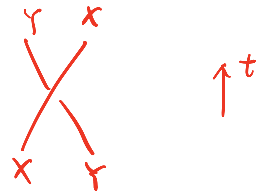
4. Modular tensor categories, Drinfeld center and anyon models.

4.1. Braided monoidal categories

Def. A <u>braided monoidal cat.</u> consists of

- a monoidal cat. $\mathcal{C}$
- a natural isomorphism call braiding that assigns to every pair of objects $X, Y \in \mathcal{C}$ an iso.

$$b_{X,Y} : X\otimes Y \to Y \otimes X$$

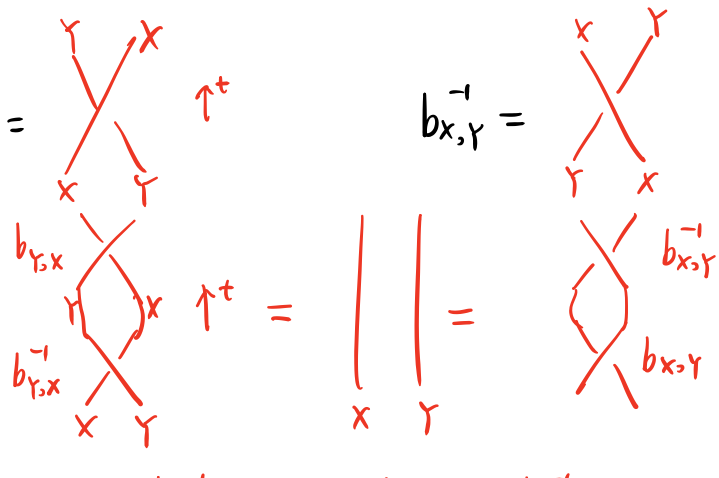


such that the hexagon eq hold:

$$\begin{tikzcd}
X\otimes(Y\otimes Z) \arrow[r, "a^{-1}_{X,Y,Z}"] \arrow[d, "b_{X,Y\otimes Z}"] & (X\otimes Y)\otimes Z \arrow[r, "b_{X,Y}\otimes id_Z"] \arrow[d, phantom, "\circlearrowleft"] & (Y\otimes X)\otimes Z \arrow[d, "a_{Y,X,Z}"] \\
(Y\otimes Z)\otimes X & Y\otimes(Z\otimes X) \arrow[l, "a^{-1}_{Y,Z,X}"] & Y\otimes(X\otimes Z) \arrow[l, "id_Y\otimes b_{X,Z}"]
\end{tikzcd}$$

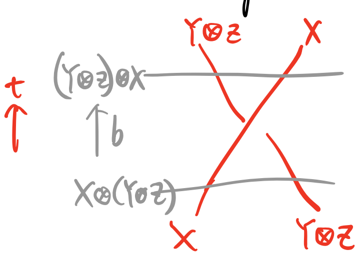
$$\begin{tikzcd}
(X\otimes Y)\otimes Z \arrow[r, "a_{X,Y,Z}"] \arrow[d, "b_{X\otimes Y,Z}"] & X\otimes(Y\otimes Z) \arrow[r, "id_X\otimes b_{Y,Z}"] \arrow[d, phantom, "\circlearrowright"] & X\otimes(Z\otimes Y) \arrow[d, "a_{X,Z,Y}"] \\
Z\otimes(X\otimes Y) & (Z\otimes X)\otimes Y \arrow[l, "a_{Z,X,Y}"] & (X\otimes Z)\otimes Y \arrow[l, "b_{X,Z}\otimes id_Y"]
\end{tikzcd}$$

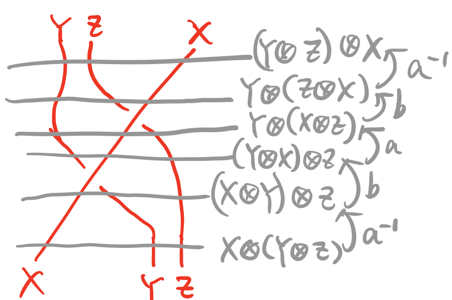
Rem. 1.



$$b_{Y,X}\circ b^{-1}_{Y,X} = id_{X\otimes Y} = b^{-1}_{X,Y}\circ b_{X,Y}$$

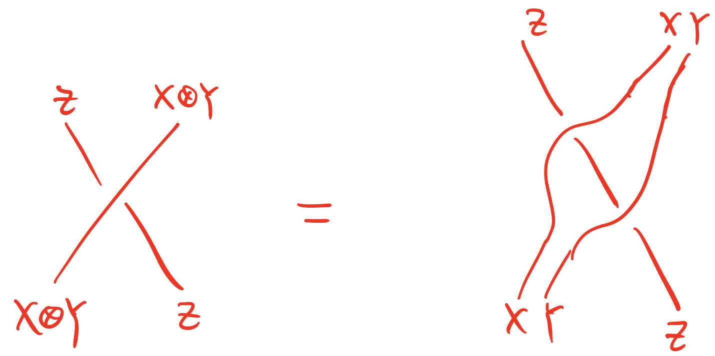
2. hexagon eq. $\Longleftrightarrow$





$$b = a^{-1} b a b a^{-1}$$

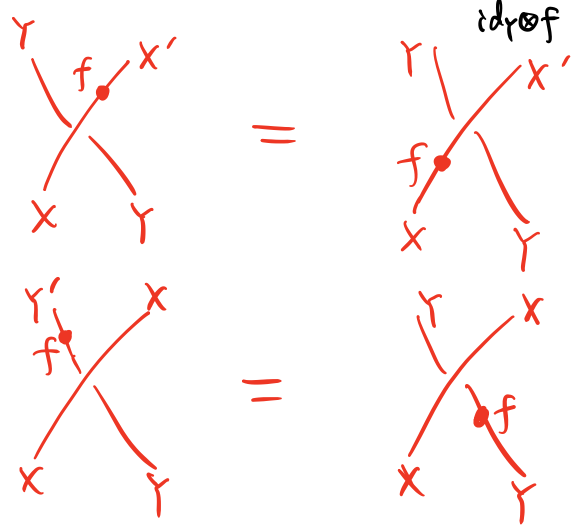
$$\Leftrightarrow aba = bab$$



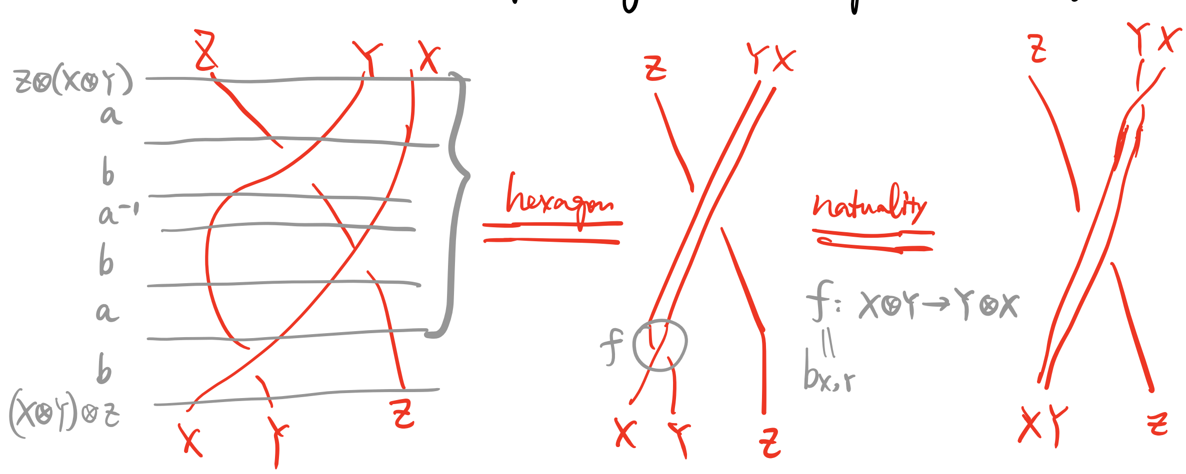
3. naturality of braiding:

$\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$

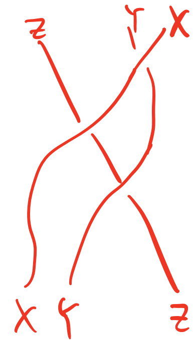
$$(X,Y) \xrightarrow{f} (X',Y) \Longrightarrow \begin{tikzcd} X\otimes Y \arrow[r, "f\otimes id_Y"] \arrow[d, "b_{X,Y}"'] \arrow[dr, phantom, "\circlearrowright"] & X'\otimes Y \arrow[d, "b_{X',Y}"] \\ Y\otimes X \arrow[r, "id_Y\otimes f"'] & Y\otimes X' \end{tikzcd}$$



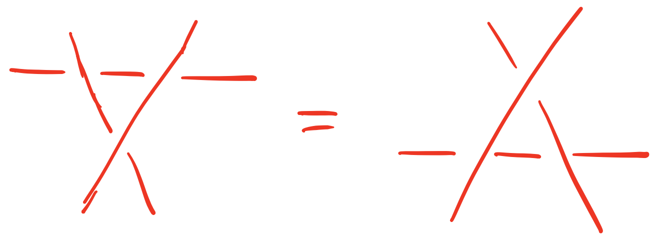
4. naturality + hexagon eq. ⇒ Yang-Baxter eq.



hexagon

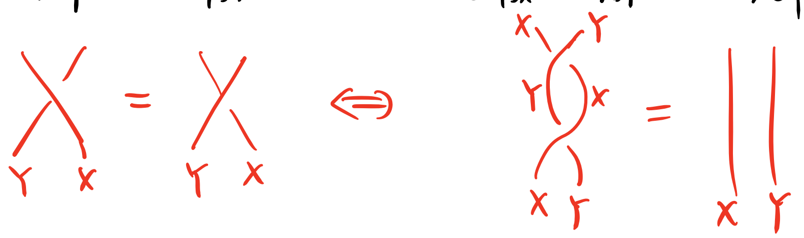


Reidemeister move III for knot $\leftrightsquigarrow$ Yang-Baxter eq.
(geometry) (algebra)



5. $\mathcal{C}$ is called <u>symmetric monoidal category</u> if

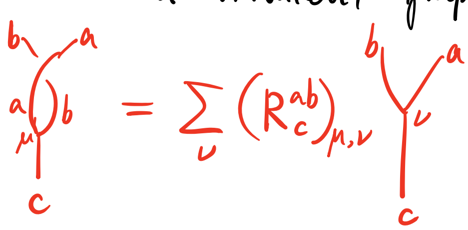
$$b_{X,Y}^{-1} = b_{Y,X} \iff b_{Y,X} \circ b_{X,Y} = id_{X\otimes Y}$$



6. String diagram.

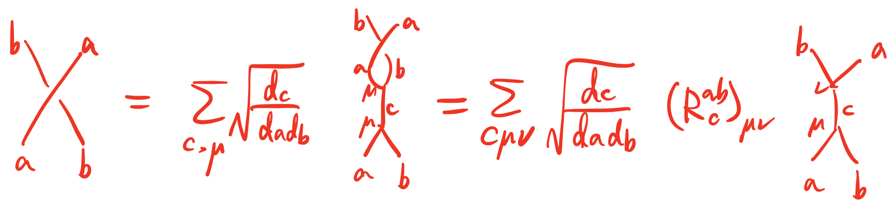
Split $\quad a \otimes b = \sum_c N_c^{ab}\, c$ in the simple obj. basis.

braiding $\longrightarrow$ a trivalent graph in 3D.



R_c^{ab} is assumed to be unitary.

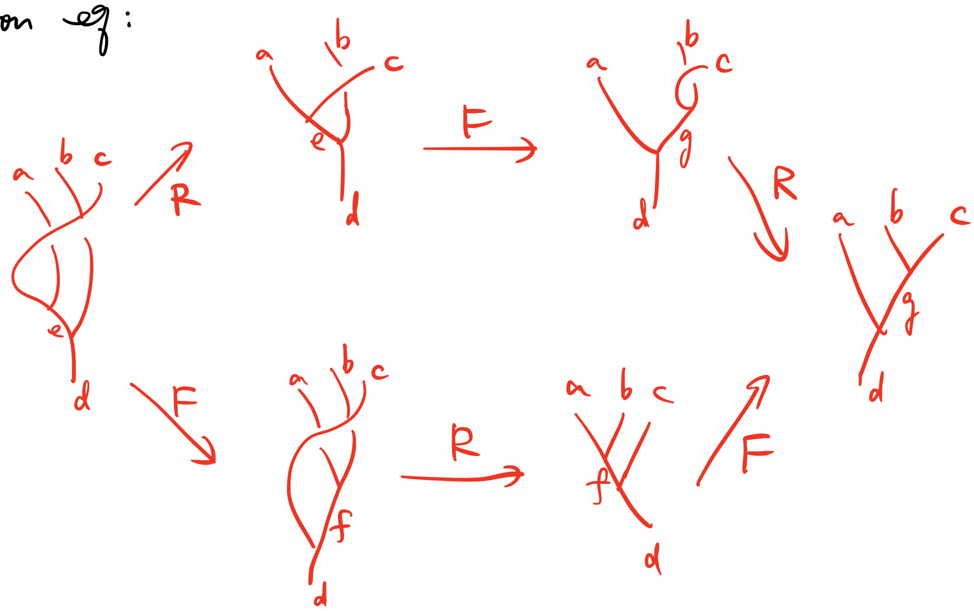
$R_{ab} =$



(3D diagram) $\longrightarrow$ (2D diagram)

naturality :

hexagon eq :



$$\sum (R^{\cdot\cdot}_{\cdot})_{\cdot\cdot} (F^{\cdot\cdot\cdot}_{\cdot})_{\cdot\cdot} (R^{\cdot\cdot}_{\cdot})_{\cdot\cdot} = \sum F R F$$

$\forall$ 3D trivalent graph $\xrightarrow{R,F}$ 2D trivalent graph $\xrightarrow{F}$ ϕ

7. modular data.

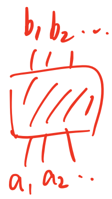
$$\theta_a = \frac{1}{d_a} \quad {}_a = \sum_{c\mu} \frac{d_c}{d_a} (R^{aa}_c)_{\mu\mu}$$

$$\begin{cases} T_{ab} = \theta_a \delta_{a,b} \\ S_{ab} := \frac{1}{D} \; {}_a \quad {}_b = \frac{1}{D} \sum_c N^c_{a\bar{b}} \frac{\theta_c}{\theta_a \theta_b} d_c \end{cases}$$

$\mathcal{C}$ is called <u>modular</u> if S, T are non-singular.

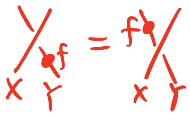
S, T generate rep of $PSL_2(\mathbb{Z})$.

8. 2+1D anyon models are described by a unitary modular tensor category (UMTC)

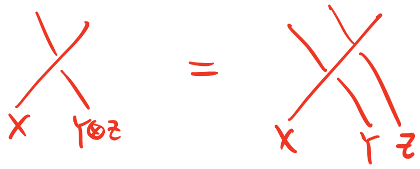
anyon	a	object
anti-particle	a^*	dual object
vacuum	1	identy obj.
fusion of anyons	$a \otimes b$	tensor product
anyon braiding	$R_{a,b}$	braiding
anyon worldline in 2+1 D		string diagram in 3D
transition amplitute		linear map: $a_1 \otimes a_2 \otimes \cdots \to b_1 \otimes b_2 \otimes \cdots$
partition function	closed 3D diagram	linear map: $1 \to 1$

4.2. Drinfeld center construction.

Let $\mathcal{C}$ be a monoidal category, a <u>half braiding</u> β_X for $X \in \mathcal{C}$ is a family $\{ \beta_X(Y) \in \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Y \otimes X) \mid Y \in \mathcal{C} \}$ of iso, natural w.r.t. Y,  satisfying $\beta_X(1) = \mathrm{id}_X$ and 



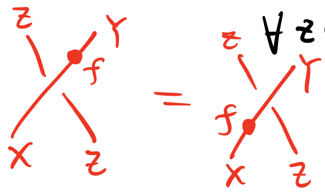
$$\beta_X(Y \otimes Z) = [\mathrm{id}_Y \otimes \beta_X(Z)] \circ [\beta_X(Y) \otimes \mathrm{id}_Z]$$



The <u>Drinfeld center $Z(\mathcal{C})$</u> of $\mathcal{C}$ has obj. (X, β_X), where $X \in \mathcal{C}$ and β_X is a half braiding for X.

The morphisms are

$$\mathrm{Hom}_{Z(\mathcal{C})}((X,\beta_X),(Y,\beta_Y)) := \{ f \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \mid [\mathrm{id}_Z \otimes f] \circ \beta_X(Z) = \beta_Y(Z) \circ [f \otimes \mathrm{id}_X], \ \forall Z \in \mathcal{C} \}$$

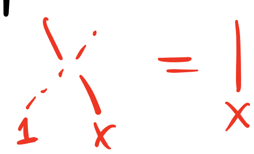


The tensor product in $Z(\mathcal{C})$ is given by

$$(X, \beta_X) \otimes (Y, \beta_Y) = (X \otimes Y, \beta_{X \otimes Y})$$

where $\beta_{X \otimes Y}(Z) := [\beta_X(Z) \otimes \mathrm{id}_Y] \circ [\mathrm{id}_X \otimes \beta_Y(Z)]$

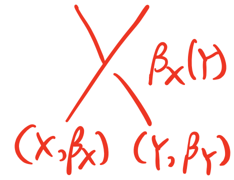
The tensor unit is $(1, \beta_1)$ where $\beta_1(X) := \mathrm{id}_X$.



The composition and tensor product of morphisms in $Z(\mathcal{C})$ are inherite from $\mathcal{C}$.

The braiding in $Z(\mathcal{C})$ is given by

$$b_{(X,\beta_X),(Y,\beta_Y)} := \beta_X(Y)$$



$\Rightarrow$ $Z(\mathcal{C})$ is a braided monoidal cat.

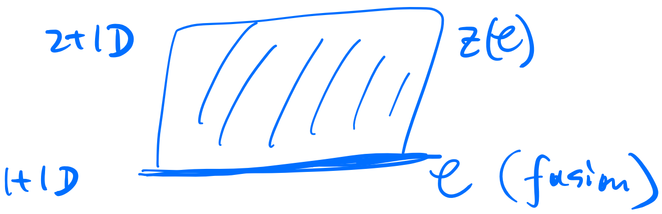
Rem. 1. $\mathcal{C}$ is monoidal $\xrightarrow{Z(\cdot)}$ $Z(\mathcal{C})$ is braided monoidal

2. If $\mathcal{C}$ is fusion, then $Z(\mathcal{C})$ is modular,

and $\dim Z(\mathcal{C}) = (\dim \mathcal{C})^2$, where $\dim \mathcal{C} := \sum_a d_a^2$

3. If $\mathcal{C}$ is modular tensor cat, then

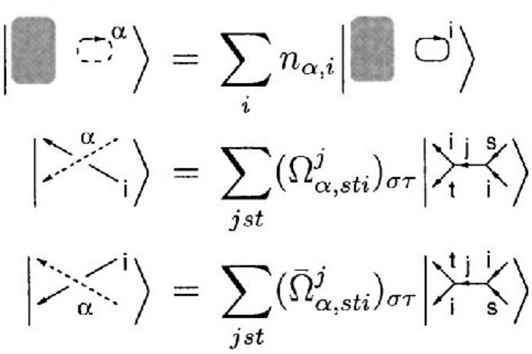
$$Z(\mathcal{C}) = \mathcal{C} \boxtimes \underbrace{\mathcal{C}^{op}}_{\text{same as } \mathcal{C} \text{ with inverse braiding}}$$

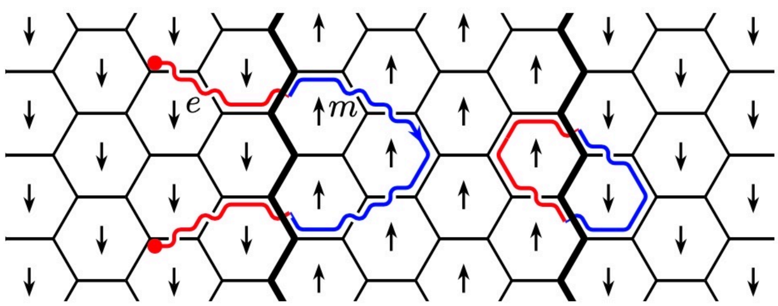


4.3. Excitation in Levin-Wen model of fusion cat. $\mathcal{C}$.

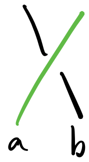
$\hookrightarrow$ described by $Z(\mathcal{C})$

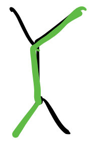
(1) diagram for ribbon operators.



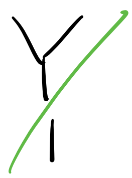
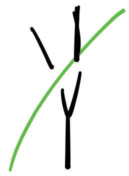


ribbon op. are labelled by obj. in $\mathcal{C}$.



$= \sum R$


$\left[\text{ribbon op.}, \begin{matrix} A_s \\ B_p \end{matrix}\right] = 0$ if $s, p \notin \partial$ ribbon.


$=$

naturality

...

$\Rightarrow$ ribbon op. are labelled by $Z(\mathcal{C})$.