Problem Set #5

Math, Jorge Barro
Bryan Chia

Problem 7.1

Let two elements of S be $\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n$ and $\alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_m y_m$. We then choose a $\beta \in (0,1)$ such that:

$$\frac{1-\beta}{2}\left[\frac{\lambda_1}{1-\beta}x_1 + ... + \frac{\lambda_n}{1-\beta}x_n\right] + \frac{\beta}{2}\left[\frac{\alpha_1}{\beta}x_1 + ... + \frac{\alpha_n}{\beta}x_n\right]$$

Since both elements are convex combinations, combination of those two will be a convex combination as well. We divide both sets by 2 so that the coefficients will all sum up to 1 which is a requirement for the elements of S.

Problem 7.2

(i) Let $p_1, p_2 \in P$. We want to show that the combination of those two points $\alpha p_1 + (1-\alpha)p_2$ is in P as well. By rules of inner product spaces,

$$\begin{aligned}\langle a, \alpha p_1 + (1-\alpha)p_2\rangle &= \alpha\langle a, p_1\rangle + (1-\alpha)\langle a, p_2\rangle \\ &= \alpha b + (1-\alpha)b \\ &= b\end{aligned}$$

Which implies that the point is in the arbitrary hyperplane. Thus, the hyperplane is convex.

(ii) Let $h_1, h_2 \in H$. We want to show that the combination of those two points $\alpha h_1 + (1-\alpha)h_2$ is in H as well. We know that $\langle a, h_1\rangle \le b$ and $\langle a, h_2\rangle \le b$.

$$\begin{aligned}\langle a, \alpha h_1 + (1-\alpha)h_2\rangle &= \alpha\langle a, h_1\rangle + (1-\alpha)\langle a, h_2\rangle \\ &\le \alpha b + (1-\alpha)b \\ &= b\end{aligned}$$

Thus, the point is in the half space H.

Problem 7.4

(i)

$$\begin{aligned}||x-y||^2 &= ||x-p+p-y||^2 \\ &= ||x-p||^2 + ||p-y||^2 + 2\langle x-p, p-y\rangle\end{aligned}$$

(ii)

$$\begin{aligned}||x-y||^2 &= ||x-p||^2 + ||p-y||^2 + 2\langle x-p, p-y\rangle \\ &= \langle x-p, p-y\rangle^2 \\ &> \langle x-p, x-p\rangle^2 \\ &= ||x-p||^2\end{aligned}$$

Thus, $||x-y|| > ||x-p||$

(iii)

$$\begin{aligned}||x-z||^2 &= ||x-\lambda y-(1-\lambda)p||^2 \\ &= ||(x-p)+\lambda(p-y)||^2 \\ &= ||x-p||^2 + \lambda^2||y-p||^2 + 2\lambda\langle x-p, p-y\rangle^2\end{aligned}$$

(iv)

$$||x-z||^2 - ||x-p||^2 = 2\lambda\langle x-p.p-y\rangle + \lambda^2||y-p||^2 \geq 0$$

since $||x-z||^2$ is larger than $||x-p||^2$ as the latter is the projection of x on C.

Part (ii) proves the theorem in the ($\Leftarrow$) direction and part (iv) proves it in the ($\Rightarrow$) direction.

Problem 7.6

Let $x_1, x_2 \in S = \{x \in \mathbb{R}^n | f(x) \leq c\}$. We wnat to show that $\alpha x_1 + (1-\alpha)x_2 \in S$, which is to say that $f(\alpha x_1 + (1-\alpha)x_2) \leq c$. Let $\alpha = \lambda$

$$\begin{aligned}f(\lambda x_1 + (1-\lambda)x_2) &\leq \lambda f(x_1) + (1-\lambda)f(x_2) \\ &\leq \lambda c + (1-\lambda)c \\ &= c\end{aligned}$$

Problem 7.7

We want to show that $f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha)f(x_2)$.

$$\begin{aligned}f(\alpha x_1 + (1-\alpha)x_2) &= \sum_{i=1}^{k} \lambda_i f_i(\alpha x_1 + (1-\alpha)x_2)\Big] \\ &\leq \sum_{i=1}^{k} \lambda_i \Big[\alpha f_i(x_1) + (1-\alpha)f_i(x_2)\Big] \\ &= f(x_1) + (1-\alpha)f(x_2)\end{aligned}$$

Problem 7.13

Let the minimum point be x_1. Choose an arbitrary point x_2. WLOG, assume that $x_2 \geq x_1$. We construct a line segment from $(x_2, f(x_2))$ to $(x_1, f(x_1))$, the minimum point. The segment of function $f(x), x \in [x_1, x_2]$ should lie between line $y = \frac{f(x_2)-f(x_1)}{x_2-x_1}(x_2 - x_1) + f(x_1)$ and the line $y = f(x_1)$.

Therefore, $\frac{f(x_2)-f(x_1)}{x_2-x_1}(x_2 - x_1) + f(x_1) \geq f(x) - f(x_1) \geq 0$ for any x and x_2 as long as $x \in [x_1, x_2]$. For an arbitrary x, the inequality holds as $x_2 \to \infty$, but we know that $\lim_{x_2 \to \infty} \frac{f(x_2)-f(x_1)}{x_2-x_1}(x - x_1) = 0$. Recall that $f(x_2), f(x_1)$ are bounded.

Thus, $0 \geq f(x) - f(x_1) \geq 0$ which implies that $f(x) = f(x_1)$.

Problem 7.20

$$
\begin{aligned}
f(\lambda x_1 + (1-\lambda)x_2) &\leq \lambda f(x_1) + (1-\lambda) f(x_2) \\
-f(\lambda x_1 + (1-\lambda)x_2) &\leq -\lambda f(x_1) - (1-\lambda) f(x_2) \\
\Rightarrow f(\lambda x_1 + (1-\lambda)x_2) &= \lambda f(x_1) + (1-\lambda) f(x_2)
\end{aligned}
$$

This proves linearity. We set $c = 0$ and thus f is affine.

Problem 7.21

$(\Rightarrow)$ If $x*$ is a local minimizer for the optimization problem, then $\phi \circ f(x*) \leq \phi \circ f(x)$ for all $x \in \mathbb{R}^n$. Since ϕ is strictly increasing, $f(x*) \leq f(x)$. $(\Leftarrow)$ We know that $f(x*) \leq f(x)$. Since ϕ is strictly increasing, we know that $\phi \circ f(x*) \leq \phi \circ f(x)$ as well.