qquad \mathcal{F}_{C_{p,q},\psi} \circ 4x_i \circ \mathcal{F}_{C_{p,q},\psi} = P_i$$

$$\mathcal{F}_{C_{p,q},\psi} \circ P_i \circ \mathcal{F}_{C_{p,q},\psi} = 4x_i \qquad \qquad \mathcal{F}_{C_{p,q},\psi} \circ X_{ij} \circ \mathcal{F}_{C_{p,q},\psi} = X_{ij}.$$

Proof. The first assertion follows immediately from Lemma 8.20 and the surjectivity statement in Proposition 8.19. The first three equalities at the end of the proposition are [KM11, Theorem 2.5.2(3)]. The last identity follows from the fact that X_{ij} is the differential of a particular element of $O_{p,q}(\mathbb{R})$ [KM11, §1.1], and hence commutes with $\mathcal{F}_{C_{p,q},\psi}$.

Let $\omega: C_{p,q} \to V_{p,q}$ be the canonical embedding and let $\Psi^{\text{ia}}_{\omega} = \mathcal{O}_{p+1,q+1}$. This group acts by conjugation on the minimal nilpotent orbit $\mathbb{O}_{p+1,q+1} \subset \mathfrak{o}_{p+1,q+1}$ and hence on its closure $\overline{\mathbb{O}}_{p+1,q+1}$ in $\mathfrak{o}_{p+1,q+1}$. We therefore obtain an action on $\overline{T^*C_{p,q}^{\circ}}$ using Proposition 7.2.

Proposition 8.22. For the canonical embedding $\omega: C_{p,q} \to V_{p,q}$ Ansatz 8.14 holds with $\Psi^{\text{ia}}_{\omega}$ and \mathfrak{x} as above.

Proof. Statement (1) is part of Proposition 8.19, and (2) is a consequence of Lemma 8.20. The symbol map is \mathbb{C} -linear on the subspaces \mathfrak{x}_i . Let us describe it explicitly. As in the proof of Proposition 7.2

$$T^*C_{p,q}^{\circ}(\mathbb{R}) = C_{p,q}^{\circ}(\mathbb{R}) \times V_{p,q}(\mathbb{R})/\mathbb{R}$$

where the implied action of \mathbb{R} is given in (7.0.8). We consider the ring of all differential operators on $V_{p,q}$ (resp., the ring of all functions $T^*V_{p,q} \cong V_{p,q} \times V_{p,q}$), and we may restrict to the subring of such operators which are tangent to $C_{p,q}$ (resp., those subring of functions that descend to $T^*C_{p,q}^{\circ}$). Under this identification, the principal symbol of the operator $\partial/\partial x_i$ is v_i , the corresponding coordinate function on $V_{p,q}$. Therefore

(8.4.12)
$$\sigma(P_i) = \epsilon_i x_i \mathcal{Q}_{p,q}(v) - 2(x \cdot v) v_i$$

(8.4.13)
$$\sigma\left(E + \frac{p+q-2}{2}\right) = x \cdot v$$

(8.4.14)
$$\sigma(X_{ij}) = \epsilon_i \epsilon_j x_i v_j - x_j v_i$$

$$(8.4.15) \sigma(x_i) = x_i,$$

where \cdot denotes the standard dot product. We let \mathfrak{x}' denote the \mathbb{R} -span of (8.4.12)-(8.4.15). As a Lie algebra under the Poisson bracket, we have by construction that $\mathfrak{x}' \cong \mathfrak{x}$, so by Proposition 8.19, $\mathfrak{x}' \cong \mathfrak{o}_{p+1,q+1}$.

Under the embedding $T^*C_{p,q}^{\circ}(\mathbb{R}) \hookrightarrow \mathbb{O}_{p+1,q+1}(\mathbb{R})$, written explicitly in (7.0.10) (with respect to a different basis) we find that (8.4.15) gives the *i*th coordinate of c, (8.4.14) gives $((-J_n v)_i c_j + (J_n c^t)_i v_j)_{ij}$, (8.4.13) gives $\langle c, v \rangle$, and (8.4.12) gives the *i*th coordinate of $J_n c^t \mathcal{Q}_n(v) - \langle c, v \rangle_n J_n v^t$. Thus the functions (8.4.12)-(8.4.15) are the pullback of the standard matrix coordinates of $\mathfrak{o}_{p+1,q+1}$ to $T^*C_{p,q}^{\circ}$ under the map $\overline{a}: T^*C_{p,q}^{\circ} \to \mathbb{O}_{p+1,q+1}$ constructed in the proof of Proposition 7.2. Since \overline{a} is an isomorphism on affinizations, and the matrix coordinates of $\mathfrak{o}_{p+1,q+1}$ generate the ring of regular functions on the closed subset $\overline{\mathbb{O}}_{p+1,q+1} \subset \mathfrak{o}_{p+1,q+1}$, we see that the principal symbols (8.4.12)-(8.4.15) generate all regular functions on $\overline{T^*C_{p,q}^{\circ}}$. This proves (3).

The ind-group-scheme $\Psi^{\text{ia}}_{\omega}$ in (4) is $O_{p+1,q+1}$, which acts on $\overline{T^*C^\circ} \cong \overline{\mathbb{O}}_{p+1,q+1}$ via the manifest conjugation action on the minimal nilpotent orbit. Using (7.0.9) we see that the small modulation group corresponds to the action of Q_i^{op} , up to a conjugation due to the fact that we are using a different quadratic form.

The action of $O_{p+1,q+1}$ on $\overline{T^*C_{p,q}^{\circ}}$ induces an action of $O_{p+1,q+1}$ on \mathfrak{x}' that is just matrix conjugation. On the other hand up to conjigation by an element of $O_{p+1,q+1}(\mathbb{C})$ the action of $\Psi_{\omega}\{\mathbb{R}\}$ on \mathcal{D}_C is induced by the conjugation action on $\mathfrak{x} = (\mathfrak{o}_{p+1,q+1})_{\mathbb{C}}$ by Lemma 8.20. Assertion (5) follows.

9. The boundary of the Schwartz space and small modulation groups

Assume that we are in the situation of §3.5, with a Schwartz space satisfying (S1), (S2), and (S4). We assume that X admits an open dense H-orbit $X^{\circ} \subset X$.

Definition 9.1. The **boundary** of the Schwartz space $\mathcal{S}(X(F),\mathcal{L}^{1/2})$ is the quotient

$$\mathcal{S}(X(F),\mathcal{L}^{1/2})/\mathcal{S}(X^{\circ}(F),\mathcal{L}^{1/2}).$$

Kazhdan suggested to one of us (Getz) that the quotient $S(X(F), \mathcal{L}^{1/2})/S(X^{\circ}(F), \mathcal{L}^{1/2})$ is the local avatar of boundary terms in the Poisson summation formula; we refer to §10 below for more details.

In §3.3 we constructed an action of $\Psi^{\rm s}_{\omega}$ on $T^*X^{\rm sm}$. It extends to the affine closure $\overline{T^*X^{\rm sm}}^{\rm aff}$. We pose the following imprecise conjecture:

Conjecture 9.2. Assume ω is a closed immersion. Then there is a correspondence between $\Psi^{\rm s}_{\omega}$ -orbits on $\overline{T^*X^{\rm sm}}$ and subquotients of $\mathcal{S}(X(F),\mathcal{L}^{1/2})$ as a $\Psi^{\rm s}_{\omega}(F)$ -module.

We do not have a conjectural formulation of the correspondence at the moment. We content ourselves with discussing a pair of examples.

9.1. **Vector spaces.** We consider the setting of §6.1. Thus let GL_V act on V. As in (6.1.1) $\Psi_{id}^s = V^{\vee} \rtimes GL_V$.

One can directly verify the following two lemmas:

Lemma 9.3. One has a decomposition

$$T^*V\coloneqq V\oplus V^\vee=V^\circ\oplus V^\vee | \ |\{0\}\oplus V^\vee$$

into $\Psi^{\rm s}_{\rm id}$ -orbits.

Lemma 9.4. One has an exact sequence

$$0 \longrightarrow \mathcal{S}(V^{\circ}(F)) \longrightarrow \mathcal{S}(V(F)) \longrightarrow \mathbb{C} \longrightarrow 0$$

of
$$\Psi^{\rm s}_{\rm id}(F)$$
-modules.

This example is suggestive, but the reader may complain that if we wanted to find a geometric analogue of the decomposition in Lemma 9.4 then one could just use the orbits of GL_V on V. This naïve approach does not work in the singular case, as we will demonstrate by example in the next subsection.

9.2. Quadric cones. We now turn to the setting of §7. We assume for simplicity that the quadratic form Q_n is split. Thus C_n is the zero locus of Q_n inside the space \mathbb{G}_a^{2n} , and $\omega: C_n \to \mathbb{G}_a^{2n}$ is the inclusion.

The following was proved in [Tom25]:

Theorem 9.5. For n > 1 there is a $\Psi_{\omega}^{\mathbf{s}}$ -equivariant decomposition into four subschemes

$$(9.2.1) \overline{T^*(C_n^\circ)}^{\text{aff}} = T^*(C_n^\circ) \sqcup C_n^\circ$$

(9.2.2)
$$\sqcup \left(\bigcup_{k=1}^{n-1} T^*(C_k) \times \mathbb{G}_a^{2(n-k)} \cup \bigcup_{k=1}^{n-1} C_2^{\circ} \times \mathbb{G}_a^{2(n-k)} \right) \sqcup \{0\}.$$

Thus $\overline{T^*(C_n^\circ)}^{\text{aff}}$ has a nested structure. Specifically, the complement of $T^*(C_n^\circ)$ in its affine closure contains nested cotangent spaces $T^*(C_k^\circ)$ and nested smooth loci C_k° for all $1 \le k \le n$ together with the origin.

The Schwartz space admits an analogous decomposition [GK23, Theorem 1.2]:

Theorem 9.6. Assume F is non-Archimedean and that $n \geq 3$. One has an exact sequence of $\Psi^s_{\omega}(F)$ -modules

$$0 \longrightarrow \mathcal{S}(C_n^{\circ}(F)) \longrightarrow \mathcal{S}(C_n(F)) \longrightarrow \mathcal{S}(C_{n-1}(F)) \oplus \mathbb{C} \longrightarrow 0.$$

We suspect that $\mathcal{S}(C_n^{\circ}(F))$ should correspond to $T^*(C_n^{\circ})$ in (9.2.1), $\mathcal{S}(C_{n-1}(F))$ should correspond to the terms in (9.2.2), and the constant quotient \mathbb{C} should correspond to C_n° in (9.2.1). Unfortunately we do not yet know what we mean by "correspond."

10. The Poisson summation conjecture

In this section we discuss the global theory, so F is a number field. We place ourselves in the general setting of $\S 3$ and assume (M1), (M2), (M3), (M4).

We assume for simplicity that there is an open dense H-orbit $X^{\circ} \subset X$. We assume moreover that X° admits a nowhere vanishing section of the canonical bundle that transforms under the action of H by a character. For each place v the nonvanishing section gives rise to an $H(F_v)$ -eigenmeasure dx_v on $X^{\circ}(F_v)$. Using this we identify the local Schwartz spaces $S(X(F), \mathcal{L}^{1/2})$ with spaces of functions $S(X(F_v)) < L^2(X(F_v)) := L^2(X(F_v), dx_v)$ as in Remark 3.1.

We assume Ansatz 3.14. Thus we have local Fourier transforms $\mathcal{F}_X: \mathcal{S}(X(F)) \to \mathcal{S}(X(F))$. Let S be a finite set of places of F including the infinite places. By an \mathcal{O}_F^S -model of a scheme Z of finite type over F we mean a flat \mathcal{O}_F^S -scheme of finite type whose generic fiber is Z. By abuse of notation, we continue to denote the model by Z. We choose \mathcal{O}_F^S models of all of the schemes appearing in §3 and assume that relevant morphisms all extend over \mathcal{O}_F^S . Thus, in particular, we have an action $X \times H \longrightarrow X$ over \mathcal{O}_F^S . We extend the open H-orbit $X^{\circ} \subset X$ over \mathcal{O}_F^S by taking the schematic closure in X.

For $v \notin S$ we assume the existence of basic functions

$$(10.0.1) b_{X_{F_v}} \in \mathcal{S}(X(F_v))^{H(\mathcal{O}_{F_v})}$$

such that

- (b1) $X^{\circ}(\mathcal{O}_{F_v}) \subseteq \operatorname{supp}(b_{X_{F_v}}) \subseteq X(\mathcal{O}_{F_v})$
- $(b2) \ b_{X_{F_n}}|_{X^{\circ}(\mathcal{O}_{F_n})} = 1$
- (b3) $\mathcal{F}_X(b_{X_{F_v}}) = b_{X_{F_v}}$.

We assume that for $v \mid \infty$ the space $\mathcal{S}(X(F_v))$ is a Fréchet space. We define

(10.0.2)
$$S(X(\mathbb{A}_F)) = \widehat{\otimes}_{v|\infty} S(X(F_v)) \otimes \otimes'_{v+\infty} S(X(F_v))$$

where the hat denotes the completed projective tensor product and the restricted tensor product is with respect to the basic functions $b_{X_{F_n}}$. Throughout this section we assume that

$$\sum_{\gamma \in X^{\circ}(F)} |f(\gamma)| < \infty$$

for all $f \in \mathcal{S}(X(\mathbb{A}_F))$.

Ansatz 10.1 (The Poisson summation formula). Let v_1 and v_2 be places of F (not necessarily distinct). If $f = f_{v_1} f_{v_2} f^{v_1 v_2} \in \mathcal{S}(X(\mathbb{A}_F))$ where $f_{v_1} \in \mathcal{S}(X^{\circ}(F_{v_1}))$ and $\mathcal{F}_X(f_{v_2}) \in \mathcal{S}(X^{\circ}(F_{v_2}))$

then

$$\sum_{\gamma \in X^{\circ}(F)} f(\gamma) = \sum_{\gamma \in X^{\circ}(F)} \mathcal{F}_X(f)(\gamma).$$

Conjecture 10.2 (Braverman and Kazhdan). Ansatz 10.1 is true when X is a reductive monoid as in $\S4$.

As stated in [BK00], this conjecture implies the functional equation of the Langlands L-function attached to ρ . Combining this with the converse theorem [CPS99], we see that Conjecture 10.2 implies much of Langlands functoriality.

The assumption on f in Ansatz 10.1 is unnatural. For every $f \in \mathcal{S}(X(\mathbb{A}_F))$ one should have

(10.0.3)
$$\sum_{\gamma \in X^{\circ}(F)} f(\gamma) + \operatorname{BT}_{X}(f) = \sum_{\gamma \in X^{\circ}(F)} \mathcal{F}_{X}(f)(\gamma) + \operatorname{BT}_{X}(\mathcal{F}_{X}(f))$$

where $BT_X : \mathcal{S}(X(\mathbb{A}_F)) \to \mathbb{C}$ is a linear functional that is reasonably explicit and related to the geometry of X. We refer to the linear functionals BT_X as **boundary terms**. These linear functionals remain mysterious at the moment, and it is an important problem to describe them geometrically. We refer loosely to an identity of the form (10.0.3) with an explicit, geometric description of the boundary terms $BT_X(f)$ as a **full Poisson summation formula**. The full Poisson summation conjecture is known for horospherical varieties as in §5 when G is a classical group or G_2 [BK02, CG25, Hsu21]. Apart from the case of matrices, the only case where we have a full Poisson summation formula with a completely geometric description of the boundary terms is the case of the cone considered in §7. We discuss this in §10.1.2 below.

Let us explain the relationship between global boundary terms and the boundary of the Schwartz space, as explained to one of us (Getz) by Kazhdan. Let v be a place of F. Assuming Conjecture 10.2, for any fixed $f^v \in \mathcal{S}(X(\mathbb{A}_F^v))$ one has a well-defined functional

$$S(X(F_v))/S(X^{\circ}(F_v)) \cap \mathcal{F}_X^{-1}(S(X^{\circ}(F_v))) \longrightarrow \mathbb{C}$$

$$f_v \longmapsto \sum_{\gamma \in X^{\circ}(F)} f_v f^v(\gamma) - \sum_{\gamma \in X^{\circ}(F)} \mathcal{F}_X(f_v f^v)$$

Studying this functional is equivalent to studying the functional $f_v \mapsto \operatorname{BT}_X(\mathcal{F}_{\rho}(f_v f^v)) - \operatorname{BT}_X(f_v f^v)$. On the other hand we have a series of inclusions

$$S(X(F_v)) \ge S(X^{\circ}(F_v)) \ge S(X^{\circ}(F_v)) \cap \mathcal{F}_X^{-1}(S(X^{\circ}(F_V))).$$

Thus understanding the structure of the boundary $S(X(F_v))/S(X^{\circ}(F_v))$ would at least shed some light on the boundary terms.

10.1. Boundary terms in our two examples.

10.1.1. Vector spaces. If X = V then one defines

$$\mathrm{BT}_V(f)\coloneqq f(0).$$

The full Poisson summation formula (10.0.3) is just the usual Poisson summation formula.

10.1.2. Quadric cones. We briefly recall the main theorem of [Get25]. In loc. cit. the first author defined operators

$$(10.1.1) I: \mathcal{S}(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \longrightarrow C^{\infty}(C_i^{\circ}(\mathbb{A}_F))$$

for i > 0, together with operators

$$c_i: \mathcal{S}(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \longrightarrow \mathbb{C}$$
$$d_{i,i'}: \mathcal{S}(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \longrightarrow \mathcal{S}(V_{i'}(\mathbb{A}_F) \oplus \mathbb{A}_F^2)$$

for $i > i' \ge 0$. Briefly, $c_i(f)$ is the regularized value of I(f) at $0 \in C_i(F)$. On the other hand $d_{i,i-1}$ is given by a partial Fourier transform and then restriction to the complement of a hyperbolic plane. One then sets $d_{i,i'} = d_{i'+1,i'} \circ \cdots \circ d_{i,i-1}$. By convention, $d_{i,i}$ is the identity. We let

$$\mathcal{F}_{\wedge}:\mathcal{S}(\mathbb{A}^2_F)\longrightarrow\mathcal{S}(\mathbb{A}^2_F)$$

be the usual $SL_2(\mathbb{A}_F)$ -equivariant Fourier transform.

The main theorem of [Get25] follows:

Theorem 10.3. For $f \in \mathcal{S}(V_n(\mathbb{A}_F) \oplus \mathbb{A}_F^2)$ the sum

(10.1.2)
$$\sum_{\xi \in C_n^{\circ}(F)} I(f)(\xi) + c_n(f)$$

$$(10.1.3) + \sum_{i=1}^{n-1} \left(c_i(d_{n,i}(f)) + \sum_{\xi \in C_i^{\circ}(F)} I(d_{n,i}(f))(\xi) \right) + \kappa d_{n,0}(f)(0_{V_0}, 0, 0)$$

is invariant under $f \longmapsto (1_{\mathcal{S}(V_i(\mathbb{A}_F))} \otimes \mathcal{F}_{\wedge})(f)$. Here κ is a suitable constant.

There is some description of the boundary terms in the setting of Braverman-Kazhdan spaces and generalized Schubert varieties in [CG25], but Theorem 10.3 is the only case beyond vector spaces where one has a manifestly geometric expression for the boundary terms.

There is a qualitative connection between the various terms in Theorem 10.3 and the subschemes in Theorem 9.5. The sum in (10.1.2) corresponds to $T^*C_n^{\circ}$, $c_n(f)$ corresponds to C_n° , and (10.1.3) corresponds to (9.2.2). Just as in the local setting, we do not yet have a conjectural understanding of what we should mean by "corresponds."

11. Automorphic representations of modulation groups

We continue to work under the assumptions of the previous section. Choose a character $\chi: \Psi^s_{\omega}(\mathbb{A}_F) \to \mathbb{C}^{\times}$ trivial on $\Psi^s_{\omega}(F)$. We assume that $\chi|_{\Psi^s_{\omega}(\widehat{\mathcal{O}}_F^S)} = 1$. For $v \notin S$ let

$$(11.0.1) K_v := \langle \mathcal{F}_{\omega}, (\mathcal{R}_{\omega} \otimes \chi)(\Psi_{\omega}^{\mathrm{s}}(\mathcal{O}_{F_v})) \rangle \leq \mathrm{Aut}(L^2(X(F_v)), \mathcal{L}^{1/2}).$$

We then define the restricted direct product

(11.0.2)
$$\Psi_{\omega}\{\mathbb{A}_F\} \coloneqq \prod_{v}' \Psi_{\omega}\{F_v\}$$

with respect to the K_v for $v \notin S$. We also have a subgroup

(11.0.3)
$$\Psi_{\omega}\{F\} = \langle \mathcal{F}_X, \Psi_{\omega}^{\mathbf{s}}(F) \rangle \leq \Psi_{\omega}\{\mathbb{A}_F\}.$$

Here the implicit map from $\Psi^{\mathbf{s}}_{\omega}(F)$ into $\Psi^{\mathbf{s}}_{\omega}(\mathbb{A}_F)$ is the diagonal embedding.

We point out that for almost all v the group K_v acts trivially on the basic function $b_{X_{F_v}} \in \mathcal{S}(X(F_v))$. Thus we have a representation

(11.0.4)
$$\mathcal{R}_{\omega}: \Psi_{\rho}\{\mathbb{A}_F\} \times \mathcal{S}(X(\mathbb{A}_F)) \longrightarrow \mathcal{S}(X(\mathbb{A}_F)).$$

Proposition 11.1. Assume that one has a $\Psi^s_{\omega}(F)$ -invariant linear functional

$$\mathrm{BT}:\mathcal{S}(X(\mathbb{A}_F))\longrightarrow\mathbb{C}.$$

For $(h, f) \in \Psi_{\omega}(\mathbb{A}_F) \times \mathcal{S}(X(\mathbb{A}_F))$ consider the function

(11.0.5)
$$\Theta_f(h) \coloneqq \sum_{\gamma \in X^{\circ}(F)} \mathcal{R}_{\omega}(h) f(\gamma) + \mathrm{BT}(\mathcal{R}_{\omega}(h)(f)).$$

Then $\Theta_f(h)$ is left $\Psi_{\omega}\{F\}$ -invariant if and only if

(11.0.6)
$$\sum_{\gamma \in X^{\circ}(F)} f(\gamma) + \mathrm{BT}(f) = \sum_{\gamma \in X^{\circ}(F)} \mathcal{F}_{\omega}(f)(\gamma) + \mathrm{BT}(\mathcal{F}_{\omega}(f)).$$

that is, the full Poisson summation conjecture holds.

Proof. If $\Theta_f(h)$ is $\Psi_{\omega}\{F\}$ -invariant then we obtain (11.0.6) by taking h to be the identity and then $h = \mathcal{F}_{\omega}$.

Conversely, suppose that (11.0.6) holds. Then $\Theta_f(h)$ is invariant under the subgroup generated by H(F) and \mathcal{F}_{ω} , and the term $\mathrm{BT}(\mathcal{R}_{\omega}(h)f)$ is invariant under $\Psi^{\mathrm{s}}_{\omega}$ by assumption. The sum $\sum_{\gamma \in G(F)} \mathcal{R}_{\omega}(h)(f)(\gamma)$ is obviously invariant under H(F), and it is invariant under the action of $V^{\vee}(F)$ because $\psi(\lambda(\omega(\gamma))) = 1$ for $(\lambda, \gamma) \in V^{\vee}(F) \times X^{\circ}(F)$.

Thus assuming the full Poisson summation conjecture it is reasonable to regard $\Theta_f(h)$ as an automorphic function on $\Psi_{\omega}\{F\}\setminus\Psi_{\omega}\{A_F\}$, and it is reasonable to view

$$\{\Theta_f(g): f \in \mathcal{S}(M_\rho(\mathbb{A}_F))\}$$

as an automorphic representation of $\Psi_{\omega}\{\mathbb{A}_F\}$. Of course the notion of an automorphic representation of $\Psi_{\omega}\{\mathbb{A}_F\}$ has yet to be defined.

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