

Quantum Double Models (QDM)

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QDM: generalization of \mathbb{Z}_2 toric code
to lattice G gauge theory.

→ finite (may be non-Abelian) group.

$$D(G), DG = \underset{\substack{\downarrow \\ \text{flux}}}{G} \times \underset{\substack{\downarrow \\ \text{charge}}}{\hat{G}} \text{ for Abelian } G.$$

2.1. The model based on a group algebra.

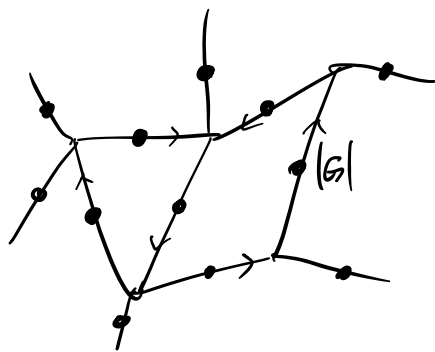
G .

$\mathcal{H} := \mathbb{C}[G] = \left\{ \sum_g a_g \cdot g \mid g \in G, a_g \in \mathbb{C} \right\}$ be the group algebra.
 \downarrow
 Hilbert space with orthonormal basis $|g\rangle$
 $\dim \mathcal{H} = |G|$

$$\text{Def: } \begin{aligned} L_+^g |z\rangle &= |gz\rangle & T_+^h |z\rangle &= \delta_{h,z} |z\rangle \\ L_-^g |z\rangle &= |zg^{-1}\rangle & T_-^h |z\rangle &= \delta_{h^{-1},z} |z\rangle \end{aligned}$$

$$\text{Satisfy: } \begin{aligned} L_+^g T_+^h &= T_+^{gh} L_+^g, & L_+^g L_-^h &= T_-^{hg^{-1}} L_+^g \\ L_-^g T_+^h &= T_+^{hg^{-1}} L_-^g, & L_-^g T_-^h &= T_-^{gh} L_-^g \end{aligned}$$

Hilbert space:



$|G|$ -dim vector space on
each link of a 2D lattice.

$$\overrightarrow{|g\rangle} = \overleftarrow{|g^{-1}\rangle}, \quad g \in G.$$

$$\text{QDM: } H = - \sum_s A_s - \sum_p B_p$$

$$A_s^g = \begin{array}{c} \begin{array}{c} |g_2\rangle \\ \swarrow \\ |g_1\rangle \end{array} \begin{array}{c} |g_3\rangle \\ \nearrow \\ |g_4\rangle \end{array} \\ \downarrow |g_5\rangle \end{array} = \begin{array}{c} \begin{array}{c} |gg_2\rangle \\ \swarrow \\ |gg_1g^{-1}\rangle \end{array} \begin{array}{c} |gg_3\rangle \\ \nearrow \\ |gg_4\rangle \end{array} \\ \downarrow |gg_5g^{-1}\rangle \end{array}$$

$$A_s^g := \prod_{\text{edge } l \text{ of } s} L_{\pm}^g, \quad A_s := \frac{1}{|G|} \sum_{g \in G} A_s^g$$

↪ gauge transf.

$$B_p^h = \begin{array}{c} h_5 \quad h_4 \\ \swarrow \quad \nearrow \\ h_6 \quad h_3 \\ \searrow \quad \swarrow \\ h_1 \quad h_2 \end{array} \begin{array}{c} p \\ \bullet \end{array} = \delta_{h_4 h_2 \dots h_6, h} \begin{array}{c} h_5 \quad h_4 \\ \swarrow \quad \nearrow \\ h_6 \quad h_3 \\ \searrow \quad \swarrow \\ h_1 \quad h_2 \end{array} \begin{array}{c} p \\ \bullet \end{array}$$

$$B_p^h = \sum_{h_1 \dots h_6 = h} \prod_{m=1}^6 T_{\pm}^{h_m}, \quad B_p := B_p^{h=1}$$

↪ zero flux condition for p.

$$\left\{ \begin{array}{l} A_s^2 = A_s \\ B_p^2 = B_p \\ [A_s, A_{s'}] = 0 \\ [B_p, B_{p'}] = 0 \\ \boxed{[A_s, B_p] = 0} \end{array} \right\} \text{ projectors}$$

$$A_s = \begin{array}{c} g_2 \downarrow \\ \bullet \\ s \end{array} \begin{array}{c} \nearrow g_1 \\ \searrow g_3 \end{array} \xrightarrow{\quad \phi \quad} \begin{array}{c} g_2 g^{-1} \downarrow \\ \bullet \end{array} \begin{array}{c} \nearrow (g_2 g^{-1}) \cdot (g_3) = g_2 \cdot g_1 \\ \searrow g_3 \end{array}$$

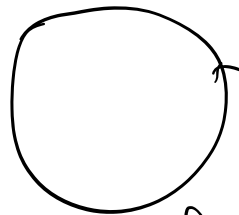
Ground state : $A_s |\psi\rangle = B_p |\psi\rangle = |\psi\rangle$, $\forall s, p$
↓ gauge transf. ↓ flat connection condition

$\Rightarrow |\psi\rangle$: flat connections up to gauge equivalence.

T.C. $|\psi\rangle = |0\rangle + |1\rangle + \dots$

$$g = e^{iA} \in U(1)$$

flat :



$$\Phi_c := \int_c A_\mu dx^\mu = 0$$

$$g_{ij} = e^{iA_{ij}}$$

$$\prod g_1 \dots g_N = 1 \Leftrightarrow B_p^{h=1}$$

Gauss law

$$\nabla \cdot \vec{A} = \rho$$

\parallel

$$A(\vec{r} + \hat{x}) - A(\vec{r})$$

$$+ A(\vec{r} + \hat{y}) - A(\vec{r})$$

\downarrow

$$g_{\vec{r}+\hat{x}} g_{\vec{r}}^{-1} g_{\vec{r}+\hat{y}} g_{\vec{r}}^{-1}$$