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## 1 Morphism between coverings is covering:

For  $F_i \to E_i \xrightarrow{p_i} B$ : coverings in  $Cov_0(B)$  with  $E_i$ : connected and B: path connected and locally path connected, the following diagram commutes:

$$E_1 \xrightarrow{f} E_2 \qquad e_2 = f(e_1),$$
 $b = p_1(e_1) = p_2(e_2),$ 

To show that f is itself a covering, we need only verify that f is locally trivial with some discrete fiber F. In fact, given any  $e_2 \in E_2$  and  $b = p_2(e_2)$ , there exists some neighborhood  $U \subset B$  that the following diagram holds (by restriction):

$$U \times F_1 \xrightarrow{f} U \times F_2 \qquad e_1 = (b, k_1), \\ e_2 = (b, k_2(b, k_1)), \quad k_i \in F_i$$

Generally,  $k_2 = k_2(b, k_1)$  depends on the base point  $b \in B$ . However, since B is locally path connected, we can restrict U to be path connected, while  $k_2 \in F_2$ : discrete. Since continuous maps preserve path connectedness,  $k_2$  is in fact independence of b, i.e.  $k_2 = \varphi(k_1)$ .

On the other hand,  $\forall e_2 = (b, k_2) \in U \times \{k_2\} \subset E_2$ , we have its preimage  $f^{-1}(e_2) = \{b\} \times \varphi^{-1}(k_2)$ . Note that  $E_2$  is connected while  $\varphi^{-1}(k_2) \in F_1$  is discrete; for the same reasoning as above,  $\varphi^{-1}(k_2) = F$  is in fact independent of  $k_2$ . This is the discrete fiber F we have been looking for. Hence f is also a covering map<sup>1</sup>.

# 2 Cylinder with ends pinched — $\pi_1$ and universal cover:

$$Y = (X \times I)/(X \times \partial I) , \quad I = [0, 1]$$
 (1)

Note that Y is homeomorphic to two cones<sup>2</sup>  $CX_1 \coprod CX_2$  with "bases"  $X_i \subset CX_i$  and "vertices"  $v_i$  respectively identified:  $X_1 \sim X_2$ ,  $v_1 \sim v_2 \equiv v$ . X is path connected and so is Y, hence we are free to choose  $\pi_1(Y) = \pi_1(Y, y_0)$ .

First note that paths that do not pass through the vertex v are all homotopic, since they are contained in a cone and cones are contractible<sup>3</sup>. Therefore all contributions to  $\pi_1(Y)$  are loop classes that do pass through the vertex v. In other words, morphisms in  $\Pi_1 Y$  are in one-to-one correspondence with morphisms in:

$$\Pi_1([0,1]/_{0\sim 1}) = \Pi_1 S^1 \tag{2}$$

Therefore, 
$$\pi_1(Y) \cong \pi_1(S^1) = \mathbb{Z}$$
.

<sup>&</sup>lt;sup>1</sup>Reference: math.stackexchange.com/a/109774.

 $<sup>^2 \</sup>mathrm{See}$  discussions from Problem Set №1.

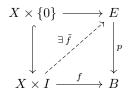
 $<sup>{}^{3}[\</sup>gamma_{1}] = [\gamma_{2} \star \gamma_{2}^{-1} \star \gamma_{1}] = [\gamma_{2}].$ 

The universal cover  $\tilde{Y}$  of Y can be constructed by assigning an induced topology to the space of path classes, same as in the general proof of its existence. Since Y is "degenerate" at its vertex, this is equivalent to "cutting open" Y at its vertex v, and joining  $\mathbb{Z}$  copies them end-to-end. More explicitly, it can be written as:

$$\tilde{Y} = (X \times \mathbb{R}) / \sim, \quad (x, n) \sim (x', n), \ \forall \ x \in X, \ n \in \mathbb{Z}$$
 (3)

While the covering map:  $\tilde{Y} \ni [x,t] \mapsto [x,t-\lfloor t \rfloor] \in Y$ , here  $\lfloor t \rfloor$  is the integer part of  $t \in \mathbb{R}$ .

## $3 \pi_1$ of fiber in fibration:



For  $F \to E \xrightarrow{p} B$ : fibration, by homotopy lifting property (HLP), any homotopy in B can be uniquely lifted to path class in E, provided some "initial condition"  $X \times \{0\}$ . This leads to the following results:

(a) For B: simply-connected, take any loop class  $[\tilde{\gamma}] \in \pi_1(E, e)$  as initial condition; its projection  $[p \circ \tilde{\gamma}] \in \pi_1(B, b) = \{[\mathbb{1}_b]\}$  is trivial, i.e.  $p \circ \tilde{\gamma} \simeq \mathbb{1}_b$ . By HLP, such homotopy can be lifted into E, i.e.

$$p \circ \tilde{\gamma} \simeq \mathbb{1}_b \quad \xrightarrow{\text{lift}} \quad \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}' = \mathbb{1}_b$$
 (4)

In other words,  $\tilde{\gamma} \simeq \tilde{\gamma}' \subset p^{-1}(b)$ , i.e. any loop in E is homotopic to some loop in  $p^{-1}(b) \cong F$ . This implies a surjective group homomorphism  $\pi_1(p^{-1}(b), e) \to \pi_1(E, e)$ , i.e. an epimorphism.  $\Box$ 

(b) For E: simply-connected, take any loop class  $[\gamma] \in \pi_1(B, b)$  and consider its lifting  $[\tilde{\gamma}]$ . Note that in general  $\tilde{\gamma}$  is *not* a loop; however, we have  $p \circ \tilde{\gamma} = \gamma$ , hence  $\tilde{\gamma}(0), \tilde{\gamma}(1) \in p^{-1}(b)$ . In general, we have:

$$\gamma \simeq \gamma' \xrightarrow{\text{lift}} \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}^{(\prime)} = \gamma^{(\prime)}$$
 (5)

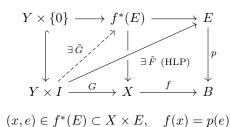
By continuity,  $\tilde{\gamma}(0)$ ,  $\tilde{\gamma}'(0) \in F_0$ : a path component of  $p^{-1}(b)$ ; similarly,  $\tilde{\gamma}(1)$ ,  $\tilde{\gamma}'(1) \in F_1$ . In other words, the start and end points of  $\tilde{\gamma}$  are confined in path components  $F_0$  and  $F_1$ , respectively. Hence a loop class in  $\pi_1(B,b)$  maps to transport between path components:

$$T_{(\cdot)}(e) \colon \pi_1(B,b) \longrightarrow \pi_0(p^{-1}(b))$$

$$[\gamma] \longmapsto T_{[\gamma]}(e)$$
(6)

As a matter of fact,  $T_{(\cdot)}(e)$  is a bijection. For  $T_{[\gamma]} = T_{[\gamma']}$ , they are characterized by two lifted paths  $\tilde{\gamma}, \tilde{\gamma}'$ ; since E is simply connected, they are always homotopic:  $\tilde{\gamma} \simeq \tilde{\gamma}'$ , hence  $[\gamma] = [\gamma']$  by projection p. This means that T is injective. Surjectivity also follows from projection  $\gamma = p \circ \gamma'$ . Therefore,  $T_{(\cdot)}(e)$  gives a bijection between  $\pi_1(B,b)$  and  $\pi_0(p^{-1}(b))$ .

#### 4 Pull-back of fibration is fibration:



We need only verify that  $f^*(E) \to X$  also has HLP, i.e. the existence of  $\tilde{F}$  in the above diagram<sup>4</sup>. By HLP of  $E \xrightarrow{p} B$ ,  $\exists \tilde{F} \colon Y \times I \to E$  as shown above. We can use  $\tilde{F}$  to construct  $\tilde{G}$  explicitly; in fact, first consider:

$$\tilde{G} \colon Y \times I \longrightarrow X \times E$$

$$(y,t) \longmapsto (G(y,t), \tilde{F}(y,t))$$
(7)

Note that  $f \circ G = p \circ \tilde{F}$ ; compared with the definition of  $f^*(E)$ , this implies that the image of  $\tilde{G}$  lies within  $f^*(E) \subset X \times E$ , hence after restriction of its codomain,  $\tilde{G}$  becomes a well-defined lifting of G into  $f^*(E)$ . Therefore,  $f^*(E) \to X$  has HLP, i.e. it is also a fibration.

### 5 More properties of fibration:

- (a) By HLP, given any initial condition  $e \in p^{-1}(b_1)$ , lifting of any path  $b_1 \xrightarrow{\gamma} b_2$  exists. The lifted path with dependence of e can then be written as  $F : p^{-1}(b_1) \times I \to E$ . This is just a generalization of 3 for non-loop paths.
- (b) Similarly, transport  $T_{[\gamma]}$  defined in 3 can be generalized for non-loop paths.  $T_{[\gamma]}$  is well-defined for path class  $[\gamma]$ , since by HLP homotopic paths can be lifted to homotopy in E. Therefore, the transport is fixed up to homotopy, i.e.

$$T \colon \operatorname{Hom}_{\Pi_1 B}(b_0, b_1) \longrightarrow \operatorname{Hom}_{\underline{\mathbf{hTop}}} \left( p^{-1}(b_0), p^{-1}(b_1) \right)$$

$$[\gamma] \longmapsto T_{[\gamma]} \tag{8}$$

Note that T defined in this way is also independent of the choice of F, since F simply specifies the starting point of the lifted path; no matter which F we choose, the lifted paths will always be homotopic in E. Hence T is well-defined in the above sense.

(c) T defined above is a functor:  $\Pi_1 B \to \underline{\mathbf{hTop}}$ . To verify this, we need only check that it is compatible with composition and maps identity morphisms to identity morphisms. Indeed,  $T_{[\mathbb{1}_b]} = [\mathbb{1}_{p^{-1}(b)}]$ , and  $T_{[\gamma']\star[\gamma]} = T_{[\gamma'\star\gamma]} = T_{[\gamma']}\circ T_{[\gamma]}$  by joining two lifted paths (up to homotopy).

<sup>&</sup>lt;sup>4</sup>Notice that  $f^*(E)$  is the limit of the diagram, hence this is automatically true by the universal property of  $f^*(E)$ . I would like to thank 刘逸华 for pointing this out. For now, we will stick to a more traditional proof.

(d) For B: path connected, there exists an isomorphism between any two objects in  $\Pi_1 B$  (a path connecting any two points in B), which is mapped to isomorphisms between fibers  $p^{-1}(b)$  in  $\underline{\mathbf{hTop}}$ . Hence any two fibers of  $E \stackrel{p}{\to} B$  have the same homotopy type.