LECTURE II-7: ABELIAN DUALITY

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1. Introduction

Today, we will discuss abelian duality in two and three dimensions (with a brief mention of four dimensions, which will be further developed in the next lecture). In two dimensions, abelian duality is often referred to as the "R goes to $1/R$" equivalence. In a certain supersymmetric version, it leads to a linear version of mirror symmetry.

In three dimensions, after studying the duality we will give an application to Polyakov's model of confinement.

Abelian duality in four dimensions will eventually have an application to Donaldson theory, that is, to $N = 2$ supersymmetric theories in dimension 4. This again gives a model of confinement. In fact, these two applications of duality are the most concrete models of the phenomenon of confinement which are known.

We begin with the classical statements of duality. In two dimensions, consider a theory which involves fields ϕ, σ, both obeying the Laplace equation

$$\nabla^2\phi = 0; \qquad \nabla^2\sigma = 0, \tag{1.1}$$

and which are related by

$$d\phi = * d\sigma. \tag{1.2}$$

Classically, either of these fields can be taken as the fundamental field of the theory. For example, if we begin with σ such that $\nabla^2\sigma = * d * d\sigma = 0$ then $d * d\sigma = 0$ so locally we can write $* d\sigma = d\phi$ for some field ϕ, i.e., σ determines ϕ (locally and up to an additive constant).

Likewise in three dimensions, consider a theory with a field ϕ obeying the Laplace equation, as well as a connection A on some line bundle $\mathscr{L}$ with curvature $F = dA$ obeying Maxwell's equations

$$dF = d * F = 0, \tag{1.3}$$

whose duality relationship is

$$* d\phi = F. \tag{1.4}$$

Again, as above, either of these fields can be taken as fundamental.

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In four dimensions, the analogue is two connections A and B on line bundles $\mathscr{L}_A$ and $\mathscr{L}_B$, each satisfying Maxwell's equations, and related by

$$F_A = * F_B. \tag{1.5}$$

For abelian duality, we could keep going to higher dimensions if we wish, but we quickly run out of field theory applications. (There are some applications in string theory.)

2. Duality in two dimensions

We wish to study quantum versions of these classical statements. We begin with the two-dimensional case. We identify S^1 with $\mathbb{R}/2\pi\mathbb{Z}$, and use additive coordinates on the circle. We take ϕ to be a map $\phi : \Sigma \to S^1$ where Σ is a compact oriented surface equipped with a Riemannian metric $g_{\alpha\beta}$. (We will consider a variant later on, in which ϕ is not required to be defined at some specified points P_i, that is, ϕ will map $\Sigma - \{P_i\}$ to S^1.) Our Lagrangian[1] is

$$\mathcal{L}(\phi) = \frac{R^2}{4\pi}\int d^2x\,\sqrt{g}\,\partial_\alpha\phi\,\partial^\alpha\phi = \frac{R^2}{4\pi}\int d\phi \wedge * d\phi. \tag{2.1}$$

The equations of motion $dd\phi = d * d\phi = 0$ reproduce the classical theory discussed above.

We will study this theory in various ways. The usual trick is to introduce new variables with the property that integrating them out would lead back to the original theory, and then integrate out the old variables instead of the new ones in order to produce a dual formulation of the theory.

So we actually wish to study a different theory, one which will contain fields ϕ and A, with ϕ a section of a trivial S^1-bundle $\mathscr{S}$ and A a connection on $\mathscr{S}$. Choosing a trivialization ϕ_0 of $\mathscr{S}$ (so that $A = \phi_0^*\tilde{A}$ for some 1-form $\tilde{A}$ on the total space of the bundle), we can define a covariant derivative

$$D_A\phi = d\phi + A, \tag{2.2}$$

and introduce a new Lagrangian

$$\mathcal{L}(\phi, A) = \frac{R^2}{4\pi}\int d^2x\,\sqrt{g}(\partial_\alpha\phi + A_\alpha)(\partial^\alpha\phi + A^\alpha) = \frac{R^2}{4\pi}\int D_A\phi \wedge * D_A\phi. \tag{2.3}$$

While we can recover the old Lagrangian by setting A to zero, there is no mechanism which enforces this, unless we introduce yet a third field σ which plays the role of a Lagrange multiplier. We take σ to be a map from Σ to S^1, and write a Lagrangian

$$\mathcal{L}(\phi, A, \sigma) = \frac{R^2}{4\pi}\int D_A\phi \wedge * D_A\phi - \frac{i}{2\pi}\int \sigma \wedge F_A, \tag{2.4}$$

where F_A is the curvature of A.

This last term requires some comment. Since we are assuming that the circle bundle $\mathscr{S}$ is trivial, we can globally write $F_A = dA$ and the last term should be interpreted as $\frac{i}{2\pi}\int d\sigma \wedge A$ (after integration by parts); this step is needed because σ is not single-valued.

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[1] All Lagrangians in this lecture are written in Euclidean signature.

More generally, to define such a term even when $\mathscr{S}$ is nontrivial, one can use a bit of topology to define $\exp(\frac{i}{2\pi}\int \sigma\wedge F_A)$, similar to defining a Chern–Simons form.

The point of writing this Lagrangian is that $\mathcal{L}(\phi, A, \sigma)$ is equivalent to $\mathcal{L}(\phi)$, as we will now show. A naïve analysis goes as follows: σ appears without derivatives, and its equation of motion is $F_A = 0$; imposing this, we can then go to a gauge where $A = 0$ and recover the original theory.

More globally, we consider the path integral

$$Z = \frac{1}{\mathrm{vol}(G)} \int \mathcal{D}\phi\, \mathcal{D}A\, \mathcal{D}\sigma\, \exp\left[-\frac{R^2}{4\pi}\int D_A\phi \wedge * D_A\phi + \frac{i}{2\pi}\int \sigma\wedge F_A\right]. \tag{2.5}$$

(We focus on the partition function for now, but path integrals with operator insertions can also be treated this way, as we will discuss later.) In light of the standard formula

$$\int \frac{dx}{2\pi} e^{ixy} = \delta(y), \tag{2.6}$$

we would like to say that doing the σ-integral will set F_A to zero. We should treat this statement with care, since we are studying circle-valued functions.

Bearing in mind our identification of S^1 with $\mathbb{R}/2\pi\mathbb{Z}$, our map $\sigma : \Sigma \to S^1$ can be locally written as a real-valued function, but it might not be globally single-valued; however, $d\sigma$ will be a (single-valued) real 1-form on Σ. Choose a circle-valued function $\sigma_h : \Sigma \to S^1$ such that $d\sigma_h$ is the *harmonic* representative in the de Rham cohomology class of $d\sigma$. (We normalize the choice of σ_h by picking some point $P \in \Sigma$ and demanding that $\sigma_h(P) \in 2\pi\mathbb{Z}$.) Then we can write $\sigma = \sigma_h + \sigma_{\mathbb{R}}$, with $\sigma_{\mathbb{R}}$ a single-valued real function.

Notice that $\frac{1}{2\pi}d\sigma$, or equivalently $\frac{1}{2\pi}d\sigma_h$, must have integral periods. In particular, if we choose a basis λ_j of integral harmonic 1-forms, and write $d\sigma_h = \sum 2\pi m_j \lambda_j$, then $m_j \in \mathbb{Z}$.

Now we compute:

$$\int \mathcal{D}\sigma\, e^{\frac{i}{2\pi}\int \sigma\wedge F_A} = \int \mathcal{D}\sigma_{\mathbb{R}}\, e^{\frac{i}{2\pi}\int \sigma_{\mathbb{R}}\wedge F_A} \sum_{d\sigma_h \in H^1(\Sigma, 2\pi\mathbb{Z})} e^{-\frac{i}{2\pi} d\sigma_h \wedge A} \tag{2.7}$$

$$= \delta(F_A) \prod_j \left(\sum_{m_j \in \mathbb{Z}} e^{-i m_j \int \lambda_j \wedge A} \right). \tag{2.8}$$

The first factor tells us that A is a flat connection. Moreover, among flat connections the gauge equivalence classes $[A]$ are labeled by the holonomies, or equivalently by the quantities $\int \lambda_j \wedge A$. Since the remaining part of the integrand is gauge invariant, we can gauge fix (omitting the factor of $(\det G)^{-1}$) and integrate over the space of gauge equivalence classes (a finite dimensional integral):

$$Z = \int \mathcal{D}\phi\, \mathcal{D}[A]\, e^{-\frac{R^2}{4\pi}\int D_A\phi\wedge * D_A\phi} \delta(F_A) \prod_j \left(\sum_{m_j\in\mathbb{Z}} e^{-i m_j \int \lambda_j\wedge A}\right) \tag{2.9}$$

$$= \int \mathcal{D}\phi\, e^{-\frac{R^2}{4\pi}\int D_A\phi\wedge * D_A\phi} \delta(F_A)\, \delta\Big(\int \lambda_i \wedge A = 0 \mod 2\pi\mathbb{Z}\Big). \tag{2.10}$$

So there are delta functions setting the holonomies as well as curvature to zero. So A is zero modulo gauge transformations, and our new theory is indeed equivalent to the original theory, with partition function

$$Z = \int \mathcal{D}\phi \, e^{-\frac{R^2}{4\pi}\int d\phi\wedge * d\phi}. \tag{2.11}$$

Now let us integrate out in the opposite order, integrating out ϕ and A but keeping σ. To integrate out ϕ, we fix the gauge in such a way that $\phi = 0$, and suppress the factor of $(\mathrm{vol}(G))^{-1}$ from the path integral. Then the path integral (2.5) reduces to

$$\int \mathcal{D}A \, \mathcal{D}\sigma \, \exp\left[-\frac{R^2}{4\pi}\int A\wedge * A + \frac{i}{2\pi}\int \sigma \wedge F_A\right]. \tag{2.12}$$

An exercise you might enjoy is verifying that the Faddeev-Popov determinant associated with this gauge fixing is

$$\int \mathcal{D}c \, \mathcal{D}\bar{c} \, \exp\left[-\frac{R^2}{4\pi}\int d^2x \, \bar{c}c\right]. \tag{2.13}$$

To do the integral over A, we need to complete the square, thinking of the second term in the exponent as $-\frac{i}{2\pi}\int A\wedge d\sigma$, i.e., it is the term linear in A. We make a change of variables $A' = A + \frac{i}{R^2} * d\sigma$; the result, including the Faddeev-Popov integral, is

$$\left(\int \mathcal{D}c \, \mathcal{D}\bar{c} \, e^{-\frac{R^2}{4\pi}\int d^2x \, \bar{c}c}\right)\left(\int \mathcal{D}A' e^{-\frac{R^2}{4\pi}\int A'\wedge * A'}\right)\left(\int \mathcal{D}\sigma e^{-\frac{1}{4\pi R^2}\int d\sigma\wedge * d\sigma}\right). \tag{2.14}$$

(The Gaussian integral over A', like the Faddeev-Popov determinant, can be thought of as a normalization factor.)

The Faddeev-Popov integral gives a factor of $(\int dt \, e^{-tR^2/4\pi})^{-1}$ for each 0-form and each 2-form on Σ; the integral over A' gives a factor of $\int dt \, e^{-tR^2/4\pi}$ for each 1-form on Σ. Thus, if n_j denotes the "number of j-forms on Σ", those terms combine to give an overall factor of

$$\left(\sqrt{\pi/(R^2/4\pi)}\right)^{(-n_0+n_1-n_2)} = (R/2\pi)^{\chi(\Sigma)}. \tag{2.15}$$

That is, the transformation rule which relates these dual formulations is

$$\int \mathcal{D}\phi \, \exp\left[-\frac{R^2}{4\pi}\int d\phi\wedge * d\phi\right] = (R/2\pi)^{\chi(\Sigma)}\int \mathcal{D}\sigma \, \exp\left[-\frac{1}{4\pi R^2}\int d\sigma \wedge * d\sigma\right]. \tag{2.16}$$

(The factor of $(R/2\pi)^{\chi(\Sigma)}$ did not show up in previous explicit calculations we have done because they were done in genus 1.) A coupling of R in the first theory maps to a coupling of $1/R$ in the second theory, which is why this duality is sometimes referred to as "R goes to $1/R$."

Now we would like to follow the operators in this theory through the duality transformation, and determine the effect on correlation functions as well as the partition function. The easy case is $d\phi$, which we expect to map to $* \, d\sigma$, as in the classical theory (actually it will map to a multiple of $* \, d\sigma$, when normalizations are taken into account).

We need to repeat the above calculation with an insertion of operators $\prod \mathcal{O}_i(\phi)$, one of which is $d\phi$. In order to do this, we need a gauge-invariant extension of the operator $d\phi$, which is provided by the covariant derivative:

$$d\phi(x) \text{ in first theory} \longrightarrow D_A\phi(x) \text{ in big theory.} \tag{2.17}$$

One then easily checks that this maps back to $d\phi$ when we integrate out σ (and gauge fix A to zero).

The difficult thing about mapping operators in general will be finding the appropriate extension to an operator in the larger theory, which reduces to the original operator when σ has been integrated out.

How will our calculation be modified? Earlier, when we completed the square, we made a change of variables $A' = A + \frac{i}{R^2} * d\sigma$. This means that if $D_A\phi$ has been inserted in the path integral, it now becomes $D_{A'}\phi - \frac{i}{R^2} * d\sigma$. Thus, when A' is integrated out, we are left with an insertion of $-\frac{i}{R^2} * d\sigma$ in the dual theory.

Some care must be used in manipulating these mappings between operator insertions. For example, even though $d\phi$ maps to $-\frac{i}{R^2} * d\sigma$, $(d\phi)^2$ will *not* map to $(\frac{i}{R^2} * d\sigma)^2$ in the dual theory, due to nonlinearities introduced when we complete the square.

What made this case easy was that the covariant derivative is a natural gauge-invariant extension of the ordinary derivative.

The hard operator insertion to deal with is $e^{i\phi}$. There is no gauge-invariant version of this. On the other hand, we don't really need it, because $\langle e^{i\phi}(P)\rangle = 0$. On the other hand, the two-point correlators $\langle e^{i\phi}(P)\, e^{-i\phi}(Q)\rangle$ are not zero in general, so we should try to dualize such insertions, when $P \neq Q$.

We are going to do something rather strange. It is impossible to construct a gauge-invariant extension *locally* for this pair of operator insertions, so we work non-locally during the duality (i.e., in the intermediate theory), introducing a term in the path integral of the form

$$e^{i\phi}(P)e^{-i\phi}(Q)e^{\frac{i}{2\pi}\int \theta\wedge A} \tag{2.18}$$

where θ is a 1-form such that

$$\frac{1}{2\pi}d\theta = \delta_P - \delta_Q \tag{2.19}$$

and all periods of θ are 0 modulo $2\pi\mathbb{Z}$.

To find such a θ, take a path ℓ from P to Q and let θ be Poincaré dual to ℓ (regarded as a distribution). We can rewrite our term (2.18) as

$$e^{i\phi}(P)e^{\frac{i}{2\pi}\int_\ell A}e^{-i\phi}(Q), \tag{2.20}$$

from which we see that this is a gauge-invariant expression. (This is an example of a choice of θ, but we allow more general θ's than that.) We will eventually think of θ as a connection form on a trivial S^1-bundle on $\Sigma - P - Q$.

Thus, we have found a gauge-invariant nonlocal object (2.18). Suppose we try to insert this into the path integral. First, if we set A to zero, we recover our original operator insertion $e^{i\phi}(P)e^{-i\phi}(Q)$, so we have correctly extended this pair of operators to the gauge theory.

To integrate out the other way, we gauge fix ϕ to zero, which reduces (2.18) to $e^{\frac{i}{2\pi}\int \theta\wedge A}$. This is a new contribution to the term linear in A in the exponent of the path integral, of the form $-\frac{i}{2\pi}\int A\wedge\theta$, so to complete the square we now must make the change of variables $A' = A + \frac{i}{R^2} * (d\sigma + \theta)$, and this makes our dualized path integral take the form

$$\int \mathcal{D}\sigma \, \exp\left[-\frac{1}{4\pi R^2}\int (d\sigma + \theta) \wedge *(d\sigma + \theta)\right]. \tag{2.21}$$

That is, we have obtained the same theory as before, but now formulated on $\Sigma - P - Q$, and now performing the path integral over maps to S^1 that do *not* extend over P or Q. That is, we write $\widetilde{\sigma} = \sigma + \alpha$, with $d\alpha = \theta$, α being a map from $\Sigma - P - Q$ to S^1, and then $\widetilde{\sigma}$ is the map of $\Sigma - P - Q$ to S^1 that does not extend over P or Q. In fact, it has winding numbers 1 and -1 around P and Q, respectively.

There are two kinds of operators now, one local in one picture, and the other local in the other picture. The operator $e^{in\phi}(P)$ in one description is mapped to an instruction "delete P from Σ, and perform the path integral over σ's that have winding number n around P" in the other description. Conversely, there is an ordinary local operator $e^{in\sigma}(P)$ in the second description, that corresponds to a nonlocal recipe in the first description.

3. Duality in three dimensions

We now turn to three-dimensional theories. Our initial theory is the theory of an S^1-valued function ϕ on a (fixed) compact 3-manifold M (possibly with boundary), governed by the Lagrangian

$$\mathcal{L} = \frac{\Lambda}{4\pi}\int d^3x \sqrt{g}\, \partial_\alpha\phi\, \partial^\alpha\phi = \frac{\Lambda}{4\pi}\int d\phi \wedge * d\phi. \tag{3.1}$$

One novel feature of three dimensions is that the prefactor Λ has the dimensions of mass. Since Λ is a dimensionful quantity rather than a constant, it will be impossible for anything special to happen at a numerical value of Λ.

We reinterpret ϕ as a section of a trivial S^1-bundle $\mathscr{L}_B$ with a connection B, and write a Lagrangian which includes the covariant derivative $D_B\phi = d\phi + B$:

$$\mathcal{L}(\phi, B) = \frac{\Lambda}{4\pi}\int d^3x \, (\partial_\alpha\phi + B_\alpha)(\partial^\alpha\phi + B^\alpha) = \frac{\Lambda}{4\pi}\int D_B\phi \wedge * D_B\phi \tag{3.2}$$

This would be a trivial theory if left as it is. To get somewhere, we introduce a line bundle $\mathscr{L}$ with connection A, and the Lagrangian

$$\mathcal{L}(\phi, B, A) = \frac{\Lambda}{4\pi}\int D_B\phi \wedge * D_B\phi - \frac{i}{2\pi}\int A \wedge F_B, \tag{3.3}$$

interpreting the last term as $\frac{-i}{2\pi}\int F_A \wedge B$ (after integrating by parts) which makes sense since $\mathscr{L}_B$ is trivial. As before, the extension to the case of $\mathscr{L}_B$ not being trivial is a term in the path integral $e^{-\frac{i}{2\pi}\int A\wedge F_B}$, the Chern–Simons form for the structure group $U(1)\times U(1)$ of the bundle $\mathscr{L}\oplus\mathscr{L}_B$.

In order to carry out the duality transformation, we need to sum over line bundles $\mathscr{L}$, producing a path integral

$$\frac{1}{\mathrm{vol}(G)\,\mathrm{vol}(G')}\sum_{\mathscr{L}}\int \mathcal{D}\phi\,\mathcal{D}A\,\mathcal{D}B\,\exp\left[-\frac{\Lambda}{4\pi}\int D_B\phi\wedge * D_B\phi + \frac{i}{2\pi}\int F_A\wedge B\right]. \tag{3.4}$$

where G and G' are gauge groups for $\mathscr{L}$ and $\mathscr{L}_B$.

We would like to integrate out A. The crude statement is that we get $\delta(F_B)$, which implies that B is a flat connection. As in the computation we did in two dimensions, the complications arise from the possibility of nontrivial holonomies for B. In fact, if $F_B = 0$ we can regard B as an element of $H^1(M, \mathbb{R}/\mathbb{Z})$ and define for each $x = c_1(\mathscr{L})$, the quantity $e^{i\int x\wedge[B]}$. When this is summed over $[x]$ in doing the path integral, the result is $\delta([B])$, showing that B is gauge-equivalent to the trivial flat connection. Thus, we reduce back to the original theory; we've learned that the extended theory with A and $\mathcal{L}$ is equivalent to the original theory.

Now we do the integral in the opposite direction, by doing the ϕ and A integrals. As in the two-dimensional case, we set ϕ to zero by a gauge transformation, removing the normalization factor $(\mathrm{vol}(G))^{-1}$. (Unlike in two dimensions, the Faddeev-Popov determinant does not contribute anything interesting, so we will not explicitly include it.) The square can be completed in the resulting path integral

$$\frac{1}{\mathrm{vol}(G')}\sum_{\mathscr{L}}\int \mathcal{D}A\,\mathcal{D}B\,\exp\left[-\frac{\Lambda}{4\pi}\int B\wedge * B + \frac{i}{2\pi}\int B\wedge F_A\right] \tag{3.5}$$

$$= \frac{1}{\mathrm{vol}(G')}\sum_{\mathscr{L}}\int \mathcal{D}A\,\mathcal{D}B'\,\exp\left[-\frac{\Lambda}{4\pi}\int B'\wedge * B' - \frac{1}{4\pi\Lambda}\int F_A\wedge * F_A\right], \tag{3.6}$$

where we shifted $B' = B - \frac{i}{\Lambda} * F_A$.
The integral in B' is Gaussian, and can be absorbed into the normalization of the path integral, leaving us with

$$\frac{1}{\mathrm{vol}(G')}\sum_{\mathscr{L}}\int \mathcal{D}A\,\exp\left[-\frac{1}{4\pi\Lambda}\int F_A\wedge * F_A\right], \tag{3.7}$$

the familiar path-integral from Yang–Mills theory! (With gauge coupling $\mathrm{e} = \sqrt{\Lambda}$.)

Now we want to map operators. An analogous calculation to the one we did in two dimensions shows that $d\phi$ extends to the covariant derivative $D_B\phi = d\phi + B$ in the larger theory, and becomes $\frac{i}{\Lambda} * F_A$ in the dual theory. This is the meaning in the quantum theory of the duality between solutions of the Laplace equation and solutions of Maxwell's equation.

What about operators such as $e^{i\phi}(P)$? We will not do this, though I will state the answer. $e^{in\phi}(P)$ is mapped, in the dual gauge theory, to the instruction: delete the point P from the 3-manifold M and consider the path integral for connections on a line bundle $\mathcal{L}$ whose first Chern class evaluates to n on a small sphere around P. The argument leading to this is quite along the lines of what we did in two dimensions.

I will not present that argument (which I recommend as an exercise) and instead discuss what is roughly the reverse. We consider Wilson loop operators in the gauge theory, and try to map those back to the scalar field theory.

Let C be a circle in M which is a boundary, and let $\lambda \in \mathbb{R}$. We have an operator $\exp(i\lambda \oint_C A)$ in the gauge theory. Let us insert this into the path integral of the big theory (the one with Lagrangian $\mathcal{L}(\phi, B, A)$). (More generally, we can do this even if C is not a boundary, but then λ must be restricted to be an integer. C being a boundary, say of a two-surface D, lets us write $\exp(i\lambda \oint_C A) = \exp(i\lambda \int_D F)$, showing gauge-invariance for any λ.)

When we integrate out A with this insertion, instead of getting $\delta(F_B = 0)$ we get $\delta(F_B = 2\pi\lambda[C])$ with $[C]$ being the Poincaré dual to C. In other words, performing the A integral determines B to be such that $F_B = 2\pi\lambda[C]$, and the modified Lagrangian after integrating out is $\int D_B\phi \wedge * D_B\phi$. Here B is a flat $U(1)$-connection with monodromy $e^{2\pi i\lambda}$ around the circle. So the Wilson loop operator for a circle C is dual to a recipe "delete C and interpret ϕ as a section of a flat bundle with monodromy around C." In other words, we modify ϕ to $\widetilde{\phi}$ such that $d\widetilde{\phi} = d\phi + B$, and interpret $\widetilde{\phi}$ as a section of a trivial S^1-bundle on $M - C$ with a flat connection of monodromy $e^{2\pi i\lambda}$ around C. (To get such a trivial S^1-bundle, we either need to assume that C is a boundary, or that λ is an integer.)

4. Application to the Polyakov model

We will describe a model constructed by Polyakov around 1975. It was the first use of duality in a nonlinear relativistic theory. The model exhibits the phenomenon known as confinement.

We work in three dimensions, and study $SO(3)$ gauge theory with a scalar field s in the 3-dimensional representation, governed by the Lagrangian

$$\mathscr{L} = \frac{1}{4\pi e^2} \int d^3x \ \mathrm{Tr}\,|F|^2 + \int d^3x \,(D\vec{s})^2 + \int d^3x \,\lambda(\vec{s}^{\,2} - a^2)^2. \tag{4.1}$$

This model exhibits the Higgs mechanism at tree level.

Classically, the vacuum can be rotated to

$$\langle \vec{s} \rangle = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \tag{4.2}$$

by a gauge transformation. In the vacuum state, the $SO(3)$ symmetry of the Lagrangian is broken to an $SO(2) = U(1)$ symmetry, and the only massless field is the $U(1)$-connection. (This is the so-called Higgs mechanism.) The low-energy theory looks like the $U(1)$ theory in three dimensions.

First Step. This theory looks classically, at low energies, like a $U(1)$ gauge theory with Lagrangian

$$\mathcal{L} = \frac{1}{4\mathrm{e}^2} \int F \wedge * F. \tag{4.3}$$

Is that the answer? Is the Gaussian fixed point of the free $U(1)$ theory stable?

One possible source of instability is a Chern–Simons interaction

$$\frac{-in}{2\pi} \int A \wedge F_A \tag{4.4}$$

which could be added to the original Lagrangian $\mathcal{L}$. (In the flow to the infrared, this term is more relevant than those appearing in $\mathcal{L}$, hence the instability.)

There are two obstacles to this being a source of instability in our problem. The first is that the coefficient n in the Chern–Simons term must be an integer. This implies that even if you can only approximately calculate the theory, you can determine n if the approximation can be made arbitrarily accurate. One can, in particular, calculate n in perturbation theory, and – as higher loop terms would involve positive powers of e – it could only arise from a one-loop term. So the effective n in any three-dimensional gauge theory can be determined by an explicit one-loop computation. In the specific example we are considering here, one can simply notice that the Chern–Simons term is odd under parity (i.e., it depends on a choice of orientation of the 3-manifold) whereas our original theory was not. The parity-invariance is a symmetry of the one-loop determinants (whether or not parity ultimately is spontaneously broken at low energies) and ensures that $n = 0$.

Thus, anything which could make this theory unstable will be hard to describe in terms of A. However, we know that the $U(1)$ gauge theory is equivalent (dual) to a scalar theory

$$\mathcal{L} = \frac{\mathrm{e}^2}{4\pi} \int d^3x \, \partial_\alpha \phi \partial^\alpha \phi \tag{4.5}$$

In this theory, we could add a term

$$g \int d^3x \, \cos n\phi \tag{4.6}$$

for some n. We have chosen the potential to be periodic as ϕ is really a map to a circle. Notice that the theory with $g \neq 0$ has a mass gap, the theory with $g = 0$ does not. Thus, we have identified something which qualitatively changes the physics, but can only be conveniently interpreted by means of the dual variables. This is strange, because doing the duality requires going to a low energy description to begin with!

How do we see this effect in the original $SO(3)$ theory? Consider the Feynman diagrams of the $SO(3)$ theory. We represent these with massless modes given by wavy lines, massive ones by solid lines. A typical Feynman diagram such as

will be completely tame: we have massive propagators, which are analytic in the momenta at low momentum. Such diagrams merely give corrections to the effective action for the $U(1)$ gauge field A which can be described as additional local, gauge-invariant terms in the Lagrangian.

We need something completely different: *instantons*. Let us work on $\mathbb{R}^3$. We have an $SO(3)$ bundle on $\mathbb{R}^3$ whose structure group has been reduced to $U(1)$ at infinity. On the S^2 at infinity, a $U(1)$ bundle $\mathscr{L}$ can have a nontrivial first Chern class, $c_1(\mathscr{L}) \neq 0$.

Pick a trivialization of the $SO(3)$ bundle, so that we can identify the spatial $\mathbb{R}^3$ with the bundle $\mathbb{R}^3$, and simply treat our section s as a map $s : \mathbb{R}^3 \to \mathbb{R}^3$. To construct such a map with nontrivial topology on the S^2 at infinity, we take s of the form

$$s(\vec{x}) = a\frac{\vec{x}}{|x|}f(|x|), \tag{4.7}$$

with $f(|x|)$ an increasing function satisfying $f(0) = 0$, $\lim_{|x|\to\infty} f(|x|) = a$, e.g.,

Such a section is invariant under combined rotations of space and gauge rotations.

We would like the minimum action solution s in this class. It should be spherically symmetric, with asymptotic behavior $Ds \sim \frac{1}{|x|^2}$ for $|x| \to \infty$. This leads to an ODE, which has a unique solution. The action can be written as (I/e^2) for some constant I, so it will diverge as the coupling goes to 0.

Note that since $\vec{s}$ vanishes precisely at the origin, the structure group is reduced to $U(1)$ away from the origin, but this reduction does not extend over the origin. In fact, over a sphere surrounding the origin, the line bundle has a nonzero first Chern class (which actually is 2 if we work in $SO(3)$; that is, the adjoint bundle of $SO(3)$ decomposes over a two-sphere surrounding the origin as $\mathcal{O} \oplus \mathcal{L} \oplus \mathcal{L}^{-1}$ where $\mathcal{L}$ has degree 2).

From this point of view, the difference between a pure $U(1)$ theory and a theory that looks like a $U(1)$ theory only at long distances is that the latter can have bundles that over a large $\mathbf{S}^2$ at infinity have a nonzero first Chern class. The Chern class can be any multiple of the 2 that was found in the explicit solution that we just described.

Note that in abelian gauge theory, it was not possible to use a bundle $\mathscr{L}$ for which $c_1(\mathscr{L}) \neq 0$ over a sphere around the origin. The qualitative difference between an $SO(3)$ theory broken to $U(1)$ and a $U(1)$ theory is that the former admits singularities where the $U(1)$ description breaks down in this way.

If we did a similar thing in 3+1 dimensions, we would find a time-independent solution of finite energy (rather than finite action). This solution looks like a particle sitting there, and in fact is a magnetic monopole. (The nonzero magnetic charge comes from the fact the $c_1(\mathscr{L}) \neq 0$ which implies that the magnetic field integrates to a nonzero amount.) The point of making this analogy is that we can think of the instanton as behaving like a zero-time slice of a magnetic monopole.

What do instantons look like at long distances? A monopole has a field which behaves like

$$F = \text{const}\, \frac{\vec{x}}{|x|^3} = \text{const}\, \frac{\hat{x}}{|x|^2} \tag{4.8}$$

(by Maxwell's equations).

The contribution to the path integral from each instanton is e^{-I/e^2}, which is small.

To return to our theory, we now ask how it behaves at long distance. For example, how does a two-point function $\langle F(x)\, F(0)\rangle$ behave as $|x| \to \infty$?

The answer in the free theory, by dimensional analysis, is $-1/|x|^3$. (One sees that the curvature F has dimension 3/2 in three-dimensional theories by considering the basic term $\frac{1}{4\pi \mathrm{e}^2}\int F \wedge * F$ in the Lagrangian.) A bit more precisely, we are asserting that

$$\langle F_{ij}(x)\, F_{k\ell}(0)\rangle = \frac{1}{|x|^3}(\delta_{ik}\delta_{j\ell} + \dots). \tag{4.9}$$

The Feynman diagrams don't affect this asymptotic behavior: there is a slight renormalization of e^2, plus other corrections which are unimportant at big distances.

How does an instanton affect this analysis? Consider an instanton localized near y, and its effect on the two-point function between x and 0.

The leading approximation to the path integral in the instanton sector is

$$e^{-I/\mathrm{e}^2} \int d^3y\, \frac{(\vec{x} - \vec{y})}{|x-y|^3}\, \frac{\vec{y}}{|y|^3}. \tag{4.10}$$

Thus, the overall behavior is

$$\langle F(\vec{x})\, F(\vec{0})\rangle \cong \frac{1}{Z}\Big(\text{pert. theory} + e^{-I/\mathrm{e}^2}(\text{instanton sector})\Big) \tag{4.11}$$

$$\cong \frac{1}{Z}\left((1+\dots)\frac{1}{|x|^3} + e^{-I/\mathrm{e}^2}(\frac{1}{|x|}+\dots)\right). \tag{4.12}$$

Since the instanton contribution is more important in the infrared than the free theory term ($1/|x|$ compared to $1/|x|^3$), the instanton triggers an instability.

To see in more detail what is happening in the infrared, we first note that instantons are *rare*: the probability to have an instanton in a small volume V_0 is proportional to $V_0 e^{-I/\mathrm{e}^2}$, so the volume of space per instanton is given by $V \sim e^{I/\mathrm{e}^2}$, and the spatial separation between two of them is $R \sim V^{1/3} \sim e^{I/3\mathrm{e}^2}$.

In the infrared, though, if we consider a large enough volume of space, we will get lots of instantons, which we can treat as a gas of particles of definite size. The particles are *charged*, so we can't ignore the interactions between them, given by Coulomb potentials. The picture is as follows

We have labeled the positions of instantons (which are positively charged) with s_i's, and the positions of anti-instantons (which are negatively charged) by t_j's. The sum over all of these takes the form

$$\sum_{n,m=0}^{\infty} \frac{1}{n!m!} \int d^3 s_i|_{i=1}^{n} \int d^3 t_j|_{j=1}^{m} e^{-\frac{I}{\mathrm{e}^2}(n+m)} e^{\sum_{i<j}\left(\frac{1}{|s_i-s_j|}+\frac{1}{|t_i-t_j|}\right)-\sum_{i,j}\frac{1}{|s_i-t_j|}}. \tag{4.13}$$

We also need an operator insertion, of

$$F(x) = \sum \frac{x-s_i}{|x-s_i|^3} - \sum \frac{x-t_j}{|x-t_j|^3}, \tag{4.14}$$

and similarly for $F(0)$.

The physics involved is the classical statistical mechanics of a plasma in space, with chemical potential I/e^2; energy E given by the Coulomb potential between the instantons and anti-instantons, and temperature $T = 4\pi\mathrm{e}^2$.

The phenomenon we need is known as "Debye screening"—a plasma screens external charges. As a result of this screening, the system will have a mass gap.

Here is a quick mathematical derivation of Debye screening (in this context). We go back to our low-energy theory, writing in dual variables—a scalar theory with $F = *\,d\phi$. We will add a term to the Lagrangian to account for the instanton effect: the term we will use (justified by the results of the calculation to come) is $\int e^{-I/\mathrm{e}^2}(e^{2i\phi}+e^{-2i\phi})$. (The 2 is present in the exponent because the basic instanton has first Chern class 2.) The operator insertions of F become insertions of $*\,d\phi$, and the quantity we are calculating can be written:

$$\Omega = \int \mathcal{D}\phi\ (*\,d\phi(x))\,(*\,d\phi(0))\exp\left[-\frac{1}{4\pi\mathrm{e}^2}\int |d\phi|^2 + \int e^{-I/\mathrm{e}^2}(e^{2i\phi}+e^{-2i\phi})\right]. \tag{4.15}$$

First we want to show that this is a correct description in the dual variables, then we will analyze this version.

We will expand Ω in perturbation theory:

$$\Omega = \sum_{n,m=0}^{\infty}\frac{1}{n!m!}\int \mathcal{D}\phi\, e^{-\frac{1}{4\pi\mathrm{e}^2}\int d\phi\wedge *\,d\phi}\,(*\,d\phi(x))\,(*\,d\phi(0))\left(\int e^{-I/\mathrm{e}^2}e^{2i\phi(y)}d^3y\right)^n\left(\int e^{-I/\mathrm{e}^2}e^{-2i\phi(z)}d^3z\right)^n. \tag{4.16}$$

We expand further, using the principle that $(\int dy\, f(y))^n = \int dy_1\ldots dy_n\, f(y_1)\ldots f(y_n)$. Thus, we can rewrite (4.16) as

$$\Omega = \sum_{n,m=0}^{\infty}\int\mathcal{D}\phi\int d^3s_i|_{i=1}^n d^3t_j|_{j=1}^m\,(*\,d\phi(x))\,(*\,d\phi(0))\,e^{-\frac{I}{\mathrm{e}^2}(n+m)}e^{2i\sum(\phi(s_i)-\phi(t_j))}e^{-\frac{1}{4\pi\mathrm{e}^2}\int|d\phi|^2}. \tag{4.17}$$

The ϕ-integral is Gaussian after the square has been completed; that integral will contribute

$$e^{\sum_i\sum_j G(s_i,t_j)} \tag{4.18}$$

to the overall answer, where $G(s,t) = 1/|s-t|$ is the Green's function for the Laplacian. We thus recover the previous formula. Note that the $i=j$ terms contributed a divergence which is renormalized, so they do not appear.

The lesson we have learned is this: our original problem ($SO(3)$ gauge theory) can be described in the infrared, using dual variables, by means of the Lagrangian

$$\frac{1}{4\pi\mathrm{e}^2}\int|d\phi|^2 + e^{-I/\mathrm{e}^2}\int d^3x\,\cos 2\phi. \tag{4.19}$$

The second term in unrenormalizable, but is well-behaved in the infrared. Expanding around the minimum of the potential, we see that ϕ has a mass. Thus, what used to be a massless $U(1)$ photon was (i) dualized and reinterpreted as a scalar, and (ii) got a mass that was more easily described in that language.

Here is another way to understand the above derivation. Start with the massless scalar field and dualize it, as in section 3 of this lecture, to a massless gauge field A. Now perturb the theory of the massless scalar by adding a weak perturbation $\int d^3x\epsilon(e^{22i\phi}+e^{-2i\phi})$. In terms of the gauge field, the operator $e^{2i\phi(P)}$ becomes, as we noted at the end of section 3, an instruction "delete the point P, and consider bundles with first Chern class 2 on a small sphere around P." In other words, from the point of view of an $SO(3)$ that looks like $U(1)$ at low energies, the instruction is "include an instanton at P." If we take the interaction term $\int d^3x\epsilon(e^{2i\phi}+e^{-2i\phi})$, and expand in powers of ϵ, we simply generate the instanton gas, with each insertion of $e^{2i\phi}$ or $e^{-2i\phi}$ corresponding to an instanton or antiinstanton.

Now we want to study confinement. An important preliminary is to note the symmetries of the problem. In general, in the duality from an abelian gauge theory, ϕ is an angular variable, a map to a circle, so ϕ is equivalent to $\phi+2\pi$. However, because the instanton-induced interaction is a trigonometric function of 2ϕ, there is a symmetry under $\phi \to \phi + \pi$.

In the original description, we could consider a curve $C \subset M$

and the associated Wilson line operator

$$\langle \mathrm{Tr}_R \,\mathrm{Hol}(A, C)\rangle \tag{4.20}$$

where R is the 2-dimensional representation of $SU(2)$ and Hol denotes the holonomy. What does this operator translate to in our infrared description? As we have seen at the end of section 3, it translates into a recipe "delete C from spacetime and interpret ϕ as a section of a flat circle bundle with monodromy around C." The monodromy is given by the angle $2\pi\lambda$ where λ, introduced in the discussion in section three, is the $U(1)$ charge appearing in the Wilson loop, modulo 1. We have chosen a Wilson line in the two-dimensional representation of $SU(2)$; the $U(1)$ charges are $\pm 1/2$ in units of the weight lattice of $SO(3)$. Hence our example has $\lambda = 1/2$. The monodromy around the circle is thus $\phi \to \phi + \pi$, which, as we noted a moment ago, is a symmetry of the theory even with the instanton-induced interaction.

The problem is now classical: we want to find the minimum of the action

$$\frac{1}{4\pi \mathrm{e}^2}\int |d\phi + B|^2 - e^{-I/\mathrm{e}^2}\int d^3x\,\cos 2\phi \tag{4.21}$$

with ϕ required to have monodromy π around C. Thus, in particular, ϕ cannot be constant. In fact, the least action solution will be given approximately in terms of a surface Σ with $\partial\Sigma = C$ of least area; ϕ will be constant far from Σ, and will jump by π in crossing Σ.

The easiest case is when C is a large curve in the plane, say given by $x = 0$, with x a linear function on $\mathbf{R}^3$. We want ϕ to jump by π in going from negative x to positive x. In the limit that C is very large, ϕ, if observed somewhere deep in the interior of C, becomes a function of x only. What needs to be minimized is then

$$\int dx\,((\frac{d\phi}{dx}+B(x))^2-(\cos 2\phi-1)) \tag{4.22}$$

with the boundary conditions that $\phi \to 0$ for $x \to -\infty$ and $\phi \to \pi$ for $x \to +\infty$. This variational problem, with the boundary conditions, has a solution that is unique up to translation of x, with ϕ approaching its asymptotic value exponential fast (because of the mass gap) and with some action L.

Now in general, if D is the minimal area surface with boundary C, and $A(D)$ is its area, the minimum action ϕ that is a section of the appropriate flat bundle has the property that ϕ is very near zero or π except near D, jumps by π in crossing D, and looks in profile near D just like the solution of the idealized one-dimensional problem discussed in the last paragraph. Its action is very nearly $A(D)L$, so the expectation value of the Wilson line is approximately $e^{-A(D)L}$, showing the area law and confinement.