A Modern Take on Renormalization

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The continuum limit $\Lambda \to \infty$ is **not** well defined. Renormalization provides a way to *define* the theory when $\Lambda \to \infty$.

My belief: The only way to fully understand renormalization is through Wilson's arguments; all other "interpretations" of renormalization are only *heuristic*.

Key words:

- Renormalization group
- Counterterms
- Regularization and cutoff

Contents

1 References

Ranked by importance:

- David Skinner's note:
  - https://www.damtp.cam.ac.uk/user/dbs26/AQFT.html
    * https://www.damtp.cam.ac.uk/user/dbs26/AQFT/Wilsonchap.pdf
    * https://www.damtp.cam.ac.uk/user/dbs26/AQFT/chap5.pdf
- Schwartz, Chapter 15
- Peskin & Schroeder, Chapter 10 & 12
- Hollowood's book:
  - https://arxiv.org/abs/0909.0859
  - https://link.springer.com/book/10.1007/978-3-642-36312-2

2 Wilson's picture

Seed theory parameters (i.e. *bare parameters*): (g_0, Λ_0), $g = (m, \lambda, \cdots)$ a collection of all possible couplings.

$$\begin{aligned} \phi_{\Lambda_0}(x) \sim \int^{\Lambda_0} \mathrm{d}p e^{ip\cdot x}\tilde{\phi}(p) &= \int^{\Lambda} \mathrm{d}p e^{ip\cdot x}\tilde{\phi}(p) + \int_{\Lambda}^{\Lambda_0} \mathrm{d}p e^{ip\cdot x}\tilde{\phi}(p) \\ &=: \phi_\Lambda(x) + \chi(x) \end{aligned} \tag{1}$$

$$\begin{aligned} \mathcal{D}\phi_{\Lambda_0}(x) \sim \prod_{\|p\|<\Lambda_0} \mathrm{d}\tilde{\phi}(p) &= \prod_{\|p\|<\Lambda} \mathrm{d}\tilde{\phi}(p) \prod_{\Lambda<\|p\|<\Lambda_0} \mathrm{d}\tilde{\phi}(p) \\ &\sim \mathcal{D}\phi_\Lambda(x)\, \mathcal{D}\chi(x) \end{aligned} \tag{2}$$

$$\begin{aligned} \mathcal{Z}(g_0, \Lambda_0) &= \int \mathcal{D}\phi_{\Lambda_0}\, e^{iS[\phi_{\Lambda_0}]} \\ &= \int \mathcal{D}\phi_\Lambda \int \mathcal{D}\chi\, e^{iS[\phi_\Lambda + \chi]} \\ &=: \int \mathcal{D}\phi_\Lambda\, e^{iS_{\text{eff}}[\phi_\Lambda]} =: \mathcal{Z}(g(\Lambda), \Lambda) \end{aligned} \tag{3}$$

Renormalized parameters: (g, Λ).

> **Subtlety:** the notation above is only schematic; in practice we first Wick-rotate to Euclidean signature, so that the momentum cutoff is easily imposed: $\|p\| = \sqrt{p_0^2 + \mathbf{p}^2} < \Lambda$. In Lorentzian signature, it's hard to define a covariant cutoff since $p_\mu p^\mu = -p_0^2 + \mathbf{p}^2$. This process can be made rigorous; just think of the 8-shaped contour in loop integrals.

Effective action:

$$S_{\text{eff}}[\phi] = -i \ln \int \mathcal{D}\chi\, e^{iS[\phi+\chi]} \tag{4}$$

$\phi = \phi_\Lambda$ is treated as constant when doing $\int \mathcal{D}\chi$.

$$\begin{aligned} \mathcal{L}[\phi+\chi] &= -\frac{1}{2}\partial_\mu(\phi+\chi)\,\partial^\mu(\phi+\chi) - \frac{1}{2}m^2(\phi+\chi)^2 - \frac{1}{4!}\lambda\,(\phi+\chi)^4 \\ &= \cdots \\ &= \mathcal{L}[\phi] + \Delta\mathcal{L}[\phi, \chi] \end{aligned} \tag{5}$$

$$S_{\text{eff}}[\phi_\Lambda] = S[\phi_\Lambda] - i \ln \int \mathcal{D}\chi\, e^{i\,\Delta S[\phi_\Lambda + \chi]} \tag{6}$$

If $\Lambda \lesssim \Lambda_0$, then S_{eff} is almost the same as the original S, with minor corrections from the $\int \mathcal{D}\chi$ term. Note that in our regularization scheme there will be no mixed terms of ϕ and χ in the effective action, since they have orthogonal Fourier modes.

Note that $\mathcal{Z}$ clearly does not depend on the intermediate scale Λ, we have:

$$0 = \Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda}\,\mathcal{Z}(g(\Lambda), \Lambda) = \left(\Lambda \frac{\partial}{\partial\Lambda} + \Lambda \frac{\partial g^{(i)}}{\partial\Lambda} \frac{\partial}{\partial g^{(i)}}\right)\mathcal{Z}(g(\Lambda), \Lambda) \tag{7}$$

This is an example of a **RG Equation**.

3 Perturbative Renormalization

However, in naïve perturbation theory, we wish to complete the entire path integral $\mathcal{Z}(g_0, \Lambda_0)$. We can think of this as integrating out more and more high energy modes, until we reach the IR scale $\Lambda \to 0$.

When $\Lambda \ll \Lambda_0$, we have no reason to believe the renormalized couplings $g(\Lambda)$ are close to the original couplings g_0 at Λ_0. In fact, they may differ by a large (but finite) renormalization factor Z: $g_0 = Zg$.

3.1 Counterterms

In the above analysis, the theory flows from UV to IR. However, in reality, the IR results are known from experiments, and we are trying to *extrapolates* from IR to UV.

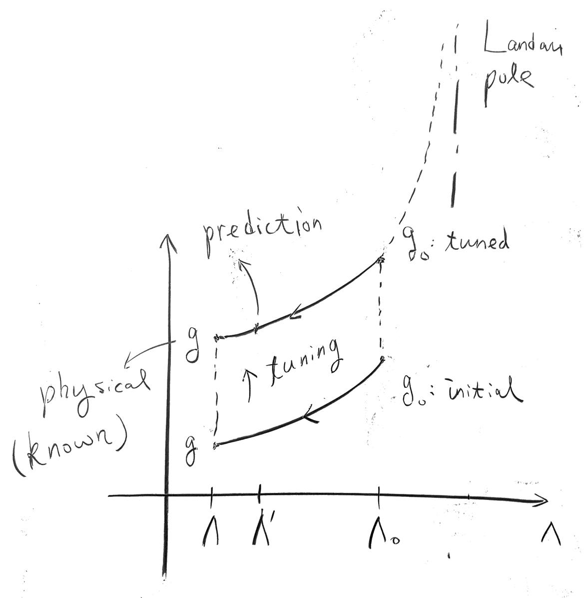
We achieve this by *tuning* the bare parameters (g_0, Λ_0) so that after RG flow, the IR results fit our experimental observations. If the IR couplings $g(\Lambda)$ are finite and small, then since $\Lambda \ll \Lambda_0$, we expect g_0 to be very large.

We often *split* g_0 into 2 parts for convenience:

$$g_0 = g + \delta g = g + (Z - 1)\, g \tag{8}$$

δg is the so-called *counterterm*; intuitively, it's the (large) correction that gets integrated out when we go from Λ_0 all the way to IR.

Basically, we have the following procedure:



0. Select some UV parameters (g_0, Λ_0)
1. Perform the RG flow: $(g_0, \Lambda_0) \to (g, \Lambda)$
2. Tune (redefine) g_0 so that (g, Λ) matches with experiments
3. Use the tuned data to predicts phenomena at a different scale (g', Λ')

Note that the tuning of UV parameters g_0 is *far from unique!* This is easy to understand: many UV theories might flow to the same IR theory. For this reason, some would say that RG is a *semi-group.*

However, for a *renormalizable* theory, we can restrict the tuning to a *finite* dimensional subspace formed by the *relevant* couplings, since most other parameters $g^{(i)}$ are *irrelevant* and get suppressed by Λ/Λ_0 in the IR. After such restriction to a relevant subspace, the RG flow is a group, and we can reverse the flow to *extrapolate* towards UV.

3.2 Perturbation

The above results are non-perturbative and should always hold. Perturbation theory is only a way to calculate the RG flow from UV to IR; it is reliable only if the IR coupling g is sufficiently small. In this case we can tune (g_0, Λ_0) with the following recursive / iterative algorithm:

1. Perturbative calculation of RG flow: $(g_0, \Lambda_0) \to (g, \Lambda)$ at $\mathcal{O}(g^n)$
2. Tune (redefine) (g_0, Λ_0) by adding counterterms, so that (g, Λ) matches with experiments
3. Increase order n and go to step 1

The (non-)renormalizability of a theory is evident in the perturbative expansion, by counting the **superficial degrees of divergence** D of the Feynman diagrams. Basically,

- Interaction vertices create loops, and loops create UV divergences. Higher order interaction vertices create more loops, which lead to more divergences.
- External lines suppress UV divergences by factors like $\frac{1}{\not{p}}$ or $\frac{1}{p^2}$.

For a renormalizable theory, there will be no divergence for diagrams with a sufficient number E of external legs; for a non-renormalizable theory, however, there will always be divergences, no matter how large E is.

3.3 Renormalization Schemes

There is a subtlety in the above procedure: how do we actually relate IR parameters (g, Λ) with actual physical quantities, e.g. amplitudes $\mathcal{M}(\mu)$?

In fact, we've assumed that $(g, \Lambda)_{\Lambda \to 0}$ gives the physical couplings that we are familiar with, e.g. mass, electric charge and so on. This is not quite true, since physical quantities are actually defined with scattering amplitudes. There are different choices of relating g with physical observables; this lead to various renormalization schemes:

- On-shell / pole-mass scheme
- Minimal subtraction (MS) & modified MS ($\overline{\text{MS}}$)

4 Renormalizability

Most parameters $g^{(i)}$ are, in fact, *irrelevant* — such terms in the Lagrangian get suppressed by Λ/Λ_0 in IR. If the IR theory has only *relevant* couplings, then one should be able to recover their physical values by tuning a finite amount of relevant couplings in the UV, and usually the tuning is unique. This is the defining characteristic of a **renormalizable** theory. Basically, this means that we can naturally obtain a UV theory by extrapolation.

> However, the tuning process described above might encounter some serious obstruction: the tuned g_0 could blow up at some finite Λ_{UV}; this is the so-called *Landau pole*. This tells us that the theory only works under some Λ_{UV}, i.e. it is not *UV complete*; it's only an *effective* theory. One have to "embed" this Lagrangian into a bigger theory that works beyond Λ_{UV}; this is the non-trivial *UV completion* of an effective theory.

On the other hand, a theory is **non-renormalizable** if it contains irrelevant couplings in the IR. In this case the IR parameters g depend sensitively on small perturbations of the UV parameters g_0, and one has to tune infinitely many bare parameters to obtain the physical IR values. Such theory is hardly *fundamental*, since it depends on infinitely many parameters; but it's a good *effective theory* nonetheless.

5 Effective Action

Furthermore, after ϕ_Λ is completely integrated out, we have:

$$S_{\mathrm{eff}}[\phi] \to W, \quad Z(g_0, \Lambda_0) = e^{iW} \tag{9}$$

Note that W no longer has any ϕ dependence, but it is a function of (g_0, Λ_0), which in turn is tuned by physical (g, Λ). W in fact contains all information about the seed theory, labeled by (g_0, Λ_0). To extract this information, we usually perturb the original action $S[\phi]$ with a source term; then we have:

$$S[\phi, J] = S[\phi] + \int \mathrm{d}x\ J(x)\,\phi(x), \quad W \to W[J] \tag{10}$$

Expand $W[J]$ in terms of J-modes, and its coefficient gives us physical coupling constants in the IR. More precisely, we can define the Legendre-transformed $\Gamma[\varphi]$.