String Theory №3 [WIP]

Bryan

Compiled @ 2020/04/15

1 Strings on Curved Space:

$$S = \frac{1}{4\pi\alpha'} \int_M \mathrm{d}^2\sigma \sqrt{g} \left(i\epsilon^{ab} B_{\mu\nu}(X)\, \partial_a X^\mu \partial_b X^\nu + \cdots \right), \tag{1}$$

$$T^a{}_a = -\frac{1}{2\alpha'} \beta^G_{\mu\nu}\, g^{ab} \partial_a X^\mu \partial_b X^\nu + \cdots , \tag{2}$$

$$\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} - \frac{1}{4} \alpha' H_{\mu\lambda\omega} H_\nu{}^{\lambda\omega} + \cdots + \mathcal{O}(\alpha'^2) \tag{3}$$

We want to verify the coefficient of $\alpha' H^2$ term in $\beta^G_{\mu\nu}$; for convenience we've omitted non-related terms in the above expressions.

Note that at $\mathcal{O}(\alpha')$ such term does not depend on the metric $G_{\mu\nu}$, and it depends only on the field strength $H = \mathrm{d}B$, not the potential B, hence it's safe to assume:

$$G_{\mu\nu} = \eta_{\mu\nu}, \quad B_{\mu\nu} = \frac{1}{3} H_{\mu\nu\rho} X^\rho, \quad H = \text{const}, \tag{4}$$

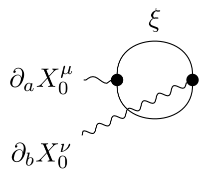
$$i\epsilon^{ab} B_{\mu\nu}(X)\, \partial_a X^\mu \partial_b X^\nu = \frac{i}{3} H_{\mu\nu\rho}\, X^\rho \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu, \tag{5}$$

We consider small perturbation away from the classical saddle: $X = X_0 + \xi$, then the 1-loop effective action is obtained by integrating over $\mathcal{O}(\xi^2)$ terms in the perturbed action[1]:

$$\Gamma^{(1)}[X_0] = -\ln \int \mathscr{D}\xi\, e^{-S^{(2)}[X_0,\xi]}, \tag{6}$$

$$\begin{aligned}
\mathcal{L}^{(2)} &= \frac{i}{3} H_{\mu\nu\rho}\, \epsilon^{ab} \left(\xi^\rho\, \partial_a X_0^\mu\, \partial_b \xi^\nu + \xi^\rho\, \partial_a \xi^\mu\, \partial_b X_0^\nu + X_0^\rho\, \partial_a \xi^\mu\, \partial_b \xi^\nu \right) \\
&\sim \frac{i}{3} H_{\mu\nu\rho}\, \epsilon^{ab} \left(\xi^\rho\, \partial_a X_0^\mu\, \partial_b \xi^\nu - \xi^\rho\, \partial_a X_0^\nu\, \partial_b \xi^\mu - \xi^\mu\, \partial_a X_0^\rho\, \partial_b \xi^\nu \right) \\
&= \frac{i}{3} H_{\mu\nu\rho}\, \epsilon^{ab} \cdot 3 \xi^\rho\, \partial_a X_0^\mu\, \partial_b \xi^\nu \\
&= i H_{\mu\nu\rho}\, \epsilon^{ab}\, \partial_a X_0^\mu\, (\xi^\rho \partial_b \xi^\nu)
\end{aligned} \tag{7}$$

Here we've used the anti-symmetric properties of $H_{\mu\nu\rho}$, ϵ^{ab}, and ignored any total derivative after integration by parts. This term introduces a cubic interaction vertex in the free background; therefore, $\Gamma^{(1)}$ can be expressed in the following diagram[2]:



$$\sim \frac{1}{2!} \left(\frac{1}{\alpha'} \right)^2 \int \mathrm{d}^2 p \left(i H_{\mu\nu\rho}\, \epsilon^{ab}\, \partial_a X_0^\mu\, i p_b \right) \frac{2}{p^4} \left(-\frac{\alpha'}{2} \right)^2 \left(i H_{\mu'}{}^{\nu\rho}\, \epsilon^{a'b'}\, \partial_{a'} X_0^{\mu'}\, i p_{b'} \right) \tag{8}$$

---

[1] Reference: Prof. Xi Yin's String Notes, see also `arXiv:0812.4408`.

[2] References:

- David Tong, *String Theory*;
- Callan & Thorlacius, *Sigma Models and String Theory*;
- Timo Weigand, *Introduction to String Theory*.

$$= \frac{2}{2!}\left(\frac{1}{\alpha'}\right)^2\left(-\frac{\alpha'}{2}\right)^2 H_{\mu\lambda\omega}H_{\nu}{}^{\lambda\omega}\partial_a X_0^\mu\,\partial_b X_0^\nu \int \mathrm{d}^2p\,\frac{p^2g^{ab}-p^ap^b}{p^4} \tag{9}$$

$$= \frac{2}{2!}\left(-\frac{1}{2}\right)^2 H_{\mu\lambda\omega}H_{\nu}{}^{\lambda\omega}\partial_a X_0^\mu\,\partial_b X_0^\nu \left(\frac{1}{2}\,g^{ab}\right)\int \mathrm{d}^2p\,\frac{1}{p^2} \tag{10}$$

$$= \frac{2}{2!}\left(-\frac{1}{2}\right)^2\left(\frac{1}{2}\right) H_{\mu\lambda\omega}H_{\nu}{}^{\lambda\omega}\partial_a X_0^\mu\,\partial_b X_0^\nu\, g^{ab}\int \mathrm{d}^2p\,\frac{1}{p^2} \tag{11}$$

$$= \frac{1}{8}H_{\mu\lambda\omega}H_{\nu}{}^{\lambda\omega}g^{ab}\partial_a X_0^\mu\,\partial_b X_0^\nu \int \mathrm{d}^2p\,\frac{1}{p^2} \tag{12}$$

Here the $\left(\frac{1}{\alpha'}\right)^2$ coefficient comes from the vertices, while $\left(-\frac{\alpha'}{2}\right)^2$ comes from the propagators. The p^ap^b integral provides an additional $(\frac{1}{2})$ factor. The overall normalization is chosen to match the $\alpha' R_{\mu\nu}$ coefficient in $\beta^G_{\mu\nu} \subset T^a{}_a$, which is $\frac{1}{1!}\times(-\frac{1}{2})\times 1 = -\frac{1}{2}$. Therefore, we have:

$$T^a{}_a \supset \frac{1}{8}H_{\mu\lambda\omega}H_{\nu}{}^{\lambda\omega}g^{ab}\partial_a X_0^\mu\,\partial_b X_0^\nu, \tag{13}$$

$$\beta^G_{\mu\nu} \supset -\frac{1}{4}\,\alpha' H_{\mu\lambda\omega}H_{\nu}{}^{\lambda\omega} \tag{14}$$

■

$\boxed{2}$ **Classical Solutions of 11D SUGRA:** Following the convention of *Polchinski*, we have bosonic action:

$$S = \frac{1}{2\kappa^2}\int \left(\mathrm{d}^{11}x\,\sqrt{-g}\,\mathcal{R} - \frac{1}{2}\,G\wedge *G - \frac{1}{6}\,C\wedge G\wedge G\right), \tag{15}$$

Here $G = \mathrm{d}C$: a 4-form field. In components, the numerical coefficients would be $\frac{1}{2} \mapsto \frac{1}{2\times 4!} = \frac{1}{48}$, and $\frac{1}{6} \mapsto \frac{1}{6\times 3!\times 4!\times 4!} = \frac{1}{20736}$.

Variation of the action yields the EOMs of our theory[3]; Note that:

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\,\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu} \tag{16}$$

$\frac{\delta S}{\delta g^{\mu\nu}}$ is easier to compute in components; note that the $C\wedge G\wedge G$ term does not depend on $g^{\mu\nu}$, therefore it does not contribute to the EOM. We have the usual Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{17}$$

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{\kappa^2}\left(\frac{4}{48}G_{\mu\sigma_1\sigma_2\sigma_3}G_{\nu}{}^{\sigma_1\sigma_2\sigma_3} - \frac{1}{2}\,g_{\mu\nu}\cdot\frac{1}{48}\,G^{\sigma_1\sigma_2\sigma_3\sigma_4}G_{\sigma_1\sigma_2\sigma_3\sigma_4}\right) \\ &= \frac{1}{12\kappa^2}\left(G_{\mu\sigma_1\sigma_2\sigma_3}G_{\nu}{}^{\sigma_1\sigma_2\sigma_3} - \frac{1}{8}\,g_{\mu\nu}\,G^{\sigma_1\sigma_2\sigma_3\sigma_4}G_{\sigma_1\sigma_2\sigma_3\sigma_4}\right) \end{aligned} \tag{18}$$

[3] Reference: arXiv:hep-th/9912164. I would like to thank *Lucy Smith* for many helpful discussions.

On the other hand, $\frac{\delta S}{\delta C}$ is best carried out using differential forms:

$$\begin{aligned}
0 = \delta_C S &= -\frac{1}{2\kappa^2}\int \left(\delta G \wedge * G + \frac{1}{6}\Big(\delta C \wedge G \wedge G - 2C \wedge \delta G \wedge G\Big)\right) \\
&= -\frac{1}{2\kappa^2}\int \left(\delta(\mathrm{d}C) \wedge * G + \frac{1}{6}\big(\delta C \wedge G \wedge G + 2\,\delta(\mathrm{d}C) \wedge C \wedge G\big)\right) \\
&= -\frac{1}{2\kappa^2}\int \left(-(-1)^3\,\delta C \wedge \mathrm{d} * G + \frac{1}{6}\big(\delta C \wedge G \wedge G - 2\,(-1)^3\,\delta C \wedge \mathrm{d}(C \wedge G)\big)\right) \\
&= -\frac{1}{2\kappa^2}\int \delta C \wedge \left(\mathrm{d} * G + \frac{1}{6}\Big(G \wedge G + 2\big(G \wedge G - C \wedge \cancel{\mathrm{d}^2 C}\big)\Big)\right) \\
&= -\frac{1}{2\kappa^2}\int \delta C \wedge \left(\mathrm{d} * G + \frac{1}{2}\,G \wedge G\right),
\end{aligned} \tag{19}$$

$$\mathrm{d} * G + \frac{1}{2}\,G \wedge G = 0 \tag{20}$$

(a) We hope to find a spacetime solution which is *maximally symmetric* in *some* directions; assume that these directions form a d-dimensional sub-manifold $\mathcal{M}_d$ with:

$$\begin{aligned}
\text{Coordinates:} &\quad x^{\mu'},\ \mu' \in \Delta \subset \{0, 1, \cdots, 11\}, \\
\text{Induced metric:} &\quad g' = g|_{\mathcal{M}_d}
\end{aligned} \tag{21}$$

The entire spacetime is then a direct product: $\mathcal{M}_d \times \widetilde{\mathcal{M}}_{11-d}$. For $\mathcal{M}_d$ to be maximally symmetric, we expect that $\kappa^2 T_{\mu'\nu'} = -\Lambda g'_{\mu'\nu'}$, i.e. the G-field serves as a cosmological constant Λ. By staring at (18) we find that this can be achieved with[4]:

$$d = 4, \quad G_{\sigma_1\sigma_2\sigma_3\sigma_4} = \alpha\sqrt{|g'|}\,\epsilon_{\sigma_1\sigma_2\sigma_3\sigma_4}, \quad G^{\sigma_1\sigma_2\sigma_3\sigma_4} = \alpha\,\frac{\operatorname{sgn} g'}{\sqrt{|g'|}}\,\epsilon^{\sigma_1\sigma_2\sigma_3\sigma_4}, \quad \{\sigma_i\} \subset \Delta, \tag{22}$$

$$G_{\cdots\,\sigma\,\cdots} = 0, \quad \sigma \notin \Delta, \tag{23}$$

$$T_{\mu\nu} = (\operatorname{sgn} g')\,\frac{\alpha^2}{12\kappa^2}\left(3!\,g'_{\mu\nu} - \frac{4!}{8}\,g_{\mu\nu}\right) = (\operatorname{sgn} g')\,\frac{\alpha^2}{2\kappa^2}\left(g'_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\right), \tag{24}$$

$$\Lambda g_{\mu\nu} = \mp(\operatorname{sgn} g')\,\frac{\alpha^2}{4\kappa^2}\,g_{\mu\nu}, \quad \begin{cases} -:\ \ \mu = \mu', \nu = \nu' \in \Delta, & \sim \mathcal{M}_4 \\ +:\ \ \mu, \nu \notin \Delta, & \sim \widetilde{\mathcal{M}}_7 \end{cases} \tag{25}$$

Matter EOM is trivially satisfied due to anti-symmetricity. We see that the other component $\widetilde{\mathcal{M}}_7$ is also maximally symmetric, but with an opposite sign in its cosmological constant.

The field equations in $\mathcal{M}_4$ and $\widetilde{\mathcal{M}}_7$ are both of the form $R_{\mu\nu} \propto g_{\mu\nu}$. For $\operatorname{sgn} g' = -1$ i.e. Lorentzian signature, the solution is flat, AdS or dS, depending on the sign of Λ; for $\operatorname{sgn} g' = -1$, the solution is flat, spherical or hyperbolic. Therefore, we have:

$$\begin{aligned}
\operatorname{sgn} g' = -1, &\quad \Lambda_{4,7} = \pm\frac{\alpha^2}{4\kappa^2}, \quad \mathcal{M}_4 = \mathrm{AdS}_{3,1}, \quad \widetilde{\mathcal{M}}_7 = S^7 \\
\operatorname{sgn} g' = +1, &\quad \Lambda_{4,7} = \mp\frac{\alpha^2}{4\kappa^2}, \quad \mathcal{M}_4 = S^4, \quad \widetilde{\mathcal{M}}_7 = \mathrm{AdS}_{6,1}
\end{aligned} \tag{26}$$

[4]This is in fact the famous *Freund–Robin ansatz*; see `Wikipedia:` *Freund–Rubin compactification*, and also the original paper: Freund & Robin, *Dynamics of Dimensional Reduction*, 1980.

(b) Global supersymmetries of a theory with the above $\mathrm{AdS}_{4/7} \times S^{4/7}$ background are given by the solutions of:

$$0 = \delta_\eta \psi^\mu \equiv D^\mu \eta(x), \quad \eta\colon \text{spinor}, \tag{27}$$

$$\begin{aligned} D^\mu &= \nabla^\mu + \frac{1}{288}\, G_{\nu\rho\sigma\lambda}\big(\Gamma^{\mu\nu\rho\sigma\lambda} - 8g^{\mu\nu}\Gamma^{\rho\sigma\lambda}\big) \\ &= \nabla^\mu + \frac{1}{288}\, G_{\nu'\rho'\sigma'\lambda'}\big(\Gamma^{\mu\nu'\rho'\sigma'\lambda'} - 8g^{\mu\nu'}\Gamma^{\rho'\sigma'\lambda'}\big) \\ &= \nabla^\mu + \alpha \begin{cases} \dfrac{-8\times 3!}{288}\,(-\Gamma^\mu\gamma_5) = \dfrac{1}{6}\,\Gamma^\mu\gamma_5, & \mu = \mu' \in \Delta, \quad \sim \mathcal{M}_4 \\ \dfrac{4!}{288}\,(-\Gamma^\mu) = -\dfrac{1}{12}\,\Gamma^\mu, & \mu \notin \Delta, \quad \sim \widetilde{\mathcal{M}}_7 \end{cases} \end{aligned} \tag{28}$$

Note that we've replaced the G indices with $\mathcal{M}_4$ indices, since G vanish in $\widetilde{\mathcal{M}}_7$ directions; due to anti-symmetricity, the G-term can be reduced to simple Γ^μ multiplications according to the μ-direction[5]. Furthermore, the spin connection in ∇^μ is also block diagonalized, same as $g_{\mu\nu}$; hence there is a natural separation of variable[6]:

$$\eta = \eta'(x')\,\eta''(x''), \quad D_{\mu'}\eta' = 0, \quad D_{\mu''}\eta'' = 0, \tag{29}$$

$$\mu', \eta', x' \sim \mathcal{M}_4, \quad \mu'', \eta'', x'' \sim \widetilde{\mathcal{M}}_7, \tag{30}$$

Due to the presence of an additional Γ, $D_{\mu'}\eta' = 0$ has only 4 linearly independent solutions labeled by μ', while $D_{\mu''}\eta'' = 0$ is Spin(8) (or Spin(7,1), depending on the signature) invariant, and has $\frac{8\times 7}{2} = 28$ linearly independent solutions[7]. Hence the total number of SUSYs is $4 + 28 = 32$, for $\mathrm{AdS}_{4/7} \times S^{4/7}$ background.

$\boxed{3}$ SUSY Sigma Models via Superspace:

$$\begin{aligned} D_{\bar\theta}\mathbf{X}^\nu &= \big(\partial_{\bar\theta} + \bar\theta\partial_{\bar z}\big)\big(X^\nu + i\theta\psi^\nu + i\bar\theta\tilde\psi^\nu + \theta\bar\theta F^\nu\big) \\ &= i\tilde\psi^\nu - \theta F^\nu + \bar\theta\,\bar\partial X^\nu - i\theta\bar\theta\,\bar\partial\psi^\nu, \\ D_\theta\mathbf{X}^\mu &= i\psi^\mu + \bar\theta F^\mu + \theta\,\partial X^\mu + i\theta\bar\theta\,\partial\tilde\psi^\mu, \end{aligned} \tag{31}$$

$$\begin{aligned} D_{\bar\theta}\mathbf{X}^\nu D_\theta\mathbf{X}^\mu &= \Big(i\tilde\psi^\nu - \theta F^\nu + \bar\theta\,\bar\partial X^\nu - i\theta\bar\theta\,\bar\partial\psi^\nu\Big)\Big(i\psi^\mu + \bar\theta F^\mu + \theta\,\partial X^\mu + i\theta\bar\theta\,\partial\tilde\psi^\mu\Big) \\ &= -\tilde\psi^\nu\psi^\mu - i\theta\Big(\tilde\psi^\nu\partial X^\mu + \psi^\mu F^\nu\Big) + i\bar\theta\Big(\psi^\mu\bar\partial X^\nu - \tilde\psi^\nu F^\mu\Big) \\ &\quad - \theta\bar\theta\Big(\bar\partial X^\nu\partial X^\mu + \tilde\psi^\nu\partial\tilde\psi^\mu - (\bar\partial\psi^\nu)\psi^\mu + F^\nu F^\mu\Big), \end{aligned} \tag{32}$$

$$\begin{aligned} G_{\mu\nu}(\mathbf{X}) &= G_{\mu\nu} + \Big(i\theta\psi^\lambda + i\bar\theta\tilde\psi^\lambda + \theta\bar\theta F^\lambda\Big)\,\partial_\lambda G_{\mu\nu} + \frac{1}{2}\Big\{i\theta\psi^\rho\partial_\rho,\, i\bar\theta\tilde\psi^\sigma\partial_\sigma\Big\}\, G_{\mu\nu} \\ &= G_{\mu\nu} + \Big(i\theta\psi^\lambda + i\bar\theta\tilde\psi^\lambda\Big)\, G_{\mu\nu,\lambda} + \theta\bar\theta\Big(F^\lambda G_{\mu\nu,\lambda} + \psi^\rho\tilde\psi^\sigma G_{\mu\nu,\rho\sigma}\Big), \end{aligned} \tag{33}$$

---

[5] Reference for Γ-matrices and spinors: *Polchinski* Vol. II, Appendix B. I'm a bit confused about all the complicated conventions, therefore the coefficients might be off by some factors...

[6] See arXiv:hep-th/9912164 for more detailed discussions.

[7] Reference: Achilleas Passias, *Aspects of Supergravity in Eleven Dimensions*.

Note that $\int \mathrm{d}^2\theta = \partial_\theta \partial_{\bar\theta}$, hence we need only focus on the $\theta\bar\theta$ term in the Lagrangian:

$$
\begin{aligned}
4\pi S_G &= \int \mathrm{d}^2 z\, \mathrm{d}^2\theta\, G_{\mu\nu}(\mathbf{X})\, D_{\bar\theta}\mathbf{X}^\mu D_\theta \mathbf{X}^\nu = \int \mathrm{d}^2 z\, \mathrm{d}^2\theta\, (-\theta\bar\theta)\Big(G_{\mu\nu}\big(\partial X^\mu \bar\partial X^\nu + \cdots\big) + \cdots\Big) \\
&= \int \mathrm{d}^2 z\, \bigg(G_{\mu\nu}\Big(\partial X^\mu \bar\partial X^\nu + \tilde\psi^\nu \partial \tilde\psi^\mu - (\bar\partial \psi^\nu)\psi^\mu + F^\nu F^\mu\Big) \\
&\qquad + \tilde\psi^\nu \psi^\mu \Big(F^\lambda G_{\mu\nu,\lambda} + \psi^\rho \tilde\psi^\sigma G_{\mu\nu,\rho\sigma}\Big) \\
&\qquad - G_{\mu\nu,\lambda}\Big(\psi^\lambda\big(\psi^\mu \bar\partial X^\nu - \tilde\psi^\nu F^\mu\big) + \tilde\psi^\lambda\big(\tilde\psi^\nu \partial X^\mu + \psi^\mu F^\nu\big)\Big)\bigg)
\end{aligned} \tag{34}
$$

Similar result holds for the B contribution S_B. We see that there is no ∂F term in the action, hence F is not dynamical and can be integrated out; we have:

$$0 = \delta_F S = \delta_F(S_G + S_B), \tag{35}$$

$$
\begin{aligned}
4\pi\, \delta S_G &= \int \mathrm{d}^2 z\, \Big(2G_{\mu\nu}F^\mu\, \delta F^\nu + G_{\mu\nu,\lambda}\big(\tilde\psi^\nu \psi^\mu\, \delta F^\lambda - \tilde\psi^\nu \psi^\lambda\, \delta F^\mu - \tilde\psi^\lambda \psi^\mu\, \delta F^\nu\big)\Big) \\
&= \int \mathrm{d}^2 z\, \Big(2F_\lambda + \big(G_{\mu\nu,\lambda} - G_{\lambda\mu,\nu} - G_{\lambda\nu,\mu}\big)\, \tilde\psi^\nu \psi^\mu\Big)\, \delta F^\lambda \\
&= \int \mathrm{d}^2 z\, \Big(2F_\lambda - 2\Gamma_{\lambda\mu\nu}\tilde\psi^\nu \psi^\mu\Big)\, \delta F^\lambda\,,
\end{aligned} \tag{36}
$$

$$4\pi\, \delta S_B = \int \mathrm{d}^2 z\, \Big(0 + \big(B_{\mu\nu,\lambda} + B_{\lambda\mu,\nu} + B_{\nu\lambda,\mu}\big)\, \tilde\psi^\nu \psi^\mu\Big)\, \delta F^\lambda = \int \mathrm{d}^2 z\, H_{\lambda\mu\nu} \tilde\psi^\nu \psi^\mu\, \delta F^\lambda\,,$$

$$F_\lambda = \left(\Gamma_{\lambda\mu\nu} - \frac{1}{2}H_{\lambda\mu\nu}\right)\tilde\psi^\nu \psi^\mu, \tag{37}$$

$$F^\lambda = \left(\Gamma^\lambda_{\mu\nu} - \frac{1}{2}H^\lambda_{\mu\nu}\right)\tilde\psi^\nu \psi^\mu, \tag{38}$$

Here we've used the (anti-)symmetry of $G_{\mu\nu}$ and $B_{\mu\nu}$, and we adopt the convention that the Levi-Civita connection $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} = G^{\lambda\lambda'}\Gamma_{\lambda'\mu\nu}$; similar holds for $B_{\mu\nu}$ and $H^\lambda_{\mu\nu}$.

Substitute F_λ into S, collect the $\psi^0, \psi^2, \tilde\psi^2$ and $\psi^2\tilde\psi^2$ terms respectively, and we have:

$$
\begin{aligned}
4\pi S &= \int \mathrm{d}^2 z\, \bigg(\left(G_{\mu\nu} + B_{\mu\nu}\right) \partial X^\mu \bar\partial X^\nu \\
&\qquad + \big(G_{\mu\nu} + \cancel{B_{\mu\nu}}\big)\Big(\tilde\psi^\mu \partial \tilde\psi^\nu - (\bar\partial \psi^\mu)\psi^\nu\Big) \\
&\qquad - \left(G_{\mu\nu,\lambda} + B_{\mu\nu,\lambda}\right)\Big(\psi^\lambda \psi^\mu \bar\partial X^\nu + \tilde\psi^\lambda \tilde\psi^\nu \partial X^\mu\Big) \\
&\qquad + G_{\mu\nu}F^\mu F^\nu - 2\left(\Gamma_{\lambda\mu\nu} - \frac{1}{2}H_{\lambda\mu\nu}\right)\tilde\psi^\nu \psi^\mu F^\lambda \\
&\qquad + \left(G_{\mu\nu,\rho\sigma} + B_{\mu\nu,\rho\sigma}\right) \tilde\psi^\nu \psi^\mu \psi^\rho \tilde\psi^\sigma\bigg) \\
&= \int \mathrm{d}^2 z\, \bigg(\left(G_{\mu\nu} + B_{\mu\nu}\right) \partial X^\mu \bar\partial X^\nu \\
&\qquad + G_{\mu\nu}\Big(\tilde\psi^\mu \partial \tilde\psi^\nu + \psi^\mu \bar\partial \psi^\nu\Big) - \left(G_{\mu\nu,\lambda} + B_{\mu\nu,\lambda}\right)\Big(\psi^\lambda \psi^\mu \bar\partial X^\nu + \tilde\psi^\lambda \tilde\psi^\nu \partial X^\mu\Big) \\
&\qquad - F_\lambda F^\lambda + \left(G_{\mu\nu,\rho\sigma} + B_{\mu\nu,\rho\sigma}\right) \psi^\mu \psi^\rho \tilde\psi^\nu \tilde\psi^\sigma\bigg)
\end{aligned} \tag{39}
$$

Here we've performed some integration by parts to clean up the result. Note that some terms involving $B_{\mu\nu}$ vanish conveniently (up to integration by parts) due to anti-symmetricity.

The $\psi^2, \tilde{\psi}^2$ terms in the integrand can be further simplified as follows:

$$\begin{aligned}
\mathcal{L}_{\psi^2} &= G_{\mu\nu}\psi^\mu\bar{\partial}\psi^\nu - (G_{\mu\nu,\lambda} + B_{\mu\nu,\lambda})\,\psi^\lambda\psi^\mu\bar{\partial}X^\nu \\
&= G_{\mu\nu}\psi^\mu\bar{\partial}\psi^\nu - \left(G_{\mu[\nu,\lambda]} + B_{\mu[\nu,\lambda]}\right)\psi^\lambda\psi^\mu\bar{\partial}X^\nu \\
&= G_{\mu\nu}\psi^\mu\bar{\partial}\psi^\nu - \left(-\Gamma_{\lambda\mu\nu} + \frac{1}{2}H_{\lambda\mu\nu}\right)\psi^\lambda\psi^\mu\bar{\partial}X^\nu \\
&= G_{\mu\nu}\psi^\mu\left(\bar{\partial}\psi^\nu + \left(\Gamma^\nu_{\rho\sigma} - \frac{1}{2}H^\nu_{\rho\sigma}\right)\psi^\rho\bar{\partial}X^\sigma\right) \\
&= G_{\mu\nu}\psi^\mu\left(\bar{\partial}\psi^\nu + \left(\Gamma^\nu_{\rho\sigma} + \frac{1}{2}H^\nu_{\rho\sigma}\right)\psi^\sigma\bar{\partial}X^\rho\right) = G_{\mu\nu}\psi^\mu\bar{\mathcal{D}}\psi^\nu, \\
\mathcal{L}_{\tilde{\psi}^2} &= G_{\mu\nu}\tilde{\psi}^\mu\partial\tilde{\psi}^\nu - (G_{\mu\nu,\lambda} + B_{\mu\nu,\lambda})\,\tilde{\psi}^\lambda\tilde{\psi}^\nu\partial X^\mu \\
&= G_{\mu\nu}\tilde{\psi}^\mu\left(\partial\tilde{\psi}^\nu + \left(\Gamma^\nu_{\rho\sigma} - \frac{1}{2}H^\nu_{\rho\sigma}\right)\tilde{\psi}^\sigma\partial X^\rho\right) = G_{\mu\nu}\tilde{\psi}^\mu\mathcal{D}\tilde{\psi}^\nu,
\end{aligned} \tag{40}$$

For the $\psi^2\tilde{\psi}^2$ term, recall that $R_{\mu\nu\rho\sigma} = e_\mu[\nabla_\rho, \nabla_\sigma]\,e_\nu, \nabla_\sigma e_\nu = e_\lambda\Gamma^\lambda_{\sigma\nu}$, and we have:

$$\begin{aligned}
\mathcal{L}_{\psi^2\tilde{\psi}^2} &= \psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma\left(G_{\mu\rho,\nu\sigma} + B_{\mu\rho,\nu\sigma} + \left(\Gamma_{\lambda\mu\rho} - \frac{1}{2}H_{\lambda\mu\rho}\right)\left(\Gamma^\lambda_{\nu\sigma} - \frac{1}{2}H^\lambda_{\nu\sigma}\right)\right) \\
&= \psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma\left(G_{\mu\rho,\nu\sigma} + \Gamma_{\lambda\mu\rho}\Gamma^\lambda_{\nu\sigma} + B_{\mu\rho,\nu\sigma} - \frac{1}{2}\left(\Gamma^\lambda_{\mu\rho}H_{\lambda\nu\sigma} + \Gamma^\lambda_{\nu\sigma}H_{\lambda\mu\rho}\right) + \frac{1}{4}H^\lambda_{\mu\rho}H_{\lambda\nu\sigma}\right) \\
&= \mathcal{L}_G + \mathcal{L}_B + \frac{1}{4}H^\lambda_{\mu\rho}H_{\lambda\nu\sigma}\,\psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma,
\end{aligned} \tag{41}$$

$$\begin{aligned}
\mathcal{L}_G &= \psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma\left(G_{\mu\rho,\nu\sigma} + \Gamma_{\lambda\mu\rho}\Gamma^\lambda_{\nu\sigma}\right) \\
&= \psi^{[\mu}\psi^{\nu]}\tilde{\psi}^{[\rho}\tilde{\psi}^{\sigma]}\left(G_{\mu\rho,\nu\sigma} + \Gamma_{\lambda\mu\rho}\Gamma^\lambda_{\nu\sigma}\right) \\
&= \frac{1}{2}\,\psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma\left\{\left(\frac{1}{2}\left(G_{\mu\rho,\nu\sigma} - G_{\mu\sigma,\nu\rho}\right) + \Gamma_{\lambda\mu\rho}\Gamma^\lambda_{\nu\sigma}\right) - \left(\cdots\right)_{\rho\leftrightarrow\sigma}\right\} \\
&= \frac{1}{2}R_{\mu\nu\rho\sigma}\,\psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma, \\
\mathcal{L}_B &= \frac{1}{2}\,\nabla_\rho H_{\mu\nu\sigma}\,\psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma,
\end{aligned} \tag{42}$$

Therefore, the total action is:

$$\begin{aligned}
S = \frac{1}{4\pi}\int \mathrm{d}^2 z\,\Bigg(&(G_{\mu\nu} + B_{\mu\nu})\,\partial X^\mu\bar{\partial}X^\nu \\
&+ G_{\mu\nu}\left(\tilde{\psi}^\mu\mathcal{D}\tilde{\psi}^\nu + \psi^\mu\bar{\mathcal{D}}\psi^\nu\right) \\
&+ \left(\frac{1}{2}R_{\mu\nu\rho\sigma} + \frac{1}{2}\,\nabla_\rho H_{\mu\nu\sigma} + \frac{1}{4}H^\lambda_{\mu\rho}H_{\lambda\nu\sigma}\right)\psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma\Bigg)
\end{aligned} \tag{43}$$

$\boxed{4}$ Mixed Anomaly Between Diffeomorphism and Axial $U(1)$ Symmetry:

✒ PAST WORK, AS TEMPLATE ✒

$\boxed{5}$ **BRST Quantization of Bosonic String**:

$$S = S^X + S^{bc}, \tag{44}$$

$$S^X = \frac{1}{2\pi\alpha'}\int \mathrm{d}^2 z\, \partial X^\mu \bar{\partial} X_\mu, \quad S^{bc} = \frac{1}{2\pi}\int \mathrm{d}^2 z \left(b\,\bar{\partial} c + \tilde{b}\,\partial\tilde{c}\right) \tag{45}$$

This is the gauge fixed action. The corresponding BRST transformation is listed in *Polchinski*; for each of the subsystems, we have its energy-momentum:

$$T^X(z) = -\frac{1}{\alpha'}:\partial X^\mu \partial X_\mu:, \quad \tilde{T}^X(\bar{z}) = -\frac{1}{\alpha'}:\bar{\partial} X^\mu \bar{\partial} X_\mu:, \tag{46}$$

$$T^{bc}(z) = :(\partial b)\, c: - \,2\,\partial(:bc:), \quad \tilde{T}^{bc}(\bar{z}) = :(\bar{\partial}\tilde{b})\,\tilde{c}: - \,2\,\bar{\partial}(:\tilde{b}\tilde{c}:), \tag{47}$$

(a) To get the energy-momentum of S, let's visit each of the subsystems respectively; first, BRST transformation of X is given by:

$$\delta X^\mu = i\epsilon\left(c\partial + \tilde{c}\bar{\partial}\right)X^\mu \tag{48}$$

Compared with the conformal transformation[8]: $\delta X^\mu = -\epsilon\left(v\partial + \tilde{v}\bar{\partial}\right)X^\mu$, we see that they are in fact identical under the equivalence $-\epsilon v \sim i\epsilon c$, $-\epsilon\tilde{v} \sim i\epsilon\tilde{c}$, hence we can simply follow the derivation of conformal current and write down δS^X's contribution to the conserved current:

$$j^X = c(z)\,T^X(z) \tag{49}$$

The transformation of b, c is less obvious; for holomorphic current, we need only focus on the holomorphic part of S^{bc}; on-shell variation yields:

$$0 = \delta S^{bc} = \left(\frac{1}{2\pi}\int \mathrm{d}^2 z\left(-\bar{\partial} c\;\delta b - \bar{\partial} b\;\delta c\right)\right)_{=0} + \frac{1}{2\pi}\int \mathrm{d}^2 z\,\bar{\partial}(b\,\delta c) = \frac{1}{2\pi}\int \mathrm{d}^2 z\,\bar{\partial}\epsilon\left(-ibc\,\partial c\right) \tag{50}$$

Here we've plugged in $\delta c = i\epsilon(z,\bar{z})\,c\partial c$, and we have moved $\bar{\partial}\epsilon$ to the beginning of the expression, while respecting the anti-commuting nature of ϵ. With a conventional i coefficient (which agrees with the convention of j^X), we have bc's contribution to the conserved current:

$$j^{bc} = i\left(-ibc\,\partial c\right) = bc\,\partial c \tag{51}$$

Note that j^{bc} is, in fact, related to the energy-momentum (at least classically):

$$\frac{1}{2}\,cT^{bc} = \frac{1}{2}\,c\,(\partial b)\,c - c\,\partial(bc) = -c\,\partial(bc) = -cb\,\partial c = bc\,\partial c = j^{bc} \tag{52}$$

Hence we have the classical BRST current:

$$j(z) = c(z)\left(T^X + \frac{1}{2}\,T^{bc}\right) \tag{53}$$

□

[8] We follow the convention of *Polchinski* unless otherwise stated.

For a quantum version, redefine $j(z)$ with normal ordering[9], and we have:

$$T(z)\, j(0) \sim T^X(z)\, T^X(0)\, c(0) + T^{bc}(z)\, cT^X(0) + T^{bc}(z) :\! bc\, \partial c \!:_{(0)}, \tag{54}$$

$$\text{where} \quad T^X(z)\, T^X(0)\, c(0) \sim \left(\frac{D}{2z^4} + \frac{2}{z^2}\, T^X(0) + \frac{1}{z}\, \partial T^X(0) \right) c(0), \tag{55}$$

Here we've used the fact that X and b, c is de-coupled in the gauge-fixed action, hence their OPE is trivial. Also, we've expanded the first term using TT OPE of the free boson. Additionally, note that $c(z)$ is primary with weight $(-1, 0)$, we have:

$$\begin{aligned} T^{bc}(z)\, cT^X(0) &\sim \left\{ T^{bc}(z)\, c(0) \right\} T^X(0) \\ &\sim \left(\frac{-1}{z^2}\, c(0) + \frac{1}{z}\, \partial c(0) \right) T^X(0), \end{aligned} \tag{56}$$

The last term in (54) can be brute-forced as follows:

$$T^{bc}(z) :\! bc\, \partial c \!:_{(0)} = \big(:\!(\partial b)\, c\!: - \, 2\, \partial (:\! bc \!:) \big)_{(z)} :\! bc\, \partial c \!:_{(0)}\,, \tag{57}$$

$$\begin{aligned} :\!(\partial b)\, c\!:_{(z)} :\! bc\, \partial c \!:_{(0)} \sim\; & :\!(\partial b)\, c_{(z)}\, bc\, \partial c_{(0)}\!: + :\!(\partial b)\, c_{(z)}\, bc\, \partial c_{(0)}\!: + :\!(\partial b)\, c_{(z)}\, bc\, \partial c_{(0)}\!: \\ & + :\!(\partial b)\, c_{(z)}\, bc\, \partial c_{(0)}\!: + :\!(\partial b)\, c_{(z)}\, bc\, \partial c_{(0)}\!: \\ \sim\; & \frac{-1}{z^2}\, (+1) :\! c_{(z)}\, b\, \partial c_{(0)}\!: + \frac{-2}{z^3}\, (-1) :\! c_{(z)}\, bc_{(0)}\!: + \frac{1}{z}\, (+1) :\! \partial b_{(z)}\, c \partial c_{(0)}\!: \\ & + \frac{-1}{z^2} \cdot \frac{1}{z}\, (+1)\, \partial c(0) + \frac{-2}{z^3} \cdot \frac{1}{z}\, (-1)\, c(0) \\ \sim\; & \frac{-1}{z^2} \left(-j^{bc}(0) + \mathcal{O}(z^2) \right) + \frac{2}{z^3} \left(z\, j^{bc}(0) + \frac{z^2}{2} :\! bc\, \partial^2 c\!:_{(0)} + \mathcal{O}(z^3) \right) \\ & + \frac{1}{z} \left(:\!(\partial b)\, c\, \partial c\!:_{(0)} + \mathcal{O}(z) \right) + \frac{-1}{z^3}\, \partial c(0) + \frac{2}{z^4}\, c(0) \\ \sim\; & \frac{4}{2z^4}\, c(0) + \frac{-1}{z^3}\, \partial c(0) + \frac{3}{z^2}\, j^{bc}(0) + \frac{1}{z} :\! \big(bc\, \partial^2 c + (\partial b)\, c\, \partial c \big)\!:_{(0)}\,, \\ \sim\; & \frac{4}{2z^4}\, c(0) + \frac{-1}{z^3}\, \partial c(0) + \frac{3}{z^2}\, j^{bc}(0) + \frac{1}{z}\, \partial j^{bc}(0), \end{aligned} \tag{58}$$

$$\begin{aligned} :\! bc\!:_{(z)} :\! bc\, \partial c\!:_{(0)} \sim\; & :\! bc_{(z)}\, bc\, \partial c_{(0)}\!: + :\! bc_{(z)}\, bc\, \partial c_{(0)}\!: + :\! bc_{(z)}\, bc\, \partial c_{(0)}\!: \\ & + :\! bc_{(z)}\, bc\, \partial c_{(0)}\!: + :\! bc_{(z)}\, bc\, \partial c_{(0)}\!: \\ \sim\; & \frac{1}{z}\, (+1) :\! c_{(z)}\, b\, \partial c_{(0)}\!: + \frac{1}{z^2}\, (-1) :\! c_{(z)}\, bc_{(0)}\!: + \frac{1}{z}\, (+1) :\! b_{(z)}\, c \partial c_{(0)}\!: \\ & + \frac{1}{z} \cdot \frac{1}{z}\, (+1)\, \partial c(0) + \frac{1}{z^2} \cdot \frac{1}{z}\, (-1)\, c(0) \end{aligned}$$

---

[9] Normal ordering between ≥ 3 operators is in fact *not* associative; this directly leads to the ambiguity we are about to discover. See *Di Francesco et al* for more detailed discussions. Naïvely, $:\! bc\, \partial c\!:_{(0)}$ is *defined* as $b(0)\, c(z_1)\, \partial c(z_2)$ while $z_1, z_2 \to 0$, with singular terms subtracted; however, different ways of taking the limit might lead to different results. For example, we can first take $z_1 \to 0$ then $z_2 \to 0$, or we can first take $z_1 \to z_2$ then $z_2 \to 0$. This two procedures will differ by $\frac{3}{2}\, \partial^2 c(z)$, which is precisely the correction we are about to find out. *I suppose this is somehow related to topology, e.g. braid group?*

$$
\begin{aligned}
:bc:_{(z)}\, :bc\,\partial c:_{(0)} &\sim \frac{1}{z}\left(-j^{bc}(0)\right) + \frac{-1}{z^2}\left(z\, j^{bc}(0)\right) + \frac{1}{z}\left(j^{bc}(0)\right) + \frac{1}{z^2}\,\partial c(0) + \frac{-1}{z^3}\, c(0) \\
&\sim \frac{-1}{z^3}\, c(0) + \frac{1}{z^2}\,\partial c(0) + \frac{-1}{z}\, j^{bc}(0),
\end{aligned} \tag{59}
$$

$$
\partial(:bc:)_{(z)}\, :bc\,\partial c:_{(0)} \sim \frac{6}{2z^4}\, c(0) + \frac{-2}{z^3}\,\partial c(0) + \frac{1}{z^2}\, j^{bc}(0), \tag{60}
$$

$$
T^{bc}(z)\, :bc\,\partial c:_{(0)} \sim \frac{-8}{2z^4}\, c(0) + \frac{3}{z^3}\,\partial c(0) + \frac{1}{z^2}\, j^{bc}(0) + \frac{1}{z}\,\partial j^{bc}(0), \tag{61}
$$

$$
T(z)\, j(0) \sim \big((55) + (56) + (61)\big) \sim \frac{D-8}{2z^4}\, c(0) + \frac{3}{z^3}\,\partial c(0) + \frac{1}{z^2}\, j(0) + \frac{1}{z}\,\partial j(0), \tag{62}
$$

We see that $j(z)$ defined with naïve normal ordering is *almost* but *not quite* a primary. It differs from primary OPE at $\mathcal{O}\big(\frac{1}{z^4}\big)$ and $\mathcal{O}\big(\frac{1}{z^3}\big)$. However, it is possible to make it into a primary by adding extra terms that do not interfere with current conservation $\bar{\partial} j = 0$. To cancel the $\frac{3}{z^3}\,\partial c(0)$ term, notice that $b(z)\,\partial^2 c(0) \sim \frac{2}{z^3}$, therefore it may be helpful to look at:

$$
\begin{aligned}
T(z)\,\partial^2 c(0) \sim T^{bc}(z)\,\partial^2 c(0) &\sim \partial_w^2 \big(T^{bc}(z)\, c(w)\big)_{w\to 0} \\
&\sim \partial_w^2 \left(\frac{-1}{(z-w)^2}\, c(w) + \frac{1}{z-w}\,\partial c(w)\right)_{w\to 0} \\
&\sim \frac{-12}{2z^4}\, c(0) + \frac{-2}{z^3}\,\partial c(0) + \frac{1}{z^2}\,\partial^2 c(0) + \frac{1}{z}\,\partial^3 c(0),
\end{aligned} \tag{63}
$$

Again we've used Tc OPE of the primary $c(w)$. We see that indeed, the $\frac{1}{z^3}\,\partial c(0)$ term can be canceled by shifting $j(z)$:

$$
j(z) \;\longmapsto\; j(z) + \frac{3}{2}\,\partial^2 c(z), \quad j(z) = cT^X + :bc\,\partial c: + \frac{3}{2}\,\partial^2 c, \tag{64}
$$

$$
T(z)\, j(0) \sim \frac{D-26}{2z^4}\, c(0) + \frac{1}{z^2}\, j(0) + \frac{1}{z}\,\partial j(0), \tag{65}
$$

We see that $j(z)$ defined in this way is a primary of weight $(1,0)$ in $D = 26$. This is the quantum BRST current. □

(b) For jj OPE, we have:

$$
j = cT^X + j', \quad j' \equiv j^{bc} + \frac{3}{2}\,\partial^2 c, \quad j^{bc} = \frac{1}{2}{:}cT^{bc}{:} = :bc\,\partial c:, \tag{66}
$$

$$
\begin{aligned}
j_z j_0 &\sim :\!\big\{T_z^X T_0^X\big\}\, c_z c_0\!: + :\!\{c_z j_0'\}\, T_z^X\!: + :\!\{j_z' c_0\}\, T_0^X\!: + j_z' j_0' \\
&\sim :\!\big\{T_z^X T_0^X\big\}\, c_z c_0\!: + :\!\big\{c_z j_0^{bc}\big\}\, T_z^X\!: + :\!\big\{j_z^{bc} c_0\big\}\, T_0^X\!: + j_z' j_0',
\end{aligned} \tag{67}
$$

From now on, for convenience and clarity, we will use subscripts to denote variable dependence: $c_z = c(z)$. Let's compute this term by term. We have:

$$
\begin{aligned}
:\!\big\{T_z^X T_0^X\big\}\, c_z c_0\!: &\sim :\!\left(\frac{D}{2z^4} + \frac{2}{z^2}\, T_0^X + \frac{1}{z}\,\partial T_0^X\right)\left(z\,\partial c_0 + \frac{z^2}{2}\,\partial^2 c_0 + \frac{z^3}{6}\,\partial^3 c_0\right) c_0\!: \\
&\sim -\left(\frac{D}{2z^3}\, c\,\partial c_0 + \frac{D}{4z^2}\, c\,\partial^2 c_0 + \frac{D}{12z}\, c\,\partial^3 c_0 + \frac{2}{z}\, :T^X c\,\partial c_0:\right),
\end{aligned} \tag{68}
$$

$$\begin{aligned} j_z^{bc} c_0 &\sim \frac{1}{2} :cT^{bc}:_z c_0 \sim \frac{1}{2} c_z \{ :T^{bc}:_z c_0 \} \sim \frac{1}{2} c_z \{ T_z \, c_0 \} \\ &\sim -\frac{1}{2} \left(\frac{-1}{z^2} c_0 + \frac{1}{z} \partial c_0 \right) (c_0 + z \, \partial c_0) \sim 0, \end{aligned} \tag{69}$$

$$j_0^{bc} c_z \sim 0, \tag{70}$$

$$\begin{aligned} j'_z j'_0 &\sim j_z^{bc} j_0^{bc} + \frac{3}{2} j_z^{bc} \, \partial^2 c_0 + \frac{3}{2} \partial^2 c_z \, j_0^{bc} \\ &\sim \frac{1}{2} :cT^{bc}:_z j_0^{bc} + \frac{3}{2} \left(j_z^{bc} \, \partial^2 c_0 + \partial^2 c_z \, j_0^{bc} \right), \end{aligned} \tag{71}$$

The task is now reduced to calculating terms in the above $j'j'$ OPE, which can be laboriously computed following a similar procedure as before. Note that there will be a $\frac{1}{z} :cT^{bc}: \partial c_0$ term which combines with the $\frac{2}{z} :cT^X: \partial c_0$ term in (68). In total, we obtain the final jj OPE:

$$j_z j_0 \sim -\frac{D-18}{2z^3} \, c \, \partial c_0 - \frac{D-18}{4z^2} \, c \, \partial^2 c_0 - \frac{D-26}{12z} \, c \, \partial^3 c_0 \tag{72}$$

(c) Following the convention of *Polchinski*, expand X^μ, b, c into modes α_n^μ, b_n, c_n, then a generic level 2 state of an open string can be created as[10]:

$$\begin{aligned} |\psi\rangle = \big(& e_{\mu\nu} \alpha_{-1}^\mu \alpha_{-1}^\nu + \beta_\mu \alpha_{-1}^\mu b_{-1} + \gamma_\mu \alpha_{-1}^\mu c_{-1} \\ & + \eta \, b_{-1} c_{-1} + e_\mu \alpha_{-2}^\mu + \beta \, b_{-2} + \gamma \, c_{-2} \big) \, |k; 0\rangle \end{aligned} \tag{73}$$

Here $e_{\mu\nu}$ is chosen to be symmetric since $\alpha_{-1}^\mu \alpha_{-1}^\nu$ commutes. By acting on L_0 (expanded in modes), we find that $m^2 = -k^2 = \frac{1}{\alpha'} = l_s$: massive.

The BRST charge $Q = \frac{1}{2\pi i} \oint \left(\mathrm{d}z \, j(z) - \mathrm{d}\bar{z} \, \tilde{j}(z) \right)$ can also be expanded in modes; note that:

$$Q^2 = \frac{1}{2} \{ Q, Q \} \propto \oint \frac{\mathrm{d}z}{2\pi i} \operatorname*{Res}_{z' \to z} j(z') \, j(z) + (\text{conjugate}) \tag{74}$$

Compared with the jj OPE, we see that Q is nilpotent iff. $D = 26$, i.e. the critical dimension of bosonic string theory. This condition is necessary for consistent BRST quantization.

The physical states are firstly, Q-closed; i.e.

$$Q_B \, |\psi\rangle = 0 \implies 4 l_s \, k^\mu e_{\mu\nu} + l_s \, k_\nu \eta + e_\nu = 0, \quad 2\sqrt{2} \, l_s \, k^\mu + e_\nu^\nu e_\mu = 0, \quad \beta_\mu = \beta = 0, \tag{75}$$

This is also the negative-norm states.

On the other hand, Q-exact states generate gauge transformations; this gives:

$$\gamma_\nu \mapsto \gamma_\nu + \gamma'_\nu, \quad \gamma \mapsto \gamma + \gamma', \quad \eta \mapsto \eta + \eta', \quad e_{\mu\nu} \mapsto e_{\mu\nu} + l_s \left(\beta'_\mu k_\nu + \beta'_\nu k_\mu \right), \tag{76}$$

Here $\beta'_\mu, \gamma'_\nu, \gamma', \eta'$ are arbitrary gauge parameters. For closed string the result can be obtained by the doubling trick, i.e. by introducing anti-holomorphic modes $\tilde{\alpha}, \tilde{b}, \tilde{c}$ and imposing reality conditions. The result is similar. ■

[10] Reference: Bram M. Wouters, *BRST quantization and string theory spectra*.

$\boxed{6}$ Linear Dilaton CFT:

For $z \mapsto z + \epsilon(z)$, we have:

$$\delta X^\mu = -\epsilon \partial X^\mu - \bar{\epsilon}\bar{\partial} X^\mu - \frac{\alpha' V^\mu}{2}\left(\partial\epsilon + \bar{\partial}\bar{\epsilon}\right) \tag{77}$$

Note that the α' term has no dependence on X.

(a) For simplicity, assume for now $X = X(z)$: holomorphic. Note that the α' term comes from the transformation of *internal* degrees of freedom. We have:

$$X'(z') - X(z) = -\frac{\alpha' V}{2}\,\partial\epsilon + \mathcal{O}\big(\epsilon^2\big), \tag{78}$$

This is a first order approximation of the finite transformation $z \mapsto z' = w(z)$.

To obtain a full expression, notice that for the above transformation to be a *symmetry*, the action should be invariant under $X(z) \mapsto X'(z)$:

$$S' = S \implies \int \mathrm{d}^2 z\, \partial X'^\mu \bar{\partial} X'_\mu(z,\bar{z}) \overset{(*)}{=\!=} \int \mathrm{d}^2 z'\, \partial' X'^\mu \bar{\partial}' X'_\mu(z',\bar{z}') = \int \mathrm{d}^2 z\, \partial X^\mu \bar{\partial} X_\mu(z,\bar{z}) \tag{79}$$

Here at $(*)$ we use the diff-invariant property of the action. Restore the anti-holomorphic component and insert (78), then we find that in order to cancel the $\mathcal{O}\big(\epsilon^2\big)$ terms, we have to fix:

$$X'(z') - X(z) = -\frac{\alpha' V}{2}\left(\partial\epsilon - \frac{1}{2}\left(\partial\epsilon\right)^2 + \mathcal{O}\big(\epsilon^3\big)\right) \tag{80}$$

The above process can be done order by order, in the end we obtain that[11]:

$$\begin{aligned} X'(z',\bar{z}') - X(z,\bar{z}) &= -\frac{\alpha' V}{2}\left(\partial\epsilon - \frac{1}{2}\left(\partial\epsilon\right)^2 + \frac{1}{3}\left(\partial\epsilon\right)^2 - \cdots\right) + \text{(conjugate)} \\ &= -\frac{\alpha' V}{2}\,\ln(1+\partial\epsilon) + \text{(conjugate)} \\ &= -\frac{\alpha' V}{2}\,\ln\left(\frac{\mathrm{d}z'}{\mathrm{d}z}\frac{\mathrm{d}\bar{z}'}{\mathrm{d}\bar{z}}\right) \end{aligned} \tag{81}$$

(b) Perform the usual Noether's procedure on the free boson action, and we have:

$$\delta\mathcal{L} \propto \frac{1}{\alpha'}\big(\partial\,\delta X^\mu\,\bar{\partial} X_\mu + \partial X^\mu \bar{\partial}\,\delta X_\mu\big) \sim \bar{\partial}\epsilon\bigg(V^\mu \partial^2 X^\mu - \frac{1}{\alpha'}\partial X^\mu \bar{\partial} X_\mu\bigg) \tag{82}$$

Here we've plugged in the holomorphic part of δX^μ, used integration by parts to move $\bar{\partial}$ before ϵ, and collected the $\bar{\partial}\epsilon$ coefficients. This gives:

$$T(z) = -\frac{1}{\alpha'}\,{:}\partial X^\mu \bar{\partial} X_\mu{:} + V^\mu \partial^2 X^\mu \tag{83}$$

[11] I would like to thank Lucy Smith for helpful discussions. A better recipe to find finite transformations is to consider its properties under composition, which will lead to some constraints that can be solved to obtain the result.

With $X^\mu_z X^\nu_0 \sim -\frac{\alpha'}{2}\,\eta^{\mu\nu}\ln|z|^2$ unchanged, the TT OPE can be calculated following the usual procedure, as shown in great detail before. Here we can use the known result from free boson theory to speed up our calculation:

$$\begin{aligned} T_z T_0 &\sim \left(V_\mu \partial^2 X^\mu + T'\right)_z \left(V_\mu \partial^2 X^\mu + T'\right)_0 \\ &\sim V_\mu V_\nu\, \partial^2 X^\mu_z\, \partial^2 X^\nu_0 + V_\mu\, \partial^2 X^\mu_z\, T'_0 + V_\mu T'_z\, \partial^2 X^\mu_0 + T'_z\, T'_0 \end{aligned} \tag{84}$$

Here T' is the usual free boson stress tensor. Combining all terms yields:

$$T_z T_0 \sim \frac{D + 6\alpha' V^2}{2z^4} + \frac{2}{z^2}\, T_0 + \frac{1}{z}\, \partial T_0, \quad c = D + 6\alpha' V^2 \tag{85}$$

■

$\boxed{7}$ Bosonic Strings on S^3:

For bosonic strings moving on S^3 (radius R) with background dilaton $\Phi = \text{const.}$ and B-field:

$$B = R^2 \sin\theta\, (\psi - \sin\psi\cos\psi)\; \mathrm{d}\theta \wedge \mathrm{d}\phi \tag{86}$$

The corresponding β-functions and trace anomaly can be computed using the formulae given in *Polchinski*; here (ψ, θ, ϕ) is the usual spherical coordinates on S^3.

In fact, field strength:

$$H = \mathrm{d}B = 2R^2 \sin\theta \sin\psi\, \mathrm{d}\psi \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi \tag{87}$$

While the spacetime curvature for a maximally symmetric sphere[12]: $\mathscr{R}_{\mu\nu} = \frac{2}{R^2}\, g_{\mu\nu}$, $\mathscr{R} = \frac{6}{R^2}$. Plug in these results, and we have:

$$\beta^G = \beta^B = 0, \quad T^a{}_a \simeq -\frac{1}{2}\,\beta^\Phi \mathcal{R} = -\frac{D - 26 - \alpha'\mathscr{R}}{12}\,\mathcal{R} \tag{88}$$

(a) Compared with the trace anomaly formula of a CFT: $T^a{}_a = -\frac{1}{12}\, c\mathcal{R}$, where $\mathcal{R}$ is the world-sheet Ricci scalar, we see that our theory is indeed conformally invariant with Weyl anomaly. Its central charge is given by:

$$c \simeq D - 26 - \alpha'\mathscr{R} = 3 - 26 - \frac{6\alpha'}{R^2} \tag{89}$$

This includes ghost contribution (-26). If we do not gauge the conformal symmetry, then there will not be ghost contribution, and we will have $c \simeq 3 - \frac{6\alpha'}{R^2}$.

(b) The background B field given above is not single-valued on the ψ circle. Note that we've encountered such difficulty in electromagnetism with a multi-valued $A^\mu(x)$. In fact, the resolution for this issue is very similar to Dirac's quantization of the magnetic monopole[13]: by allowing the action S to be invariant modulo 2π, since $e^{-(S+2\pi i)} = e^{-S}$.

[12] I would like to thank 林般 for some very helpful hints.

[13] Reference: J. J. Sakurai, *Modern Quantum Mechanics*.

More specifically, for $\psi \mapsto \psi + 2\pi$, we have:

$$2\pi i\, n = \Delta S = \frac{i}{2\pi\alpha'}\,\Delta\int_\Sigma X^*B = \frac{i}{2\pi\alpha'}\,\Delta\int_{X(\Sigma)} B = \frac{i}{2\pi\alpha'}\,\Delta\int_M H \tag{90}$$

B is a 2-form in S^3, X^*B denotes its pullback to the worldsheet, and $X(\Sigma) \subset S^3$ denotes the embedding of Σ into S^3. Note that H is proportional to the volume form in S^3, hence we have:

$$\Delta\int_M H = 2R^2\,\Delta\mathrm{Vol}(M) = 2R^2\,\mathbb{Z}\,\mathrm{Vol}(S^3) = 2R^2\,2\pi^2\,\mathbb{Z} = 4\pi^2 R^2\,\mathbb{Z} \tag{91}$$

This leads to the following quantization:

$$\frac{R^2}{\alpha'} = n \in \mathbb{Z}, \quad R \geq \sqrt{\alpha'} \geq \left(\alpha'/\ell\right)^{1/3} \tag{92}$$

In particular, in string units: $\alpha' = 1$, we have $R \geq 1$. ■

$\boxed{8}$ Anomalous Currents:

(a) For a conserved current in flat worldsheet to be anomalous in curved worldsheet, then its deviation from conservation must be proportional to the Ricci scalar:

$$\nabla_a j^a = QR, \quad Q = \text{const.} \tag{93}$$

The logic here is similar to the Weyl anomaly[14]: $\nabla_a j^a$ is diff- and Poincaré-invariant with dimension 2, because we have preserved these symmetries, and it vanishes in the flat case; this leaves only one possibility — $\nabla_a j^a \propto R$: the Ricci scalar.

For conformal transformation $z \mapsto z + \epsilon(z)$, $\bar{z} \mapsto \bar{z} + \bar{\epsilon}(\bar{z})$, we have:

$$\delta_\epsilon j(0) = -\operatorname*{Res}_{z\to 0}\epsilon(z)\,T(z)\,j(0) - \operatorname*{Res}_{\bar{z}\to 0}\bar{\epsilon}(\bar{z})\,\tilde{T}(\bar{z})\,j(0) \tag{94}$$

Hence the $z^{-3}, \bar{z}^{-3}$ coefficients of the OPE reflect the $\epsilon = z^2$, $\bar{\epsilon} = \bar{z}^2$ transformation of j. By comparing the Weyl transformations[15], this yields a total coefficient of $4Q$.

(b) For bc CFT with $j = :cb:$, the anomaly can be explicitly calculated using our results in (a), i.e. by calculating Tj OPE. Following the standard procedure[16], we obtain that:

$$T_z j_0 \sim \frac{1-2\lambda}{z^3} + \mathcal{O}\left(\frac{1}{z^2}\right) \tag{95}$$

Note that the anti-holomorphic part is zero, therefore, we have: $Q = \frac{1}{4}(1-2\lambda)$. ■

[14] See *Polchinski* for reference.

[15] Note that (*Conformal*) = (*Weyl*) + (*Translation*).

[16] For more detailed discussions, see Blumenhagen et al, *Basic Concepts of String Theory*.