Problem Set #2

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Problem 3.1

(i)

$$
\begin{aligned}
\frac{1}{4}\Big(||x+y||^2 - ||x-y||^2\Big) &= \frac{1}{4}\Big(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle\Big) \\
&= \frac{1}{4}\Big(\langle x+y, x\rangle + \langle x+y, y\rangle - \langle x-y, x\rangle - \langle x-y, -y\rangle\Big) \\
&= \frac{1}{4}\Big(\langle x, x+y\rangle + \langle y, x+y\rangle - \langle x, x-y\rangle - \langle -y, x-y\rangle\Big) \\
&= \frac{1}{4}\Big(\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle - \langle x, x\rangle \\
&\quad - \langle x, -y\rangle - \langle -y, x\rangle - \langle -y, -y\rangle\Big) \\
&= \frac{1}{4}\Big(4\langle x, y\rangle\Big) \\
&= \langle x, y\rangle
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\frac{1}{2}\Big(||x+y||^2 + ||x-y||^2\Big) &= \frac{1}{4}\Big(\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle + \langle x, x\rangle \\
&\quad + \langle x, -y\rangle + \langle -y, x\rangle + \langle -y, -y\rangle\Big) \\
&= \frac{1}{2}\Big(2\langle x, x\rangle + 2\langle y, y\rangle\Big) \\
&= ||x||^2 + ||y||^2
\end{aligned}
$$

Problem 3.2

$$\begin{aligned}
&\frac{1}{4}\Big(||x+y||^2 - ||x-y||^2 + i||x-iy||^2 - i||x+iy||^2\Big) \\
&= \frac{1}{4}\Big(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i\langle x-iy, x-iy\rangle - i\langle x+iy, x+iy\rangle\Big) \\
&= \frac{1}{4}\Big(\langle x+y, x\rangle + \langle x+y, y\rangle - \langle x-y, x\rangle + \langle x-y, y\rangle \\
&\quad + i\langle x-iy, x\rangle + (i)(-i)\langle x-iy, y\rangle - i\langle x+iy, x\rangle - i(i)\langle x+iy, y\rangle\Big) \\
&= \frac{1}{4}\Big(\langle\overline{x, x+y}\rangle + \langle\overline{y, x+y}\rangle - \langle\overline{x, x-y}\rangle + \langle\overline{y, x-y}\rangle \\
&\quad + i\langle\overline{x, x-iy}\rangle + (i)(-i)\langle\overline{y, x-iy}\rangle - i\langle\overline{x, x+iy}\rangle - i(i)\langle\overline{y, x+iy}\rangle\Big) \\
&= \frac{1}{4}\Big(\langle\overline{x,x}\rangle + \langle\overline{x,y}\rangle + \langle\overline{y,x}\rangle + \langle\overline{y,y}\rangle - \langle\overline{x,x}\rangle + \langle\overline{x,y}\rangle + \langle\overline{y,x}\rangle - \langle\overline{y,y}\rangle \\
&\quad + i\langle\overline{x,x}\rangle - \langle\overline{x,y}\rangle + \langle\overline{y,x}\rangle + i\langle\overline{y,y}\rangle - i\langle\overline{x,x}\rangle - \langle\overline{x,y}\rangle + \langle\overline{y,x}\rangle - i\langle\overline{y,y}\rangle\Big) \\
&= \frac{1}{4}\Big(4\langle\overline{y,x}\rangle\Big) \\
&= \langle x, y\rangle
\end{aligned}$$

Problem 3.3

(i)

$$\begin{aligned}
cos\theta &= \frac{\langle x, x^5\rangle}{||x|| * ||x^5||} \\
&= \frac{\int_0^1 x^6 dx}{\int_0^1 x^2 dx * \int_0^1 x^{10} dx} \\
&= \frac{\frac{1}{7}}{\sqrt{\frac{1}{33}}} \\
&= 0.82065
\end{aligned}$$

$$cos^{-1} 0.82605 = 0.6082$$

(ii)

$$\begin{aligned}
cos\theta &= \frac{\langle x^2, x^4\rangle}{||x^2|| * ||x^4||} \\
&= \frac{\int_0^1 x^6 dx}{\int_0^1 x^4 dx * \int_0^1 x^8 dx} \\
&= \frac{\frac{1}{7}}{\sqrt{\frac{1}{45}}} \\
&= 0.9581
\end{aligned}$$

$$cos^{-1}0.9581 = 0.28975$$

Problem 3.8

(i) First, we prove that each of the elements' inner product spaces with other elements are equal to 0. Then, we show that each of the inner product spaces of elements with themselves are equal to 1.

$$\begin{aligned}
\langle cost, sint\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} costsint \quad dt\\
&= 0\\
\langle cost, sin2t\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} costsin2t \quad dt\\
&= 0\\
\langle cos2t, sin2t\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} cos2tsin2t \quad dt\\
&= 0\\
\langle cos2t, sint\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} cos2tsint \quad dt\\
&= 0\\
\langle cost, cost\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} costcost \quad dt\\
&= 1\\
\langle cos2t, cos2t\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} cos2tcos2t \quad dt\\
&= 1\\
\langle sint, sint\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} sintsint \quad dt\\
&= 1\\
\langle sin2t, sin2t\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} sin2tsin2t \quad dt\\
&= 1
\end{aligned}$$

(ii)

$$
\begin{aligned}
||t|| &= \sqrt{\langle t, t\rangle} \\
&= \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} t * t \quad dt} \\
&= \sqrt{\frac{1}{\pi}\frac{2\pi^3}{3}} \\
&= \sqrt{\frac{2}{3}\pi}
\end{aligned}
$$

(iii)

$$
\begin{aligned}
Proj_x(cos3t) &= \sum_{i=1}^{m} \langle x_i, v\rangle x_i \\
&= \langle cos3t, sint\rangle sint + \langle cost, cos3t\rangle cost \\
&\quad + \langle cos2t, cos3t\rangle cos2t + \langle cos3t, sin2t\rangle sin2t \\
&= 0
\end{aligned}
$$

(iv)

$$
\begin{aligned}
Proj_x(cos3t) &= \sum_{i=1}^{m} \langle x_i, v\rangle x_i \\
&= \langle cos2t, t\rangle cos2t + \langle cost, t\rangle cost \\
&\quad + \langle sint, t\rangle sint + \langle sin2t, t\rangle sin2t \\
&= 0 + 0 + \frac{2\pi}{\pi} sint + \frac{-\pi}{\pi} sin2t \\
&= 2sint - sin2t
\end{aligned}
$$

Problem 3.9

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then we have that $\langle x, y\rangle = \begin{bmatrix} x_1 & x_2 \end{bmatrix} * \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1y_1 + x_2y_2$.

We now take a transformation of x and y by R_θ. We get $\begin{bmatrix} cos\theta x_1 - sin\theta x_2 \\ sin\theta x_1 + cos\theta x_2 \end{bmatrix}$ and $\begin{bmatrix} cos\theta y_1 - sin\theta y_2 \\ sin\theta y_1 + cos\theta y_2 \end{bmatrix}$.

We next do matrix multiplication and we get the following:

$$
\begin{aligned}
&\begin{bmatrix} cos\theta x_1 - sin\theta x_2 & sin\theta x_1 + cos\theta x_2 \end{bmatrix} \begin{bmatrix} cos\theta y_1 - sin\theta y_2 \\ sin\theta y_1 + cos\theta y_2 \end{bmatrix} \\
&= (cos\theta x_1 - sin\theta x_2)(cos\theta y_1 - sin\theta y_2) + (sin\theta x_1 + cos\theta x_2)(sin\theta y_1 + cos\theta y_2) \\
&= cos^2\theta x_1 y_1 + sin^2\theta x_2 y_2 - sin\theta cos\theta x_2 y_1 - sin\theta cos\theta x_1 y_2 \\
&\quad + sin^2\theta x_1 y_1 + cos^2\theta x_2 y_2 + cos\theta sin\theta x_2 y_1 + sin\theta cos\theta x_1 y_2 \\
&= x_1 y_1 + x_2 y_2
\end{aligned}
$$

□

Problem 3.10

(i) *Proof.* ($\Leftarrow$) Suppose $Q^H Q = QQ^H = I$, then

$$
\begin{aligned}
\langle Qx, Qy \rangle &= \overline{Qx^T} Qy \\
&= \overline{x^T Q^T} Qy \\
&= \overline{x^T Q^T} Qy \\
&= \overline{x^T} Q^H Qy \\
&= \overline{x^T} y \\
&= x^H y \\
&= \langle x, y \rangle
\end{aligned}
$$

($\Rightarrow$) Suppose Q is orthonormal, then

$$
\begin{aligned}
\langle Qx, Qy \rangle &= Qx^H Qy \\
&= \overline{(Qx)^T} Qy \\
&= \overline{x^T Q^T} Qy \\
&= x^H Q^H Qy
\end{aligned}
$$

Since $(Qx)^H Q^y = x^H y = x^H I y$, $Q^H Q = I$. We can substitute Q for Q^H to get the same result for $QQ^H = I$ □

(ii) *Proof.*

$$
\begin{aligned}
||Qx|| &= \langle Qx, Qx \rangle \\
&= \langle x, x \rangle \\
&= ||x||
\end{aligned}
$$

□

(iii) *Proof.*

$$\begin{aligned}\langle Q^{-1}x, Q^{-1}y\rangle &= (Q^{-1}x)^H Q^{-1}y \\ &= x^H (Q^{-1})^H Q^{-1} y \\ &= x^H I y \\ &= x^H y \\ &= \langle x, y\rangle\end{aligned}$$

□

(iv) *Proof.* We want to show that each of the columns' inner products are 0 with another column and 1 with itself.
We get the any column i of Q by multiplying Q by a vector e_i in the standard basis. Since Q is an orthonormal matrix, then we know that $\langle Qe_i, Qe_j\rangle = \langle e_i, e_j\rangle$ and since for any $\langle e_i, e_j\rangle = 0$ when i and $\langle e_i, e_i\rangle = 1$, we know that the columns of Q are orthonormal as well.

□

(v) *Proof.* $1 = det(I) = det(Q^H Q) = det(Q^H)det(Q) = detQ^2$.
Thus $|detQ| = 1$ □

(vi) *Proof.* We know that $\langle Q_1 x, Q_1 y\rangle = \langle x, y\rangle$ and $\langle Q_2 x, Q_2 y\rangle = \langle x, y\rangle$ implying that $(Q_1 x)^H Q_1 y = x^H y$ and $(Q_2 x)^H Q_2 y = x^H y$.

$$\begin{aligned}\langle Q_1 Q_2 x, Q_1 Q_2 y\rangle &= (Q_1 Q_2 x)^H Q_1 Q_2 y \\ &= x^H Q_2^H Q_1^H Q_1 Q_2 y \\ &= x^H y\end{aligned}$$

□

Problem 3.11

When we apply the Gram Schmidt orthonormalization process to a collection of linearly dependent vectors, suppose we have two linearly dependent vectors x and y, the projection of x onto the unit vector of y will still be x. Thus, it will be cancelled off eventually as it is subtracted from itself in the orthonormalization process.

Problem 3.16

(i) Example: $I = II = (-I)(-I)$

(ii) Suppose $A = Q_1 R_1 = Q_2 R_2$, then $Q_2^H Q_1 = R_2 R_1^{-1} = A$.
Since A is orthorgonal, $A^H A = I, A^H = A^{-1}$.
Since A is triangular and orthorgonal, its columns and rows are all orthorgonal unit vectors meaning that it has to be the identity matrix. Thus, $R_2 R_1^{-1} = A \Rightarrow R_2 = R_1$ and $Q_2^H Q_1 = A \Rightarrow Q_1 = Q_2$ since Q is orthorgonal. Thus, the decomposition of A is unique.

Problem 3.17

$$
\begin{aligned}
A^H A x &= A^H b \\
(\hat{Q}\hat{R})^H \hat{Q}\hat{R}x &= (\hat{Q}\hat{R})^H b \\
\hat{R}^H \hat{Q}^H \hat{Q}\hat{R}x &= \hat{R}^H \hat{Q}^H b \\
\hat{R}^H \hat{R}x &= \hat{R}^H \hat{Q}^H b \\
(\hat{R}^H)^{-1}\hat{R}^H \hat{R}x &= (\hat{R}^H)^{-1}\hat{R}^H \hat{Q}^H b \\
\hat{R}x &= \hat{Q}^H b
\end{aligned}
$$

Problem 3.23

Proof. Using the triangle inequality,

$$
\begin{aligned}
||x|| = ||x-y+y|| &\le ||x-y|| + ||y|| \\
||x|| - ||y|| &\le ||x-y|| \\
||y|| = ||y-x+x|| &\le ||y-x|| + ||x|| \\
||y|| - ||x|| &\le ||y-x|| = ||x-y||
\end{aligned}
$$

Thus, $||x|| - ||y|| \ge \Big| ||x|| - ||y|| \Big|$

$\square$

Problem 3.24

(i) *Proof.* Positivity: $0 \le \left| \int_a^b f(t)dt \right| \le \int_a^b \left|f(t)\right|dt$ and $\left|f(t)\right|dt = 0$ if $f(t) = 0$.

S.P.:

$$
\begin{aligned}
||\alpha f||_{L1} &= \int_a^b |\alpha f(t)|dt \\
&= |\alpha| \int_a^b f(t)dt = |\alpha| ||f||_{L1}
\end{aligned}
$$

Triangle Inequality:

$$
\begin{aligned}
||f+g||_{L1} &= \int_a^b |f(t)+g(t)|dt \\
&\le \int_a^b |f(t)| + |g(t)|dt \\
&= \int_a^b |f(t)|dt + \int_a^b |g(t)|dt \\
&= ||f||_{L1} + ||g||_{L1}
\end{aligned}
$$

$\square$

(ii) *Proof.* Positivity: $0 \leq \left| \int_a^b f(t)^2 dt \right|^{\frac{1}{2}} \leq (\int_a^b \left| f(t)^2 \right| dt)^{\frac{1}{2}} = (\int_a^b \left| f(t) \right|^2 dt)^{\frac{1}{2}}$ and $(\int_a^b \left| f(t) \right|^2 dt)^{\frac{1}{2}}$ if $f(t) = 0$.

S.P.:

$$\begin{aligned} ||\alpha f||_{L2} &= \left(\int_a^b |\alpha f(t)|^2 \right)^{\frac{1}{2}} dt \\ &= |\alpha| \left(\int_a^b |f(t)|^2 dt \right)^{\frac{1}{2}} = |\alpha| ||f||_{L2} \end{aligned}$$

Triangle Inequality:

$$\begin{aligned} ||f+g||_{L2}^2 &= \int_a^b |f(t)+g(t)|^2 dt \\ &\leq \int_a^b |f(t)+g(t)||f(t)+g(t)| dt \\ &\leq \int_a^b (|f(t)|+|g(t)|)|f(t)+g(t)| dt \\ &= \int_a^b |f(t)||f(t)+g(t)| + |g(t)||f(t)+g(t)| dt \\ &\leq \left((\int_a^b |f(t)|^2)^{\frac{1}{2}} + (\int_a^b |g(t)|^2)^{\frac{1}{2}} \right) \left(\int_a^b |f(t)+g(t)|^2) \right)^{\frac{1}{2}} \text{by Holder's Inequality} \\ &= \left(||f||_{L2} + ||g||_{L2} \right) ||f+g||_{L2} \end{aligned}$$

Thus, $||f+g||_{L2} \leq ||f||_{L2} + ||g||_{L2}$ since $||f+g||_{L2} \geq 0$ □

(iii) *Proof.* Positivity: $0 \leq \sup_{[a,b]} |f(x)|$ since $|f(x) \geq 0$ and $0 = \sup_{[a,b]} |f(x)|$ if $f(t) = 0$.

S.P.:

$$\begin{aligned} ||\alpha f||_{L^\infty} &= \sup |\alpha f| \\ &= \sup\{\alpha f_1, \alpha f_2, ..., \alpha f_n\} \\ &= \alpha \sup\{f_1, f_2, ..., f_n\} \\ &= \alpha f||_{L^\infty} \end{aligned}$$

Triangle Inequality:

$$\begin{aligned} ||f+g||_{L^\infty} &= \sup |f+g| \\ &\leq \sup |f| + \sup |g| \\ &= ||f||_{L^\infty} + ||g||_{L^\infty} \end{aligned}$$

$\square$

Problem 3.26

Proof. We first prove that topological equivalence is an equivalence relation.

1. $|| \cdot || \; ||_a \cdot ||_a$: Choose $0 < m < 1, M > 1$, then $m||x||_a \leq ||x||_a \leq M||x||_a$ holds.

2. $|| \cdot ||_a \; || \cdot ||_b \to || \cdot ||_b \; || \cdot ||_a$: Suppose $m||x||_a \leq ||x||_b \leq M||x||_a$ holds, then we know that $\frac{1}{M}||x||_b \leq ||x||_a \leq \frac{1}{m}||x||_b$, implying that $||x||_b \; ||x||_a$.

3. $||\cdot||_a \; ||\cdot||_b, ||\cdot||_b \; ||\cdot||_c \to ||\cdot||_a \; ||\cdot||_c$: This implies that there exists m_1, m_2, M_1, M_2 such that $m_1||x||_a \leq ||x||_b \leq M_1||x||_a$ and $m_2||x||_b \leq ||x||_c \leq M_2||x||_b$. Thus, $m_1 m_2||x||_a \leq m_2||x||_b \leq ||x||_c$ and $||x||_c \leq M_2||x||_b \leq M_1 M_2||x||_a$. Thus, $m_1 m_2||x||_a \leq ||x||_c \leq M_1 M_2||x||_a$

Thus, we have an equivalence relation.

$\square$

For this question, all sums are finite sums from 1 to n.

(i) *Proof.* We first show that $||x||_1 \leq ||x||_2\sqrt{2}$. By the Cauchy Schwatz inequality, we know that $|\langle x, y\rangle| \leq ||x||_2||y||_2$. Let $x = \{|x_1|, |x_2|, ...|x|_n\}$ and $x = \{1, 1, 1, ..., 1\}$ Then,

$$
\begin{aligned}
|x^T y| &\leq ||x||_2||y||_2 \\
\left|\sum |x_i|\right| &\leq \sqrt{\sum |x_i|^2}\sqrt{\sum 1} \\
\left|\sum |x_i|\right| &\leq ||x||_2\sqrt{n} \\
||x||_1 &\leq ||x||_2\sqrt{n}
\end{aligned}
$$

$$
\begin{aligned}
||x||_2^2 = \sum |x_i|^2 &\leq \sum |x_i|^2 + 2\sum |x_i||x_j| \\
&= ||x_1||^2
\end{aligned}
$$

This implies that $||x||_2 \leq ||x||_1$. The absolute values are irrelevant since the sums are all positive. $\square$

(ii) *Proof.* By remark 3.66, since $||x||_i \leq ||x||_p$ for any i and p, and $||x||_\infty = \sup\{|x_i|\}$, $||x||_\infty \leq ||x||_2$.

$||x||_2^2 = \sum |x_i|^2 \leq n \sup\{|x_i|^2\} \leq n\big(\sup\{|x_i|\}\big)^2 = n||x||_\infty^2$.
Thus, $||x||_2 \leq \sqrt{n}||x||_\infty$ $\square$

Problem 3.28

For this question, all sup is over all vectors $x \neq 0$. We also use inequalities from Problem 3.26.

(i) *Proof.* $||A||_2 = \sup \frac{||Ax||_2}{||x||_2} \geq \sup \frac{||Ax||_2}{||x||_1} \geq \frac{1}{\sqrt{n}}||Ax||_1 = \sqrt{n}||A||_1$
Thus, $||A||_1 \leq \sqrt{n}||A||_2$

$||A||_2 = \sup \frac{||Ax||_2}{||x||_2} \leq \sup \frac{||Ax||_2}{\frac{1}{\sqrt{n}}||x||_1} = \sqrt{n}||A||_1$
Thus, $\frac{1}{\sqrt{n}}||A||_2 \leq ||A||_1$

□

(ii) *Proof.* $||A||_2 = \sup \frac{||Ax||_2}{||x||_2} \leq \sup \frac{\sqrt{n}||Ax||_\infty}{||x||_2} \leq \sup \frac{\sqrt{n}||Ax||_\infty}{||x||_\infty} = \sqrt{n}||A||_\infty$
Thus, $||A||_2 \leq \sqrt{n}||A||_\infty$

$||A||_2 = \sup \frac{||Ax||_2}{||x||_2} \geq \sup \frac{||A||_\infty}{||A||_2} \geq \sup \frac{1}{\sqrt{n}} \frac{||Ax||_\infty}{||x||_2} = \frac{1}{\sqrt{n}}||A||_\infty$
Thus, $||A||_2 \geq \frac{1}{\sqrt{n}}||A||_\infty$

□

Problem 3.29

Proof. $||Q||_2 = \sup_{x\neq 0} \frac{||Qx||_2}{||x||_2} = \sup_{x\neq 0} \frac{||x||_2}{||x||_2} = 1$ by Theorem 3.2.15(ii).

$||R_x||_2 = \sup_{A\neq 0} \frac{||Ax||_2}{||A||_2} = \sup_{A\neq 0} \frac{||Ax||_2}{\sup_{y\neq 0} \frac{||Ay||_2}{||y||_2}} \leq \sup_{A\neq 0} \frac{||Ax||_2}{\frac{||Ax||_2}{||x||_2}} = ||x||_2$

If $||x||_2 = 1$, then $||R_x||_2 = ||x||_2 = 1$ by our above equations since it is part of an orthnormal basis. If it is not part of an orthonormal basis then $||R_x||_2 \geq ||x||_2$ and so $||R_x||_2 = ||x||_2$

□

Problem 3.30

1. Positivity
   $||A||_S = ||SAS^{-1}|| = sup_{x\neq 0}\frac{||SAS^{-1}x||}{||x||}$ is positive by definition. Also, if $SAS^{-1} = 0$, $||A||_s = 0$.

2. Scale Preservation
   $||\alpha A||_S = ||S\alpha AS^{-1}|| = sup_{x\neq 0}|\alpha|\frac{||SAS^{-1}x||}{||x||} = \alpha||A||_S$.

3. Triangle Inequality

$$\begin{aligned}
||A+B||_S &= ||S(A+B)S^{-1}||_S \\
&= ||SAS^{-1}+SB)S^{-1}||_S \\
&= \sup_{x\neq 0} \frac{||(SAS^{-1}+SBS^{-1})x||}{||x||} \\
&\leq \sup_{x\neq 0} \frac{||(SAS^{-1}||}{||x||} + \sup_{x\neq 0} \frac{||(SBS^{-1}||}{||x||} \\
&= ||A||_S + ||B||_S
\end{aligned}$$

4. Subadditivity For all $v \in V$, we have that:

   $||AB||_S = ||SABS^{-1}|| = sup_{x\neq 0} \frac{||SABS^{-1}x||}{||x||} \leq sup_{x\neq 0} \frac{||SAS^{-1}x||}{||x||} + sup_{x\neq 0} \frac{||SBS^{-1}x||}{||x||} = ||A||_S ||B||_S$

Problem 3.37

Let $p(x) = ax^2 + bx + c$, such that we can express p in terms of the power basis $[a \quad b \quad c]^T$. We need q to be a vector that when multiplied by p gives us $L[p] = p'(1) = 2a + b$. We find that $q = [2 \quad 1 \quad 0]$ gives us the desired vector.

Problem 3.38

We want a matrix representation of $D.D[1] = 0, D[x] = 1, D[x^2] = 2x$.

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

For each column of hte adjoint, we can calculate $\langle b_i, D[v]\rangle = c_0 + c_1x^2 + c_2x^2, a_0 + a_1x + a_2x^2\rangle$ for all $b_i \in \{1, x, x^2\}$ where the second polynomial will yield a system of equations whose solution will yield the i^{th} column of the matrix representation of the adjoint. Note $c_0 + c_1x^2 + c_2x^2, a_0 + a_1x + a_2x^2\rangle = \int_0^1 (c_0 + c_1x^2 + c_2x^2)(a_0 + a_1x + a_2x^2)dx = \frac{1}{60}(10a_0(6c_0 + 3c_1 + 2c_2) + 5a_1(6c_0 + 4c_1 + 3c_2) + a_2(20c_0 + 15c_1 + 12c_2))$ and $\langle 1, a_1 + 2a_2x\rangle = a_1 + a_2$, $\langle x, a_1 + 2a_2x\rangle = \frac{a_1}{2} + \frac{2a_2}{3}$ and $\langle x, a_1 + 2a_2x\rangle = \frac{a_1}{3} + \frac{a_2}{2}$. Solving this system gives us:

$$D^* = \begin{bmatrix} -6 & 2 & 3 \\ 12 & -24 & -26 \\ 0 & 30 & 30 \end{bmatrix}$$

Problem 3.39

(i) *Proof.*

$$\begin{aligned}(S+T)^*(w) &= \langle (S+T)^*(w), v\rangle_V \\ &= \langle w, (S+T)(v)\rangle_W \\ &= \langle w, S(v)\rangle_W + \langle w, T(v)\rangle_W \text{ since L is linear} \\ &= \langle S^*(w), v\rangle_V + \langle T^*(w), v\rangle_V \\ &= S^* + T^*\end{aligned}$$

$$\begin{aligned}(\alpha T)^*(w) &= \langle \alpha T^*(w), v\rangle_V \\ &= \langle \alpha w, T(v)\rangle_W \\ &= \bar{\alpha}\langle w, T(v)\rangle_W \\ &= \bar{\alpha}\langle T^*(w), v\rangle_V \\ &= \alpha T^*\end{aligned}$$

□

(ii) *Proof.*

$$\begin{aligned}(S^*(S)^*(w)) &= S * \big(\langle (S^*(w), v\rangle_V\big) \\ &= \langle (S^*(S^*(w)), S*(v)\rangle_W \\ &= \langle S*(w), v\rangle_V \\ &= \langle w, S(v)\rangle_W \\ &= S\end{aligned}$$

□

(iii) *Proof.*

$$\begin{aligned}(ST)^*(w) &= *\langle (ST)^*(w), v\rangle_V \\ &= \langle w, ST(v)\rangle_V \\ &= \langle S*(w), T(v)\rangle_V \\ &= \langle T^*S^*(w), v\rangle_V\end{aligned}$$

□

(iv) *Proof.*

$$\begin{aligned}(T^{-1})^*(w) &= *\langle (T^{-1})^*(w), v\rangle_V \\ &= \langle w, T^{-1}(v)\rangle_V \\ &= \langle T(w), v\rangle_V \\ &= \langle w, T*v\rangle_V \\ &= \langle (T*)^{-1}w, v\rangle_V \\ &= (T^*)^{-1}\end{aligned}$$

□

Problem 3.40

(i) *Proof.*

$$\begin{aligned} \langle B, AB \rangle &= tr(B^H AB) \\ &= tr[(A^H B)^H B] \\ &= \langle A^H B, B \rangle \end{aligned}$$

Thus, $A^* = A^H$. □

(ii) *Proof.*

$$\begin{aligned} \langle A_2, A_3 A_1 \rangle &= tr(A_2^H A_3 A_1) \\ &= tr(A_2^H A_1 A_3) \\ &= tr(A_1 A_2^H A_3) \\ &= tr((A_2 A_1^H)^H A_3) \\ &= tr((A_2 A_1^*)^H A_3) \\ &= \langle A_2 A_1^*, A_3 \rangle \end{aligned}$$

□

(iii) *Proof.*

$$\begin{aligned} \langle T_A^*(Y), X \rangle &= \langle A^* Y - Y A^*, X \rangle \\ &= \langle A^* Y, X \rangle - \langle Y A^*, X \rangle \\ &= \langle Y, AX \rangle - \langle Y, XA \rangle \\ &= \langle Y, AX - XA \rangle \\ &= \langle Y, T_A(X) \rangle \\ &= \langle T_A^*(Y), X \rangle \end{aligned}$$

Thus, $(T_A)^* = T_A^*$

□

Problem 3.44

Proof. We prove this by contradiction. Suppose $0 = 0^H x = (y^H A)x = Y^H Ax = y^H b \neq 0$ which is a contradiction. Thus, both statements cannot be true at the same time.

□

Problem 3.45

Proof. Let $A \in Sym_n(\mathbb{R})$ and $B \in Skew_n(\mathbb{R})$. We want to show that A and B are orthorgonal which then proves that $Sym_n(\mathbb{R})^\perp = Skew_n(\mathbb{R})$ since both A and B are arbitrary.

We know that $Tr(AB) = Tr(BA), Tr(A^T B) = Tr(B^T A)$. Thus $\langle A, B\rangle = \langle A, B\rangle$.

$\langle A, B\rangle = Tr(A^T B) = Tr(AB) = Tr(BA) = Tr(-B^T A) = \langle -B, A\rangle = -\langle A, B\rangle$
$\langle A, B\rangle + \langle A, B\rangle = 0$
. Thus, $\langle A, B\rangle = 0$. Thus, A and B are orthorgonal. □

Problem 3.46

(i) *Proof.* $x \in N(A^H A) \Rightarrow A^H Ax = 0 \Rightarrow A^H(Ax) = 0 \Rightarrow Ax \in N(A^H)$ □

Proof.

$$
\begin{aligned}
A^H A &= 0 \\
x^H A^H Ax &= x^H 0 = 0 \\
(Ax)^H Ax &= 0 \\
\langle Ax, Ax\rangle &= 0 \Rightarrow Ax = 0
\end{aligned}
$$

For any linear transformation $A : V \to W$, we define the Range of A to be the set of vectors $w \in W$ such that there exists a $v \in V$ where $Av = w$. Since $Ax = 0$, $A0 = 0$ and since $0 \in V$, there exists a vector in V such that $Ax = w$ for some $w \in W$. Thus, $Ax \in R(A)$ □

(ii) *Proof.* ($\subseteq$) Let $x \in N(A^H A)$. Then $AxA = 0$.

$$
\begin{aligned}
A^H A &= 0 \\
x^H A^H Ax &= x^H 0 = 0 \\
(Ax)^H Ax &= 0 \\
\langle Ax, Ax\rangle &= 0 \Rightarrow Ax = 0
\end{aligned}
$$

Thus, $x \in N(A)$.
($\supseteq$) Let $x \in N(A)$, $\Rightarrow Ax = 0$. Then $A^H Ax = A^H 0 = 0$. Thus $x \in N(A^H A)$. □

(iii) A and $A^H A$ have the same rank. We have shown that $N(A^H A) = N(A)$. This implies that $dim(N(A^H A)) = dim(N(A))$ and by the rank nullity theorem that $rk(A) + nul(A) = n$ for an m by n matrix, $rk(A^H A) = rk(A)$.

(iv) If A has linearly independent columns, it has full rank, meaning that $A^H A$ also has full rank and is linearly independent by (iii). This implies that it is also invertible or non singular.

Problem 3.47

(i) *Proof.*

$$\begin{aligned}[A(A^HA)^{-1}A^H][A(A^HA)^{-1}A^H] &= A(A^HA)^{-1}A^HA(A^HA)^{-1}A^H \\ &= A(A^HA)^{-1}A^H\end{aligned}$$

□

(ii) *Proof.*

$$[A(A^HA)^{-1}A^H]^H = A(A^HA)^{-1}A^H$$

□

(iii) We use the following lemma: Suppose A_{m*n} and B_{n*k} of rank n, then the $rk(AB) = rk(A)$.

Proof.

$$rk(P) = rk[A(A^HA)^{-1}A^H] = rkA(A^HA)^{-1}A = rk(A) = n$$

□

Problem 3.48

(i) *Proof.*

$$\begin{aligned}P(cA) &= \frac{cA + cA^T}{2} \\ &= c\frac{A + A^T}{2} \\ &= cP(A)\end{aligned}$$

$$\begin{aligned}P(A+B) &= \frac{A + B + (A+B)^T}{2} \\ &= \frac{A + B + A^T + B^T}{2} \\ &= P(A) + P(B)\end{aligned}$$

□

(ii) *Proof.*

$$\begin{aligned}P^2(A) &= \frac{\frac{A+A^T}{2} + (\frac{A+A^T}{2})^T}{2} \\ &= \frac{\frac{A+A^T}{2} + (\frac{A^T+A}{2})}{2} \\ &= \frac{A + A^T}{2} \\ &= P(A)\end{aligned}$$

□

(iii) *Proof.*

$$\begin{aligned}
\langle A, \frac{A+A^T}{2}\rangle &= \langle P^*(A), A\rangle \\
tr(A^H\frac{A+A^T}{2}) &= tr(P^*(A)^H A) \\
tr(\frac{A^H A + A^H A^T}{2}) &= tr(P^*(A)^H A)
\end{aligned}$$

Suppose $P^*(A) = P(A)$,

$$\begin{aligned}
tr(\frac{A^H A + A^H A^T}{2}) &= tr(\frac{A+A^T}{2}^H A) \\
tr(\frac{A^H A + A^H A^T}{2}) &= tr(\frac{A^H + A}{2} A) \\
tr(\frac{A^H A + A^H A^T}{2}) &= tr(\frac{A^H A}{2}) + tr(\frac{AA}{2}) \\
tr(\frac{A^H A}{2}) + tr(\frac{(AA)^T}{2}) &= tr(\frac{A^H A}{2}) + tr(\frac{AA}{2}) \\
tr(\frac{A^H A}{2}) + tr(\frac{AA}{2}) &= tr(\frac{A^H A}{2}) + tr(\frac{AA}{2})
\end{aligned}$$

which is true. □

(iv) *Proof.* ($\subseteq$) Let $A \in N(P)$: $\frac{A+A^T}{2} = 0, A + A^T = 0, A = -A^T$. Thus, $A \in Skew_n(\mathbb{R})$.

($\supseteq$) Let $A \in Skew_n(\mathbb{R}) : A = -A^T, A + A^T = 0, \frac{A+A^T}{2} = 0$. Thus, $A \in N(P)$.

□

(v) *Proof.* ($\subseteq$) Let $B \in R(P)$: There exists an $A \in M_n(\mathbb{R})$ such that $\frac{A+A^T}{2} = B = \frac{A^T+A}{2} = \left(\frac{A^T+A}{2}\right)^T = B^T$. And $B \in Sym_n(\mathbb{R})$

($\supseteq$) Let $B \in Sym_n(\mathbb{R})$. Let $A = B, P(B) = B$. Thus, $\frac{B^T+B}{2} = \frac{B+B}{2} = B$, implying that $B \in R(P)$.

□

(vi) *Proof.*

$$\begin{aligned}
||A - P(A)||_F &= \sqrt{\left\langle A - P(A), A - P(A) \right\rangle} \\
&= \sqrt{\langle A - \frac{A + A^T}{2}, A - \frac{A + A^T}{2} \rangle} \\
&= \sqrt{\langle \frac{A + A^T}{2}, \frac{A + A^T}{2} \rangle} \\
&= \sqrt{tr\left(\left(\frac{A + A^T}{2}\right)^T \left(\frac{A + A^T}{2}\right)\right)} \\
&= \sqrt{\frac{tr(A^T A) + tr(A^T A) - tr(A^2) - tr((AA)^T)}{4}} \\
&= \sqrt{\frac{tr(A^T A) - tr(A^2))}{2}}
\end{aligned}$$

$\square$

Problem 3.50
$sy^2 = 1 - rx^2$
$y^2 = \frac{1}{s} - \frac{r}{s}x^2$
Thus we have the following:

$$b = \begin{bmatrix} y_1^2 \\ y_2^2 \\ . \\ . \\ . \\ y_n^2 \end{bmatrix} \quad , \quad A = \begin{bmatrix} x_1^2 & 1 \\ x_2^2 & 1 \\ . & \\ . & \\ . & \\ x_n^2 & 1 \end{bmatrix} \quad , \quad x = \begin{bmatrix} -\frac{r}{s} \\ -\frac{1}{s} \end{bmatrix}$$