# String Theory №5

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## 1 Partition Function for Compact Scalars

(a) Mode expansion of $X$ CFT is[1]:

$$\partial X(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\frac{\alpha_m}{z^{m+1}}, \quad \bar{\partial} X(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\frac{\tilde{\alpha}_m}{\bar{z}^{m+1}}, \tag{1}$$

$$X = x - i\sqrt{\frac{\alpha'}{2}}\left(\alpha_0 \ln z + \tilde{\alpha}_0 \ln \bar{z}\right) + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\left(\frac{\alpha_m}{z^m} + \frac{\tilde{\alpha}_m}{\bar{z}^m}\right), \tag{2}$$

Momentum $p$ is the charge for *spacetime* translation; we have:

$$X \mapsto X + \text{const}, \quad j_a = \frac{i}{\alpha'}\partial_a X, \tag{3}$$

$$p = \frac{1}{2\pi i}\oint_C \left(\mathrm{d}z\, j - \mathrm{d}\bar{z}\,\tilde{j}\right) = \frac{1}{\alpha'}\sqrt{\frac{\alpha'}{2}}\left(\alpha_0 + \tilde{\alpha}_0\right) = \sqrt{\frac{1}{2\alpha'}}\left(\alpha_0 + \tilde{\alpha}_0\right) \tag{4}$$

Additionally, for compact free boson, $X$ is only defined modulo $2\pi R$; therefore, states after $X + 2\pi R$ translation should be identical to the original states, i.e.

$$e^{ip\,(2\pi R)} = \mathbb{1}, \quad p = \frac{n}{R}, \quad n \in \mathbb{Z} \tag{5}$$

This, in fact, holds for any field theory[2] defined for $X \in S^1$, including the ordinary quantum mechanics (a classical field theory) on $S^1$.

On the other hand, there are additional constraints in string theory: for the state of a *single* closed string, there is a discrete translational symmetry on the *worldsheet*:

$$X(\sigma^1 + 2\pi) \cong X(\sigma^1), \quad X(\sigma^1 + 2\pi) = X(\sigma^1) + 2\pi R w, \quad w \in \mathbb{Z} \tag{6}$$

With some definite winding number $w$. In $(z, \bar{z})$ coordinates, we have:

$$2\pi R w = X\left(z\, e^{2\pi i}, \bar{z}\, e^{-2\pi i}\right) - X(z, \bar{z}) = -i\sqrt{\frac{\alpha'}{2}}\, 2\pi i\left(\alpha_0 - \tilde{\alpha}_0\right) = 2\pi\sqrt{\frac{\alpha'}{2}}\left(\alpha_0 - \tilde{\alpha}_0\right), \tag{7}$$

$$p = \frac{p_L + p_R}{2}, \quad p_L = \sqrt{\frac{2}{\alpha'}}\,\alpha_0, \quad p_R = \sqrt{\frac{2}{\alpha'}}\,\tilde{\alpha}_0, \tag{8}$$

$$p_{L,R} = \frac{n}{R} \pm \frac{wR}{\alpha'}, \tag{9}$$

$$X = x - i\,\frac{\alpha'}{2}\left(p_L \ln z + p_R \ln \bar{z}\right) + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\left(\frac{\alpha_m}{z^m} + \frac{\tilde{\alpha}_m}{\bar{z}^m}\right), \tag{10}$$

[1] Again we follow the convention of *Polchinski*.

[2] Reference: discussions in *Polchinski*, Chapter 8.

For the oscillator expressions for $L_0$, recall that:

$$T(z) = -\frac{1}{\alpha'} :\partial X\, \partial X: = \sum_m \frac{L_m}{z^{m+2}}, \tag{11}$$

$$L_{m\neq 0} = \frac{1}{2}\sum_l \alpha_{m-l}\alpha_l, \quad L_0 = \frac{1}{2} :\sum_l \alpha_{-l}\alpha_l: \sim \frac{\alpha' p_L^2}{4} + \sum_{l>0} \alpha_{-l}\alpha_l, \tag{12}$$

The $L_0$ expression may be off by some normal ordering constant; this ambiguity can be resolved by considering:

$$2L_0\,|0,0;n=w=0\rangle = (L_1L_{-1} - L_{-1}L_1)\,|0,0;p_L=p_R=0\rangle = 0-0=0 \tag{13}$$

Therefore the normal ordering constant is, in fact, trivial, and we have:

$$L_0 = \frac{\alpha' p_L^2}{4} + \sum_{l>0}\alpha_{-l}\alpha_l, \quad \tilde{L}_0 = \frac{\alpha' p_R^2}{4} + \sum_{l>0}\tilde{\alpha}_{-l}\tilde{\alpha}_l, \tag{14}$$

(b) The torus partition function is given by:

$$\langle \mathbb{1}\rangle_{T^2} \equiv Z(\tau = \tau_1 + i\tau_2) = \int \mathcal{D}X e^{-S} = \mathrm{Tr}\, e^{-(2\pi\tau_2)H} e^{i\,(2\pi\tau_1)P} \tag{15}$$

Here $P$ generates *worldsheet* translation along $\sigma^1$, not to be confused with $p$ which generates *spacetime* translation; with $z = e^{-iw},\ w = \sigma^1 + i\sigma^2$,

$$\begin{aligned} T^0{}_1 &= \eta^{00}\,(\partial_0\sigma^2)\,T_{21} = -iT_{12} = -i\left(T_{ww}\,(\partial_1 w)(\partial_2 w) + T_{\bar{w}\bar{w}}\,(\partial_1\bar{w})(\partial_2\bar{w})\right) \\ &= T_{ww} - T_{\bar{w}\bar{w}} \\ &= \left(T_{zz}\,(\partial_w z)^2 + \tfrac{c}{24}\right) - \left(T_{\bar{z}\bar{z}}\,(\partial_{\bar{w}}\bar{z})^2 + \tfrac{\tilde{c}}{24}\right) \\ &= T(z)\,(-iz)^2 - \tilde{T}(\bar{z})\,(+i\bar{z})^2 + \frac{c-\tilde{c}}{24}, \end{aligned} \tag{16}$$

$$\begin{aligned} P = \int \frac{\mathrm{d}\sigma_1}{2\pi}\left(-T^0{}_1\right) &= -\int \frac{\mathrm{d}\sigma_1}{2\pi}\,T(z)\,(-iz)^2 + \int \frac{\mathrm{d}\sigma_1}{2\pi}\,\tilde{T}(\bar{z})\,(+i\bar{z})^2 - \frac{c-\tilde{c}}{24} \\ &= +\oint \frac{\mathrm{d}z}{2\pi\,(-iz)}\,T(z)\,(-iz)^2 + \oint \frac{\mathrm{d}\bar{z}}{2\pi\,(+i\bar{z})}\,\tilde{T}(\bar{z})\,(+i\bar{z})^2 - \frac{c-\tilde{c}}{24} \\ &= \oint \frac{\mathrm{d}z}{2\pi i}\,zT(z) - \oint \frac{\mathrm{d}\bar{z}}{2\pi i}\,\bar{z}\tilde{T}(\bar{z}) - \frac{c-\tilde{c}}{24} \\ &= L_0 - \tilde{L}_0 - \frac{c-\tilde{c}}{24} \\ &= \left(L_0 - \tfrac{c}{24}\right) - \left(\tilde{L}_0 - \tfrac{\tilde{c}}{24}\right), \end{aligned} \tag{17}$$

$$\begin{aligned} H = \int \frac{\mathrm{d}\sigma_1}{2\pi}\,T^0{}_0 &= \int \frac{\mathrm{d}\sigma_1}{2\pi}\,T_{22} \\ &= L_0 + \tilde{L}_0 - \frac{c+\tilde{c}}{24} \\ &= \left(L_0 - \tfrac{c}{24}\right) + \left(\tilde{L}_0 - \tfrac{\tilde{c}}{24}\right), \end{aligned}$$

Here we've used the fact that $\oint \frac{\mathrm{d}\bar{z}}{\bar{z}} = \oint \frac{\mathrm{d}\bar{z}}{\bar{z}} = 2\pi i$. Therefore,

$$Z(\tau) = \mathrm{Tr}\, e^{-(2\pi\tau_2)H} e^{i\,(2\pi\tau_1)P} = \mathrm{Tr}\, q^{L_0 - \frac{c}{24}}\,\bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}}, \quad q = e^{2\pi i\tau} \tag{18}$$

Using the expressions in (a), we find that $L_0$ action on a state $|\psi\rangle$ created by $\alpha_{-l}, \tilde{\alpha}_{-l}$ yields the sum of occupation numbers $N_l$ weighted by $l$:

$$L_0 \, |\psi\rangle = \left( \frac{\alpha' k_L^2}{4} + \sum_{l>0} l \cdot N_l \right) |\psi\rangle \tag{19}$$

With $c = \tilde{c} = 1$, we obtain:

$$\begin{aligned} Z(\tau) &= (q\bar{q})^{-\frac{1}{24}} \sum_{n,w} e^{-2\pi\tau_2\alpha' \frac{k_L^2+k_R^2}{4}} e^{2\pi i\tau_1\alpha' \frac{k_L^2-k_R^2}{4}} \sum_{(N_l),(\tilde{N}_l)} q^{\sum_{l>0} l\cdot N_l} \bar{q}^{\sum_{l>0} l\cdot \tilde{N}_l} \\ &= (q\bar{q})^{-\frac{1}{24}} \sum_{n,w} e^{-\pi\tau_2\left(\frac{\alpha' n^2}{R^2} + \frac{w^2R^2}{\alpha'}\right) + 2\pi i\tau_1 nw} \sum_{(N_l),(\tilde{N}_l)} \prod_{l>0} q^{l\cdot N_l} \bar{q}^{l\cdot \tilde{N}_l} \\ &= |\eta(\tau)|^{-2} \sum_{n,w} e^{-\pi\tau_2\left(\frac{\alpha' n^2}{R^2} + \frac{w^2R^2}{\alpha'}\right) + 2\pi i\tau_1 nw} \end{aligned} \tag{20}$$

We've simplified the contributions from the oscillator modes using $\eta(\tau)$, since they are identical to the oscillator contributions of the non-compact $X \in \mathbb{R}^1$:

$$\begin{aligned} (q\bar{q})^{-\frac{1}{24}} \sum_{(N_l),(\tilde{N}_l)} \prod_{l>0} q^{l\cdot N_l} \bar{q}^{l\cdot \tilde{N}_l} &= (q\bar{q})^{-\frac{1}{24}} \prod_{l>0} \sum_{N_l,\tilde{N}_l=0}^{\infty} q^{l\cdot N_l} \bar{q}^{l\cdot \tilde{N}_l} \\ &= (q\bar{q})^{-\frac{1}{24}} \prod_{l>0} \frac{1}{1-q^l} \frac{1}{1-\bar{q}^l} = |\eta(\tau)|^{-2} \end{aligned} \tag{21}$$

In the $R \to \infty$ limit, only the $w = 0$ modes survive; all other modes are exponentially suppressed by the $e^{-\pi\tau_2 w^2 R^2/\alpha'}$ factor; i.e.

$$\begin{aligned} Z(\tau) &= |\eta(\tau)|^{-2} \sum_{n,w} \exp\left\{ -\pi\tau_2 \left( \frac{\alpha' n^2}{R^2} + \frac{w^2R^2}{\alpha'} \right) + 2\pi i\tau_1 nw \right\} \\ &\to |\eta(\tau)|^{-2} \sum_{n} \exp\left\{ -\pi\tau_2 \frac{\alpha' n^2}{R^2} \right\}, \quad k = \frac{n}{R} \\ &\to |\eta(\tau)|^{-2} \, V \int \frac{\mathrm{d}k}{2\pi} \exp\left\{ -\pi\tau_2 \, \alpha' k^2 \right\} \\ &= V \, |\eta(\tau)|^{-2} \left(4\pi^2\alpha'\tau_2\right)^{-\frac{1}{2}} \\ &\equiv V \cdot Z_X(\tau) = 2\pi R \, Z_X(\tau) \end{aligned} \tag{22}$$

We recover the partition function $V \cdot Z_X(\tau)$ for non-compact $X$, as expected.

(c) Using the Poisson resummation formula, we find that:

$$Z(\tau) = 2\pi R \, Z_X(\tau) \sum_{m,w} \exp\left( -\frac{\pi R^2 |m - w\tau|^2}{\alpha'\tau_2} \right) \tag{23}$$

$Z_X(\tau)$ is modular invariant by the properties of the Dedekind $\eta(\tau)$ function, as is demonstrated for the non-compact $X$ in *Polchinski*.

The sum, on the other hand, is naturally invariant under $T\colon \tau \mapsto \tau + 1$, by making a change of variables $m \mapsto m + w$. It is also invariant under $S\colon \tau \mapsto -1/\tau$ with $m \mapsto -w, w \mapsto m$[3]. Therefore, $Z(\tau)$ is modular invariant.

## $\boxed{2}$ $\mathbb{Z}_2$ Orbifold

The $\mathbb{Z}_2$ orbifold is constructed by imposing an additional identification on $X \in S^1$:

$$X \cong -X \tag{24}$$

The target space is then reduced to $S^1/\mathbb{Z}_2 \cong [0, \pi R]$.

(a) The first contributions to the orbifold partition function comes from the states that are invariant reflection $r$; we have:

$$\mathrm{Tr}_{S^1/\mathbb{Z}_2} = \mathrm{Tr}_{S^1} \frac{1+r}{2} = \frac{1}{2}\,\mathrm{Tr}_{S^1} + \frac{1}{2}\,\mathrm{Tr}_{S^1} \circ r \tag{25}$$

Acting on $q^{L_0 - \frac{c}{24}}\,\bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}}$, the first term gives $\frac{1}{2}\,Z_{S^1}(\tau)$ where $Z_{S^1}$ is the $S^1$ partition function we've obtained in $\boxed{1}$.

For the second term, note that:

$$r\colon \quad \Big|(N_l), (\tilde{N}_l); n, w\Big\rangle \longmapsto (-1)^{\sum_l (N_l + \tilde{N}_l)} \Big|(N_l), (\tilde{N}_l); -n, -w\Big\rangle \tag{26}$$

In particular, it reverses $n, w$, hence $r$ insertion gives vanishing amplitude unless $n = w = 0$; the summation is very much similar to the $Z_{S^1}$ case, i.e. we have:

$$\begin{aligned} \frac{1}{2}\,\mathrm{Tr}_{S^1}\Big(r\, q^{L_0 - \frac{c}{24}}\,\bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}}\Big) &= \frac{1}{2}\,(q\bar{q})^{-\frac{1}{24}} \prod_{l>0} \sum_{N_l, \tilde{N}_l = 0}^{\infty} (-1)^{N_l + \tilde{N}_l}\, q^{l \cdot N_l}\, \bar{q}^{\,l \cdot \tilde{N}_l} \\ &= \frac{1}{2}\,(q\bar{q})^{-\frac{1}{24}} \prod_{l>0} \frac{1}{1-(-q^l)} \frac{1}{1-(-\bar{q}^{\,l})} = \left|\frac{\eta(\tau)}{\theta_2(\tau)}\right| \end{aligned} \tag{27}$$

Where we've used the fact that[4]: $q^{-\frac{1}{24}} \prod_{l>0} \frac{1}{1-(-q^l)} = \sqrt{2}\sqrt{\frac{\eta(\tau)}{\theta_2(\tau)}}$. Therefore, the total contributions from $r$–invariant states are:

$$\frac{1}{2}\,Z_{S^1}(\tau) + \left|\frac{\eta(\tau)}{\theta_2(\tau)}\right| \tag{28}$$

(b) With $X \cong -X$, new possibilities emerge as the boundary condition along $\sigma^1$:

$$X(\sigma^1 + 2\pi) \cong X(\sigma^1), \quad X(\sigma^1 + 2\pi) = \pm X(\sigma^1) + 2\pi R w, \quad w \in \mathbb{Z} \tag{29}$$

The "$-$" sign corresponds to the *twisted states*. Due to the anti-periodicity, $\partial X$ has a half-integer mode expansion:

$$\partial X\big(z\, e^{2\pi i}\big) = -\partial X(z), \tag{30}$$

[3]Reference: *Polchinski*.

[4]Reference: Blumenhagen & Plauschinn, *Introduction to CFT*, and also *Polchinski*.

$$\partial X(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\frac{\alpha_{m-\frac{1}{2}}}{z^{m+\frac{1}{2}}}, \quad \bar{\partial}X(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\frac{\tilde{\alpha}_{m-\frac{1}{2}}}{\bar{z}^{m+\frac{1}{2}}}, \tag{31}$$

$$X = x + i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\frac{1}{m+\frac{1}{2}}\left(\frac{\alpha_{m+\frac{1}{2}}}{z^{m+\frac{1}{2}}} + \frac{\tilde{\alpha}_{m+\frac{1}{2}}}{\bar{z}^{m+\frac{1}{2}}}\right), \tag{32}$$

Apply the boundary condition on $X$, and we find that $x = \pi R w'$; however, due to the identification $X + 2\pi R \cong X \cong -X$, there are only two inequivalent choices: $x = 0$ and $x = \pi R$, which correspond to the string localized around either of the two fixed points of the $\mathbb{Z}_2$ action.

Much similar to the case in $\boxed{1}$, we have:

$$\left[\alpha_{\frac{1}{2}+l}, \alpha_{-\frac{1}{2}-l}\right] = \frac{1}{2} + l, \tag{33}$$

$$L_{m\neq 0} = \frac{1}{2}\sum_{l}\alpha_{m-\frac{1}{2}-l}\alpha_{\frac{1}{2}+l}, \quad L_0 = \frac{1}{2}:\sum_{l}\alpha_{-\frac{1}{2}-l}\alpha_{\frac{1}{2}+l}: \sim \sum_{l\geq 0}\alpha_{-\frac{1}{2}-l}\alpha_{\frac{1}{2}+l} \tag{34}$$

We can use the same trick to fix the normal ordering constant in $L_0$; this time it is non-trivial:

$$L_{-1} = \frac{1}{2}\alpha^2_{-\frac{1}{2}} + \sum_{l\geq 0}\alpha_{-\frac{1}{2}-l}\alpha_{\frac{1}{2}+l}, \quad L_1 = \frac{1}{2}\alpha^2_{\frac{1}{2}} + \sum_{l>0}\alpha_{\frac{1}{2}-l}\alpha_{\frac{1}{2}+l}, \tag{35}$$

$$\begin{aligned} L_0\left|0,0;x\right\rangle &= \frac{1}{2}\left(L_1L_{-1} - L_{-1}L_1\right)\left|0,0;x\right\rangle \\ &= \frac{1}{2}\times\frac{1}{4}\alpha^2_{\frac{1}{2}}\alpha^2_{-\frac{1}{2}}\left|0,0;x\right\rangle - 0 \\ &= \frac{1}{16}\left|0,0;x\right\rangle, \end{aligned} \tag{36}$$

$$L_0 = \frac{1}{16} + \sum_{l\geq 0}\alpha_{-\frac{1}{2}-l}\alpha_{\frac{1}{2}+l} = \frac{1}{16} + \sum_{l\geq 0}\left(l+\frac{1}{2}\right)N_{l+\frac{1}{2}} = \frac{1}{16} + \sum_{l>0}\left(l-\frac{1}{2}\right)N_{l-\frac{1}{2}}, \tag{37}$$

The trace can then be computed, following the same recipe as before:

$$\begin{aligned} \mathrm{Tr}_{S^1}\left(\frac{1+r}{2}q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}\right) &= (q\bar{q})^{-\frac{1}{24}+\frac{1}{16}}\prod_{l+\frac{1}{2}\in\mathbb{Z}^+}\sum_{N_l,\tilde{N}_l=0}^{\infty}\frac{1+(-1)^{N_l+\tilde{N}_l}}{2}q^{l\cdot N_l}\bar{q}^{l\cdot\tilde{N}_l}\times 2 \\ &= \frac{1}{2}(q\bar{q})^{+\frac{1}{48}}\left\{\prod_{l>0}\left|\frac{1}{1-q^{l-\frac{1}{2}}}\right|^2 + \prod_{l>0}\left|\frac{1}{1+q^{l-\frac{1}{2}}}\right|^2\right\}\times 2 \\ &= \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right| \end{aligned} \tag{38}$$

There is an extra factor of 2 from the number of twisted sectors: $x = 0$ and $x = \pi R$.

(c) The full partition function is therefore:

$$Z(\tau) = \frac{1}{2}Z_{S^1}(\tau) + \left|\frac{\eta(\tau)}{\theta_2(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right| \tag{39}$$

The first term is modular invariant, as is proved in $\boxed{1}$.

The remaining terms are also modular invariant, due to the transformational properties of $\eta$ and $\theta$ functions[5]:

$$T \circlearrowright \left|\frac{\eta(\tau)}{\theta_2(\tau)}\right| \overset{S}{\longleftrightarrow} \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right| \overset{T}{\longleftrightarrow} \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right| \circlearrowleft S \tag{40}$$

Therefore, the full partition function is modular invariant.

## $\boxed{3}$ Torus 4-point function in $bc$ CFT

$$\left\langle c(w_1)\, b(w_2)\, \tilde{c}(\bar{w}_3)\, \tilde{b}(\bar{w}_4)\right\rangle = \int \mathcal{D}b\,\mathcal{D}\tilde{b}\,\mathcal{D}c\,\mathcal{D}\tilde{c}\; c(w_1)\, b(w_2)\, \tilde{c}(\bar{w}_3)\, \tilde{b}(\bar{w}_4)\, e^{-S'} \equiv Z' \tag{41}$$

First we argue that only the zero modes of the insertions survive the path integral[6]. In fact, as anti-commuting replacements of the gauge degrees of freedom, ghost modes are *defined* to be the eigenvalues of $P^\dagger P$, where $P$ is the conformal Killing differential[7]. More specifically, given a conformal Killing vector (CKV) $\delta\sigma^a$, the conformal Killing equation can be written as:

$$P\,\delta\sigma = 0 \tag{42}$$

While $P^\dagger \delta' g = 0$ gives moduli variation $\delta' g_{ab}$ of the metric. Roughly speaking, $P$ captures the variation of gauge fixing under an arbitrary gauge transformation; naturally, CKV's are given by $(\ker P)$, while $(\det P) \sim \Delta_{FP}$ is the Faddeev–Popov functional measure near the gauge slice. $(\det P)$ can then be calculated with:

$$\delta\sigma^a \mapsto c^a, \quad \delta' g_{ab} \mapsto b_{ab}, \quad \Delta_{FP} \sim \det P \sim \int \mathcal{D}b\,\mathcal{D}\tilde{b}\,\mathcal{D}c\,\mathcal{D}\tilde{c}\; e^{-S'}, \tag{43}$$

$$S' = \frac{1}{2\pi}\int \mathrm{d}^2\sigma\, g^{1/2}\, b_{ab}\,(P\cdot c)^{ab} = \frac{1}{2\pi}\int \mathrm{d}^2 w\,\left(b\,\bar{\partial}_w c + \tilde{b}\,\partial_w \tilde{c}\right) \tag{44}$$

In the end we have chosen conformal gauge, such that[8] $P \sim (\bar{\partial}_w, \partial_w)$, $P^\dagger P \sim -\bar{\partial}_w \partial_w = -\nabla^2$. In the $w = \sigma^1 + i\sigma^2$ coordinates, CKV's are simple translations: $c^a = \text{const}$; with $z = e^{-iw}$, it gets mapped to $c^z = c^w\,\partial_w z = c^w\,(-iz)$, which agrees with the zero mode $c_0$ in the $c(z)$ expansion:

$$c(z) = \sum_{m=-\infty}^{\infty} \frac{c_m}{z^{m+1-\lambda}} = c_0\, z + \sum_{m\neq 0} \frac{c_m}{z^{m-1}}, \quad \lambda = 2 \tag{45}$$

Now we are finally ready to prove our argument: for anti-commuting variables like $c(z)$,

$$\int \mathcal{D}c \sim \prod_m \int \mathrm{d}c_m \sim \prod_m \frac{\partial}{\partial c_m} \tag{46}$$

Since $c_0$ corresponds to a CKV, $P\cdot c_0 = 0$, therefore it vanishes in $S' = \int \mathrm{d}^2\sigma\,(b\cdot P\cdot c)$; for the path integral to be non-zero, there has to be some additional $c_0$ insertions, i.e.

$$Z' \sim \int \mathcal{D}b\,\mathcal{D}\tilde{b}\,\mathcal{D}c\,\mathcal{D}\tilde{c}\; c_0 b_0 \tilde{c}_0 \tilde{b}_0\, e^{-S'} \sim \left(\frac{1}{\sqrt{\tau_2}}\right)^4 \int \mathcal{D}'b\,\mathcal{D}'\tilde{b}\,\mathcal{D}'c\,\mathcal{D}'\tilde{c}\; e^{-S'}, \quad \int \mathcal{D}'c \sim \prod_{m\neq 0}\int \mathrm{d}c_m \tag{47}$$

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[5] Reference: *Blumenhagen & Plauschinn.*

[6] I would like to thank 谷夏 for some very helpful discussions about this problem.

[7] Reference: *Polchinski*, Chapter 3 & 5.

[8] References:

- Nakahara, *Geometry, Topology and Physics*;
- Blumenhagen et al, *Basic Concepts of String Theory.*

Note the additional $\left(\frac{1}{\sqrt{\tau_2}}\right)^4$ factor coming from the zero modes[9]; this has to do with the normalization of the zero modes, each contributing a factor of $\frac{1}{\sqrt{A}}$, where $A \sim \tau_2$ is the volume (surface area) of the torus. On a different note, since it is very difficult, if not impossible, to keep track of various (often divergent) constant factors in the path integral, we have been and will be calculating $Z'$ up to an overall constant coefficient.

Now we have to deal with the path integral over non-zero modes. Note that the holomorphic mode expansion (45) is incomplete for our purpose: it gives the *on-shell* mode expansion, while our path integral should go over all possible configurations, including the off-shell modes, which is *not* holomorphic. However, on $T^2 = S^1 \times S^1$, the full modes are simple[10]:

$$-\nabla^2 \psi_{n_1,n_2} = \lambda_{n_1,n_2} \psi_{n_1,n_2}, \tag{48}$$

$$\begin{aligned} \psi_{n_1,n_2} &= \exp\left(i\left(n_1\tilde{\sigma}^1 + n_2\tilde{\sigma}^2\right)\right), \quad \tilde{\sigma}^2 = \frac{\sigma^2}{\tau_2}, \quad \tilde{\sigma}^1 = \sigma^1 - \sigma^2\frac{\tau_1}{\tau_2}, \\ &= \exp\left\{i\left(n_1\sigma^1 + \frac{n_2 - n_1\tau_1}{\tau_2}\,\sigma^2\right)\right\}, \end{aligned} \tag{49}$$

Here we first use the "rectangular" coordinates $(\tilde{\sigma}^1, \tilde{\sigma}^2) \in [0, 2\pi]^2$ to write down the obvious eigenfunctions $\psi_{n_1,n_2}$, and then relate them back to the $(\sigma^1, \sigma^2)$ coordinates. Therefore, we have:

$$\begin{aligned} \lambda_{n_1,n_2} &= \left\{n_1^2 + \left(\frac{n_2 - n_1\tau_1}{\tau_2}\right)^2\right\} \\ &= \frac{1}{\tau_2^2}\left\{(n_1\tau_2)^2 + \left(n_1\tau_1 - n_2\right)^2\right\} \\ &= \frac{1}{\tau_2^2}\left|n_1\tau - n_2\right|^2, \end{aligned} \tag{50}$$

$$\det{}' P \sim \left(\prod_{n_1,n_2}{}' \sqrt{\lambda_{n_1,n_2}}\right)^2 \sim \prod_{n_1,n_2}{}' \lambda_{n_1,n_2} \tag{51}$$

The determinant can be computed with $\zeta$-function regularization, as is performed in detail in *Di Francesco*; the result can be nicely summarized using the Eisenstein series, as shown in *Nakahara*:

$$E(\tau, s) = \sum_{n_1,n_2}{}' \frac{\tau_2^s}{\left|n_1\tau - n_2\right|^{2s}}, \tag{52}$$

$$\det{}' P \sim \prod_{n_1,n_2}{}' \frac{1}{\tau_2^2}\left|n_1\tau - n_2\right|^2 \sim \tau_2 \exp\left\{-\partial_s E'(\tau, s)_{s=0}\right\} = \tau_2^2\left|\eta(\tau)\right|^4 \tag{53}$$

Finally, we have:

$$Z' \sim \tau_2^{-2}\det{}' P \sim \tau_2^{-2}\tau_2^2\left|\eta(\tau)\right|^4 \sim \left|\eta(\tau)\right|^4 \tag{54}$$

[9]Reference: *Di Francesco et al.*

[10]References: (1) *Nakahara*, (2) *Di Francesco et al.*, and (3) http://theory.uchicago.edu/~sethi/Teaching/P483-W2018/p483-sol3.pdf.

## $\boxed{4}$ Torus Propagator as a Trace

$$w' \to 0, \quad \left\langle \partial_w X(w)\, \partial_{w'} X(w') \right\rangle = \mathrm{Tr}\left( \partial_w X(w)\, \partial_{w'} X(w')\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} \right) \tag{55}$$

Here we've dropped the time ordering in the $w' \to 0$ limit. Recall the mode expansion of $\partial X$ in $\boxed{1}$; we see that only the "diagonal" components of $\partial X(w)\, \partial X(w')$ survive in the trace, i.e.

$$\begin{aligned}
\partial_w X(w)\, \partial_{w'} X(w') &= (\partial_w z)(\partial_{w'} z')\, \partial_z X(z)\, \partial_{z'} X(z'), \quad z = e^{-iw}, \quad 1 \le |z| \le e^{2\pi\tau_2} \\
&\sim -\frac{\alpha'}{2} \sum_{n=-\infty}^{\infty} \frac{\alpha_{-n}\alpha_n}{z^{-n+1} z'^{n+1}} (-iz)(-iz') \\
&= \frac{\alpha'}{2} \left( \alpha_0^2 + \sum_{n>0} \left( \left(\frac{z}{z'}\right)^n + \left(\frac{z'}{z}\right)^n \right) \alpha_{-n}\alpha_n + \sum_{n>0} n \left(\frac{z'}{z}\right)^n \right) \\
&= \frac{\alpha'}{2} \left( \alpha_0^2 + \sum_{n>0} \left( \left(\frac{z}{z'}\right)^n + \left(\frac{z'}{z}\right)^n \right) \alpha_{-n}\alpha_n + \frac{zz'}{(z-z')^2} \right)
\end{aligned} \tag{56}$$

The last term is a normal ordering constant; here it is naturally regularized by $\left(\frac{z'}{z}\right)^n$.

The $\alpha_0^2$ term can be substituted with spacetime momentum $p$; we have:

$$p = \sqrt{\frac{1}{2\alpha'}} \left(\alpha_0 + \tilde{\alpha}_0\right) = \sqrt{\frac{1}{2\alpha'}}\, 2\alpha_0 = \sqrt{\frac{2}{\alpha'}}\, \alpha_0, \tag{57}$$

$$\partial_w X(w)\, \partial_{w'} X(w') \sim \frac{\alpha'}{2} \left( \frac{\alpha' p^2}{2} + \sum_{n>0} \left( \left(\frac{z}{z'}\right)^n + \left(\frac{z'}{z}\right)^n \right) n N_n \right) \tag{58}$$

On the other hand, the partition function is:

$$\begin{aligned}
Z(\tau) = \langle \mathbb{1} \rangle &= (q\bar{q})^{-\frac{1}{24}}\, V \int \frac{\mathrm{d}k}{2\pi}\, e^{-\pi\tau_2\, \alpha' k^2} \sum_{(N_l),(\tilde{N}_l)} q^{\sum_{l>0} l\cdot N_l}\, \bar{q}^{\sum_{l>0} l\cdot \tilde{N}_l} \\
&= (q\bar{q})^{-\frac{1}{24}}\, V \int \frac{\mathrm{d}k}{2\pi}\, e^{-\pi\tau_2\, \alpha' k^2} \sum_{(N_l),(\tilde{N}_l)} \prod_{l>0} q^{l\cdot N_l}\, \bar{q}^{l\cdot \tilde{N}_l} \\
&= |\eta(\tau)|^{-2}\, V \int \frac{\mathrm{d}k}{2\pi}\, e^{-\pi\tau_2\, \alpha' k^2}
\end{aligned} \tag{59}$$

We can work out $Z^{-1}\langle \partial X \partial X \rangle$ by considering term by term insertion of the $\partial X \partial X$ mode expansion into the above expression. For the $\frac{\alpha' p^2}{2}$ term, we have a contribution of:

$$\frac{\displaystyle\int \frac{\mathrm{d}k}{2\pi} \frac{\alpha' k^2}{2}\, e^{-\pi\tau_2\, \alpha' k^2}}{\displaystyle\int \frac{\mathrm{d}k}{2\pi}\, e^{-\pi\tau_2\, \alpha' k^2}} = \frac{\alpha'}{2} \frac{1}{2\cdot \pi\alpha'\tau_2} = \frac{1}{4\pi\tau_2} \tag{60}$$

For the $nN_n$ insertion, we have a contribution of:

$$\frac{\sum\limits_{(N_l)} nN_n\, q^{\sum\limits_{l>0} l\cdot N_l}}{\sum\limits_{(N_l)} q^{\sum\limits_{l>0} l\cdot N_l}} = \frac{\sum\limits_{(N_l)} nN_n \prod\limits_{l>0} q^{l\cdot N_l}}{\sum\limits_{(N_l)} \prod\limits_{l>0} q^{l\cdot N_l}} = \frac{\sum\limits_{N_n=0}^{\infty} nN_n\, q^{n\cdot N_n}}{\sum\limits_{N_n=0}^{\infty} q^{n\cdot N_n}} = \frac{nq^n \frac{\partial}{\partial(q^n)} \sum\limits_{N_n=0}^{\infty} q^{n\cdot N_n}}{\sum\limits_{N_n=0}^{\infty} q^{n\cdot N_n}}$$
$$= \frac{nq^n \frac{\partial}{\partial(q^n)} \frac{1}{1-q^n}}{\frac{1}{1-q^n}} = \frac{nq^n}{1-q^n} \tag{61}$$

Therefore, the complete result is given by:

$$\frac{1}{Z(\tau)}\langle \partial_w X(w)\, \partial_{w'} X(w')\rangle = \frac{\alpha'}{2}\left(\frac{1}{4\pi\tau_2} + \sum_{n>0}\left(\left(\frac{z}{z'}\right)^n + \left(\frac{z'}{z}\right)^n\right)\frac{nq^n}{1-q^n} + \frac{zz'}{(z-z')^2}\right)$$
$$\xrightarrow[z'\to 1]{w'\to 0} \frac{\alpha'}{2}\left(\frac{1}{4\pi\tau_2} + \sum_{n>0}\left(z^n + z^{-n}\right)\frac{nq^n}{1-q^n} + \frac{z}{(z-1)^2}\right) \tag{62}$$

On the other hand, the torus propagator is given by:

$$G'(w,\bar{w};w',\bar{w}') = -\frac{\alpha'}{2}\,\ln|f(w-w',\tau)|^2 + \frac{\alpha'}{4\pi\tau_2}\left(\mathrm{Im}\,(w-w')\right)^2, \tag{63}$$

$$f(w,\tau) \equiv \theta_1\left(\frac{w}{2\pi}\,\middle|\,\tau\right) = 2\,e^{\frac{i\pi\tau}{4}}\sin\frac{w}{2}\prod_{m>0}^{\infty}(1-q^m)(1-z^{-1}q^m)(1-zq^m), \quad z = e^{-iw} \tag{64}$$

We find that $\partial_w\partial_{w'}G'$ contains the same zero mode contribution $\frac{\alpha'}{8\pi\tau_2}$ and normal ordering contribution $\frac{\alpha'}{2}\frac{z}{(z-1)^2}$ as in (62):

$$\partial_w\partial_{w'}G'(w,\bar{w};w',\bar{w}')_{w'=0} = \frac{\alpha'}{8\pi\tau_2} + \frac{\alpha'}{2}\partial_w^2 \ln f(w,\tau), \tag{65}$$

$$\partial_w^2 \ln f(w,\tau) = \partial_w^2 \ln\sin\frac{w}{2} + \partial_w^2 \sum_{m>0}\Big(\ln(1-zq^m) + \ln(1-z^{-1}q^m)\Big), \tag{66}$$

$$\partial_w^2 \ln\sin\frac{w}{2} = \partial_w^2 \ln\sin\frac{w}{2} = -\frac{1}{4\sin^2\frac{w}{2}} = \frac{1}{2(\cos w - 1)} = \frac{1}{z+z^{-1}-2} = \frac{z}{(z-1)^2}, \tag{67}$$

The remaining parts come from oscillator modes; they also match with (62), but the equivalence is less obvious: we have[11]:

$$\partial_w^2 \sum_{m>0}\ln(1-zq^m) = \partial_w^2 \sum_{m>0}\sum_{n>0} -\frac{1}{n}\left(zq^m\right)^n$$
$$= \sum_{n>0}\partial_w^2\left(-\frac{1}{n}z^n\right)\sum_{m>0} q^{mn}, \quad \partial_w = -iz\,\partial_z$$
$$= \sum_{n>0} -\frac{(-in)^2}{n}z^n \cdot \frac{q^n}{1-q^n}$$
$$= \sum_{n>0} z^n \frac{nq^n}{1-q^n}, \tag{68}$$

[11]Reference: http://theory.uchicago.edu/~sethi/Teaching/P483-W2018/p483-sol3.pdf. I would like to thank Lucy Smith for providing this hint.

$$\partial_w^2 \sum_{m>0} \ln\left(1 - z^{-1} q^m\right) = \sum_{n>0} z^{-n} \frac{n q^n}{1 - q^n}, \tag{69}$$

This is precisely the contribution from oscillator modes in (62). Therefore, we have:

$$\frac{1}{Z(\tau)} \big\langle \partial_w X(w)\, \partial_{w'} X(w') \big\rangle_{w'=0} = \partial_w \partial_{w'} G'(w, \bar{w}; w', \bar{w}')_{w'=0} \tag{70}$$