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Correlations of Pauli Spinor Fields

The Lagrangian of a nonrelativistic free particle is given by:

$$\mathcal{L} = \bar{\psi} \left(i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} \right) \psi \tag{1}$$

The Euclidean action is then given by:

$$(-S) = \int d^4x \, \mathcal{L}_{t=-i\tau} = -\int d^4x \, \bar{\psi} \left(\frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m} \right) \psi = -\beta H$$
 (2)

Note that we have $N = \int d^3 \mathbf{x} \, \bar{\psi} \psi = \int d^3 \mathbf{x} \, n$ as a conserved charge, therefore it is natural to include N in the partition function:

$$Z = \operatorname{Tr} e^{-\beta(H - \mu N)} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \int d^4x \left\{ -\bar{\psi} \left(\frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m} \right) \psi + \mu n \right\}$$
 (3)

For a spin- $\frac{1}{2}$ free fermion, ψ is given by a 2-component Pauli spinor: $\psi \sim (\psi_+, \psi_-)$, and $\bar{\psi} = \psi^{\dagger}$ is the conjugate transpose of ψ . Relevant observables in the Heisenberg picture are then given by:

$$n = \psi^{\dagger} \psi, \quad \tilde{n} = n - \langle n \rangle, \quad s_i = \psi^{\dagger} \sigma_i \psi,$$
 (4)

$$D_n = \left\langle \mathcal{T}_{\tau} \, \tilde{n}(x) \, \tilde{n}(x') \right\rangle, \quad D_{ij} = \left\langle \mathcal{T}_{\tau} \, s_i(x) \, s_j(x') \right\rangle \tag{5}$$

To compute the **density correlation** D_n , define the generating functional:

$$Z[j] = \int \mathcal{D}\bar{\psi} \,\mathcal{D}\psi \,\exp \int \mathrm{d}^4x \left\{ -\bar{\psi} \left(\frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m} \right) \psi + \mu n(x) + j(x) \, n(x) \right\} \tag{6}$$

To remove the non-zero $\langle n \rangle$, consider:

$$W[j] = \ln Z[j], \quad D_n = \left\langle \mathcal{T}_\tau \, \tilde{n}(x) \, \tilde{n}(x') \right\rangle = \left. \frac{\delta^2 W[j]}{\delta j(x) \, \delta j(x')} \right|_{i=0} = \left. \frac{\delta^2 W^{(2)}[j]}{\delta j(x) \, \delta j(x')} \right|_{i=0} \tag{7}$$

$$W^{(2)} \sim \mathcal{O}(j^2) \tag{8}$$

For free theory, $W^{(2)}$ can be computed explicitly by mode expansions:

$$\psi = \frac{1}{\sqrt{V}} \sum_{k} e^{ik \cdot x} \psi_{k}, \quad j = \sum_{q} e^{-iq \cdot x} j_{q}, \quad \sum_{k} e^{ik \cdot x} = V \int \frac{\mathrm{d}^{3} \mathbf{k}}{(2\pi)^{3}} e^{i\mathbf{k} \cdot \mathbf{x}} \sum_{n \in \mathbb{Z}} e^{i\omega_{n}\tau}$$

$$Z[j] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \int \mathrm{d}^{4}x \left\{ -\bar{\psi} \left(\frac{\partial}{\partial \tau} - \frac{\nabla^{2}}{2m} - \mu - j(x) \right) \psi \right\}$$

$$= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left\{ -\beta \sum_{k,k'} \bar{\psi}_{k} \left(\left(i\omega + \frac{\mathbf{k}^{2}}{2m} - \mu \right) \delta_{k,k'} - j_{k-k'} \right) \psi_{k'} \right\}$$

$$\sim \left\{ \det_{k,k',\bullet} \left(\beta D_{0}^{-1}(k) \delta_{k,k'} - \beta j_{k-k'} \right) \right\}^{+1}, \quad D_{0}^{-1}(k) = i\omega + \frac{\mathbf{k}^{2}}{2m} - \mu,$$

$$= \left\{ \det_{k,k'} \left(\beta D_{0}^{-1}(k) \delta_{k,k'} - \beta j_{k-k'} \right) \right\}^{+2}$$

$$(10)$$

Note that the determinant should go over the implicit spinor indices as well, which results in a power of 2 in the final expression.

We can then compute W[j] using Jacobi's formula:

$$W[j] = \ln Z[j] = 2 \ln \det_{k,k'} \left(\beta D_0^{-1}(k) \, \delta_{k,k'} - \beta j_{k-k'} \right)$$

$$= 2 \operatorname{Tr}_{k,k'} \ln \left(\beta D_0^{-1}(k) \, \delta_{k,k'} - \beta j_{k-k'} \right)$$

$$= 2 \operatorname{Tr}_{k,k'} \left\{ \delta_{k,k'} \ln \left(\beta D_0^{-1}(k) \right) + \ln \left(\mathbb{1}_{k,k'} - j_{k-k'} D_0(k) \right) \right\}$$
(11)

The first term is the vacuum contribution with no dependence of j, hence it is irrelevant in $W^{(2)} \sim \mathcal{O}(j^2)$. Expansion of the matrix log in the second term reveals that:

$$\ln\left(\mathbb{1}_{k,k'} - j_{k-k'}D_0(k)\right) = -\sum_{n=1}^{\infty} \frac{1}{n} \left(j_{k-k'}D_0(k)\right)_{k,k'}^n$$

$$W^{(2)} = 2 \operatorname{Tr}_{k,k'} \frac{-1}{2} \left(j_{k-k'}D_0(k)\right)_{k,k'}^2$$

$$= -\operatorname{Tr}_{k,k'} \sum_{q} \left(j_{k-q}D_0(k)\right) \left(j_{q-k'}D_0(q)\right)$$

$$= -\sum_{k,q} j_{k-q}D_0(k) j_{q-k}D_0(q)$$

$$= -\sum_{k} \sum_{q} j_{k-q}D_0(k) j_{q-k}D_0(q-k+k)$$

$$(13)$$

$$= -\sum_{q}^{k} \sum_{k}^{q-k} j_{-q} \Big(D_0(k) D_0(q+k) \Big) j_q$$
$$= \frac{\beta V}{2} \sum_{k}^{q-k} j_{-k} D_n(k) j_k,$$

$$D_{n}(k) = -\frac{2}{\beta V} \sum_{q} D_{0}(q) D_{0}(k+q)$$

$$= -\frac{2}{\beta V} \sum_{q} \frac{1}{i\omega_{q} + E_{q}} \frac{1}{i(\omega_{q} + \omega_{k}) + E_{q+k}}, \quad E_{q} = \frac{\mathbf{q}^{2}}{2m} - \mu,$$

$$= -\frac{2}{\beta V} \sum_{q} \left(\frac{1}{i\omega_{q} + E_{q}} - \frac{1}{i\omega_{q} + i\omega_{k} + E_{q+k}} \right) \frac{1}{i\omega_{k} + E_{q+k} - E_{k}}$$

$$= -2 \int \frac{d^{3}\mathbf{q}}{(2\pi)^{3}} \frac{1}{i\omega_{k} + E_{q+k} - E_{q}} T \sum_{\omega_{q}} \left(\frac{1}{i\omega_{q} + E_{q}} - \frac{1}{i\omega_{q} + i\omega_{k} + E_{q+k}} \right)$$

$$= -2 \int \frac{d^{3}\mathbf{q}}{(2\pi)^{3}} \frac{1}{i\omega_{k} + E_{q+k} - E_{q}} \left((1 - n(E_{q})) - (1 - n(E_{q+k} + i\omega_{k})) \right)$$

$$= -2 \int \frac{d^{3}\mathbf{q}}{(2\pi)^{3}} \frac{1}{i\omega_{k} + E_{q+k} - E_{q}} \left(\frac{1}{-e^{\beta E_{q+k}} + 1} - \frac{1}{e^{\beta E_{q}} + 1} \right)$$
(14)

Here we've completed the Matsubara sum of the fermionic frequencies, using:

$$T\sum_{\omega_q} \frac{1}{i\omega_q + E_q} = 1 - n(E_q), \quad n(E_q) = \frac{1}{e^{\beta E_q} + 1}, \quad n(E_q + i\omega_k) = \frac{1}{-e^{\beta E_q} + 1}$$
(15)

The retarded propagator and the spectral density is obtained by analytic continuation:

$$D_n^R(\omega, \mathbf{k}) = D_n(\omega \to i\omega - \epsilon, \mathbf{k})$$

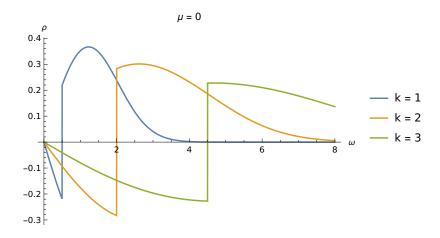
$$= -2 \int \frac{\mathrm{d}^3 \mathbf{q}}{(2\pi)^3} \frac{1}{\omega + i\epsilon - (E_{q+k} - E_q)} \left(\frac{1}{e^{\beta E_{q+k}} - 1} + \frac{1}{e^{\beta E_q} + 1} \right), \tag{16}$$

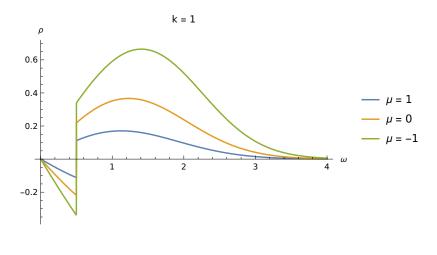
$$\rho_{n} = 2 \operatorname{Im} D_{n}^{R}(\omega, \mathbf{k}), \quad \frac{1}{x \pm i\epsilon} = \mathcal{P} \frac{1}{x} \mp i\pi \delta(x)
= 4\pi \int \frac{\mathrm{d}^{3}\mathbf{q}}{(2\pi)^{3}} \, \delta \Big(\omega - (E_{q+k} - E_{q}) \Big) \Big(\frac{1}{e^{\beta E_{q+k}} - 1} + \frac{1}{e^{\beta E_{q}} + 1} \Big)
= 4\pi \cdot \frac{1}{(2\pi)^{3}} \cdot 2\pi \int_{0}^{\pi} \sin \theta \, \mathrm{d}\theta \int_{0}^{\infty} q^{2} \, \mathrm{d}q \, \delta \Big(\omega - (E_{q+k} - E_{q}) \Big) \Big(\frac{1}{e^{\beta (E_{q} + \omega)} - 1} + \frac{1}{e^{\beta E_{q}} + 1} \Big)
= \frac{1}{\pi} \int_{0}^{\pi} \sin \theta \, \mathrm{d}\theta \, \frac{m}{k \cos \theta} \int_{0}^{\infty} q^{2} \, \mathrm{d}q \, \delta \Big(q - \frac{2m\omega - k^{2}}{2k \cos \theta} \Big) \Big(\frac{1}{e^{\beta (E_{q} + \omega)} - 1} + \frac{1}{e^{\beta E_{q}} + 1} \Big)
= \frac{1}{\pi} \int_{-1}^{1} \mathrm{d}x \, \frac{m}{kx} \left(\frac{q_{0}}{x} \right)^{2} \Big(\frac{1}{e^{\beta (E_{0}/x^{2} - \mu + \omega)} - 1} + \frac{1}{e^{\beta (E_{0}/x^{2} - \mu)} + 1} \Big) \theta \Big(\frac{q_{0}}{x} \Big)$$

We have $k = \|\mathbf{k}\|$, $x = \cos \theta$, $q_0 = \frac{2m\omega - k^2}{2k}$, $E_0 = \frac{q_0^2}{2m}$, and the Heaviside θ -function $\theta\left(\frac{q_0}{x}\right)$ which enforces that $q_0/x > 0$. With $d\left(\frac{1}{x^2}\right) = -\frac{2}{x^3} dx$, we can rewrite the above integral into:

$$\rho_{n} = \frac{1}{\pi} \left(-\operatorname{sgn} q_{0} \right) \left(-\frac{mq_{0}^{2}}{2k} \right) \int_{1}^{\infty} dy \left(\frac{1}{e^{\beta(E_{0}y - \mu + \omega)} - 1} + \frac{1}{e^{\beta(E_{0}y - \mu)} + 1} \right), \quad y = \frac{1}{x^{2}},
= \left(\operatorname{sgn} q_{0} \right) \frac{mq_{0}^{2}}{2\pi k} \cdot \frac{1}{\beta E_{0}} \left\{ -\ln\left(1 - e^{-\beta(E_{0} - \mu + \omega)} \right) + \ln\left(1 + e^{-\beta(E_{0} - \mu)} \right) \right\}
= \left(\operatorname{sgn} q_{0} \right) \frac{m^{2}T}{k\pi} \ln \frac{1 + e^{-\beta(E_{0} - \mu)}}{1 - e^{-\beta(E_{0} - \mu + \omega)}}, \quad k = \|\mathbf{k}\|, \quad q_{0} = \frac{2m\omega - k^{2}}{2k}, \quad E_{0} = \frac{q_{0}^{2}}{2m}$$
(18)

We see that the result grows linearly in T, and flips sign at $q_0 = 0$ or $\omega = \frac{k^2}{2m}$. Note that this is precisely the dispersion relation of ψ . We can interpret this result as particle pairs being created at such frequencies. For m = 1, T = 1, plots of ρ_n as a function of ω with various μ, k is shown below.





For the **spin correlations** D_{ij} , the above analysis still holds, and we need only change the source term in (6) from j(x) n(x) to $J^{i}(x) s_{i}(x)$. We have:

$$Z[j] \sim \left\{ \det_{k,k',\bullet} \left(\beta D_0^{-1}(k) \, \delta_{k,k'} - \beta J_{k-k'}^i \sigma_i \right) \right\}^{+1}, \quad D_0^{-1}(k) = i\omega + \frac{\mathbf{k}^2}{2m} - \mu$$
 (19)

There are now no-trivial dependence on the spinor indices. After completing the trace of Pauli matrices, we find that the result is basically the same as before, but with an additional δ_{ij} factor:

$$W^{(2)} = -\frac{1}{2} \prod_{k,k',\bullet} \sum_{q} \left(J_{k-q}^{i} \sigma_{i} D_{0}(k) \right) \left(J_{q-k'}^{j} \sigma_{j} D_{0}(q) \right)$$

$$= -\frac{1}{2} \sum_{q} \sum_{k} J_{-q}^{i} \left(\operatorname{tr}(\sigma_{i} \sigma_{j}) D_{0}(k) D_{0}(q+k) \right) J_{q}^{j}$$

$$= \frac{\beta V}{2} \sum_{k} J_{-k}^{i} D_{ij}(k) J_{k}^{j},$$

$$D_{ij}(k) = \frac{1}{2} \operatorname{tr}(\sigma_{i} \sigma_{j}) D_{n}(k) = \delta_{ij} D_{n}(k), \quad \rho_{ij} = \delta_{ij} D_{ij}(k)$$
(21)

This means that the spin correlation along a same direction is identical with the density correlation, while there is no correlation between spins in different directions.