String Theory №7 [WIP]

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1 T-duality of Heterotic Strings[1]

We use $d \leq 10$ to denote the number of noncompact dimensions; the remaining $m \geq d$ dimensions are compactified. For heterotic strings, the $I \geq 10$ dimensions of the left-moving sector are already compactified on a lattice Γ_{16} or $\Gamma_8 \times \Gamma_8$. Here we use the label I to index the 16 internal dimensions.

(a) Generally, if we compactify an open string on the x^m direction: $x^m \cong x^m + 2\pi R$ with constant backgrounds A_m, then its zero mode spectrum, *with winding* $w = 0$, can be obtained from canonical quantization of the effective point particle action, with an additional gauge action term in the form of a Wilson line[2]:

$$-W_q = -iq \int \mathrm{d}x^m \, A_m \sim -iq \int \mathrm{d}\tau \, A_m \dot{X}^m \tag{1}$$

By imposing that the canonical momentum to be periodic along x^m, we find that:

$$k_m = \frac{n_m}{R} - qA_m \tag{2}$$

To obtain the winding states, we have to reproduce the above action from the world-sheet description. For heterotic strings with $m < 10 \leq I \leq 26$, this can be achieved by adding the following term to the usual world-sheet action[3]:

$$S_A \propto \int \mathrm{d}^2\sigma \, \epsilon^{ab} A^I_\mu \, \partial_a X^\mu \, \partial_b X_I \tag{3}$$

With proper normalizations to match the result in (2).

Canonical quantization then produces[4]:

$$k_m = \frac{n_m}{R} \pm \frac{w_m R}{\alpha'} - q_I A^I_m - \frac{w_n R}{2} A^n_I A^I_m, \tag{4}$$

$$k^I_L = \sqrt{\frac{2}{\alpha'}} \left(q^I + w^m R A^I_m\right), \tag{5}$$

The "$\pm$" signs in k_m correspond to the left and right-moving sector, respectively. Only the left-moving sector has an additional 16 dimensional internal torus, therefore k^I is labeled with an "L".

Note that the charge q^I now takes value on the Γ_{16} or $\Gamma_8 \times \Gamma_8$ lattice, and:

$$l \circ l' = \frac{\alpha'}{2} \left(k^I_L \, k'_{L,I} + k^m_L \, k'_{L,m} - k^m_R \, k'_{R,m}\right) = q^I q'_I + 2nw \tag{6}$$

We can then see that the new "extended" lattice indeed satisfies the even and self-dual conditions, which follows from the even and self-dual properties of Γ_{16} or $\Gamma_8 \times \Gamma_8$.

[1] I would like to thank Lucy Smith for help with this problem.

[2] Reference: *Polchinski*, Chapter 8.

[3] Reference: *Blumenhagen et al*, Basic Concepts of String Theory.

[4] Reference: *Polchinski*, Chapter 11.

(b) With $m = 9$ and $G_{dd} = 1$, we have:

$$W_q = \exp\left(-iq_I\theta^I\right), \quad A_9^I = -\frac{\theta^I}{2\pi R} \tag{7}$$

Note that W_q captures the phase change of the paths that wind around x^9; the extra phase from a non-trivial Wilson line might affect the boundary condition of some states while leaving others intact, thus breaking the original gauge symmetry. Our discussions here closely follow *Polchinski*, Chapter 11.

For the SO(32) theory with:

$$RA_9^I = \text{diag}\left((\tfrac{1}{2})^8, 0^8\right) \tag{8}$$

Adjoint states are labeled by a pair of indices valued in $1, \cdots, 32$; those with one index from $1 \leq A \leq 16$ and one from $17 \leq A \leq 32$ are antiperiodic due to the additional phase $e^{i\pi} = -1$ from the Wilson line, so the gauge symmetry is reduced to SO(16) $\times$ SO(16).

Similarly, for the $E_8 \times E_8$ theory with:

$$R' A_9^I = \text{diag}\left(1, 0^7, 1, 0^7\right) \tag{9}$$

Note that Γ_8, the root lattice of E_8, is basically the root lattice union an additional spinor weight lattice of SO(16). With the above Wilson line, the integer-charged states from the SO(16) root lattice in each E_8 remain periodic, while the half-integer charged states from the SO(16) spinor lattices become antiperiodic, due to the additional phase $e^{i\frac{1}{2}\cdot 2\pi} = -1$. Again the gauge symmetry is broken down to SO(16) $\times$ SO(16).

In summary, with the above Wilson line, the SO(32) and $E_8 \times E_8$ theory shares a unbroken gauge of SO(16) $\times$ SO(16). Consider he spectrum of the SO(16) $\times$ SO(16) *neutral states*, i.e. those with internal momentum:

$$k_L^I = \sqrt{\frac{2}{\alpha'}}\left(q^I + wRA_9^I\right) = 0 \tag{10}$$

For the SO(32) theory, since $q^I \in \Gamma_{16}$ while $RA_9^I = \text{diag}\left((\tfrac{1}{2})^8, 0^8\right)$, we must have $w = 2m$ for this to hold. The same goes for the $E_8 \times E_8$ theory; therefore, we have:

$$k_{L,R} = \frac{\tilde{n}}{R} \pm \frac{2mR}{\alpha'}, \quad k'_{L,R} = \frac{\tilde{n}'}{R'} \pm \frac{2m'R'}{\alpha'}, \tag{11}$$

$$\tilde{n} = n + 2m, \quad \tilde{n}' = n' + 2m' \tag{12}$$

(c) If the two theories are related by T-duality, then we should expect:

$$(k_L, k_R) \longleftrightarrow (k'_L, -k'_R), \tag{13}$$

Under suitable mapping of parameters. Indeed, it is straightforward to verify that $(\tilde{n}, m) \leftrightarrow (m', \tilde{n}')$ realizes this, along with $RR' = \alpha'/2$. The above arguments can then be generalized to higher levels, by acting on fermionized left-moving fields λ^A and carefully organizing representations. We see that the two heterotic string theories are equivalent under T-duality.

$\boxed{2}$ String Junction[5]

For a string junction to be mechanically stable, the tension force exerted on the junction must cancel each other; this is a Newtonian mechanics problem, but with (p,q)-string tension given by the BPS bound:

$$\tau_{(p,q)} = \frac{\sqrt{p^2+q^2/g^2}}{2\pi\alpha'} \tag{14}$$

Stability of the system implies that the BPS bound should be saturated.

From Newtonian mechanics, we know that three forces cannot cancel each other unless they are co-planar. Therefore, a 3-string junction must be co-planar in order to be stable. Suppose they lie in the (X^1, X^2) plane, then the tension exerted on the junction can expressed as:

$$\vec{T}_i = \tau_{(p_i,q_i)}\,(\cos\theta_i, \sin\theta_i), \quad i = 1,2,3, \tag{15}$$

$\sum_i \vec{T}_i = 0$ gives two equations, and we have two independent unknowns (the angle between two pairs of strings); therefore if a solution exists, it should be unique up to rotations and reflections.

In fact, a solution can be found by simple observations:

$$\cos\theta_i = \frac{p_i}{\sqrt{p^2+q^2/g^2}}, \quad \sin\theta_i = \frac{q_i/g}{\sqrt{p^2+q^2/g^2}}, \tag{16}$$

It satisfies $\sum_i \vec{T}_i = 0$ since that total (p,q) vanishes at each junction.

To find the remaining supersymmetries of this system, we start from the original supersymmetries of a (p,q) string (which saturates the BPS bound) extended along the $\hat{X} = (\cos\theta, \sin\theta)$ direction:

$$\frac{1}{2L\tau_{(p,q)}}\left\{\begin{bmatrix} Q_\alpha \\ \tilde{Q}_\alpha \end{bmatrix}, \begin{bmatrix} Q^\dagger_\beta & \tilde{Q}^\dagger_\beta \end{bmatrix}\right\} = \delta_{\alpha\beta}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (\Gamma^0\Gamma^\theta)_{\alpha\beta}\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}, \tag{17}$$

$$\Gamma^\theta = \hat{X}\cdot\vec{\Gamma} = \Gamma^1\cos\theta + \Gamma^2\sin\theta \tag{18}$$

We see that the algebra depends on θ, i.e. it is different for strings in different directions. However, if we can find a (maximal) subalgebra that is independent of θ, then we would have found the remaining supersymmetries of the full system[6].

We begin with diagonalizing the matrix on the RHS with:

$$U(\tfrac{\theta}{2}) = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}, \tag{19}$$

$$\frac{1}{2L\tau_{(p,q)}}\left\{U^{\mathrm{T}}\begin{bmatrix} Q_\alpha \\ \tilde{Q}_\alpha \end{bmatrix}, \begin{bmatrix} Q^\dagger_\beta & \tilde{Q}^\dagger_\beta \end{bmatrix}U\right\} = \begin{bmatrix} (\mathbb{1}+\Gamma^0\Gamma^\theta)_{\alpha\beta} & 0 \\ 0 & (\mathbb{1}-\Gamma^0\Gamma^\theta)_{\alpha\beta} \end{bmatrix}, \tag{20}$$

Note that $(\mathbb{1}+\Gamma^0\Gamma^\theta)(\mathbb{1}-\Gamma^0\Gamma^\theta) = 0$ and $(\mathbb{1}+\Gamma^0\Gamma^\theta)+(\mathbb{1}-\Gamma^0\Gamma^\theta) = 2\mathbb{1}$, i.e. they are orthogonal to each other; acting $(\mathbb{1}\pm\Gamma^0\Gamma^\theta)$ on both sides, we find the following combinations, which gives the 16 SUSYs of a (p,q) string:

$$(\mathbb{1}-\Gamma^0\Gamma^\theta)\Big(\cos\tfrac{\theta}{2}\,Q + \sin\tfrac{\theta}{2}\,\tilde{Q}\Big)_\alpha = 0 = (\mathbb{1}+\Gamma^0\Gamma^\theta)\Big(-\sin\tfrac{\theta}{2}\,Q + \cos\tfrac{\theta}{2}\,\tilde{Q}\Big)_\beta \tag{21}$$

[5] Reference: arXiv:0812.4408.

[6] The $\tau_{(p,q)}$ factor can be absorbed by rescaling generators, hence does not matter in our discussions.

For further simplification, we can isolate the θ dependence in Γ^θ by working in a specific representation of the Clifford algebra, e.g. the Dirac representation given by *Polchinski*. Then α is given by 10 D spinor components: $\alpha = (s_0, s_1, s_2, s_3, s_4),\ s_i = \pm$, with additional chirality constraints from both Q and $\tilde{Q}$: $\prod_i s_i = +$. In the end, we have 16 independent components as expected.

Details of the expansion is given in arXiv:0812.4408. When the dust settles, we find that the 16 SUSYs in (21) is given by:

$$\sin\tfrac{\theta}{2}\left(\cos\tfrac{\theta}{2}\,Q + \sin\tfrac{\theta}{2}\,\tilde{Q}\right)_{(++s)} + \cos\tfrac{\theta}{2}\left(\cos\tfrac{\theta}{2}\,Q + \sin\tfrac{\theta}{2}\,\tilde{Q}\right)_{(--s)}, \tag{22a}$$

$$\sin\tfrac{\theta}{2}\left(\cos\tfrac{\theta}{2}\,Q + \sin\tfrac{\theta}{2}\,\tilde{Q}\right)_{(+-s)} - \cos\tfrac{\theta}{2}\left(\cos\tfrac{\theta}{2}\,Q + \sin\tfrac{\theta}{2}\,\tilde{Q}\right)_{(-+s)}, \tag{22b}$$

$$\cos\tfrac{\theta}{2}\left(-\sin\tfrac{\theta}{2}\,Q + \cos\tfrac{\theta}{2}\,\tilde{Q}\right)_{(++s)} - \sin\tfrac{\theta}{2}\left(-\sin\tfrac{\theta}{2}\,Q + \cos\tfrac{\theta}{2}\,\tilde{Q}\right)_{(--s)}, \tag{22c}$$

$$\cos\tfrac{\theta}{2}\left(-\sin\tfrac{\theta}{2}\,Q + \cos\tfrac{\theta}{2}\,\tilde{Q}\right)_{(+-s)} + \sin\tfrac{\theta}{2}\left(-\sin\tfrac{\theta}{2}\,Q + \cos\tfrac{\theta}{2}\,\tilde{Q}\right)_{(-+s)}, \tag{22d}$$

$$s = (s_2 s_3 s_4), \qquad \prod_i s_i = + \tag{23}$$

By trial and error, we can find the 8 linear combinations that are independent of θ; they are:

$$(a) + (c) \quad \Longrightarrow \quad \tilde{Q}_{(++s)} + Q_{(--s)}, \tag{24}$$

$$(b) + (d) \quad \Longrightarrow \quad \tilde{Q}_{(+-s)} - Q_{(-+s)}, \tag{25}$$

Therefore, the string junction is $\frac{8}{32} = \frac{1}{4}$ BPS.

3 Two and Three-Point Functions in AdS/CFT