Lsn14

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Warmup Problem:

Let $X_1 \sim Exp(\beta = 1)$ and $X_2 \sim Exp(\beta = 1/2)$, find the PDF of $Z = \frac{X_1}{X_2}$

The second method we will talk about using for finding the pdf of a transformed random variable is the method of transformations. The method come straight from what's commonly called U-substitution in Calculus.

$P(U \leq u) = P(h(Y) \leq u) = P(Y \leq h^{-1}(u))$

It's probably worth a note here quick to talk about inverse functions. While I know we've seen these before, recall that if $h(Y) = z$ is a function mapping Y to Z, $h^{-1}(Z) = Y$. For example if $h(.) = \exp(.)$ then $h^{-1}(.) = \log(.)$.

Our instincts can fail us sometimes on inverse functions though. For example if $h(x) = x^2 = z$, the $h^{-1}(z) = \sqrt{z}$ *only* for $x > 0$. If $x < 0$ then $h^{-1}(z) = -\sqrt{z}$.

Our text accounts for this with the following caveat on pg. 313. Let Y have probabilty density function $f_y(y)$. If $h(y)$ is **either increasing or decreasing** for all y in the support of Y, then $U = h(Y)$ has density function

$$f_U(u) = f_Y[h^{-1}(u)]|\frac{dh^{-1}}{du}|$$

So, to summarize, we need to first recognize that $h(.)$ is strictly increasing or decreasing on the support of y. Then we need to find the inverse of $h(.)$, substitute it into the pdf of y and multiply by $|\frac{dh^{-1}}{du}|$.

For example: Let $Z \sim Exp(2)$, find the density function of $U = Z^2$

First we note that the support of Z is $\mathbb{R}^+$. So, on the support of Z, $h(Z) = (Z)^2$ is an increasing function we can employ the transformation method.

Next we note that $h^{-1}(.) =?$

and $|\frac{dh^{-1}}{du}| =?$.

We can also calculate:

$f_Y(h^{-1}(u)) = f_Y(\sqrt{u}) =?$

So, all together

$f_U(u) =?$

As it turns out, this is what's called a Weibull distribution.

Note that if our question had been $Z \sim (\mu, \sigma)$ and we were asked to find the density function of $U = Z^2$ we could not use the above technique. (why?)

Consider a random variable Y that has a uniform distribution on the interval (1,5). Find the density function of $U = 2Y^2 + 3$

The same thought process can be extended to multi-variate transformations (though as we will see in Section 6.6, I really like to think of these a bit differently). In this case, we will let $U = f(y_1, y_2)$ and let $G = y_1$. We then find the joint density of U and $G = y_1$ and integrate out y_1. Let me show this with an example:

Consider $f(y_1, y_2) = \frac{1}{8} y_1 \exp(-(y_1 + y_2)/2)\mathbb{1}(0 \leq y_1)\mathbb{1}(0 \leq y_2)$ and say we want to find the density of $U = \frac{Y_2}{Y_1}$. Here we let $G = Y_1$ and note that U is a decreasing function for Y_2.Note now that we are only thinking of U as a function of Y_2. So, $U = h(Y_2)$ and therefore $h^{-1}(u) = uy_1$.

We need a few more pieces. It follows from above that $Y_2 = U * G$, $h^{-1}(u) = U * G$, and clearly $Y_1 = G$. Using this, we can re-write

$$f(y_1, y_2) = f(U, G) =$$

After we do this, we next want to marginalize over G by integrating over the support of G.

We could do this via Mathematica (Nothing wrong with this!). OR, we could squint our eyes and realize that this is *almost* a Gamma distribution with $\alpha = 3$ and $\beta = \frac{1+U}{2}$

As I mentioned above, I don't really like to do this for bivariate transformations. The reason is it's not technically correct. We actually need the determinant of the Jacobian of $\boldsymbol{h}^{-1}(U, G) = (G, UG) = (Y_1, Y_2)$. Recall that a Jacobian is:

We will talk more about this in future lessons, but if we take the determinant of the Jacobian we end up with: