e is a transcendental number

Basic definitions

- For any polynomial $f \in \mathbb{Z}[X] = a_0 + a_1 X + \cdots + a_n X^n$, $\bar{f} := |a_0| + |a_1|X + \cdots + |a_n|X^n$. See in `f_bar`.
- For any prime number p and natural number n we can define a polynomial $f_{p,n} \in \mathbb{Z}[X]$ as $X^{p-1}(X-1)^p \cdots (X-n)^p$. If p and n is clear from context, we are going to ignore the subscript.
- $f_{p,n}$ has degree $(n+1)p-1$.
- With f defined and any nonnegative real number t, we associate f with an integral $I(f,t)$ to be

$$\int_0^t e^{t-x} f(x) \mathrm{d}x$$

- If f has degree n, then using integrating by part n times we have

$$I(f,t) = e^t \sum_{i=0}^{n} f^{(i)}(0) - \sum_{i=0}^{n} f^{(i)}(t)$$

- For any polynomial $g \in \mathbb{Z}$ with degree d and coefficient g_i, $J_p(g)$ is defined to be

$$J_p(g) = \sum_{i=0}^{d} g_i I(f_{p,d}, i)$$

So if $g(e) = 0$, we will have

$$\begin{aligned}
J_p(g) &= \sum_{i=0}^{d} g_i I(f_{p,d}, i) \\
&= \sum_{i=0}^{d} g_i \left(e^i \sum_{j=0}^{(n+1)p-1} f_{p,d}^{(j)}(0) - \sum_{j=0}^{(n+1)p-1} f_{p,d}^{(j)}(i) \right) \\
&= \sum_{j=0}^{(n+1)p-1} f_{p,d}^{(j)}(0) \sum_{i=0}^{d} g_i e^i - \sum_{j=0}^{(n+1)p-1} \sum_{i=0}^{d} g_i f_{p,d}^{(j)}(i) \\
&= - \sum_{j=0}^{(n+1)p-1} \sum_{i=0}^{d} g_i f_{p,d}^{(j)}(i)
\end{aligned}$$

We are going to deduce two contradictory bounds for $J_p(g)$ with a large prime p. To evaluate the sum, we will split the big sum $\sum_{j=0}^{(n+1)p-1}$ to three sums: $j < p-1$, $j = p-1$ and $j > p-1$.

Lower bound

Using the notation as above, for any prime p and natural number n, we have the followings :

- If $j < p-1$ then in this case, in fact all the summand is zero;
- If $j = p-1$ then $f_{p,n}^{(j)}(0) = (p-1)!(-1)^{np}n!^p$ and $f_{p,n}^{(j)}(i) = 0$ for all $i > 0$.
- If $j > p-1$ then $p!|f_{p,n}^{(j)}(k)$ for all $k = 0, \cdots, n$.

Then if $g \in \mathbb{Z}$ is any polynomial with degree n and coefficient g_i with $g_0 \neq 0$ and e as a root then, from above we can show that there is some $M \in \mathbb{Z}$ such that

$$J_p(g) = -g_0(p-1)!(-1)^{np}n!^p + M \times p!$$

So if we choose p to be a prime number such that $p > n$ and $p > |g_0|$, then $|J_p(g)| = (p-1)! \left|-g_0(-1)^{np}n!^p + Mp\right|$. So $(p-1)! \leq J_p(g)$. Because otherwise $|-g_0(-1)^{np}n!^p + Mp| = 0$. So $p|g_0n!^p$, then either $p|g_0$ or $p|n!^p$. The first case cannot happen as we chose $p > |g_0|$. The second happens if and only if $p|n!$ but we chose $p > n$.

Upper bound

This time we utilize the integral definition of I. For a prime p and $g \in \mathbb{Z}$ is any polynomial with degree n and coefficient g_i and e as a root then: let A be the maximum of g_i

$$\begin{aligned}
|J_p(g)| &= \left|\sum_{i=0}^{d} g_i I(f_{p,d}, i)\right| \\
&\leq \sum_{i=0}^{d} |g_i| \, |I(f_{p,d}, i)| \\
&\leq \sum_{i=0}^{d} |g_i| \, |I(f_{p,d}, i)| \\
&= \sum_{i=0}^{d} |g_i| \left|\int_0^t e^{i-x} f_{p,d}(x) \mathrm{d}x\right| \\
&\leq \sum_{i=0}^{d} |g_i| i e^i \bar{f}_{p,d}(i) \\
&\leq \sum_{i=0}^{d} |g_i| (d+1) e^{d+1} (2(d+1))^{p+pd} \\
&\leq A^p (d+1)^p (e^{d+1})^p ((2(d+1))^{d+1})^p
\end{aligned}$$

The point is for some real number c (independent of p, depending on g), $|J_p(g)| \leq c^p$.

The desired contradiction

Assume e is algebraic and $g \in \mathbb{Z}[X]$ admits e as a root with degree d and coefficient g_i. We can assume $g_0 \neq 0$ by dividing a suitable power of X if necessary. Then we know that for some real number c independent of g, we have $(p-1)! \leq J_p(g) \leq c^p$ for all $p > |g_0|$ and $p > d$. But this is not possible.