LECTURE II-8, PART I: SOLITONS

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8.1. What is a soliton?

In classical mathematical physics, by a soliton one usually means a "traveling wave" solution of a nonlinear PDE $u_t = F(u, u_x, ...)$, i.e. a solution of the form $u(x,t) = f(x - vt)$. Solitons play a very important role in the theory of integrable systems, where any solution can be approximated by a superposition of solitons, moving at different velocities. As a result, the theory of integrable systems is sometimes called soliton theory.

Today we will be interested in solitons arising in field theory (as traveling wave solutions of the classical field equations) and primarily in the role they play in quantization of field theories. This is a different point of view from the one in soliton theory. In particular, no claim is made about nonlinear superposition of solitons, and the models we consider will not, in general, be exactly integrable.

We will consider solitons for Poincare invariant field theories on Minkowski space. By a soliton for a particular field theory we will mean a traveling wave solution of the field equations (i.e. a solution which depends on $x - vt$), which is localized in space and has finite energy. By Poincare invariance, we can always assume that $v = 0$, i.e. that the solution is time-independent. We will be mostly interested in solitons which provide the global minimum for the energy in the corresponding homotopy class.

Remark. It is important to distinguish solitons from instantons. Instantons are localized in Euclidean spacetime (i.e. only exist for an instant) and have finite action, while solitons are localized in space (of Minkowski spacetime), exist eternally, and have finite energy.

8.2. Solitons and components of the space of classical solutions

Classically, existence of solitons is related with existence of different components of the space of classical solutions of finite energy. In a connected component which does not contain a zero energy solution, the minimum of energy is often attained at a soliton. As an example of this, you may recall the situation discussed in Lecture II-1: a 2-dimensional scalar field theory with the potential $U = (\phi^2 - a^2)^2$. In this case, the space of t-independent classical solutions of finite energy has 4 components: $X_{++}, X_{-+}, X_{+-}, X_{--}$, where $X_{+-} = \{\phi : \phi(-\infty) = a, \phi(\infty) = -a\}$ etc. On two of these components, X_{+-} and X_{-+}, the energy is strictly positive, and its minimum is attained at 2 solitons, $\phi = f(x)$ and $\phi = -f(x)$, where $f(x)$ is the solution of the Newton equation $f'' = U'(f)$ for the potential $-U$, with boundary values a at $-\infty$ and $-a$ at ∞ (such a solution is defined uniquely up to translations).

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Notice that solitons are not invariant under the Poincare group. But one often encounters solitons which have rotational symmetry in space, around some "center of mass". In a scalar field theory, this would mean that the group of symmetry of such a soliton is $P_s = SO(d-1)\times\mathbb{R}$. Since the Poincare group is $P = SO(d-1,1)\ltimes\mathbb{R}^{d-1,1}$, the P-orbit $\mathcal{O}$ of the soliton in the space of solutions, i.e. the quotient P/P_s, is a $d-1$-dimensional vector bundle over the upper part of a 2-sheeted hyperboloid. It is easy to check the following.

1. This bundle is naturally isomorphic to the cotangent bundle.

2. The restriction of the symplectic structure on the space of solutions $\mathcal{O}$ is nondegenerate, and thus defines a symplectic structure on $\mathcal{O}$.

3 (normalization of symplectic form) Let m be the mass (i.e. the energy in the center of mass frame) of the soliton. Then there exists a P-equivariant symplectic diffeomorphism $\mathcal{O}\to T^*\mathcal{O}_m^+$, where $\mathcal{O}_m^+$ is the upper part of the hyperboloid $x^2 = -m^2$ in $\mathbb{R}^{d-1,1}$.

8.3. Solitons and quantization

Since classically solitons correspond to components in the space of solutions, quantum mechanically they should correspond to direct summands in the Hilbert space. As an example, consider a scalar field theory with the space X of classical solutions of finite energy, in which there is a component X_s containing a soliton $s\in X_s$. Then we have a symplectic embedding $T^*\mathcal{O}_m^+\to X_s$, where m is the mass of s. Let us assume that the minimum of energy at the image of this map is nondegenerate, in the sense that there is no other solutions of energy m, and the second derivative of the energy in a direction transversal to the image is positive.

In this case, in the weak coupling region we should expect that

1. The component $\mathcal{H}_s$ of the Hilbert space $\mathcal{H}$ of the quantum theory has no vacuum (because classically there is no P-invariant solution).

2. The Hamiltonian H on $\mathcal{H}_s$ satisfies the inequality $H\geq m'$, where m' is some positive mass parameter, such that $m'\to m$ in the weak coupling limit (in general, we should expect m to get quantum corrections). If the theory has a mass gap, we should expect that the states with $H=m$ form a space which is a quantization of $T^*\mathcal{O}_m^+$, i.e. the irreducible representation of P of the form $L^2(\mathcal{O}_m^+)$. We should also expect that the spectrum near m is discrete since the second derivative is positive.

Note that if the action (or energy) functional of the theory is multiplied by a constant C, the mass of the soliton is also multiplied by C, while masses of usual particles do not change. This means that in the classical approximation ($C\to\infty$), solitons are much heavier than usual particles. Therefore, a soliton cannot be seen in perturbation theory: the contribution to the correlation functions of an intermediate state containing a soliton is exponentially small (compared to the coupling constant) in the weak coupling limit.

8.4. Solitons in theories with fermions

In theories with fermions, the orbit of a soliton under P is often not the whole space of lowest energy states in the corresponding connected component of the space of solutions. For example, if the model is supersymmetric, it is clear apriori that the orbit is not the whole space of lowest energy solutions: the space of lowest energy states is the orbit of the superPoincare and not just the Poincare group.

Let us consider an example of such a situation. Consider the theory in 2 dimensions with a scalar and a pair of fermions:

$$\mathcal{L} = \frac{1}{2\lambda}\int d^2x(|d\phi|^2+(\phi^2-a^2)^2)+i\int d^2x(\psi_+\partial_-\psi_+ +\psi_-\partial_+\psi_- - g\phi\psi_+\psi_-). \tag{8.1}$$

Remark 1. For a suitable value of g this model is supersymmetric.

Remark 2. This model has a chiral symmetry $\phi \to -\phi$, $\psi_\pm \to \pm\psi_\pm$, which prohibits a mass term $m\psi_+\psi_-$ for the fermions.

Consider the soliton $\phi(x,t) = f(x)$ for the bosonic part of the theory. Fermionic extensions of this solution are functions of finite energy $\psi(x) = \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix}$ satisfying the Euler-Lagrange equations

$$\left[\partial_x \psi + \begin{pmatrix} 0 & -gf(x) \\ -gf(x) & 0 \end{pmatrix}\right] \psi = 0. \tag{8.2}$$

Thus,

$$\psi = \begin{pmatrix} \varepsilon \\ \varepsilon \end{pmatrix} e^{g \int_0^x f(y)dy} + \begin{pmatrix} \varepsilon' \\ -\varepsilon' \end{pmatrix} e^{-g \int_0^x f(y)dy}, \tag{8.3}$$

where $\varepsilon, \varepsilon'$ are odd variables. It is easy to see that only the solutions with $\varepsilon' = 0$ are in L^2, so the space of solutions of finite energy is 1-dimensional.

Thus, each pair of fermions ψ_+, ψ_- interacting with ϕ via (8.1) creates a fermionic degree of freedom in the space of configurations of minimal energy in the connected component of $f(x)$. Namely, if the number of such pairs is n then the space of minimal energy configurations is not $T^*\mathcal{O}_m^+$, but the supermanifold $T^*\mathcal{O}_m^+ \times \mathbb{R}^{0|n}$.

Therefore, if n is even, in quantum theory the space of lowest energy states in the corresponding component of the Hilbert space is $L^2(\mathcal{O}_m^+) \otimes S$, where $S = S_+ + S_-$ is the spin representation of $Spin(n)$.

8.5. Solitons in 2+1 and 3+1 dimensions.

In spite of the difference between instantons and solitons, there is a connection between them. Namely, often an instanton in a Euclidean field theory in $d-1$ dimensions gives rise to a soliton in the Minkowski version of the same theory in d dimensions. Consider for example the 2+1-dimensional $U(1)$ gauge theory with a complex scalar:

$$\mathcal{L} = \int d^3x \left(\frac{F_A^2}{e^2} + \frac{1}{\lambda}(|d_A\phi|^2 + (|\phi|^2 - a^2)^2)\right). \tag{8.4}$$

In Lecture II-6 we saw that the 2-dimensional version of this theory has instantons with Chern classes 1 and -1. In the 2+1-dimensional theory, these instantons become solitons, and govern the lowest energy modes in the corresponding components of the Hilbert space as described above.

Now consider nonabelian gauge theory in 3+1 dimensions. Namely, consider an $SO(3)$ gauge theory with a boson in the 3-dimensional representation, and the Lagrangian

$$\mathcal{L} = \int d^4x \left(\frac{F_A^2}{4e^2} + \frac{1}{2\lambda}(|d_A\phi|^2 + (|\phi|^2 - a^2)^2)\right). \tag{8.5}$$

We considered in Lecture II-7 the 3-dimensional version of this theory. For convenience, we identified the spacetime $\mathbb{R}^3$ with the Lie algebra of the gauge group and with the space of values of ϕ (the bracket in the Lie algebra is the cross-product).

This allows to write the scalar field ϕ and infinitesimal gauge transformations as vector fields on $\mathbb{R}^3$.

We found that in 3 dimensions this theory has an instanton in which ϕ is of the form $\phi = \frac{x}{r} f(r)$, $r = |x|$ (as we have explained, we identify the spacetime and the space of values of ϕ). In 4 dimensions this instanton will become a soliton. Such solitons are called magnetic monopoles.

In fact, we have not one soliton, but infinitely many, since the center of the soliton can be any point in $\mathbb{R}^3$. So the space of time-independent solitons is at least $\mathbb{R}^3$. In fact, this space is not $\mathbb{R}^3$ but $\mathbb{R}^3 \times S^1$. The reason is that there are some gauge symmetries compatible with spherical symmetry, which allow to produce new solitons out of old ones. Let us see how it happens.

First of all, recall that any field configuration (A, ϕ) of finite energy defines an integer topological invariant – "the first Chern class at infinity" c_1. Indeed, in order for the energy to be finite, we must have $|\phi| = a$ at infinity, so ϕ defines a map from a sphere at infinity in $\mathbb{R}^3$ to the sphere of radius a, and c_1 is the degree of this map. Another definition of c_1: the section ϕ defines at infinity a splitting of our 3-dimensional vector bundle into a direct sum $\phi \oplus \phi^\perp$ of a 1-dimensional and a 2-dimensional vector bundle. Thus, ϕ defines a reduction of the structure group to $SO(2) = U(1)$ at infinity. The number c_1 is the first Chern class of this bundle restricted to the infinite 2-sphere in $\mathbb{R}^3$.

For example, the soliton configuration discussed above has $c_1 = 1$.

In physical language, this topological phenomenon means that at spacial infinity the $SO(3)$ gauge symmetry is broken to $U(1)$. However, the remaining group $U(1)$ of transformations at infinity acts nontrivially on the space of classical solutions. In particular, it produces new solitons. To understand this action, let us consider the solitons (A, ϕ) with $c_1 = 1$, discussed above. Let us represent the infinitesimal operator of the group $U(1)$ by a spherically symmetric gauge symmetry: $\varepsilon = \frac{x}{r} g(r)$, where g is some function. If the function $g(r)$ satisfies $g(r) \sim c_0 r$ at $r \to 0$ and $g(+\infty) = c$, then this formula defines a smooth gauge transformation which is "constant" at infinity (with respect to the reduction of structure group defined by the soliton). Then the action of ε is

$$A \to A - d_A \varepsilon, \phi \to \phi. \tag{8.6}$$

It is clear that $d_A \varepsilon$ cannot be identically zero, since the soliton connection A is not flat. On the other hand, if $c \in 2\pi\mathbf{Z}$, the connection $A - d_A \varepsilon$ is equivalent to A by the gauge transformation $e^{i\varepsilon}$ which vanishes at infinity, so the solutions (A, ϕ) and $(A - d_A \varepsilon, \phi)$ are the same in this case.

This shows that the space of time-dependent solitons with Chern class 1 at infinity is at least $\mathbb{R}^3 \times S^1$ (with no canonical zero on either $\mathbb{R}^3$ or S^1). One can show that in fact it is exactly that. We will denote the circle coordinate on this space by α.

By using the Poincare group transformations, we can generate solutions which are time-dependent and propagate at a constant speed. We can also perform a time dependent gauge transformation, which will make the α-coordinate time dependent, i.e. $\alpha = \alpha_0 + st$, $s \in \mathbb{R}$. As a result, the space of time dependent solitons is the product $T^*\mathcal{O} \times T^*S^1$.

8.6. The 3+1-dimensional theory with the θ-angle

Consider the 3+1-dimensional theory of the previous section with the θ-angle, i.e. let us add to the (Minkowski) Lagrangian a term

$$-\frac{\theta}{16\pi^2}\int \mathrm{Tr}(F\wedge F). \tag{8.7}$$

To give this term a topological interpretation, let us compactify the time and consider the theory on the spacetime $\mathbb{R}^3\times S^1$. In this case, for any field configuration of finite energy, besides the first Chern class c_1 at infinity (which is also called the monopole number, or the hedgehog number), we can define another integer topological invariant – the second Chern class c_2, which is given by the integral $\frac{1}{8\pi}\int Tr(F\wedge F)$. Thus, classically, term (8.7) just counts the second Chern class of the bundle. Therefore, quantum mechanically, it weights the contribution from bundles with $c_2 = k$ to the path integral with $e^{ik\theta}$.

It is clear that this theory has the same time-independent solitons as the theory without θ, since the added term is topological. However, since time has been compactified, the time dependent solitons which were discussed in the previous section have to satisfy the equality $\alpha = \alpha_0 + \frac{n}{T}t$, where n is an integer, which we will call the winding number, and T is the circumference of the compactified time axis. In other words, the parameter s defined in the previous section has to have the form $s = n/T$.

Claim. *If the monopele number of a soliton configuration is k and the winding number is n then $c_2 = kn$.*

The proof is by a direct calculation.

Now consider 3 operators in our theory:

1. $\frac{\partial}{\partial\alpha}$. (This operator is the generator of the Lie algebra of the unbroken $U(1)$).
2. The electric charge

$$Q_{el} = \frac{1}{ae^2}\int_{\mathbb{R}^3} Tr(d_A\phi\wedge *F_A). \tag{8.8}$$

3. The magnetic charge

$$Q_{mag} = \frac{1}{4\pi a}\int_{\mathbb{R}^3} Tr(d_A\phi\wedge F_A). \tag{8.9}$$

In perturbation theory, $Q_{mag} = 0$ (as nonzero c_2 requires big action), and $Q_{el} = -i\frac{\partial}{\partial\alpha}$. Thus, in perturbation theory Q_{el} has integer eigenvalues.

Let us see, however, what happens nonperturbatively, i.e. when we take into account field configurations with nonzero c_2.

We compute in the gauge where $A_0 = 0$. It is easy to compute that classically we have $\frac{\partial}{\partial\alpha} = \int Tr(d_A\phi\frac{\delta}{\delta A})$. Therefore, the operator $-i\frac{\partial}{\partial\alpha}$ is the charge for the current $J = Tr(d_A\phi\wedge\pi_A)$, where π_A is the conjugate (momentum) variable for A.

Now compute π_A:

$$\pi_A = \frac{\delta\mathcal{L}}{\delta A_t} = \frac{A_t\wedge dt}{e^2} - \frac{\theta}{8\pi^2}F_A. \tag{8.10}$$

Thus, we get

$$-i\frac{\partial}{\partial\alpha} = Q_{el} - \frac{\theta}{2\pi}Q_{mag}. \tag{8.11}$$

Since the spectrum of $-i\frac{\partial}{\partial\alpha}$ is integer, we get $Q_{el} = \frac{\theta}{2\pi}Q_{mag}$ mod $\mathbb{Z}$. But Q_{mag} is also an integer, since it is a topological invariant classically. Thus, Q_{el} need not be an integer beyond perturbation theory. In other words, there exist states (of very high energy $\sim 1/\hbar$) on which Q_{el} is not an integer. So the electric charge is discretized, but in presence of magnetic monopoles we do not expect an integral electric charge.

It is not hard to show that operators Q_{el}, Q_{mag} commute. (Classically, it is obvious, as Q_{mag} is a locally constant function). Thus, the joint spectrum of them is a lattice in $\mathbb{R}^2$. If we consider the family of theories parametrized by $\theta \in S^1$, the monodromy transformation of this lattice around the circle is given by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in the basis Q_{mag}, Q_{el}.

In the second part of this lecture we will explain (in the free theory) how to extend this monodromy representation to an action of $SL_2(\mathbb{Z})$. I.e. how to construct a family of theories parametrized by a complex parameter $\tau = \frac{\theta}{2\pi} + ir$ modulo modular transformations, such that the monodromy representation is the standard action of $SL_2(\mathbb{Z})$ on a 2-dimensional lattice.