

QCD Partition Function at $\mathcal{O}(g^2)$

$$\ln \mathcal{Z}_I^{(2)} = -\frac{1}{2} \text{(a)} - \frac{1}{2} \text{(b)} + \frac{1}{12} \text{(c)} + \frac{1}{8} \text{(d)}$$

(a) (b) (c) (d)

$$\ln \mathcal{Z}_I^{(2)} = \ln \mathcal{Z}^{(a)} + \ln \mathcal{Z}^{(b)} + \ln \mathcal{Z}^{(c)} + \ln \mathcal{Z}^{(d)} \quad (1)$$

The contribution of (a) is given by:

$$\ln \mathcal{Z}^{(a)} = \frac{1}{2!} (-1)^1 \frac{T}{V} \sum_k \frac{T}{V} \sum_p \text{Tr} \left(S(k) (g\gamma^\nu T^b) S(p) (g\gamma^\mu T^a) \right) \left(\frac{V}{T} \right) \delta_{ab} \Delta_{\mu\nu}(p-k) \quad (2)$$

Here the trace goes over spinor, color and flavor indices. $S(k)$ is the quark propagator, with suppressed spinor, color and flavor indices, while $\delta_{ab} \Delta_{\mu\nu}$ is the gluon propagator, where a, b are adjoint indices; each vertex contributes a $(g\gamma^\mu T^a)$ factor.

In our convention, $k = (\omega_n, \mathbf{k})$ stands for the Euclidean 4-momentum with ω_n : the discrete Matsubara frequency. Each \sum_k comes with a factor $\frac{T}{V}$ while each spacetime delta function comes with an inverse factor: $\frac{V}{T}$; this is due to the fact that:

$$1 = \int \frac{d^4 k}{(2\pi)^4} (2\pi)^4 \delta^4(k - k_0) \sim \frac{1}{\beta V} \sum_k \beta V \delta_{k, k_0} \quad (3)$$

Following the same recipe from QED, we can write down:

$$\begin{aligned} \ln \mathcal{Z}^{(a)} &= - \left(\text{Tr} (T^a T^b) \delta_{ab} \right) \frac{1}{2} g^2 \frac{V}{T} \cdot \frac{T}{V} \sum_k \frac{T}{V} \sum_p \text{Tr} (S(k) \gamma^\nu S(p) \gamma^\mu) \Delta_{\mu\nu}(p-k) \\ &= - \left(\frac{N_c^2 - 1}{2} N_f \right) \frac{g^2}{288} \frac{V}{T} \left(5T^4 + \frac{18}{\pi^2} T^2 \mu^2 + \frac{9}{\pi^4} \mu^4 \right) \end{aligned} \quad (4)$$

The (b) term is structurally similar to the (a) term; now the amplitude can be written down simply by replacing the propagator $S(k) \mapsto W(k)$ of the ghost, while the vertex is $(g\gamma^\mu T^a) \mapsto (-igk^\nu T^b)$ instead:

$$\ln \mathcal{Z}^{(b)} = \frac{1}{2!} (-1)^1 \frac{T}{V} \sum_k \frac{T}{V} \sum_p \text{Tr} \left(W(k) (-igp^\nu T^b) W(p) (-igk^\mu T^a) \right) \left(\frac{V}{T} \right) \delta_{ab} \Delta_{\mu\nu}(p-k) \quad (5)$$

The trace now goes over suppressed *adjoint* indices of $W(k)_{ab} = -\delta_{ab} \Delta(k)$ and $(T_a)_{bc} = f_{abc}$, where f_{abc} is the structure constant of $\text{SU}(N_c)$. Therefore¹,

$$\begin{aligned} \ln \mathcal{Z}^{(b)} &= - \left(\text{Tr} (T^a T^b) \delta_{ab} \right) \frac{1}{2} g^2 \frac{V}{T} \cdot \frac{T}{V} \sum_k \frac{T}{V} \sum_p \Delta(k) \Delta(p) (-k^\mu p^\nu) \Delta_{\mu\nu}(p-k) \\ &= - \left(\frac{N_c^2 - 1}{2} 2N_c \right) \frac{1}{2} g^2 \frac{V}{T} \cdot \frac{T}{V} \sum_k \frac{T}{V} \sum_p \Delta(k) \Delta(p) (-k^\mu p^\nu) (-g_{\mu\nu}) \Delta(p-k) \end{aligned} \quad (6)$$

¹ $\text{Tr} (T^a T^b)_{\text{ad}}$ in the adjoint representation is precisely the *Killing form* of the $\mathfrak{su}(N_c)$ algebra, which is $2N_c$ times the $\text{Tr} (T^a T^b)_0$ in the fundamental representation; see Wikipedia: [Killing form](#).

Now we compute the remaining $\sum_{k,p}$. We have:

$$\Delta(k) \Delta(p) (k \cdot p) \Delta(p - k) = \frac{k \cdot p}{k^2 p^2 (p - k)^2} \quad (7)$$

The generic method to carry out such summation is by using the mixed representation of the propagator; for some propagator $D(k)$, we have:

$$\begin{aligned} D(k) = D(w_n, \mathbf{k}) &= \int_0^\beta d\tau e^{-i\omega_n \tau} T \sum_m e^{i\omega_m \tau} D(w_n, \mathbf{k}) \\ &= \int_0^\beta d\tau e^{-i\omega_n \tau} \tilde{D}(\tau, \mathbf{k}), \end{aligned} \quad (8)$$

$$\begin{aligned} \tilde{D}(\tau, \mathbf{k}) &= T \sum_m e^{i\omega_m \tau} D(w_n, \mathbf{k}) \\ &= T \sum_m e^{i\omega_m \tau} \int \frac{d\omega}{2\pi} \frac{\rho(\omega, \mathbf{k})}{\omega + i\omega_0} \\ &= \int \frac{d\omega}{2\pi} \rho(\omega, \mathbf{k}) T \sum_m \frac{e^{i\omega_m \tau}}{\omega + i\omega_0} \\ &= \int \frac{d\omega}{2\pi} \rho(\omega, \mathbf{k}) e^{-\omega \tau} (1 \pm n_\pm(\omega)), \end{aligned} \quad (9)$$

$$\rho(\omega, \mathbf{k}) = \frac{1}{i} (D(\omega + i\epsilon) - D(\omega - i\epsilon)) = 2 \operatorname{Im} D(\omega + i\epsilon, \mathbf{k}), \quad n_\pm = \frac{1}{e^{\beta\omega} \mp 1}, \quad (10)$$

Then the Matsubara sum \sum_{ω_n} becomes a sum over exponentials like $e^{-i\omega_n \tau}$, which is easier to deal with. However, for this particular problem, there is a shortcut²; notice that the denominator of (7) is invariant under $p \mapsto k - p$, hence:

$$\begin{aligned} \sum_p \frac{k \cdot p}{k^2 p^2 (p - k)^2} &= \sum_{(k-p)} \frac{k \cdot (k - p)}{k^2 (k - p)^2 p^2} \\ &= \sum_p \frac{k \cdot (k - p)}{k^2 p^2 (p - k)^2} \\ &= \sum_p \frac{\frac{1}{2}(k \cdot p + k \cdot (k - p))}{k^2 p^2 (p - k)^2} \\ &= \sum_p \frac{1}{2p^2 (p - k)^2}, \end{aligned} \quad (11)$$

$$\begin{aligned} \frac{T}{V} \sum_k \frac{T}{V} \sum_p \Delta(k) \Delta(p) (k \cdot p) \Delta(p - k) &= \frac{1}{2} \frac{T}{V} \sum_p \frac{1}{p^2} \frac{T}{V} \sum_k \frac{1}{(p - k)^2} \\ &= \frac{1}{2} \left(\frac{T}{V} \sum_p \frac{1}{p^2} \right)^2 = \frac{1}{2} \left(\frac{T^2}{12} \right)^2, \end{aligned} \quad (12)$$

²Reference: Laine & Vuorinen, *Basics of Thermal Field Theory*.

$$\boxed{\ln \mathcal{Z}^{(b)} = -\left(\frac{N_c^2 - 1}{2} 2N_c\right) \frac{1}{2} g^2 \frac{V}{T} \cdot \frac{1}{2} \left(\frac{T^2}{12}\right)^2 = -\frac{V}{T} N_c (N_c^2 - 1) \frac{1}{4} g^2 \frac{T^4}{144}} \quad (13)$$

The (c) term is structually similar to the (b) term, but with a symmetrized 3-gluon vertex:

$$\left(\frac{1}{3!}\right) igf_{abc} (g_{\mu\nu}(k-p)_\rho + g_{\nu\rho}(p-q)_\mu + g_{\rho\mu}(q-k)_\nu) = \left(\frac{1}{3!}\right) igf_{abc} D_{\mu\nu\rho}(k, p, q) \quad (14)$$

To link the legs of two 3-gluon vertices as shown in (c), there are $3!$ possibilities. Therefore, we have:

$$\ln \mathcal{Z}^{(c)} = \frac{1}{2!} \cdot 3! \cdot \left(\frac{1}{3!}\right)^2 \frac{T}{V} \sum_k \frac{T}{V} \sum_p \text{Tr} \left(W(k) (igf_{abc} D_{\mu\nu\rho}(k, -p, p-k)) W(k) (igf^{bac} D^{\mu\nu\rho}(p, -k, k-p)) \right) \left(\frac{V}{T}\right) \delta_{ab} \Delta_\mu \quad (15)$$

☞ PAST WORK, AS TEMPLATE ☞

With the metric convention: $g \sim (-+++)$, we have:

$$\mathcal{Z} = \int \mathcal{D}A^\mu e^{-S} \delta[\partial_\mu A^\mu - f] \det[\partial^2 \delta^4(x-y)] \quad (16)$$

Here S is the Euclidean action:

$$(-S) = \int d^4x \mathcal{L}_{t=-i\tau}, \quad \int d^4x = \int_0^\beta d\tau \int d^3\mathbf{x} \quad (17)$$

For pure QED, we have:

$$\mathcal{L}_t = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (18)$$

Setting $t = i\tau$ is equivalent to carrying out a Wick rotation: $x^0 \mapsto -ix^0$, $A^0 \mapsto -iA^0$, while:

$$g_{\mu\nu} A^\mu A^\nu = g'_{\mu\nu} A'^\mu A'^\nu \implies g_{\mu\nu} \mapsto g'_{\mu\nu} = \delta_{\mu\nu} \quad (19)$$

Under this convention, the Euclidean action is formally unchanged; same applies for the gauge-fixing and the ghost term:

$$\mathcal{L}_{t=-i\tau} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad g_{\mu\nu} \mapsto \delta_{\mu\nu}, \quad (20)$$

$$\delta[\partial_\mu A^\mu - f] \implies \mathcal{L}_{gf} = -\frac{1}{2\rho} (\partial_\mu A^\mu)^2, \quad (21)$$

$$\det[\partial^2 \delta^4(x-y)] \implies \mathcal{L}_{gh} \sim (\partial^2 \bar{\eta}) \eta \sim -\partial_\mu \bar{\eta} \partial^\mu \eta, \quad (22)$$

Here we've dropped some total derivative terms in the ghost Lagrangian. The partition function is then reduced to:

$$\mathcal{Z} = \int \mathcal{D}A^\mu \mathcal{D}\bar{\eta} \mathcal{D}\eta e^{-S'}, \quad (-S') = \int d^4x (\mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{gh})_{t=-i\tau} \quad (23)$$

The action can be further simplified by partial integration and dropping boundary terms:

$$\begin{aligned} (-S') &= \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\rho} (\partial_\mu A^\mu)^2 - \partial_\mu \bar{\eta} \partial^\mu \eta \right) \\ &= \int d^4x \left(-\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2\rho} (\partial_\mu A^\mu)^2 - \partial_\mu \bar{\eta} \partial^\mu \eta \right) \\ &= \int d^4x \left(-\frac{1}{2} (\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu) - \frac{1}{2\rho} (\partial_\mu A^\mu \partial_\nu A^\nu) - \partial_\mu \bar{\eta} \partial^\mu \eta \right) \\ &\sim \int d^4x \left(-\frac{1}{2} (-A_\nu \partial^2 A^\nu + A_\mu \partial^\mu \partial_\nu A^\nu) + \frac{1}{2\rho} (A^\mu \partial_\mu \partial_\nu A^\nu) + \bar{\eta} \partial^2 \eta \right) \\ &= \int d^4x \left(-\frac{1}{2} A^\mu \left(-\delta_{\mu\nu} \partial^2 + \partial_\mu \partial_\nu - \frac{1}{\rho} \partial_\mu \partial_\nu \right) A^\nu + \bar{\eta} \partial^2 \eta \right) \\ &= -\frac{1}{2} \int d^4x \left(A^\mu \left(-\delta_{\mu\nu} \partial^2 + \left(1 - \frac{1}{\rho}\right) \partial_\mu \partial_\nu \right) A^\nu - 2\bar{\eta} \partial^2 \eta \right) \end{aligned} \quad (24)$$

With $\beta = \frac{1}{T}$, expand A^μ, η into dimensionless Fourier modes, and we have:

$$A^\mu = \frac{1}{\sqrt{TV}} \sum_k e^{ik_\nu x^\nu} A_k^\mu, \quad \sum_k e^{ik_\nu x^\nu} = V \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} \sum_{n \in \mathbb{Z}} e^{i\omega_n \tau} \quad (25)$$

$$\begin{aligned}
\sum_{p,k} \int d^4x e^{i(p+k) \cdot x} &= \sum_{p,k} (2\pi)^4 \delta^4(p+k) \\
&= V^2 \int \frac{d^3\mathbf{p} d^3\mathbf{k}}{(2\pi)^6} \sum_{m,n \in \mathbb{Z}} \beta \delta_{m,-n} \cdot (2\pi)^3 \delta^3(\mathbf{p} + \mathbf{k}) \\
&= \beta V \cdot V \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{n \in \mathbb{Z}} \int d^3\mathbf{p} \delta^3(\mathbf{p} + \mathbf{k}) \sum_{n \in \mathbb{Z}} \delta_{m,-n} \\
&= \beta V \sum_k \int d^3\mathbf{p} \delta^3(\mathbf{p} + \mathbf{k}) \sum_{n \in \mathbb{Z}} \delta_{m,-n},
\end{aligned} \tag{26}$$

$$\begin{aligned}
(-S') &= -\frac{1}{2TV} \sum_{p,k} \int d^4x e^{i(p+k) \cdot x} \left(A_p^\mu \left(-\delta_{\mu\nu}(-k^2) + \left(1 - \frac{1}{\rho}\right) (-k_\mu k_\nu) \right) A_k^\nu - 2\bar{\eta}_p(-k^2)\eta_k \right) \\
&= -\frac{\beta V}{2TV} \sum_k \left(A_{-k}^\mu \left(k^2 \delta_{\mu\nu} - \left(1 - \frac{1}{\rho}\right) k_\mu k_\nu \right) A_k^\nu + 2\bar{\eta}_{-k} k^2 \eta_k \right) \\
&= -\frac{\beta^2}{2} \sum_k \left(A_k^{\mu\dagger} D_{\mu\nu}^{-1}(k) A_k^\nu + 2\bar{\eta}_k^\dagger k^2 \eta_k \right)
\end{aligned} \tag{27}$$

Here we've used the reality condition on A, η , namely $A_k^{\mu\dagger} = A_{-k}^\mu$, and defined the k -space inverse propagator $D_{\mu\nu}^{-1}(k)$. Similar result applies for η_k , except that we have to be careful about Grassmann variables. Carry out $\int \mathcal{D}A^\mu \mathcal{D}\bar{\eta} \mathcal{D}\eta$, and we have:

$$\begin{aligned}
\mathcal{Z} &\sim \left(\det_{\mu,\nu,k} \beta^2 D_{\mu\nu}^{-1}(k) \right)^{-1/2} \left(\det_k 2\beta^2 k^2 \right)^{+1} \\
&= \prod_k \left(\beta^{2 \times 4 \times (-1/2)} \cdot \left(\det_{\mu,\nu} D_{\mu\nu}^{-1}(k) \right)^{-1/2} \cdot 2\beta^2 k^2 \right) \\
&= \prod_k \left(\beta^{-4} \left(\det_{\mu,\nu} D_{\mu\nu}^{-1}(k) \right)^{-1/2} \cdot 2\beta^2 k^2 \right)
\end{aligned} \tag{28}$$

The determinant is evaluated as follows³:

$$\begin{aligned}
\det_{\mu,\nu} D_{\mu\nu}^{-1} &= \det_{\mu,\nu} \left(k^2 \delta_{\mu\nu} - \left(1 - \frac{1}{\rho}\right) k_\mu k_\nu \right) \\
&= k^8 \det_{\mu,\nu} \left(\delta_{\mu\nu} - \left(1 - \frac{1}{\rho}\right) \frac{k_\mu k_\nu}{k^2} \right) \\
&= k^8 \left(1 - \left(1 - \frac{1}{\rho}\right) \frac{k^2}{k^2} \right) \\
&= \frac{1}{\rho} k^8,
\end{aligned} \tag{29}$$

$$\mathcal{Z} \sim \prod_k \frac{2}{\rho} \beta^{-4} k^{-4} \cdot \beta^2 k^2 \sim \prod_k \beta^{-2} k^{-2}, \tag{30}$$

$$\ln \mathcal{Z} \sim - \sum_k \ln(\beta^2 k^2) \tag{31}$$

³Reference: Wikipedia: *Determinant # Sylvester's determinant theorem*.

We see that $\ln \mathcal{Z}$ is simply twice the result of a neutral scalar field, with mass $m \rightarrow 0$, i.e.

$$\ln \mathcal{Z} \sim -2 \times \frac{1}{2} \sum_k \ln(\beta^2 k^2) \sim -2 \sum_k \left(\frac{1}{2} \beta E_k + \ln(1 - e^{-\beta E_k}) \right), \quad (32)$$

$$\mathcal{Z} \sim \prod_k \left\{ \exp \left(-\frac{1}{2} \beta E_k - \ln(1 - e^{-\beta E_k}) \right) \right\}^2, \quad (33)$$

$$\Omega = -T \ln \mathcal{Z} = 2 \sum_k \left(\frac{1}{2} E_k + T \ln(1 - e^{-E_k/T}) \right), \quad (34)$$

$$p = -\frac{\Omega}{V} = -2 \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \left(\frac{1}{2} E_k + T \ln(1 - e^{-E_k/T}) \right), \quad (35)$$

Here $E_k = \|\mathbf{k}\|$. Ignore the vacuum contribution to p , and we have:

$$p = -2T \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \ln(1 - e^{-\|\mathbf{k}\|/T}) = 2 \frac{\pi^2}{90} T^4 \quad (36)$$

Summary

1. We've completed the calculation of pure QED partition function \mathcal{Z} and thermodynamic potential Ω (energy density p) by introducing a “soft” gauge fix \mathcal{L}_{gf} . Alternatively, we can simply impose a “hard” Lorenz gauge fix $\delta[\partial_\mu A^\mu]$; this can be achieved by taking $\rho \rightarrow \infty$ in the Gaussian packet $-\frac{1}{2\rho} (\partial_\mu A^\mu)^2$, or by integrating out A^0 directly — similar to (27), we have:

$$\begin{aligned} \mathcal{Z} &\sim \int \mathcal{D}A^\mu \mathcal{D}\bar{\eta} \mathcal{D}\eta \delta[\partial_\mu A^\mu] \exp \left(-\frac{\beta^2}{2} \sum_k \left(A_k^{\mu\dagger} (k^2 \delta_{\mu\nu} - k_\mu k_\nu) A_k^\nu + 2\bar{\eta}_k^\dagger k^2 \eta_k \right) \right) \\ &\sim \int \mathcal{D}A^i \mathcal{D}\bar{\eta} \mathcal{D}\eta \exp \left(-\frac{\beta^2}{2} \sum_k \left(A_k^{\mu\dagger} (k^2 \delta_{\mu\nu} - k_\mu k_\nu) A_k^\nu + 2\bar{\eta}_k^\dagger k^2 \eta_k \right) \right)_{A^0=A^0[A^i]} \end{aligned} \quad (37)$$

$$A^0[A^i] = - \int \partial_i A^i d\tau, \quad k_\mu A_k^\mu = 0, \quad (38)$$

Here we've omitted a non-dynamical Jacobian $\det[\theta(\tau - \tau')] = \det \int^\tau d\tau'' \delta(\tau'' - \tau')$. The ghost integral gives the same contribution, while the A^i integral yields:

$$\mathcal{Z}_A = \int \mathcal{D}A^i \exp \left(-\frac{\beta^2}{2} \sum_k \left(A_k^{\mu\dagger} (k^2 \delta_{\mu\nu}) A_k^\nu \right) \right), \quad (39)$$

$$\begin{aligned} A_k^{\mu\dagger} (k^2 \delta_{\mu\nu}) A_k^\nu &= A_k^{i\dagger} (k^2 \delta_{ij}) A_k^j + A_k^{0\dagger} (k^2) A_k^0 = A_k^{i\dagger} (k^2 \delta_{ij}) A_k^j + \frac{k^2}{\omega^2} A_k^{0\dagger} (\omega^2) A_k^0 \\ &\stackrel{(38)}{=} A_k^{i\dagger} (k^2 \delta_{ij}) A_k^j + \frac{k^2}{\omega^2} A_k^{i\dagger} (k_i k_j) A_k^0 = A_k^{i\dagger} k^2 \left(\delta_{ij} + \frac{k_i k_j}{\omega^2} \right) A_k^j \\ &= A_k^{i\dagger} D_{ij}^{-1}(k) A_k^j, \end{aligned} \quad (40)$$

$$\begin{aligned} \mathcal{Z}_A &\sim \left(\det_{i,j,k} \beta^2 D_{ij}^{-1}(k) \right)^{-1/2} = \prod_k \beta^{-3} k^{-3} \left(1 + \frac{\mathbf{k}^2}{\omega^2} \right)^{-1/2} = \prod_k (\beta^{-4} k^{-4} \cdot \beta \omega) \\ &= \prod_k \beta^{-4} k^{-4} \prod_{\mathbf{k}} \prod_n 2\pi n \sim \prod_k \beta^{-4} k^{-4} \end{aligned} \quad (41)$$

We see that the result from a “hard” Lorenz gauge fixing is the same as before, up to a non-dynamical overall coefficient.

2. In our previous calculations, we notice that after functional integration, ρ is just a non-dynamical overall coefficient in \mathcal{Z} , hence it can be safely dropped from the final expression; see eq. (30). Therefore, the result is independent of parameter ρ .