Arithmetical Structures on Cycles with a Multi-Edge

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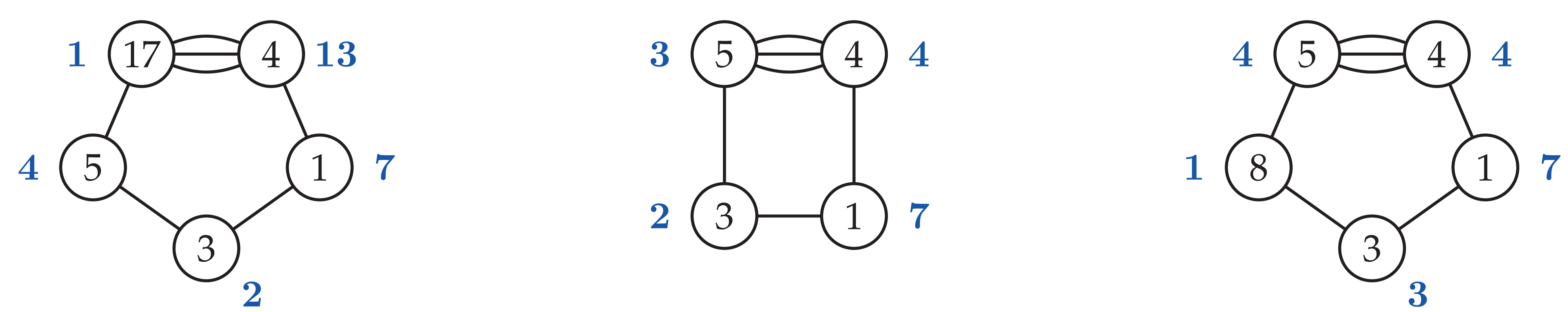
Definitions

An arithmetical structure on a finite, connected graph G is an assignment of positive integers to the vertices such that:

- The labeling of each vertex divides the sum of the labels at adjacent vertices.
- The greatest common divisor among all labels is 1.

The labels used are typically written in vector notation and denoted as the $\mathbf{r}$ vector of the arithmetical structure. The factors by which each label divides the sum of neighbors are denoted as the $\mathbf{d}$-labels.

Examples of Arithmetical Structures on $\widetilde{C}_{n,3}$



Main Theorem

Let $n \geq 2$ and fix k. Then the number of arithmetical structures on a cycle with a k-multiedge $\widetilde{C}_{n,k}$ is

$$\left|\mathrm{Arith}(\widetilde{C}_{n,k})\right| = \sum_{m=2}^{n} \sum_{j=1}^{m} B(n-2,\, n-m)\, 2^{m-j} \cdot \left|\mathrm{SSArith}(\widetilde{C}_{j,k})\right|,$$

where $|\mathrm{SSArith}(\widetilde{C}_{n,k})|$ is the set of arithmetical structures $(\mathbf{d}, \mathbf{r})$ on $\widetilde{C}_{n,k}$ such that $\min(\mathbf{d}) > 1$.

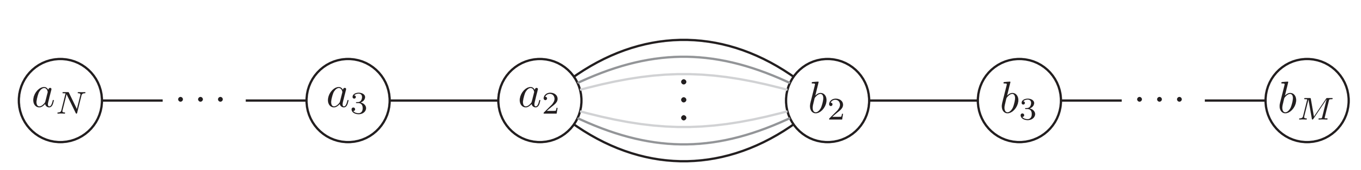
Generating Structures

Let a_1 and a_2 be integers. We define the decreasing sequence $F(a_1, a_2) = (a_1, a_2, \ldots, a_N)$ by setting

$$a_{i+1} \equiv -a_{i-1} \pmod{a_i}$$

for each $i \geq 2$ where N is the largest index such that $a_N > 0$. It is known that $a_N = \gcd(a_1, a_2)$.
Examples: $F(13,6) = (13,6,5,4,3,2,1)$, $F(20,14) = (20,14,8,2)$, $F(3 \cdot 4, 5, 3, 1)$, and $F(3 \cdot 5, 4, 1)$.

Theorem. *We can construct a pair* $(\mathbf{d}, \mathbf{r}) \in \mathrm{Arith}(\widetilde{C}_{n,k})$ *if and only if* r_1, r_2, *and* k *are pair-wise coprime.*

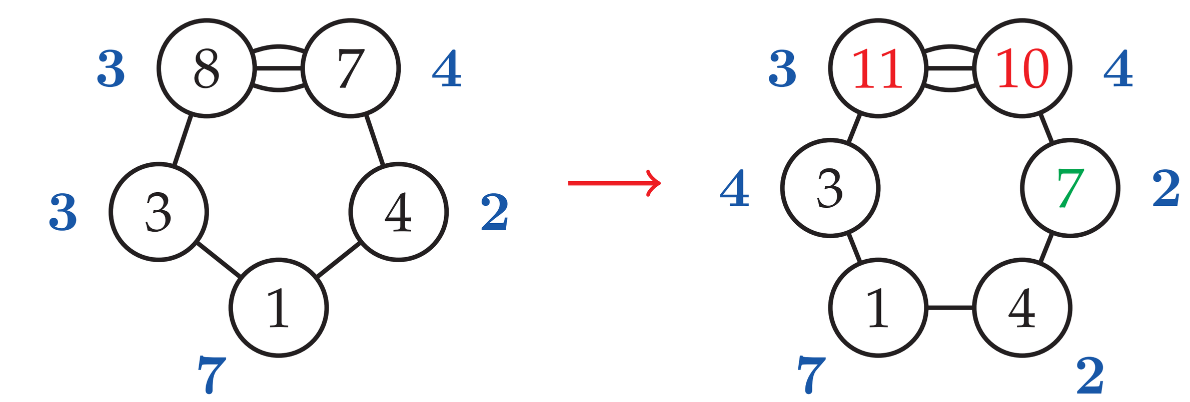


Bounds

Let $(\mathbf{d}, \mathbf{r}) \in \mathrm{SSArith}(\widetilde{C}_{n,k})$ with $r_1 > r_2$, then $d_1 \in \{2, 3, \ldots, k\}$, and $d_2 > \left\lfloor \frac{k^2}{d_1} + 1 \right\rfloor$.

Push Operation

The push operation is a operation performed on strongly smooth arithmetical structures on $\widetilde{C}_{n,k}$ graphs with $d_1 = k$ and $d_3, d_4, \ldots, d_n \neq 1$.



Flip Operation

The flip operation is an operation performed on strongly smooth arithmetical structures where $d_1 = k$ and $d_2 = k+2$. We add the difference between r_2 and r_{n-1} to r_1' and flip the positions of $r_2 \ldots r_{n-1}$.

$$[\ 31\ \ 22\ \ 17\ \ 12\ \ 7\ \ 2\ \ 3\ \ 7\ \ 11\ \ 15\ \ 19\ \ 23\ \ 27\ ]$$
$$\Downarrow$$
$$[\ 32\ \ 23\ \ 19\ \ 15\ \ 11\ \ 7\ \ 3\ \ 2\ \ 7\ \ 12\ \ 17\ \ 22\ \ 27\ ]$$

Data for the number of structures on $\widetilde{C}_{n,3}$

Strongly smooth structures with less than two 1's in their $\mathbf{r}$ vector and $r_1 > r_2$. All values are conjectured.

n	SArith	SSArith	n	SArith	SSArith	n	SArith	SSArith	n	SArith	SSArith
3	5	1	5	51	17	7	499	173	9	4759	1678
4	17	7	6	163	61	8	1549	551	10	$\geq 14,778$	$\geq 13,799$

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References

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