On Acyclic Subgraphs and Transient Length Bounds

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Abstract

An important problem in the theory of finite dynamical systems is understanding how the structure of a system determines its dynamics. In this paper, we study acyclic subgraphs contained in dependency graphs induced by boolean monomial systems. Results by O. Colón, et al., relate the number of state transitions to walks in the dependency graph of the system. We are interested in the role that acyclic subgraphs play in the behavior of the system. In particular, we have observed that for certain acyclic graphs, we can obtain a family of linear equations whose solution appears to provide insight into transient lengths.

Introduction

Systems and Dependency Graphs

Let $f = (f_1, ..., f_n) : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be a **boolean monomial parallel update system**, where the monomial functions are of the form

$$f_i = \alpha_i x_1^{\varepsilon_{1i}} \cdots x_n^{\varepsilon_{ni}}$$

where $\alpha_i \in \{0, 1\}$ and $\varepsilon_{ji} \in \{0, 1\}$. Let $f^m := f \circ f \circ \cdots \circ f$ be the m-fold composition of the self-map f. By definition we have

$$f_i^m = \alpha_i (f^{m-1}{}_1)^{\varepsilon_{1i}} \cdots (f^{m-1}{}_n)^{\varepsilon_{ni}}$$

With f we associate a **dependency graph** $\mathcal{D}_f = (V, E)$ with vertex set $V = \{a_1, ..., a_n\}$ and a set of directed edges $E \subseteq V \times V$. There is a directed edge $a_i \to a_j$ if $\alpha_i = 1$ and x_j is a factor in f_i. An example below.

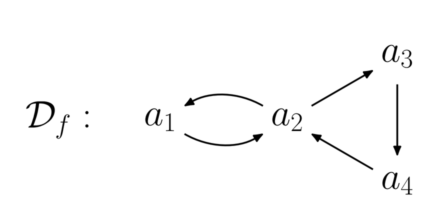

Figure 1: Here $\mathcal{D}_f$ is given by the function $f = (x_2, x_1x_3, x_4, x_2)$.

State Spaces

The dynamics of f are encoded in its **state space** $\mathcal{S}(f)$, a digraph whose vertices are the 2^n elements of $\mathbb{F}_2^n$ and with a directed edge from a to b if $f(a) = b$.

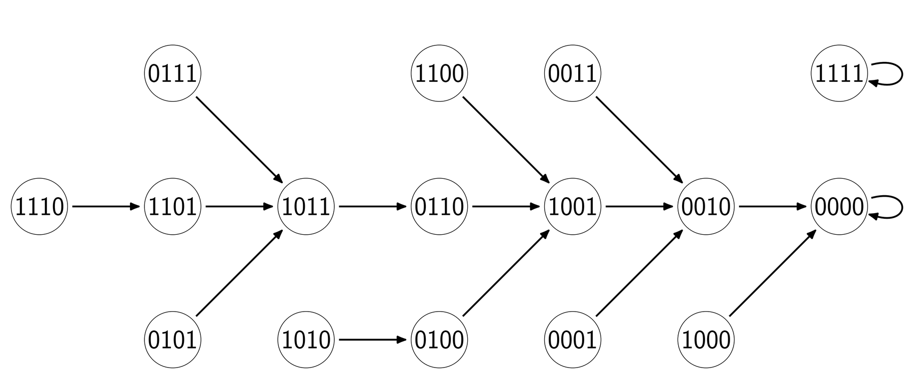

Figure 2: The system f from Fig. 1 induces the following state space.

Results

Transient of a System

Because $\mathbb{F}_2^n$ is finite, there exist minimal integers $t \geq 0$ and $p \geq 1$ such that $f^{t+p}(a) = f^t(a)$. Here, p is the period (the length of the cycle) and t is called the **transient length** of a, i.e., the smallest number of iterations needed for the orbit of a to enter its eventual cycle. Equivalently, the chain $a \longrightarrow f(a) \longrightarrow \cdots \longrightarrow f^{t-1}(a)$ constitutes the transient part of the orbit of a. One can define the system's overall transient as the maximum transient length over all states

$$T(f) = \max_{a \in \mathbb{F}_2^n} \{t(a)\}$$

The **transient states** of the system are those states in $\mathcal{S}(f)$ that do not lie on a cycle. In other words, a state a is transient if its transient length t is positive (i.e., if a is not already on a cycle).

Maximally Acyclic Subgraphs and Walks

Suppose $G = (V, E)$ is a non-empty and strongly connected digraph. We say that a subgraph $H \subseteq G$ is a **maximally acyclic subgraph** of G if H spans the vertex set of G and contains no directed cycles. That is, $V(H) = V(G)$ and for any edge $e \in G$ such that $e \notin H$ the graph $H + e$ contains a cycle.

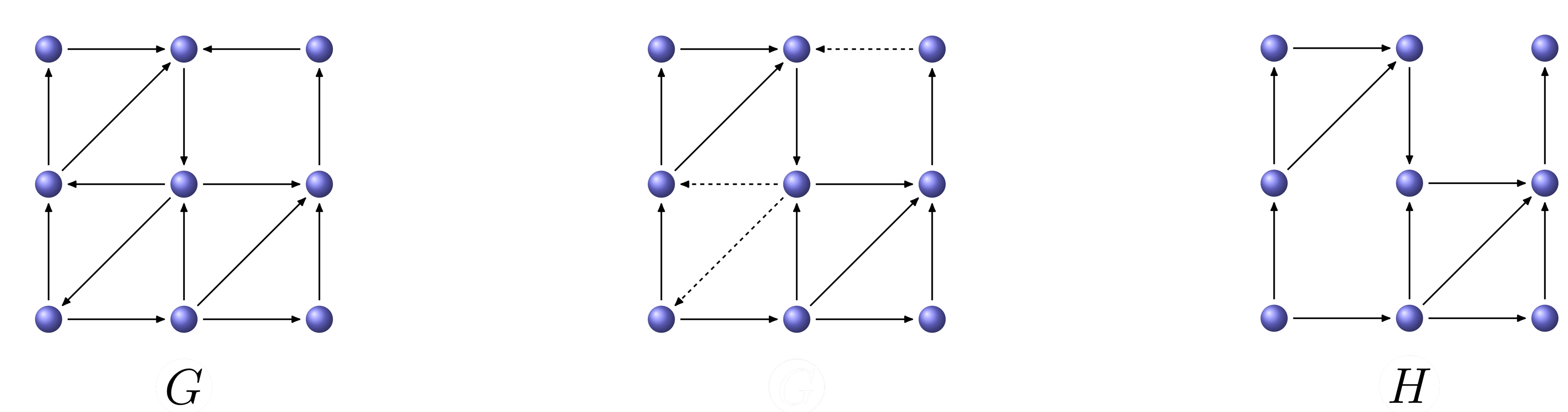

Figure 3: H is a maximally acyclic subgraph of G.

We study how these maximally acyclic subgraphs enable a length characterization for arbitrary walks in a digraph. Formally, for any strongly connected digraph G with maximally acyclic subgraph H let $c_1, c_2, \ldots, c_n$ be the distinct lengths of cycles contained in G and define $\langle c_1, c_2, \ldots, c_n \rangle = \{c_1x_1 + c_2x_2 + \cdots + c_nx_n : x_i \in \mathbb{Z}_{\geq 0}\}$. Then length of any walk $p : u \to v$ is given by the following

$$|p| = \langle c_1, c_2, \ldots, c_n \rangle + j$$

where $j = |uHv|$ for some $u - v$ path on H.

Example. Take G and let H be a maximally acyclic subgraph rooted at the vertex x. Any $x - y$ walk p attains length $|p| = \langle 2, 3, 5 \rangle + j = 2x + 3y + 5z + 3$. And any $x - z$ walk q attains length $|q| = \langle 2, 3, 5 \rangle + j = 2x + 3y + 5z + 4$.

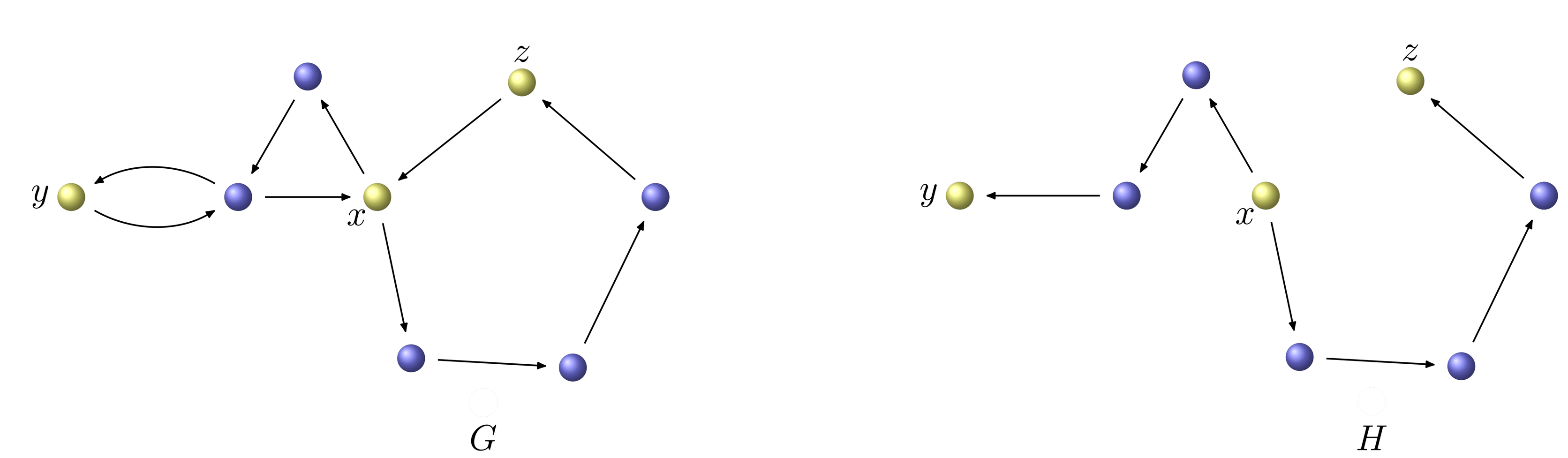

Figure 4: H is a maximally acyclic subgraph of G rooted at vertex x.

Consider the $x - y$ walk p of length 16 given by $|p| = 2(0) + 3(1) + 5(2) + 3$. Also, consider the $x - z$ walk q of length 17 given by $|q| = 2(1) + 3(2) + 5(1) + 4$. Motivated by a well-known result by O. Colón that relates the number of state transitions to walks in the dependency graph of the system. For a system f assume that f_i^m is non-zero, then there exists a walk $p : a_i \to a_j$ in $\mathcal{D}_f$ of length m if and only if x_j divides f_i^m. We relate the latter characterization of walks with the maximum transient length of a system.

Future Work

Transient Length Bounds

Consider a system f with dependency graph $\mathcal{D}_f$ and some maximally acyclic subgraph $H \subseteq \mathcal{D}_f$. Let the cycle lengths be $c_1, c_2, \ldots, c_n$ and define

$$\Gamma = \langle c_1, c_2, \ldots, c_n \rangle$$

If g is the Frobenius number of Γ, then every integer $\geq g+1$ appears as a walk length in $\mathcal{D}_f$. With H providing acyclic shifts $J \subset \mathbb{Z}$. Any walk of the form $p : u \to v$ attains length

$$|p| = \ell + j, \quad \ell \in \Gamma,\ j = |uHv|$$

We can ensure that $|p| - j = \ell \geq g+1$ is smallest for $j \in J$ when $j = \text{diam}(\mathcal{D}_f)$. Suppose v is a fixed vertex and u is any other vertex in $\mathcal{D}_f$, then if $|p|$ is minimal the transient length is bounded by

$$T(f) \geq 1 + g + \text{diam}(\mathcal{D}_f)$$

A lower bound for the maximum transient. Every state with walk length at least $1 + g + \text{diam}(\mathcal{D}_f)$ is reached in every shifted system. Similarly, we can derive an upper bound. Observe that there exists a walk $q : u \to v$ with $|q| \leq 2 \cdot \text{diam}(\mathcal{D}_f)$ for any $u, v \in V(\mathcal{D}_f)$. Assume that v is a fixed vertex and u is any other vertex. If $|q|$ is maximal then the transient length is bounded by

$$T(f) \leq g + 2 \cdot \text{diam}(\mathcal{D}_f)$$

Example. Take the system f where $\mathcal{D}_f \cong C_2 \vee C_3 \vee C_5$. Because $\text{diam}(\mathcal{D}_f) = 7$ and the Frobenius number $g = 1$ of the semigroup $\langle 2, 3, 5 \rangle$ generated by the cycle lengths bounds the transient of our system $9 \leq T(f) \leq 15$.

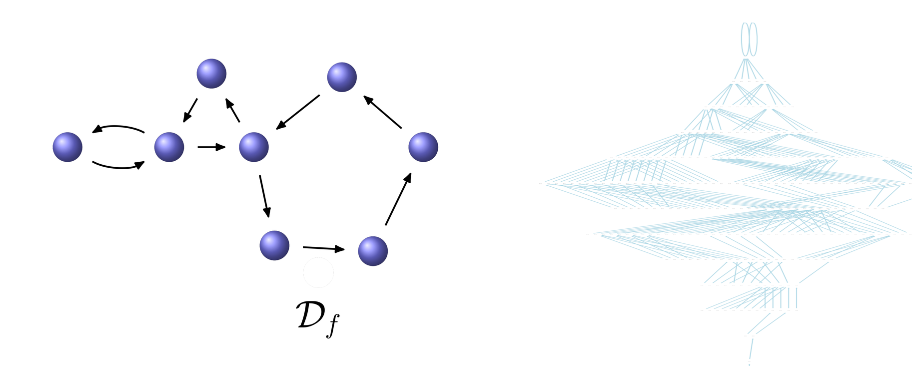

Figure 5: The dependency graph of f and its state space.

Observations

The structure of $H \subseteq \mathcal{D}_f$ and the Frobenius number of the semigroup Γ, set a boundary for transient lengths. In particular, the family of linear equations derived from any rooted H reflect how transient states emerge, offering both lower and upper bounds. A closed formula for $T(f)$ under an arbitrary system remains an open problem.

References

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