

## Correlations of Pauli Spinor Fields

The Lagrangian of a nonrelativistic free particle is given by:

$$\mathcal{L} = \bar{\psi} \left( i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} \right) \psi \quad (1)$$

The Euclidean action is then given by:

$$(-S) = \int d^4x \mathcal{L}_{t=-i\tau} = - \int d^4x \bar{\psi} \left( \frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m} \right) \psi = -\beta H \quad (2)$$

Note that we have  $N = \int d^3\mathbf{x} \bar{\psi}\psi = \int d^3\mathbf{x} n$  as a conserved charge, therefore it is natural to include  $N$  in the partition function:

$$Z = \text{Tr} e^{-\beta(H-\mu N)} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \int d^4x \left\{ -\bar{\psi} \left( \frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m} \right) \psi + \mu n \right\} \quad (3)$$

For a spin- $\frac{1}{2}$  free fermion,  $\psi$  is given by a 2-component Pauli spinor:  $\psi \sim (\psi_+, \psi_-)$ , and  $\bar{\psi} = \psi^\dagger$  is the conjugate transpose of  $\psi$ . Relevant observables in the Heisenberg picture are then given by:

$$n = \psi^\dagger \psi, \quad \tilde{n} = n - \langle n \rangle, \quad s_i = \psi^\dagger \sigma_i \psi, \quad (4)$$

$$D_n = \langle \mathcal{T}_\tau \tilde{n}(x) \tilde{n}(x') \rangle, \quad D_{ij} = \langle \mathcal{T}_\tau s_i(x) s_j(x') \rangle \quad (5)$$

To compute the **density correlation**  $D_n$ , define the generating functional:

$$Z[j] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \int d^4x \left\{ -\bar{\psi} \left( \frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m} \right) \psi + \mu n(x) + j(x) n(x) \right\} \quad (6)$$

To remove the non-zero  $\langle n \rangle$ , consider:

$$W[j] = \ln Z[j], \quad D_n = \langle \mathcal{T}_\tau \tilde{n}(x) \tilde{n}(x') \rangle = \left. \frac{\delta^2 W[j]}{\delta j(x) \delta j(x')} \right|_{j=0} = \left. \frac{\delta^2 W^{(2)}[j]}{\delta j(x) \delta j(x')} \right|_{j=0} \quad (7)$$

$$W^{(2)} \sim \mathcal{O}(j^2) \quad (8)$$

For free theory,  $W^{(2)}$  can be computed explicitly by mode expansions:

$$\psi = \frac{1}{\sqrt{V}} \sum_k e^{ik \cdot x} \psi_k, \quad j = \sum_q e^{-iq \cdot x} j_q, \quad \sum_k e^{ik \cdot x} = V \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} \sum_{n \in \mathbb{Z}} e^{i\omega_n \tau} \quad (9)$$

$$\begin{aligned} Z[j] &= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \int d^4x \left\{ -\bar{\psi} \left( \frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m} - \mu - j(x) \right) \psi \right\} \\ &= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left\{ -\beta \sum_{k,k'} \bar{\psi}_k \left( \left( i\omega + \frac{\mathbf{k}^2}{2m} - \mu \right) \delta_{k,k'} - j_{k-k'} \right) \psi_{k'} \right\} \\ &\sim \left\{ \det_{k,k',\bullet} \left( \beta D_0^{-1}(k) \delta_{k,k'} - \beta j_{k-k'} \right) \right\}^{+1}, \quad D_0^{-1}(k) = i\omega + \frac{\mathbf{k}^2}{2m} - \mu, \\ &= \left\{ \det_{k,k'} \left( \beta D_0^{-1}(k) \delta_{k,k'} - \beta j_{k-k'} \right) \right\}^{+2} \end{aligned} \quad (10)$$

Note that the determinant should go over the implicit spinor indices as well, which results in a power of 2 in the final expression.

We can then compute  $W[j]$  using Jacobi's formula:

$$\begin{aligned}
W[j] &= \ln Z[j] = 2 \ln \det_{k,k'} (\beta D_0^{-1}(k) \delta_{k,k'} - \beta j_{k-k'}) \\
&= 2 \operatorname{Tr}_{k,k'} \ln (\beta D_0^{-1}(k) \delta_{k,k'} - \beta j_{k-k'}) \\
&= 2 \operatorname{Tr}_{k,k'} \left\{ \delta_{k,k'} \ln (\beta D_0^{-1}(k)) + \ln (\mathbb{1}_{k,k'} - j_{k-k'} D_0(k)) \right\}
\end{aligned} \tag{11}$$

The first term is the vacuum contribution with no dependence of  $j$ , hence it is irrelevant in  $W^{(2)} \sim \mathcal{O}(j^2)$ . Expansion of the matrix log in the second term reveals that:

$$\ln (\mathbb{1}_{k,k'} - j_{k-k'} D_0(k)) = - \sum_{n=1}^{\infty} \frac{1}{n} \left( j_{k-k'} D_0(k) \right)_{k,k'}^n \tag{12}$$

$$\begin{aligned}
W^{(2)} &= 2 \operatorname{Tr}_{k,k'} \frac{-1}{2} \left( j_{k-k'} D_0(k) \right)_{k,k'}^2 \\
&= - \operatorname{Tr}_{k,k'} \sum_q \left( j_{k-q} D_0(k) \right) \left( j_{q-k'} D_0(q) \right) \\
&= - \sum_{k,q} j_{k-q} D_0(k) j_{q-k} D_0(q) \\
&= - \sum_k \sum_{q-k} j_{k-q} D_0(k) j_{q-k} D_0(q-k+k) \\
&= - \sum_q \sum_k j_{-q} \left( D_0(k) D_0(q+k) \right) j_q \\
&= \frac{\beta V}{2} \sum_k j_{-k} D_n(k) j_k,
\end{aligned} \tag{13}$$

$$\begin{aligned}
D_n(k) &= - \frac{2}{\beta V} \sum_q D_0(q) D_0(k+q) \\
&= - \frac{2}{\beta V} \sum_q \frac{1}{i\omega_q + E_q} \frac{1}{i(\omega_q + \omega_k) + E_{q+k}}, \quad E_q = \frac{\mathbf{q}^2}{2m} - \mu, \\
&= - \frac{2}{\beta V} \sum_q \left( \frac{1}{i\omega_q + E_q} - \frac{1}{i\omega_q + i\omega_k + E_{q+k}} \right) \frac{1}{i\omega_k + E_{q+k} - E_k} \\
&= -2 \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \frac{1}{i\omega_k + E_{q+k} - E_q} T \sum_{\omega_q} \left( \frac{1}{i\omega_q + E_q} - \frac{1}{i\omega_q + i\omega_k + E_{q+k}} \right) \\
&= -2 \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \frac{1}{i\omega_k + E_{q+k} - E_q} \left( (1 - n(E_q)) - (1 - n(E_{q+k} + i\omega_k)) \right) \\
&= -2 \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \frac{1}{i\omega_k + E_{q+k} - E_q} \left( \frac{1}{-e^{\beta E_{q+k}} + 1} - \frac{1}{e^{\beta E_q} + 1} \right)
\end{aligned} \tag{14}$$

Here we've completed the Matsubara sum of the fermionic frequencies, using:

$$T \sum_{\omega_q} \frac{1}{i\omega_q + E_q} = 1 - n(E_q), \quad n(E_q) = \frac{1}{e^{\beta E_q} + 1}, \quad n(E_q + i\omega_k) = \frac{1}{-e^{\beta E_q} + 1} \tag{15}$$

The retarded propagator and the spectral density is obtained by analytic continuation:

$$D_n^R(\omega, \mathbf{k}) = D_n(\omega \rightarrow i\omega - \epsilon, \mathbf{k})$$

$$= -2 \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \frac{1}{\omega + i\epsilon - (E_{q+k} - E_q)} \left( \frac{1}{e^{\beta E_{q+k}} - 1} + \frac{1}{e^{\beta E_q} + 1} \right), \quad (16)$$

$$\rho_n = 2 \text{Im } D_n^R(\omega, \mathbf{k}), \quad \frac{1}{x \pm i\epsilon} = \mathcal{P} \frac{1}{x} \mp i\pi \delta(x)$$

$$= 4\pi \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \delta(\omega - (E_{q+k} - E_q)) \left( \frac{1}{e^{\beta E_{q+k}} - 1} + \frac{1}{e^{\beta E_q} + 1} \right)$$

$$= 4\pi \cdot \frac{1}{(2\pi)^3} \cdot 2\pi \int_0^\pi \sin \theta d\theta \int_0^\infty q^2 dq \delta(\omega - (E_{q+k} - E_q)) \left( \frac{1}{e^{\beta(E_q + \omega)} - 1} + \frac{1}{e^{\beta E_q} + 1} \right) \quad (17)$$

$$= \frac{1}{\pi} \int_0^\pi \sin \theta d\theta \frac{m}{k \cos \theta} \int_0^\infty q^2 dq \delta\left(q - \frac{2m\omega - k^2}{2k \cos \theta}\right) \left( \frac{1}{e^{\beta(E_q + \omega)} - 1} + \frac{1}{e^{\beta E_q} + 1} \right)$$

$$= \frac{1}{\pi} \int_{-1}^1 dx \frac{m}{kx} \left(\frac{q_0}{x}\right)^2 \left( \frac{1}{e^{\beta(E_0/x^2 - \mu + \omega)} - 1} + \frac{1}{e^{\beta(E_0/x^2 - \mu)} + 1} \right) \theta\left(\frac{q_0}{x}\right)$$

We have  $k = \|\mathbf{k}\|$ ,  $x = \cos \theta$ ,  $q_0 = \frac{2m\omega - k^2}{2k}$ ,  $E_0 = \frac{q_0^2}{2m}$ , and the Heaviside  $\theta$ -function  $\theta(\frac{q_0}{x})$  which enforces that  $q_0/x > 0$ . With  $d(\frac{1}{x^2}) = -\frac{2}{x^3} dx$ , we can rewrite the above integral into:

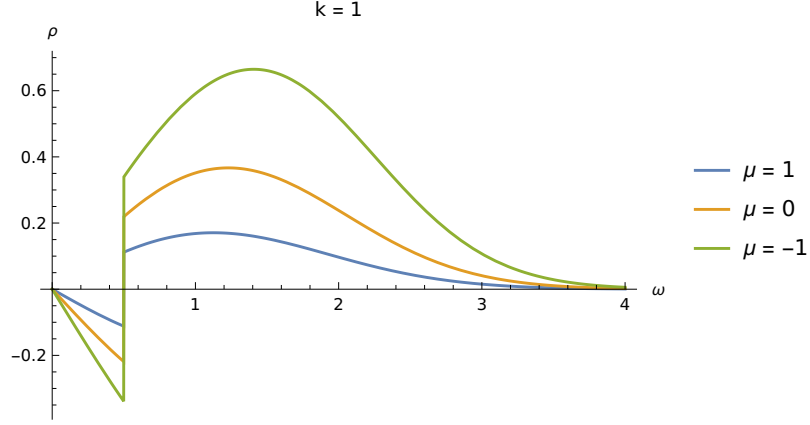
$$\rho_n = \frac{1}{\pi} (-\text{sgn } q_0) \left( -\frac{mq_0^2}{2k} \right) \int_1^\infty dy \left( \frac{1}{e^{\beta(E_0 y - \mu + \omega)} - 1} + \frac{1}{e^{\beta(E_0 y - \mu)} + 1} \right), \quad y = \frac{1}{x^2},$$

$$= (\text{sgn } q_0) \frac{mq_0^2}{2\pi k} \cdot \frac{1}{\beta E_0} \left\{ -\ln \left( 1 - e^{-\beta(E_0 - \mu + \omega)} \right) + \ln \left( 1 + e^{-\beta(E_0 - \mu)} \right) \right\} \quad (18)$$

$$= (\text{sgn } q_0) \frac{m^2 T}{k\pi} \ln \frac{1 + e^{-\beta(E_0 - \mu)}}{1 - e^{-\beta(E_0 - \mu + \omega)}}, \quad k = \|\mathbf{k}\|, \quad q_0 = \frac{2m\omega - k^2}{2k}, \quad E_0 = \frac{q_0^2}{2m}$$

We see that the result grows linearly in  $T$ , and flips sign at  $q_0 = 0$  or  $\omega = \frac{k^2}{2m}$ . Note that this is precisely the dispersion relation of  $\psi$ . We can interpret this result as particle pairs being created at such frequencies. For  $m = 1$ ,  $T = 1$ , plots of  $\rho_n$  as a function of  $\omega$  with various  $\mu, k$  is shown below.





For the **spin correlations**  $D_{ij}$ , the above analysis still holds, and we need only change the source term in (6) from  $j(x) n(x)$  to  $J^i(x) s_i(x)$ . We have:

$$Z[j] \sim \left\{ \det_{k,k',\bullet} (\beta D_0^{-1}(k) \delta_{k,k'} - \beta J_{k-k'}^i \sigma_i) \right\}^{+1}, \quad D_0^{-1}(k) = i\omega + \frac{k^2}{2m} - \mu \quad (19)$$

There are now no-trivial dependence on the spinor indices. After completing the trace of Pauli matrices, we find that the result is basically the same as before, but with an additional  $\delta_{ij}$  factor:

$$\begin{aligned} W^{(2)} &= -\frac{1}{2} \text{Tr}_{k,k',\bullet} \sum_q \left( J_{k-q}^i \sigma_i D_0(k) \right) \left( J_{q-k'}^j \sigma_j D_0(q) \right) \\ &= -\frac{1}{2} \sum_q \sum_k J_{-q}^i \left( \text{tr}(\sigma_i \sigma_j) D_0(k) D_0(q+k) \right) J_q^j \\ &= \frac{\beta V}{2} \sum_k J_{-k}^i D_{ij}(k) J_k^j, \end{aligned} \quad (20)$$

$$D_{ij}(k) = \frac{1}{2} \text{tr}(\sigma_i \sigma_j) D_n(k) = \delta_{ij} D_n(k), \quad \rho_{ij} = \delta_{ij} D_{ij}(k) \quad (21)$$

This means that the spin correlation along a same direction is identical with the density correlation, while there is no correlation between spins in different directions.