Part II. Topological insulators and superconductors

(= fermionic $\underbrace{\text{symmetry-protected topological}}_{\text{SPT}}$ phases w/o interactions)

	noninteracting	interacting
bosonic	/	bSPT
fermionic	TI/TSC	fSPT

G-SPT : states without anyons, but still can NOT be deformed into trivial product state preserving symmetry G.

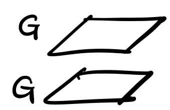
Topological phase $\longrightarrow$ Abelian monoid under stacking.

G-SPT ($\subset$ invertible phases) $\longrightarrow$ Abelian group under stacking.

∃ inverse

classified by $\mathbb{Z}$, $\mathbb{Z}_n$ or direct sum $\oplus$ of them.

Symmetry action on stacked system :



coproduct : $G \to G \times G$

$g \mapsto g \otimes g$

$U(g) \mapsto U_1(g) \otimes U_2(g)$

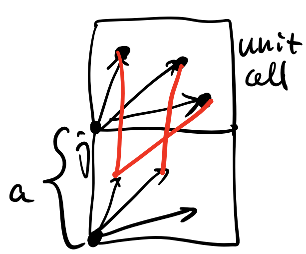
5. Integer quantum Hall effect and Chern insulators.

Noninteracting TI with symmetry group $U(1)_c = U(1)_f$

5.1. Band theory.

free fermions hopping on lattice

$$H = \sum_{j,\delta,m,n} t_\delta^{mn} \, c_{j+\delta,m}^{+} c_{j,n} + h.c.$$



Fourier transformation $\quad c_{j,n} = \frac{1}{\sqrt{N}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{j}}\, c_{kn}$

linear system size

$$\begin{cases} c_{j+L,n} = c_{j,n} \ \text{(finite system)} \Rightarrow e^{ik_j \cdot L} = 1 \Rightarrow k \in \frac{2\pi}{L}\mathbb{Z} \ \text{(momentum quantization)} \\ j \in a\mathbb{Z} \ \text{(discrete lattice)} \Rightarrow k + \frac{2\pi}{a} \sim k \Rightarrow k \in [0, \frac{2\pi}{a}) \end{cases}$$

(periodicity of momentum)

$$\Rightarrow k = 0, \frac{2\pi}{L}, \frac{2\pi}{L}\times 2, \cdots, \frac{2\pi}{L}\left(\frac{L}{a} - 1\right) = \frac{2\pi}{a} - \frac{2\pi}{L}.$$

Brillouin Zone $= T^d$ ← space dim of the lattice.

$$H = \sum_{j\delta mn} t_\delta^{mn} \frac{1}{N} \sum_{\vec{k},\vec{k}' \in BZ} e^{-i\vec{k}\cdot(\vec{j}+\vec{\delta}) + i\vec{k}'\cdot\vec{j}}\, c^\dagger_{\vec{k},m}\, c_{\vec{k}',n} + h.c.$$

$$= \sum_{\vec{k},\vec{k}'\in BZ} \sum_{mn\delta} \underbrace{\left[\frac{1}{N}\sum_{\vec{j}} e^{i(\vec{k}'-\vec{k})\cdot\vec{j}}\right]}_{\delta_{\vec{k}\vec{k}'}} t_\delta^{mn} e^{-i\vec{k}\cdot\vec{\delta}}\, c^\dagger_{\vec{k},m}\, c_{\vec{k}',n} + h.c.$$

$$= \sum_{\vec{k}\in BZ} \sum_{m,n} \underbrace{\left(\sum_{\delta} t_\delta^{mn} e^{-i\vec{k}\cdot\vec{\delta}}\right)}_{=: t_{\vec{k}}^{m,n}} c^\dagger_{k,m}\, c_{k,n} + h.c.$$

$$= \sum_{\vec{k}\in BZ} \left(c^\dagger_{k,1}, \cdots, c^\dagger_{k,M}\right) \mathcal{H}_{\vec{k}} \begin{pmatrix} c_{k,1} \\ \vdots \\ c_{k,M} \end{pmatrix}$$

$$\downarrow$$

$$(\mathcal{H}_{\vec{k}})_{m,n} = \sum_{\delta}\left(t_\delta^{mn} e^{-i\vec{k}\cdot\vec{\delta}} + t_\delta^{nm*} e^{i\vec{k}\cdot\vec{\delta}}\right)$$

$$\mathcal{H}: \quad BZ = T^d \longrightarrow \mathrm{Mat}_M(\mathbb{C})$$
$$\vec{k} \longmapsto \mathcal{H}_{\vec{k}}$$

Adding symmetries to system $\Leftrightarrow$ adding constraints on target manifold.

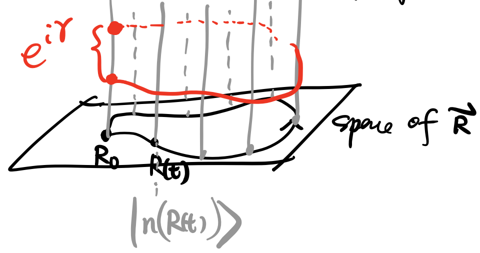
Classification of TI $\Leftrightarrow$ finding homotopy classes of $\mathcal{H}$.

5.2. Berry phase.

Assume a system depends on parameters $\vec{R} = (R_1, \cdots, R_N)$

$$H(\vec{R})\,|n(\vec{R})\rangle = E_n(\vec{R})\,|n(\vec{R})\rangle$$

$|n(\vec{R})\rangle \sim e^{i\theta}|n(\vec{R})\rangle$ are the same physical state.



If the parameter $\vec{R}(t)$ is slowly changed with time t:

$$i\frac{d}{dt}|\psi(t)\rangle = H(\vec{R}(t))\,|\psi(t)\rangle.$$

Assume $|\psi(t)\rangle = e^{i\gamma_n(t)}\, e^{-i\int_0^t E_n(R(t'))dt'}\,|n(\vec{R}(t))\rangle$, then

$$i\frac{d}{dt}|\psi(t)\rangle = -\frac{d\gamma_n(t)}{dt}|\psi(t)\rangle + \cancel{E_n(R(t))\,|\psi(t)\rangle} + e^{i\gamma_n(t)}e^{-i\int_0^t E_n(R(t'))dt'}\, i\frac{d}{dt}|n(\vec{R}(t))\rangle$$

$$= e^{i\gamma_n(t)}e^{-i\int_0^t E_n(R(t'))dt'}\,\underbrace{H(\vec{R}(t))|n(\vec{R}(t))\rangle}_{\cancel{E_n(R(t))|n(\vec{R}(t))\rangle}}$$

$$\Rightarrow -\frac{d\gamma_n(t)}{dt}|\psi(t)\rangle + e^{i\gamma_n(t)}e^{-i\int_0^t E_n(R(t'))dt'}\, i\frac{d}{dt}|n(\vec{R}(t))\rangle = 0$$

$$\Rightarrow -\frac{d\gamma_n(t)}{dt}|n(R(t))\rangle + i\frac{d}{dt}|n(\vec{R}(t))\rangle = 0$$

$$\Rightarrow \frac{d\gamma_n(t)}{dt} = i\langle n(R(t))|\frac{d}{dt}|n(R(t))\rangle$$

$$= i\frac{d\vec{R}}{dt}\cdot\langle n(R(t))|\nabla_{\vec{R}}|n(R(t))\rangle$$

$$\Rightarrow \gamma_n(L) = \int_0^t dt\,\frac{d\gamma_n(t)}{dt} = \int_0^t i\langle n(\vec{R})|\nabla_{\vec{R}}|n(\vec{R})\rangle\cdot\frac{d\vec{R}}{dt}\cdot dt$$

$$= \oint_R^L i\langle n(R)|\nabla_{\vec{R}}|n(R)\rangle\cdot d\vec{R} \longrightarrow \text{Berry phase}$$

Berry connection $A_n(\vec{R}) := i\langle n(\vec{R})|\nabla_{\vec{R}}|n(\vec{R})\rangle$

Gauge transformation: $|n'(\vec{R})\rangle := e^{i\alpha(\vec{R})}|n(\vec{R})\rangle$

$$\Rightarrow A_n'(\vec{R}) := i\langle n'(R)|\nabla_{\vec{R}}|n'(R)\rangle$$

$$= i\langle n(R)|\left(i e^{i\alpha(\vec{R})}\nabla_{\vec{R}}\alpha(\vec{R}) + e^{i\alpha(\vec{R})}\nabla_{\vec{R}}\right)|n(\vec{R})\rangle$$

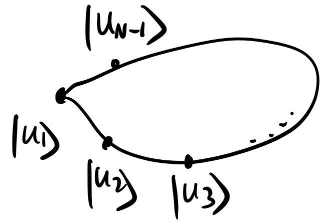
$$= -\langle n(R)|n(\vec{R})\rangle\,\nabla_{\vec{R}}\alpha(\vec{R}) + i\langle n(R)|\nabla_{\vec{R}}|n(\vec{R})\rangle$$

$$= A_n(\vec{R}) - \nabla_{\vec{R}}\, \alpha(\vec{R})$$

$$\Rightarrow \gamma_n'(L) = \oint_L A_n'(\vec{R}) = \oint_L \left[A_n(\vec{R}) - \nabla_{\vec{R}}\, \alpha(\vec{R})\right] = \gamma_n(L)$$

The Berry phase $\gamma_n(L) := \oint_L \vec{A}_n(\vec{R}) \cdot d\vec{R}$ is gauge invariant for closed loop L in the parameter space of $\vec{R}$.

- discrete $\rightarrow$ contineous



$|u_j\rangle \rightarrow |u_{j+1}\rangle$

$$\langle u_j | u_{j+1}\rangle = \left|\langle u_j | u_{j+1}\rangle\right| \cdot e^{i \arg \langle u_j | u_{j+1}\rangle}$$

phase diffenence $\arg \langle u_j | u_{j+1}\rangle = \mathrm{Im} \ln \langle u_j | u_{j+1}\rangle$

Total phase diffenence $\gamma = \sum_j \mathrm{Im} \ln \langle u_j | u_{j+1}\rangle = \mathrm{Im} \ln\left(\langle u_0|u_1\rangle\langle u_1|u_2\rangle \cdots \cdots \langle u_{N-1}|u_0\rangle\right)$

$$|u_{\vec{R}}\rangle \rightarrow |u_{\vec{R}+d\vec{R}}\rangle = |u_{\vec{R}}\rangle + d\vec{R}\cdot \nabla_{\vec{R}} |u_{\vec{R}}\rangle + \cdots$$

$$\Rightarrow \langle u_{\vec{R}} | u_{\vec{R}+d\vec{R}}\rangle \approx 1 + d\vec{R}\, \langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle$$

$$\Rightarrow \ln \langle u_{\vec{R}} | u_{\vec{R}+d\vec{R}}\rangle \approx \ln\left[1 + d\vec{R}\, \langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle\right]$$

$$\approx d\vec{R}\cdot \langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle$$

$$\Rightarrow \gamma = \mathrm{Im} \oint d\vec{R}\cdot \underline{\langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle} = \oint d\vec{R}\cdot \underbrace{i\langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle}_{\vec{A}_{\vec{R}}}$$

$$\left(\langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle\right)^* = \left(\langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle\right)^+$$

$$= \left(\nabla_{\vec{R}} |u_{\vec{R}}\rangle\right)^+ |u_{\vec{R}}\rangle = \left(\langle u_{\vec{R}} | \nabla_{\vec{R}}\right) |u_{\vec{R}}\rangle$$

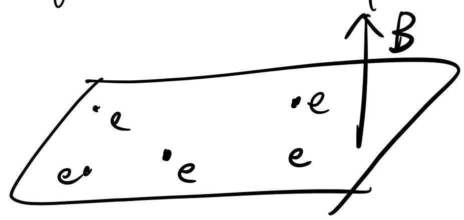
$$= - \langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle$$

$\Rightarrow \langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle$ is pure imaginary.

Non-Abelian generalization:

$$A_{mn}(\vec{R}) := i \langle m(\vec{R}) | \nabla_{\vec{R}} | n(\vec{R}) \rangle$$

5.3. Integer quantum Hall effect.



$$H = \frac{1}{2m}(-i\vec{\nabla} + e\vec{A})^2$$

$$\vec{A} \Rightarrow \vec{B} = B\hat{z}$$

Landau gauge : $\vec{A} = Bx\hat{y} = (0, Bx, 0)$

$$\Rightarrow B_z := \partial_x A_y - \partial_y A_x = \partial_x (B \cdot x) = B$$

$$\Rightarrow \vec{B} = B\hat{z} = (0, 0, B)$$

$$H = \frac{1}{2m}(-i\vec{\nabla} + e\vec{A}) = \frac{1}{2m}\left[-\partial_x^2 + (-i\partial_y + eBx)^2\right]$$

$[H, p_y] = 0 \Rightarrow H$ is invariant under y translation.

$$\psi(x,y) = e^{ik_y y}\psi_{k_y}(x)$$

$$H = \sum_{k_y} \frac{1}{2m}\left[-\partial_x^2 + (k_y + eBx)^2\right]$$

$\rightarrow$ a family of 1D harmonic oscillator parametrized by k_y.

$$H = \sum_{k_y}\left[-\frac{1}{2m}\partial_x^2 + \frac{1}{2}m\omega^2(x + k_y \ell^2)^2\right]$$

where $\begin{cases} \omega = \frac{eB}{m} & \text{is frequency of HO.} \\ \ell = \sqrt{\frac{\hbar}{eB}} & \text{is magnetic length.} \end{cases}$