Advanced QFT №3

Bryan

`Compiled @ 2020/10/20`

1 BRST Symmetry

The BRST transformation of c^a ghost is:

$$\delta c^a = \frac{1}{2} f^a{}_{bc} c^b c^c \Lambda, \quad \delta c = \delta c^a T_a = \frac{1}{2}[c, c\Lambda] \tag{1}$$

$D_\mu = \partial_\mu + A^a_\mu T_a$, T_a acts on c^a by adjoint representation: $(T_a)^c{}_b c^b = f^c{}_{ab} c^b$, i.e.

$$T_a \cdot c = (T_a)^c{}_b c^b T_c = f^c{}_{ab} T_c c^b = [T_a, T_b] c^b = [T_a, c], \tag{2}$$

$$D_\mu c = \partial_\mu c + [A_\mu, c] = [D_\mu, c], \tag{3}$$

$$D_\mu \delta c = \partial_\mu \delta c + \big[A_\mu, \delta c\big], \tag{4}$$

$$\begin{aligned}(D_\mu \delta c)^a &= \partial_\mu \delta c^a + \big[A_\mu, \delta c\big]^a \\ &= \partial_\mu \delta c^a + A^c_\mu \big[T_c, T_b\big]^a \delta c^b \\ &= \partial_\mu \delta c^a + A^c_\mu f^a{}_{cb} \delta c^b \\ &= \partial_\mu \delta c^a + A^c_\mu (T_c)^a{}_b \delta c^b \\ &= D_\mu(\delta c^a),\end{aligned} \tag{5}$$

$$\text{i.e.} \quad (D_\mu \delta c)^a - \frac{1}{2} D_\mu\big(f^a{}_{bc} c^b c^c \Lambda\big) = (D_\mu \delta c)^a + \frac{1}{2} D_\mu\big(f^a{}_{bc} c^b \Lambda c^c\big) = 0 \tag{6}$$

2 Relativistic Particle

$$L_q = L + L_{gf} + L_{gh}, \tag{7}$$

$$L = \frac{1}{2e}\left(\frac{1}{c_0}\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2 - \frac{e}{2} m^2 c_0^4, \tag{8}$$

$$L_{gh} = -e\dot{b}c \tag{9}$$

- For $t \mapsto t' = t - \xi(t)$, we have gauge transformation: $\delta X^\mu = \xi \dot{X}^\mu$, $\delta e = \frac{\mathrm{d}}{\mathrm{d}t}\big(e\xi\big)$, $\delta L = \frac{\mathrm{d}}{\mathrm{d}t}\big(\xi L\big)$, replace $\xi \mapsto c\Lambda$, and we have BRST transformation:

$$\delta X^\mu = c\Lambda \dot{X}^\mu = c\dot{X}^\mu \Lambda, \quad \delta e = \frac{\mathrm{d}}{\mathrm{d}t}\big(ec\Lambda\big) = \frac{\mathrm{d}}{\mathrm{d}t}\big(ec\big)\Lambda \tag{10}$$

- Assume nilpotency, and we have:

$$\begin{aligned}0 = \delta_\Lambda \delta_{\Lambda'} X^\mu &= \big((\delta_\Lambda c)\, \dot{X}^\mu + c\, \delta_\Lambda \dot{X}^\mu\big)\Lambda' \\ &= \Big((\delta_\Lambda c)\, \dot{X}^\mu + c\,\big(\dot{c}\dot{X}^\mu + \cancel{c\ddot{X}^\mu}\big)\Lambda\Big)\Lambda' \\ &= (\delta_\Lambda c + c\dot{c}\Lambda)\, \dot{X}^\mu \Lambda',\end{aligned} \tag{11}$$

$$\boxed{\delta_\Lambda c = -c\dot{c}\Lambda} \tag{12}$$

$$\begin{aligned}\delta_\Lambda \delta_{\Lambda'} e &= \frac{\mathrm{d}}{\mathrm{d}t}\big((\delta_\Lambda e)\, c + e\, \delta_\Lambda c\big)\Lambda' \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\big(ec\big)\Lambda c - e\, c\dot{c}\Lambda\right)\Lambda' \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big(e\dot{c}\Lambda c - e\, c\dot{c}\Lambda\big)\Lambda' = 0\end{aligned} \tag{13}$$

- The BRST transformation for c is also nilpotent:

$$\begin{aligned}\delta_\Lambda \delta_{\Lambda'} c &= -\big((\delta_\Lambda c)\, \dot{c} + c\, \delta_\Lambda \dot{c}\big)\Lambda' \\ &= -\big(-c\dot{c}\Lambda\, \dot{c} - c\, c\dot{c}\Lambda\big)\Lambda' = 0\end{aligned} \tag{14}$$

- Gauge fixing $f = e(t) - 1 = 0$ can be imposed by:

$$\delta\big[f\big] \sim \int \mathcal{D}d(t)\, \exp\left(i \int \mathrm{d}t\, d(t)\, f(t)\right), \tag{15}$$

$$L_{gf} = d(t) f(t) = d(t)\, \big(e(t) - 1\big) \tag{16}$$

The quantum action is $S_q = \int \mathrm{d}t\, L_q,\ L_q = L[X,e] + L_{gf}[e,d] + L_{gh}[e,b,c]$. We want S_q to be BRST invariant, which will help determine transformation rules for b, d; consider:

$$\begin{aligned}\delta(L_{gf} + L_{gh}) &= (e-1)\, \delta d + d\, \delta e - \delta\big(e\dot{b}c\big) \\ &= (e-1)\, \delta d + d(t)\, \frac{\mathrm{d}}{\mathrm{d}t}\big(ec\big)\, \Lambda + \big(ec\big)\, \delta\dot{b} \\ &= (e-1)\, \delta d + \frac{\mathrm{d}}{\mathrm{d}t}\big(ec\big)\, \big(d(t)\, \Lambda - \delta b\big) + \frac{\mathrm{d}}{\mathrm{d}t}\big(ec\, \delta b\big)\end{aligned} \tag{17}$$

We find a natural choice of $\delta b\,, \delta d$:

$$\delta d = 0, \quad \delta b = d(t)\, \Lambda, \quad \delta(L_{gf} + L_{gh}) = \frac{\mathrm{d}}{\mathrm{d}t}\big(ec\, \delta b\big) \tag{18}$$

- The complete quantum action is BRST invariant, since:

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}t}\big(\xi L\big)_{\xi \mapsto c\Lambda} = \frac{\mathrm{d}}{\mathrm{d}t}\big(cL\big)\Lambda, \quad \delta(L_{gf} + L_{gh}) = \frac{\mathrm{d}}{\mathrm{d}t}\big(ec\, \delta b\big), \tag{19}$$

$$\delta S_q = \int \mathrm{d}t\, \delta L_q = \int \mathrm{d}t\, \Big(\delta L + \delta(L_{gf} + L_{gh})\Big) = 0 \tag{20}$$

- Note that $\frac{\delta S_q}{\delta d} = 0 \ \Rightarrow\ f = e - 1 = 0$. Moreover,

$$\frac{\delta S_q}{\delta e} = 0 \quad \Longrightarrow \quad -\frac{1}{2}\left(\frac{1}{e^2 c_0^2}\dot{X}^2 + m^2 c_0^4\right) + d - \dot{b}c = 0, \tag{21}$$

$$d = d[X,b,c] = \frac{1}{2}\,\big(\tfrac{1}{c_0^2}\dot{X}^2 + m^2 c_0^4\big) + \dot{b}c \tag{22}$$

Therefore, it is convenient to consider the reduced Lagrangian $L_q[X,b,c] = (L_q)_{e=1,\, d=d[X]}$, where e, d are integrated out[1]. The symmetries are thus reduced to:

$$\delta X^\mu = c\dot{X}^\mu \Lambda, \quad \delta b = d[X,b,c]\Lambda, \quad \delta c = -c\dot{c}\Lambda, \tag{23}$$

[1] Reference: *Polchinski*.

$$\delta L_q = \frac{\mathrm{d}}{\mathrm{d}t}\Big(cL\Lambda + ec\,\delta b\Big)_{e=1} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(cL_{e=1} + \frac{c}{2}\left(\tfrac{1}{c_0^2}\dot{X}^2 + m^2c_0^4\right)\Big)\Lambda = \frac{\mathrm{d}}{\mathrm{d}t}\Big(c\left(\tfrac{1}{c_0}\dot{X}\right)^2\Big)\Lambda, \tag{24}$$

$$L_q = \frac{1}{2}\left(\tfrac{1}{c_0}\dot{X}\right)^2 - \frac{1}{2}m^2c_0^4 - \dot{b}c \tag{25}$$

On the other hand, the *on-shell* variation is given by:

$$\delta_0 L_q = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_q}{\partial \dot{X}^\mu}\,\delta X^\mu + \frac{\partial L_q}{\partial \dot{b}}\,\delta b\right) = \frac{\mathrm{d}}{\mathrm{d}t}\Big\{c\Big(\left(\tfrac{1}{c_0}\dot{X}\right)^2 + d[X,b,c]\Big)\Big\}\Lambda, \tag{26}$$

The $\dot{b}c$ term in $d[X,b,c]$ is killed by the c multiplication: $c\,d[X,b,c] = c\,d[X]$. Therefore, the canonical BRST charge Q is given by:

$$0 = \delta_0 L_q - \delta L_q = \frac{\mathrm{d}Q}{\mathrm{d}t}\Lambda = \frac{\mathrm{d}}{\mathrm{d}t}\big(c\,d[X]\big)\Lambda, \tag{27}$$

$$Q = c\,d[X] = \frac{c}{2}\left(\tfrac{1}{c_0^2}\dot{X}^2 + m^2c_0^4\right) \tag{28}$$

- Note that $L_{gh} = -\dot{b}c = b\dot{c} - \frac{\mathrm{d}}{\mathrm{d}t}\big(bc\big) \sim b\dot{c}$; for future convenience, let's replace $(-\dot{b}c) \mapsto b\dot{c}$ in the Lagrangian, and we have:

$$p_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = \tfrac{1}{c_0^2}\dot{X}_\mu, \quad p_c = \frac{\partial L}{\partial \dot{c}} \equiv \big(b\dot{c}\big)\frac{\overleftarrow{\partial}}{\partial \dot{c}} = b \tag{29}$$

Here we adopt the "right" derivative convention, in this case the Hamiltonian:

$$H = p_\mu \dot{X}^\mu + p_c\,\dot{c} - L_q = \frac{c_0^2}{2}\,p_\mu p^\mu + \frac{1}{2}m^2c_0^4 = \frac{1}{2}\left(p^2c_0^2 + m^2c_0^4\right) \tag{30}$$

- We have:

$$Q = \frac{c}{2}\left(\tfrac{1}{c_0^2}\dot{X}^2 + m^2c_0^4\right) = \frac{c}{2}\left(p^2c_0^2 + m^2c_0^4\right) = cH \tag{31}$$

After canonical quantization, p_μ, p_c and H are promoted to Hermitian operators, and:

$$Q^2 = cH\cdot cH = 0 \tag{32}$$

- Note that:

$$[p_\mu, X^\nu] = -i\delta^\nu_\mu, \quad [p_\mu, \mathcal{F}(X)] = -i\partial_\mu\mathcal{F}(X), \tag{33}$$

i.e. p_μ acts on $\mathcal{F}(X)$ by X–derivative; from the path integral perspective, we have:

$$\big\langle p_\mu\,\mathcal{F}(X)\big\rangle = \int \mathscr{D}p\;\mathscr{D}X\;\mathscr{D}b\;\mathscr{D}c\;e^{iS[p,X,b,c]}\,p_\mu\,\mathcal{F}(X), \tag{34}$$

$$S[p,X,b,c] = \int \mathrm{d}t\,\Big(p_\mu\dot{X}^\mu + \dot{b}c - H[p]\Big), \tag{35}$$

$$\int \mathrm{d}t'\,\frac{\delta S}{\delta X^\mu(t')} = \int \mathrm{d}t' \int \mathrm{d}t\,p_\mu\,\partial_t\,\delta(t-t') \sim -\int \mathrm{d}t'\,\dot{p}_\mu(t') = -p_\mu, \tag{36}$$

$$\begin{aligned}\big\langle p_\mu\,\mathcal{F}(X)\big\rangle &= \int \mathrm{d}t' \int \mathscr{D}p\;\mathscr{D}X\;\mathscr{D}b\;\mathscr{D}c\,\left(i\,\frac{\delta}{\delta X^\mu(t')}\,e^{iS[p,X,b,c]}\right)\mathcal{F}(X)\\ &= \int \mathscr{D}p\;\mathscr{D}X\;\mathscr{D}b\;\mathscr{D}c\;e^{iS[p,X,b,c]}\int \mathrm{d}t'\left(-i\,\frac{\delta}{\delta X^\mu(t')}\right)\mathcal{F}(X),\end{aligned} \tag{37}$$

$$p_\mu\,\mathcal{F}(X) \sim \int \mathrm{d}t'\left(-i\,\frac{\delta}{\delta X^\mu(t')}\right)\mathcal{F}(X) = -i\,\frac{\partial}{\partial X^\mu}\mathcal{F}(X) \tag{38}$$

For $e^{ik_\mu X^\mu}\left|0\right\rangle$, $\mathcal{F}(X) = e^{ik_\mu X^\mu}$, it is Q–closed iff.

$$\begin{aligned} 0 = Q\, e^{ik_\mu X^\mu}\left|0\right\rangle &= \frac{c}{2}\left(p^2c_0^2 + m^2c_0^4\right)e^{ik_\mu X^\mu}\left|0\right\rangle \\ &= \frac{c}{2}\left(k^2c_0^2 + m^2c_0^4\right)\left(e^{ik_\mu X^\mu}\left|0\right\rangle\right), \end{aligned} \tag{39}$$

$$k^2c_0^2 = k_\mu k^\mu c_0^2 = -E^2 + \mathbf{k}^2c_0^2 = -m^2c_0^4, \tag{40}$$

Or $E^2 = \mathbf{k}^2c_0^2 + m^2c_0^4$. This is the dispersion relation of a relativistic particle.