Problem Set #1

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Problem 1 - 3.6

Proof. Since $B_i \cap B_j = \emptyset \forall i, j \in I$, it is trivial to show that $(A \cap B_i) \cap (A \cap B_j) = \emptyset \forall i, j \in I$, each of the sets are disjoint.Thus, we can use the third axiom of the discrete probability measure which gives us the following:

$$\begin{aligned}
\sum_i P(A \cap B_i) &= P\Big(\bigcup_{i \in I}(A \cap B_i)\Big) \\
&= P[(A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup ...] \\
&= P[(A \cap (B_1 \cup B_2)) \cup (A \cap B_3) \cup ...] \\
&= P[(A \cap (B_1 \cup B_2 \cup B_3)) \cup ...] \\
&= P[A \cap \Omega] \\
&= P(A)
\end{aligned}$$

□

Problem 1 - 3.8

Proof.

$$\begin{aligned}
1 - \prod_{k=1}^{n}(1 - P(E_k)) &= (1 - P(E_1))(1 - P(E_2))(1 - P(E_3))...(1 - P(E_n)) \\
&= 1 - [(1 - P(E_1) - P(E_2) + P(E1)P(E2))(1 - P(E_3))...(1 - P(E_n))] \\
&= 1 - [(1 - P(E_1) - P(E_2) + P(E1 \cap E2))(1 - P(E_3))...(1 - P(E_n))] \\
&= 1 - [(1 - P(E_1 \cup E_2))(1 - P(E_3))...(1 - P(E_n))] \\
&= 1 - [(1 - P(E1 \cup E2 \cup E_3 \cup ... \cup E_n)] \\
&= P(E1 \cup E2 \cup E_3 \cup ... \cup E_n) \\
&= P\Big(\bigcup_{k=1}^{n}(E_k)\Big)
\end{aligned}$$

□

Problem 1 - 3.11

$$\begin{aligned}
P(s = crime|testedt) &= \frac{P(s = crime \cap testedt)}{P(testedt|s = crime)P(s = crime) + P(testedt|s = inn.)P(s = inn.)} \\
&= \frac{P(testedt|s = crime)P(s = crime)}{P(testedt|s = crime)P(s = crime) + P(testedt|s = inn.)P(s = inn.)} \\
&= \frac{P(testedt|s = crime)P(s = crime)}{P(testedt|s = crime)P(s = crime) + P(testedt|s = inn.)(1 - P(s = crime))} \\
&= \frac{1 * \frac{1}{250mil}}{1 * \frac{1}{250mil} + \frac{1}{3mil} * (1 - \frac{1}{250mil})} \\
&= 0.0118577
\end{aligned}$$

Problem 1 - 3.12

Proof. Let:
A_1: Car is behind door 1
A_2: Car is behind door 2
A_3: Car is behind door 3
B: Monty opens door 2
Suppose the player chooses door 1 initially, then

$$\begin{aligned}
P(A_1|B) &= \frac{P(B|A_1) * P(A_1)}{P(B)} \\
&= \frac{\frac{1}{2} * \frac{1}{3}}{\frac{1}{2} * \frac{1}{3} + 0 * \frac{1}{3} + 1 * \frac{1}{3}} \\
&= \frac{1}{3} \\
P(A_2|B) &= 0 \\
P(A_3|B) &= \frac{2}{3}
\end{aligned}$$

□

Thus, his odds are better if he switches doors. For 10 doors, the odds are $\frac{1}{10}$ if he doesn't switch doors and $\frac{9}{10}$ if he switches doors. As the probability that he chose the right door the first time doesn't change.

Problem 1 - 3.16

Proof.

$$\begin{aligned} Var[X] &= E\Big[(X-\mu)^2\Big] \\ &= E\Big[(X^2 - 2\mu X + \mu^2)\Big] \\ &= E[X^2] - 2\mu E[X] + E[\mu^2] by 3.3.12 \\ &= E[X^2] - 2E[X]^2 + \mu^2 \\ &= E[X^2] - \mu^2 \end{aligned}$$

□

Problem 1 - 3.33

Proof.

$$\begin{aligned} Var\left[\frac{B}{n}\right] &= \frac{1}{n^2} Var(B) \\ &= \frac{p(1-p)}{n} \end{aligned}$$

By Chebyshev's Inequality,

$$P\left(\left|\frac{B}{n} - p\right| \geq \varepsilon\right) \leq \frac{p(1-p)}{n\varepsilon^2}$$

□

Problem 1 - 3.36

$Var[X] = (0.801)(0.199)$ due to properties of the Bernoulli distribution. Thus, calculating the Z score, we get $\frac{5500 - 5000}{\sqrt{6242 * 0.801 * 0.199}} = 15.85$ which gives us a probability of 0%

Problem 2

(a) Let $\Omega = \{1,2,3,4,5,6,7,8\}$
Let $A = \{1,2,3,4\}$
Let $B = \{3,4,5,6\}$
Let $C = \{3,4,7,8\}$

(b) Let $\Omega = \{1,2,3,4,5,6,7,8\}$
Let $A = \{1,2,3,4\}$
Let $B = \{1,2,5,6\}$
Let $C = \{1,3,7,8\}$

Problem 3

Proof. Let $d_1 = d_2$. We want to show that $P(d_1) = P(d_2)$

$$\begin{aligned} \frac{1}{d_1} &= \frac{1}{d_2} \\ 1 + \frac{1}{d_1} &= 1 + \frac{1}{d_2} \\ log_{10}(1 + \frac{1}{d_1}) &= log_{10}(1 + \frac{1}{d_2}) \end{aligned}$$

The last line is true given that log_{10} is a 1-1 function and well-defined on $1 + \frac{1}{d}$ as long as d is discrete and within the specified range of numbers 1 to 9. □

Proof. We want to show that the probabilities of all possible events add up to 1.

$$\begin{aligned} \sum_{d=1}^{9} log_{10}\left(1 + \frac{1}{d}\right) &= log_{10}(2) + log_{10}\left(\frac{3}{2}\right) + log_{10}\left(\frac{4}{3}\right) + ... + log_{10}\left(\frac{10}{9}\right) \\ &= log_{10}\left(2 * \frac{3}{2} * \frac{4}{3} * ... * \frac{10}{9}\right) \\ &= log_{10}(10) \\ &= 1 \end{aligned}$$

□

Problem 4

(a)

$$\begin{aligned} E[X] &= \frac{1}{2} * 2 + \frac{1}{2^2} * 2^2 + ... \\ &= 1 + 1 + 1 + ... \\ &= \infty \end{aligned}$$

(b)

$$\begin{aligned} E[lnX] &= \frac{1}{2} * ln2 + \frac{1}{2^2} * ln2^2 + ... \\ &= 1.38629 \end{aligned}$$

Problem 5

For the US investor:

$$\begin{aligned} E\left[\frac{USD}{CHF}\right] &= 0.5 * \frac{5}{4} + 0.5 * \frac{4}{5} \\ &= 1.025 \end{aligned}$$

For the Swiss investor:

$$\begin{aligned} E\left[\frac{CHF}{USD}\right] &= 0.5 * \frac{5}{4} + 0.5 * \frac{4}{5} \\ &= 1.025 \end{aligned}$$

Since both expect their currency to be stronger than the other country's, both would invest in the other country right now.

Problem 6

(a) A pareto distribution of the following specification:

$$P(X > x) = \begin{cases} \frac{1}{(x+1)^2}, & \text{if } x \geq 0 \\ 1, & \text{otherwise} \end{cases}$$

(b) The pdf of X takes the form $f(x) = 2x$, while Y is a uniformly distributed variable with the following specifications: $Y = U[0.685, 0.695]$

(c) We can use three uniformly distributed variables, with the pdfs of Z nested in Y, which is nested in X. For example, $X = U[-3, 3]$, $Y = U[-2, 2]$, $Z = U[-1, 1]$.

Problem 7

(a) True.

Proof.

$$\begin{aligned} pdf(Y) &= 0.5 * \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + 0.5 * \frac{1}{\sqrt{2\pi}} e^{-\frac{(-x)^2}{2}} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\ &= pdf(X) \end{aligned}$$

Thus, Y is also normally distributed with a mean of 0 and variance 1. □

(b) True. The proof follows from above.

(c) True. Since $P(Y \geq c|X = x) = P(Y \geq c)$ as Y does not depend on X, they are independent.

(d) True.

Proof.

$$\begin{aligned} Cov[X, Y] &= E[XY] - E[X]E[Y] \\ &= E[XXZ] - 0 \\ &= E[Z]E[XX] \text{ since Z and X are independent} \\ &= 0 \text{ since the mean of Z is 0} \end{aligned}$$

□

(e) False from this problem.

Problem 8

Looking at the random variable M, we know that the cdf $F(x) = P(M \leq x)$ is simply equal to x by the property of the uniform distribution. Since each distribution is i.i.d., the probability that M remains above x for every draw is equal to x^n. Thus, $F(x) = x^n$. To find the pdf, we differentiate to get $f(x) = nx^{n-1}$. The calculated expected value using integration is $\frac{n}{n+1}$.

Looking at the random variable m, we know that the cdf of m, $F(x) = P(m < x) = 1 - P(m > x) = 1 - (1 - P(X_i > x))\forall i$, which gives us $F(x) = 1 - (1 - x)^n$. Differentiating the cdf to get the pdf gives us $f(x) = n(1 - x)^{n-1}$. Calculating the expected value, we get $\frac{n}{n+1} - 1$

Problem 9

(a)

$$\frac{510-500}{\sqrt{1000/4}} = 0.63246$$
$$\frac{490-500}{\sqrt{1000/4}} = -0.63246$$

Using a normal distribution table, we get 0.4729. Thus, 47.29% is the probability that the good states differ from 500 by at most 2%.

(b) Using the Weak Law of Large Numbers,

$$P\left(\left|\frac{X_1+X_2+...+X_n}{n} - 0.5\right| \geq 0.5 * 0.01\right) \leq \frac{0.5^2}{n(0.005)^2}$$

$$\begin{aligned} n &= \frac{0.25}{0.01 * 0.005^2} \\ &= 1,000,000 \end{aligned}$$

Problem 10

Proof. Since $e^{\theta x}$ is always convex, by the Jensen's inequality,

$$\begin{aligned} E[e^{\theta x}] - e^{\theta E[x]} &\geq 0 \\ 1 - e^{\theta E[x]} &\geq 0 \\ 1 &\geq e^{\theta E[x]} \\ ln(1) = 0 &\geq \theta E[x] \end{aligned}$$

Since $E[x] < 0$, it implies that $\theta > 0$ as $\theta \neq 0$ $\square$