6. Examples of TI/TSC

Chern insulator is protected by $U(1)_c$ charge conservation symmetry.

Fermion parity $\mathbb{Z}_2^f = \{1, P_f = (-1)^F\}$

$F = \sum_j c_j^\dagger c_j$ is the fermion number.

$$H = \sum c^\dagger c + (\Delta cc + h.c.) + V c^\dagger c^\dagger cc + \cdots$$ preserves $\mathbb{Z}_2^f$.

6.1. 1+1D Majorana chain (by Kitaev 2000)

- Majorana fermion = real fermion

complex fermion $c, c^\dagger$:

$$\begin{cases} (c^\dagger)^2 = c^2 = 0 \\ cc^\dagger + c^\dagger c = 1 \end{cases}$$

Split c and $c^\dagger$ into "real" part and "imaginary" part:

$$\begin{cases} c = \frac{1}{2}(\gamma_1 + i\gamma_2) \\ c^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_2) \end{cases} \qquad \begin{cases} \gamma_1 = c + c^\dagger \\ \gamma_2 = \frac{1}{i}(c - c^\dagger) \end{cases}$$

Satisfying:

$$\begin{cases} \gamma_1^\dagger = \gamma_1, \quad \gamma_2^\dagger = \gamma_2 \\ \gamma_1^2 = (c+c^\dagger)^2 = cc^\dagger + c^\dagger c = 1 \\ \gamma_2^2 = \frac{1}{i^2}(c-c^\dagger)^2 = cc^\dagger + c^\dagger c = 1 \\ \gamma_1\gamma_2 + \gamma_2\gamma_1 = \frac{1}{i}(c+c^\dagger)(c-c^\dagger) + \frac{1}{i}(c-c^\dagger)(c+c^\dagger) = 0 \end{cases}$$

$$\begin{cases} H = \epsilon c^\dagger c \Rightarrow \begin{cases} E = \epsilon, & |ES\rangle = |n=1\rangle, & n = c^\dagger c = 1 \\ E = 0, & |GS\rangle = |n=0\rangle, & n = c^\dagger c = 0 \end{cases} \\ \begin{cases} n = c^\dagger c = \frac{1}{4}(\gamma_1 - i\gamma_2)(\gamma_1 + i\gamma_2) = \frac{1}{4}(2 + 2i\gamma_1\gamma_2) = \frac{1}{2}(1 + i\gamma_1\gamma_2) \\ P_f = (-1)^n = 1 - 2n = -i\gamma_1\gamma_2 = \pm 1 \end{cases} \\ H = \epsilon n = \epsilon \cdot \frac{1 - P_f}{2} = \frac{\epsilon}{2}(1 + \underbrace{i\gamma_1\gamma_2}) \end{cases}$$

2 Majorana fermions $\gamma_1, \gamma_2 \longleftrightarrow$ 1 complex fermion mode c

$\gamma_1 \longrightarrow \gamma_2$ $(-i\gamma_1\gamma_2 = 1)$ $\quad P_f = -i\gamma_1\gamma_2 = 1 \quad n = c^\dagger c = 0$

$\gamma_1 \longleftarrow \gamma_2$ $(-i\gamma_2\gamma_1 = 1)$ $\quad P_f = -i\gamma_1\gamma_2 = -1 \quad n = c^\dagger c = 1$

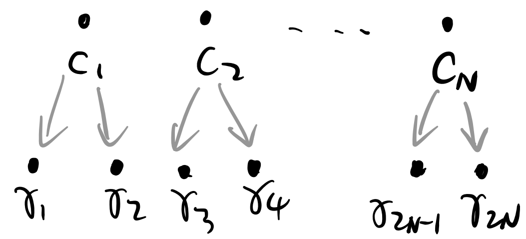
change pairing direction $\Longleftrightarrow$ change fermion parity $P_f = \pm 1$

- Majorana chain.

Consider N complex fermions $=$ $2N$ Majorana fermions.

$$c_j = \frac{1}{2}(\gamma_{2j-1} + i\gamma_{2j})$$

$(j = 1, 2, \cdots, N)$



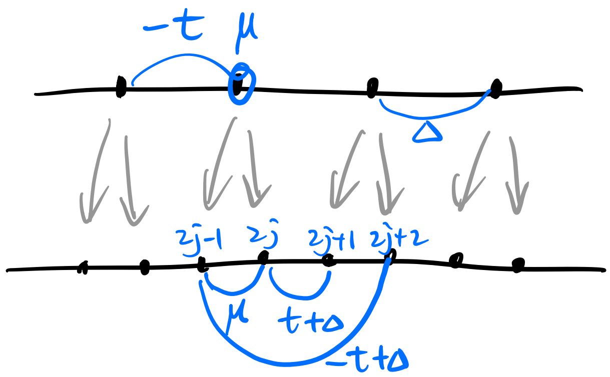
(1) p-wave superconductor chain.

$$H = \sum_j \left[-t(c_j^\dagger c_{j+1} + h.c.) - \mu c_j^\dagger c_j + \Delta(c_j c_{j+1} + h.c.) \right]$$

periodic boundary $\rightarrow -t(c_N^\dagger c_1 + h.c.)$ $\qquad t, \mu, \Delta \in \mathbb{R}$.

$$= \sum_j \left[-t \frac{1}{4}(\gamma_{2j-1} - i\gamma_{2j})(\gamma_{2j+1} + i\gamma_{2j+2}) + h.c. + \cdots \right]$$

$$= \frac{i}{2} \sum_j \left[-\mu \gamma_{2j-1}\gamma_{2j} + (t+\Delta)\gamma_{2j}\gamma_{2j+1} + (-t+\Delta)\gamma_{2j-1}\gamma_{2j+2} \right]$$



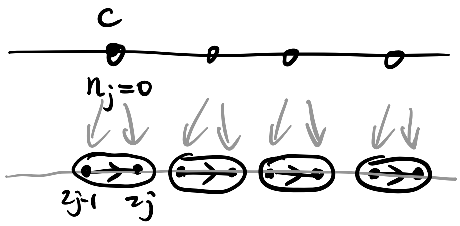
$$= \begin{cases} (\mu = -2, t = \Delta = 0) = \sum_j i\gamma_{2j-1}\gamma_{2j} = -\sum_j (P_f)_j \\ (\mu = 0, t = \Delta = 1) = \sum_j i\gamma_{2j}\gamma_{2j+1} \end{cases}$$

(2) Trivial chain $(\mu = -2, t = \Delta = 0)$

$$H = -\sum_j (P_f)_j$$

GS : $n_j = 0 \Leftrightarrow (P_f)_j = +1$

$\Leftrightarrow -i\gamma_{2j-1}\gamma_{2j} = 1$

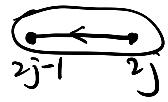


$|GS\rangle = \bigotimes_j |n_j = 0\rangle$

$|GS\rangle = \bigotimes_j |-i\gamma_{2j-1}\gamma_{2j} = 1\rangle$

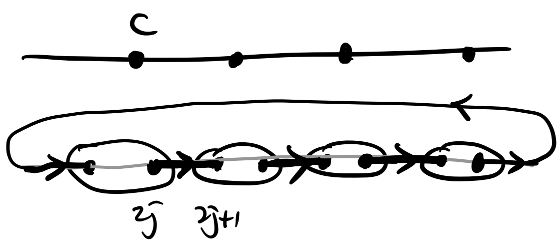
$-i\gamma_{2j-1}\gamma_{2j} = 1$

ES : $n_j = 1$ for some j.

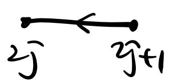


(3) Nontrivial chain ($\mu = 0$, $t = \Delta = 1$)

$$H = -\sum_j (c_j^\dagger c_{j+1} + h.c.) + \sum_j (c_j c_{j+1} + h.c.)$$
$$= \sum_j i\gamma_{2j}\gamma_{2j+1}$$



$|GS\rangle = \bigotimes_j |-i\gamma_{2j}\gamma_{2j+1} = 1\rangle$

excited state : 

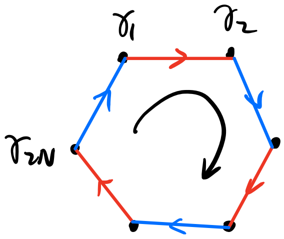
$-i\gamma_{2j}\gamma_{2j+1} = -1$.

Fermion parity of $|GS\rangle$.

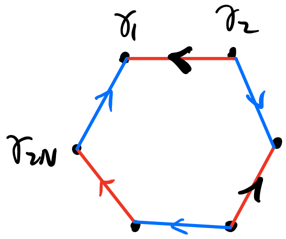
$$P_f|GS\rangle = \prod_j (-i\gamma_{2j-1}\gamma_{2j})|GS\rangle$$
$$= (-i)^N(\gamma_1\gamma_2)(\gamma_3\gamma_4)\cdots(\gamma_{2N-1}\gamma_{2N})\ |GS\rangle$$
$$= (-i)^N\gamma_1(\gamma_2\gamma_3)(\gamma_4\gamma_5)\cdots(\gamma_{2N-2}\gamma_{2N-1})\gamma_{2N}\ |GS\rangle$$
$$= (-i)^N(\gamma_2\gamma_3)\cdots(\gamma_{2N-2}\gamma_{2N-1})(\gamma_1\gamma_{2N})\ |GS\rangle$$
$$= -i\gamma_1\gamma_{2N}\ |GS\rangle$$
$$= -(-i\gamma_{2N}\gamma_1)\ |GS\rangle$$
$$= -|GS\rangle$$

$P_f = -1$ acting on $|GS\rangle$

$\Leftrightarrow$ $|GS\rangle$ has odd number of complex fermions c_j.



$$\begin{cases} |\psi_1\rangle : \ -i\gamma_1\gamma_2 = -i\gamma_3\gamma_4 = \cdots = -i\gamma_{2N-1}\gamma_{2N} = 1 \\ |\psi_2\rangle : \ -i\gamma_2\gamma_3 = -i\gamma_4\gamma_5 = \cdots = -i\gamma_{2N}\gamma_1 = 1 \\ |\psi_1\rangle \text{ and } |\psi_2\rangle \text{ have different fermion parity.} \end{cases}$$

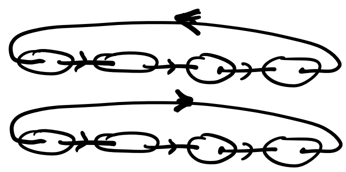


$|\psi_1\rangle$ and $|\psi_2\rangle$ have $\begin{cases}\text{same} \\ \text{different}\end{cases}$ fermion parity

$\Leftrightarrow$ number of counter clockwise arrow N_{AC} is $\begin{cases}\text{odd} \\ \text{even}\end{cases}$

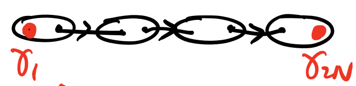
$\Leftrightarrow$ loop is $\begin{cases}\text{Kastelyn} \\ \text{non-Kastelyn}\end{cases}$ oriented

$$\begin{cases} \text{periodic boundary condition} \\ \text{antiperiodic} \quad \cdots - \cdot \\ \text{open} \quad \cdot - - - \end{cases}$$



$P_f = -1$

$P_f = 1$



edge state

γ_1, γ_{2N} are unpaired Majorana fermion.

GSD = 2

- p-wave SC in momentum space.

$$H = \sum_j \left[-t\left(c_j^\dagger c_{j+1} + h.c.\right) - \mu c_j^\dagger c_j + \Delta\left(c_j c_{j+1} + h.c.\right) \right]$$

$$\begin{cases} c_j = \frac{1}{\sqrt{N}} \sum_k e^{ikj} c_k \\ c_j^\dagger = \frac{1}{\sqrt{N}} \sum_k e^{-ikj} c_k^\dagger \end{cases}$$

$$= \sum_k \left[-t\left(e^{ik} c_k^\dagger c_k + h.c.\right) - \mu c_k^\dagger c_k + \Delta\left(\underbrace{e^{ik} c_k c_k}_{} + h.c.\right) \right]$$

$$\sum_k \Delta e^{ik} c_{-k} c_k$$

$$= \frac{1}{2}\sum_k \Delta e^{ik} c_k c_k + \frac{1}{2}\sum_k \Delta e^{-ik} c_k c_k$$

$$= \frac{1}{2}\sum_k \left(\Delta\, e^{ik} c_{-k} c_k - \Delta e^{-ik} c_{-k} c_k\right)$$
$$= \sum_k i\Delta \sin k\; c_{-k} c_k$$

$$= \sum_k \left(-2t\cos k - \mu\right) c_k^\dagger c_k + \Delta\left(i \sin k\; c_{-k} c_k + h.c.\right)$$

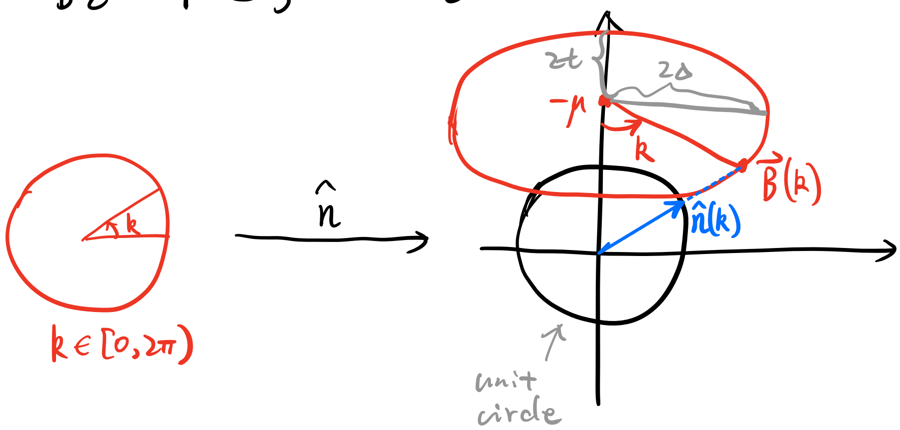
$$= \sum_k \frac{1}{2} \left(c_k^\dagger, c_{-k}\right) \underbrace{\begin{pmatrix} -2t\cos k - \mu & -i\Delta \sin k \\ 2i\Delta \sin k & 2t\cos k + \mu \end{pmatrix}}_{\mathcal{H}_k} \underbrace{\begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}}_{\text{Nambu basis}}$$

$$\mathcal{H}_k = 2\Delta \sin k\; \sigma_y - (2t\cos k + \mu)\, \sigma_z$$
$$= \vec{B}(k)\cdot\vec{\sigma} = E(k)\; \hat{n}(k)\cdot\vec{\sigma}$$

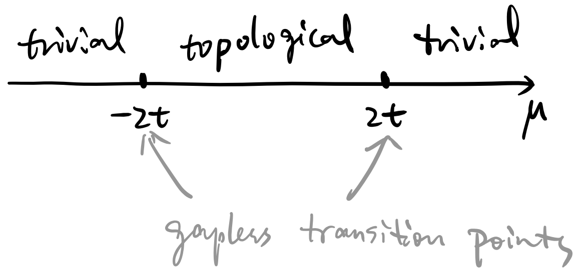
$$\vec{B}(k) := \left(2\Delta \sin k,\ -2t\cos k - \mu\right)$$

$$\hat{n}(k) := \frac{\vec{B}(k)}{|\vec{B}(k)|}$$

$$\hat{n}:\quad BZ = T^1 = S^1 \longrightarrow S^1$$



$$\text{mapping degree} = \text{winding number} = \begin{cases} 1, & \text{if } -\mu - 2t < 0 < -\mu + 2t \\ 0, & \text{otherwise} \end{cases}$$



phase diagram

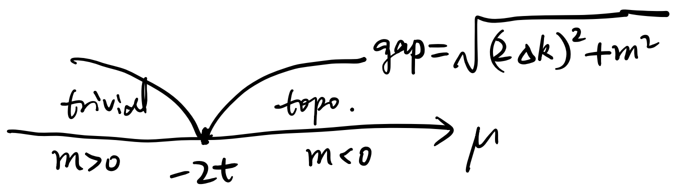
- Continuum model, field theory.

Consider $\mu = -2t - m$ with m small.

$$\mathcal{H}_k = 2\Delta \sin k\; \sigma_y + (-2t\cos k + 2t + m)\, \sigma_z$$

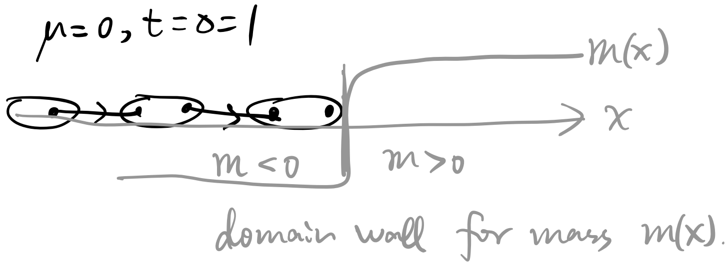
$$\xrightarrow{k\to 0} 2\Delta k\,\sigma_y + m\,\sigma_z$$

1+1 D massive Dirac Hamiltonian.



- Majorana edge mode

Consider $\mu=0,\ t=\Delta=1$



For $\mu=-2t-m\approx -2t$:

$$\mathcal{H}_k = 2\Delta\cdot k\,\sigma_y + m\,\sigma_z$$

$$\downarrow$$

$$\mathcal{H} = 2\Delta\,\sigma_y\, i\partial_x + m(x)\,\sigma_z$$

Try to solve zero energy state near $x=0$:

$$\mathcal{H}\,\Psi(x) = 0$$

$$\Leftrightarrow\ 2\Delta\,\sigma_y\, i\partial_x \Psi(x) + m(x)\,\sigma_z\,\Psi(x) = 0 \qquad v\sim\frac{\Delta}{m}$$

$$\Leftrightarrow\ \partial_x\Psi(x) = -i\frac{1}{v}\,m(x)\,\sigma_x\,\Psi(x)$$

choose $\Psi(x) = \varphi(x)\begin{pmatrix}1\\ \pm 1\end{pmatrix}$

$$\Leftrightarrow\ \partial_x\varphi(x) = \pm\frac{1}{v}\,m(x)\,\varphi(x)$$

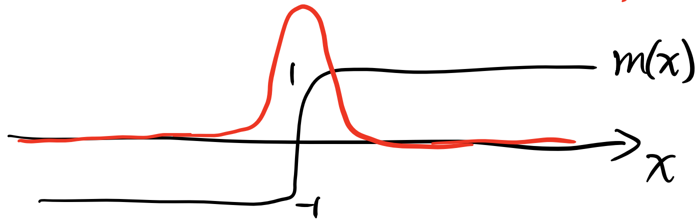
$$\Leftrightarrow\ \varphi(x) = e^{\pm\frac{1}{v}\int_0^x m(x')\,dx'}$$

$$\Leftrightarrow\ \Psi(x) = e^{\pm\frac{1}{v}\int_0^x m(x')\,dx'}\begin{pmatrix}1\\ \pm 1\end{pmatrix}$$

$$\begin{cases} ① & \Psi_+(x) = e^{+\frac{1}{v}\int_0^x m(x')\,dx'}\begin{pmatrix}1\\1\end{pmatrix} \xrightarrow[x\to-\infty]{x\to\infty} e^{\frac{1}{v}x}\begin{pmatrix}1\\1\end{pmatrix} \quad \times \\ ② & \Psi_-(x) = e^{-\frac{1}{v}\int_0^x m(x')\,dx'}\begin{pmatrix}1\\-1\end{pmatrix} \xrightarrow{x\to\pm\infty} e^{-\frac{1}{v}x}\begin{pmatrix}1\\-1\end{pmatrix} \quad \checkmark \end{cases}$$

$l = v \sim \frac{\Delta}{m}$

$\Psi_-(x) = e^{-\frac{1}{v}|x|}\begin{pmatrix}1\\-1\end{pmatrix}$: zero-energy edge mode.



domain wall fermion