LECTURE II-11: SUPERSYMMETRIC FIELD THEORIES

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11.1. General remarks on supersymmetry

Starting from today we study field theories with supersymmetry, i.e. theories whose symmetry group has a nontrivial extension to a supergroup. As usual, by even and odd (infinitesimal) symmetries we mean even, respectively odd, elements of the Lie superalgebra of this supergroup.

It often happens that a solution to the classical field equations has nontrivial odd symmetries. Examples:

gradient flowlines in Morse theory
holomorphic curves
instantons
monopoles
Seiberg-Witten solutions
hyperKahler structures
Calabi-Yau metrics
Metrics of G_2 and $Spin_7$ holonomy

In the next sections we will consider some of these examples.

11.2. Supersymmetric solitons (BPS states).

First consider a case in which the gradient flowlines of Morse theory will appear. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a Morse function (i.e. its critical points are nondegenerate and $|\nabla h|$ grows at infinity). A Morse function always has finitely many critical points.

Consider (in Minkowski signature) the theory of maps $\Phi : \mathbb{R}^{2|2} \to \mathbb{R}^n$ with the Lagrangian

$$\mathcal{L} = \int d^2x d^2\theta (\frac{1}{2} D_+\Phi D_-\Phi - h(\Phi)), \tag{11.1}$$

where $D_\pm = \frac{\partial}{\partial\theta_\pm} - \theta_\pm\partial_\pm$, $\partial_\pm = \frac{\partial}{\partial x_\pm}$, $x_\pm = \frac{1}{2}(t \pm x)$. This model has an obvious supersymmetry under $Q_\pm = \frac{\partial}{\partial\theta_\pm} + \theta_\pm\partial_\pm$. These supersymmetry operators satisfy the obvious commutation relations $Q_\pm^2 = \partial_\pm$, $\{Q_+, Q_-\} = 0$.

We have $(Q_+ \pm Q_-)^2 = 2\frac{\partial}{\partial t}$. So if we look for classical solutions with time translational symmetry (i.e. for solitons), we may in particular look for those of them which are invariant under one of the supersymmetry, say $Q_+ \pm Q_-$.

We have

$$Q_+ \pm Q_- = (\frac{\partial}{\partial\theta_+} \pm \frac{\partial}{\partial\theta_-}) + (\theta_+ \pm \theta_-)\frac{\partial}{\partial t} + (\theta_+ \mp \theta_-)\frac{\partial}{\partial x}. \tag{11.2}$$

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Thus, the supersymmetry condition for time-independent solutions is

$$[(\frac{\partial}{\partial\theta_+} \pm \frac{\partial}{\partial\theta_-}) + (\theta_+ \mp \theta_-)\frac{d}{dx}]\Phi = 0. \tag{11.3}$$

Let us look for even solutions. It is easy to show that such solutions are of the form $\Phi = \phi + \theta_+\theta_-\nabla h(\phi)$. For them, the supersymmetry condition is

$$\frac{d\phi}{dx} \mp \nabla h(\phi) = 0, \tag{11.4}$$

which is the condition for the gradient flowline. Thus, supersymmetric solitons are the flowlines of the gradient flow.

Notice that these 1-st order equations imply the 2-nd order equations of motion. Indeed, it is easy to show that the equations of motion are

$$\partial_+\partial_-\phi + \nabla(\nabla h)^2(\phi) = 0, \tag{11.5}$$

or for time-independent solutions

$$\frac{d^2\phi}{dx^2} = \nabla(\nabla h)^2(\phi), \tag{11.6}$$

which can be obtained by differentiation of (11.4) with respect to x.

Another way to see this: the Lagrangian for time-independent fields (i.e the Hamiltonian) is

$$H(\phi) = \int dx(\frac{1}{2}(\frac{d\phi}{dx})^2 + \frac{1}{2}(\nabla h(\phi))^2). \tag{11.7}$$

Rewriting H, we get

$$H(\phi) = \frac{1}{2}\int dx(\frac{d\phi}{dx} \mp \nabla(\phi))^2 \pm \int_{-\infty}^{\infty} dh(s). \tag{11.8}$$

Since the last term is locally constant on the space of fields of finite energy, a supersymmetric solution provides the global minimum for the energy in each connected component of the space of fields. The value of energy at this minimum is $S = |\int dh|$.

Definition. *Supersymmetric solitons, that is classical solutions invariant under some supersymmetries, are called classical BPS states.*

11.3. The role of BPS states in quantum theory.

We have mentioned above and used the fact that the vector fields Q_+, Q_- commute. Since the space of solutions of the classical field equations is a symplectic supermanifold, these vector fields must be (at least locally) generated by some Hamiltonian functions $\tilde{Q}_+, \tilde{Q}_-$ (defined up to adding a locally constant function). But these functions need not Poisson commute: their Poisson bracket has to be a locally constant function, not necessarily equal to zero.

In our case, the functions $\tilde{Q}_+, \tilde{Q}_-$ are easy to write down: if $\Phi = \phi + \theta_+\psi_+ + \theta_-\psi_- + \theta_+\theta_- F$, then

$$\tilde{Q}_+ = \int dx(\psi_+\partial_+\phi + \psi_-\nabla h(\phi)), \tilde{Q}_- = \int dx(\psi_-\partial_-\phi - \psi_+\nabla h(\phi)). \tag{11.9}$$

The computation of the Poisson bracket gives $\{\tilde{Q}_+, \tilde{Q}_-\} = -2S$, $S = \int dh = h(\phi(\infty)) - h(\phi(-\infty))$.

From this picture it is clear what will happen with operators Q_+, Q_- in quantum theory. We will have $Q_+^2 = P_+, Q_-^2 = P_-$ (where P_+, P_- are the corresponding momentum operators), and $(Q_+ \pm Q_-)^2 = 2(H \mp \hat{S})$, where $H = (P_+ + P_-)/2$ is the Hamiltonian, and $\hat{S}$ commutes with local operators.

In quantum theory, to every connected component X_a of the space X of fields of finite energy there corresponds a summand $\mathcal{H}_a$ of the Hilbert space. We expect that, if there is no breaking of supersymmetry, to every supersymmetric soliton $\Phi \in X_a$ there corresponds a state Ψ in $\mathcal{H}_a$ which is also supersymmetric: $(Q_+ \pm Q_-)\Psi = 0$. Then $(H \mp S_a)\Psi = 0$, where $S_a = \hat{S}|_{\mathcal{H}_a}$ is a scalar. Therefore, since $H \geq 0$, we have $H\Psi = |S_a|\Psi$. In particular, for every connected component of X there is only one supersymmetry (out of the two), for which there can be supersymmetric states in this component.

In general, on $\mathcal{H}_a$ we have $H \geq |S_a|$.

Now we want to determine whether there is a supersymmetric quantum state, that is a quantum state annihilated by $Q_+ \pm Q_-$, corresponding to the supersymmetric classical state. In the lowest order of perturbation theory one finds no bosonic zero mode except the translations and no fermionic zero mode. Hence, in that approximation the lowest energy state of given momentum is unique and nondegenerate. Also, the theory in the vacuum sector has a mass gap (classically and therefore for weak enough coupling) so the unique ground state that is found in the leading approximation is isolated from any continuum. Hence the quantum theory for weak enough coupling has an isolated and unique ground state in this sector of the Hilbert space, and it follows from the supersymmetry algebra that this state must be annihilated by $Q_+ \pm Q_-$. This massive state is called a quantum BPS state.

However, if (in a family of theories) two critical points of h collide and become a degenerate critical point, the mass of BPS paths between them goes to zero, so we can expect that the corresponding sector of Hilbert space loses its mass gap, and massless particles appear. At such a point, supersymmetric states can appear or disappear in the quantum theory. We will make this more explicit later in the context of $N = 2$ supersymmetry.

11.4. N=2 supersymmety in 2 dimensions.

Consider the space $\mathbb{R}^{2|4}$ with coordinates $x_+, x_-, \theta_+, \theta_-, \bar{\theta}_+, \bar{\theta}_-$. This space admits an action of the N=2 supersymmetry algebra, with supersymmetry generators

$$\begin{aligned} Q_+ &= \frac{\partial}{\partial\theta_+} + \bar{\theta}_+\frac{\partial}{\partial x_+}, \bar{Q}_+ = \frac{\partial}{\partial\bar{\theta}_+} + \theta_+\frac{\partial}{\partial x_+}, \\ Q_- &= \frac{\partial}{\partial\theta_-} + \bar{\theta}_-\frac{\partial}{\partial x_-}, \bar{Q}_- = \frac{\partial}{\partial\bar{\theta}_-} + \theta_-\frac{\partial}{\partial x_-}, \end{aligned} \tag{11.10}$$

Let us also introduce vector fields

$$
\begin{aligned}
D_+ &= \frac{\partial}{\partial\theta_+} - \bar\theta_+ \frac{\partial}{\partial x_+}, \bar D_+ = \frac{\partial}{\partial\bar\theta_+} - \theta_+ \frac{\partial}{\partial x_+}, \\
D_- &= \frac{\partial}{\partial\theta_-} - \bar\theta_- \frac{\partial}{\partial x_-}, \bar D_- = \frac{\partial}{\partial\bar\theta_-} - \theta_- \frac{\partial}{\partial x_-},
\end{aligned} \tag{11.11}
$$

which commute with the supersymmetry generators. Therefore, any Lagrangian which is written in terms of the D's is supersymmetric.

Recall that a chiral function (or superfield) on $\mathbb{R}^{2|4}$ is a function satisfying the equations $\bar D_+\Phi = \bar D_-\Phi = 0$. A general solution to these equations has the form

$$
\begin{aligned}
\Phi = \phi - \theta_+\bar\theta_+\partial_+\phi - -\theta_-\bar\theta_-\partial_-\phi + \theta_+\bar\theta_+\theta_-\bar\theta_-\partial_+\partial_-\phi + \theta_+\theta_- F + \\
\theta_+\psi_+ + \theta_-\psi_- - \theta_+\bar\theta_+\theta_-\partial_+\psi_- - \theta_-\bar\theta_-\theta_+\partial_-\psi_+.
\end{aligned} \tag{11.12}
$$

You can read more about chiral functions in the superhomework.

Consider the theory of chiral maps Φ of $\mathbb{R}^{2|4}$ into $\mathbb{C}^n$, with the Lagrangian

$$
\frac{1}{2}\int d^2x d^4\theta \Phi\bar\Phi + \int d^2x d\theta_+ d\theta_- W(\Phi) + \int d^2x d\bar\theta_+ d\bar\theta_- \bar W(\Phi), \tag{11.13}
$$

where W is a holomorphic function on $\mathbb{C}^n$, called the superpotential. This Lagrangian is N=2 supersymmetric. In components (for $x_+ = z, x_- = \bar z$), it looks like

$$
\frac{1}{2}\int d^2x(|d\phi|^2 - |F|^2 + W'(\phi)F + \bar W'(\phi)\bar F + \text{terms with fermions}). \tag{11.14}
$$

Setting all fermions to zero and the "dummy" field F to the stationary point $F = \bar W'(\phi)$, we get the bosonic energy functional

$$
H = \frac{1}{2}\int d^2x(|d\phi|^2 + |W'(\phi)|^2). \tag{11.15}
$$

It is easy to see that this functional coincides with (11.7), for the function $h = \mathrm{Re}(e^{-i\alpha}W)$, where α is any real number. Thus, at the classical level we are doing a special case of the previous problem. However, now we have more supersymmetry and therefore a more interesting theory.

11.5. N=2 BPS states

In the theory we are considering, there is an important symmetry called the R-symmetry. It acts according to $\theta_+ \to e^{i\beta}\theta_+, \theta_- \to e^{-i\beta}\theta_-$. If we require that Φ is unchanged under this symmetry (i.e. Bose fields are unchanged and $\psi_+ \to e^{-i\beta}\psi_+, \psi_- \to e^{i\beta}\psi_-$), then Lagrangian (11.13) is obviously invariant under the symmetry.

The commutation relations for the supersymmetry Hamiltonians are

$$
\begin{gathered}
\{Q_\pm, \bar Q_\pm\} = 2P_\pm, \\
\{Q_+, \bar Q_-\} = \{\bar Q_+, Q_-\} = 0 \text{ (by R-symmetry) }, \\
\{Q_+, Q_-\} = T,\ \{\bar Q_+, \bar Q_-\} = \bar T,
\end{gathered} \tag{11.16}
$$

where T is analogous to S in Section 11.2 – it is a locally constant function on the space of classical solutions (which is, unlike S, not necessarily real), and for brevity we drop twiddles over Q's. Also, the squares of all the Q's are zero.

In fact, the function T is easy to compute, like the function S in the previous problem. Namely,

$$T = W(\phi(\infty)) - W(\phi(-\infty)). \tag{11.17}$$

Now we will look at supersymmetric states. Choose a real number α and look for states which are invariant under two supersymmetries $Q_1(\alpha) == Q_+ + e^{i\alpha}\bar{Q}_-$, $Q_2(\alpha) = \bar{Q}_+ - e^{-i\alpha}Q_-$.

We have

$$\{Q_1, Q_2\} = 2i(H - \mathrm{Re}(e^{-i\alpha}T)), \tag{11.18}$$

which implies that supersymmetric classical states have to be time-independent.

The equation $Q_1\Phi = 0$ for an even function Φ gives (in the time-independent case):

$$\frac{\partial\phi}{\partial x} = e^{-i\alpha}\overline{W'(\phi)}, \tag{11.19}$$

and the second equation gives the same result. This implies, in particular, that

$$\bar{T} = \bar{W}(\phi(\infty)) - \bar{W}(\phi(-\infty)) = \int \bar{W}'(\phi)\frac{d\phi}{dx}dx = e^{-i\alpha}\int_{\infty}^{\infty} |W'(\phi)|^2 dx, \tag{11.20}$$

which implies that $\alpha = \arg T$. Thus, from equation (11.28) we get that for a supersymmetric solution of the classical equations we have

$$H = Re(e^{-i\alpha}T) = |T|. \tag{11.21}$$

For other states in the connected component of this solution we have $H \geq |T|$.

Now, what are the supersymmetric solutions (of finite energy) geometrically? It is clear from equation (11.19) that they are separatrices between critical points of W for the gradient flow of $h = \mathrm{Re}(e^{-i\alpha}W)$.

Now let us turn to quantum theory. From classical considerations we saw that in our theory $H \geq |T|$, and in a nondegenerate case $H = |T|$ only for BPS states. Therefore, we should hope that in quantum theory the same situation takes place, apriori with a corrected value of T.

Consider the generic situation when all zero-modes of the Hamiltonian near a classical BPS state arise from the superPoincare group. This is the case if for any 3 critical points a, b, c of the potential we have $|T_{ac}| < |T_{ab}| + |T_{bc}|$, where T_{ab} is the value of T on the component of the space of solutions which go from a to b. In other words, this is the case when the gradient flowline between a, b never passes through c.

In this case, in quantum theory, for small values of the coupling, we expect that the point $|T|$ in the spectrum of H occurs discretely. From our classical computations, we expect that the eigenspace corresponding to this eigenvalue is finite-dimensional, and is an irreducible representation of the odd part of the superPoincare algebra, in which Q_1, Q_2 act by zero. This representation is nothing but the space of sections of the equivariant vector bundle on the upper part of the hyperboloid, where the fiber is the standard 2-dimensional irreducible representation of the 4-dimensional Clifford algebra generated by the two remaining supersymmetries.

Notice that all other superPoincare representations occuring in this theory have to be not 2-dimensional but rather 4-dimensional over the ring of functions on the hyperboloid, since for levels of energy above T the Clifford algebra satisfied by the supersymmetry operators corresponds to a nondegenerate quadratic form, and the only irreducible represenation of this algebra is 4-dimensional.

11.6. N=1 Supersymmetry in 4 dimensions.

Now consider supersymmetry in 4 dimensions. We start with $N = 1$ supersymmetry. In this case the odd part of the supersymmetry algebra is similar to the $N = 2$ case in two dimensions. It is generated by $Q_\pm$, $\bar{Q}_\pm$, with relations

$$\begin{aligned} \{Q_\alpha, \bar{Q}_\beta\} &= 2P_{\alpha\beta}, \\ \{Q_\alpha, Q_\beta\} &= 0, \\ \{\bar{Q}_\alpha, \bar{Q}_\beta\} &= 0. \end{aligned} \tag{11.22}$$

where $\alpha, \beta \in \{+, -\}$, and $P_{\alpha\beta}$ is a basis of the space of complex linear functions on the spacetime (These relations exhibit the isomorphism of Poincare representations $S_+ \otimes S_- \to V_{\mathbb{C}}$, where $S_\pm$ are the spinor representations.) A central extension like in (11.16) cannot arise here because it is prohibited by the Poincare symmetry (this central extension would have to be in the representation $S^2\mathbb{C}^2$ of $SU(2)$ (where $\mathbb{C}^2$ is the standard representation), which is the spin 1 representation and contains no invariants).

The determinant of the quadratic form corresponding to the Clifford algebra (11.22) is $(P^2)^2$. The rank of this form if $P^2 = 0$ is 2. Therefore, massless representations of the superPoincare correspond to 2-dimensional representations of the Clifford algebra, and massive representations correspond to 4-dimensional representations of the Clifford algebra.

More precisely, consider a representation W of the superPoincare and the subspace W_p on which $P = p$, where $p \in \mathbb{R}^{1,3}$. Let G_p be the stabilizer of p in the group of rotations, H_p the maximal torus in G_p (always isomorphic to $U(1)$). With respect to H_p, W has a decomposition in a direct sum of representations of integer and half-integer spins. These spins are called helicities of W, and each helicity has a multiplicity.

It is easy to see that the supersymmetry operators $Q_\alpha, \bar{Q}_\alpha$ raise helicity by 1/2 (since they live in the spinor representation of the Poincare). Thus, the massless representations have helicities $j, j + 1/2$ with multiplicity 1, and the massive representations have helicities $j, j + 1/2, j + 1$ with multiplicities $1, 2, 1$.

Now let us consider massless particle multiplets which are allowed by $N = 1$ supersymmetry. There are two such basic multiplets.

1. Vector multiplet: a gauge field A and a chiral spinor λ in the adjoint representation. This is the N=1 analogue of the gauge field. In particular, in the $U(1)$ case the theory is free. In the infrared, it generates a massless vector and a massless spinor, so the helicities are $-1, 1$ (for vector) and $-1/2, 1/2$ (for spinor). In

particular, the massless representation of the SuperPoincare arising in this theory is reducible, and splits into two: the one with helicities $-1, -1/2$ and the one with helicities $1/2, 1$. However, over $\mathbb{R}$ this splitting does not exist and our representation is irreducible. A more physical version of this statement is to say that the helicity $-1, -1/2$ states are related to helicity $1/2, 1$ states by CPT conjugation, so that one pair must be present if the other is. Thus, this field configuration is the minimal supersymmetric one which contains a gauge boson.

2. Chiral multiplet: A massless complex scalar ϕ and a massless chiral spinor χ. In this case the story is analogous. The obtained massless representation has a spinor and a scalar, so it has helicities $0, 0$ (for scalar) and $-1/2, 1/2$ (for spinor). Thus this representation is again a sum of two, with helicities $-1/2, 0$ and $0, 1/2$. This decomposition is only valid over $\mathbb{C}$ and not over $\mathbb{R}$; over $\mathbb{R}$, the representation is irreducible, so this combination is the smallest supersymmetric one containing a scalar.

The massive versions of these multiplets are as follows.

1. Massive vector multiplet (or hypermultiplet). The minimal real supersymmetric representation containing a massive vector has helicities $-1, -1/2, 0, 1/2, 1$ with multiplicities $1, 2, 2, 2, 1$. This involves a massive vector, two massive spinors (chiral and antichiral), and a real massive scalar. The corresponding 8-dimensional representation of the Clifford algebra is (over $\mathbb{C}$) a sum of two 4-dimensional representations.

2. Massive chiral multiplet. Same as massless chiral multiplet (we could consider the same fields with masses, such that the mass of bosons equals the mass of fermions).

11.7. N=2 Supersymmetry in 4 dimensions.

In the N=2 case, we have two copies of operators Q: $Q^{(1)}$ and $Q^{(2)}$, and they commute in the same way as before if the indices are equal, and give zero commutator if they are not equal (as vector fields). However, now there is a possibility for a central extension: it is no longer prohibited by the Poincare symmetry.

Consider first the case when the central term is zero. In this case the quadratic form of the Clifford algebra is nondegenerate (in 8 dimensions) in the massive case, and has rank 4 in the massless case. Thus, an irreducible massive representation should be 16-dimensional, and an irreducible massless representation should be 4-dimensional. The helicities for representations are obained like in $N = 1$ case. In particular, in a massive representation the helicities are in groups of the form (j,j+1/2,j+1,j+3/2,j+2) with multiplicities (1,4,6,4,1), and in a massless representation they are in groups (j,j+1/2,j+1) with multiplicities (1,2,1).

Now let us consider the simplest $N = 2$ supersymmetric theory. It is obtained by combining fields from a (classically) massless vector multiplet and a massless chiral multiplet in adjoint representation. Thus the fields are: from vector multiplet – (A, λ), from chiral multiplet – (ϕ, χ). The Lagrangian is the minimal $N = 2$ supersymmetric Lagrangian on these fields. Such a Lagrangian exists and is uniquely determined by the minimality condition. In the $U(1)$ case the theory is free, but in the nonabelian case there are nontrivial interactions.

What does this theory do in the infrared? In the $U(1)$ case, the answer is simple: we should add together the representations for the vector and chiral multiplets. We get helicities (-1,-1/2,0,1/2,1) with multiplicities (1,2,2,2,1). This is the sum of two complex conjugate massless representations of the N=2 Clifford algebra: the one with helicities (0,1/2,1) and multiplicities (1,2,1) and the one with helicities

(-1,-1/2,0) and multiplicities (1,2,1).

But now let us consider the nonabelian case (say the gauge group G is simple). Then the bosonic fields are the gauge field A and a scalar ϕ in the complexified adjoint representation $\mathfrak{g}_{\mathbb{C}}$. The bosonic part of the Lagrangian is

$$L_{bosonic} = \int (\frac{1}{4e^2}F^2 + |d_A\phi|^2 + |[\phi, \bar{\phi}]|^2). \tag{11.23}$$

Thus the bosonic part of the space of classical vacua is the set of solutions of the equations $[\phi, \bar{\phi}] = 0$ modulo the action of G. This space can be identified with $\mathfrak{t}_{\mathbb{C}}/W$, where $\mathfrak{t}$ is the maximal commutative subalgebra in $\mathfrak{g}$ and W the Weyl group. In the case $G = SU(2)$, this quotient is identified with $\mathbb{C}$ by introducing the global coordinate $u = Tr(\phi^2)$ (the u-plane).

Recall from Lecture 2 that in this situation we have gauge symmetry breaking from G to the centralizer H of ϕ. In particular, for $SU(2)$ near $u \neq 0$ the gauge symmetry is broken classically from $SU(2)$ to $U(1)$ (Higgs mechanism). The components of the gauge field which are charged nontrivially with respect to the surviving $U(1)$ will become massive. By N=2 supersymmetry, the same will happen for the corresponding components of λ, ϕ, ψ. Thus, in the charge 2 and -2 sectors of the Hilbert space the lowest energy states will be in a massive representation with helicities as before: (-1,-1/2,0,1/2,1) with multiplicities (1,2,2,2,1). However, we know that there is no such representation without central extension. Thus, without central extension we get a contradiction, and hence the central extension must appear.

The central extension will show up in the commutation relations between $Q_\alpha^{(1)}$ and $Q_\beta^{(2)}$:

$$\{Q_\alpha^{(1)}, Q_\beta^{(2)}\} = \varepsilon_{\alpha\beta}Y, \tag{11.24}$$

where Y is an operator which commutes with all local operators (central charge). Before the central charge, the algebra had a $U(2)$ R-symmetry (action on indices 1 and 2), but the central charge breaks this symmetry down to $SU(2)$ (the chiral $U(1)$ symmetry is anomalous in our theory, because of the index problems, and it is broken to $\mathbb{Z}/2\mathbb{Z}$).

It is easy to see that in this case the massive particles considered above must have mass exactly $|Y|$ (this is where the quadratic form of the Clifford algebra becomes degenerate).

It turns out that classically there are BPS states which correspond to these massive particles. Namely if we are at the vacuum $u \in \mathbb{C}$, we have

$$\phi \sim \frac{1}{\sqrt{2}} \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}. \tag{11.25}$$

where $a = \pm u^{1/2}$. Let $a = \rho e^{i\alpha}$. We should look for BPS states (i.e. states which are time-independent and invariant under half of the supersymmetry) which satisfy the condition (11.25) at infinity. If $Y = |Y|e^{i\beta}$ then the equation of being invariant under half of the supersymmetry is

$$F = e^{-i\beta} * d_A\phi, \tag{11.26}$$

where $*$ is in $\mathbb{R}^3$. (the BPS monopole equations). Since F is real, we must have $\alpha = \beta$ modulo π.

It can be shown that such BPS states exist in sectors with charges 2 and -2 (charges are the eigenvalues of the infinitesimal operator corresponding to the unbroken $U(1)$-gauge symmetry). When these solutions are quantized, we will get the same result as in 2 dimensions. Namely, in quantum theory, we will get a representation of the superPoincare with helicities coming in groups (j,j+1/2,j+1) with multiplicities (1,2,1). Adding two copies of such groups with $j = -1$ and 0, we will get the massive hypermultiplet (=massive vector multiplet); this multiplet has the right helicities, which we found by considering the Higgs mechanism. Thus, in presence of central charge we get no contradiction.

Remark. Computing the commutators of the Q-s using currents, one can show that classically

$$Y = \int_\Sigma (\phi * F + \frac{1}{e^2}\phi F), \tag{11.27}$$

where Σ is a distant sphere in the space cycle. So it is a combination of the electric and magnetic charge for the effective $U(1)$ theory.