Problem Set #3

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Problem 4.2

We want a matrix representation of $D.D[1] = 0, D[x] = 1, D[x^2] = 2x$.

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Finding $P_A(z) = det|D| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} = \lambda^3 = 0$
$\Rightarrow \lambda = 0$
Solving for the vectors that would satisfy the null space of the matrix D gives us the eigenspace that is composed of vectors of the following form: $\{[a \quad 0 \quad 0]^T\}$. The algebraic multiplicty is 3 and the geometric multiplicty is 1.

Problem 4.4

(i)

$$\begin{aligned} Av &= \lambda v \\ (Av)^H &= (\lambda v)^H \\ v^H A^H &= \overline{\lambda} v^H \\ v^H A^H v &= \overline{\lambda} v^H v \\ v^H A v &= \overline{\lambda} v^H v \\ v^H \lambda v &= \overline{\lambda} v^H v \\ \lambda v^H v &= \overline{\lambda} v^H v \\ \lambda &= \overline{\lambda} \text{ since } v^H v = ||v|| \text{ which is } > 0 \text{ for } v \neq 0. \end{aligned}$$

Thus, it only has real eigenvalues.

(ii)

$$\begin{aligned}
Av &= \lambda v \\
(Av)^H &= (\lambda v)^H \\
v^H A^H &= \overline{\lambda} v^H \\
v^H A^H v &= \overline{\lambda} v^H v \\
v^H (-A) v &= \overline{\lambda} v^H v \\
v^H (-\lambda) v &= \overline{\lambda} v^H v \\
-\lambda v^H v &= \overline{\lambda} v^H v \\
-\lambda &= \overline{\lambda} \text{ since } v^H v = ||v|| \text{ which is } > 0 \text{ for } v \neq 0.
\end{aligned}$$

Thus, it only has imaginary eigenvalues.

Problem 4.6

Suppose A is upper triangular, then $A - \lambda I$ is also upper triangular. Thus, $P_A(z) = |A - \lambda I| = \prod(a_i i - \lambda)$ is the product of the diagonal entries by the lemma proved below. Thus the roots of the characteristic polynomial are its eigenvalues which are the diagonal entries.

Lemma: The determinant of an upper triangular matrix is the product of its diagonal entries.

Proof. We prove this inductively. Let

$$A_n = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

The above is true for $n = 1$. Now we assume our inductive hypothesis, that $det(A_n) = \prod_i^n a_{ii}$. Then, if we consider $det(A_{n+1})$, we find that using the cofactor expansion method of computing the determinant and since it is an upper triangular matrix, $det(A_{n+1}) = \prod_i^n a_{ii} * a_{n+1n+1}$.

$\square$

Problem 4.8

(i) Let $asinx + bcosx + csin2x + dcos2x = 0$.
If we sub in $x = 0$, we get $b + d = 0, b = -d$.
If we sub in $x = \pi$, we get $-b + d = 0, b = d$.
This implies that $b = d = 0$.
If we sub in $x = \frac{\pi}{2}$, we get $a = 0$.
Finally, if we sub in $x = 1$, we get $csin2 = 0 \Rightarrow c = 0$.
Thus, the polynomials in S are linearly independent. Since the question has stated that S spans V, S is a basis of V.

(ii) Using the vector $[sinx \quad cosx \quad sin2x \quad cos2x]$ as a representation of a basis, the following matrix gives us the derivative operator:

$$D = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

(iii) We can consider the two complementary bases $\{sinx, cosx\}$ and $\{sin2x, cos2x\}$.

Problem 4.13

Solving for the eigenvalues of A, we get $(1, 0.4)$. This gives us two corrresponding eigenvectors that form the columns of the following invertible matrix P.

$$P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Problem 4.15

By Theorem 3.2, A is diagonalizable and thus in the same theorem, we know that A and its diagonal matrix D have the same eigenvalues. Since D is a diagonal matrix, it's diagonal elements are simply the eigenvalues of D and A. This means that D can be written as:

$$D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

Next we note the following:

$$\begin{aligned} f(A) &= a_0 I + a_1 A + ... + a_n A^n \\ &= a_0 P^{-1} I P + a_1 P^{-1} D P + ... + a_n P^{-1} D^n P \\ &= P^{-1}(a_0 I + a_1 D + ... + a_n D^n) P \\ &= P^{-1} f(D) P \end{aligned}$$

Thus, f(A) and f(D) also have the same eigenvalues. We consider the following to characteristic polynomial of $f(A)_z$ to get the eigenvalues of $f(D)$ and hence $f(A)$.

$$
\begin{aligned}
P_{f(A)}(z) &= |zI - f(A)| \\
&= |zI - f(D)| \\
&= |zI - a_0 I - a_1 D - ... - a_n D^n| \\
&= \begin{vmatrix} z - a_0 - a_1\lambda_1 - ... - a_n\lambda_1^n & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & z - a_0 - a_1\lambda_n - ... - a_n\lambda_n^n \end{vmatrix} \\
&= \begin{vmatrix} z - f(\lambda_1) & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & z - f(\lambda_n) \end{vmatrix}
\end{aligned}
$$

Thus, the eigenvalues of A are $\{f(\lambda_1), ..., f(\lambda_n)\}$

Problem 4.16

(i)

$$
\begin{aligned}
\lim_{k\to\infty} A^k &= \lim_{k\to\infty} PD^kP^{-1} \\
&= \lim_{k\to\infty} P\left(\begin{bmatrix} 1 & 0 \\ 0 & \frac{6}{15} \end{bmatrix}\right)^k P^{-1} \\
&= P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \\
&= \frac{1}{3}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}
\end{aligned}
$$

Checking the limits:

$$
\begin{aligned}
||\lim_{k\to\infty} A^k - B|| &= ||\lim_{k\to\infty} P^{-1}D^kP - B|| \\
&= \left|\left|\begin{bmatrix} \frac{2}{3}1^k + \frac{1}{3}\frac{6}{15}^k & \frac{1}{3}1^k + \frac{1}{3}\frac{6}{15}^k \\ \frac{2}{3}1^k - \frac{2}{3}\frac{6}{15}^k & \frac{1}{3}1^k + \frac{2}{3}\frac{6}{15}^k \end{bmatrix} - \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}\right|\right| \\
&= \text{Largest column sum} \\
&= \frac{1}{3}\left(\frac{6}{15}\right)^k \leq \frac{1}{3}\left(\frac{6}{15}\right)^N \leq \epsilon \text{ is what we want}
\end{aligned}
$$

Let

$$
\begin{aligned}
ln(\frac{3}{4}\epsilon) &= Nln(\frac{6}{15}) \\
N &= ln\left(\frac{3}{4}\epsilon\right)/ln\left(\frac{6}{15}\right)
\end{aligned}
$$

Thus we choose N as above.

(ii) For the infinity norm, we use the largest row sum that would give us an N of:

$$N = ln\left(\epsilon\right)/ln\left(\frac{6}{15}\right)$$

For the Frobenius norm, we use the trace of our differentiated matrix conjugate tranposed multiplied by the same matrix to get an N of:

$$N = ln\left(\frac{9}{10}\epsilon\right)/ln\left(\frac{6}{15}\right)$$

(iii) Using the Semisimple Spectral Mapping Theorem,
$f(\lambda_i) = \lambda_i^3 + 5\lambda_i + 3$
$f(1) = 1 + 5 + 3$
$f(0.4) = 0.4^3 + 5 * 0.4 = 3 = 5.064$

Problem 4.18

Proof.

$$\begin{aligned} Ax - \lambda x &= 0 \\ (A-) &= x \\ x^T(A^T - \lambda I^T) &= 0 \\ x^T A^T &= \lambda x^T \end{aligned}$$

□

Problem 4.18

Proof. Suppose $B = U^H AU$, then:
$B^H = (U^H AU)^H = U^H A^H U = U^H AU = B$.
Thus, B is also Hermetian.

□

Problem 4.24

Proof.

$$\begin{aligned} \rho(x) &= \frac{\langle x, Ax\rangle}{||x||^2} \\ &= \frac{x^H Ax}{||x||^2} \\ &= \frac{x^H \lambda x}{||x||^2} \\ &= \frac{\lambda\langle x, x\rangle}{||x||^2} \\ &= \frac{\lambda||x||^2}{||x||^2} \\ &= \lambda \end{aligned}$$

And λ is real by Corollary 4.49 $\square$

Proof.

$$\begin{aligned}
\rho(x) &= \frac{\langle x, Ax \rangle}{||x||^2} \\
&= \frac{x^H A x}{||x||^2} \\
&= \frac{x^H \lambda x}{||x||^2} \\
&= \frac{\lambda \langle x, x \rangle}{||x||^2} \\
&= \frac{\lambda ||x||^2}{||x||^2} \\
&= \lambda
\end{aligned}$$

And λ is imaginary by 4.4(ii) which we proved for all matrices and not just 2 by 2 matrices $\square$

Problem 4.25

(i) *Proof.*

$$\begin{aligned}(x_1x_1^H + ... + x_nx_n^H)x_j &= x_1x_1^Hx_j + ... + x_nx_n^Hx_j \\ &= x_1 * 0 + ... + x_j * 1 + ... + x_n * 0 \\ &= x_j\end{aligned}$$

Thus, $(x_1x_1^H + ... + x_nx_n^H) = I$ ☐

(ii) *Proof.*

$$\begin{aligned}I &= (x_1x_1^H + ... + x_nx_n^H) \\ A &= A(x_1x_1^H + ... + x_nx_n^H) \\ A &= Ax_1x_1^H + ... + Ax_nx_n^H \\ &= \lambda x_1x_1^H + ... + \lambda x_nx_n^H\end{aligned}$$

☐

Problem 4.28

(i) *Proof.* We first show that $tr(AB) \geq 0$. Since A and B are positive semdefinite, there exists matrices X and Y such that $A = X^HX$ and $B = Y^HY$.

$$\begin{aligned}tr(AB) &= tr(X^HXY^HY) \\ &= tr((YX^H)(XY^H)) \\ &= tr((XY^H)^H(XY^H)) \geq 0\end{aligned}$$

We next show that $tr(A)tr(B) \geq tr(AB)$.
Since A and B are positive semidefinite, they are Hermetian.

$$\begin{aligned}tr(AB) &= \langle A, B\rangle \\ &\leq ||A||||B|| \\ &= \sqrt{tr(A^HA)}\sqrt{tr(B^HB)} \\ &= \sqrt{tr(A^2)}\sqrt{tr(B^2)} \\ &\leq \sqrt{tr(A)^2tr(B)^2} \\ = tr(A)tr(B)\end{aligned}$$

$tr(A^2) \leq tr(A)^2$ is true since the former are the squared eigenvalues added togehter but the latter are the eigenvalues added together squared. Since it only has non-negative eigenvalues, this inequality holds true. ☐

(ii) (a)

$$\langle A, A\rangle = \sqrt{tr(A^HA)}\sqrt{tr(B^HB)} \geq 0$$

(b)

$$\sqrt{tr(\alpha A^HA)} = \sqrt{\alpha tr(A^HA)} \text{ by Matrix Rules}$$

(c)

$$\begin{aligned}||A+B||_F^2 &= tr((A+B)^H(A+B))\\ &= tr(A^HA) + tr(B^HB) + tr(A^HB) + tr(B^HA)\\ &= tr(A^HA) + tr(B^HB) + 2\langle A, B\rangle\\ &= ||A||_F^2 + ||B||_F^2 + 2\langle A, B\rangle\end{aligned}$$

Using Cauchy-Schwartz, we have that

$$\begin{aligned}||A||_F^2 + ||B||_F^2 + 2\langle A, B\rangle &\leq ||A||_F^2 + ||B||_F^2 + 2||A||_F||B||_F\\ &= (||A||_F + ||B||_F)^2\\ ||A+B||_F &\leq ||A||_F + ||B||_F\end{aligned}$$

(d)

$$\begin{aligned}||AB||_F^2 &= tr[(AB)^H(AB)]\\ &= tr(B^HA^HAB)\\ &= tr(BB^HA^HA)\\ &= tr((B^HB)^HA^HA)\\ &\leq tr(B^HB)tr * A^HA)\\ &= ||B||_F^2||A||_F^2\end{aligned}$$

Thus, $||AB||_F \leq ||A||_F||B||_F$

Problem 4.31

(i)

$$\begin{aligned}
||A||_2 &= \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2} \\
&= \sup_{x\neq 0} \frac{||U\Sigma V^H x||_2}{||x||_2} \\
&= \sup_{x\neq 0} \frac{||\Sigma V^H x||_2}{||x||_2} \\
&= \sup_{y\neq 0} \frac{||\Sigma y||_2}{||Vy||_2} \\
&= \sup_{y\neq 0} \frac{\left(\sum_{i=1}^n \sigma_i^2 |y_i|^2\right)^{\frac{1}{2}}}{\left(\sum_{i=1}^n |y_i|^2\right)^{\frac{1}{2}}} \\
&\leq \sigma_1
\end{aligned}$$

We choose $y = [1 \quad ... \quad 0]^T$, and this will give us an equality of σ_1.

(ii)

$$\begin{aligned}
||A^{-1}||_2 &= \sup_{x\neq 0} \frac{||A^{-1}x||_2}{||x||_2} \\
&= \sup_{x\neq 0} \frac{||(U\Sigma V^H)^{-1} x||_2}{||x||_2} \\
&= \sup_{x\neq 0} \frac{||(V^H)^{-1}\Sigma^{-1}U^{-1} x||_2}{||x||_2} \\
&= \sup_{x\neq 0} \frac{||\Sigma^{-1} U^H x||_2}{||x||_2} \\
&= \sup_{y\neq 0} \frac{||\Sigma^{-1} y||_2}{||Uy||_2} \\
&= \sup_{y\neq 0} \frac{\left(\sum_{i=1}^n (\frac{1}{\sigma_i})^2 |y_i|^2\right)^{\frac{1}{2}}}{\left(\sum_{i=1}^n |y_i|^2\right)^{\frac{1}{2}}} \\
&\leq \frac{1}{\sigma_1}
\end{aligned}$$

We choose $y = [0 \quad ... \quad 1]^T$, and this will give us an equality of $\frac{1}{\sigma_1}$.

(iii) We first prove the equality of $||A^H A||_2 = ||A||_2^2$.

(iv)

$$
\begin{aligned}
||A_H A||_2 &= \sup_{x \neq 0} \frac{||A^H A x||_2}{||x||_2} \\
&= \sup_{x \neq 0} \frac{||(U\Sigma V^H)^H (U\Sigma V^H) x||_2}{||x||_2} \\
&= \sup_{x \neq 0} \frac{||((V^H)^H \Sigma^H U^H U \Sigma V^H) x||_2}{||x||_2} \\
&= \sup_{x \neq 0} \frac{||(V \Sigma^H \Sigma V^H) x||_2}{||x||_2} \\
&= \sup_{x \neq 0} \frac{||(\Sigma^H \Sigma V^H) x||_2}{||x||_2} \\
&= \sup_{y \neq 0} \frac{\left(\sum_{i=1}^n \sigma_i^4 |y_i|^2 \right)^{\frac{1}{2}}}{\left(\sum_{i=1}^n |y_i|^2 \right)^{\frac{1}{2}}} \\
&\leq \sigma_1^2
\end{aligned}
$$

We choose $y = [1 \quad \ldots \quad 0]^T$, and this will give us an equality of σ_1^2 and thus have equality with $||A||_2^2$.

(v)

$$
\begin{aligned}
||A^H||_2^2 &= \sup_{x \neq 0} \frac{{||A^H x||_2}^2}{||x||_2} \\
&= \sup_{x \neq 0} \frac{{||(U\Sigma V^H)^H x||_2}^2}{||x||_2} \\
&= \sup_{x \neq 0} \frac{{||(V^H)^H \Sigma^H U^H x||_2}^2}{||x||_2} \\
&\leq \overline{\sigma_1}^2
\end{aligned}
$$

We know that σ_1 is the positive square root of the non-zero eigenvalues of $A^H A$, which is positive semidefinite. Thus, $\overline{\sigma_1}^2 = \sigma_1^2 = ||A||_2^2$.

To show that $||A^T||_2^2 = ||A^H||_2^2$ is thus trivial. Thus, all of them are equivalent.

(vi)

$$
\begin{aligned}
||UAV||_2 = \sup_{x \neq 0} \frac{||UUAV^HVx||_2}{||x||_2} \\
&= \sup_{x \neq 0} \frac{||Ax||_2}{||x||_2} \\
&= ||A||_2
\end{aligned}
$$

Problem 4.31

(i)

$$
\begin{aligned}
||UAV||_F^2 &= tr[(UAV)^H(UAV)] \\
&= tr[V^HA^HU^HUAV] \\
&= tr[V^HA^HAV] \\
&= tr[VV^HA^HA] \\
&= tr[A^HA]
\end{aligned}
$$

Thus, $||UAV||_F = ||A||_F$

(ii)

$$
\begin{aligned}
||A||_F^2 &= tr(A^HA) \\
&= tr((U\Sigma V^H)^H U\Sigma V) \\
&= tr(\Sigma^H\Sigma) \\
&= \sigma_1^2 + .. + \sigma_n^2
\end{aligned}
$$

Thus, $||A||_F = \sqrt{\sigma_1^2 + .. + \sigma_n^2}$

Problem 4.33

Proof. We know that $||A||_2 = \sigma_1$ where σ_1 is the greatest singular value of A.
$A = U\Sigma V^H \Rightarrow \Sigma = U^HAV$.
$\sup_{||x||_2=1,||y||_2=1} |y^HAx| = \sigma_1$ holds if we select $y^H = (u_i)^H$ and $x = v_i$ that corresponds to σ_1, the greatest singular value, which is u_1, the first column of U and v_1, the first column of V. □

Problem 4.36

A possible matrix is: $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

Problem 4.38

(i)

$$\begin{aligned}AA^\dagger A &= U_1\Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H A\\ &= A\end{aligned}$$

(ii)

$$\begin{aligned}A^\dagger A A^\dagger &= U_1\Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H A^\dagger\\ &= A^\dagger\end{aligned}$$

(iii)

$$\begin{aligned}(AA^\dagger)^H &= (U_1\Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H)^H\\ &= I^H\\ &= U_1\Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H\\ &= AA^\dagger\end{aligned}$$

(iv)

$$\begin{aligned}(A^\dagger A)^H &= (V_1 \Sigma_1^{-1} U_1^H U_1\Sigma_1 V_1^H)^H\\ &= I^H\\ &= V_1 \Sigma_1^{-1} U_1^H U_1\Sigma_1 V_1^H\\ &= A^\dagger A\end{aligned}$$