Finite Temperature Field Theory №4

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1 Ward Identity with Hard Thermal Loop

The hard thermal loop (HTL) approximation of the photon self-energy is given by:

$$\Pi_{\mu\nu}(Q) = m_E^2 \left(\frac{q^0}{q} \int \frac{\mathrm{d}^2\Omega\,(\hat{\mathbf{k}})}{4\pi} \frac{\hat{K}_\mu \hat{K}_\nu}{\hat{\mathbf{k}}\cdot\hat{\mathbf{q}} + q^0/q} - g_{\mu 0} g_{\nu 0} \right) \tag{1}$$

Where $m_E^2 = \frac{e^2T^2}{3}$. Here we've chosen the Lorentzian signature $g \sim (-+++)$, and $Q^\mu \sim (q^0, \mathbf{q})$, where $q^0 = -i\omega,\ q = \|\mathbf{q}\|$, while the normalized $\hat{K}_\mu \sim (1, \hat{\mathbf{k}}),\ \hat{K}^\mu \sim (-1, \hat{\mathbf{k}})$. We have:

$$Q^\mu \hat{K}_\mu = q^0 + \hat{\mathbf{k}}\cdot\mathbf{q} = q\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{q}} + q^0/q\right), \quad Q^\mu g_{\mu 0} = q_0 = -q^0, \tag{2}$$

$$Q^\mu \Pi_{\mu\nu} = m_E^2\, q^0 \left(\int \frac{\mathrm{d}^2\Omega\,(\hat{\mathbf{k}})}{4\pi} \hat{K}_\nu + g_{\nu 0} \right) = 0 \tag{3}$$

More specifically, we have $Q^\mu \Pi_{\mu 0} \propto \left(\int \frac{\mathrm{d}^2\Omega(\hat{\mathbf{k}})}{4\pi} 1\right) - 1 = 0$, and also $Q^\mu \Pi_{\mu 0} \propto \int \frac{\mathrm{d}^2\Omega(\hat{\mathbf{k}})}{4\pi} \hat{K}_i = 0$. Therefore, the self-energy is transverse.

2 Decomposition of $\Pi_{\mu\nu}$

As we know, $Q^\mu \Pi_{\mu\nu} = 0$, hence it can be nicely decomposed by various projections related to Q^μ; recall that:

$$E_{\mu\nu} = \frac{Q_\mu Q_\nu}{Q^2} = \hat{Q}_\mu \hat{Q}_\nu, \quad P_{\mu\nu} = g_{\mu\nu} - E_{\mu\nu}, \tag{4}$$

$$N^\mu = P^{\mu\nu}(0, \mathbf{q})_\nu = P^{\mu j} q_j = -P^{\mu 0} q_0 = P^{\mu 0} q^0, \quad Q_\mu N^\mu = 0, \tag{5}$$

$$P_L^{\mu\nu} = \hat{N}^\mu \hat{N}^\nu, \quad P_T^{\mu\nu} = P^{\mu\nu} - P_L^{\mu\nu}, \tag{6}$$

Note that $P_T^{\mu\nu}$ annihilates both Q^μ and N^μ, by linearity it also annihilates both $(1, \mathbf{0})$ and $(0, \mathbf{q})$, i.e.

$$P_T^{\mu 0} q_0 = 0, \quad P_T^{\mu 0} = 0, \tag{7}$$

$$P_T^{ij} = \delta^{ij} - \hat{Q}^i \hat{Q}^j - \hat{N}^i \hat{N}^j = \delta^{ij} - \hat{q}^i \hat{q}^j \tag{8}$$

The transverse self-energy can then be decomposed as:

$$\Pi^{\mu\nu} = \Pi_L P_L^{\mu\nu} + \Pi_T P_T^{\mu\nu}, \tag{9}$$

$$\Pi^\mu{}_\mu = \Pi_L + 2\Pi_T, \quad \Pi^{00} = \Pi_L P_L^{00}, \tag{10}$$

$$N^i = P^{ij} q_j = \left(1 - \frac{q^2}{Q^2}\right) q^i = -\frac{q_0^2}{Q^2}\, q^i, \quad N^0 = \left(0 - \frac{q^2}{Q^2}\right) q^0 = -\frac{q^2}{Q^2} q^0, \quad N^2 = -\frac{q_0^2 q^2}{Q^2} \tag{11}$$

$$P_L^{00} = \frac{N_0^2}{N^2} = -\frac{q^2}{Q^2} \tag{12}$$

We see that $\Pi_{\mu\nu}$ is entirely determined by Π^{00} and $\Pi^{\mu}{}_{\mu}$. More explicitly, we have[1]:

$$\begin{aligned}\Pi_L &= -\frac{Q^2}{q^2}\,\Pi^{00}\\ \Pi_T &= \frac{1}{2}\left(\Pi^{\mu}{}_{\mu} + \frac{Q^2}{q^2}\,\Pi^{00}\right) = \frac{1}{2}\left(-\frac{q_0^2}{q^2}\,\Pi^{00} + \Pi^{i}{}_{i}\right)\end{aligned} \tag{13}$$

$\Pi^{00}, \Pi^{i}{}_{i}$ is computed explicitly as follows:

$$\Pi_{ij} = m_E^2\,\frac{q^0}{q}\int\frac{\mathrm{d}^2\Omega\,(\hat{\mathbf{k}})}{4\pi}\,\frac{\hat{k}_i\hat{k}_j}{\hat{\mathbf{k}}\cdot\hat{\mathbf{q}} + q^0/q}, \quad \Pi^{00} = m_E^2\left(\frac{q^0}{q}\int\frac{\mathrm{d}^2\Omega\,(\hat{\mathbf{k}})}{4\pi}\,\frac{1}{\hat{\mathbf{k}}\cdot\hat{\mathbf{q}} + q^0/q} - 1\right) \tag{14}$$

$$\begin{aligned}\Pi^{i}{}_{i} &= m_E^2\,\frac{q^0}{q}\int\frac{\mathrm{d}^2\Omega\,(\hat{\mathbf{k}})}{4\pi}\,\frac{\hat{k}^i\hat{k}_i}{\hat{\mathbf{k}}\cdot\hat{\mathbf{q}} + q^0/q}, \quad \hat{k}^i\hat{k}_i = 1,\\ &= m_E^2\,\frac{q^0}{q}\int_0^{\pi}\frac{2\pi\sin\theta\,\mathrm{d}\theta}{4\pi}\,\frac{1}{\cos\theta + q^0/q}\\ &= m_E^2\,\frac{q^0}{q}\int_{-1}^{1}\frac{\mathrm{d}z}{2}\,\frac{1}{z + q^0/q}\\ &= m_E^2\,\frac{q^0}{q}\,H\!\left(\frac{q^0}{q}\right), \quad H(x) = \frac{1}{2}\ln\frac{x+1}{x-1},\end{aligned} \tag{15}$$

$$\Pi^{00} = \Pi^{i}{}_{i} - m_E^2, \tag{16}$$

$$\begin{aligned}\Pi_L &= -m_E^2\,\frac{Q^2}{q^2}\left(\frac{q^0}{q}H\!\left(\frac{q^0}{q}\right) - 1\right), \quad \frac{Q^2}{q^2} = 1 - \frac{q_0^2}{q^2},\\ \Pi_T &= \frac{1}{2}\left(-\frac{q_0^2}{q^2}\,\Pi^{00} + \Pi^{i}{}_{i}\right) = \frac{1}{2}\left(\frac{Q^2}{q^2}\,\Pi^{i}{}_{i} + \frac{q_0^2}{q^2}\,m_E^2\right)\\ &= \frac{1}{2}\,m_E^2\,\frac{q^0}{q}\left(\frac{Q^2}{q^2}\,H\!\left(\frac{q^0}{q}\right) + \frac{q^0}{q}\right)\end{aligned} \tag{17}$$

3 HTL in QCD

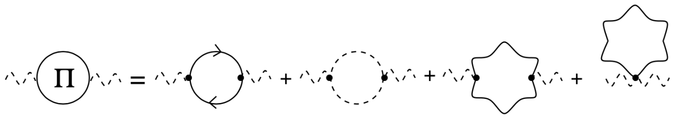
The gluon self-energy at 1-loop is similar to the QED situation, except that now we should include the extra degrees of freedom in the summation.

The quark 1-loop correction is structurally identical to the QED fermion 1-loop correction, but enhanced N_f fold due to the flavor degrees of freedom. Also, $\mathrm{Tr}\big(T^aT^b\big) = \frac{1}{2}\,\delta^{ab}$ gives an extra $\frac{1}{2}$ factor. In the end, the quark loop contribution is given by the overall factor:

$$m_E^2 \longmapsto m_{\text{quark}}^2\,\delta^{ab} = \frac{1}{2}N_f\,\delta^{ab} \tag{18}$$

The rest is identical to the QED case.

[1] Note that the metric convention here might result in signs that may differ from the textbook.



The ghost loop is also similar, except now we have a factor of N_c instead of $\frac{1}{2}N_f$, and with a frequency summation given by:

$$\begin{aligned} &\frac{T}{V}\sum_K \Delta(K)\left(-i\,(K-Q)^\nu\right)\Delta(K-Q)\left(-iK^\mu\right), \quad \Delta(K)=\frac{1}{K^2} \\ =& -\frac{T}{V}\sum_K \frac{(K-Q)^\nu K^\mu}{K^2(K-Q)^2} \simeq -\frac{T}{V}\sum_K \frac{K^\mu K^\nu}{K^2(K-Q)^2} \end{aligned} \tag{19}$$

We've computed a similar Matsubara summation before, while treating the QED fermionic loop; the method still applies here but with bosonic frequencies. This produces an additional factor of $(-\frac{1}{8})$, relative to the fermionic case[2]. In the end, we have:

$$m_E^2 \longmapsto m_{\text{ghost}}^2\,\delta^{ab} = -\frac{1}{8}N_c\,\delta^{ab} \tag{20}$$

[2] Reference: Laine & Vuorinen, *Basics of Thermal Field Theory.*