

## 1 Morphism between coverings is covering:

For  $F_i \rightarrow E_i \xrightarrow{p_i} B$ : coverings in  $\text{Cov}_0(B)$  with  $E_i$ : connected and  $B$ : path connected and locally path connected, the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & \searrow p_1 & \swarrow p_2 \\ & B & \end{array} \quad \begin{array}{l} e_2 = f(e_1), \\ b = p_1(e_1) = p_2(e_2), \end{array}$$

To show that  $f$  is itself a covering, we need only verify that  $f$  is locally trivial with some discrete fiber  $F$ . In fact, given any  $e_2 \in E_2$  and  $b = p_2(e_2)$ , there exists some neighborhood  $U \subset B$  that the following diagram holds (by restriction):

$$\begin{array}{ccc} U \times F_1 & \xrightarrow{f} & U \times F_2 \\ & \searrow p_1 & \swarrow p_2 \\ & U & \end{array} \quad \begin{array}{l} e_1 = (b, k_1), \\ e_2 = (b, k_2(b, k_1)), \quad k_i \in F_i \end{array}$$

Generally,  $k_2 = k_2(b, k_1)$  depends on the base point  $b \in B$ . However, since  $B$  is locally path connected, we can restrict  $U$  to be path connected, while  $k_2 \in F_2$ : discrete. Since continuous maps preserve path connectedness,  $k_2$  is in fact independence of  $b$ , i.e.  $k_2 = \varphi(k_1)$ .

On the other hand,  $\forall e_2 = (b, k_2) \in U \times \{k_2\} \subset E_2$ , we have its preimage  $f^{-1}(e_2) = \{b\} \times \varphi^{-1}(k_2)$ . Note that  $E_2$  is connected while  $\varphi^{-1}(k_2) \in F_1$  is discrete; for the same reasoning as above,  $\varphi^{-1}(k_2) = F$  is in fact independent of  $k_2$ . This is the discrete fiber  $F$  we have been looking for. Hence  $f$  is also a covering map<sup>1</sup>. ■

## 2 Cylinder with ends pinched — $\pi_1$ and universal cover:

$$Y = (X \times I) / (X \times \partial I), \quad I = [0, 1] \quad (1)$$

Note that  $Y$  is homeomorphic to two cones<sup>2</sup>  $CX_1 \amalg CX_2$  with “bases”  $X_i \subset CX_i$  and “vertices”  $v_i$  respectively identified:  $X_1 \sim X_2$ ,  $v_1 \sim v_2 \equiv v$ .  $X$  is path connected and so is  $Y$ , hence we are free to choose  $\pi_1(Y) = \pi_1(Y, y_0)$ .

First note that paths that do *not* pass through the vertex  $v$  are all homotopic, since they are contained in a cone and cones are contractible<sup>3</sup>. Therefore all contributions to  $\pi_1(Y)$  are loop classes that *do* pass through the vertex  $v$ . In other words, morphisms in  $\Pi_1 Y$  are in one-to-one correspondence with morphisms in:

$$\Pi_1([0, 1] / \sim) = \Pi_1 S^1 \quad (2)$$

Therefore,  $\pi_1(Y) \cong \pi_1(S^1) = \mathbb{Z}$ . □

<sup>1</sup>Reference: [math.stackexchange.com/a/109774](https://math.stackexchange.com/a/109774).

<sup>2</sup>See discussions from Problem Set №1.

<sup>3</sup> $[\gamma_1] = [\gamma_2 \star \gamma_2^{-1} \star \gamma_1] = [\gamma_2]$ .

The universal cover  $\tilde{Y}$  of  $Y$  can be constructed by assigning an induced topology to the space of path classes, same as in the general proof of its existence. Since  $Y$  is “degenerate” at its vertex, this is equivalent to “cutting open”  $Y$  at its vertex  $v$ , and joining  $\mathbb{Z}$  copies them end-to-end. More explicitly, it can be written as:

$$\tilde{Y} = (X \times \mathbb{R}) / \sim, \quad (x, n) \sim (x', n), \quad \forall x \in X, n \in \mathbb{Z} \quad (3)$$

While the covering map:  $\tilde{Y} \ni [x, t] \mapsto [x, t - [t]] \in Y$ , here  $[t]$  is the integer part of  $t \in \mathbb{R}$ . ■

**3  $\pi_1$  of fiber in fibration:**

$$\begin{array}{ccc} X \times \{0\} & \longrightarrow & E \\ \downarrow & \nearrow \exists \tilde{f} & \downarrow p \\ X \times I & \xrightarrow{f} & B \end{array}$$

For  $F \rightarrow E \xrightarrow{p} B$ : fibration, by homotopy lifting property (HLP), any homotopy in  $B$  can be uniquely lifted to path class in  $E$ , provided some “initial condition”  $X \times \{0\}$ . This leads to the following results:

- (a) For  $B$ : simply-connected, take any loop class  $[\tilde{\gamma}] \in \pi_1(E, e)$  as initial condition; its projection  $[p \circ \tilde{\gamma}] \in \pi_1(B, b) = \{[\mathbb{1}_b]\}$  is trivial, i.e.  $p \circ \tilde{\gamma} \simeq \mathbb{1}_b$ . By HLP, such homotopy can be lifted into  $E$ , i.e.

$$p \circ \tilde{\gamma} \simeq \mathbb{1}_b \xRightarrow{\text{lift}} \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}' = \mathbb{1}_b \quad (4)$$

In other words,  $\tilde{\gamma} \simeq \tilde{\gamma}' \subset p^{-1}(b)$ , i.e. any loop in  $E$  is homotopic to some loop in  $p^{-1}(b) \cong F$ . This implies a surjective group homomorphism  $\pi_1(p^{-1}(b), e) \rightarrow \pi_1(E, e)$ , i.e. an epimorphism. □

- (b) For  $E$ : simply-connected, take any loop class  $[\gamma] \in \pi_1(B, b)$  and consider its lifting  $[\tilde{\gamma}]$ . Note that in general  $\tilde{\gamma}$  is *not* a loop; however, we have  $p \circ \tilde{\gamma} = \gamma$ , hence  $\tilde{\gamma}(0), \tilde{\gamma}(1) \in p^{-1}(b)$ . In general, we have:

$$\gamma \simeq \gamma' \xRightarrow{\text{lift}} \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}^{(i)} = \gamma^{(i)} \quad (5)$$

By continuity,  $\tilde{\gamma}(0), \tilde{\gamma}'(0) \in F_0$ : a path component of  $p^{-1}(b)$ ; similarly,  $\tilde{\gamma}(1), \tilde{\gamma}'(1) \in F_1$ . In other words, the start and end points of  $\tilde{\gamma}$  are confined in path components  $F_0$  and  $F_1$ , respectively. Hence a loop class in  $\pi_1(B, b)$  maps to *transport* between path components:

$$\begin{aligned} T_{(\cdot)}(e): \pi_1(B, b) &\longrightarrow \pi_0(p^{-1}(b)) \\ [\gamma] &\longmapsto T_{[\gamma]}(e) \end{aligned} \quad (6)$$

As a matter of fact,  $T_{(\cdot)}(e)$  is a bijection. For  $T_{[\gamma]} = T_{[\gamma']}$ , they are characterized by two lifted paths  $\tilde{\gamma}, \tilde{\gamma}'$ ; since  $E$  is simply connected, they are always homotopic:  $\tilde{\gamma} \simeq \tilde{\gamma}'$ , hence  $[\gamma] = [\gamma']$  by projection  $p$ . This means that  $T$  is injective. Surjectivity also follows from projection  $\gamma = p \circ \tilde{\gamma}$ . Therefore,  $T_{(\cdot)}(e)$  gives a bijection between  $\pi_1(B, b)$  and  $\pi_0(p^{-1}(b))$ . ■

**4 Pull-back of fibration is fibration:**

$$\begin{array}{ccccc}
 Y \times \{0\} & \longrightarrow & f^*(E) & \longrightarrow & E \\
 \downarrow & \nearrow \exists \tilde{G} & \downarrow & \nearrow \exists \tilde{F} \text{ (HLP)} & \downarrow p \\
 Y \times I & \xrightarrow{G} & X & \xrightarrow{f} & B
 \end{array}$$

$$(x, e) \in f^*(E) \subset X \times E, \quad f(x) = p(e)$$

We need only verify that  $f^*(E) \rightarrow X$  also has HLP, i.e. the existence of  $\tilde{F}$  in the above diagram<sup>4</sup>. By HLP of  $E \xrightarrow{p} B$ ,  $\exists \tilde{F}: Y \times I \rightarrow E$  as shown above. We can use  $\tilde{F}$  to construct  $\tilde{G}$  explicitly; in fact, first consider:

$$\begin{aligned}
 \tilde{G}: Y \times I &\longrightarrow X \times E \\
 (y, t) &\longmapsto (G(y, t), \tilde{F}(y, t))
 \end{aligned} \tag{7}$$

Note that  $f \circ G = p \circ \tilde{F}$ ; compared with the definition of  $f^*(E)$ , this implies that the image of  $\tilde{G}$  lies within  $f^*(E) \subset X \times E$ , hence after restriction of its codomain,  $\tilde{G}$  becomes a well-defined lifting of  $G$  into  $f^*(E)$ . Therefore,  $f^*(E) \rightarrow X$  has HLP, i.e. it is also a fibration.  $\blacksquare$

**5 More properties of fibration:**

(a) By HLP, given any initial condition  $e \in p^{-1}(b_1)$ , lifting of any path  $b_1 \xrightarrow{\gamma} b_2$  exists. The lifted path with dependence of  $e$  can then be written as  $F: p^{-1}(b_1) \times I \rightarrow E$ . This is just a generalization of [3](#) for non-loop paths.  $\square$

(b) Similarly, transport  $T_{[\gamma]}$  defined in [3](#) can be generalized for non-loop paths.  $T_{[\gamma]}$  is well-defined for path class  $[\gamma]$ , since by HLP homotopic paths can be lifted to homotopy in  $E$ . Therefore, the transport is fixed up to homotopy, i.e.

$$\begin{aligned}
 T: \text{Hom}_{\Pi_1 B}(b_0, b_1) &\longrightarrow \text{Hom}_{\mathbf{hTop}}(p^{-1}(b_0), p^{-1}(b_1)) \\
 [\gamma] &\longmapsto T_{[\gamma]}
 \end{aligned} \tag{8}$$

Note that  $T$  defined in this way is also independent of the choice of  $F$ , since  $F$  simply specifies the starting point of the lifted path; no matter which  $F$  we choose, the lifted paths will always be homotopic in  $E$ . Hence  $T$  is well-defined in the above sense.  $\square$

(c)  $T$  defined above is a functor:  $\Pi_1 B \rightarrow \mathbf{hTop}$ . To verify this, we need only check that it is compatible with composition and maps identity morphisms to identity morphisms. Indeed,  $T_{[\mathbb{1}_b]} = [\mathbb{1}_{p^{-1}(b)}]$ , and  $T_{[\gamma'] \star [\gamma]} = T_{[\gamma' \star \gamma]} = T_{[\gamma']} \circ T_{[\gamma]}$  by joining two lifted paths (up to homotopy).  $\square$

<sup>4</sup>Notice that  $f^*(E)$  is the limit of the diagram, hence this is automatically true by the universal property of  $f^*(E)$ . I would like to thank 刘逸华 for pointing this out. For now, we will stick to a more traditional proof.

(d) For  $B$ : path connected, there exists an isomorphism between any two objects in  $\Pi_1 B$  (a path connecting any two points in  $B$ ), which is mapped to isomorphisms between fibers  $p^{-1}(b)$  in  $\underline{\mathbf{hTop}}$ . Hence any two fibers of  $E \xrightarrow{p} B$  have the same homotopy type. ■