Advanced QFT №1

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1 Symmetry & Noether's Theorem

1.1 2D σ-Model

$$\mathcal{L} = -\frac{1}{2}\,\eta_{\alpha\beta}\,\eta_{\mu\nu}\,\partial^\alpha X^\mu\,\partial^\beta X^\nu = -\frac{1}{2}\,\partial^\alpha X_\mu\,\partial_\alpha X^\mu, \quad X^\mu \in \mathbb{R}^{1,D-1} \tag{1}$$

- For $\delta X^\mu = a^\mu + \lambda^\mu{}_\nu X^\nu$, the Lagrangian (density) transforms as follows:

$$\begin{aligned}
\delta\mathcal{L} &= -\partial^\alpha X_\mu\,\partial_\alpha\,\delta X^\mu \\
&= -\partial^\alpha X_\mu\,\partial_\alpha(a^\mu + \lambda^\mu{}_\nu X^\nu) \\
&= -\partial^\alpha X_\mu\,(\partial_\alpha a^\mu + X^\nu\,\partial_\alpha\lambda^\mu{}_\nu + \lambda^\mu{}_\nu\,\partial_\alpha X^\nu) \\
&= -\partial^\alpha X_\mu\,\partial_\alpha a^\mu - \partial^\alpha X^\mu\,\partial_\alpha X^\nu\,\lambda_{\mu\nu} - X^\nu\,\partial^\alpha X^\mu\,\partial_\alpha\lambda_{\mu\nu} \\
&= -\partial^\alpha X_\mu\,\partial_\alpha a^\mu - \partial^\alpha X^\mu\,\partial_\alpha X^\nu\,\lambda_{(\mu\nu)} - X^\nu\,\partial^\alpha X^\mu\,\partial_\alpha\lambda_{\mu\nu}
\end{aligned} \tag{2}$$

Since a^μ and $\lambda^\mu{}_\nu$ are independent, imposing $\delta L = 0$ yields $\partial_\alpha a^\mu = 0$, $a = \text{const}$. Furthermore, if $\delta L = 0$ is to hold for arbitrary X^μ fields, then $\partial_\alpha\lambda_{\mu\nu} = 0$, $\lambda_{(\mu\nu)} = 0$, i.e. $\lambda_{\mu\nu}$ is constant and anti-symmetric over its indices.

- Promote $\delta X \mapsto \epsilon(x)\,\delta X = \epsilon(x)\,(a^\mu + \lambda^\mu{}_\nu X^\nu)$, with $\epsilon(x)$ some localized bump function; using (2) and considering *on-shell* variation, we have:

$$\begin{aligned}
0 = \delta S &= -\int \mathrm{d}^2x\,(\partial^\alpha X_\mu\,a^\mu\,\partial_\alpha\epsilon + X^\nu\,\partial^\alpha X^\mu\,\lambda_{\mu\nu}\,\partial_\alpha\epsilon) \\
&= -\int \mathrm{d}^2x\,\Big(\partial^\alpha X_\mu\,a^\mu\; + X_{[\nu}\,\partial^\alpha X_{\mu]}\,\lambda^{[\mu\nu]}\Big)\,\partial_\alpha\epsilon
\end{aligned} \tag{3}$$

It is evident (after partial integration) that the following currents are conserved; they are the Noether currents associated with a^μ and $\lambda^{[\mu\nu]}$:

$$j^\alpha_\mu = -\partial^\alpha X_\mu, \quad j^\alpha_{\mu\nu} = -X_{[\nu}\,\partial^\alpha X_{\mu]} = \frac{1}{2}\,(X_\mu\,\partial^\alpha X_\nu - X_\nu\,\partial^\alpha X_\mu) \tag{4}$$

Conserved charge $Q = \int \mathrm{d}^2x\,j^0(x)$, we have:

$$P_\mu = -\int \mathrm{d}x^1\,\partial^0 X_\mu = \int \mathrm{d}x^1\,\partial_0 X_\mu, \quad M_{\mu\nu} = \frac{1}{2}\int \mathrm{d}x^1\,(X_\nu\,\partial_0 X_\mu - X_\mu\,\partial_0 X_\nu) \tag{5}$$

They can be interpreted as spacetime momentum and spacetime angular momentum. ■

1.2 Real Scalar in $(3+1)$ D

$$\mathcal{L} = -\frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}\,m^2\phi^2 \tag{6}$$

- For ϕ: scalar, under $x' = \lambda \circ x$, $\phi(x) \mapsto \phi'(x)$, while:

$$\phi'(x') = \phi(x) \implies \phi'(x) = \phi(\lambda^{-1} \circ x) \tag{7}$$

For $\lambda \sim \lambda^{\mu}{}_{\nu}$: Lorentz transformation, $\eta_{\mu\nu}\lambda^{\mu}{}_{\rho}\lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$, or equivalently, $(\lambda^{-1})^{\mu}{}_{\nu} = \lambda_{\nu}{}^{\mu}$. Therefore,

$$\phi'(x^{\mu}) = \phi(\lambda^{-1} \circ x^{\mu}) = \phi(x^{\nu}\lambda_{\nu}{}^{\mu}) \tag{8}$$

- Under $x'^{\mu} = \lambda^{\mu}{}_{\nu}x^{\nu}$, we have:

$$\begin{aligned}
\mathcal{L}'(x') &= -\frac{1}{2}\,\partial'^{\mu}\phi'(x')\,\partial'_{\mu}\phi'(x') - \frac{1}{2}\,m^2\phi'^2(x') \\
&= -\frac{1}{2}\,\partial'^{\mu}\phi(x)\,\partial'_{\mu}\phi(x) - \frac{1}{2}\,m^2\phi^2(x) \\
&= -\frac{1}{2}\,\eta^{\mu\nu}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\,\partial_{\rho}\phi(x)\,\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\,\partial_{\sigma}\phi(x) - \frac{1}{2}\,m^2\phi^2(x) \\
&= -\frac{1}{2}\,\eta^{\rho\sigma}\partial_{\rho}\phi(x)\,\partial_{\sigma}\phi(x) - \frac{1}{2}\,m^2\phi^2(x) \\
&= \mathcal{L}(x)
\end{aligned} \tag{9}$$

Here we've used $\eta^{\mu\nu}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} = \eta^{\mu\nu}\lambda_{\mu}{}^{\rho}\lambda_{\nu}{}^{\sigma} = \eta^{\rho\sigma}$. Furthermore, $S' = \int \mathrm{d}^4x\,\mathcal{L}'(x) = \int \mathrm{d}^4x'\,\mathcal{L}'(x') = \int \mathrm{d}^4x'\,\mathcal{L}(x) = \int \mathrm{d}^4x\,\mathcal{L}(x) = S$, hence the action is invariant under Lorentz transformation.

- Consider an infinitesimal Lorentz transformation: $\lambda \sim \mathbb{1} + \omega$, then $\eta_{\mu\nu}\lambda^{\mu}{}_{\rho}\lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$ implies that $\omega_{\mu\nu}$ is anti-symmetric: $\omega_{\mu\nu} + \omega_{\nu\nu} = 0$. For $\delta x^{\mu} = \omega^{\mu}{}_{\nu}x^{\nu}$, we have:

$$\delta\phi = -\frac{\partial\phi}{\partial x^{\mu}}\,\delta x^{\mu} = -\omega^{\mu}{}_{\nu}x^{\nu}\,\partial_{\mu}\phi \tag{10}$$

To obtain the corresponding Noether charges, we can simply repeat the operations done in our previous problem; alternatively, we can try to derive a general recipe[1]: for $\mathcal{L} = \mathcal{L}(\phi, \partial_{\mu}\phi)$ and $S = \int \mathrm{d}^4x\,\mathcal{L}$, we have:

$$\begin{aligned}
\delta S &= \int \mathrm{d}^4x\,\delta\mathcal{L} \\
&= \int \mathrm{d}^4x\left(\frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\,\delta\partial_{\mu}\phi\right) \\
&= \int \mathrm{d}^4x\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\right)\delta\phi + \int \mathrm{d}^4x\,\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\,\delta\phi\right)
\end{aligned} \tag{11}$$

If we vary S w.r.t. a symmetry of the system, we will have $\delta\mathcal{L} = \partial_{\mu}K^{\mu}$ some total derivative; when on-shell, such variation gives the conserved current with boundary term K^{μ}:

$$j^{\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\,\delta\phi - K^{\mu} \tag{12}$$

[1] References: arXiv:1601.03616 and *Tong:* http://damtp.cam.ac.uk/user/tong/qft.html

Back to our Lorentz transformation $\delta\phi = -\omega^{\mu}{}_{\nu} x^{\nu} \partial_{\mu}\phi$, we have symmetry variation:

$$\delta\mathcal{L} = -\omega^{\mu}{}_{\nu} x^{\nu} \partial_{\mu}\mathcal{L} = -\partial_{\mu}(\omega^{\mu}{}_{\nu} x^{\nu} \mathcal{L}) \tag{13}$$

This gives a boundary term $K^{\mu} = -\omega^{\mu}{}_{\nu} x^{\nu}\mathcal{L}$, and the Noether current and its corresponding conserved charge can be calculated as follows:

$$j^{\mu} = -\omega^{\sigma}{}_{\nu} x^{\nu} \left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\,\partial_{\sigma}\phi - \delta^{\mu}_{\sigma}\mathcal{L} \right), \tag{14}$$

$$Q = \int \mathrm{d}^3x\, j^0 = -\omega^{\sigma}{}_{\nu} \int \mathrm{d}^3x\, x^{\nu} \big(\partial_0\phi\,\partial_{\sigma}\phi - \delta^{0}_{\sigma}\mathcal{L} \big), \tag{15}$$

Note that $\omega^{\mu}{}_{\nu}$ is arbitrary, therefore Q can be decomposed into independent charges:

$$Q = \omega_{\mu\nu} M^{\mu\nu}, \quad M^{\mu\nu} = -\int \mathrm{d}^3x\, x^{[\mu} \Big(\partial_0\phi\,\partial^{\nu]}\phi - \eta^{\nu]0}\mathcal{L} \Big), \tag{16}$$

The indices of $M^{\mu\nu}$ are anti-symmetrized to match the degrees of freedom in $\omega_{\mu\nu}$. Note that the $\mathcal{L}$ term only appears when one of the indices is 0.

Canonical quantization:

$$\dot{\phi} = \partial_0\phi, \quad \Pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}, \quad [\phi(\mathbf{x}), \Pi(\mathbf{y})] = [\phi, \dot{\phi}](\mathbf{x}) = i\delta(\mathbf{x} - \mathbf{y}) \tag{17}$$

Other equal-time commutators between ϕ, Π all just vanish. We have:

$$M^{0i} = -M^{i0} = -\frac{1}{2} \int \mathrm{d}^3x \left(\dot{\phi}\left(x^0\partial^i - x^i\partial^0 \right)\phi + x^i\mathcal{L} \right), M^{ij} = -\frac{1}{2} \int \mathrm{d}^3x\, \dot{\phi}\left(x^i\partial^j - x^j\partial^i \right)\phi \tag{18}$$

Notice that $x^{[\mu}\partial^{\nu]} = \frac{1}{2}\left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu} \right) = \frac{1}{2}D^{ij}$ is the Killing vector fields of $\mathbb{R}^{3,1}$, hence they naturally follow the commutation relations of $\mathfrak{so}(3,1)$ (up to a constant coefficient)[2]. We have:

$$\begin{aligned} [M^{ij}, M^{kl}] &= \frac{1}{4} \int \mathrm{d}^3x \int \mathrm{d}^3y \left[\dot{\phi}D^{ij}\phi(x), \dot{\phi}D^{kl}\phi(y) \right] \\ &= \frac{1}{4} \int \mathrm{d}^3x\, \dot{\phi}\left[D^{ij}, D^{kl} \right]\phi \end{aligned} \tag{19}$$

Similar holds for M^{i0}. Therefore, $M^{\mu\nu}$'s indeed form the Lie algebra $\mathfrak{so}(3,1)$. ■

[2] I would like to thank 林般 for pointing this out.