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## 1 Gravity

$$ds^{2} = -f(r) dt^{2} + \frac{1}{f(r)} dr^{2} + r^{2} d\Omega^{2}$$
(1)

$$f(r) = 1 - \frac{GM}{r} + \frac{Q^2}{r^2} = \frac{(r - r_+)(r - r_-)}{r^2}$$
 (2)

- 1. Event horizon(s): f(r) = 0, we have:
- (a)  $M > |Q|, r_{\pm} = M \pm \sqrt{M^2 Q^2}, 2$  event horizons;
- (b)  $M = |Q|, r_{\pm} = M, 1$  event horizon;
- (c) M < |Q|, no event horizon! "Naked" singularity.
- 2. New coordinate:  $v = t + r^*$ ,

$$r^* = r + \frac{1}{2k_+} \ln \frac{|r - r_+|}{r_+} + \frac{1}{2k_-} \ln \frac{|r - r_-|}{r_-}, \quad k_{\pm} = \frac{r_{\pm} - r_{\mp}}{2r_{\pm}^2}$$
 (3)

We have:

$$dt = dv - dr^* = dv - \left(1 + \frac{1}{2k_+} \frac{1}{r - r_+} + \frac{1}{2k_-} \frac{1}{r - r_-}\right) dr$$

$$= dv - \left(1 + \frac{1}{r^2 f(r)} \frac{r_+^2 (r - r_-) - r_-^2 (r - r_+)}{r_+ - r_-}\right) dr$$

$$= dv - \left(1 + \frac{1}{r^2 f(r)} \left((r_+ + r_-) r - r_+ r_-\right)\right) dr$$

$$= dv - \frac{1}{f(r)} dr$$
(5)

Therefore,

(a) The new metric:

$$ds^{2} = -f(r)\left(dv - \frac{1}{f(r)}dr\right)^{2} + \frac{1}{f(r)}dr^{2} + r^{2}d\Omega^{2}$$

$$= -f(r)dv^{2} + 2dvdr + r^{2}d\Omega^{2}$$
(6)

It is only singular at r = 0.

**Note:** during the exam I panicked when I saw (3), and I made a very stupid mistake in step (4). However, I knew what this new coordinate is trying to achieve — it's aiming to eliminate the coordinate singularities in  $\frac{1}{f} dr^2$  by absorbing it into  $dv^2$ , so I guessed the result (5) correctly and carried on. I hope they gave me some points for getting the right answer, despite with some wrong process (>\_<).

(b)  $\frac{\partial}{\partial v}$  is a Killing vector field, for the metric components are all v-independent. More precisely, since  $\frac{\partial}{\partial v}$  itself is a coordinate basis, we have the Lie derivative:

$$\mathcal{L}_{\frac{\partial}{\partial v}}g_{\mu\nu} = \partial_v g_{\mu\nu} = 0 \tag{7}$$

2 QFT

(c)  $\left\| \frac{\partial}{\partial v} \right\|^2 = g_{\mu\nu} \delta_v^{\mu} \delta_v^{\nu} = g_{vv} = -f(r)$ , therefore, for M > |Q| we have:

•  $\frac{\partial}{\partial v}$  timelike:  $r > r_+$  and  $r < r_-$ 

• spacelike:  $r_- < r < r_+$ 

• null:  $r = r_+$  and  $r = r_-$ 

## 2 QFT

We shall restore the reasonable convention:  $\eta_{\mu\nu} \sim (-,+,+,+)$ .

1. 1PI: diagrammatic correction to the (1-particle) propagator that cannot be split into 2 disconnected parts by cutting one line; e.g.

2. Consider the following Lagrangian:

$$\mathcal{L} = -\frac{1}{2}Z(\partial\phi_r)^2 - \frac{1}{2}m^2Z\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 - \frac{1}{2}\delta_Z(\partial\phi_r)^2 - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4$$
 (8)

The convention here is rather bizarre; normally we write down the UV Lagrangian  $\mathcal{L}_{UV}$  and split it into 2 parts, one is the effective IR Lagrangian  $\mathcal{L}_{IR}$  and the other one is the counterterm:

$$\mathcal{L}_{\text{UV}} = -\frac{1}{2}Z(\partial\phi_r)^2 - \frac{1}{2}m^2Z\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 
= \left(-\frac{1}{2}(\partial\phi_r)^2 - \frac{1}{2}m_p^2\phi_r^2 - \frac{\lambda_p}{4!}\phi_r^4\right) - \left(-\frac{1}{2}\delta_Z(\partial\phi_r)^2 - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4\right) 
= \mathcal{L}_{\text{IR}} + \mathcal{L}_{\text{ct}}$$
(9)

Normally, we use  $\mathcal{L}$  to denote the UV Lagrangian  $\mathcal{L}_{\mathrm{UV}}$ ; this is the convention adopted by numerous standard textbooks, incl. *Peskin & Schroeder* [1], *Weinberg*, and also *Srednicki*. However, the Lagrangian in (8) seems to be  $\mathcal{L}_{\mathrm{IR}}$  instead of  $\mathcal{L}_{\mathrm{UV}}$ . Anyway, we have:

$$Z + \delta_Z = 1, \quad m^2 Z + \delta_m = m_p^2, \quad \lambda + \delta_\lambda = \lambda_p$$
 (10)

Where  $m_p, \lambda_p$  is the physical IR couplings, fixed by the renormalization scheme. The convention here is really confusing and somewhat inconsistent; e.g. if we choose to write the UV mass term as  $-\frac{1}{2}m^2Z\phi_r^2$ , then the corresponding UV interaction term should look like  $-\frac{\lambda}{4!}Z^2\phi_r^4$ , but here we do not have the  $Z^2$  factor. Also, we usually use  $m_0, \lambda_0$  to denote bare couplings, but here it seems that they are denoted by  $m, \lambda$ .

We can write down the renormalized Feynman rules nonetheless, despite some sign issues due to the conventions; to avoid further confusion, we will adopt the usual notation:  $m_0, \lambda_0$  for bare couplings, and  $m = m_p, \lambda = \lambda_p$  for physical couplings. We have:

• Renormalized propagator:  $\frac{-i}{p^2+m^2-i\epsilon}$ 

• Renormalized vertex:  $-i\lambda$ 

• Counterterm  $\phi^2$  vertex:  $+i(\delta_Z(-p^2)+\delta_m)$ ,  $-\otimes$ 

• Counterterm  $\phi^4$  vertex:  $+i\delta_{\lambda}$ 

 $2 ext{ } QFT$ 

3. The sum of all two point 1PI diagrams (no propagator on external legs) is given by:

$$-iM(p^2) = \tag{11}$$

The full propagator is thus:

With  $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ , we get:

$$G(p^2) = \frac{-i}{p^2 + m^2} \cdot \frac{1}{1 - (-iM)\frac{-i}{p^2 + m^2}} = \frac{-i}{p^2 + m^2 + M(p^2)}$$
(13)

Here we've suppressed the  $(-i\epsilon)$  prescription in the above expressions, but it's presence is always implied.

4. On-shell renormalization scheme — the full propagator:

$$G(p^2) = \frac{-i}{p^2 + m^2 + M(p^2) - i\epsilon} \xrightarrow{p^2 \to -m^2} \frac{-i}{p^2 + m^2 - i\epsilon}$$
 (14)

This means that  $M(p^2 = -m^2) = 0$ . Furthermore,  $M(p^2) \sim \#(p^2 + m^2) + \mathcal{O}(p^4)$ , to ensure that the residue is 1 at the pole, we should have  $\# \sim 0$ , i.e.

$$M(p^2)\big|_{p^2=-m^2} = 0, \quad \frac{\partial}{\partial(p^2)}M(p^2)\big|_{p^2=-m^2} = 0$$
 (15)

5. At 1-loop  $\mathcal{O}(\lambda)$ , if we do not include counterterm contributions, then there is only one diagram contributing to  $M(p^2)$ :

$$= (-i\lambda) \cdot \frac{1}{2} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{-i}{k^2 + m^2 - i\epsilon}$$
 (16)

Here  $\frac{1}{2}$  is the symmetry factor of the diagram; alternative, we can count the distinct ways of connection the 4 legs of the  $\phi^4$  vertex and divide it by 4!, which is indeed  $\frac{4\times3}{4!} = \frac{1}{2}$ .

The  $p^0$  integral has poles at  $p_0^2 = \mathbf{p}^2 + m^2 - i\epsilon$ , i.e.  $p^0 = \pm \sqrt{\mathbf{p}^2 + m^2} \mp i\epsilon$ , and it's regular everywhere else; we can thus compute the  $p^0$  integral on the  $\mathbb C$  plane using a right-tilted 8-shaped contour, which does not enclose the poles. Effectively, we've performed a Wick rotation  $p^0 \mapsto ip^0$  so that the integral happens in Euclidean p space:

$$\frac{-i\lambda}{2} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{k^2 + m^2} = \frac{-i\lambda}{2} \frac{A(S^d)}{(2\pi)^D} \int \frac{k^d \, \mathrm{d}k}{k^2 + m^2}$$
(17)

Here D = d + 1, d is the spatial dimension. There are many ways to regularize this integral; if we continue to work in general D = d + 1 dimensions, then dimensional regularization is automatically implied. We have:

$$A(S^d) = \frac{2\pi^{D/2}}{\Gamma(D/2)}, \quad \int \frac{k^d \, \mathrm{d}k}{k^2 + m^2} = \frac{m^D}{m^2} \int \frac{t^d \, \mathrm{d}t}{1 + t^2}$$
 (18)

References 4

The t-integral is related to Beta functions; consider  $t\mapsto \frac{t^2}{1+t^2}$ , and we have:

$$\int_0^\infty \frac{t^d dt}{1+t^2} = \frac{1}{2} \int_0^1 t^{\frac{D}{2}-1} (1-t)^{-\frac{D}{2}} dt = \frac{\Gamma(\frac{D}{2}) \Gamma(1-\frac{D}{2})}{2\Gamma(1)} = \frac{1}{2} \Gamma(\frac{D}{2}) \Gamma(1-\frac{D}{2}) = \frac{\pi}{2 \sin \frac{\pi D}{2}}$$
(19)

The last line is Euler's reflection formula, but here we actually don't need that since the  $\Gamma(\frac{D}{2})$  factor is canceled by  $A(S^d)$ . In the end we have:

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{k^2 + m^2} = \frac{\pi^{D/2}}{(2\pi)^D} \Gamma(1 - \frac{D}{2}) m^{D-2} = \frac{1}{(4\pi)^{D/2}} \Gamma(1 - \frac{D}{2}) m^{D-2}, \tag{20}$$

$$= \frac{-i\lambda}{2} \frac{1}{(4\pi)^{D/2}} \Gamma(1 - \frac{D}{2}) m^{D-2}$$
 (21)

We then have to include counterterm contributions so that the renormalization condition (15) is satisfies; we have:

$$-iM(p^{2}) \sim \frac{1}{2} + \frac{-i\lambda}{2} \frac{1}{(4\pi)^{D/2}} \Gamma(1 - \frac{D}{2}) m^{D-2} + i(\delta_{Z}(-p^{2}) + \delta_{m})$$

$$\sim 0 + 0 \cdot (p^{2} + m^{2}) + \mathcal{O}(p^{4})$$
(22)

Therefore,

$$\delta_Z = 0, \quad \delta_m = \frac{\lambda}{2} \frac{1}{(4\pi)^{D/2}} \Gamma(1 - \frac{D}{2}) m^{D-2}$$
 (23)

Alternatively, if we are working in D=4=3+1 dimensions, it's easier to impose a naïve cutoff  $\Lambda$ , which gives:

$$\int_{-\infty}^{\Lambda} \frac{k^{d} dk}{k^{2} + m^{2}} \sim \int_{-\infty}^{\Lambda} k^{d-2} dk + \int_{-\infty}^{\Lambda} k^{d} dk \left( \frac{1}{k^{2} + m^{2}} - \frac{1}{k^{2}} \right) 
= \int_{-\infty}^{\Lambda} k^{d-2} dk - m^{2} \int_{-\infty}^{\Lambda} \frac{k^{d-2} dk}{k^{2} + m^{2}}, \quad d = D - 1 = 3 
= \frac{\Lambda^{2}}{2} - \frac{m^{2}}{2} \ln \left( 1 + \frac{\Lambda^{2}}{m^{2}} \right),$$
(24)

Similarly, with  $A(S^3) = 2\pi^2$ , we have:

$$\delta_Z = 0, \quad \delta_m = \frac{\lambda}{2} \frac{2\pi^2}{(2\pi)^4} \left\{ \frac{\Lambda^2}{2} - \frac{m^2}{2} \ln\left(1 + \frac{\Lambda^2}{m^2}\right) \right\}$$

$$= \frac{\lambda}{32\pi^2} \left\{ \Lambda^2 - m^2 \ln\left(1 + \frac{\Lambda^2}{m^2}\right) \right\}$$
(25)

## References

Michael E. Peskin and Daniel V. Schroeder. An Introduction to Quantum Field Theory.
 Addison-Wesley, Reading, USA, 1995. ISBN: 978-0-201-50397-5.