

1 Morphism between coverings is covering:

For $F_i \rightarrow E_i \xrightarrow{p_i} B$: coverings in $\text{Cov}_0(B)$ with E_i : connected and B : path connected and locally path connected, the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & \searrow p_1 & \swarrow p_2 \\ & B & \end{array} \quad \begin{array}{l} e_2 = f(e_1), \\ b = p_1(e_1) = p_2(e_2), \end{array}$$

To show that f is itself a covering, we need only verify that f is locally trivial with some discrete fiber F . In fact, given any $e_2 \in E_2$ and $b = p_2(e_2)$, there exists some neighborhood $U \subset B$ that the following diagram holds (by restriction):

$$\begin{array}{ccc} U \times F_1 & \xrightarrow{f} & U \times F_2 \\ & \searrow p_1 & \swarrow p_2 \\ & U & \end{array} \quad \begin{array}{l} e_1 = (b, k_1), \\ e_2 = (b, k_2(b, k_1)), \quad k_i \in F_i \end{array}$$

Generally, $k_2 = k_2(b, k_1)$ depends on the base point $b \in B$. However, since B is locally path connected, we can restrict U to be path connected, while $k_2 \in F_2$: discrete. Since continuous maps preserve path connectedness, k_2 is in fact independence of b , i.e. $k_2 = \varphi(k_1)$.

On the other hand, $\forall e_2 = (b, k_2) \in U \times \{k_2\} \subset E_2$, we have its preimage $f^{-1}(e_2) = \{b\} \times \varphi^{-1}(k_2)$. Note that E_2 is connected while $\varphi^{-1}(k_2) \in F_1$ is discrete; for the same reasoning as above, $\varphi^{-1}(k_2) = F$ is in fact independent of k_2 . This is the discrete fiber F we have been looking for. Hence f is also a covering¹. ■

2 Cylinder with ends pinched — π_1 and universal cover:

$$Y = (X \times I) / (X \times \partial I), \quad I = [0, 1] \quad (1)$$

Note that Y is homeomorphic to two cones² $CX_1 \amalg CX_2$ with “bases” $X_i \subset CX_i$ and “vertices” v_i respectively identified: $X_1 \sim X_2$, $v_1 \sim v_2 \equiv v$. X is path connected and so is Y , hence we are free to choose $\pi_1(Y) = \pi_1(Y, y_0)$.

First note that paths that do *not* pass through the vertex v are all homotopic, since they are contained in a cone and cones are contractible³. Therefore all contributions to $\pi_1(Y)$ are loop classes that *do* pass through the vertex v . In other words, morphisms in $\Pi_1 Y$ are in one-to-one correspondence with morphisms in:

$$\Pi_1([0, 1] / 0 \sim 1) = \Pi_1 S^1 \quad (2)$$

Therefore, $\pi_1(Y) \cong \pi_1(S^1) = \mathbb{Z}$. □

¹Reference: math.stackexchange.com/a/109774.

²See discussions from Problem Set №1.

³ $[\gamma_1] = [\gamma_2 \star \gamma_2^{-1} \star \gamma_1] = [\gamma_2]$.

The universal cover \tilde{Y} of Y can be constructed by assigning an induced topology to the space of path classes, same as in the general proof of its existence. Since Y is “degenerate” at its vertex, this is equivalent to “cutting open” Y at its vertex v , and joining \mathbb{Z} copies them end-to-end. More explicitly, it can be written as:

$$\tilde{Y} = (X \times \mathbb{R}) / \sim, \quad (x, n) \sim (x', n), \quad \forall x \in X, n \in \mathbb{Z} \quad (3)$$

While the covering map: $\tilde{Y} \ni [x, t] \mapsto [x, t - \lfloor t \rfloor] \in Y$, here $\lfloor t \rfloor$ is the integer part of $t \in \mathbb{R}$. \blacksquare

3 π_1 of fiber in fibration:

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\quad} & E \\ \downarrow & \searrow \exists \tilde{f} & \downarrow p \\ X \times I & \xrightarrow{\quad f \quad} & B \end{array}$$

For $F \rightarrow E \xrightarrow{p} B$: fibration, by homotopy lifting property (HLP), any homotopy in B can be uniquely lifted to path class in E , provided some “initial condition” $X \times \{0\}$. This leads to the following results:

- (a) For B : simply-connected, take any loop class $[\tilde{\gamma}] \in \pi_1(E, e)$ as initial condition; its projection $[p \circ \tilde{\gamma}] \in \pi_1(B, b) = \{[\mathbb{1}_b]\}$ is trivial, i.e. $p \circ \tilde{\gamma} \simeq \mathbb{1}_b$. By HLP, such homotopy can be lifted into E , i.e.

$$p \circ \tilde{\gamma} \simeq \mathbb{1}_b \xrightarrow{\text{lift}} \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}' = \mathbb{1}_b \quad (4)$$

In other words, $\tilde{\gamma} \simeq \tilde{\gamma}' \subset p^{-1}(b)$, i.e. any loop in E is homotopic to some loop in $p^{-1}(b) \cong F$. This implies a surjective group homomorphism $\pi_1(p^{-1}(b), e) \rightarrow \pi_1(E, e)$, i.e. an epimorphism. \square

- (b) For E : simply-connected, take any loop class $[\gamma] \in \pi_1(B, b)$ and consider its lifting $[\tilde{\gamma}]$. Note that in general $\tilde{\gamma}$ is *not* a loop; however, we have $p \circ \tilde{\gamma} = \gamma$, hence $\tilde{\gamma}(0), \tilde{\gamma}(1) \in p^{-1}(b)$. In general, we have:

$$\gamma \simeq \gamma' \xrightarrow{\text{lift}} \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}' = \gamma' \quad (5)$$

By continuity, $\tilde{\gamma}(0), \tilde{\gamma}'(0) \in F_0$: a path component of $p^{-1}(b)$; similarly, $\tilde{\gamma}(1), \tilde{\gamma}'(1) \in F_1$. In other words, the start and end points of $\tilde{\gamma}$ are confined in path components F_0 and F_1 , respectively. Hence a loop class in $\pi_1(B, b)$ maps to *transport* between path components:

$$\begin{aligned} T_{(\cdot)}(e): \pi_1(B, b) &\longrightarrow \pi_0(p^{-1}(b)) \\ [\gamma] &\longmapsto T_{[\gamma]}(e) \end{aligned} \quad (6)$$

As a matter of fact, $T_{(\cdot)}(e)$ is a bijection. For $T_{[\gamma]} = T_{[\gamma']}$, they are characterized by two lifted paths $\tilde{\gamma}, \tilde{\gamma}'$; since E is simply connected, they are always homotopic: $\tilde{\gamma} \simeq \tilde{\gamma}'$, hence $[\gamma] = [\gamma']$ by projection p . This means that T is injective. Surjectivity also follows from projection $\gamma = p \circ \tilde{\gamma}$. Therefore, $T_{(\cdot)}(e)$ gives a bijection between $\pi_1(B, b)$ and $\pi_0(p^{-1}(b))$. \blacksquare

4 Pull-back of fibration is fibration:

$$\begin{array}{ccccc}
 Y \times \{0\} & \longrightarrow & f^*(E) & \longrightarrow & E \\
 \downarrow & \nearrow \exists \tilde{G} & \downarrow & \nearrow \exists \tilde{F} \text{ (HLP)} & \downarrow p \\
 Y \times I & \xrightarrow{G} & X & \xrightarrow{f} & B
 \end{array}$$

$$(x, e) \in f^*(E) \subset X \times E, \quad f(x) = p(e)$$

We need only verify that $f^*(E) \rightarrow X$ also has HLP, i.e. the existence of \tilde{F} in the above diagram⁴. By HLP of $E \xrightarrow{p} B$, $\exists \tilde{F}: Y \times I \rightarrow E$ as shown above. We can use \tilde{F} to construct \tilde{G} explicitly; in fact, first consider:

$$\begin{aligned}
 \tilde{G}: Y \times I &\longrightarrow X \times E \\
 (y, t) &\longmapsto (G(y, t), \tilde{F}(y, t))
 \end{aligned} \tag{7}$$

Note that $f \circ G = p \circ \tilde{F}$; compared with the definition of $f^*(E)$, this implies that the image of \tilde{G} lies within $f^*(E) \subset X \times E$, hence after restriction of its codomain, \tilde{G} becomes a well-defined lifting of G into $f^*(E)$. Therefore, $f^*(E) \rightarrow X$ has HLP, i.e. it is also a fibration. \blacksquare

5 More properties of fibration:

(a) By HLP, given any initial condition $e \in p^{-1}(b_1)$, lifting of any path $b_1 \xrightarrow{\gamma} b_2$ exists. The lifted path with dependence of e can then be written as $F: p^{-1}(b_1) \times I \rightarrow E$. This is just a generalization of [3](#) for non-loop paths. \square

(b) Similarly, transport $T_{[\gamma]}$ defined in [3](#) can be generalized for non-loop paths. $T_{[\gamma]}$ is well-defined for path class $[\gamma]$, since by HLP homotopic paths can be lifted to homotopy in E . Therefore, the transport is fixed up to homotopy, i.e.

$$\begin{aligned}
 T: \text{Hom}_{\Pi_1 B}(b_0, b_1) &\longrightarrow \text{Hom}_{\mathbf{hTop}}(p^{-1}(b_0), p^{-1}(b_1)) \\
 [\gamma] &\longmapsto T_{[\gamma]}
 \end{aligned} \tag{8}$$

Note that T defined in this way is also independent of the choice of F , since F simply specifies the starting point of the lifted path; no matter which F we choose, the lifted paths will always be homotopic in E . Hence T is well-defined in the above sense. \square

(c) T defined above is a functor: $\Pi_1 B \rightarrow \mathbf{hTop}$. To verify this, we need only check that it is compatible with composition and maps identity morphisms to identity morphisms. Indeed, $T_{[\mathbb{1}_b]} = [\mathbb{1}_{p^{-1}(b)}]$, and $T_{[\gamma'] \star [\gamma]} = T_{[\gamma' \star \gamma]} = T_{[\gamma']} \circ T_{[\gamma]}$ by joining two lifted paths (up to homotopy). \square

⁴Notice that $f^*(E)$ is the limit of the diagram, hence this is automatically true by the universal property of $f^*(E)$. I would like to thank 刘逸华 for pointing this out. For now, we will stick to a more traditional proof.

(d) For B : path connected, there exists an isomorphism between any two objects in $\Pi_1 B$ (a path connecting any two points in B), which is mapped to isomorphisms between fibers $p^{-1}(b)$ in \mathbf{hTop} . Hence any two fibers of $E \xrightarrow{p} B$ have the same homotopy type. ■