Lsn9

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Admin/HW

Recall that for a general PDF, $f(y)$ we could find $E[g(Y)]$ through

The same concepts can be applied to joint PDFs, $f(y_1, y_2)$ to find $E[g(Y_1, Y_2)]$. For example, if our PDF is $f(y_1, y_2) = 6(1 - y_2)\mathbb{1}(0 \leq y_1 \leq y_2)\mathbb{1}(y_1 \leq y_2 \leq 1)$ we can find $E[Y_1 - 3Y_2]$

Note that even if $g(Y_1, Y_2) = Y_1$ the same process would apply. However, if we marginal PDF for Y_1, say $f_1(y_1)$ we could integrate with respect to the marginal PDF only. To see this note:

Let's use this to find $E[Y_1]$ and $E[Y_2]$ for our above joint PDF.

The same properties for expectation that we previously found still apply (Theorems 5.6-5.8) but there's one new theorem that will be extremely important in our future work, namely that if Y_1 and Y_2 are **independent** then $E[g(Y_1)h(Y_2)] = E[g(Y_1)]E[h(Y_2)]$ which is a direct consequence for the fact that $f(y_1, y_2) = f_1(y_1)f_2(y_2)$ for independent random variables.

This, in some cases, makes our life much easier. For example, let's consider problem 5.74 in our text. The joint PDF here is $f(y_1, y_2) = 1\mathbb{1}(0 \leq y_1 1)\mathbb{1}(0 \leq y_2 \leq 1)$ Without much thought, I can immediately say that y_1 and y_2 are independent. Why? What do you think the marginal densities for y_1 and y_2 are?

Now that I know this, what is $E[Y_1 Y_2]$? Given that the second moment for a uniform random variable is $\frac{1}{3}(\theta_1 + \theta_2)^2$ what is $E[(Y_1 Y_2)^2]$.

Covariances/Correlations

A common use for expectations for jointly distributed random variables is in the calculation of covariances or correlations. We can think of a correlation as quantifying how one variable behaves as another variable changes. For example, if our Correlation is close to 1, high values for X would suggest likely high values for Y if X and Y have joint pdf $f(x, y)$. By far the most common way to calculate Covariance is by:

And correlation by:

If X and Y are independent, we have $E[XY] = E[X]E[Y]$ from above, so therefore $Cov[X, Y] =?$

Answering problems about Correlation/covariance often just take book keeping and pushing through lots of integrals. For example, let's look at $f(y_1, y_2) = 6(1 - y_2)\mathbb{1}(0 \leq y_1 \leq y_2)\mathbb{1}(y_1 \leq y_2 \leq 1)$ and say we want to find the Correlation between Y_1 and Y_2 what expected values do we need?

Here's one of those instances where I'd probably just use Mathematica or another tool (not that I have to, I've got some awesome Calculus skills!)

Let Y_1 and Y_2 be uncorrelated random variables and consider $U_1 = Y_1 + Y_2$ and $U_2 = Y_1 - Y_2$. Find $\text{Cov}(U_1, U_2)$ in terms of the Variance of Y_1 and Variance of Y_2

Conditional Expectation

This is not covered in our syllabus (though it is in WMS section 5.11), but I think it's important so we'll just touch on it a bit here. The same thing we did above we can do for conditional PDFs. Recall we found $f(y_1|y_2) = \frac{f(y_1,y_2)}{f_2(y_2)}$. We can use this conditional PDF in our expectation function by finding

$$E[X|Y = y_2] = \int_{-\infty}^{\infty} X f(x|y) dy$$

For example, our book has $f(x, y) = \frac{1}{2}\mathbb{1}(0 \leq x \leq 2)\mathbb{1}(x \leq y \leq 2)$

and finds that $E[Y_1|Y_2 = y_2] = \frac{y_2}{2}\mathbb{1}(0 \leq y_2 \leq y_2)$

One of the coolest uses of this is in what's called the law of iterated expectation. Often times we are presented with a problem where we say to ourselves, it sure would be easier to solve this problem if I had more information.

For example:

A quality control plan for an assembly line involves sampling $n = 10$ finsihed items per day and counting Y, the number of dectives. If p denotes the probability of observing a defect, then $Y \sim \text{Binom}(n, p)$. But p varies from day to day and is assumed to have a distribution $p \sim \text{Unif}(0, 1/4)$. What is the expected value of Y?

In this problem it would be **much** easier to find $E[Y]$ if we *knew* p. However we don't... One problem solving strategy is to use what is called iterated expectation using the fact $E[Y] = E[E[Y|p]]$. That is, we first take the expected value of Y assuming p is known, then find the expected value of p. For this case, if we assume p is known, then $E[Y|p] = np$ by properties of the Binomial distribution. Then we take the outer expectation $E[np]$ relying on the distribution of p we know $E[p] = \frac{1}{8}$. So therefore, $E[Y] = E[E[Y|p]] = E[np] = n\frac{1}{8} = 1.25$.