LECTURE II-1: SYMMETRY BREAKING

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In this semester we will continue the discussion of quantum field theory, but now mostly dynamics rather than perturbation theory and purely formal things. The topics we will discuss are much more difficult to deal with rigorously than the formal theory we studied in the fall semester. We will try to do it when possible, but it will not always be possible. In general, we will try to form an intuitive picture of what is going on, and illustrate it by considering concrete examples.

1.0. Theories and realizations. [This section is explanatory and was written by the preparers of these notes in order to create a false sense of security i.e. an unsubstantiated feeling that we understand what we are talking about in the rest of the lecture. Unfortunately, this is not the case, at least if "understand" means what it usually means among mathematicians.]

For the purposes of the present and forthcoming lectures, it will be important for us to distinguish theories and their realizations. So let us explain what we mean by a theory and what we mean by its realization. This explaination is not a mathematical definition (in fact, it is hard to give a definition which is both rigorous and useful), but we hope that it will make clear what we are talking about.

Recall that a classical physical system Σ is usually described by defining the space of states X of Σ (a symplectic manifold, maybe infinite-dimensional) and a 1-parameter group g^t of time translations which preserves symplectic structure. Then g^t produces a Hamiltonian flow, which is defined (ignoring topological problems) by a Hamiltonian function H. This function is called the Hamiltonian, or energy function of the system. One should remember that H is defined only up to adding a (locally) constant function.

In most examples, H is bounded from below. In such a case, H is always normalized in such a way that the infinum of H on (each connected component of) X is zero.

In this situation, by **a theory** we mean a pair (X, H), where X is a symplectic manifold, and $H : X \to \mathbb{R}$ the energy function, defined up to a (locally) constant function. By **a vacuum state of this theory** we mean a lowest energy equilibrium state $x \in X$ of the system Σ, i.e. a state where the energy functional H attains a global minimum.

The same theory can have different vacuum states. For example, if we have a particle on the line with potential energy $U(x) = \frac{g}{4!}(x^2 - a^2)^2$, then the space X is the phase plane $\mathbb{R}^2$ with coordinates (x, p), the Hamiltonian is $\frac{p^2}{2} + U(x)$, and there are two vacuum states $(a, 0)$ and $(-a, 0)$.

Often a classical system Σ can be described by a Lagrangian $\mathcal{L}$, defined on some space of fields S on the spacetime V. In this case its space of states is the space $X \subset S$ of extremals of $\mathcal{L}$. As we have seen before, the space X carries a natural closed 2-form ω, which is nondegenerate, and thus defines a symplectic structure on X. Also, the group of time translations acts on X and preserves ω. Therefore, the flow on X generated by this 1-parameter group is Hamiltonian, and is defined by a Hamiltonian function H.

However, one should remember that the description of a theory by a Lagrangian is not intrinsic, since different Lagrangians defined on different spaces of fields S may define the same theory. Indeed, consider, for example, the Lagrangian $\mathcal{L}_1 = \int x'(t)^2 dt/2$ defined on $S_1 = C^\infty(\mathbb{R}, M)$, where M is a Riemannian manifold. This Lagrangian defines the geodesic flow. The space of states in this case is T^*M, and the Hamiltonian function is $p^2/2$. On the other hand, we can write the Lagrangian $\mathcal{L}_2 = \int(-x'(t)p(t)+p^2(t)/2)dt$ defined on the space $S_2 = C^\infty(\mathbb{R}, T^*M)$. It is easy to see that these two Lagrangians define the same theory.

We will consider relativistically invariant theories (X, H), i.e. theories with an action of the Poincare group $\mathbf{P}$ on X, which preserves the symplectic structure (in the case when the system is defined by a Lagrangian, the group $\mathbf{P}$ acts on S and preserves the Lagrangian, and therefore, it acts on X preserving the Poisson structure). The group of time translations is a subgroup of $\mathbf{P}$, so $\mathbf{P}$ also preserves the energy function H. In this case, when we talk about vacuum states of the theory (X, H), we mean a vacuum state invariant under $\mathbf{P}$.

Recall that to define a symplectic manifold X is the same thing as to define the Poisson algebra $A = C^\infty(X)$ (X can be reconstructed as the spectrum of A). Therefore, we may say that a theory is a pair (A, H), where A is a Poisson algebra and $H \in A$. The Poisson algebra A is called the algebra of observables of the system.

Now let us consider quantum systems. The definition of a theory in this case is similar to the classical case. Namely, we will define **a quantum theory** to be a pair (A, H), where A is a *-algebra (not necessarily commutative), and H is a selfadjoint element of A, defined up to adding a real number. The algebra A is called the algebra of quantum observables (operators). The element H, as before, is called the Hamiltonian. Everything here is dependent on a real positive parameter $\hbar$ (the Planck constant).

By **a realization (or solution)** of a quantum theory (A, H) we will mean an irreducible *-representation of the algebra A in some Hilbert space $\mathcal{H}$, such that the spectrum of the operator H is bounded from below (representations are considered up to an isomorphism which preserves H). We will always normalize H so that the lowest point of its spectrum is zero. The space $\mathcal{H}$ is called the quantum space of states of the system (in this realization). Of course, as in the classical case, the same theory can have different realizations, as the same algebra can have different representations.

As we have mentioned, we will be interested in relativiatically invariant theories, i.e. theories with an action of the Poincare group on A, so that the subgroup of time translations acts by $a \to e^{itH/\hbar} a e^{-itH/\hbar}$. When we talk about realizations of such a theory, we will assume that $\mathbf{P}$ acts in $\mathcal{H}$ by unitary operators, with the group of time translations acting by $e^{itH/\hbar}$.

By a vacuum state we mean a vector $\Omega \in \mathcal{H}$ such that $Hv = 0$. In a relativistically invariant situation, a vacuum state is the same thing as a $\mathbf{P}$-invariant vector.

Remark. In general, as we will see, an irreducible realization of a theory can have many vacuum states. Therefore, the notions of a realization and of a vacuum state are not equivalent. However, if the algebra of observables is commutative (i.e. in the classical theory), each irreducible representation of this algebra is 1-dimensional, and there is no real difference between the notions of a realization and a vacuum state. Therefore, the word "realization" is not usually used when one refers to the classical theory.

Suppose that we have a classical theory (A_0, H_0) which has been quantized, and the corresponding $\hbar$-dependent family of quantum theories is (A, H) (here by an $\hbar$-dependent family we mean a family depending on the dimensionless parameter $\hbar/S_0$, where S_0 is a characteristic scale of action). This means that we have a quantization map – some linear map $A_0 \to A$, given by $a \to \hat{a}$, such that $\hat{H}_0 = H$, and $[\hat{a}, \hat{b}] = i\hbar\widehat{\{a, b\}} + o(\hbar)$, $\hbar \to 0$. In this case, we will say that a state $v \in \mathcal{H}$ of norm 1 is localized near a classical solution $x \in X = \mathrm{Spec}A_0$ if for any $a \in A_0$ $\langle v, \hat{a}v\rangle \to a(x)$, $\hbar \to 0$.

Let us now explain the connection between the classical and the quantum notions of a vacuum state. Suppose we have a quantum vacuum state Ω of norm 1 which is localized near a classical state x. In this case x is a classical vacuum state. Indeed, $H_0(x) = \lim_{\hbar\to 0}\langle\Omega, H\Omega\rangle = 0$, and for any $F \in A_0$ $\{F, H_0\}(x) = \lim_{\hbar\to 0} \frac{1}{i\hbar}\langle\Omega, [\hat{F}, H]\Omega\rangle = 0$, so x is a stationary point of H_0.

Remark. Note any quantum vacuum state is localized near a classical vacuum state. Sometimes a quantum vacuum state is "spread" with some density over the set of classical vacuum states. We will see examples of this in today's lecture.

Given a quantum theory (A, H), it is convenient to represent its realizations by correlation functions. Namely, given a realization $\mathcal{H}$ of this system, and a vacuum state $\Omega \in \mathcal{H}$, we can define correlation functions $\langle\Omega, L_1...L_n\Omega\rangle$, where $L_i \in A$. Since the action of A in $\mathcal{H}$ is irreducible, the realization $\mathcal{H}$ can be completely reconstructed from these correlation functions.

Remark. The irreducibility condition is not always satisfied in physically interesting examples. But here for simplicity we will assume that it is satisfied.

Sometimes a quantum system can be defined by a Lagrangian. Of course, as in the classical case, this is not always possible, and if possible, not in a unique way. However, such a presentation is very convenient for understanding the behaviour of the system. So let us explain (on examples) how to pass from a Lagrangian to the Hamiltonian and the operator algebra.

We will start with the case of quantum mechanics, when the spacetime is just the time line. Consider a Lagrangian for one boson:

$$\mathcal{L} = \int [\frac{(\phi')^2}{2} - U(\phi)]dt.$$

Then, by definition, the operator algebra is generated by operators ϕ_0, ϕ_0', H, with the canonical commutation relations

$$[\phi_0, \phi_0'] = i\hbar,\ [H, \phi_0] = -i\hbar\phi_0',\ [H, \phi_0'] = i\hbar U'(\phi_0).$$

Define the local operators $\phi(t) = e^{iHt/\hbar}\phi_0 e^{-iHt/\hbar}$, $\phi'(t) = e^{iHt/\hbar}\phi_0' e^{-iHt/\hbar}$. It follows from the above definition that $d\phi/dt = \phi'$, and ϕ satisfies the Newton's differential equation

$$\phi''(t) = -U'(\phi(t)).$$

The operators of the form $F(\phi(t), \phi'(t))$, where F is a polynomial, are called local operators at t (we order products in such a way that ϕ' stands on the right from ϕ).

The operator algebra is spanned (topologically) by operators $\phi(t_1)...\phi(t_n)$. Thus, a realization of the theory is defined by prescribing expectation values of these operators – the correlation functions.

In any realization of the theory, the Hamiltonian is given by the following explicit formula:

$$H = \frac{1}{2}(\phi')^2 + U(\phi) + C.$$

Indeed, the difference of the left and the right hand sides of this equation commutes with ϕ and ϕ', so by irreducibility it acts by a scalar.

Now consider quantum field theory. We first consider the theory in a spacetime $V = L \times \mathbb{R}$, where L is a lattice (finite or infinite). Let ∇_L be the discrete gradient operator on the lattice. Consider a Lagrangian

$$\mathcal{L} = \sum_{x \in L} \int dt [\frac{1}{2}(\phi_t^2(x,t) - (\nabla_L \phi(x,t))^2) - U(\phi(x,t))]$$

In this case the operator algebra is generated by operators $\phi(x,0), \phi_t(x,0)$, $x \in L$, and H, satisfying the commutation relations

$$[\phi(x,0), \phi_t(x,0)] = i\hbar, \ [H, \phi(x,0)] = -i\hbar\phi_t(x,0),$$
$$[H, \phi_t(x,0)] = i\hbar[-\Delta_L \phi(x,0) + U'(\phi(x,0))],$$

(where Δ_L is the lattice Laplacian) and such that $\phi(x,0), \phi'(x,0)$ commute with $\phi(y,0), \phi'(y,0)$ if $x \neq y$ (causality). The local operators $\phi(x,t), \phi'(x,t)$ are defined as above. (Observe that since H does not commute with ϕ, for $t_1 \neq t_2$ the operators $\phi(x_1,t_1)$, $\phi(x_2,t_2)$, in general, do not commute).

As before, the operator algebra is spanned by the operators $\phi(x_1,t_1)...\phi(x_n,t_n)$. Thus, a realization of the theory is determined by expectation values of these operators – the correlation functions.

As in the case of quantum mechanics, in any realization we can compute the Hamiltonian explicitly. Namely, the Hamiltonian is of the form $H = \sum_{x \in L} H_x$, where $H_x = \frac{1}{2}(\phi_t^2(x,0) + (\nabla_L \phi(x,0))^2) + U(\phi(x,0)) + C_x$.

Remark. Of course, if the lattice L is infinite, the definition of H can be problematic, since the sum over L may be divergent. However, since commutators of H with other operators are well defined, one may hope that the constants C_x can in fact be adjusted in such a way that the sum converges. This is indeed true in many situations.

Now let us consider field theory in continuous spacetime. In this case the operator algebra and the Hamiltonian are defined similarly to the case of discrete space, which was considered above. Namely, the operator algebra will be generated by the operators $\phi(x,0)$, $\phi_t(x,0)$, and also H_b, $b \in \mathfrak{p}$, where $\mathfrak{p}$ is the Lie algebra of the Poincare group $\mathbf{P}$, with the commutation relations between ϕ, ϕ_t and H_b similar to the above.

However, we will face an additional problem – now expressions like $\phi^2(x,t)$ may not be well defined, because of ultraviolet divergences. This problem can be cured by the ultraviolet renormalization theory, which we discussed last semester, if the Lagrangian we started with was renormalizable. In this case, the algebra of local operators is not quite an algebra, but an OPE algebra (an algebra with operator product expansion). It is almost never possible to compute the structure constants of this algebra exactly (rational conformal field theory in 2 dimensions is the main exception), but it is possible to compute them in perturbation expansion to any finite order. In general, for continuous space we have additional analytic difficulties (compared to the case of discrete space), but they will not be very important in the present lecture, so we will not discuss them here. We will just need the rough general picture, which has been outlined in this introduction.

Finally, let us say what we will mean by a **vacuum** for a quantum theory (A, H). We will mean by a vacuum for (A, H) one of two, roughly equivalent, things:

1. A linear functional $\langle , \rangle : A \to \mathbb{C}$ on the operator algebra (the expectation value), which satisfies some field theory axioms (e.g. axioms for Wightman functions);

2. A realization $\mathcal{H}$ of (A, H) together with a vacuum state Ω, normalized to unity.

The passage from 2 to 1 is trivial, and the passage from 1 to 2 is a part of the general formalism of field theory (see Kazhdan's lecture 1).

In general, a vacuum is not the same thing as a realization, since the same realization can have different vacua. For example, in the theory of Dirac operator on a manifold the space of vacua is the space of harmonic spinors. However, in a Wightman field theory in infinite volume, one can show that any realization has exactly one vacuum state.

1.1. What is symmetry breaking, and why it does not happen in quantum mechanics.

Suppose we have some classical physical theory (A, H), which has a symmetry group G. Let us ask the following question: does this theory have a G-invariant vacuum state?

If we have a quantum theory (A, H), which has a symmetry group G, then the correct analogue of this equestion is: does this theory have a G-invariant realization?

If the answer is no, one says that the symmetry is broken. If the answer is yes, one says that the symmetry is preserved.

In classical mechanics and classical field theory symmetry breaking can easily happen. For example, consider a classical particle of mass 1 on the line whose potential energy is $U(x) = g(x^2 - a^2)^2/4!$, where $a > 0, g > 0$. The space of states of this particle is the plane with coordinates x, p, and its Hamiltonian is $\frac{p^2}{2} + U(x)$. There is an action of the group $G = \mathbb{Z}/2\mathbb{Z}$ on the space of states, by $(x, p) \to (-x, -p)$, which preserves the equations of motion, and there are two lowest energy states: $s_+ = (a, 0), s_- = (-a, 0)$, which are permuted by G. But there is no lowest energy state which is G-invariant.

In quantum mechanics, symmetry breaking does not occur. This is a simple, but nontrivial and very important result. Let us show why this is true in the case of quantum mechanics of bosons, with a real Lagrangian. In this case, the operator algebra is generated by operators x_i, p_i, $i = 1, ..., n$, satisfying the Heisenberg algebra relations, and has a Hamiltonian $H = \frac{p^2}{2} + U(x)$. There is a realization of the theory in $\mathcal{H} = L^2(\mathbb{R}^n)$, with x_i acting by multiplication by the coordinate functions, and $p_i = -i\hbar \frac{\partial}{dx_i}$. It is well known this representation is irreducible. Any symmetry group G of the potential U also acts in $\mathcal{H}$. Thus, symmetry breaking does not occur.

Remark. In fact, according to the Stone-von-Neumann theorem, $\mathcal{H}$ is the unique realization of the theory.

In the case of bosons with real Lagrangian we can in fact make a stronger statement. Namely, not only is the realization unique, but the vacuum is also unique (and therefore invariant under any symmetry group of U). For simplicity we will show it in the case of only one boson on the line, but the argument we will give generalizes to any number of bosons in a space of any dimension.

We will consider a single boson on the line, in a field with potential $U(x)$ as above.

Theorem 1.1. *Let $H = -\frac{1}{2}\frac{d^2}{dx^2} + U(x)$ be any Schrödinger operator, such that the potential $U(x)$ tends to $+\infty$ at infinity (so that H has discrete spectrum). Let E_0 be the smallest eigenvalue of H. Then there exists a unique, up to a factor, function $\psi \in L^2(\mathbb{R})$ (called the vacuum state wave function) such that $H\psi = E_0\psi$.*

Proof. For any $f \in L^2(\mathbb{R})$ we have

$$(f, Hf) = \int_{-\infty}^{\infty} \left(\frac{1}{2}|f'(x)|^2 + U(x)|f(x)|^2 \right) dx \tag{1.1}$$

Thus, ψ is defined by the condition that it is a global minimum point for the energy functional $E(f) := (f, Hf)$ on the sphere $||f|| = 1$. The proof of uniqueness of ψ rests on the following Lemma.

Lemma. If ψ is a real global minimum point of (1.1) than ψ has constant sign.

Proof of the Lemma. Let $E(\psi) = E_0$. Suppose that ψ changes sign at the point x_0. Since ψ satisfies the Euler-Lagrange (=Schrödinger) equation $H\psi = E_0\psi$, we have $\psi'(x_0) \neq 0$. Consider the function $|\psi|$. It is clear that $E(|\psi|) = E(\psi) = E_0$, but $|\psi|$ is not smooth, so it does not satisfy the Euler-Lagrange equation $Hf = E_0 f$, and thus cannot be the global minimum of E. So, the smallest value of E on the sphere is less than E_0 – a contadiction.

Now it is easy to prove the theorem. If the space of solutions of $H\psi = E_0\psi$ is more than 1-dimensional, then there exist two linearly independent, orthogonal real solutions ψ_1, ψ_2. On the other hand, both of them have to be of constant sign, so $(\psi_1, \psi_2) \neq 0$ – a contradiction. □

It is useful to consider how symmetry breaking, which is absent in the quantum theory, arises in the quasiclassical limit. For this purpose, we should introduce the Planck's constant $\hbar$, and consider the $\hbar$-dependent Hamiltonian

$$H = -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + U(x), \tag{1.2}$$

where $U(x) = g(x^2 - a^2)^2/4!$. Let E_0, E_1 be the lowest eigenvalues of H in the space of even and odd functions, respectively, and ψ_0, ψ_1 the corresponding eigenvectors. We assume that ψ_0, ψ_1 have unit norm, and are normalized in such a way that $\psi_0(0) > 0$, $\psi_1'(0) > 0$. It is easy to show in the same way as above that ψ_1 does not change sign in the regions $x > 0$, $x < 0$. Define $\psi_+ = \frac{1}{\sqrt{2}}(\psi_0 + \psi_1)$, $\psi_- = \frac{1}{\sqrt{2}}(\psi_0 - \psi_1)$.

Then it is possible to prove the following.

Theorem 1.2. *(i) As $\hbar \to 0$, $E_0, E_1 \sim a\hbar\sqrt{g/4!}$, and $E_1 - E_0 \sim Ce^{-S_0/\hbar}$, where S_0 is a positive constant.*

(ii) In the sense of distributions,

$$\lim_{\hbar \to 0} |\psi_\pm|^2 = \delta(x \mp a). \tag{1.3}$$

This theorem shows that for very small $\hbar$, although there is only one lowest energy state ψ_0 with energy E_0, there is another stationary state ψ_1 with energy E_1 almost indistinguishable from the lowest one, and in the 2-dimensional space spanned by ψ_0, ψ_1, there are two orthogonal states ψ_+, ψ_- localized near the classical equilibrium states $a, -a$. The states ψ_+, ψ_- are not stationary (i.e. are not eigenvectors of H), but their failure to be stationary is indistinguishable to any finite order in $\hbar$ (in fact, the angle between $H\psi_\pm$ and $\psi_\pm$ is dominated by const$e^{-S_0/\hbar}$. In particular, symmetry restoration in quantum theory is not seen at the perturbation theory level.

Thus, we have seen how symmetry is lost in the quasiclassical limit. One can consider this effect from a slightly different prospective, by looking how symmetry appears in the process of quantization.

Recall that we have two classical lowest-energy states $a, -a$, near which the operator H looks approximately like a harmonic oscillator. Therefore, we can look for eigenfunctions of H perturbatively, in the form

(1.4)
$$\tilde{\psi}_\pm(x) = f_\pm\left(\frac{x \mp a}{\sqrt{\hbar}}\right),\ f_\pm(z) = (\pi\hbar)^{-1/4} g^{1/8} e^{-\sqrt{g}z^2/2}(1 + u_1(z)\hbar^{1/2} + u_2(z)\hbar + ...)$$

(the 0-th term of this expansion is the lowest eigenfunction of the harmonic oscillator). It is easy to see that the real "eigenfunction" of the form (1.4), normalized to have unit norm, is unique for each sign.

Now, one can see the following.

(i) The formal series $f_\pm(z)$ do not converge. However, they represent asymptotic expansions of actual functions $\psi_\pm(z\sqrt{\hbar} \pm a)$ (where $\psi_\pm$ are as above), which are smooth from the right in $\hbar$ on $[0, \infty)$, but not analytic.

(ii) The formal series $\tilde{\psi}_\pm(x)$ are eigenfunctions of H with the same eigenvalue. On the other hand, the actual functions $\psi_\pm(x)$ are not eigenvectors of H, although their failure to be ones is exponentally small. The actual lowest eigenvector of H is unique up to a factor, and equals $\psi_0 = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-)$. In particular, it is G-invariant, unlike ψ_+, ψ_-.

(iii) Let $W_+^n(t_1, ..., t_n)$, $W_-^n(t_1, ..., t_n)$ be the correlation functions computed perturbatively (by Feynman calculus) using the formal vacua $\tilde{\psi}_+$, $\tilde{\psi}_-$. Then W_+^n, W_-^n do not serve as small $\hbar$ asymptotic expansions of the correlation functions of any realization of our quantum theory. However, the averages $\frac{1}{2}(W_+^n + W_-^n)$ do serve as such asymptotic expansions.

This shows how symmetry appears in quantization, when one goes from the perturbative to the nonperturbative setting.

1.2. Still no symmetry breaking in quantum field theory in finite volume.

Before going over to quantum field theory, we will give one more argument, at the physical level of rigor, which shows why there is no symmetry breaking in quantum mechanics (this argument is more of an explanation than a proof). It is based on the path integral approach. It does not use representation theory of the Heisenberg algebra, nor the positivity of the vacuum wave function, and has the advantage that it also works for quantum field theory on a spacetime with a "space" part of finite volume.

As before, we will consider the $\mathbb{Z}/2\mathbb{Z}$-symmetric quartic potential $U(x)$. Suppose that symmetry breaking were the case. Then the two perturbative vacua $\tilde{\psi}_+, \tilde{\psi}_-$ do indeed exist nonperturbatively, i..e serve as asymptotic expansions of actual lowest eigenstates ψ_+, ψ_- of the Hamiltonian, in two different realizations of the theory, $\mathcal{H}_+, \mathcal{H}_-$. Consider the space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. The vectors $\psi_+, \psi_- \in \mathcal{H}$ are "localized" near $a, -a$, and are orthogonal to each other. Thus, the inner product $(\psi_+, e^{-Ht/\hbar}\psi_-)$ must vanish. Let us now compute the same inner product using path integrals.

Recall Feynman-Kac formula:

$$(1.5) \qquad (\delta_{x_1}, e^{-Ht/\hbar}\delta_{x_2}) = \int_{\phi:[0,t]\to\mathbb{R}, \phi(0)=x_1, \phi(t)=x_2} e^{-S(\phi)/\hbar} D\phi,$$

where

$$(1.6) \qquad S(\phi) = \int_0^t [\frac{1}{2}(\phi')^2 + U(\phi)]ds.$$

The states ψ_+, ψ_- are "localized" near $a, -a$ (i.e. are close to delta-functions at $a, -a$ after a suitable normalization). So, if we believe the Feynman-Kac formula in this situation, we can substitute $\delta(x-a)$, $\delta(x+a)$ instead of them, and apply the Feynman-Kac formula. Using the small $\hbar$ stationary phase estimate on the right hand side of (1.5), we will get

$$(1.7) \qquad (\psi_+, e^{-tH/\hbar}\psi_-) \sim Ce^{-S_*(t)/\hbar},$$

where $S_*(t)$ is the least possible action of a path $\phi : [0,t] \to \mathbb{R}$ such that $\phi(0) = -a, \phi(t) = a$.

The least action $S_*(t)$ is attained at a classical trajectory $\phi = \phi_*(\tau)$, which is a solution of the Euler-Lagrange differential equation, i.e. the Newton's equation $\phi'' = \frac{g}{6}\phi(\phi^2 - a^2)$ with boundary conditions $\phi(0) = -a$, $\phi(t) = a$. (Such a solution exists and is unique).

Remark. The function $x = \phi_*(\tau)$ describes the motion of a ball in the potential field with potential $-U(x)$ (a camel's back), from one hump to the other. The initial velocity of the ball is such that the time needed to go from the top of one hump to the top of the other is t. The reason that the potential $U(x)$ is replaced here by $-U(x)$ is that we are doing a Euclidean path integral, which means that we performed a Wick rotation $t \to it$. This rotation transforms the Newton's equation for the potential U into the Newton's equation for the potential $-U$.

Formula (1.7) contradicts the fact that $(\psi_+, e^{-Ht/\hbar}\psi_-)$ vanishes. So our assumption that there are two vacua was false.

Remark. Formula (1.7) actually gives the correct estimate of the inner product $(\psi_+, e^{-tH}\psi_-)$. In particular, $S_0 = S_*(\infty)$. This estimate can be confirmed by rigorous methods.

In this argument, we have never used the fact that ψ_+, ψ_- are functions on the real line. All we used is that they are "localized" near classical equilibrium states $a, -a$, i.e. that for any local observable A the expectation value of A on $\psi_\pm$ is close to the value of the corresponding classical observable at the point $(\pm a, 0)$ in the phase space. Thus, our argument is independent of the realization of the space of states as $L^2(\mathbb{R})$. This makes it easy to generalize this argument to the case of field theory.

Consider a spacetime $M \times \mathbb{R}$, where M is the "space". We will assume that we have already performed a Wick rotation, so that the metric on the spacetime is a Riemannian product metric. We will assume that the volume of M is finite and equals V.

Consider the field theory on $M \times R$ with one scalar Bose field ϕ, described by the Euclidean Lagrangian

$$\mathcal{L} = \int (\frac{1}{2}(\nabla\phi)^2 + U(\phi))dx, \tag{1.8}$$

where U is the quartic potential as above. As before, we have two equilibrium states $a, -a$.

Now consider this theory quantum mechanically. Then we can see that there is still no symmetry breaking. The simplest way to see it is to consider first a discrete space M. In this case, the operator algebra is a finite tensor product of Heisenberg algebras corresponding to points of M, so it is a Heisenberg algebra itself, and the representation-theoretic argument the we used in the case of quantum mechanics shows that symmetry breaking does not occur.

However, it is more instructive to use the path integral argument. As before, assume that symmetry breaking occurs. Then we have two realizations $\mathcal{H}_+$, $\mathcal{H}_-$, and two orthogonal quantum vacua $\Omega_\pm \in \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ whic are "localized" near the equilibrium points $a, -a$ for small $\hbar$, in the sense that $\langle \Omega_\pm, \phi(x_1)...\phi(x_n)\Omega_\pm\rangle \to (\pm a)^n$, $\hbar \to 0$.

As in the quantum mechanics case, the inner product $(\Omega_+, e^{-tH/\hbar}\Omega_-)$ vanishes. On the other hand, computing it using the Feynman-Kac formula, we will get

$$(\Omega_+, e^{-tH/\hbar}\Omega_-) \sim e^{-S_*(t)V/\hbar}, \tag{1.9}$$

where $S_*(t)$ is as above. The reason is that the least action is attained on the space-independent classical solution $\phi_*^M(\mu,\tau) = \phi_*(\tau)$, $\mu \in M$, $\tau \in \mathbb{R}$.

Since (1.9) is nonzero (here it is essential that the volume V is finite), we get a contradiction.

As in quantum mechanics, we can define two sets of correlation functions W_+^n, W_-^n, evaluated by using perturbation theory near the lowest energy points $a, -a$ (of course, we can only define them in the renormalizable case, i.e. in 4 dimensions and below; also, one should remember that renormalization is not uniquely determined). Our reasonings show that W_+^n, W_-^n are not small $\hbar$ expansions of the correlation functions of a realization of our quantum theory. On the other hand, the functions $W_0^n = \frac{1}{2}(W_+^n + W_-^n)$ have a chance to be asymptotic expansions of the actual (nonperturbative) correlation functions.

For $\dim(M) = 1, 2$ the existence of the quantum theory $\mathcal{H}$ has been established in constructive field theory. In this case, it is possible to show that there exists a G-invariant vacuum Ω_0, such that the correlation functions of the theory with respect to Ω_0 indeed have the asymptotic expansion given by W_0^n.

1.3. Symmetry breaking in quantum field theory in infinite volume.

When the volume V of the space M becomes infinite, the arguments of the previous section fail. The representation-theoretic argument fails, because the canonical representation of the operator algebra, whioch we used in the case of finite volume, is now an infinite tensor product of spaces corresponding to points; so we have to make sense of it, and there may be no G-invariant way of doing so. The path integral argument also fails. Indeed, the right hand side of (1.9) vanishes, so both computations of $(\Omega_+, e^{-tH/\hbar}\Omega_-)$ give the same answer, and we can derive no contradiction. Moreover, if we assume that we have a representation $\mathcal{H}$ of the operator algebra with two vacua Ω_+, Ω_-, then using the formula

$$\langle \Omega_+, \phi(x_1^*)....\phi(x_n^*)\Omega_-\rangle = \int_{\phi:X\to\mathbb{R},\phi\to\pm a,t\to\pm\infty} \phi(x_1)...\phi(x_n)e^{-S(\phi)/\hbar}D\phi, \tag{1.10}$$

where $x^* = (\mu, it)$ for $x = (\mu, t)$, we infer that the inner product (1.10) vanishes for any $x_1, ..., x_n$. This shows that the space $\mathcal{H}$ splits in an orthogonal direct sum: $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where the spaces $\mathcal{H}_\pm$ are the Hilbert spaces of separate realizations generated by the vacua Ω_+, Ω_-. This means, symmetry breaking could occur, like in the classical theory: quantum effects are not strong enough to restore symmetry.

This is what in fact happens in quantum field theory. More precisely, the situation is the following.

(i) If the symmetry group G is a finite group, symmetry can be broken in infinite volume starting with spacetime dimension 2. For example, in the example we considered symmetry breaking does occur.

(ii) If the symmetry group G is a connected Lie group, symmetry breaking can occur starting with spacetime dimension 3. The fact that it does not occur in dimension 2 is a remarkable and a very important fact, which we will discuss in the next section. This fact was proved by S.Coleman in 1973 (see S.Coleman's paper, CMP, vol. 31, page 259).

Thus, symmetry breaking is an "infrared" effect, associated with the behavior of the theory at large distances.

1.4. Infinite volume asymptotics of correlation functions.

Let us see how symmetry is broken in the infinite volume limit of finite volume quantum field theories. We will assume that M is a torus $T_r = (\mathbb{R}/r\mathbb{Z})^{d-1}$, and $r \to \infty$. In the limit, we hope to recover a Poincare invariant field theory corresponding to our Lagrangian. However, as we know, there are two such theories: $\mathcal{H}_+$ and $\mathcal{H}_-$. So which of them do we recover?

Let us formulate the question more precisely. Consider the correlation functions W_r^n, corresponding to the field theory with space being the torus T_r. Let also W_+^n, W_-^n be the correlation functions of the theories $\mathcal{H}_+$, $\mathcal{H}_-$. The question is, what is the asymptotics of W_r^n as $r \to \infty$, in terms of W_+^n, W_-^n?

The answer is: there exists an r-dependent normalization constant $C(r)$ such that

$$\lim_{r\to\infty} C(r)W_r^n = \frac{1}{2}(W_+^n + W_-^n). \tag{1.11}$$

Now consider a field theory defined by a Lagrangian $\mathcal{L} = \int(\frac{1}{2}(\nabla\phi)^2) + U(\phi))d^n x$, where $U(\phi)$ is a general potential, with a finite symmetry group G. Suppose that

$a_1, ..., a_m$ are the global minimum points of U, transitively acted on by G, and $U(a_i) = 0$. Consider first the quantum field theory with space being the torus of volume $V = r^{d-1}$. We have seen that there is one realization $\mathcal{H}$ of this theory, and it has certain correlation functions W_r^n. On the other hand, in infinite volume we will have m different realizations $\mathcal{H}_1, ..., \mathcal{H}_m$, with correlation functions $W_1^n, ..., W_m^n$.

Let us consider the asymptotics of W_r^n as $r \to \infty$. The answer is the following: for a suitable V-dependent normalization constant $C(r)$,

$$\lim_{r\to\infty} C(r) W_r^n = \frac{1}{m} \sum W_i^n, \tag{1.12}$$

This shows that the limit of normalized finite volume correlation functions may, in general, fail to satisfy the cluster decomposition axiom, (see Kazhdan's lectures).

This story can be slightly generalized. Namely, we can make M a ball of radius r, and impose some boundary conditions B on fields at the boundary of the spacetime. This means, we will define correlation functions by the formula

$$W_r^n(B)(x_1, ..., x_n) = \int_{\phi \text{satisfying boundary conditions}} \phi(x_1)...\phi(x_n) e^{-S(\phi)} D\phi.$$

Then the infinite volume asymptotics of $W_r^n(B)$ looks like

$$\lim_{r\to\infty} C(r) W_r^n(B) = \sum p_i W_i^n, \tag{1.12}$$

The collection of numbers p_i represents the "density" with which the quantum vacuum in finite volume is spread over the set of classical minima (=quantum vacua in infinite volume). This density depends on the boundary conditions. The word "density" should not be taken literally, however, because the numbers p_i are in general complex numbers.

If we take the simplest boundary condition $\phi = a_i$, then in the limit we will get the correlation functions W_i^n (i.e. $p_i = 1, p_j = 0$, $i \neq j$), so we get purely the i-th vacuum. However, if we impose some other boundary condition, we will, in general, get a mixture of vacua.

For example, in the case of $\mathbb{Z}/2\mathbb{Z}$-invariant quartic potential, we can impose boundary conditions $\phi = a$, $\phi = -a$, or $\phi = 0$ (do not worry that the action of all fields in the third case is infinite; since we are considering a normalized path integral, i.e. divided by the partition function, this infinity will cancel). Let $W_r^n(s)$ are the corresponding correlation functions $s = a, -a, 0$. Then in the first case $p_+ = 1, p_- = 0$ (as it it hard to get from the boundary anywhere except a with a small action), in the second case $p_+ = 0, p_- = 1$ (for a similar reason), and in the third case $p_+ = p_- = 1/2$.

1.5. Continuous symmetry breaking.

Now we will consider symmetry breaking in quantum field theory, when the symmetry group is a connected Lie group. The typical example is a complex valued Bose field ϕ, and the Lagrangian

$$\mathcal{L}(\phi) = \int d^d x \left(\frac{1}{2} |\nabla \phi|^2 + (|\phi|^2 - a^2)^2 \right). \tag{1.13}$$

This Lagrangian has a $U(1)$-symmetry, acting by $\phi \to e^{i\theta}\phi$. In the classical theory, we have lowest energy states $\phi = ae^{i\theta}$, $\theta \in [0, 2\pi)$, so we have symmetry breaking.

We will try to find out whether symmetry breaking exists also in the quantum theory.

Above we showed that symmetry breaking does not occur in the 1-dimensional case (quantum mechanics). We did it for the case of a real boson ϕ, but for the complex boson the argument (with path integral) works even better. For instance, for Lagrangian (1.13) the set of classical minima of energy is the circle $|\phi| = a$, which is connected. Therefore, to go from one classical minimum, $ae^{i\theta_1}$, to another, $ae^{i\theta_2}$, one does not need to go over the "hump" of the potential, and can go along the circle of minima, so one can do it with even less action than before. Therefore, the path integral computation described above would show that in the complex case there is even more linking between the states ψ_+, ψ_- than in the real case. This effect might make us think that perhaps in field theory, continuous symmetry breaking does not happen as easily as discrete symmetry breaking. And indeed, it turns out that continuous symmetry breaking cannot happen in two dimensions, and can happen only in dimensions ≥ 3.

Unfortunately, the path integral method is too crude to show that continuous symmetry breaking does not occur in 2 dimensions. Indeed, it is easy to see that all paths in the integral which computes $(\Omega_+, e^{-tH}\Omega_-)$ have infinite action in infinite volume, so we can derive no contradiction. So let us demonstrate why symmetry is preserved in 2 dimensions and broken above 2 dimensions by considering the simplest example.

The simplest example is the theory of a free massless real scalar Bose field in d dimensions, defined by the Lagrangian

$$\mathcal{L}_0(\phi) = \int d^d x (\frac{1}{2}(\nabla\phi)^2). \tag{1.14}$$

This theory has a translation symmetry $\phi \to \phi + c$. Let us show that this symmetry is broken for $d > 2$, and preserved for $d = 2$.

For $d > 2$, symmetry breaking is obvious. Indeed, in this case we have a Wightman field theory generated by an elementary field ϕ, satisfying Wightman axioms (see Kazhdan's lectures). In this theory, the 1-point function of the operator ϕ is zero, while the 1-point function of the operator $\phi + c$ is c. Therefore, the transformation $\phi \to \phi + c$ does not preserve the 1-point function. This shows, that in the case $d > 2$ we in fact have not a single realization, but a family of realizations $\mathcal{H}_c$ parametrized by points c of the line. The theory $\mathcal{H}_c$ is defined by the condition that $\langle \Omega_c, \phi\Omega_c \rangle = c$, where Ω_c is the vacuum of $\mathcal{H}_c$. The vacuum Ω_c is "localized" near the classical equilibrium state c. The map $\phi \to \phi + c$ transforms $\mathcal{H}_b$ to $\mathcal{H}_{b+c}$.

For $d = 2$, the situation is not the same. The problem is that the operator ϕ is not defined in 2 dimensions, although its derivatives are. Indeed, in 2 dimensions, we have

$$\langle \partial\phi(x)\partial\phi(y) \rangle = -\partial_x\partial_y \ln|x - y|, \tag{1.15}$$

so if the operator ϕ was defined in some way, we would have

$$\langle \phi(x)\phi(y) \rangle = -\ln|x - y| + C, \tag{1.16}$$

which contradicts positivity (the function on the RHS of (1.16) is not positive).

Let us say this more precisely. What we have in 2 dimensions is a quantum field theory generated by a Wightman map ϕ from Schwarz functions to operators, which is defined not on the whole space of Schwarz functions $S(V)$ but only on the space $S_0(V)$ of Schwarz functions on V with integral zero. Then derivatives of ϕ can then be defined on all Schwarz functions by

$$\partial_i \phi(f) = -\phi(\partial_i f)$$

(the right hand side makes sense since $\int \partial_i f = 0$).

However, in the theory defined by such ϕ the question of symmetry breaking does not arise, since there is no symmetry to begin with: on the space $S_0(V)$, the maps ϕ and $\phi + c$ are the same. So, in order to raise the question about symmetry breaking, we should consider an extension of our operator algebra, which will have a nontrivial action of symmetry. The most reasonable way of doing so is the following.

Instead of considering the theory of an $\mathbb{R}$-valued massless scalar ϕ, we will consider the theory of a circle-valued field ϕ with the same Lagrangian. We take the circle to be $S_\lambda = \mathbb{R}/2\pi\lambda\mathbb{Z}$. The Lagrangian is the same as before. In this case, the local functional ϕ is not defined, but instead we have local functionals $e^{ik\phi/\lambda}$, $k \in \mathbb{Z}$. The local functionals in this theory are Laurent polynomials in $e^{i\phi/\lambda}$ whose coefficients are differential polynomials in the derivatives of the field ϕ, but not in ϕ itself. The translational symmetry in this theory is the $U(1)$-symmetry: for a complex number z with $|z| = 1$, $z \circ e^{ik\phi/\lambda} = z^k e^{ik\phi/\lambda}$, and z acts trivially on the derivatives of ϕ. Classically, this theory has a family of vacua (equilibrium states), given by $\phi = c$, $c \in S_\lambda$.

Let us show that in this system symmetry is not broken quantum-mechanically. It is enough to show that the 1-point function $\langle \Omega, \mathcal{O}(0)\Omega\rangle$ vanishes for any local operator $\mathcal{O}$ of the form $\mathcal{O} = P(\phi)e^{ik\phi/\lambda}$ for $k \neq 0$. Let us show this in the case $P = 1$ (in general, the proof is analogous). Proof: From the OPE in the free theory (cf. Witten's lecture 3 from the fall term) we get

$$\langle \Omega, \mathcal{O}(x)\mathcal{O}^*(0)\Omega\rangle = |x|^{-k^2/\lambda^2},$$

so this 2-point function vanishes at infinity. But by clustering, the limit of this function at infinity is $|\langle \Omega, \mathcal{O}(0)\Omega\rangle|^2$. So, $\langle \Omega, \mathcal{O}(0)\Omega\rangle = 0$.

Another way to see that there is no symmetry breaking is as follows. The Hilbert space of the quantized theory is of the form $\mathcal{H} = F \otimes F^* \otimes l_2(\lambda\mathbb{Z}) = \oplus_{k\in\mathbb{Z}}(F \otimes F^*)_k$, where F is the Fock space. The operators corresponding to derivatives of ϕ respect this decomposition, while the operator $e^{ik\phi/\lambda}$ maps the space $(F \otimes F_*)_{k'}$ to $(F \otimes F^*)_{k+k'}$. The vacuum vector Ω belongs to the zero component $(F \otimes F^*)_0$. This implies that all correlation functions of operators in this theory are invariant under the action of $U(1)$. In particular, $\langle e^{ik\phi/\lambda}\rangle = \delta_{0k}$, which shows that the vacuum Ω is not "localized" near any classical vacuum but is "spread" uniformly over the space S_λ of classical vacua. Thus, symmetry under $U(1)$ is not broken.

Remark 1. It is instructive to see why our argument that there is symmetry breaking for $d > 2$ fails in finite volume, where, as we know, there should be no symmetry breaking. The problem is that in finite volume the field ϕ is not defined, although its derivatives are. Indeed, since the spacetime is the product of a time line with a compact space, at large distances in looks simply like a line, the 2-point function of ϕ (which is the Green's function of the spacetime) at large $|t|$

looks like the 1-dimensional Green's function, i.e. $-|t|+C+o(1)$. This function is not bounded from below, so it violates positivity. Moreover, as follows from considering the case $d=2$, the situation is the same if all spatial directions but one are compactified.

Remark 2. Above we have considered the theory of a free massless scalar ϕ in dimension $d>2$ and found that its space of quantum vacua is the space of values of ϕ (i.e. the target space $\mathbb{R}$). More generally, if one considers the sigma-model with spacetime $\mathbb{R}^d$, $d\geq 2$, and target space M (a Riemannian manifold), the space of quantum vacua, as well as the space of classical vacua, will be M. (Here you should forget for a moment that this sigma-model for nonlinear M is not renormalizable, and so it is not clear what this statement means. We will clarify this point later.) In particular, M as a Riemannian manifold can be recovered from the quantum theory as moduli space of quantum vacua (see Remark 3 below).

On the other hand, we saw that for $d=2$ the theory of a circle-valued field ϕ (which is the same as the 2-dimensional sigma-model with target space S_λ) has only one vacuum Ω. This is the case for 2-dimensional sigma-model in general: its moduli space of quantum vacua is generally very small, and does not coincide with the space of classical vacua (=the target space). In particular, the target space cannot be recovered intrinsically from the quantum theory. For example, the circle S_λ cannot be recovered intrinsically from the theory of maps into S_λ considered above. This, in fact, happens for a good reason – one can show that the theories attached to the circles S_λ and $S_{1/\lambda}$ are equivalent (for example, their partition functions coincide – see Gawedzki's lecture 1, formula (9)). This is the starting point for the theory of mirror symmetry.

Remark 3. Consider a field theory with the Lagrangian $\mathcal{L}=\int d^dx(\frac{1}{2}(\nabla\phi)^2+U(\phi))$, where ϕ takes values in some Riemannian manifold M, and U is a potential function on M ($U\geq 0$). Let $M(0)$ is the set of zeros of U. Assume that $M(0)$ is nonempty and smooth, and that d^2U is nondegenerate on $T_xM/T_xM(0)$ for $x\in M(0)$. Suppose that there is a Lie group G which acts by isometries on M, fixes U, and acts transitively on M_0. In this case, one can show (at the physical level of rigor) that the "infrared behavior" of the theory described by $\mathcal{L}$ is the same as the "infrared behavior" of the sigma-model with target space being the space $M(0)$ of classical vacua. The precise meaning of this statement is explained in Section 1.7. (We can ignore nonrenormalizability problems by defining the theories by a cutoff path integral, where integration is taken over fields defined on a lattice with step Λ^{-1}, or over fields having only Fourier modes with $|k|<\Lambda$, with respect to some coordinate system; in this setting, the cutoff Λ is not sent to infinity). This fact can be explained heuristically: if a function ϕ has only low Fourier modes, it cannot oscillate rapidly, so in order to have a small action and thus give a noticeable contribution to the path integral, it has to stay closely to the minimum locus $M(0)$, i.e. has to be close to a map into $M(0)$.

Thus, continuous symmetry breaking, being an infrared effect, will occur in the theory described by $\mathcal{L}$ iff it occurs in the corresponding sigma-model. So, as follows from remark 2, symmetry breaking tends not to occur in dimension 2, but tends to occur in dimension >2.

These are, however, mostly heuristic arguments. In the next section we will treat the issue of continuous symmetry breaking in a more systematic way, using Goldstone's theorem.

1.6. Goldstone's theorem.

Recall the standard formalism of Noether's theorem and currents in classical field theory. Suppose we have a Lagrangian $\mathcal{L} = \mathcal{L}(\phi)$ in a d-dimensional spacetime V. Denote the space of solutions of the corresponding Euler-Lagrange equations by X.

Let G^s be a 1-parameter symmetry group of this Lagrangian. Let $D_\phi := \frac{d}{ds}|_{s=0} G^s \phi$. We assume that D_ϕ is a local functional of ϕ.

Let $\eta \in \Omega^1(X, \Omega^{n-1}(V))$ be the canonical 1-form on the space of solutions that was discussed in Bernstein's lectures and in Witten's problem sets (the canonical 2-form on X was defined as $\int_C d_X \eta$, where C is an $n-1$-dimensional cycle). For instance, the formula for η for the free theory of a massless scalar is

$$\eta(\delta\phi)(x) = \delta\phi(x) * d\phi(x). \tag{1.17}$$

Let $J \in \Omega^0(X, \Omega^{d-1}(V))$ be defined by the formula $J = \eta(D_\phi)$. Since D_ϕ is local, so is J. Thus, J is a local functional on X with values in $\Omega^{d-1}(V)$. For instance, if $\mathcal{L}$ is the Lagrangian of the theory of a free massless scalar, and $G^s\phi = \phi + s$, then $D_\phi = 1$, so $J(x) = *d\phi(x)$. The local functional J is called the *current* corresponding to the symmetry G^s. The main property of the current is that it is *conserved*, i.e. $d_V J = 0$. This follows form the fact that $d_V \eta = 0$.

In finite volume it is useful to define the charge functional $Q = \int_C J(x)$, where C is some spacelike cycle (e.g. $t = \text{const}$). Since the current is conserved, this quantity is independent of the choice of the cycle, as long as it represents the fundamental homology class of "space". If $\Omega = \int_C d_X \eta$ is a nondegenerate 2-form on X, then Q is a Hamiltonian which defines the symmetry group G^s, in the sense that $\frac{d}{ds}|_{s=0}(G^s)^* F = \{F, Q\}$, where $\{,\}$ is the Poisson bracket on S, and F any local functional on X.

In infinite volume, the functional Q is not necessarily defined, since the integral does not converge. In this case, it is convenient to set $C = C_0 = \{t = 0\}$ (here we have chosen a time coordinate on the spacetime), and define the "cutoff charge functional"

$$Q_f = \int_{C_0} f(x) J(x), \tag{1.18}$$

where $f : C_0 \to \mathbb{R}$ is a Schwarz function. The limit $\lim_{f\to 1} Q_f$ in this case does not exist, but for any local functional F

$$\lim_{f\to 1} \{F, Q_f\} = \frac{d}{ds}|_{s=0}(G^s)^* F. \tag{1.19}$$

Remark. By $f \to 1$ we mean that f converges to 1 uniformly on any compact set, and all derivatives of f go to zero uniformly on the whole space.

In quantum theory, the story is the same, except that (local) functionals are replaced with local operators, and Poisson bracket with commutator times i. That is, to any one-parameter symmetry G^s there corresponds a $d-1$-form-valued local operator $J(x)$, which is conserved, and the charge operator $Q = \int_C J(x)$ (in finite volume) has the property

$$[F, Q] = -i \frac{d}{ds}|_{s=0}(G^s)^* F. \tag{1.20}$$

The case of infinite volume is dealt with in the same way as in the classical theory, by considering cutoff operators Q_f.

Now we will discuss Goldstone's theorem. Suppose that the symmetry G^s in the theory defined by $\mathcal{L}$ is broken not only classically but also quantum mechanically. In this case, if we have a solution of the theory $\mathcal{H}$ (a Hilbert space with an action of the operator algebra), then there exists a scalar local operator ϕ whose 1-point function is not invariant under symmetry.

Remark. Strictly speaking, we only know that some n-point function is not invariant; but in all known situations with symmetry breaking there is also a non-invariant 1-point function.

Let Q_f be the cutoff charge operator for the symmetry G^s. We have

$$\langle\Omega|[Q_f, \phi(0)]|\Omega\rangle \neq 0, \tag{1.21}$$

for f sufficiently close to 1. Thus, $\langle\Omega|[J(x)\phi(0)]|\Omega\rangle \neq 0$ for some x.

Consider the 2-point functions

$$M_+(x) := \langle\Omega|J(x)\phi(0)|\Omega\rangle, M_-(x) := \langle\Omega|\phi(0)J(x)|\Omega\rangle \tag{1.22}$$

(these are $d-1$-forms on V). Since the symmetry is broken, $M_+ \neq M_-$, although by space-like separation, they coincide if x is spacelike.

Let $\mathcal{H} = \int_{p\in V_+} \mathcal{H}_p$ be the spectral decomposition of $\mathcal{H}$ with respect to the action of the translation group (here V_+ is the positive part of the full light cone). Using the decomposition of the inner product in a sum over intermediate states, $(\langle J(x)\Omega, \phi(0)\Omega\rangle = \sum_n \langle J(x)\Omega, n\rangle\langle n, \phi(0)\Omega\rangle)$, we get

$$M_\pm(x) = \int_{V_+} K_\pm(x,p)dp, \tag{1.23}$$

where $K_+(x,p), K_-(x,p)$ are the contributions to M_+, M_- from intermediate states of 4-momentum p ($K_\pm$ are vector-valued distributions in p).

Because of Lorentz invariance, the distributions K_+, K_- look like

$$K_\pm = pe^{\pm ipx}\rho_\pm(-p^2), \tag{1.24}$$

where $\rho_\pm(s)$ are distributions on the half-line $s \geq 0$.

This yields

$$M_\pm(x) = i * d_V \int_0^\infty \rho_\pm(m^2)W_m(\pm x)dm^2, \tag{1.25}$$

where $W_m(x) = \int_{\mathcal{O}_m^+} e^{ipx}dp$ is the Klein-Gordon propagator with mass m defined in Lecture 1 last term ($\mathcal{O}_m^+$ is the upper sheet of the hyperboloid $p^2 = -m^2$).

Let $M(x) = M_+(x) - M_-(x)$. Since $W_m(x) = W_m(-x)$ when x is spacelike, for spacelike x (1.25) yields

$$M(x) = i * d_V \int_0^\infty (\rho_+(m^2) - \rho_-(m^2))W_m(x)dm^2. \tag{1.26}$$

However, as we have mentioned, by spacelike separation $M(x)$ vanishes for spacelike x. This implies that $\rho_+ = \rho_- = \rho$, and so (1.25) yields

$$M(x) = i * d_V \int_0^\infty \rho(m^2)(W_m(x) - W_m(-x))dm^2. \tag{1.27}$$

Differentiating both sides of (1.27), at a point x such that $x^2 > 0$, and using the conservation of the current and the Klein-Gordon equation $\nabla^2 W_m = -m^2 W_m$, we get

$$\int_0^\infty m^2 \rho(m^2)(W_m(x) - W_m(-x))dm^2 = 0. \tag{1.28}$$

Taking the Fourier transform, we get

$$p^2 \rho(-p^2) = 0. \tag{1.29}$$

Thus, $\rho(m^2) = c\delta(m^2)$, where c is a constant. The constant c cannot vanish, otherwise we will prove that $[J(x), \phi(0)]$ has a zero expectation value at the vacuum, which contradicts the assumption of symmetry breaking.

This argument shows that all contributions to the 2-point function $\langle \Omega, J(x)\phi(0)\Omega\rangle$ comes from intermediate states of zero mass. This implies that $L^2(\mathcal{O}_0^+)$ is contained in the discrete spectrum of $\mathcal{H}$ (i.e. as an honest subrepresentation of the Poincare group). Such a subrepresentation is interpreted in quantum field theory as a massless particle of zero spin. Thus, we have proved the following statement, which goes under the name of Goldstone's theorem:

Theorem. *In the Hilbert space of a realization of a field theory with continuous symmetry breaking, there is a massless scalar (i.e. a subrepresentation of the Poincare group isomorphic to $L^2(\mathcal{O}_0^+)$) which is created by the current of the broken symmetry.*

Remark. "Created" means that $J(x)\Omega$ is not orthogonal to the subrepresentation.

Definition. The massless scalar which we found is called the Goldstone boson corresponding to the broken symmetry G^s.

Remark 1. If the symmetry with respect to G^s was not broken, then $\lim_{f\to 1} Q_f\Omega$ would be zero. On the other hand, when symmetry breaking occurs, Q_f creates Goldstone bosons from the vacuum. Thus, Goldstone bosons "measure" the failure of symmetry.

Remark 2. The Goldstone boson can be created not only by the current operator of the symmetry, but also by other local operators. In fact, as we saw in the proof of Goldstone's theorem, it will be created by any scalar local operator whose 1-point function is not invariant under the symmetry.

Remark 3. There is no claim in Goldstone's theorem that the Goldstone boson is free, i.e. that it can be created by a free field $\phi(x)$. In fact, as we will see, this is often not the case.

Remark 4. If continuous symmetry breaking occurs classically, the Goldstone boson can already be seen in perturbation theory. As an example consider Lagrangian (1.13). Consider the classical vacuum state $\phi = a$. This vacuum state is degenerate. Therefore, if we introduce real variables $\phi_1 = \mathrm{Re}\phi - a$, $\phi_2 = \mathrm{Im}\phi$, and rewrite the Lagrangian in terms of these variable, then because of the degeneracy of the minimum the field ϕ_2 will be classically massless. Therefore, if we compute the 2-point function of ϕ_2 it will have a pole at $k^2 = 0$ (modulo the perturbation parameters). Of course, in principle loop terms might shift this pole. In other words, the classically massless ϕ_2 may get nonzero mass quantum-mechanically. What Goldstone theorem tells us is that this will not happen if symmetry is broken.

Corollary from Goldstone theorem. *Symmetry breaking does not happen in 2 dimensions.*

Proof. Otherwise, by Goldstone's theorem Goldstone bosons would have to exist. But in a 2-dimensional quantum field theory, there can be no massless particles created by a local operator. Indeed, the 2-point function of this operator in momentum space equals $w(k^2) = \int_0^\infty \frac{d\mu(m^2)}{k^2+m^2}$, where μ is the spectral measure. If there is a massless particle, this measure will have an atom at $m = 0$. But in this case the 2-point function $W(x)$ in position space will behave like $-C \ln x^2$ at infinity, i.e. would violate the positivity axiom. □

Now assume that we have a Lagrangian which has a Lie group G of symmetries. Assume that $\mathcal{H}$ is a realization of the quantum field theory defined by this Lagrangian, whose stabilizer is $H \subset G$. In this case one says that in the realization $\mathcal{H}$, the G-symmetry is spontaneously broken to H.

Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of G, H. Goldstone's theorem implies

Corollary. *$\mathcal{H}$ contains in its discrete spectrum a subrepresentation isomorphic to $L^2(\mathcal{O}_0^+) \otimes (\mathfrak{g}/\mathfrak{h})$.*

Proof. The proof is clear: if this is not so than there exists an element $Y \in \mathfrak{g}$, $Y \notin \mathfrak{h}$, such that $Q_f^Y \Omega \to 0$ (weakly) when $f \to 1$. This means that the symmetry with respect to Y is not broken – a contradiction.

The corollary means that Goldstone bosons corresponding to linearly independent broken infinitesimal symmetries are also linearly independent.

Example. Let $\phi : \mathbb{R}^d \to \mathbb{R}^N$ be a scalar field, and consider the Lagrangian

$$\mathcal{L}(\phi) = \int d^d x (\frac{1}{2}(\nabla\phi)^2 + \frac{g}{4!}(\phi^2 - a^2)^2). \tag{1.30}$$

This Lagrangian has an $SO(N)$-symmetry, and the space of its classical vacua is S^{N-1}. Therefore, classically the $SO(N)$-symmetry is broken to $SO(N-1)$. As we know, if $d > 2$, the same will happen quantum mechanically (for a weakly coupled theory), and so any realization (solution) of the theory has $N-1$ independent Goldstone bosons.

Let P be the classical minimum with coordinates $(0, 0, ..., 0, a)$. Consider the realization $\mathcal{H}_P$ of the theory, where the vacuum Ω is localized near P. Let $Y_i \in \mathfrak{so}_N$, $i = 1, ..., N-1$, be the infinitesimal rotations in the planes generated by basis vectors e_i, e_N of $\mathbb{R}^N$. The 1-point function of the operator ϕ_i is not invariant under Y_i, so ϕ_i creates the Goldstone boson corresponding to Y_i.

We can construct low energy (non-vacuum) states localized near other classical vacua than P. Indeed, let P' be another classical vacuum, and $Y \in \mathfrak{so}(N)$ is an element such that $e^Y P = P'$. Let J_Y be the current corresponding to Y, and Q_f^Y the corresponding cutoff charge. Then the state $e^{iQ_f^Y}\Omega$ is a low energy state localized near P'.

At long distances the theory will behave as a sigma-model into the space of classical vacua. This is an interesting statement for $N \geq 3$, because in this case the target (S^{N-1}) is not flat, so the sigma-model is not free. More precisely, at low energies (or long distances), the theory of bosons ϕ_i, $i = 1, ..., N-1$, will be free in the zero approximation, but in the first approximation it will not be free but will be described by the Lagrangian of the sigma-model into the sphere.

1.7. Infrared behavior of purely non-renormalizable field theories.

In this section we will clarify the meaning of the statement that for $d > 2$ a quantum field theory behaves in the infrared limit as a sigma-model into the space of classical vacua.

Suppose we have a purely nonrenormalizable field theory described by a Lagrangian $\mathcal{L}$. We will call a Lagrangian purely nonrenormalizable if all its couplings have negative dimension. An example of such a Lagrangian is the Lagrangian of a sigma-model for $d > 2$. Such Lagrangians are not good for perturbative renormalization in the UV limit, but create no problem in the IR limit, since all their interactions are IR irrelevant from the point of view of the Wilsonian renormalization group flow. Namely, if we introduce an UV momentum cutoff Λ (which is now not being sent to infinity), we can define correlation functions of $\mathcal{L}$ perturbatively: the correlation function is the sum of amplitudes of all Feynman diagrams, which are evaluated as usual, with integration carried out with cutoff $|q| < \Lambda$. Because $\mathcal{L}$ has no mass terms, there will be some IR divergences, but they can be dealt with in the same way as we dealt with UV divergences in Lectures 1-3 last semester. Moreover, since the theory is purely non-renormalizable, only finitely many graphs will be divergent for each number of external legs (like in a superrenormalizable theory in UV renormalization).

Now suppose that we want to compute the asymptotic expansion of the n-point function in momentum representation, around the point $k_i = 0$. We will have (for $\sum k_i = 0$):

$$G_n(k_1, ..., k_n) = k_1^{-2} ... k_n^{-2} G_n^0(k_1, ..., k_n), \tag{1.31}$$

where G_n^0 is a certain series, having a limit at $k_i = 0$ (In general, this will not be a power series; it may contain terms of the form $k^4 \ln k^2$).

The key property of this series, which follows from pure nonrenormalziability, is that modulo terms of any finite power, it is determined by finitely many Feynman graphs. Thus, we can obtain the IR asymptotics of the correlation functions to any order in k_i without having to sum the perturbation series.

Now suppose we have an actual quantum field theory, given by some renormalizable Lagrangian $\mathcal{L}'$. When we say that the theory defined by $\mathcal{L}'$ is described in the IR limit by a purely nonrenormalizable Lagrangian $\mathcal{L}$ (on the same fields), we mean that to a certain order in k_i (near $k_i = 0$), the functions G_n^0 given by (1.31) are the same for $\mathcal{L}$ as for $\mathcal{L}'$.

For instance, when at the end of the previous section we said that at low energies (or long distances), the theory of bosons ϕ_i, $i = 1, ..., N-1$, is free in the zero approximation, and described by the Lagrangian of the sigma-model into the sphere,

$$\mathcal{L}_\sigma = \int d^d x \left(\frac{1}{2} \sum (\nabla \phi_i)^2 + R(\sum \phi_i^2)(\sum (\nabla \phi_i)^2)\right)$$

in the first approximation (where R is proportional to the curvature), we meant that the functions $G_n^0(k_1, ..., k_n)$ for ϕ_i are the same as in the free theory modulo $o(1)$, and the same as in the sigma-model modulo $o(k^2)$.

Computing higher terms of the k-expansion, one can construct a purely non-renormalizable low energy effective theory, which will describe our theory in the infrared to any required accuracy.

Remark. The restriction of the function $G_0^n(k_1, ..., k_n)$ to the locus $k_i^2 = 0$ (but k_i is not necessarily zero) has a physical meaning: it is the scattering amplitude (S-matrix) of n Goldstone bosons. Thus, the statement is that scattering matrix of n Goldstone bosons in model (1.30) is like in the free theory for $N = 2$ (as the circle is flat), but has a quadratic correction due to curvature for $N > 2$.