LECTURE II-2, PART I: INFRARED BEHAVIOR OF QUANTUM FIELD THEORIES

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Given a quantum field theory X, we want to solve it, that is, to learn the most interesting things about it. A big piece of "solving" a theory is determining what it flows to in the infrared. Fairly often, the answer is: "nothing," that is, X flows to a trivial theory. This happens precisely when X has a mass gap, for then all (Euclidean) correlation functions decay exponentially. Showing that a given theory flows to a trivial theory may, however, be a rather deep result.

Very often, the infrared limit is not trivial but is a free theory of massless particles, together with an irrelevant interaction which goes to zero in the infrared. In fact, this happens in most of the simplest examples that we will meet. Note that an irrelevant interaction would, in the ultraviolet, be considered "unrenormalizable"; the perturbations that are ill-behaved in the ultraviolet are just the ones that vanish as one flows to the infrared limit.

When a theory is free in the infrared, the question then becomes: *which* massless particles is it a free theory of? They might not be related to the ones in the original Lagrangian. In fact, as we shall see in the second part of the lecture, the answer to this question may depend on the vacuum state we are in.

For the infrared limit to be trivial is a special case of the infrared limit being free; it is the case that there are no massless particles at all in the physical spectrum.

We begin with an example, and then discuss several general features of infrared limits.

1. Example from last time

Consider a theory which breaks $SO(3)$ to $SO(2) = U(1)$. We have three real scalars which can be combined to a 3-component object $\vec{\phi}$ which transforms in the 3-dimensional representation of $SO(3)$. The Lagrangian is

$$\mathcal{L} = \frac{1}{\lambda} \int d^n x \left[\frac{1}{2} (\partial_\mu \vec{\phi})^2 + V(|\vec{\phi}|) \right],$$

where, letting $\rho = |\vec{\phi}|$, $V(\rho)$ is a potential which has a nondegenerate minimum away from the origin.

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The simplest such potential is $V(\rho) = \frac{1}{8}(\rho^2 - \rho_0^2)^2$, with a minimum at ρ_0. If we let $\Omega = \phi/\rho \in S^2$, we can rewrite the Lagrangian as

$$\mathcal{L} = \frac{1}{\lambda}\int d^n x \left[\frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}\rho^2(\partial_\mu \Omega)^2 + V(\rho)\right].$$

The term $\frac{1}{2}\rho^2(\partial\Omega)^2$ represents the round metric on S^2.

When λ is small, we can hope that the classical approximation will be good. Since ρ is a massive field, we can integrate it out of the theory by setting it equal to its minimum value ρ_0 and studying fluctuations

$$\rho = \rho_0 + w,$$

where w is now the quantum field which will appear in our path integrals. To first approximation, we would have an effective Lagrangian

$$\mathcal{L}_{\text{eff}}(\Omega) = \frac{\rho_0^2}{2\lambda}\int d^n x\, (\partial\Omega)^2,$$

describing a nonlinear sigma model of maps to G/H, which in the present case is S^2. The massless fields are described by Ω.

Can this be an answer? In other words, could *any* quantum field theory flow to this sigma model in the infrared? In fact, it *is* a possible answer for spacetime dimension $n > 2$, because the nonlinear sigma-model is non-renormalizable above two dimensions, so the interaction we get is irrelevant. That is, if a_i are Riemann normal coordinates on near a point $P \in S^2$, we can expand schematically near P

$$(\partial\Omega)^2 = (da)^2 + Ra^2(da)^2 + \ldots$$

with R being the Riemann tensor of S^2. From this we see that the interaction is irrelevant above two dimensions. In fact, to give $(da)^2$ dimension n, we must assign dimension $(n-2)/2$ to a, whence the interaction has dimension $2n-2$, which exceeds n for $n > 2$. The fact that the sigma model is a possible answer for $n > 2$ but not for $n \le 2$ is an aspect of the fact, already discussed last week, that spontaneous breaking of a continuous symmetry is possible for $n > 2$ but not for $n \le 2$.

Is the sigma model the *correct* answer for the infrared behavior of our particular problem, at least for sufficiently small λ? We claim that it is. To study this point, let us treat the effects of w perturbatively. The interactions of w with the Goldstone boson come from the interaction term in the Lagrangian:

$$\frac{1}{2\lambda}\rho^2(\partial\Omega)^2 = \frac{1}{2\lambda}(\rho_0 + w)^2(\partial\Omega)^2 = \frac{1}{2\lambda}(\rho_0^2 + 2\rho_0 w + w^2)(\partial\Omega)^2.$$

The operator $\partial\Omega$ is highly nonlinear, and can be thought of as emitting an arbitrary number of Goldstone bosons. We need to calculate Feynman diagrams involving w's, in order to find the effective Lagrangian. A typical diagram is

representing w with a solid line and a's with dotted lines. The w-propagator is $\frac{1}{k^2+m_w^2}$; since we are interested in small momenta k, we expand in powers of k. The leading term is

$$\text{const}\,(\partial\Omega)^2\frac{1}{m_w^2}(\partial\Omega)^2.$$

This has $SO(3)$ symmetry and is an irrelevant interaction. It is even more irrelevant than the terms, sketched above, that come by expanding $(\partial\Omega)^2$ in powers of a.

In fact, while it is instructive to study these diagrams, just to show that the sigma model is infrared-stable, we do not need the details of the diagrams. All we need to know is that the effective action has $SO(3)$ symmetry. So what possible terms could be generated in the effective action?

- A potential $V(\Omega)$ is not possible, because there is no $SO(3)$-invariant function on S^2. This is the basic reason that, given the $SO(3)$ symmetry, the sigma model is infrared stable. A potential function would change the picture completely. For instance, a generic potential would have an isolated, nondegenerate minimum, giving us a unique vacuum with an infrared-trivial massive theory, in contrast to a continuous family of vacua associated with spontaneously broken $SO(3)$.
- The only term with only two derivatives that respects all the symmetries of the problem is $(\partial\Omega)^2$ itself. So quantum corrections due to diagrams with w fields should definitely be expected to modify the coefficient of this term.
- Other possible terms like $((\partial\Omega^2)^2$ and $\partial\Omega\,\nabla^2(\partial\Omega)$ have more than two derivatives and are more and more irrelevant.

So the leading infrared behavior is determined by an effective action of the form $\frac{1}{f^2}(\partial\Omega)^2$ with

$$\frac{1}{f^2}=\frac{\rho_0^2}{2\lambda}+\text{ loop corrections.}$$

When this is expanded in Riemann normal coordinates about a given vacuum, that is a given point $P\in G/H$, one gets interactions (of which the first was sketched above) that involve the Riemann tensor of G/H and its covariant derivatives. If one uses these interactions to compute scattering amplitudes involving Goldstone bosons with small momenta of order k, the tree level amplitudes are all proportional to k^2 for k near zero, as the interaction terms all contain precisely two derivatives. Loop contributions are smaller for $k \to 0$, since the interactions are irrelevant in the infrared. To be more precise, loop amplitudes all either (i) renormalize the constant f in the $SO(3)$-invariant Lagrangian, or (ii) give corrections to the scattering amplitudes that vanish faster than k^2 for $k \to 0$.

Hence, the terms of order k^2 in the Goldstone boson scattering are all completely determined by the one constant f (or more generally by the choice of a G-invariant metric on the homogeneous space G/H). In the 1960's, it was discovered that the low energy scattering of pions beautifully fits such a description, with $G = SU(2) \times SU(2)$ and H a diagonal $SU(2)$. This is how it was discovered that the strong interactions have a spontaneously broken approximate chiral symmetry; the discovery played a very major role in the subsequent development of physics.

What happens if one wants to compute terms in the Goldstone boson scattering of higher order than k^2? It is clear that in order k^4, new constants will enter that can only be determined from microscopic calculations (or experiment), since there are G-invariant interactions with four derivatives (such as the $((\partial\Omega)^2)^2$ term found above from the explicit tree diagram considered). However, interestingly, in four spacetime dimensions, the lowest order correction to the k^2 amplitude for Goldstone bosons is not of order k^4 but of order $k^4 \ln k$. It comes from a loop diagram

with vertices drawn from the two-derivative part of the Lagrangian, and hence is uniquely detemined in terms of the same constant f that controls the k^2 terms in the scattering amplitudes. The analysis of low-energy Goldstone boson interactions via the ideas I have explained is known as "current algebra." In particular, via current algebra relations, one can deduce from experiment what is the broken symmetry group G, and many of the parameters in the G-invariant effective Lagrangian.

One final comment about symmetry-breaking examples such as this one: if we begin with a G-invariant microscopic Lagrangian $\mathcal{L}_{\rm micro}$ which we perturb to

$$\mathcal{L}_{\rm micro} + \varepsilon(\delta\mathcal{L})$$

with the term $\delta\mathcal{L}$ not being G-invariant, then in the infrared we will get

$$\frac{1}{f^2}(\partial\Omega)^2 + \varepsilon V(\Omega) + O(\varepsilon^2),$$

with $V(\Omega)$ being a non-G-invariant operator – of which the most relevant part is of course a potential with no derivatives, as suggested in the notation V. $V(\Omega)$ is highly constrained by the fact that it must transform under G the same way that $\delta\mathcal{L}$ does. For example, in the case of strong interactions, a small $\delta\mathcal{L}$ term, breaking $SU(2) \times SU(2)$ to a diagonal $SU(2)$, is actually present; it selects a unique vacuum from what would otherwise be a continuous family, and gives small masses to the pions. In current algebra studies of pions, one really takes the momentum k to be of order the pion mass.

2. Which spins?

Now we consider in a general way infrared-free theories in 4 dimensions. (The considerations that follow generalize above 4 dimensions but become trivial below dimension 4). The general discussion seems to suggest that infrared-free theories might have massless particles of any spin. But in practice, in all interesting examples I am familiar with, one can argue *a priori* that any massless particles will have spins 0, 1/2 or 1.

Most theories of interest can be formulated not just on flat $\mathbf{R}^4$, but on a more general curved 4-manifold M^4 with a general metric g. In fact, any theory that is part of the description of nature has this property, since general relativity is part of nature and in nature, space-time is curved! In quantum field theory, the ability to work on a curved space-time implies the existence of a very special operator, called the stress tensor or energy-momentum tensor $T_{\mu\nu}(x)$. It measures the response to a change in the metric tensor g. We suppose that a theory is formulated with a general g by a Lagrangian $\mathcal{L}(\phi_i; g)$, which is invariant under diffeomorphisms acting both on the ϕ_i and on g. g is not one of the fields of the theory – it is arbitrary but is held fixed in studying the classical or quantum dynamics of the ϕ_i – and this diffeomorphism invariance means that the theory, if formulated in a spacetime (M, g), really depends on g only up to diffeomorphism. In this setup, the stress tensor is defined as

$$T_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}.$$

This implies obviously that T is a symmetric tensor

$$T^{\mu\nu} = T^{\nu\mu}.$$

T can also be shown to obey

$$D_\mu T^{\mu\nu} = 0$$

by virtue of diffeomorphism invariance. If our theory is actually *conformally invariant*, then T is traceless, that is $g^{\mu\nu}T_{\mu\nu} = 0$.

Having such a stress tensor leads to powerful statements even if one specializes to the case that M is flat Euclidean space. For instance, last fall, when we axiomatized quantum field theory, we required Poincaré invariance, with conserved charges $Q(K)$ for every Killing vector field K in spacetime. The existence of a conserved, symmetric stress tensor is a local statement that leads to Poincaré invariance globally. Given any Killing vector field K, one uses the Killing vector equation (which reads $D_\mu K_\nu + D_\nu K_\mu = 0$) plus symmetry and conservation of T to prove that the current

$$J^\nu(K) = K_\mu T^{\mu\nu}$$

is conserved. Then one obtains the conserved charge

$$Q(K) = \int_{\Sigma^{n-1}} *J(K)$$

where the integral is taken over an initial value hypersurface (such as time zero).

I have presented this as if one needs to have a Lagrangian so as to deduce the existence of a stress tensor $T = \delta\mathcal{L}/\delta g$. Though this is a powerful approach, one can also argue more abstractly. Consider any theory which can be formulated for any metric on M. To define an operator $T(y)$, we must give a definition, for any specified n points $x_1, \ldots, x_n$ on M^4 distinct from each other and from y and operators $\mathcal{O}_1, \ldots, \mathcal{O}_n$, the correlation function $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) T(y) \rangle_g$ (here the subscript serves to emphasize that the correlation function depends on a metric g). We define this as the derivative of $\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)\rangle_g$ with respect to g:

$$\frac{\delta}{\delta g(y)} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_g = \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) T(y) \rangle_g.$$

The reader should verify that this definition agrees with the previous one in case a Lagrangian exists. T as defined in this way is obviously symmetric; it is conserved if the theory depends only on the diffeomorphism class of the metric g. Many of the properties of a local quantum field operator follow readily from this definition of T, and it is plausible to believe that they all do in general.

In any event,[1] one normally considers in practice theories that have a local, conserved, symmetric (and of course gauge-invariant) stress tensor. As I have essentially already noted, any theory that appears in nature has this property, since T appears directly in the Einstein equations!

Existence of T leads[2] to sharp restrictions on possible massless particles. The possible spins of a massless particle in a theory with a stress tensor are $0, 1/2$, and 1. A further and analogous restriction is the following. Let J^μ be a conserved current associated with a "global symmetry." Thus, J transforms as a vector under Poincaré, and the conserved charge

$$Q(J) = \int *J$$

is Poincaré invariant. Then Q annihilates any massless particle of spin 1.

The proofs are so similar that we consider the two cases together. In four dimensions, denote by $|p, j\rangle$, a massless one-particle state of momentum p and spin j (in general $j \in \mathbf{Z}/2$, and let $|p', j\rangle$ be a state of different momentum in the same Poincaré representation. Let $|p, j\rangle$ be an eigenstate of Q with eigenvalue q. Consider the matrix elements $\langle p', j | J^\nu | j, p\rangle$ and $\langle p', j | T^{\mu\nu} | j, p\rangle$. The latter cannot vanish at all, and the former cannot vanish unless $q = 0$. For in the limit that $p' \to p$, we have by Lorentz invariance

(1) $\langle p', j | J^\nu | j, p\rangle \sim p^\nu$, and
(2) $\langle p', j | T^{\mu\nu} | j, p\rangle \sim p^\mu p^\nu$

[1]Except in studying quantum gravity, which has a very different flavor from quantum field theory in a fixed spacetime, which is the subject of our lectures this spring.

[2]S. Weinberg and E. Witten, *Limits on massless particles*, Phys. Lett. B **96** (1980), 59–62.

The proportionality constant is q in the first case and 1 in the second, since (as Q and the momentum operators are obtained by integration of J or T) matrix elements of J or T with identical initial and final states measure the charge or momentum of the state.

On the other hand, one can prove using Lorentz invariance that for all $p' \neq p$, these matrix elements vanish in the first case for spin greater than 1/2, and in the second case for spin greater than 1. The proof goes by noting first for for $p' \neq p$, the subgroup of the Lorentz group that leaves fixed both p and p' is a copy of $SO(2)$ (or $SO(n-2)$ if we are in n spacetime dimensions; the present considerations degenerate below four dimensions as $SO(n-2)$ is then trivial). One simply shows that $SO(2)$ invariance of $\langle p', j|J^\nu|j,p\rangle \sim Qp^\nu$ and $\langle p', j|T^{\mu\nu}|j,p\rangle$, assuming that these matrix elements are nonzero, implies that the spin is in absolute value $\leq 1/2$ or ≤ 1, in the two cases. A convenient way to perform this computation is to go to a Lorentz frame in which (writing the time coordinate first), $p = (1,1,0,0)$ and $p' = (1,-1,0,0)$. The $SO(2)$ that leaves fixed both p and p' is the rotation of the last two coordinates. Under the generator of this $SO(2)$, the states $|j,p\rangle$ and $|j,p'\rangle$ have respectively eigenvalue j and $-j$. The minus sign for the $|j,p'\rangle$ state, which is crucial, arises because it describes a particle moving in the opposite direction from $|j,p\rangle$; they each have the same spin relative to their own directions of motion, but opposite spins if referred to a fixed axis. So $\langle p', j|J^\nu|j,p\rangle$ or $\langle p', j|T^{\mu\nu}|j,p\rangle$ can be nonzero only if, in the $SO(2)$ action on J or T, there is a term with spin or eigenvalue $-2j$. As the components of J transform under $SO(2)$ with spin ≤ 1 in absolute value, while for T one has components of spin ≤ 2, we get $|j| \leq 1/2$ or $|j| \leq 1$ in the two cases, as was claimed above.

We can actually be somewhat more precise about this result. We have so far used only representation theory, but in quantum field theory one has also a CPT theorem, which implies in four dimensions that every massless particle of spin j is accompanied by one of spin $-j$. So spins $\pm 1/2$ will go together, and likewise spins ± 1.

In general dimension n, similar reasoning gives the following result. The spin of a massless particle is classified by a representation of the "little group" $SO(n-2)$. If a stress tensor exists, the allowed representations for massless particles are the spinor representation(s), and exterior powers of the fundamental $n-2$-dimensional representation (including the trivial representation).[3] Global charges vanish except for massless particles transforming in the trivial or spinor representation. This n-dimensional formulation is related to the statement in four dimensions as follows: $j = \pm 1/2$ correspond to the two spinor representations, while $j = 0$ and $j = \pm 1$ come from the exterior powers of the fundamental representation.

3. Why are particles massless?

If the couplings in a theory are generic, massless particles must be massless for a reason. One possible reason is supersymmetry, but we won't discuss that now. Other possible reasons are as follows.

[3] For $n-2$ divisible by four, the middle exterior power can be decomposed into self-dual or anti-self-dual pieces which are each *real*; one can be present without the other.

Spin 0 particles are massless when they are Goldstone bosons, that is, when there is a broken symmetry. Spin 1/2 particles are massless when they are chiral fermions; their masslessness is due to an *unbroken* chiral symmetry. This means simply the following: the CPT theorem says that if one has n massless particles of spin 1/2, one also has n such particles of spin $-1/2$. If there is an unbroken symmetry group G and the massless particles of spin 1/2 transform in a representation R of G, then the massless particles of spin $-1/2$ transform, by the CPT theorem, in the representation $\overline{R}$ (the complex conjugate of R). If R and $\overline{R}$ are distinct. this spectrum cannot be perturbed in a G-invariant way to a give masses to the fermions.

The reasons just mentioned are really the only known reasons to have massless particles of spin 0 or 1/2 without supersymmetry and without adjusting some parameters to make particles massless. For spin 1 the situation is somewhat different. The Poincaré representation of a massless spin 1 particle, in four (or more) dimensions, simply cannot be perturbed to give a mass to the particle, unless there is a massless spin 0 particle that can combine with it in a Higgs mechanism as will be discussed in the second half of lecture. If our massless spin 0 particles are Goldstone bosons, then the broken symmetries that shift them ensure that they cannot participate and disappear from the massless spectrum in a Higgs mechanism. So as long as the massless spin 0 particles are Goldstone bosons, and *as long as the theory is infrared-free*, the existence of massless spin 1 particles is stable just from Poincaré symmetry.

This depends heavily on the theory being infrared-free since we applied group theory to one-particle states. If interactions are important even at low energies, the states with different numbers of particles can "mix," and we cannot draw a conclusion just by applying Poincaré invariance to the one-particle states.

What interactions can massless particles of these types have? In the case of spin 0 particles, which we assume to be Goldstone bosons, there are no relevant interactions. We have already seen at the beginning of this lecture that there are no relevant or marginal interactions of Goldstone bosons only. There are likewise no relevant or marginal interactions of fermions only above two dimensions – we explored such questions in the fall term – and with similar arguments and a little more care, one can show that there are no relevant or marginal couplings of Goldstone bosons to fermions.

Spin 1 particles are of a different nature, since they correspond to gauge fields, and gauge fields *can* have relevant interactions in the infrared. If G is the gauge group and A is the connection, the Lagrangian

$$\int \frac{1}{\mathrm{g}^2}\,|F_A|^2 d^4x$$

is nonlinear. It contains couplings that in four dimensions are marginal classically. Whether the interactions are relevant or irrelevant quantum mechanically depends on the behavior of the β-function. An irrelevant nonlinearity in the infrared will allow the gauge theory to function as an effective description. On the other hand, if the nonlinearity is relevant in the infrared, then the gauge theory is *not* the answer. (In the intermediate case when $\beta = 0$ we would get a non-free theory in the infrared.)

Goldstone bosons are always invariant under gauge symmetries – a gauge group acting on them would violate the global symmetry that leads to having Goldstone bosons in the first place. So if we see massless spin 1 fields in the infrared without adjusting parameters to make it so, we should expect that either G must be abelian, or else there must be enough fermions in large enough representations of G so that $\beta > 0$. To explain their masslessness, the fermions are chiral (that is, the states of $j = 1/2$ and $j = -1/2$ transform differently) either under the gauge group itself, or under some unbroken global symmetry group that commutes with the gauge group.