String Theory №6 [WIP]

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1 Type 0 Superstrings

A closed superstring theory consists of sectors labeled by the boundary conditions $(-1)^{\alpha}$ of $(\psi, \tilde{\psi})$ along with suitable GSO projections $(-1)^F = \pm 1$. Here we follow the discussions of *Polchinski*, with R: $\alpha = 1$ and NS: $\alpha = 0$.

There are also some consistency conditions: by modular invariance, there must be at least one left-moving R sector and at least one right-moving R sector; on the other hand, the OPE must close, and since $\mathrm{R} \times \mathrm{R} = \mathrm{NS}$ there must be some corresponding NS sector for each R sector.

If we includes only the (NS, NS) and the (R, R) sectors, then both must exist due to the above conditions. In fact, closure of OPE implies that the (NS+, NS+) sector must exist. In addition, NS− sector must be paired with another NS− sector due to the level matching condition of the closed string, i.e. it is possible (but not required) to have a (NS−, NS−) sector.

The full possiblities can then be generated by enumerating all possible (R, R) sectors (there are $2 \times 2 = 4$ of them), while applying an extra consistency check that all pairs of vertex operators O_1, O_2 are mutually local, i.e.

$$\exp i\pi\Big(F_1\alpha_2 - F_2\alpha_1 - \tilde{F}_1\tilde{\alpha}_2 + \tilde{F}_2\tilde{\alpha}_1\Big) = 1 \tag{1}$$

If $O_1 \in$ (NS+, NS+), then we have $\alpha_1 = \tilde{\alpha}_1 = 0 = F_1 = \tilde{F}_1$, hence the above factor is always trivial; for $O_1 \in$ (R, R), however, $\alpha_1 = \tilde{\alpha}_1 = 1$, which yields a non-trivial constraint for the second operator: $F_2 - \tilde{F}_2 = F_1\alpha_2 - \tilde{F}_1\tilde{\alpha}_2 = \alpha_2\big(F_1 - \tilde{F}_1\big) \pmod 2$, assuming $\alpha_2 = \tilde{\alpha}_2$. With $\alpha_2 = 0$ this gives $F_2 = \tilde{F}_2$, and with $\alpha_2 = 1$ this gives $F_2 - \tilde{F}_2 = F_1 - \tilde{F}_1$; this means that all (R, R) sectors have the same sign difference between F and $\tilde{F}$. The possible solutions can then be narrowed down to:

$$\text{0A}\colon\ (\text{NS}+, \text{NS}+),\ (\text{NS}-, \text{NS}-),\ (\text{R}+, \text{R}-),\ (\text{R}-, \text{R}+), \tag{2}$$

$$\text{0B}\colon\ (\text{NS}+, \text{NS}+),\ (\text{NS}-, \text{NS}-),\ (\text{R}+, \text{R}+),\ (\text{R}-, \text{R}-), \tag{3}$$

$$\text{And additionally, } (\text{NS}+, \text{NS}+) \text{ with any } \textit{single one} \text{ of the 4 possible (R, R) sectors.} \tag{4}$$

If there are two (R, R) sectors, then there must be an acompanying (NS−, NS−) sector due to the closure of OPE. It is straight-foward to check that these possiblilities are all valid under the above constaints: (0) level matching of closed strings, (1) mutual locality, (2) closure of OPE, and (3) (apparent) modular invariance (not sufficient yet, to be checked below).

(a) The torus partition function of the theory breaks up into a product of independent sums over the bosonic X and fermionic $(\psi, \tilde{\psi})$ oscillators. The bosonic part is identical to the bosonic string situation, therefore modular invariant; to check the total partition function for modular invariance, we will look at the fermionic contributions $Z = Z_{\psi,\tilde{\psi}}$ explicitly.

Similar to the Type II case, the building block of Z is given by:

$$Z^{\alpha}{}_{\beta} = \mathrm{Tr}_{\alpha}\left[(-1)^{\beta F} q^H\right], \quad q = e^{2\pi i\tau} \tag{5}$$

Where α, β labels the periodicity in the spatial and temporal directions (σ^1, σ^2); note that for fermionic fields, anti-periodicity in the time direction gives the simple trace, while the periodic path integral gives the trace weighted by $(-1)^F$, as is explained in *Polchinski*, Appendix A.

In 10 D, $\mu = 1, \cdots, 10$, in total there are $N = 10-2 = 4\times 2 = 8$ real, *transverse* spinor components in $(\psi^\mu, \tilde{\psi}^\mu)$; pairing them into complex chiral spinors like $\psi^1 \pm \psi^2$, each one of them contributes a factor of $Z^\alpha{}_0$ in the total partition function.

Note that for type II theories, the boundary conditions and GSO projections (α, F) for the left and right movers are "decoupled"; any possible (α, F) can be paired with any possible $(\tilde{\alpha}, \tilde{F})$, hence the left and right contributions can be calculated separately. For type 0 theories, however, the left and right (α, F)'s are coupled, hence we have to calculate their contributions together. With the above considerations, we have:

$$\begin{aligned}
\mathrm{Tr}_{(\mathrm{NS,NS})}\left[\frac{1+(-1)^{F-\tilde{F}}}{2}\,q^H\right] &= \frac{1}{2}\left\{\mathrm{Tr}_{(\mathrm{NS,NS})}\,q^H + \mathrm{Tr}_{(\mathrm{NS,NS})}\left[(-1)^{F-\tilde{F}}q^H\right]\right\} \\
&= \frac{1}{2}\left\{\left|(Z^0{}_0)^{N/2}\right|^2 + \left|(Z^0{}_1)^{N/2}\right|^2\right\} \\
&= \frac{1}{2}\left\{\left|Z^0{}_0\right|^N + \left|Z^0{}_1\right|^N\right\},
\end{aligned} \tag{6}$$

$$\begin{aligned}
\mathrm{Tr}_{(\mathrm{R,R})}\left[\frac{1\mp(-1)^{F-\tilde{F}}}{2}\,q^H\right] &= \frac{1}{2}\left\{\mathrm{Tr}_{(\mathrm{R,R})}\,q^H \mp \mathrm{Tr}_{(\mathrm{R,R})}\left[(-1)^{F-\tilde{F}}q^H\right]\right\} \\
&= \frac{1}{2}\left\{\left|Z^1{}_0\right|^N \mp \left|Z^1{}_1\right|^N\right\},
\end{aligned}$$

$$Z^{0\mathrm{A|B}} = \frac{1}{2}\left\{\left|Z^0{}_0\right|^N + \left|Z^0{}_1\right|^N + \left|Z^1{}_0\right|^N \mp \left|Z^1{}_1\right|^N\right\} \tag{7}$$

Similarly, for the situation in (4) with no (NS−, NS−) sector, depending on the GSO projections $(F, \tilde{F})$ in the single (R, R) sector, we have:

$$\begin{aligned}
Z' &= \left|\frac{1}{2}\left((Z^0{}_0)^{N/2} + (Z^0{}_1)^{N/2}\right)\right|^2 + \frac{1}{2}\left((Z^1{}_0)^{N/2} + (-1)^F (Z^1{}_1)^{N/2}\right)\cdot\frac{1}{2}\left((Z^1{}_0)^{N/2} + (-1)^{\tilde{F}}(Z^1{}_1)^{N/2}\right)^* \\
&= \frac{1}{2}\left\{Z^{0A|B} + \mathrm{Re}\,(Z^0{}_0\overline{Z^0{}_1})^{N/2} + (-1)^{\tilde{F}}(\mathrm{Re}\,|\,i\,\mathrm{Im})\,(Z^1{}_0\overline{Z^1{}_1})^{N/2}\right\}
\end{aligned} \tag{8}$$

To check for modular invariance, note that[1]:

$$Z^\alpha{}_\beta(\tau) = Z^\beta{}_{-\alpha}(-\tfrac{1}{\tau}) = Z^\alpha{}_{\alpha+\beta-1}(\tau+1)\,\cdot\exp\left(-i\pi\,\frac{3\alpha^2-1}{12}\right) \tag{9}$$

We see that $Z^{0A|B}$ is indeed modular invariant, while $Z' = \frac{1}{2}Z^{0A|B} + (\cdots)$ is *not* modular invariant, due to the extra "mixing" terms in $(\cdots)$.

(b) Consider the ground states in the type 0 theories; the NS ground state is tachyonic:

$$m^2 = -k^2 = -\frac{1}{2\alpha'} \tag{10}$$

With $(-1)^F = -1$, while the level 1 states are massless and form a vector representation $\mathbf{8}_v$ of the massless little group SO(8). After GSO projections, the NS ground state becomes the (NS−) ground state, while the level 1 massless states become the (NS+) ground states.

[1] See *Polchinski*, Chapter 10. Note that the factor $\exp\left(-i\pi\,\frac{3\alpha^2-1}{12}\right)$ comes from a global gravitational anomaly, but does not matter in $Z^{0A|B}$ since we are taking absolute values.

On the other hand, the R ground state is massless. In general, 10 D Dirac spinors form a representation $\mathbf{32}_{\text{Dirac}}$ of SO(9, 1); however, in the massless case it can be further reduced into two Weyl spinors $\mathbf{16}+\mathbf{16}'$, labeled by chirality $\Gamma^{11}=(-1)^F$. They are spinor representations of SO(8). The on-shell condition (i.e. the Dirac equation) further reduces the representation into $\mathbf{8}$ and $\mathbf{8}'$, one for (R+) and one for (R−).

The closed string spectrum is then obtained by tensor product of the left and right moving part. For type 0 theories, we see that there is a tachyonic state: the (NS−, NS−) ground state is a $\mathbf{1}\times\mathbf{1}=\mathbf{1}$ scalar tachyon; with a momentum rescale $k\mapsto k/2$, the mass is now given by $m^2=-2/\alpha'$. The remaining massless states are:

$$\text{(NS+, NS+)}: \qquad \mathbf{8}_v\times\mathbf{8}_v=[0]+[2]+(2) \tag{11}$$
$$\text{(R}\pm\text{, R}\pm\text{)}: \qquad \mathbf{8}^{(\prime)}\times\mathbf{8}^{(\prime)}=[0]+[2]+[4]_\pm \tag{12}$$
$$\text{(R+, R}-\text{)}: \qquad \mathbf{8}\times\mathbf{8}'=[1]+[3] \tag{13}$$

Where we've listed the irreducible decompositions of the various $\mathbf{8}\times\mathbf{8}$ tensor product, following the notations of *Polchinski*.

$\boxed{2}$ Kaluza–Klein Mechanism

The $D=d+1$ dimensional metric can be parameterized as follows:

$$\mathrm{d}s^2=G^D_{MN}\,\mathrm{d}x^M\,\mathrm{d}x^N=G_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu+e^{2\sigma}\left(\mathrm{d}x^d+A_\mu\,\mathrm{d}x^\mu\right)^2, \tag{14}$$
$$G^D_{\mu\nu}=G_{\mu\nu}+e^{2\sigma}A_\mu A_\nu, \quad x^d\cong x^d+2\pi R \tag{15}$$

Where $\mu=0,1,\cdots,(d-1)$ labels the noncompact directions, and the x^d direction is compactified. $G_{\mu\nu},\sigma$ and A_μ should depend only on the noncompact coordinates x^μ, $A^\mu=G^{\mu\nu}A_\nu$.

G^D_{MN} can be inverted by solving $\delta^L_N=G_D^{LM}G^D_{MN}$, or in components:

$$0=G_D^{\mu\nu}\,e^{2\sigma}A_\nu+G_D^{\mu d}\,e^{2\sigma} \quad\Longrightarrow\quad G_D^{\mu d}=-G_D^{\mu\nu}A_\nu, \tag{16}$$
$$1=G_D^{\mu d}\,e^{2\sigma}A_\mu+G_D^{dd}\,e^{2\sigma} \quad\Longrightarrow\quad G_D^{dd}=e^{-2\sigma}-G_D^{\mu d}A_\mu=e^{-2\sigma}+G_D^{\mu\nu}A_\mu A_\nu, \tag{17}$$
$$\delta^\mu_\rho=G^{\mu\nu}G_{\nu\rho}=G_D^{\mu\nu}G^D_{\nu\rho}+G_D^{\mu d}\,e^{2\sigma}A_\rho, \tag{18}$$

Contract the last equation with A^ρ, and we can solve for $G_D^{\mu d}$ and then all other components. Alternatively, we can use the inversion formula for a block matrix[2]; either way, we obtain a nice and clean result:

$$G_D^{\mu d}=-A^\mu, \quad G_D^{dd}=e^{-2\sigma}+A^2, \quad G_D^{\mu\nu}=G^{\mu\nu}, \tag{19}$$

There is also a formula[3] for the determinant G_D; we have:

$$G_D^{-1}=G_d^{-1}\big(e^{-2\sigma}+A^2-A^2\big)=G_d^{-1}e^{-2\sigma}, \quad G_D=G_d\,e^{2\sigma} \tag{20}$$

[2]See e.g. `Wikipedia:` *Block matrix # Block matrix inversion*.

[3]See e.g. `Wikipedia:` *Determinant # Block matrices*.

(a) The Christoffel symbols can hence be calculated explicitly, using the G^D_{MN} components; the Ricci scalar can then be computed with brute force[4]; in the end, we have:

$$R_D = R_d - 2e^{-\sigma}\nabla^2 e^\sigma - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}, \tag{21}$$

$$\begin{aligned} S &= \frac{1}{2\kappa_0^2}\int \mathrm{d}^D x\,\sqrt{-G_D}\,R_D \\ &= \frac{1}{2\kappa_0^2}\cdot 2\pi R\int \mathrm{d}^d x\,\sqrt{-G_d}\,e^\sigma R_D \\ &\sim \frac{\pi R}{\kappa_0^2}\int \mathrm{d}^d x\,\sqrt{-G_d}\,e^\sigma\Big(R_d - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}\Big) \end{aligned} \tag{22}$$

Here we've dropped the $\nabla^2 e^\sigma$ term in the Einstein–Hilbert action, for it is a total derivative:

$$\nabla^2 e^\sigma = \frac{1}{\sqrt{-G_d}}\,\partial_\mu\Big(\sqrt{-G_d}\,G_d^{\mu\nu}\,\partial_\nu e^\sigma\Big) \tag{23}$$

However, if there is a D-dimensional dilaton Φ coupled to gravity: $\mathcal{L}_D \sim e^{-2\Phi}R_D$, then the $e^{-2\Phi}\nabla^2 e^\sigma$ term cannot be dropped, since it will contribute a Φ–σ coupling term. Here we are setting $\Phi \equiv 0$.

The e^σ factor before R_d can be absorbed by rescaling; first we eliminate the zero mode of σ by rescaling the coupling $\kappa_0 \to \kappa$:

$$\sigma = \sigma_0 + \sigma', \quad \langle\sigma\rangle = \sigma_0, \quad \langle\sigma'\rangle = 0, \tag{24}$$

$$\frac{1}{\kappa_0^2}\,e^\sigma = \frac{1}{\kappa^2}\,e^{\sigma'}, \quad \kappa = \kappa_0\,e^{-\sigma_0/2}, \tag{25}$$

Then we work on the remaining $\sigma' = \sigma - \sigma_0$. Note that:

$$G'_{\mu\nu} = e^{2\omega(x)}G_{\mu\nu}, \quad G' = e^{2\omega\cdot d}G, \quad G'^{\mu\nu} = e^{-2\omega}G^{\mu\nu}, \tag{26}$$

$$R'_d = e^{-2\omega}\Big(R_d - 2\,(d-1)\nabla^2\omega - (d-2)(d-1)\,\partial_\mu\omega\,\partial^\mu\omega\Big), \tag{27}$$

$$\sqrt{-G}\,e^{\sigma'}R_d \sim \sqrt{-G'}\,R'_d \sim \sqrt{-G}\,e^{(d-2)\omega}R_d, \quad \omega = \frac{\sigma'}{d-2}, \tag{28}$$

Before we proceed, let's first work out the Weyl transformation of the Laplacian:

$$\begin{aligned} \nabla'^2\sigma' &= \frac{1}{\sqrt{-G'}}\,\partial_\mu\Big(\sqrt{-G'}G'^{\mu\nu}\partial_\nu\sigma'\Big) \\ &= \frac{1}{\sqrt{-G}}\,e^{-\omega d}\,\partial_\mu\Big(\sqrt{-G}\,e^{+\omega d}e^{-2\omega}\,G^{\mu\nu}\partial_\nu\sigma'\Big) \\ &= e^{-\omega d}\,(\partial_\mu e^{\sigma'})\,G^{\mu\nu}\partial_\nu\sigma' + e^{-2\omega}\nabla^2\sigma' \\ &= G'^{\mu\nu}\partial_\mu\sigma'\partial_\nu\sigma' + e^{-2\omega}\nabla^2\sigma' \end{aligned} \tag{29}$$

The transformed Ricci scalar can then be rewritten as:

$$\begin{aligned} R'_d &= e^{-2\omega}R_d - 2\,\frac{d-1}{d-2}\,e^{-2\omega}\nabla^2\sigma' - \frac{d-1}{d-2}\,\partial_\mu\sigma'\,\partial'^\mu\sigma' \\ &= e^{-2\omega}R_d - 2\,\frac{d-1}{d-2}\Big(\nabla'^2\sigma' - \partial_\mu\sigma'\,\partial'^\mu\sigma'\Big) - \frac{d-1}{d-2}\,\partial_\mu\sigma'\,\partial'^\mu\sigma' \\ &= e^{-2\omega}R_d - 2\,\frac{d-1}{d-2}\,\nabla'^2\sigma' + \frac{d-1}{d-2}\,\partial_\mu\sigma'\,\partial'^\mu\sigma' \end{aligned} \tag{30}$$

[4] Reference: www.weylmann.com/kaluza.pdf, and *Polchinski*, Chapter 8.

Again, the $\nabla'^2\sigma'$ term is a total derivative and can be dropped in the action. In the end, we get:

$$S \sim \frac{\pi R}{\kappa^2} \int \mathrm{d}^d x \sqrt{-G'_d} \left(R'_d - \frac{d-1}{d-2}\, \partial_\mu \sigma' \,\partial'^\mu \sigma' - \frac{1}{4} e^{2(\sigma+\omega)} F_{\mu\nu} F'^{\mu\nu} \right) \tag{31}$$

This is the effective d-dimensional theory that we have been looking for, with a gauge field $F_{\mu\nu}$ and a massless dilaton σ'. Roughly speaking, the dilaton σ' can be treated as a Goldstone boson due to the breaking of scale invariance by compactification[5].

Following the convention of *Polchinski*, we define $A_\mu = R\tilde{A}_\mu,\ \rho = Re^\sigma,\ \rho_0 = \langle\rho\rangle = Re^{\sigma_0}$, then the gravitational and gauge couplings are given by:

$$\frac{1}{2\kappa_d^2} = \frac{\pi R}{\kappa^2}, \quad -\frac{1}{4g_d^2} = -\frac{1}{4}\, e^{2\langle\sigma+\omega\rangle} R^2 \cdot \frac{\pi R}{\kappa^2} = -\frac{1}{4}\, e^{2\sigma_0} R^2 \cdot \frac{1}{2\kappa_d^2}, \tag{32}$$

$$\therefore \quad \kappa_d^2 = \frac{\kappa^2}{2\pi R} = \frac{\kappa_0^2}{2\pi\rho_0}, \quad g_d^2 = \frac{2\kappa_d^2}{\rho_0^2} = \frac{\kappa_0^2}{\pi\rho_0^3}, \quad \rho_0 = Re^{\sigma_0} \tag{33}$$

(b) The above mechanism provides a natural theory of gravity and electromagnetism in $d = 4$. Note that the gravitational and gauge couplings are related with the radius of the compact dimension:

$$\frac{g_d^2}{\kappa_d^2} = \frac{2}{\rho_0^2} \tag{34}$$

In reality gravity is much weaker than electromagnetism, which means that $\rho_0 \to 0$, or $R \to 0$ if we gauge-fix $\sigma_0 \equiv 0$. In other words, the radius is constrained by the ratio of the couplings:

$$R \sim \sqrt{2}\,\frac{\kappa_d}{g_d} \tag{35}$$

$\boxed{3}$ Fiberwise T-Duality and the Dilaton

(a) For a bosonic string moving in a general background of massless fields in $D = d + 1 = 26$, its worldsheet action is given by:

$$S = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \sqrt{g} \left\{ \left(g^{ab} G_{MN}(X) + i\epsilon^{ab} B_{MN}(X) \right) \partial_a X^M \partial_b X^N + \alpha' \mathcal{R}\, \Phi(X) \right\} \tag{36}$$

Where Φ is the worldsheet Ricci scalar. The X^{25} direction is to be compactified, and the background fields G_{MN}, B_{MN} and Φ depends only on X^μ, $\mu = 0, 1, \cdots, (d-1) = 24$.

G_{MN} can be further split into $G_{\mu\nu}, G_{\mu d}$ and G_{dd} with $d = 25$, in a way similar to (14), but here we are using a simpler convention, with $G^D_{\mu\nu} = G_{\mu\nu}$ instead of (15). Similar goes for B_{MN}, with $B_{\mu\nu}, B_{\mu d}$, and $B_{dd} = 0$, due to anti-symmetricity.

(b) After replacing $\partial_a X^d \mapsto \partial_a X^d + A_a$ where A_a is an auxilary abelian gauge field on the worldsheet, the X^d related parts in the Lagrangian become:

$$\mathcal{L}_d[X^d, A_a] = \frac{\sqrt{g}}{4\pi\alpha'} \left\{ 2 \left(g^{ab} G_{\mu d} + i\epsilon^{ab} B_{\mu d} \right) (\partial_a X^d + A_a)\, \partial_b X^\mu + g^{ab} G_{dd} \,(\partial_a X^d + A_a)\, (\partial_b X^d + A_b) \right\} \tag{37}$$

[5]For a more careful discussion, see *Polchinski*. See also physics.stackexchange.com/q/138537.

Consider a translation $X^d \mapsto X^d + \lambda$, where λ depends on $X^\mu = X^\mu(\sigma)$ and hence depends on the worldsheet coordinates σ; it is clear that:

$$\partial_a\big(X^d + \lambda\big) + (A_a - \partial_a\lambda) = \partial_a X^d + A_a, \quad \mathcal{L}_d\big[\big(X^d + \lambda\big), (A_a - \partial_a\lambda)\big] = \mathcal{L}_d\big[X^d, A_a\big] \tag{38}$$

i.e. X^d translation is equivalent to a local gauge transformation $A_a \mapsto A_a - \partial_a\lambda$.

In fact, we would like A_a to be "pure gauge", capturing only the X^d translational symmetry and nothing more; this can be achieved by adding yet another auxilary field $\phi(\sigma)$ and an extra term:

$$\mathcal{L}_d \longmapsto \mathcal{L}_d + i\epsilon^{ab}F_{ab}\,\phi, \quad F_{ab} = \partial_a A_b - \partial_b A_a, \quad \int \mathcal{D}\phi\, e^{-i\epsilon^{ab}F_{ab}\,\phi} \sim \delta\big[\epsilon^{ab}F_{ab}\big] \tag{39}$$

Which forces $F_{12} \equiv 0$ in the remaining path integral. Note that the only non-zero independent component of F_{ab} in 2D is F_{12}, therefore $F_{12} \equiv 0$ implies that $F_{ab} \equiv 0$, or $F = \mathrm{d}A = 0$. On the plane, this implies that $A = \mathrm{d}\lambda$, i.e. it is indeed pure gauge[6].

We can then proceed to integrate out A_a. Since $A = \mathrm{d}\lambda$, we can gauge fix $A \equiv 0$, and the action reduces to the original one:

$$S'[X, \phi = 0, A_a = 0] = S[X] \tag{40}$$

Following the Faddeev–Popov procedure, we find that the path integral also reduces to the original one, up to some additional gauge volume determinant Δ_{FP}, which is independent of X. This implies that the theory for the fields (X, ϕ, A_a) is, indeed, equivalent to that of the original string theory which has only the X fields.

(c) Following our discussions in (b), we see that:

$$\partial_a X^d + A_a = 0 + A_a - \partial_a\lambda, \quad \lambda = -\partial_a X^d, \tag{41}$$

Before completing the path integral, we perform a gauge transformation $A_a \mapsto A_a - \partial_a\lambda$, with $\lambda = -X^d$. Assuming that there is no anomaly, we can ignore the functional Jacobian of the transformation, and the path integral shall be gauge invariant; in this case, the $\partial_a X^d$ term is canceled precisely by the gauge transformation, which is equivalent to setting $X^d = 0$ in the action:

$$S''[X^\mu, \phi, A] = S'\big[X^\mu, X^d = 0, \phi, A\big] \tag{42}$$

[6]However, if there are punctures on the worldsheet, then there is non-trivial cohomology, and A need not be $\mathrm{d}\lambda$. I have not yet understood how this would affect our result; this problem is also acknowledged in *Blumenhagen et al*, Chapter 14, yet it seems to simply ignore such issue.

✒ PAST WORK, AS TEMPLATE ✒

(a) Mode expansion of X CFT is[7]:

$$\partial X(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\frac{\alpha_m}{z^{m+1}}, \quad \bar\partial X(\bar z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\frac{\tilde\alpha_m}{\bar z^{m+1}}, \tag{43}$$

$$X = x - i\sqrt{\frac{\alpha'}{2}}\big(\alpha_0 \ln z + \tilde\alpha_0 \ln \bar z\big) + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\left(\frac{\alpha_m}{z^m} + \frac{\tilde\alpha_m}{\bar z^m}\right), \tag{44}$$

Momentum p is the charge for *spacetime* translation; we have:

$$X \mapsto X + \text{const}, \quad j_a = \frac{i}{\alpha'}\,\partial_a X, \tag{45}$$

$$p = \frac{1}{2\pi i}\oint_C \big(\mathrm{d}z\, j - \mathrm{d}\bar z\, \tilde j\,\big) = \frac{1}{\alpha'}\sqrt{\frac{\alpha'}{2}}\,\big(\alpha_0 + \tilde\alpha_0\big) = \sqrt{\frac{1}{2\alpha'}}\,\big(\alpha_0 + \tilde\alpha_0\big) \tag{46}$$

Additionally, for compact free boson, X is only defined modulo $2\pi R$; therefore, states after $X + 2\pi R$ translation should be identical to the original states, i.e.

$$e^{ip\,(2\pi R)} = \mathbb{1}, \quad p = \frac{n}{R}, \quad n \in \mathbb{Z} \tag{47}$$

This, in fact, holds for any field theory[8] defined for $X \in S^1$, including the ordinary quantum mechanics (a classical field theory) on S^1.

On the other hand, there are additional constraints in string theory: for the state of a *single* closed string, there is a discrete translational symmetry on the *worldsheet*:

$$X(\sigma^1 + 2\pi) \cong X(\sigma^1), \quad X(\sigma^1 + 2\pi) = X(\sigma^1) + 2\pi R w, \quad w \in \mathbb{Z} \tag{48}$$

With some definite winding number w. In $(z, \bar z)$ coordinates, we have:

$$2\pi R w = X\big(z\, e^{2\pi i}, \bar z\, e^{-2\pi i}\big) - X(z, \bar z) = -i\sqrt{\frac{\alpha'}{2}}\,2\pi i\,\big(\alpha_0 - \tilde\alpha_0\big) = 2\pi\sqrt{\frac{\alpha'}{2}}\,\big(\alpha_0 - \tilde\alpha_0\big), \tag{49}$$

$$p = \frac{p_L + p_R}{2}, \quad p_L = \sqrt{\frac{2}{\alpha'}}\,\alpha_0, \quad p_R = \sqrt{\frac{2}{\alpha'}}\,\tilde\alpha_0, \tag{50}$$

$$p_{L,R} = \frac{n}{R} \pm \frac{wR}{\alpha'}\,, \tag{51}$$

$$X = x - i\,\frac{\alpha'}{2}\,\big(p_L \ln z + p_R \ln \bar z\big) + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\left(\frac{\alpha_m}{z^m} + \frac{\tilde\alpha_m}{\bar z^m}\right), \tag{52}$$

For the oscillator expressions for L_0, recall that:

$$T(z) = -\frac{1}{\alpha'}:\!\partial X\,\partial X\!: = \sum_m \frac{L_m}{z^{m+2}}, \tag{53}$$

$$L_{m\neq 0} = \frac{1}{2}\sum_l \alpha_{m-l}\alpha_l, \quad L_0 = \frac{1}{2}:\sum_l \alpha_{-l}\alpha_l: \sim \frac{\alpha' p_L^2}{4} + \sum_{l>0}\alpha_{-l}\alpha_l, \tag{54}$$

[7] Again we follow the convention of *Polchinski*.

[8] Reference: discussions in *Polchinski*, Chapter 8.

The L_0 expression may be off by some normal ordering constant; this ambiguity can be resolved by considering:

$$2L_0\,|0,0;n=w=0\rangle = (L_1L_{-1}-L_{-1}L_1)\,|0,0;p_L=p_R=0\rangle = 0-0=0 \tag{55}$$

Therefore the normal ordering constant is, in fact, trivial, and we have:

$$L_0 = \frac{\alpha' p_L^2}{4} + \sum_{l>0}\alpha_{-l}\alpha_l, \quad \tilde{L}_0 = \frac{\alpha' p_R^2}{4} + \sum_{l>0}\tilde{\alpha}_{-l}\tilde{\alpha}_l, \tag{56}$$

(b) The torus partition function is given by:

$$\langle \mathbb{1}\rangle_{T^2} \equiv Z(\tau=\tau_1+i\tau_2) = \int \mathcal{D}X e^{-S} = \mathrm{Tr}\, e^{-(2\pi\tau_2)H} e^{i\,(2\pi\tau_1)P} \tag{57}$$

Here P generates *worldsheet* translation along σ^1, not to be confused with p which generates *spacetime* translation; with $z=e^{-iw},\ w=\sigma^1+i\sigma^2$,

$$\begin{aligned} T^0{}_1 = \eta^{00}\,(\partial_0\sigma^2)\,T_{21} = -iT_{12} &= -i\left(T_{ww}\,(\partial_1 w)(\partial_2 w) + T_{\bar{w}\bar{w}}\,(\partial_1\bar{w})(\partial_2\bar{w})\right)\\ &= T_{ww} - T_{\bar{w}\bar{w}}\\ &= \left(T_{zz}\,(\partial_w z)^2 + \tfrac{c}{24}\right) - \left(T_{\bar{z}\bar{z}}\,(\partial_{\bar{w}}\bar{z})^2 + \tfrac{\tilde{c}}{24}\right)\\ &= T(z)\,(-iz)^2 - \tilde{T}(\bar{z})\,(+i\bar{z})^2 + \frac{c-\tilde{c}}{24}, \end{aligned} \tag{58}$$

$$\begin{aligned} P = \int \frac{\mathrm{d}\sigma_1}{2\pi}\,(-T^0{}_1) &= -\int\frac{\mathrm{d}\sigma_1}{2\pi}\,T(z)\,(-iz)^2 + \int\frac{\mathrm{d}\sigma_1}{2\pi}\,\tilde{T}(\bar{z})\,(+i\bar{z})^2 - \frac{c-\tilde{c}}{24}\\ &= +\oint \frac{\mathrm{d}z}{2\pi\,(-iz)}\,T(z)\,(-iz)^2 + \oint \frac{\mathrm{d}\bar{z}}{2\pi\,(+i\bar{z})}\,\tilde{T}(\bar{z})\,(+i\bar{z})^2 - \frac{c-\tilde{c}}{24}\\ &= \oint \frac{\mathrm{d}z}{2\pi i}\,zT(z) - \oint\frac{\mathrm{d}\bar{z}}{2\pi i}\,\bar{z}\tilde{T}(\bar{z}) - \frac{c-\tilde{c}}{24}\\ &= L_0 - \tilde{L}_0 - \frac{c-\tilde{c}}{24}\\ &= \left(L_0 - \tfrac{c}{24}\right) - \left(\tilde{L}_0 - \tfrac{\tilde{c}}{24}\right),\\ H = \int\frac{\mathrm{d}\sigma_1}{2\pi}\,T^0{}_0 &= \int\frac{\mathrm{d}\sigma_1}{2\pi}\,T_{22}\\ &= L_0 + \tilde{L}_0 - \frac{c+\tilde{c}}{24}\\ &= \left(L_0 - \tfrac{c}{24}\right) + \left(\tilde{L}_0 - \tfrac{\tilde{c}}{24}\right), \end{aligned} \tag{59}$$

Here we've used the fact that $\displaystyle\oint \frac{\mathrm{d}\bar{z}}{\bar{z}} = \oint \frac{\mathrm{d}\bar{z}}{\bar{z}} = 2\pi i$. Therefore,

$$Z(\tau) = \mathrm{Tr}\, e^{-(2\pi\tau_2)H} e^{i\,(2\pi\tau_1)P} = \mathrm{Tr}\, q^{L_0-\frac{c}{24}}\,\bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}, \quad q=e^{2\pi i\tau} \tag{60}$$

Using the expressions in (a), we find that L_0 action on a state $|\psi\rangle$ created by $\alpha_{-l},\tilde{\alpha}_{-l}$ yields the sum of occupation numbers N_l weighted by l:

$$L_0\,|\psi\rangle = \left(\frac{\alpha' k_L^2}{4} + \sum_{l>0} l\cdot N_l\right)|\psi\rangle \tag{61}$$

With $c = \tilde{c} = 1$, we obtain:

$$
\begin{aligned}
Z(\tau) &= (q\bar{q})^{-\frac{1}{24}} \sum_{n,w} e^{-2\pi\tau_2\alpha' \frac{k_L^2+k_R^2}{4}} e^{2\pi i\tau_1\alpha' \frac{k_L^2-k_R^2}{4}} \sum_{(N_l),(\tilde{N}_l)} q^{\sum_{l>0} l\cdot N_l} \bar{q}^{\sum_{l>0} l\cdot\tilde{N}_l} \\
&= (q\bar{q})^{-\frac{1}{24}} \sum_{n,w} e^{-\pi\tau_2\left(\frac{\alpha' n^2}{R^2}+\frac{w^2R^2}{\alpha'}\right)+2\pi i\tau_1 nw} \sum_{(N_l),(\tilde{N}_l)} \prod_{l>0} q^{l\cdot N_l}\bar{q}^{l\cdot\tilde{N}_l} \\
&= |\eta(\tau)|^{-2} \sum_{n,w} e^{-\pi\tau_2\left(\frac{\alpha' n^2}{R^2}+\frac{w^2R^2}{\alpha'}\right)+2\pi i\tau_1 nw}
\end{aligned}
\tag{62}
$$

We've simplified the contributions from the oscillator modes using $\eta(\tau)$, since they are identical to the oscillator contributions of the non-compact $X \in \mathbb{R}^1$:

$$
\begin{aligned}
(q\bar{q})^{-\frac{1}{24}} \sum_{(N_l),(\tilde{N}_l)} \prod_{l>0} q^{l\cdot N_l}\bar{q}^{l\cdot\tilde{N}_l} &= (q\bar{q})^{-\frac{1}{24}} \prod_{l>0} \sum_{N_l,\tilde{N}_l=0}^{\infty} q^{l\cdot N_l}\bar{q}^{l\cdot\tilde{N}_l} \\
&= (q\bar{q})^{-\frac{1}{24}} \prod_{l>0} \frac{1}{1-q^l}\frac{1}{1-\bar{q}^l} = |\eta(\tau)|^{-2}
\end{aligned}
\tag{63}
$$

In the $R \to \infty$ limit, only the $w = 0$ modes survive; all other modes are exponentially suppressed by the $e^{-\pi\tau_2 w^2R^2/\alpha'}$ factor; i.e.

$$
\begin{aligned}
Z(\tau) &= |\eta(\tau)|^{-2} \sum_{n,w} \exp\left\{-\pi\tau_2\left(\frac{\alpha' n^2}{R^2}+\frac{w^2R^2}{\alpha'}\right)+2\pi i\tau_1 nw\right\} \\
&\to |\eta(\tau)|^{-2} \sum_{n} \exp\left\{-\pi\tau_2\,\frac{\alpha' n^2}{R^2}\right\}, \quad k = \frac{n}{R} \\
&\to |\eta(\tau)|^{-2}\, V\!\int \frac{\mathrm{d}k}{2\pi} \exp\left\{-\pi\tau_2\,\alpha' k^2\right\} \\
&= V\,|\eta(\tau)|^{-2}\left(4\pi^2\alpha'\tau_2\right)^{-\frac{1}{2}} \\
&\equiv V\cdot Z_X(\tau) = 2\pi R\, Z_X(\tau)
\end{aligned}
\tag{64}
$$

We recover the partition function $V \cdot Z_X(\tau)$ for non-compact X, as expected.

(c) Using the Poisson resummation formula, we find that:

$$
Z(\tau) = 2\pi R\, Z_X(\tau) \sum_{m,w} \exp\left(-\frac{\pi R^2|m-w\tau|^2}{\alpha'\tau_2}\right)
\tag{65}
$$

$Z_X(\tau)$ is modular invariant by the properties of the Dedekind $\eta(\tau)$ function, as is demonstrated for the non-compact X in *Polchinski*.

The sum, on the other hand, is naturally invariant under $T\colon \tau \mapsto \tau+1$, by making a change of variables $m \mapsto m+w$. It is also invariant under $S\colon \tau \mapsto -1/\tau$ with $m \mapsto -w, w \mapsto m$ [9]. Therefore, $Z(\tau)$ is modular invariant.

[9] Reference: *Polchinski*.

$\boxed{4}$ $\mathbb{Z}_2$ Orbifold

The $\mathbb{Z}_2$ orbifold is constructed by imposing an additional identification on $X \in S^1$:

$$X \cong -X \tag{66}$$

The target space is then reduced to $S^1/\mathbb{Z}_2 \cong [0, \pi R]$.

(a) The first contributions to the orbifold partition function comes from the states that are invariant reflection r; we have:

$$\mathrm{Tr}_{S^1/\mathbb{Z}_2} = \mathrm{Tr}_{S^1} \frac{1+r}{2} = \frac{1}{2}\,\mathrm{Tr}_{S^1} + \frac{1}{2}\,\mathrm{Tr}_{S^1} \circ r \tag{67}$$

Acting on $q^{L_0-\frac{c}{24}}\,\bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}$, the first term gives $\frac{1}{2}\,Z_{S^1}(\tau)$ where Z_{S^1} is the S^1 partition function we've obtained in $\boxed{1}$.

For the second term, note that:

$$r:\quad \Big|(N_l), (\tilde{N}_l); n, w\Big\rangle \longmapsto (-1)^{\sum_l (N_l+\tilde{N}_l)}\Big|(N_l), (\tilde{N}_l); -n, -w\Big\rangle \tag{68}$$

In particular, it reverses n, w, hence r insertion gives vanishing amplitude unless $n = w = 0$; the summation is very much similar to the Z_{S^1} case, i.e. we have:

$$\begin{aligned}\frac{1}{2}\,\mathrm{Tr}_{S^1}\left(r\,q^{L_0-\frac{c}{24}}\,\bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}\right) &= \frac{1}{2}\,(q\bar{q})^{-\frac{1}{24}}\prod_{l>0}\sum_{N_l,\tilde{N}_l=0}^{\infty}(-1)^{N_l+\tilde{N}_l}q^{l\cdot N_l}\bar{q}^{l\cdot\tilde{N}_l} \\ &= \frac{1}{2}\,(q\bar{q})^{-\frac{1}{24}}\prod_{l>0}\frac{1}{1-(-q^l)}\frac{1}{1-(-\bar{q}^l)} = \left|\frac{\eta(\tau)}{\theta_2(\tau)}\right|\end{aligned} \tag{69}$$

Where we've used the fact that[10]: $q^{-\frac{1}{24}}\prod_{l>0}\frac{1}{1-(-q^l)} = \sqrt{2}\sqrt{\frac{\eta(\tau)}{\theta_2(\tau)}}$. Therefore, the total contributions from r–invariant states are:

$$\frac{1}{2}\,Z_{S^1}(\tau) + \left|\frac{\eta(\tau)}{\theta_2(\tau)}\right| \tag{70}$$

(b) With $X \cong -X$, new possibilities emerge as the boundary condition along σ^1:

$$X(\sigma^1+2\pi) \cong X(\sigma^1), \quad X(\sigma^1+2\pi) = \pm X(\sigma^1) + 2\pi R w, \quad w \in \mathbb{Z} \tag{71}$$

The "$-$" sign corresponds to the *twisted states*. Due to the anti-periodicity, ∂X has a half-integer mode expansion:

$$\partial X\big(z\,e^{2\pi i}\big) = -\partial X(z), \tag{72}$$

$$\partial X(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\frac{\alpha_{m-\frac{1}{2}}}{z^{m+\frac{1}{2}}}, \quad \bar{\partial}X(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\frac{\tilde{\alpha}_{m-\frac{1}{2}}}{\bar{z}^{m+\frac{1}{2}}}, \tag{73}$$

$$X = x + i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\frac{1}{m+\frac{1}{2}}\left(\frac{\alpha_{m+\frac{1}{2}}}{z^{m+\frac{1}{2}}} + \frac{\tilde{\alpha}_{m+\frac{1}{2}}}{\bar{z}^{m+\frac{1}{2}}}\right), \tag{74}$$

Apply the boundary condition on X, and we find that $x = \pi R w'$; however, due to the identification $X + 2\pi R \cong X \cong -X$, there are only two inequivalent choices: $x = 0$ and $x = \pi R$, which correspond to the string localized around either of the two fixed points of the $\mathbb{Z}_2$ action.

---

[10]Reference: Blumenhagen & Plauschinn, *Introduction to CFT*, and also *Polchinski*.

Much similar to the case in $\boxed{1}$, we have:

$$\left[\alpha_{\frac{1}{2}+l}, \alpha_{-\frac{1}{2}-l}\right] = \frac{1}{2} + l, \tag{75}$$

$$L_{m\neq 0} = \frac{1}{2}\sum_l \alpha_{m-\frac{1}{2}-l}\alpha_{\frac{1}{2}+l}, \quad L_0 = \frac{1}{2} : \sum_l \alpha_{-\frac{1}{2}-l}\alpha_{\frac{1}{2}+l} : \sim \sum_{l\geq 0}\alpha_{-\frac{1}{2}-l}\alpha_{\frac{1}{2}+l} \tag{76}$$

We can use the same trick to fix the normal ordering constant in L_0; this time it is non-trivial:

$$L_{-1} = \frac{1}{2}\,\alpha^2_{-\frac{1}{2}} + \sum_{l\geq 0}\alpha_{-\frac{1}{2}-l}\alpha_{\frac{1}{2}+l}, \quad L_1 = \frac{1}{2}\,\alpha^2_{\frac{1}{2}} + \sum_{l>0}\alpha_{\frac{1}{2}-l}\alpha_{\frac{1}{2}+l}, \tag{77}$$

$$\begin{aligned} L_0\,|0,0;x\rangle &= \frac{1}{2}\left(L_1L_{-1} - L_{-1}L_1\right)|0,0;x\rangle \\ &= \frac{1}{2}\times\frac{1}{4}\,\alpha^2_{\frac{1}{2}}\alpha^2_{-\frac{1}{2}}\,|0,0;x\rangle - 0 \\ &= \frac{1}{16}\,|0,0;x\rangle\,, \end{aligned} \tag{78}$$

$$L_0 = \frac{1}{16} + \sum_{l\geq 0}\alpha_{-\frac{1}{2}-l}\alpha_{\frac{1}{2}+l} = \frac{1}{16} + \sum_{l\geq 0}\left(l+\frac{1}{2}\right)N_{l+\frac{1}{2}} = \frac{1}{16} + \sum_{l>0}\left(l-\frac{1}{2}\right)N_{l-\frac{1}{2}}, \tag{79}$$

The trace can then be computed, following the same recipe as before:

$$\begin{aligned} \mathrm{Tr}_{S^1}\left(\frac{1+r}{2}\,q^{L_0-\frac{c}{24}}\,\bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}\right) &= (q\bar{q})^{-\frac{1}{24}+\frac{1}{16}}\prod_{l+\frac{1}{2}\in\mathbb{Z}^+}\sum_{N_l,\tilde{N}_l=0}^{\infty}\frac{1+(-1)^{N_l+\tilde{N}_l}}{2}\,q^{l\cdot N_l}\bar{q}^{l\cdot\tilde{N}_l}\times 2 \\ &= \frac{1}{2}\,(q\bar{q})^{+\frac{1}{48}}\left\{\prod_{l>0}\left|\frac{1}{1-q^{l-\frac{1}{2}}}\right|^2 + \prod_{l>0}\left|\frac{1}{1+q^{l-\frac{1}{2}}}\right|^2\right\}\times 2 \\ &= \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right| \end{aligned} \tag{80}$$

There is an extra factor of 2 from the number of twisted sectors: $x = 0$ and $x = \pi R$.

(c) The full partition function is therefore:

$$Z(\tau) = \frac{1}{2}\,Z_{S^1}(\tau) + \left|\frac{\eta(\tau)}{\theta_2(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right| \tag{81}$$

The first term is modular invariant, as is proved in $\boxed{1}$.

The remaining terms are also modular invariant, due to the transformational properties of η and θ functions[11]:

$$T \circlearrowright \left|\frac{\eta(\tau)}{\theta_2(\tau)}\right| \xleftrightarrow{S} \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right| \xleftrightarrow{T} \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right| \circlearrowleft S \tag{82}$$

Therefore, the full partition function is modular invariant.

[11] Reference: *Blumenhagen & Plauschinn.*

$\boxed{5}$ Torus 4-point function in bc CFT

$$\left\langle c(w_1)\, b(w_2)\, \tilde{c}(\bar{w}_3)\, \tilde{b}(\bar{w}_4)\right\rangle = \int \mathscr{D}b\,\mathscr{D}\tilde{b}\,\mathscr{D}c\,\mathscr{D}\tilde{c}\; c(w_1)\, b(w_2)\, \tilde{c}(\bar{w}_3)\, \tilde{b}(\bar{w}_4)\, e^{-S'} \equiv Z' \tag{83}$$

First we argue that only the zero modes of the insertions survive the path integral[12]. In fact, as anti-commuting replacements of the gauge degrees of freedom, ghost modes are *defined* to be the eigenvalues of $P^\dagger P$, where P is the conformal Killing differential[13]. More specifically, given a conformal Killing vector (CKV) $\delta\sigma^a$, the conformal Killing equation can be written as:

$$P\,\delta\sigma = 0 \tag{84}$$

While $P^\dagger \delta' g = 0$ gives moduli variation $\delta' g_{ab}$ of the metric. Roughly speaking, P captures the variation of gauge fixing under an arbitrary gauge transformation; naturally, CKV's are given by $(\ker P)$, while $(\det P) \sim \Delta_{FP}$ is the Faddeev–Popov functional measure near the gauge slice. $(\det P)$ can then be calculated with:

$$\delta\sigma^a \mapsto c^a, \quad \delta' g_{ab} \mapsto b_{ab}, \quad \Delta_{FP} \sim \det P \sim \int \mathscr{D}b\,\mathscr{D}\tilde{b}\,\mathscr{D}c\,\mathscr{D}\tilde{c}\; e^{-S'}, \tag{85}$$

$$S' = \frac{1}{2\pi}\int \mathrm{d}^2\sigma\, g^{1/2}\, b_{ab}\,(P\cdot c)^{ab} = \frac{1}{2\pi}\int \mathrm{d}^2 w\,\left(b\,\bar{\partial}_w c + \tilde{b}\,\partial_w \tilde{c}\right) \tag{86}$$

In the end we have chosen conformal gauge, such that[14] $P \sim (\bar{\partial}_w, \partial_w)$, $P^\dagger P \sim -\bar{\partial}_w \partial_w = -\nabla^2$. In the $w = \sigma^1 + i\sigma^2$ coordinates, CKV's are simple translations: $c^a = \text{const}$; with $z = e^{-iw}$, it gets mapped to $c^z = c^w\,\partial_w z = c^w\,(-iz)$, which agrees with the zero mode c_0 in the $c(z)$ expansion:

$$c(z) = \sum_{m=-\infty}^{\infty} \frac{c_m}{z^{m+1-\lambda}} = c_0\, z + \sum_{m\neq 0} \frac{c_m}{z^{m-1}}, \quad \lambda = 2 \tag{87}$$

Now we are finally ready to prove our argument: for anti-commuting variables like $c(z)$,

$$\int \mathscr{D}c \sim \prod_m \int \mathrm{d}c_m \sim \prod_m \frac{\partial}{\partial c_m} \tag{88}$$

Since c_0 corresponds to a CKV, $P\cdot c_0 = 0$, therefore it vanishes in $S' = \int \mathrm{d}^2\sigma\,(b\cdot P\cdot c)$; for the path integral to be non-zero, there has to be some additional c_0 insertions, i.e.

$$Z' \sim \int \mathscr{D}b\,\mathscr{D}\tilde{b}\,\mathscr{D}c\,\mathscr{D}\tilde{c}\; c_0 b_0 \tilde{c}_0 \tilde{b}_0\, e^{-S'} \sim \left(\frac{1}{\sqrt{\tau_2}}\right)^4 \int \mathscr{D}'b\,\mathscr{D}'\tilde{b}\,\mathscr{D}'c\,\mathscr{D}'\tilde{c}\; e^{-S'}, \quad \int \mathscr{D}'c \sim \prod_{m\neq 0} \int \mathrm{d}c_m \tag{89}$$

Note the additional $\left(\frac{1}{\sqrt{\tau_2}}\right)^4$ factor coming from the zero modes[15]; this has to do with the normalization of the zero modes, each contributing a factor of $\frac{1}{\sqrt{A}}$, where $A \sim \tau_2$ is the volume (surface

[12] I would like to thank 谷夏 for some very helpful discussions about this problem.

[13] Reference: *Polchinski*, Chapter 3 & 5.

[14] References:

- Nakahara, *Geometry, Topology and Physics*;
- Blumenhagen et al, *Basic Concepts of String Theory*.

[15] Reference: *Di Francesco et al.*

area) of the torus. On a different note, since it is very difficult, if not impossible, to keep track of various (often divergent) constant factors in the path integral, we have been and will be calculating Z' up to an overall constant coefficient.

Now we have to deal with the path integral over non-zero modes. Note that the holomorphic mode expansion (87) is incomplete for our purpose: it gives the *on-shell* mode expansion, while our path integral should go over all possible configurations, including the off-shell modes, which is *not* holomorphic. However, on $T^2 = S^1 \times S^1$, the full modes are simple[16]:

$$-\nabla^2 \psi_{n_1,n_2} = \lambda_{n_1,n_2} \psi_{n_1,n_2}, \tag{90}$$

$$\begin{aligned} \psi_{n_1,n_2} &= \exp\left(i\left(n_1\tilde{\sigma}^1 + n_2\tilde{\sigma}^2\right)\right), \quad \tilde{\sigma}^2 = \frac{\sigma^2}{\tau_2}, \quad \tilde{\sigma}^1 = \sigma^1 - \sigma^2\frac{\tau_1}{\tau_2}, \\ &= \exp\left\{i\left(n_1\sigma^1 + \frac{n_2 - n_1\tau_1}{\tau_2}\,\sigma^2\right)\right\}, \end{aligned} \tag{91}$$

Here we first use the "rectangular" coordinates $(\tilde{\sigma}^1, \tilde{\sigma}^2) \in [0, 2\pi]^2$ to write down the obvious eigenfunctions ψ_{n_1,n_2}, and then relate them back to the (σ^1, σ^2) coordinates. Therefore, we have:

$$\begin{aligned} \lambda_{n_1,n_2} &= \left\{n_1^2 + \left(\frac{n_2 - n_1\tau_1}{\tau_2}\right)^2\right\} \\ &= \frac{1}{\tau_2^2}\left\{(n_1\tau_2)^2 + \left(n_1\tau_1 - n_2\right)^2\right\} \\ &= \frac{1}{\tau_2^2}\left|n_1\tau - n_2\right|^2, \end{aligned} \tag{92}$$

$$\det{}' P \sim \left(\prod_{n_1,n_2}{}' \sqrt{\lambda_{n_1,n_2}}\right)^2 \sim \prod_{n_1,n_2}{}' \lambda_{n_1,n_2} \tag{93}$$

The determinant can be computed with ζ-function regularization, as is performed in detail in *Di Francesco*; the result can be nicely summarized using the Eisenstein series, as shown in *Nakahara*:

$$E(\tau, s) = \sum_{n_1,n_2}{}' \frac{\tau_2^s}{\left|n_1\tau - n_2\right|^{2s}}, \tag{94}$$

$$\det{}' P \sim \prod_{n_1,n_2}{}' \frac{1}{\tau_2^2}\left|n_1\tau - n_2\right|^2 \sim \tau_2 \exp\left\{-\partial_s E'(\tau, s)_{s=0}\right\} = \tau_2^2\left|\eta(\tau)\right|^4 \tag{95}$$

Finally, we have:

$$Z' \sim \tau_2^{-2} \det{}' P \sim \tau_2^{-2}\tau_2^2\left|\eta(\tau)\right|^4 \sim \left|\eta(\tau)\right|^4 \tag{96}$$

6 Torus Propagator as a Trace

$$w' \to 0, \quad \left\langle \partial_w X(w)\, \partial_{w'} X(w')\right\rangle = \mathrm{Tr}\left(\partial_w X(w)\, \partial_{w'} X(w')\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}}\right) \tag{97}$$

[16] References: (1) *Nakahara*, (2) *Di Francesco et al.*, and (3) http://theory.uchicago.edu/~sethi/Teaching/P483-W2018/p483-sol3.pdf.

Here we've dropped the time ordering in the $w' \to 0$ limit. Recall the mode expansion of ∂X in $\boxed{1}$; we see that only the "diagonal" components of $\partial X(w)\,\partial X(w')$ survive in the trace, i.e.

$$
\begin{aligned}
\partial_w X(w)\,\partial_{w'} X(w') &= (\partial_w z)(\partial_{w'} z')\,\partial_z X(z)\,\partial_{z'} X(z'), \quad z = e^{-iw}, \quad 1 \le |z| \le e^{2\pi\tau_2} \\
&\sim -\frac{\alpha'}{2} \sum_{n=-\infty}^{\infty} \frac{\alpha_{-n}\alpha_n}{z^{-n+1} z'^{n+1}} (-iz)(-iz') \\
&= \frac{\alpha'}{2}\left(\alpha_0^2 + \sum_{n>0}\left(\left(\frac{z}{z'}\right)^n + \left(\frac{z'}{z}\right)^n\right)\alpha_{-n}\alpha_n + \sum_{n>0} n\left(\frac{z'}{z}\right)^n\right) \\
&= \frac{\alpha'}{2}\left(\alpha_0^2 + \sum_{n>0}\left(\left(\frac{z}{z'}\right)^n + \left(\frac{z'}{z}\right)^n\right)\alpha_{-n}\alpha_n + \frac{zz'}{(z-z')^2}\right)
\end{aligned}
\tag{98}
$$

The last term is a normal ordering constant; here it is naturally regularized by $\left(\frac{z'}{z}\right)^n$.

The α_0^2 term can be substituted with spacetime momentum p; we have:

$$
p = \sqrt{\frac{1}{2\alpha'}}\,(\alpha_0 + \tilde{\alpha}_0) = \sqrt{\frac{1}{2\alpha'}}\,2\alpha_0 = \sqrt{\frac{2}{\alpha'}}\,\alpha_0,
\tag{99}
$$

$$
\partial_w X(w)\,\partial_{w'} X(w') \sim \frac{\alpha'}{2}\left(\frac{\alpha' p^2}{2} + \sum_{n>0}\left(\left(\frac{z}{z'}\right)^n + \left(\frac{z'}{z}\right)^n\right) nN_n\right)
\tag{100}
$$

On the other hand, the partition function is:

$$
\begin{aligned}
Z(\tau) = \langle \mathbb{1} \rangle &= (q\bar{q})^{-\frac{1}{24}}\, V\!\int \frac{\mathrm{d}k}{2\pi}\, e^{-\pi\tau_2\,\alpha' k^2} \sum_{(N_l),(\tilde{N}_l)} q^{\sum_{l>0} l\cdot N_l}\, \bar{q}^{\sum_{l>0} l\cdot \tilde{N}_l} \\
&= (q\bar{q})^{-\frac{1}{24}}\, V\!\int \frac{\mathrm{d}k}{2\pi}\, e^{-\pi\tau_2\,\alpha' k^2} \sum_{(N_l),(\tilde{N}_l)} \prod_{l>0} q^{l\cdot N_l}\bar{q}^{l\cdot\tilde{N}_l} \\
&= |\eta(\tau)|^{-2}\, V\!\int \frac{\mathrm{d}k}{2\pi}\, e^{-\pi\tau_2\,\alpha' k^2}
\end{aligned}
\tag{101}
$$

We can work out $Z^{-1}\langle \partial X \partial X\rangle$ by considering term by term insertion of the $\partial X \partial X$ mode expansion into the above expression. For the $\frac{\alpha' p^2}{2}$ term, we have a contribution of:

$$
\frac{\displaystyle\int \frac{\mathrm{d}k}{2\pi}\,\frac{\alpha' k^2}{2}\, e^{-\pi\tau_2\,\alpha' k^2}}{\displaystyle\int \frac{\mathrm{d}k}{2\pi}\, e^{-\pi\tau_2\,\alpha' k^2}} = \frac{\alpha'}{2}\,\frac{1}{2\cdot\pi\alpha'\tau_2} = \frac{1}{4\pi\tau_2}
\tag{102}
$$

For the nN_n insertion, we have a contribution of:

$$
\begin{aligned}
\frac{\displaystyle\sum_{(N_l)} nN_n\, q^{\sum_{l>0} l\cdot N_l}}{\displaystyle\sum_{(N_l)} q^{\sum_{l>0} l\cdot N_l}} &= \frac{\displaystyle\sum_{(N_l)} nN_n \prod_{l>0} q^{l\cdot N_l}}{\displaystyle\sum_{(N_l)}\prod_{l>0} q^{l\cdot N_l}} = \frac{\displaystyle\sum_{N_n=0}^{\infty} nN_n\, q^{n\cdot N_n}}{\displaystyle\sum_{N_n=0}^{\infty} q^{n\cdot N_n}} = \frac{\displaystyle nq^n \frac{\partial}{\partial(q^n)} \sum_{N_n=0}^{\infty} q^{n\cdot N_n}}{\displaystyle\sum_{N_n=0}^{\infty} q^{n\cdot N_n}} \\
&= \frac{nq^n \frac{\partial}{\partial(q^n)}\frac{1}{1-q^n}}{\frac{1}{1-q^n}} = \frac{nq^n}{1-q^n}
\end{aligned}
\tag{103}
$$

Therefore, the complete result is given by:

$$\begin{aligned}\frac{1}{Z(\tau)}\big\langle\partial_w X(w)\,\partial_{w'}X(w')\big\rangle &= \frac{\alpha'}{2}\left(\frac{1}{4\pi\tau_2}+\sum_{n>0}\left(\left(\frac{z}{z'}\right)^n+\left(\frac{z'}{z}\right)^n\right)\frac{nq^n}{1-q^n}+\frac{zz'}{(z-z')^2}\right)\\ &\xrightarrow[z'\to 1]{w'\to 0}\frac{\alpha'}{2}\left(\frac{1}{4\pi\tau_2}+\sum_{n>0}\left(z^n+z^{-n}\right)\frac{nq^n}{1-q^n}+\frac{z}{(z-1)^2}\right)\end{aligned} \tag{104}$$

On the other hand, the torus propagator is given by:

$$G'(w,\bar{w};w',\bar{w}') = -\frac{\alpha'}{2}\,\ln|f(w-w',\tau)|^2+\frac{\alpha'}{4\pi\tau_2}\left(\mathrm{Im}\,(w-w')\right)^2, \tag{105}$$

$$f(w,\tau)\equiv\theta_1\left(\frac{w}{2\pi}\,\middle|\,\tau\right)=2\,e^{\frac{i\pi\tau}{4}}\sin\frac{w}{2}\prod_{m>0}^{\infty}(1-q^m)(1-z^{-1}q^m)(1-zq^m),\quad z=e^{-iw} \tag{106}$$

We find that $\partial_w\partial_{w'}G'$ contains the same zero mode contribution $\frac{\alpha'}{8\pi\tau_2}$ and normal ordering contribution $\frac{\alpha'}{2}\frac{z}{(z-1)^2}$ as in (104):

$$\partial_w\partial_{w'}G'(w,\bar{w};w',\bar{w}')_{w'=0}=\frac{\alpha'}{8\pi\tau_2}+\frac{\alpha'}{2}\partial_w^2\ln f(w,\tau), \tag{107}$$

$$\partial_w^2\ln f(w,\tau)=\partial_w^2\ln\sin\frac{w}{2}+\partial_w^2\sum_{m>0}\Big(\ln(1-zq^m)+\ln(1-z^{-1}q^m)\Big), \tag{108}$$

$$\partial_w^2\ln\sin\frac{w}{2}=\partial_w^2\ln\sin\frac{w}{2}=-\frac{1}{4\sin^2\frac{w}{2}}=\frac{1}{2\,(\cos w-1)}=\frac{1}{z+z^{-1}-2}=\frac{z}{(z-1)^2}, \tag{109}$$

The remaining parts come from oscillator modes; they also match with (104), but the equivalence is less obvious: we have[17]:

$$\begin{aligned}\partial_w^2\sum_{m>0}\ln(1-zq^m) &= \partial_w^2\sum_{m>0}\sum_{n>0}-\frac{1}{n}\,(zq^m)^n\\ &=\sum_{n>0}\partial_w^2\left(-\frac{1}{n}\,z^n\right)\sum_{m>0}q^{mn},\quad \partial_w=-iz\,\partial_z\\ &=\sum_{n>0}-\frac{(-in)^2}{n}\,z^n\cdot\frac{q^n}{1-q^n}\\ &=\sum_{n>0}z^n\frac{nq^n}{1-q^n},\end{aligned} \tag{110}$$

$$\partial_w^2\sum_{m>0}\ln(1-z^{-1}q^m)=\sum_{n>0}z^{-n}\frac{nq^n}{1-q^n}, \tag{111}$$

This is precisely the contribution from oscillator modes in (104). Therefore, we have:

$$\frac{1}{Z(\tau)}\big\langle\partial_w X(w)\,\partial_{w'}X(w')\big\rangle_{w'=0}=\partial_w\partial_{w'}G'(w,\bar{w};w',\bar{w}')_{w'=0} \tag{112}$$

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[17]Reference: `http://theory.uchicago.edu/~sethi/Teaching/P483-W2018/p483-sol3.pdf`. I would like to thank Lucy Smith for providing this hint.