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1 Symmetry & Noether's Theorem

1.1 2D σ -Model

$$\mathcal{L} = -\frac{1}{2} \eta_{\alpha\beta} \eta_{\mu\nu} \partial^{\alpha} X^{\mu} \partial^{\beta} X^{\nu} = -\frac{1}{2} \partial^{\alpha} X_{\mu} \partial_{\alpha} X^{\mu}, \quad X^{\mu} \in \mathbb{R}^{1,D-1}$$
 (1)

• For $\delta X^{\mu} = a^{\mu} + \lambda^{\mu}_{\ \nu} X^{\nu}$, the Lagrangian (density) transforms as follows:

$$\delta \mathcal{L} = -\partial^{\alpha} X_{\mu} \, \partial_{\alpha} \, \delta X^{\mu}
= -\partial^{\alpha} X_{\mu} \, \partial_{\alpha} (a^{\mu} + \lambda^{\mu}_{\nu} X^{\nu})
= -\partial^{\alpha} X_{\mu} \, (\partial_{\alpha} a^{\mu} + X^{\nu} \, \partial_{\alpha} \lambda^{\mu}_{\nu} + \lambda^{\mu}_{\nu} \, \partial_{\alpha} X^{\nu})
= -\partial^{\alpha} X_{\mu} \, \partial_{\alpha} a^{\mu} - \partial^{\alpha} X^{\mu} \, \partial_{\alpha} X^{\nu} \, \lambda_{\mu\nu} - X^{\nu} \, \partial^{\alpha} X^{\mu} \, \partial_{\alpha} \lambda_{\mu\nu}
= -\partial^{\alpha} X_{\mu} \, \partial_{\alpha} a^{\mu} - \partial^{\alpha} X^{\mu} \, \partial_{\alpha} X^{\nu} \, \lambda_{(\mu\nu)} - X^{\nu} \, \partial^{\alpha} X^{\mu} \, \partial_{\alpha} \lambda_{\mu\nu}$$
(2)

Since a^{μ} and λ^{μ}_{ν} are independent, imposing $\delta L = 0$ yields $\partial_{\alpha} a^{\mu} = 0$, a = const. Furthermore, if $\delta L = 0$ is to hold for arbitrary X^{μ} fields, then $\partial_{\alpha} \lambda_{\mu\nu} = 0$, $\lambda_{(\mu\nu)} = 0$, i.e. $\lambda_{\mu\nu}$ is constant and anti-symmetric over its indices.

• Promote $\delta X \mapsto \epsilon(x) \, \delta X = \epsilon(x) \, (a^{\mu} + \lambda^{\mu}_{\nu} X^{\nu})$, with $\epsilon(x)$ some localized bump function; using (2) and considering *on-shell* variation, we have:

$$0 = \delta S = -\int d^2 x \left(\partial^{\alpha} X_{\mu} a^{\mu} \partial_{\alpha} \epsilon + X^{\nu} \partial^{\alpha} X^{\mu} \lambda_{\mu\nu} \partial_{\alpha} \epsilon \right)$$
$$= -\int d^2 x \left(\partial^{\alpha} X_{\mu} a^{\mu} + X_{[\nu} \partial^{\alpha} X_{\mu]} \lambda^{[\mu\nu]} \right) \partial_{\alpha} \epsilon$$
(3)

It is evident (after partial integration) that the following currents are conserved; they are the Noether currents associated with a^{μ} and $\lambda^{[\mu\nu]}$:

$$j^{\alpha}_{\mu} = -\partial^{\alpha} X_{\mu}, \quad j^{\alpha}_{\mu\nu} = -X_{[\nu} \,\partial^{\alpha} X_{\mu]} = \frac{1}{2} \left(X_{\mu} \,\partial^{\alpha} X_{\nu} - X_{\nu} \,\partial^{\alpha} X_{\mu} \right) \tag{4}$$

Conserved charge $Q = \int d^2x \, j^0(x)$, we have:

$$P_{\mu} = -\int dx^{1} \,\partial^{0} X_{\mu} = \int dx^{1} \,\partial_{0} X_{\mu}, \quad M_{\mu\nu} = \frac{1}{2} \int dx^{1} \left(X_{\nu} \,\partial_{0} X_{\mu} - X_{\mu} \,\partial_{0} X_{\nu} \right) \tag{5}$$

They can be interpreted as spacetime momentum and spacetime angular momentum.

1.2 Real Scalar in (3+1) D

$$\mathcal{L} = -\frac{1}{2} \partial^{\mu} \phi \, \partial_{\mu} \phi - \frac{1}{2} \, m^2 \phi^2 \tag{6}$$

• For ϕ : scalar, under $x' = \lambda \circ x$, $\phi(x) \mapsto \phi'(x)$, while:

$$\phi'(x') = \phi(x) \implies \phi'(x) = \phi(\lambda^{-1} \circ x) \tag{7}$$

For $\lambda \sim \lambda^{\mu}_{\ \nu}$: Lorentz transformation, $\eta_{\mu\nu}\lambda^{\mu}_{\ \rho}\lambda^{\nu}_{\ \sigma} = \eta_{\rho\sigma}$, or equivalently, $(\lambda^{-1})^{\mu}_{\ \nu} = \lambda_{\nu}^{\ \mu}$. Therefore,

$$\phi'(x^{\mu}) = \phi(\lambda^{-1} \circ x^{\mu}) = \phi(x^{\nu} \lambda_{\nu}^{\ \mu}) \tag{8}$$

• Under $x'^{\mu} = \lambda^{\mu}_{\ \nu} x^{\nu}$, we have:

$$\mathcal{L}'(x') = -\frac{1}{2} \, \partial'^{\mu} \phi'(x') \, \partial'_{\mu} \phi'(x') - \frac{1}{2} \, m^2 \phi'^2(x')
= -\frac{1}{2} \, \partial'^{\mu} \phi(x) \, \partial'_{\mu} \phi(x) - \frac{1}{2} \, m^2 \phi^2(x)
= -\frac{1}{2} \, \eta^{\mu\nu} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \, \partial_{\rho} \phi(x) \, \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \, \partial_{\sigma} \phi(x) - \frac{1}{2} \, m^2 \phi^2(x)
= -\frac{1}{2} \, \eta^{\rho\sigma} \partial_{\rho} \phi(x) \, \partial_{\sigma} \phi(x) - \frac{1}{2} \, m^2 \phi^2(x)
= \mathcal{L}(x)$$
(9)

Here we've used $\eta^{\mu\nu} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} = \eta^{\mu\nu} \lambda_{\mu}^{\rho} \lambda_{\nu}^{\sigma} = \eta^{\rho\sigma}$. Furthermore, $S' = \int d^4x \, \mathcal{L}'(x) = \int d^4x' \, \mathcal{L}'(x') = \int d^4x' \, \mathcal{L}(x) = \int d^4x' \, \mathcal{L}(x) = \int d^4x' \, \mathcal{L}(x') = \int d$

• Consider an infinitesimal Lorentz transformation: $\lambda \sim 1 + \omega$, then $\eta_{\mu\nu}\lambda^{\mu}_{\ \rho}\lambda^{\nu}_{\ \sigma} = \eta_{\rho\sigma}$ implies that $\omega_{\mu\nu}$ is anti-symmetric: $\omega_{\mu\nu} + \omega_{\nu\nu} = 0$. For $\delta x^{\mu} = \omega^{\mu}_{\ \nu} x^{\nu}$, we have:

$$\delta\phi = -\frac{\partial\phi}{\partial x^{\mu}} \,\delta x^{\mu} = -\omega^{\mu}_{\ \nu} x^{\nu} \,\partial_{\mu}\phi \tag{10}$$

To obtain the corresponding Noether charges, we can simply repeat the operations done in our previous problem; alternatively, we can try to derive a general recipe¹: for $\mathcal{L} = \mathcal{L}(\phi, \partial_{\mu}\phi)$ and $S = \int d^4x \, \mathcal{L}$, we have:

$$\delta S = \int d^4 x \, \delta \mathcal{L}$$

$$= \int d^4 x \left(\frac{\partial \mathcal{L}}{\partial \phi} \, \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \, \delta \partial_\mu \phi \right)$$

$$= \int d^4 x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \delta \phi + \int d^4 x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \, \delta \phi \right)$$
(11)

If we vary S w.r.t. a symmetry of the system, we will have $\delta \mathcal{L} = \partial_{\mu} K^{\mu}$ some total derivative; when on-shell, such variation gives the conserved current with boundary term K^{μ} :

$$j^{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \,\delta\phi - K^{\mu} \tag{12}$$

¹References: arXiv:1601.03616 and *Tong:* http://damtp.cam.ac.uk/user/tong/qft.html

Back to our Lorentz transformation $\delta\phi = -\omega^{\mu}_{\ \nu}x^{\nu}\partial_{\mu}\phi$, we have symmetry variation:

$$\delta \mathcal{L} = -\omega^{\mu}_{\ \nu} x^{\nu} \partial_{\mu} \mathcal{L} = -\partial_{\mu} (\omega^{\mu}_{\ \nu} x^{\nu} \mathcal{L}) \tag{13}$$

This gives a boundary term $K^{\mu} = -\omega^{\mu}_{\nu} x^{\nu} \mathcal{L}$, and the Noether current and its corresponding conserved charge can be calculated as follows:

$$j^{\mu} = -\omega^{\sigma}_{\ \nu} x^{\nu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \, \partial_{\sigma} \phi - \delta^{\mu}_{\sigma} \mathcal{L} \right), \tag{14}$$

$$Q = \int d^3x \, j^0 = -\omega^{\sigma}_{\ \nu} \int d^3x \, x^{\nu} \left(\partial_0 \phi \, \partial_{\sigma} \phi - \delta^0_{\sigma} \mathcal{L} \right), \tag{15}$$

Note that $\omega^{\mu}_{\ \nu}$ is arbitrary, therefore Q can be decomposed into independent charges:

$$Q = \omega_{\mu\nu} M^{\mu\nu}, \quad M^{\mu\nu} = -\int d^3x \, x^{[\mu} \Big(\partial_0 \phi \, \partial^{\nu]} \phi - \eta^{\nu]0} \mathcal{L} \Big), \tag{16}$$

The indices of $M^{\mu\nu}$ are anti-symmetrized to match the degrees of freedom in $\omega_{\mu\nu}$. Note that the \mathcal{L} term only appears when one of the indices is 0.

Canonical quantization:

$$\dot{\phi} = \partial_0 \phi, \quad \Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}, \quad [\phi(\mathbf{x}), \Pi(\mathbf{y})] = [\phi, \dot{\phi}](\mathbf{x}) = i\delta(\mathbf{x} - \mathbf{y})$$
 (17)

Other equal-time commutators between ϕ , Π all just vanish. We have:

$$M^{0i} = -M^{i0} = -\frac{1}{2} \int d^3x \left(\dot{\phi} \left(x^0 \partial^i - x^i \partial^0 \right) \phi + x^i \mathcal{L} \right), M^{ij} = -\frac{1}{2} \int d^3x \, \dot{\phi} \left(x^i \partial^j - x^j \partial^i \right) \phi$$
(18)

Notice that $x^{[\mu}\partial^{\nu]} = \frac{1}{2}(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}) = \frac{1}{2}D^{ij}$ is the Killing vector fields of $\mathbb{R}^{3,1}$, hence they naturally follow the commutation relations of $\mathfrak{so}(3,1)$ (up to a constant coefficient)². We have:

$$[M^{ij}, M^{kl}] = \frac{1}{4} \int d^3x \int d^3y \left[\dot{\phi} D^{ij} \phi(x), \dot{\phi} D^{kl} \phi(y) \right]$$
$$= \frac{1}{4} \int d^3x \, \dot{\phi} \left[D^{ij}, D^{kl} \right] \phi$$
(19)

Similar holds for M^{i0} . Therefore, $M^{\mu\nu}$'s indeed form the Lie algebra $\mathfrak{so}(3,1)$.

 $^{^2\}mathrm{I}$ would like to thank 林般 for pointing this out.