String Theory №1

Bryan Compiled @ 2020/10/06

1 Read *Polchinski* Sections 1.3 and 1.4:

Read, *mostly* understood. □

2 Spinning Closed String in AdS Space:

For a classical spinning string, we have Nambu–Goto action:

$$S_{NG} = -T \int \mathrm{d}\tau\, \mathrm{d}\sigma \sqrt{-\det \gamma_{ab}}, \quad \gamma_{ab} = G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \tag{1}$$

Here $G_{\mu\nu}$ is the spacetime metric. γ_{ab} can be treated as the induced metric on the worldsheet.

In AdS space we have:

$$ds^2 = R^2 \left(-\cosh^2\rho\, \mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\rho\, \mathrm{d}\Omega^2\right) \tag{2}$$

Where $\mathrm{d}\Omega^2$ is the metric of a unit $(d-2)$–sphere S^{d-2}. For convenience let's define unit S^{d-2} metric G^1_{ij}, and raise or lower the $i, j, \cdots$ indices using G^1_{ij} instead of G_{ij}, i.e.,

$$G^1_{ij} = G_{ij} / \left(R^2 \sinh^2\rho\right), \quad i, j = 2, \cdots, d-1 \tag{3}$$

Furthermore, we consider the special case that the closed string is *folded*, like a rubber band stretched along a line; in this case we can choose the worldsheet parameter $(\tau, \sigma) = (t, \rho)$ while $\Omega = \Omega(t, \rho) = \Omega(\tau, \sigma)$, which leads to the following decomposition:

$$\partial_a X^\mu = \delta^\mu_a + \delta^\mu_i\, \partial_a \Omega^i, \quad a = 0, 1, \quad i = 2, \cdots, d-1, \tag{4}$$

$$\begin{aligned}
\gamma_{ab} &= G_{\mu\nu}\, \partial_a X^\mu \partial_b X^\nu \\
&= G_{ab} + G_{ij}\, \partial_a \Omega^i\, \partial_b \Omega^j \\
&= G_{ab} + R^2 \sinh^2\rho\, G^1_{ij}\, \partial_a \Omega^i\, \partial_b \Omega^j \\
&= R^2 \left\{ \begin{pmatrix} -\cosh^2\rho & \\ & 1 \end{pmatrix} + \sinh^2\rho \begin{pmatrix} (\partial_a\Omega)^2 & \partial_a\Omega \cdot \partial_b\Omega \\ \partial_b\Omega \cdot \partial_a\Omega & (\partial_b\Omega)^2 \end{pmatrix} \right\}
\end{aligned} \tag{5}$$

Here $\partial_a\Omega \cdot \partial_b\Omega \equiv \partial_a\Omega^i\, \partial_b\Omega_i \equiv G^1_{ij}\, \partial_a\Omega^i\, \partial_b\Omega^j$, and we have:

$$\begin{aligned}
\det \gamma_{ab} &= (R^2)^2 \Big\{ \sinh^4\rho \det\left(\partial_a\Omega^i \partial_b\Omega_i\right) \\
&\qquad + \sinh^2\rho \left((\partial_a\Omega)^2 - (\partial_b\Omega)^2 \cosh^2\rho\right) \\
&\qquad - \cosh^2\rho \Big\}, \\
\sqrt{-\det \gamma_{ab}} &= R^2 \Big\{ \cosh^2\rho - \sinh^2\rho \left((\partial_a\Omega)^2 - (\partial_b\Omega)^2 \cosh^2\rho\right) \\
&\qquad - \sinh^4\rho \det\left(\partial_a\Omega^i \partial_b\Omega_i\right) \Big\}^{1/2}
\end{aligned} \tag{6}$$

Mark the end points of the string with $\rho = r(t)$, then the total length of such closed folded string is $\ell = 4r$. We then have:

$$S = -4TR^2 \int \mathrm{d}t \int_0^r \mathrm{d}\rho \sqrt{\cosh^2\rho - \sinh^2\rho \left((\partial_a\Omega)^2 - (\partial_b\Omega)^2 \cosh^2\rho\right) - \sinh^4\rho \det\left(\partial_a\Omega^i \partial_b\Omega_i\right)} \tag{7}$$

Further simplification comes from the fact that, due to rotational symmetry, the string's motion can be restricted in a plane where its position is characterized by some angle $\theta = \Omega^{i_0} \in \{\Omega^i\}_i$. In this case other angle parameters $\Omega^i|_{i\neq i_0} = 0$, and the action is further reduced to:

$$S = -4TR^2 \int \mathrm{d}t \int_0^r \mathrm{d}\rho \sqrt{\cosh^2\!\rho - \sinh^2\!\rho \left((\partial_a\theta)^2 - (\partial_b\theta)^2 \cosh^2\!\rho\right)} = \int \mathrm{d}t \int_0^r \mathrm{d}\rho\, \mathcal{L}, \tag{8}$$

$$\mathcal{L} = -4TR^2 \sqrt{\cosh^2\!\rho - \omega^2 \sinh^2\!\rho}, \quad \omega = \partial_t\theta,\ \partial_\rho\theta = 0 \tag{9}$$

We consider the special solution $\theta = \omega t$, while in general the endpoint $r = r(t)$ could be dynamical; variation of the action w.r.t. $r(t)$ gives[1]:

$$0 = \delta S = -4TR^2 \int \mathrm{d}t \int_r^{r+\delta r} \mathrm{d}\rho \sqrt{\cosh^2\!\rho - \omega^2 \sinh^2\!\rho} = -4TR^2 \int \mathrm{d}t \sqrt{\cosh^2\!r - \omega^2 \sinh^2\!r}\, \delta r\,, \tag{10}$$

$$\omega^2 = \frac{\cosh^2\!r}{\sinh^2\!r} = \coth^2\!r \tag{11}$$

Note that if ω is constant, then r must be fixed by (11). Taking θ as the only dynamical variable, it is then straight-forward to write the energy E and angular momentum J for such folded closed string:

$$\omega = \dot{\theta}, \quad \Pi = \frac{\partial \mathcal{L}}{\partial \omega} = 4TR^2 \frac{\omega \sinh^2\!\rho}{\sqrt{\cosh^2\!\rho - \omega^2 \sinh^2\!\rho}}, \tag{12}$$

$$J = \int_0^r \mathrm{d}\rho\, \Pi = 4TR^2 \int_0^r \mathrm{d}\rho \frac{\omega \sinh^2\!\rho}{\sqrt{\cosh^2\!\rho - \omega^2 \sinh^2\!\rho}}, \tag{13}$$

$$E = \int_0^r \mathrm{d}\rho\, (\Pi\omega - \mathcal{L}) = 4TR^2 \int_0^r \mathrm{d}\rho \frac{\cosh^2\!\rho}{\sqrt{\cosh^2\!\rho - \omega^2 \sinh^2\!\rho}}, \tag{14}$$

In the large string limit, $r \to \infty$, $\omega = \coth r \to 1$. Expand in terms of $\epsilon = \omega - 1 > 0$, we find that $r = \frac{1}{2} \ln\left(1 + \frac{2}{\epsilon}\right) \sim \frac{1}{2} \ln \frac{2}{\epsilon}$, or alternatively, $e^{2r} \cdot \epsilon \sim 2$. With some help from Mathematica™, we get:

$$\begin{aligned} E - J &= 4TR^2 \int_0^r \mathrm{d}\rho \frac{\cosh^2\!\rho - \omega \sinh^2\!\rho}{\sqrt{\cosh^2\!\rho - \omega^2 \sinh^2\!\rho}} = 4TR^2 \int_0^r \mathrm{d}\rho \left(1 + \frac{\epsilon^2}{8} \sinh^2(2\rho) + \mathcal{O}(\epsilon^3)\right) \\ &= 4TR^2 \left(r\left(1 - \frac{\epsilon^2}{16} + \mathcal{O}(\epsilon^3)\right) + \mathcal{O}(1)\right) = \left(2TR^2 \ln \frac{2}{\epsilon}\right)\left(1 - \frac{\epsilon^2}{16} + \mathcal{O}(\epsilon^3)\right) \\ &\sim 2TR^2 \left(\ln \frac{2}{\epsilon}\right) \end{aligned} \tag{15}$$

Similarly, $J \sim 4TR^2 \int_0^r \mathrm{d}\rho \sinh^2\!\rho \sim TR^2\left(\frac{2}{\epsilon}\right)$, this gives:

$$E - J \sim 2TR^2 \ln \frac{J}{TR^2} \tag{16}$$

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[1] The above reasoning is confirmed in e.g. arXiv:hep-th/0204051.

$\boxed{3}$ Special Conformal Transformations:

$$x^\mu \xmapsto{K(a)} \tilde{x}^\mu = \frac{x^\mu + x^2 a^\mu}{1 + 2a\cdot x + a^2 x^2} \tag{17}$$

(a) Under special conformal transformation $K(a)$, metric $\delta_{\mu\nu} \mapsto g_{\mu\nu}$ while:

$$g_{\alpha\beta}\,\mathrm{d}\tilde{x}^\alpha\,\mathrm{d}\tilde{x}^\beta = \delta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,, \quad g_{\alpha\beta} = \delta_{\mu\nu}\frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial x^\nu}{\partial \tilde{x}^\beta} \tag{18}$$

To calculate this we have to know the inverse transformation $x = K^{-1}(a)\,\tilde{x}$. First, notice the following decomposition[2] of $K(a)$:

$$\tilde{x}^\mu = \frac{\frac{x^\mu}{x^2} + a^\mu}{\frac{1}{x^2} + \frac{2a\cdot x}{x^2} + a^2} = \frac{\frac{x^\mu}{x^2} + a^\mu}{\left|\frac{x^\mu}{x^2} + a^\mu\right|^2}, \tag{19}$$

$$\text{i.e.}\quad K(a)\colon\ x^\mu \xmapsto{I} \frac{x^\mu}{x^2} \xmapsto{T(a)} y^\mu = \frac{x^\mu}{x^2} + a^\mu \xmapsto{I} \tilde{x}^\mu = \frac{y^\mu}{y^2}, \tag{20}$$

$$\text{i.e.}\quad \frac{\tilde{x}^\mu}{\tilde{x}^2} = \frac{y^\mu}{y^2}\Big/\frac{1}{y^2} = y^\mu = \frac{x^\mu}{x^2} + a^\mu \tag{21}$$

From (21), we see that the transformation parameter a^μ composes linearly: $K(b)\,K(a) = K(a+b)$, therefore $K^{-1}(a) = K(-a)$, and we have:

$$x^\mu = K(-a)\,\tilde{x}^\mu = \frac{\tilde{x}^\mu - \tilde{x}^2 a^\mu}{1 - 2a\cdot\tilde{x} + a^2\tilde{x}^2} = \frac{\tilde{y}^\mu}{y^2}, \tag{22}$$

$$\begin{aligned}\frac{\partial x^\mu}{\partial \tilde{x}^\alpha} &= \frac{\partial x^\mu}{\partial \tilde{y}^\sigma}\frac{\partial \tilde{y}^\sigma}{\partial \tilde{x}^\alpha} = \left(\frac{\partial}{\partial \tilde{y}^\sigma}\frac{\tilde{y}^\mu}{\tilde{y}^2}\right)\frac{\partial}{\partial \tilde{x}^\alpha}\left(\frac{\tilde{x}^\sigma}{\tilde{x}^2} - a^\sigma\right) = \left(\frac{\partial}{\partial \tilde{y}^\sigma}\frac{\tilde{y}^\mu}{\tilde{y}^2}\right)\left(\frac{\partial}{\partial \tilde{x}^\alpha}\frac{\tilde{x}^\sigma}{\tilde{x}^2}\right)\\ &= \left(\tilde{y}^2\delta^\mu_\sigma - 2\tilde{y}^\mu\tilde{y}_\sigma\right)\left(\tilde{x}^2\delta^\sigma_\alpha - 2\tilde{x}^\sigma\tilde{x}_\alpha\right)\Big/\left(\tilde{y}^4\tilde{x}^4\right),\end{aligned} \tag{23}$$

$$\begin{aligned} g_{\alpha\beta} &\xlongequal{(18)} \delta_{\mu\nu}\left(\tilde{y}^2\delta^\mu_\sigma - 2\tilde{y}^\mu\tilde{y}_\sigma\right)\left(\tilde{x}^2\delta^\sigma_\alpha - 2\tilde{x}^\sigma\tilde{x}_\alpha\right)\left(\tilde{y}^2\delta^\nu_\rho - 2\tilde{y}^\nu\tilde{y}_\rho\right)\left(\tilde{x}^2\delta^\rho_\beta - 2\tilde{x}^\rho\tilde{x}_\beta\right)\Big/\left(\tilde{y}^8\tilde{x}^8\right)\\ &\xlongequal{\sum_{\mu,\nu}} \tilde{y}^{-4}\delta_{\sigma\rho}\left(\tilde{x}^2\delta^\sigma_\alpha - 2\tilde{x}^\sigma\tilde{x}_\alpha\right)\left(\tilde{x}^2\delta^\rho_\beta - 2\tilde{x}^\rho\tilde{x}_\beta\right)\Big/\tilde{x}^8\\ &\xlongequal{\sum_{\sigma,\rho}} \tilde{y}^{-4}\tilde{x}^{-4}\delta_{\alpha\beta}\end{aligned} \tag{24}$$

We see that $g_{\alpha\beta} = f(x)\,\delta_{\alpha\beta}$, with coefficient:

$$f(x) = \tilde{y}^{-4}\tilde{x}^{-4} \xlongequal{(20)} \frac{x^4}{\tilde{x}^4} \xlongequal{(21)} \left(1 + 2a\cdot x + a^2x^2\right)^2 \tag{25}$$

$\square_{(a)}$

(b) In 2D with $z = x^1 + ix^2$, $x^\mu \sim (z, \bar{z})$, we see from (21) that:

$$\frac{x^\mu}{x^2} \sim \frac{z}{|z|^2} = \frac{1}{\bar{z}} \longmapsto \frac{1}{\bar{z}} + a, \quad \text{i.e.}\quad z \longmapsto w = \frac{1}{\frac{1}{z} + \bar{a}} = \frac{z}{1 + z\bar{a}} \tag{26}$$

Expand in the $\bar{a} \to 0$ limit, we find that $w = z\,(1 - z\bar{a} - \cdots) \sim z - z^2\bar{a}$, i.e. it is generated by:

$$K_{\bar{z}} = -z^2\partial_z = -z^2\partial, \quad \partial \equiv \partial_z \tag{27}$$

[2]See *Di Francesco et al*, and also `github.com/davidsd/ph229`.

Note that when considering non-holomorphic functions, we have to consider $(z, \bar{z})$ as *two* independent variables; hence the anti-holomorphic transformation $\bar{z} \mapsto \bar{w} = \frac{\bar{z}}{1+\bar{z}a} \sim \bar{z} - \bar{z}^2 a$ provides another degree of freedom, namely:

$$K_\mu \sim \left(K_{\bar{z}} = -z^2 \partial,\ K_z = -\bar{z}^2 \bar{\partial}\right), \qquad \partial \equiv \partial_z,\ \bar{\partial} \equiv \partial_{\bar{z}} \tag{28}$$

Similarly, for translation $z \mapsto z + a$ and its conjugate, we have $P_\mu \sim \left(P_z = \partial, P_{\bar{z}} = \bar{\partial}\right)$. However, dilation and rotation are both encoded in a complex rescaling $z \mapsto \lambda z,\ \lambda = re^{i\theta} \in \mathbb{C}$; we have:

$$z \mapsto \lambda z, \quad \lambda = re^{i\theta} \in \mathbb{C}, \quad \begin{array}{l} \delta r \longleftrightarrow D = z\,\partial + \bar{z}\,\bar{\partial}, \\ \delta\theta \longleftrightarrow M = i\left(z\,\partial - \bar{z}\,\bar{\partial}\right), \end{array} \tag{29}$$

In summary, we have $\mathrm{span}_{\mathbb{R}}\{P_\mu, K_\mu, D, M\} = \mathfrak{so}(3,1)$ generating the "global" transformation subgroup of the 2D conformal group; here, the $\mathfrak{so}(3,1)$ boost is a linear combination[3] of P_μ and K_μ. More specifically, in 2D any holomorphic or anti-holomorphic function gives a conformal transformation, hence the (classical) 2D conformal group is generated by:

$$\ell_m = z^{m+1}\partial, \quad \bar{\ell}_m = \bar{z}^{m+1}\bar{\partial}, \quad m \in \mathbb{Z} \tag{30}$$

i.e. the *Witt algebra* (or Virasoro algbera $\mathbf{Vir}_c$ with $c = 0$). It is clear that a (complexified) $\mathfrak{so}(3,1)$ lives inside $\mathbf{Vir}_c$, i.e.,

$$\begin{aligned} \mathfrak{so}(3,1)^{\mathbb{C}} &= \mathrm{span}_{\mathbb{C}}\{P_\mu, K_\mu, D, M\} \\ &= \mathrm{span}_{\mathbb{C}}\left\{\ell_m, \bar{\ell}_m \,\middle|\, m = 0, \pm 1\right\} = \mathfrak{sl}(2,\mathbb{R})^{\mathbb{C}} \oplus_{\mathbb{C}} \mathfrak{sl}(2,\mathbb{R})^{\mathbb{C}} \subset \mathbf{Vir}_c \end{aligned} \tag{31}$$

■

$\boxed{4}$ *bc* **CFT**:

$$S = \frac{1}{2\pi}\int \mathrm{d}^2 z\, b\,\bar{\partial} c \tag{32}$$

Stress tensor of a theory can be obtained via variation over the metric, or equivalently, over the fields ϕ^i with $\delta\phi$ induced by some *local* spacetime translation $x^\mu \mapsto x^\mu + \delta x^\mu$, $\delta x^\mu = \epsilon(x)\, a^\mu$. Here $\epsilon(x)$ is any compactly supported bump function, centered around some point x_0.

In 2D, we have $\mu = z, \bar{z}$; for $\phi(z, \bar{z})$ with conformal weight $(h, \bar{h})$, consider $z \mapsto z'$, $\bar{z} \mapsto \bar{z}'$. For convenience, let's first consider a generic variation $\delta z = \epsilon(z, \bar{z})$ before restricting to spacetime translation; we have:

$$\phi'(z', \bar{z}') = \left(\frac{\mathrm{d}z'}{\mathrm{d}z}\right)^{-h}\left(\frac{\mathrm{d}\bar{z}'}{\mathrm{d}\bar{z}}\right)^{-\bar{h}}\phi(z, \bar{z}), \tag{33}$$

$$\tilde{\delta}\phi = \left(-h\,\partial\epsilon - \bar{h}\,\bar{\partial}\bar{\epsilon}\right)\phi, \tag{34}$$

$$\delta\phi = \tilde{\delta}\phi - \frac{\partial\phi}{\partial x^\mu}\,\delta x^\mu = \left(-h\,\partial\epsilon - \bar{h}\,\bar{\partial}\bar{\epsilon}\right)\phi - \epsilon\,\partial\phi - \bar{\epsilon}\,\bar{\partial}\phi, \tag{35}$$

Here we use $\tilde{\delta}\phi$ to denote the "internal" variation related to the conformal weights.

[3] See e.g. `github.com/davidsd/ph229`.

Note that $\phi = b, c$ are anti-commuting Grassmann numbers, variation of the action gives:

$$\begin{aligned}\delta S\,[b, c, \delta b\,, \delta c] &= \frac{1}{2\pi}\int \mathrm{d}^2 z\,\big(\delta b\;\bar\partial c + b\,\bar\partial\,\delta c\big)\\ &= \frac{1}{2\pi}\int \mathrm{d}^2 z\,\big(-\bar\partial c\;\delta b - \bar\partial b\;\delta c\big) + \frac{1}{2\pi}\int \mathrm{d}^2 z\,\bar\partial(b\,\delta c)\end{aligned}\tag{36}$$

For *unknown* b, c and arbitary $\delta b\,, \delta c$, the second term is reduced to a boundary term at infinity and can be dropped; imposing $\delta S = 0$ gives the equation of motion (EOM): $\bar\partial b = \bar\partial c = 0$.

On the other hand, for *on-shell* b, c and compactly supported $\phi = \delta b\,, \delta c$ given in (35), the first term in (36) vanishes while $\delta S_0 = 0$ still holds; this gives:

$$\begin{aligned}0 = \delta S_0 &= \frac{1}{2\pi}\int \mathrm{d}^2 z\,\bar\partial(b\,\delta c) = \frac{1}{2\pi}\int \mathrm{d}^2 z\,\bar\partial\,\big(-(1-\lambda)\,bc\,\partial\epsilon - b\,\partial c\,\epsilon\big)\\ &= \frac{1}{2\pi}\int \mathrm{d}^2 z\,\big(-(1-\lambda)\,bc\,\bar\partial\partial\epsilon - b\,\partial c\,\bar\partial\epsilon\big)\end{aligned}\tag{37}$$

Here we've distributed the $\bar\partial$ operator and dropped all terms that vanish automatically by EOM. Next we shall collect the $\partial\epsilon, \bar\partial\epsilon$ terms; integrating by parts on the first integrand gives:

$$\begin{aligned}0 = \delta S_0 &= \frac{1}{2\pi}\int \mathrm{d}^2 z\,\big((1-\lambda)\,\partial(bc)\; - b\,\partial c\big)\,\bar\partial\epsilon\\ &= \frac{1}{2\pi}\int \mathrm{d}^2 z\,\big((\partial b)\,c - \lambda\,\partial(bc)\big)\,\bar\partial\epsilon\\ &= -\frac{1}{2\pi}\int \mathrm{d}^2 z\,\epsilon(z,\bar z)\,\partial_{\bar z}\big((\partial b)\,c - \lambda\,\partial(bc)\big)\end{aligned}\tag{38}$$

Notice that we have obtained a conserved current using a generic $\delta z = \epsilon(z, \bar z), \delta\bar z = \bar\epsilon(z, \bar z)$; by setting $\epsilon = \epsilon(z)$, we get *a* energy momentum tensor[4]:

$$T(z) = \,:(\partial b)\,c:\, - \lambda\,\partial(:bc:)\tag{39}$$

Normal ordering is added manually to remove singular terms.

To compute TT OPE, we need the OPE of $b(z)\,c(0)$; this is obtained by examining the following path integral, which is zero since the integrand is a total functional derivative:

$$0 = \int \mathcal{D}b\,\mathcal{D}c\,\frac{\delta}{\delta\phi}\big(e^{-S}\,\psi\big)\tag{40}$$

Taking $\phi, \psi = b, c$, this generates operator equations such as $\bar\partial\,b(z)c(0) = 2\pi\delta^2(z, \bar z)$. Note that $\bar\partial(\frac{1}{z}) = 2\pi\delta^2(z, \bar z)$, which gives:

$$b(z)\,c(0) \sim c(z)\,b(0) \sim \frac{1}{z}, \quad b(z)\,b(0) \sim 0 \sim c(z)\,c(0)\tag{41}$$

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[4]Note that the energy momentum tensor obtained in this way is generally *not* unique: it can be off by a boundary term; see Luboš' comment at physics.stackexchange.com/a/96100, also arXiv:1601.03616. However, it is possible to fix this redundancy by considering Tb OPE and match its conformal dimension. I would like to thank 林般 for pointing this out.

With the bc OPE in hand, the TT OPE is computed directly with brute force, by repeatedly applying Wick's theorem. This gives:

$$T(z)\,T(0) \sim \frac{-6\lambda^2+6\lambda-1}{z^4}+\cdots \tag{42}$$

In general we have $-6\lambda^2+6\lambda-1=\frac{c}{2}$; for $\lambda=2$ this gives $c=-26$. ■

5 Free Fermion CFT:

$$S=\int \mathrm{d}^2z\,\psi_i\,\bar\partial\psi^i, \quad \psi^i=\psi_i^*, \quad \psi_i=\psi_i(z) \tag{43}$$

(a) Mode expansion of such chiral fermion is given by:

$$\psi_i=\sum_{k\in\mathbb{Z}+\frac{1}{2}}\frac{b_{ik}}{z^{k+\frac{1}{2}}}, \quad b_{ik}=\frac{1}{2\pi i}\oint \mathrm{d}z\,z^{k-\frac{1}{2}}\psi_i \tag{44}$$

Canonical quantization is achieved by simply imposing anti-commutation relations; this is justified by mapping the system onto a cylinder, then b_{ik}'s indeed map to modes on the spatial circle[5]. The only non-zero commutators are:

$$\left\{b_{ik},b_q^{j\dagger}\right\}=\delta_{k+q,0}\,\delta_i^j \tag{45}$$

This gives the only non-zero 2-point functions:

$$\begin{aligned}\langle\psi_i(z)\,\psi^j(w)\rangle &= \sum_{k,q\in\mathbb{Z}+\frac{1}{2}}\frac{1}{z^{k+\frac{1}{2}}}\frac{1}{w^{q+\frac{1}{2}}}\langle b_{ik}b_q^{j\dagger}\rangle \\ &= \sum_{k,q\in\mathbb{Z}+\frac{1}{2}}\frac{1}{z^{k+\frac{1}{2}}}\frac{1}{w^{q+\frac{1}{2}}}\,\langle 0|\left\{b_{ik},b_q^{j\dagger}\right\}|0\rangle=\frac{\delta_i^j}{z-w}\end{aligned} \tag{46}$$

Note that $b_k^i\,|0\rangle=0,\ \forall\ k\geq\frac{1}{2}$.

(b)(c) Combining two ψ expansions gives the mode expansion of $J_i{}^j=\,:\psi_i(z)\,\psi^j(z):$, namely:

$$J_i{}^j(z)=\sum_{k\in\mathbb{Z}}\frac{(J_i{}^j)_k}{z^{k+1}}, \quad (J_i{}^j)_k=\sum_{q\in\mathbb{Z}+\frac{1}{2}}:b_{iq}\,b_{k-q}^{j\dagger}: \tag{47}$$

It is in fact more convenient to obtain the JJ OPE first, and then use it to find the $[J_0,J_0]$ mode commutator[6]; note that $\psi_i(z)\,\psi^j(w)$ contraction gives $\frac{\delta_i^j}{z-w}$, we have:

$$J_i{}^j(z)\,J_k{}^l(0)\sim\frac{\delta_i^l\delta_k^j}{z^2}+\frac{\delta_k^jJ_i{}^l(0)-\delta_i^lJ_k{}^j(0)}{z}, \tag{48}$$

$$\left[(J_i{}^j)_0,(J_k{}^l)_0\right]=\frac{1}{(2\pi i)^2}\oint_0\mathrm{d}w\oint_w\mathrm{d}z\,J_i{}^j(z)\,J_k{}^l(w)=\delta_i^l\,(J_k{}^j)_0-\delta_k^j\,(J_i{}^l)_0 \tag{49}$$

[5]This can be proven rigorously by considering operator equations like in the bc CFT problem.

[6]I would like to thank 谷夏 for providing this hint.

(d) Similar to *bc* CFT, we have:

$$T(z) = \frac{1}{2}\left(:\psi_i\,\partial\psi^i: - :\partial\psi_i\,\psi^i:\right), \quad T(z)\,T(w) \sim \frac{n/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{50}$$

With each (complex) field contributing $\frac{1}{2} \times 2$ central charge[7].

(e) For real fermions, there is an additional reality condition:

$$\psi^i = \psi_i^\star = \psi_i \tag{51}$$

The canonical quantization still holds without the extra adjoint, same as the 2-point function:

$$\langle\psi_i(z)\,\psi_j(w)\rangle = \frac{\delta_{ij}}{z-w} \tag{52}$$

Similar holds for $J_{ij} = :\psi_i\psi_j:$ and its OPE, but we no longer need to distinguish upper/lower indices; we have:

$$J_{ij}(z)\,J_{kl}(0) \sim \frac{-\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{z^2} + \frac{-\delta_{ik}J_{jl}(0) + \delta_{il}J_{jk}(0) + \delta_{jk}J_{il}(0) - \delta_{jl}J_{ik}(0)}{z} \tag{53}$$

$$[(J_{ij})_0, (J_{kl})_0] = -\delta_{ik}(J_{jl})_0 + \delta_{il}(J_{jk})_0 + \delta_{jk}(J_{il})_0 - \delta_{jl}(J_{ik})_0 \tag{54}$$

This is precisely the $\mathfrak{o}(n)$ algebra. ■

[7] In fact a complex (Dirac) fermion can be "treated like" (*dual to*) a boson; this is *bosonization*.