Quantum Double Models (QDM)

Kitaev, Annals of phys. 303, 2 (2003)

arXiv 1997

QDM: generalization of $\mathbb{Z}_2$ toric code
to lattice G gauge theory.
↳ finite (may be non-Abelian) group.

$$D(G),\ DG = \underset{\text{flux}}{G} \times \underset{\text{charge}}{\hat{G}} \quad \text{for Abelian } G.$$

2.1. The model based on a group algebra.

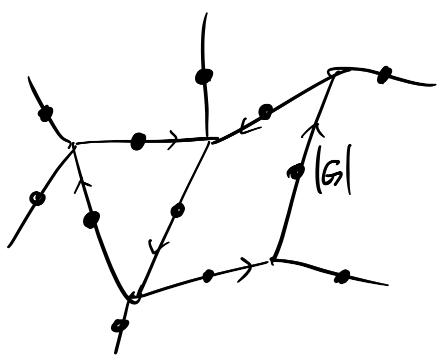
G.

$\mathcal{H} := \mathbb{C}[G] = \{ \sum_g a_g \cdot g \mid g \in G,\ a_g \in \mathbb{C} \}$ be the group algebra.
↓
Hilbert space with orthonormal basis $|g\rangle$
$\dim \mathcal{H} = |G|$

Def: $L_+^g |z\rangle = |gz\rangle \qquad T_+^h |z\rangle = \delta_{h,z} |z\rangle$
$\qquad L_-^g |z\rangle = |zg^{-1}\rangle \qquad T_-^h |z\rangle = \delta_{h^{-1},z} |z\rangle$

Satisfy: $L_+^g T_+^h = T_+^{gh} L_+^g, \quad L_+^g L_-^h = T_-^{hg^{-1}} L_+^g$
$\qquad L_-^g T_+^h = T_+^{hg^{-1}} L_-^g, \quad L_-^g T_-^h = T_-^{gh} L_-^g$

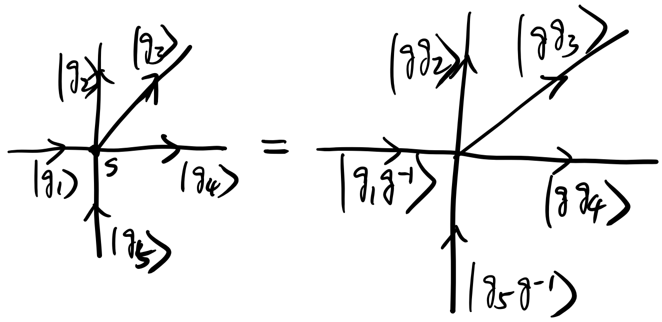
Hibert space:



|G|-dim vector space on each link of ∀ 2D lattice.

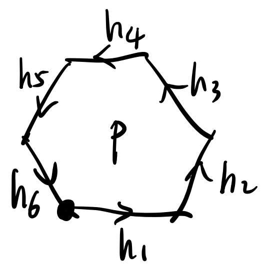
, $g \in G$.

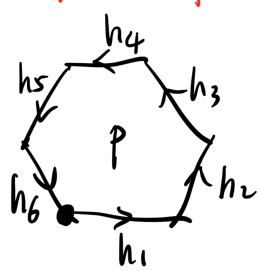
QDM : $H = -\sum_s A_s - \sum_p B_p$

A_s^g 

$$A_s^g := \prod_{\substack{\text{edge } l \\ \text{of } s}} L_\pm^g , \qquad A_s := \frac{1}{|G|} \sum_{g \in G} A_s^g$$

↳ gauge transf.

B_p^h 

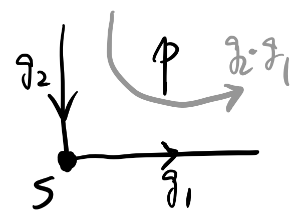
$= \delta_{h_1 h_2 \cdots h_6 , h}$ 

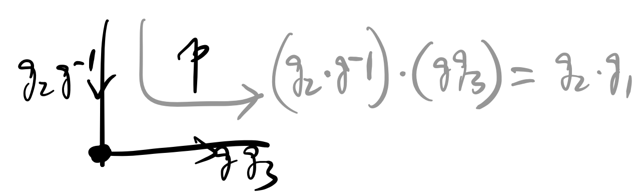
$$B_p^h = \sum_{h_1 \cdots h_6 = h} \prod_{m=1}^{6} T_\pm^{h_m} , \qquad B_p := B_p^{h=1}$$

↳ zero flux condition for p.

$$\begin{cases} A_s^2 = A_s \\ B_p^2 = B_p \end{cases} \text{projectors}$$

$$\begin{cases} A_s^2 = A_s \\ B_p^2 = B_p \\ [A_s , A_{s'}] = 0 \\ [B_p , B_{p'}] = 0 \\ \boxed{[A_s , B_p] = 0} \end{cases}$$

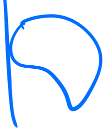
A_s 

$=$ 

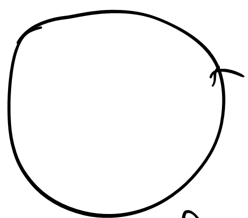
Ground state : $A_s|\Psi\rangle = B_p|\Psi\rangle = |\Psi\rangle \,, \; \forall s, p$

A_s ↓ gauge transf. $\quad B_p$ ↓ flat connection condition

$\Rightarrow |\Psi\rangle$: flat connections up to gauge equivalence.

TC. $|\Psi\rangle = \Big|$  $\Big\rangle + \Big|$  $\Big\rangle + \cdots$

$g = e^{iA} \in U(1)$



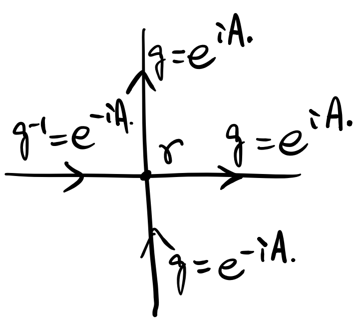
flat :

$$\Phi_c := \int_c A_\mu dx^\mu = 0$$

$$\boxed{g_{ij} = e^{iA_{ij}}}$$

$$\downarrow$$

$$\prod g_1 \cdots g_N = 1 \Leftrightarrow B_p^{h=1}$$



Gauss law

$$\nabla \cdot \vec{A} = \rho$$

$$\|$$

$$A(\vec{r}+\hat{x}) - A(\vec{r}) + A(\vec{r}+\hat{y}) - A(\vec{r})$$

$$\downarrow$$

$$g_{\vec{r}+\hat{x}}\, g_{\vec{r}}^{-1}\, g_{\vec{r}+\hat{y}}\, g_{\vec{r}}^{-1}$$