String Theory №1

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$\boxed{1}$ Read *Polchinski* Sections 1.3 and 1.4:

Read, *mostly* understood. $\square$

$\boxed{2}$ Spinning Closed String in AdS Space:

For a classical spinning string, we have Nambu–Goto action:

$$S_{NG} = -T \int \mathrm{d}\tau\, \mathrm{d}\sigma\, \sqrt{-\det \gamma_{ab}}, \quad \gamma_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \tag{1}$$

Here $G_{\mu\nu}$ is the spacetime metric. γ_{ab} can be treated as the induced metric on the worldsheet.

In AdS space we have:

$$ds^2 = R^2\left(-\cosh^2\!\rho\, \mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\!\rho\, \mathrm{d}\Omega^2\right) \tag{2}$$

Where $\mathrm{d}\Omega^2$ is the metric of a unit $(d-2)$–sphere S^{d-2}. For convenience let's define unit S^{d-2} metric G^1_{ij}, and raise or lower the $i, j, \cdots$ indices using G^1_{ij} instead of G_{ij}, i.e.,

$$G^1_{ij} = G_{ij} / \left(R^2 \sinh^2\!\rho\right), \quad i, j = 2, \cdots, d-1 \tag{3}$$

Furthermore, we consider the special case that the closed string is *folded*, like a rubber band stretched along a line; in this case we can choose the worldsheet parameter $(\tau, \sigma) = (t, \rho)$ while $\Omega = \Omega(t, \rho) = \Omega(\tau, \sigma)$, which leads to the following decomposition:

$$\partial_a X^\mu = \delta^\mu_a + \delta^\mu_i\, \partial_a \Omega^i, \quad a = 0, 1, \quad i = 2, \cdots, d-1, \tag{4}$$

$$\begin{aligned}
\gamma_{ab} &= G_{\mu\nu}\, \partial_a X^\mu \partial_b X^\nu \\
&= G_{ab} + G_{ij}\, \partial_a \Omega^i\, \partial_b \Omega^j \\
&= G_{ab} + R^2 \sinh^2\!\rho\, G^1_{ij}\, \partial_a \Omega^i\, \partial_b \Omega^j \\
&= R^2 \left\{ \begin{pmatrix} -\cosh^2\!\rho & \\ & 1 \end{pmatrix} + \sinh^2\!\rho \begin{pmatrix} (\partial_a \Omega)^2 & \partial_a \Omega \cdot \partial_b \Omega \\ \partial_b \Omega \cdot \partial_a \Omega & (\partial_b \Omega)^2 \end{pmatrix} \right\}
\end{aligned} \tag{5}$$

Here $\partial_a \Omega \cdot \partial_b \Omega \equiv \partial_a \Omega^i\, \partial_b \Omega_i \equiv G^1_{ij}\, \partial_a \Omega^i\, \partial_b \Omega^j$, and we have:

$$\begin{aligned}
\det \gamma_{ab} = (R^2)^2 \Big\{ &\sinh^4\!\rho\, \det\left(\partial_a \Omega^i \partial_b \Omega_i\right) \\
&+ \sinh^2\!\rho \left((\partial_a \Omega)^2 - (\partial_b \Omega)^2 \cosh^2\!\rho\right) \\
&- \cosh^2\!\rho \Big\}, \\
\sqrt{-\det \gamma_{ab}} = R^2 \Big\{ &\cosh^2\!\rho - \sinh^2\!\rho \left((\partial_a \Omega)^2 - (\partial_b \Omega)^2 \cosh^2\!\rho\right) \\
&- \sinh^4\!\rho\, \det\left(\partial_a \Omega^i \partial_b \Omega_i\right) \Big\}^{1/2}
\end{aligned} \tag{6}$$

Mark the end points of the string with $\rho = r(t)$, then the total length of such closed folded string is $\ell = 4r$. We then have:

$$S = -4TR^2 \int \mathrm{d}t \int_0^r \mathrm{d}\rho\, \sqrt{\cosh^2\!\rho - \sinh^2\!\rho \left((\partial_a \Omega)^2 - (\partial_b \Omega)^2 \cosh^2\!\rho\right) - \sinh^4\!\rho\, \det\left(\partial_a \Omega^i \partial_b \Omega_i\right)} \tag{7}$$

Further simplification comes from the fact that, due to rotational symmetry, the string's motion can be restricted in a plane where its position is characterized by some angle $\theta = \Omega^{i_0} \in \{\Omega^i\}_i$. In this case other angle parameters $\Omega^i|_{i\neq i_0} = 0$, and the action is further reduced to:

$$S = -4TR^2 \int \mathrm{d}t \int_0^r \mathrm{d}\rho \sqrt{\cosh^2\rho - \sinh^2\rho \left((\partial_a\theta)^2 - (\partial_b\theta)^2 \cosh^2\rho\right)} = \int \mathrm{d}t \int_0^r \mathrm{d}\rho\, \mathcal{L}, \tag{8}$$

$$\mathcal{L} = -4TR^2 \sqrt{\cosh^2\rho - \omega^2 \sinh^2\rho}, \quad \omega = \partial_t\theta,\ \partial_\rho\theta = 0 \tag{9}$$

We consider the special solution $\theta = \omega t$, while in general the endpoint $r = r(t)$ could be dynamical; variation of the action w.r.t. $r(t)$ gives[1]:

$$0 = \delta S = -4TR^2 \int \mathrm{d}t \int_r^{r+\delta r} \mathrm{d}\rho \sqrt{\cosh^2\rho - \omega^2 \sinh^2\rho} = -4TR^2 \int \mathrm{d}t \sqrt{\cosh^2 r - \omega^2 \sinh^2 r}\, \delta r\,, \tag{10}$$

$$\omega^2 = \frac{\cosh^2 r}{\sinh^2 r} = \coth^2 r \tag{11}$$

Note that if ω is constant, then r must be fixed by (11). Taking θ as the only dynamical variable, it is then straight-forward to write the energy E and angular momentum J for such folded closed string:

$$\omega = \dot\theta, \quad \Pi = \frac{\partial\mathcal{L}}{\partial\omega} = 4TR^2 \frac{\omega \sinh^2\rho}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}}, \tag{12}$$

$$J = \int_0^r \mathrm{d}\rho\, \Pi = 4TR^2 \int_0^r \mathrm{d}\rho \frac{\omega \sinh^2\rho}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}}, \tag{13}$$

$$E = \int_0^r \mathrm{d}\rho\, (\Pi\omega - \mathcal{L}) = 4TR^2 \int_0^r \mathrm{d}\rho \frac{\cosh^2\rho}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}}, \tag{14}$$

In the large string limit, $r \to \infty$, $\omega = \coth r \to 1$. Expand in terms of $\epsilon = \omega - 1 > 0$, we find that $r = \frac{1}{2}\ln\left(1 + \frac{2}{\epsilon}\right) \sim \frac{1}{2}\ln\frac{2}{\epsilon}$, or alternatively, $e^{2r}\cdot\epsilon \sim 2$. With some help from Mathematica™, we get:

$$\begin{aligned} E - J &= 4TR^2 \int_0^r \mathrm{d}\rho \frac{\cosh^2\rho - \omega\sinh^2\rho}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}} = 4TR^2 \int_0^r \mathrm{d}\rho \left(1 + \frac{\epsilon^2}{8}\sinh^2(2\rho) + \mathcal{O}(\epsilon^3)\right) \\ &= 4TR^2\left(r\left(1 - \frac{\epsilon^2}{16} + \mathcal{O}(\epsilon^3)\right) + \mathcal{O}(1)\right) = \left(2TR^2 \ln\frac{2}{\epsilon}\right)\left(1 - \frac{\epsilon^2}{16} + \mathcal{O}(\epsilon^3)\right) \\ &\sim 2TR^2\left(\ln\frac{2}{\epsilon}\right) \end{aligned} \tag{15}$$

Similarly, $J \sim 4TR^2 \int_0^r \mathrm{d}\rho \sinh^2\rho \sim TR^2\left(\frac{2}{\epsilon}\right)$, this gives:

$$E - J \sim 2TR^2 \ln\frac{J}{TR^2} \tag{16}$$

■

[1] The above reasoning is confirmed in e.g. arXiv:hep-th/0204051.

$\boxed{3}$ Special Conformal Transformations:

$$x^\mu \xmapsto{K(a)} \tilde{x}^\mu = \frac{x^\mu + x^2 a^\mu}{1 + 2a\cdot x + a^2 x^2} \tag{17}$$

(a) Under special conformal transformation $K(a)$, metric $\delta_{\mu\nu} \mapsto g_{\mu\nu}$ while:

$$g_{\alpha\beta}\, \mathrm{d}\tilde{x}^\alpha\, \mathrm{d}\tilde{x}^\beta = \delta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu\,, \quad g_{\alpha\beta} = \delta_{\mu\nu}\frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial x^\nu}{\partial \tilde{x}^\beta} \tag{18}$$

To calculate this we have to know the inverse transformation $x = K^{-1}(a)\,\tilde{x}$. First, notice the following decomposition[2] of $K(a)$:

$$\tilde{x}^\mu = \frac{\frac{x^\mu}{x^2} + a^\mu}{\frac{1}{x^2} + \frac{2a\cdot x}{x^2} + a^2} = \frac{\frac{x^\mu}{x^2} + a^\mu}{\left|\frac{x^\mu}{x^2} + a^\mu\right|^2}, \tag{19}$$

$$\text{i.e.}\quad K(a)\colon\ x^\mu \xmapsto{I} \frac{x^\mu}{x^2} \xmapsto{T(a)} y^\mu = \frac{x^\mu}{x^2} + a^\mu \xmapsto{I} \tilde{x}^\mu = \frac{y^\mu}{y^2}, \tag{20}$$

$$\text{i.e.}\quad \frac{\tilde{x}^\mu}{\tilde{x}^2} = \frac{y^\mu}{y^2}\Big/\frac{1}{y^2} = y^\mu = \frac{x^\mu}{x^2} + a^\mu \tag{21}$$

From (21), we see that the transformation parameter a^μ composes linearly: $K(b)\,K(a) = K(a{+}b)$, therefore $K^{-1}(a) = K(-a)$, and we have:

$$x^\mu = K(-a)\,\tilde{x}^\mu = \frac{\tilde{x}^\mu - \tilde{x}^2 a^\mu}{1 - 2a\cdot\tilde{x} + a^2\tilde{x}^2} = \frac{\tilde{y}^\mu}{\tilde{y}^2}, \tag{22}$$

$$\begin{aligned}\frac{\partial x^\mu}{\partial \tilde{x}^\alpha} &= \frac{\partial x^\mu}{\partial \tilde{y}^\sigma}\frac{\partial \tilde{y}^\sigma}{\partial \tilde{x}^\alpha} = \left(\frac{\partial}{\partial \tilde{y}^\sigma}\frac{\tilde{y}^\mu}{\tilde{y}^2}\right)\frac{\partial}{\partial \tilde{x}^\alpha}\left(\frac{\tilde{x}^\sigma}{\tilde{x}^2} - a^\sigma\right) = \left(\frac{\partial}{\partial \tilde{y}^\sigma}\frac{\tilde{y}^\mu}{\tilde{y}^2}\right)\left(\frac{\partial}{\partial \tilde{x}^\alpha}\frac{\tilde{x}^\sigma}{\tilde{x}^2}\right)\\ &= \left(\tilde{y}^2\delta^\mu_\sigma - 2\tilde{y}^\mu\tilde{y}_\sigma\right)\left(\tilde{x}^2\delta^\sigma_\alpha - 2\tilde{x}^\sigma\tilde{x}_\alpha\right)\Big/\left(\tilde{y}^4\tilde{x}^4\right),\end{aligned} \tag{23}$$

$$\begin{aligned} g_{\alpha\beta} &\overset{(18)}{=\!=} \delta_{\mu\nu}\left(\tilde{y}^2\delta^\mu_\sigma - 2\tilde{y}^\mu\tilde{y}_\sigma\right)\left(\tilde{x}^2\delta^\sigma_\alpha - 2\tilde{x}^\sigma\tilde{x}_\alpha\right)\left(\tilde{y}^2\delta^\nu_\rho - 2\tilde{y}^\nu\tilde{y}_\rho\right)\left(\tilde{x}^2\delta^\rho_\beta - 2\tilde{x}^\rho\tilde{x}_\beta\right)\Big/\left(\tilde{y}^8\tilde{x}^8\right)\\ &\overset{\sum_{\mu,\nu}}{=\!=} \tilde{y}^{-4}\delta_{\sigma\rho}\left(\tilde{x}^2\delta^\sigma_\alpha - 2\tilde{x}^\sigma\tilde{x}_\alpha\right)\left(\tilde{x}^2\delta^\rho_\beta - 2\tilde{x}^\rho\tilde{x}_\beta\right)\Big/\tilde{x}^8\\ &\overset{\sum_{\sigma,\rho}}{=\!=} \tilde{y}^{-4}\tilde{x}^{-4}\delta_{\alpha\beta}\end{aligned} \tag{24}$$

We see that $g_{\alpha\beta} = f(x)\,\delta_{\alpha\beta}$, with coefficient:

$$f(x) = \tilde{y}^{-4}\tilde{x}^{-4} \overset{(20)}{=\!=} \frac{x^4}{\tilde{x}^4} \overset{(21)}{=\!=} \left(1 + 2a\cdot x + a^2x^2\right)^2 \tag{25}$$

$\square_{(a)}$

(b) In 2D with $z = x^1 + ix^2$, $x^\mu \sim (z,\bar{z})$, we see from (21) that:

$$\frac{x^\mu}{x^2} \sim \frac{z}{|z|^2} = \frac{1}{\bar{z}} \longmapsto \frac{1}{\bar{z}} + a, \quad \text{i.e.}\quad z \longmapsto w = \frac{1}{\frac{1}{z} + \bar{a}} = \frac{z}{1 + z\bar{a}} \tag{26}$$

Expand in the $\bar{a} \to 0$ limit, we find that $w = z\,(1 - z\bar{a} - \cdots) \sim z - z^2\bar{a}$, i.e. it is generated by:

$$K_{\bar{z}} = -z^2\partial_z = -z^2\partial, \quad \partial \equiv \partial_z \tag{27}$$

[2] See *Di Francesco et al*, and also github.com/davidsd/ph229.

Note that when considering non-holomorphic functions, we have to consider $(z, \bar{z})$ as *two* independent variables; hence the anti-holomorphic transformation $\bar{z} \mapsto \bar{w} = \frac{\bar{z}}{1+\bar{z}a} \sim \bar{z} - \bar{z}^2 a$ provides another degree of freedom, namely:

$$\begin{gathered} K_\mu \;\sim\; \left(K_{\bar{z}} = -z^2 \partial,\; K_z = -\bar{z}^2 \bar{\partial}\right), \\ \partial \equiv \partial_z,\; \bar{\partial} \equiv \partial_{\bar{z}} \end{gathered} \tag{28}$$

Similarly, for translation $z \mapsto z + a$ and its conjugate, we have $P_\mu \sim \left(P_z = \partial, P_{\bar{z}} = \bar{\partial}\right)$. However, dilation and rotation are both encoded in a complex rescaling $z \mapsto \lambda z,\ \lambda = re^{i\theta} \in \mathbb{C}$; we have:

$$z \mapsto \lambda z, \quad \lambda = re^{i\theta} \in \mathbb{C}, \quad \begin{aligned} \delta r \;&\longleftrightarrow\; D = z\,\partial + \bar{z}\,\bar{\partial}, \\ \delta\theta \;&\longleftrightarrow\; M = i\left(z\,\partial - \bar{z}\,\bar{\partial}\right), \end{aligned} \tag{29}$$

In summary, we have $\mathrm{span}_{\mathbb{R}} \{P_\mu, K_\mu, D, M\} = \mathfrak{so}(3,1)$ generating the "global" transformation subgroup of the 2D conformal group; here, the $\mathfrak{so}(3,1)$ boost is a linear combination[3] of P_μ and K_μ. More specifically, in 2D any holomorphic or anti-holomorphic function gives a conformal transformation, hence the (classical) 2D conformal group is generated by:

$$\ell_m = z^{m+1}\partial, \quad \bar{\ell}_m = \bar{z}^{m+1}\bar{\partial}, \quad m \in \mathbb{Z} \tag{30}$$

i.e. the *Witt algebra* (or Virasoro algbera $\mathbf{Vir}_c$ with $c = 0$). It is clear that a (complexified) $\mathfrak{so}(3,1)$ lives inside $\mathbf{Vir}_c$, i.e.,

$$\begin{aligned} \mathfrak{so}(3,1)^{\mathbb{C}} &= \mathrm{span}_{\mathbb{C}} \{P_\mu, K_\mu, D, M\} \\ &= \mathrm{span}_{\mathbb{C}} \left\{\ell_m, \bar{\ell}_m \,\middle|\, m = 0, \pm 1\right\} = \mathfrak{sl}(2,\mathbb{R})^{\mathbb{C}} \oplus_{\mathbb{C}} \mathfrak{sl}(2,\mathbb{R})^{\mathbb{C}} \subset \mathbf{Vir}_c \end{aligned} \tag{31}$$

■

$\boxed{\mathbf{4}}$ *bc* **CFT**:

$$S = \frac{1}{2\pi} \int \mathrm{d}^2 z\, b\, \bar{\partial} c \tag{32}$$

Stress tensor of a theory can be obtained via variation over the metric, or equivalently, over the fields ϕ^i with $\delta\phi$ induced by some *local* spacetime translation $x^\mu \mapsto x^\mu + \delta x^\mu$, $\delta x^\mu = \epsilon(x)\, a^\mu$. Here $\epsilon(x)$ is any compactly supported bump function, centered around some point x_0.

In 2D, we have $\mu = z, \bar{z}$; for $\phi(z, \bar{z})$ with conformal weight $(h, \bar{h})$, consider $z \mapsto z'$, $\bar{z} \mapsto \bar{z}'$. For convenience, let's first consider a generic variation $\delta z = \epsilon(z, \bar{z})$ before restricting to spacetime translation; we have:

$$\phi'(z', \bar{z}') = \left(\frac{\mathrm{d}z'}{\mathrm{d}z}\right)^{-h} \left(\frac{\mathrm{d}\bar{z}'}{\mathrm{d}\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}), \tag{33}$$

$$\tilde{\delta}\phi = \left(-h\,\partial\epsilon - \bar{h}\,\bar{\partial}\bar{\epsilon}\right)\phi, \tag{34}$$

$$\delta\phi = \tilde{\delta}\phi - \frac{\partial\phi}{\partial x^\mu}\,\delta x^\mu = \left(-h\,\partial\epsilon - \bar{h}\,\bar{\partial}\bar{\epsilon}\right)\phi - \epsilon\,\partial\phi - \bar{\epsilon}\,\bar{\partial}\phi, \tag{35}$$

Here we use $\tilde{\delta}\phi$ to denote the "internal" variation related to the conformal weights.

[3] See e.g. `github.com/davidsd/ph229`.

Note that $\phi = b, c$ are anti-commuting Grassmann numbers, variation of the action gives:

$$\begin{aligned}\delta S\,[b, c, \delta b\,, \delta c] &= \frac{1}{2\pi}\int \mathrm{d}^2 z\,\big(\delta b\;\bar\partial c + b\,\bar\partial\,\delta c\big)\\ &= \frac{1}{2\pi}\int \mathrm{d}^2 z\,\big(-\bar\partial c\;\delta b - \bar\partial b\;\delta c\big) + \frac{1}{2\pi}\int \mathrm{d}^2 z\,\bar\partial(b\,\delta c)\end{aligned} \tag{36}$$

For *unknown* b, c and arbitary $\delta b\,, \delta c$, the second term is reduced to a boundary term at infinity and can be dropped; imposing $\delta S = 0$ gives the equation of motion (EOM): $\bar\partial b = \bar\partial c = 0$.

On the other hand, for *on-shell* b, c and compactly supported $\varphi = \delta b\,, \delta c$ given in (35), the first term in (36) vanishes while $\delta S_0 = 0$ still holds; this gives:

$$\begin{aligned}0 = \delta S_0 &= \frac{1}{2\pi}\int \mathrm{d}^2 z\,\bar\partial(b\,\delta c) = \frac{1}{2\pi}\int \mathrm{d}^2 z\,\bar\partial\,\big(-(1-\lambda)\,bc\,\partial\epsilon - b\,\partial c\,\epsilon\big)\\ &= \frac{1}{2\pi}\int \mathrm{d}^2 z\,\big(-(1-\lambda)\,bc\,\bar\partial\partial\epsilon - b\,\partial c\,\bar\partial\epsilon\big)\end{aligned} \tag{37}$$

Here we've distributed the $\bar\partial$ operator and dropped all terms that vanish automatically by EOM. Next we shall collect the $\partial\epsilon, \bar\partial\epsilon$ terms; integrating by parts on the first integrand gives:

$$\begin{aligned}0 = \delta S_0 &= \frac{1}{2\pi}\int \mathrm{d}^2 z\,\big((1-\lambda)\,\partial(bc)\; - b\,\partial c\big)\,\bar\partial\epsilon\\ &= \frac{1}{2\pi}\int \mathrm{d}^2 z\,\big((\partial b)\,c - \lambda\,\partial(bc)\big)\,\bar\partial\epsilon\\ &= -\frac{1}{2\pi}\int \mathrm{d}^2 z\,\epsilon(z,\bar z)\,\partial_{\bar z}\big((\partial b)\,c - \lambda\,\partial(bc)\big)\end{aligned} \tag{38}$$

Notice that we have obtained a conserved current using a generic $\delta z = \epsilon(z, \bar z), \delta\bar z = \bar\epsilon(z, \bar z)$; by setting $\epsilon = \epsilon(z)$, we get *a* energy momentum tensor[4]:

$$T(z) = {:}(\partial b)\,c{:} - \lambda\,\partial({:}bc{:}) \tag{39}$$

Normal ordering is added manually to remove singular terms.

To compute TT OPE, we need the OPE of $b(z)\,c(0)$; this is obtained by examining the following path integral, which is zero since the integrand is a total functional derivative:

$$0 = \int \mathscr{D}b\,\mathscr{D}c\,\frac{\delta}{\delta\phi}\big(e^{-S}\,\phi\big) \tag{40}$$

Taking $\phi = b, c$, this generates operator equations such as $\bar\partial b(z)\,c(0) = 2\pi\delta^2(z, \bar z)$. Note that $\bar\partial(\frac{1}{z}) = 2\pi\delta^2(z, \bar z)$, which gives:

$$b(z)\,c(0) \sim c(z)\,b(0) \sim \frac{1}{z} \tag{41}$$

[4]Note that the energy momentum tensor obtained in this way is generally *not* unique: it can be off by a boundary term; see Luboš' comment at physics.stackexchange.com/a/96100, also arXiv:1601.03616. However, it is possible to fix this redundancy by considering Tb OPE and match its conformal dimension. I would like to thank 林般 for pointing this out.

With the bc OPE in hand, the TT OPE is computed directly with blunt force. This gives:

$$T(z)\,T(0) \sim \frac{-6\lambda^2+6\lambda-1}{z^4} + \cdots \tag{42}$$

In general we have $-6\lambda^2+6\lambda-1 = \frac{c}{2}$; for $\lambda = 2$ this gives $c = -26$. ■

$\boxed{5}$ **Free Fermion CFT:**

$$S = \int \mathrm{d}^2 z\, \psi_i\, \bar{\partial}\psi^i, \quad \psi^i = \psi_i^*, \quad \psi_i = \psi_i(z) \tag{43}$$

(a) Mode expansion of such chiral fermion is given by:

$$\psi_i = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \frac{b_{ik}}{z^{k+\frac{1}{2}}}, \quad b_{ik} = \frac{1}{2\pi i}\oint \mathrm{d}z\, z^{k-\frac{1}{2}}\psi_i \tag{44}$$

Canonical quantization is achieved by simply imposing anti-commutation relations; this is justified by mapping the system onto a cylinder, then b_{ik}'s indeed map to modes on the spatial circle[5]. The only non-zero commutators are:

$$\left\{b_{ik}, b_q^{j\dagger}\right\} = \delta_{k+q,0}\,\delta_i^j \tag{45}$$

This gives the only non-zero 2-point functions:

$$\begin{aligned}\langle\psi_i(z)\,\psi^j(w)\rangle &= \sum_{k,q\in\mathbb{Z}+\frac{1}{2}} \frac{1}{z^{k+\frac{1}{2}}}\frac{1}{w^{q+\frac{1}{2}}}\langle b_{ik} b_q^{j\dagger}\rangle \\ &= \sum_{k,q\in\mathbb{Z}+\frac{1}{2}} \frac{1}{z^{k+\frac{1}{2}}}\frac{1}{w^{q+\frac{1}{2}}}\langle 0|\left\{b_{ik}, b_q^{j\dagger}\right\}|0\rangle = \frac{\delta_i^j}{z-w}\end{aligned} \tag{46}$$

Note that $b_k^i\,|0\rangle = 0,\ \forall\ k \geq \frac{1}{2}$.

(b)(c) Combining two ψ expansions gives the mode expansion of ${J_i}^j = :\psi_i(z)\,\psi^j(z)::$

$${J_i}^j(z) = \sum_{k\in\mathbb{Z}} \frac{({J_i}^j)_k}{z^{k+1}}, \quad ({J_i}^j)_k = \sum_{q\in\mathbb{Z}+\frac{1}{2}} :b_{iq}\, b_{k-q}^{j\dagger}: \tag{47}$$

It is in fact more convenient to obtain the JJ OPE first, and then use it to find the $[J_0, J_0]$ mode commutator[6]; note that $\psi_i(z)\,\psi^j(w)$ contraction gives $\frac{\delta_i^j}{z-w}$, we have:

$${J_i}^j(z)\,{J_k}^l(0) \sim \frac{\delta_i^l\delta_k^j}{z^4} + \frac{\delta_k^j {J_i}^l(0) - \delta_i^l {J_k}^j(0)}{z}, \tag{48}$$

$$\left[({J_i}^j)_0, ({J_k}^l)_0\right] = \frac{1}{(2\pi i)^2}\oint_0 \mathrm{d}w \oint_w \mathrm{d}z\, {J_i}^j(z)\,{J_k}^l(w) = \delta_i^l\,({J_k}^j)_0 - \delta_k^j\,({J_i}^l)_0 \tag{49}$$

[5]This can be proven rigorously by considering operator equations like in the bc CFT problem.

[6]I would like to thank 谷夏 for providing this hint.

(d) Similar to bc CFT, we have:

$$T(z) = \frac{1}{2} :\psi_i \, \partial\psi^i: - :\partial\psi_i \, \psi^i: , \quad T(z)\, T(w) \sim \frac{n/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{50}$$

With each field contributing $\frac{1}{2}$ central charge.

(e) For real fermions, there is an additional reality condition:

$$\psi^i = \psi_i^\star = \psi_i \tag{51}$$

The canonical quantization still holds without the extra adjoint, same as the 2-point function:

$$\langle \psi_i(z)\, \psi_j(w) \rangle = \frac{\delta_{ij}}{z-w} \tag{52}$$

Similar holds for $J_{ij} = :\psi_i \psi_j:$ and its OPE, but we no longer need to distinguish upper/lower indices; we have:

$$J_{ij}(z)\, J_{kl}(0) \sim \frac{-\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{z^4} + \frac{-\delta_{ik} J_{jl}(0) + \delta_{il} J_{jk}(0) + \delta_{jk} J_{il}(0) - \delta_{jl} J_{ik}(0)}{z} \tag{53}$$

$$[(J_{ij})_0, (J_{kl})_0] = -\delta_{ik} (J_{jl})_0 + \delta_{il} (J_{jk})_0 + \delta_{jk} (J_{il})_0 - \delta_{jl} (J_{ik})_0 \tag{54}$$

This is precisely the $\mathfrak{o}(n)$ algebra. ■