

## 1 Symmetry & Noether's Theorem

### 1.1 2D $\sigma$ -Model

$$\mathcal{L} = -\frac{1}{2} \eta_{\alpha\beta} \eta_{\mu\nu} \partial^\alpha X^\mu \partial^\beta X^\nu = -\frac{1}{2} \partial^\alpha X_\mu \partial_\alpha X^\mu, \quad X^\mu \in \mathbb{R}^{1,D-1} \quad (1)$$

- For  $\delta X^\mu = a^\mu + \lambda^\mu{}_\nu X^\nu$ , the Lagrangian (density) transforms as follows:

$$\begin{aligned} \delta \mathcal{L} &= -\partial^\alpha X_\mu \partial_\alpha \delta X^\mu \\ &= -\partial^\alpha X_\mu \partial_\alpha (a^\mu + \lambda^\mu{}_\nu X^\nu) \\ &= -\partial^\alpha X_\mu (\partial_\alpha a^\mu + X^\nu \partial_\alpha \lambda^\mu{}_\nu + \lambda^\mu{}_\nu \partial_\alpha X^\nu) \\ &= -\partial^\alpha X_\mu \partial_\alpha a^\mu - \partial^\alpha X^\mu \partial_\alpha X^\nu \lambda_{\mu\nu} - X^\nu \partial^\alpha X^\mu \partial_\alpha \lambda_{\mu\nu} \\ &= -\partial^\alpha X_\mu \partial_\alpha a^\mu - \partial^\alpha X^\mu \partial_\alpha X^\nu \lambda_{(\mu\nu)} - X^\nu \partial^\alpha X^\mu \partial_\alpha \lambda_{\mu\nu} \end{aligned} \quad (2)$$

Since  $a^\mu$  and  $\lambda^\mu{}_\nu$  are independent, imposing  $\delta L = 0$  yields  $\partial_\alpha a^\mu = 0$ ,  $a = \text{const.}$  Furthermore, if  $\delta L = 0$  is to hold for arbitrary  $X^\mu$  fields, then  $\partial_\alpha \lambda_{\mu\nu} = 0$ ,  $\lambda_{(\mu\nu)} = 0$ , i.e.  $\lambda_{\mu\nu}$  is constant and anti-symmetric over its indices.

- Promote  $\delta X \mapsto \epsilon(x) \delta X = \epsilon(x) (a^\mu + \lambda^\mu{}_\nu X^\nu)$ , with  $\epsilon(x)$  some localized bump function; using (2) and considering *on-shell* variation, we have:

$$\begin{aligned} 0 = \delta S &= - \int d^2x (\partial^\alpha X_\mu a^\mu \partial_\alpha \epsilon + X^\nu \partial^\alpha X^\mu \lambda_{\mu\nu} \partial_\alpha \epsilon) \\ &= - \int d^2x \left( \partial^\alpha X_\mu a^\mu + X_{[\nu} \partial^\alpha X_{\mu]} \lambda^{[\mu\nu]} \right) \partial_\alpha \epsilon \end{aligned} \quad (3)$$

It is then evident (after partial integration) that the following currents are conserved; they are the Noether currents associated with  $a^\mu$  and  $\lambda^{[\mu\nu]}$ :

$$j_\mu^\alpha = -\partial^\alpha X_\mu, \quad j_{\mu\nu}^\alpha = -X_{[\nu} \partial^\alpha X_{\mu]} = \frac{1}{2} (X_\mu \partial^\alpha X_\nu - X_\nu \partial^\alpha X_\mu) \quad (4)$$

Conserved charge  $Q = \int d^2x j^0(x)$ , we have:

$$P_\mu = - \int dx^1 \partial^0 X_\mu = \int dx^1 \partial_0 X_\mu, \quad M_{\mu\nu} = \frac{1}{2} \int dx^1 (X_\nu \partial_0 X_\mu - X_\mu \partial_0 X_\nu) \quad (5)$$

They can be interpreted as spacetime momentum and spacetime angular momentum. ■

### 1.2 Real Scalar in $(3+1)$ D

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 \quad (6)$$

- For  $\phi$ : scalar, under  $x' = \lambda \circ x$ ,  $\phi(x) \mapsto \phi'(x)$ , while:

$$\phi'(x') = \phi(x) \implies \phi'(x) = \phi(\lambda^{-1} \circ x) \quad (7)$$

For  $\lambda \sim \lambda^\mu{}_\nu$ : Lorentz transformation,  $\eta_{\mu\nu} \lambda^\mu{}_\rho \lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ , or equivalently,  $(\lambda^{-1})^\mu{}_\nu = \lambda_\nu{}^\mu$ . Therefore,

$$\phi'(x^\mu) = \phi(\lambda^{-1} \circ x^\mu) = \phi(x^\nu \lambda_\nu{}^\mu) \quad (8)$$

- Under  $x'^\mu = \lambda^\mu{}_\nu x^\nu$ , we have:

$$\begin{aligned} \mathcal{L}'(x') &= -\frac{1}{2} \partial'^\mu \phi'(x') \partial'_\mu \phi'(x') - \frac{1}{2} m^2 \phi'^2(x') \\ &= -\frac{1}{2} \partial'^\mu \phi(x) \partial'_\mu \phi(x) - \frac{1}{2} m^2 \phi^2(x) \\ &= -\frac{1}{2} \eta^{\mu\nu} \frac{\partial x^\rho}{\partial x'^\mu} \partial_\rho \phi(x) \frac{\partial x^\sigma}{\partial x'^\nu} \partial_\sigma \phi(x) - \frac{1}{2} m^2 \phi^2(x) \\ &= -\frac{1}{2} \eta^{\rho\sigma} \partial_\rho \phi(x) \partial_\sigma \phi(x) - \frac{1}{2} m^2 \phi^2(x) \\ &= \mathcal{L}(x) \end{aligned} \quad (9)$$

Here we've used  $\eta^{\mu\nu} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} = \eta^{\mu\nu} \lambda_\mu{}^\rho \lambda_\nu{}^\sigma = \eta^{\rho\sigma}$ . Furthermore,  $S' = \int d^4x' \mathcal{L}'(x') = \int d^4x \mathcal{L}(x) = S$ , hence the action is invariant under Lorentz transformation.

- Consider an infinitesimal Lorentz transformation:  $\lambda \sim \mathbb{1} + \omega$ , then  $\eta_{\mu\nu} \lambda^\mu{}_\rho \lambda^\nu{}_\sigma = \eta_{\rho\sigma}$  implies that  $\omega_{\mu\nu}$  is anti-symmetric:  $\omega_{\mu\nu} + \omega_{\nu\mu} = 0$ . For  $\delta x^\mu = \omega^\mu{}_\nu x^\nu$ , we have:

$$\delta\phi = -\frac{\partial\phi}{\partial x^\mu} \delta x^\mu = -\omega^\mu{}_\nu x^\nu \partial_\mu \phi \quad (10)$$

To obtain the corresponding Noether charges, we can simply repeat the operations done in our previous problem; alternatively, we can try to derive a general recipe<sup>1</sup>: for  $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$  and  $S = \int d^4x \mathcal{L}$ , we have:

$$\begin{aligned} \delta S &= \int d^4x \delta \mathcal{L} \\ &= \int d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\partial_\mu \phi \right) \\ &= \int d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \int d^4x \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) \end{aligned} \quad (11)$$

If we vary  $S$  w.r.t. a symmetry of the system, we will have  $\delta\mathcal{L} = \partial_\mu K^\mu$  some total derivative; when on-shell, such variation gives the conserved current with boundary term  $K^\mu$ :

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - K^\mu \quad (12)$$

Back to our Lorentz transformation  $\delta\phi = -\omega^\mu{}_\nu x^\nu \partial_\mu \phi$ , we have symmetry variation:

$$\delta\mathcal{L} = -\omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = -\partial_\mu (\omega^\mu{}_\nu x^\nu \mathcal{L}) \quad (13)$$

We can write this down without explicit calculations, since we know  $\mathcal{L}$  itself is a Lorentz scalar, and that's how scalar transforms under Lorentz transformations.

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<sup>1</sup>References: [arXiv:1601.03616](#) and Tong: <http://damtp.cam.ac.uk/user/tong/qft.html>

This gives a boundary term  $K^\mu = -\omega^\mu{}_\nu x^\nu \mathcal{L}$ , and the Noether current and its corresponding conserved charge can be calculated as follows:

$$j^\mu = -\omega^\sigma{}_\nu x^\nu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\sigma \phi - \delta^\mu_\sigma \mathcal{L} \right), \quad (14)$$

$$Q = \int d^3x j^0 = -\omega^\sigma{}_\nu \int d^3x x^\nu (\partial_0 \phi \partial_\sigma \phi - \delta^\sigma_0 \mathcal{L}), \quad (15)$$

Note that  $\omega^\mu{}_\nu$  is arbitrary, therefore  $Q$  can be decomposed into independent charges:

$$Q = \frac{1}{2} \omega_{\mu\nu} M^{\mu\nu}, \quad M^{\mu\nu} = - \int d^3x 2x^{[\mu} (\partial_0 \phi \partial^{\nu]} \phi - \eta^{\nu]0} \mathcal{L}), \quad (16)$$

The indices of  $M^{\mu\nu}$  are anti-symmetrized to match the degrees of freedom in  $\omega_{\mu\nu}$ . Note that the  $\mathcal{L}$  term only appears when one of the indices is 0.

Note that the canonical momentum:

$$\varpi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} = \partial_0 \phi \quad (17)$$

It is thus natural to re-organize  $M^{0i}$  in the following way:

$$\begin{aligned} M^{0i} &= -M^{i0} = - \int d^3x (\varpi (x^0 \partial^i - x^i \partial^0) \phi - x^i \mathcal{L}) \\ &= - \int d^3x (x^0 \varpi \partial^i \phi + x^i (\varpi \dot{\phi} - \mathcal{L})) \\ &= - \int d^3x (x^0 \varpi \partial^i \phi + x^i \mathcal{H}) \end{aligned} \quad (18)$$

We've obtained an interesting result: the expression for the boost generator  $M^{i0}$  contains the Hamiltonian density  $\mathcal{H}$ , weighted by the radial distance  $x^i$ . This is natural since a boost does indeed contains time evolution for excitations away from the origin. It's an important result utilized by the so-called *Rindler decomposition*; in fact,  $M^{i0}$  becomes the Hamiltonian for an accelerated observer in the Rindler patch<sup>2</sup>.

For  $M^{ij}$ , we have:

$$M^{ij} = - \int d^3x \dot{\phi} (x^i \partial^j - x^j \partial^i) \phi \quad (19)$$

This is interpreted as the angular momentum of the field  $\phi$ . Suppose  $\phi$  is a wave packet localized around  $\mathbf{x}$  with momentum  $\approx \mathbf{k}$ , then we have the classical angular momentum up to some factor:

$$M^{ij} \sim (x^i k^j - x^j k^i) \int d^3x E \phi^2, \quad E = \sqrt{\mathbf{k}^2 + m^2} \quad (20)$$

The  $\int d^3x E \phi^2$  factor in the above expression is an  $\mathcal{O}(1)$  normalization constant for a particle-like wave packet; to see this, note that  $\phi \in \mathbb{R}$  has a phase factor  $\phi \sim a e^{+ik \cdot x} + a^\dagger e^{-ik \cdot x} \sim \cos(k \cdot x)$ ,

$$E = \int d^3x \mathcal{H} = \int d^3x \left( \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 + \dots \right) \sim (E^2 + \mathbf{k}^2 + m^2 + \dots) \int d^3x \frac{1}{2} \phi^2, \quad (21)$$

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<sup>2</sup>See the lecture notes of Tom Hartman: [hartmanhep.net/topics2015/gravity-lectures.pdf](http://hartmanhep.net/topics2015/gravity-lectures.pdf) or Daniel Harlow [arXiv:1409.1231](https://arxiv.org/abs/1409.1231).

$$E \int d^3x \phi^2 \sim 1, \quad (22)$$

Indeed, we have:  $M^{ij} \sim (x^i k^j - x^j k^i)$ .

Canonical quantization:

$$[\phi(\mathbf{x}), \varpi(\mathbf{y})] = [\phi(\mathbf{x}), \dot{\phi}(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}) \quad (23)$$

Other equal-time commutators between  $\phi, \varpi$  all just vanish. Operator products are then regularized by normal ordering:  $M \mapsto :M:$ , which can be explicitly implemented by normal ordering of the oscillator modes:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} (a_k e^{ik \cdot x} + a_k^\dagger e^{-ik \cdot x}), \quad [a_k, a_{k'}^\dagger] = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \quad (24)$$

The  $k$  dependence in  $a_k, E_k$  is, in fact, only a  $\mathbf{k}$  dependence; we've dropped the boldface in the subscripts simply for convenience.

For example, the first term in  $M^{0i} = -M^{i0}$  can be expanded as:

$$-x^0 \int d^3x \varpi \partial^i \phi = x^0 \int \frac{d^3k}{(2\pi)^3} k^i a_k^\dagger a_k = x^0 P^i \quad (25)$$

Here  $P^\mu$  is the momentum operator on the Hilbert space, promoted from the classical  $-i\partial^\mu$ .  $M^{0i}$  is thus further reduced to:

$$M^{0i} = -M^{i0} = x^0 P^i - \int d^3x x^i \mathcal{H} \quad (26)$$

We note that this result is almost the classical  $x^0 P^i - x^i P^0$ , but here  $x^i P^0$  is replaced with the integral over energy density  $\mathcal{H}$ . The result can be nicely re-written with the stress tensor  $T^{\mu\nu}$ ; just run the Noether's procedure with  $\delta x^\mu = \epsilon^\mu$ , then we shall obtain:

$$\begin{aligned} j'^\mu &= \epsilon_\nu T^{\mu\nu}, \quad T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi + \eta^{\mu\nu} \mathcal{L}, \\ Q' &= \epsilon_\mu P^\mu, \quad P^\mu = \int d^3x T^{\mu 0} = \int d^3x (\partial^0 \phi \partial^\mu \phi + \eta^{0\mu} \mathcal{L}) \\ M^{\mu\nu} &= \int d^3x 2x^{[\mu} T^{\nu]0} \end{aligned} \quad (27)$$

The quantization  $M \mapsto :M:$  is thus reduced to the quantization of  $T^{\nu 0}$ , weighted by a  $x^\mu$  factor.

First let's look at  $T^{00} = \mathcal{H}$ ; note that:

$$\int d^3x x^i e^{i\mathbf{k} \cdot \mathbf{x}} = (2\pi)^3 \left( -i \frac{\partial}{\partial k_i} \right) \delta(\mathbf{k}) \quad (28)$$

One can then check explicitly with mode expansion that, up to normal ordering, we have<sup>3</sup>:

$$H = \int d^3x \mathcal{H} = \int \frac{d^3k}{(2\pi)^3} E_k a_k^\dagger a_k, \quad (29)$$

$$\mathcal{O}^i = \int d^3x x^i \mathcal{H} = \int \frac{d^3k}{(2\pi)^3} E_k a_k^\dagger \left( +i \frac{\partial}{\partial k_i} \right) a_k, \quad (30)$$

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<sup>3</sup>See a similar result in <https://physics.stackexchange.com/q/27906>. Moreover, the charge can be computed at arbitrary time slice  $t$ , but the  $t$ -dependence ( $\sim e^{\pm iEt}$ ) drops out in the final result, due to the on-shell condition  $E_k^2 = \mathbf{k}^2 + m^2$  and symmetries, e.g.  $\int d^3k k^i = 0$ . Note that  $\frac{\partial E_k}{\partial k_i} = \frac{k^i}{E_k}$ .

At first glance, derivative of  $a_k = a_{\mathbf{k}}$  with respect to  $k^i$  seems puzzling; however, note that  $(+i \frac{\partial}{\partial k^i})$  is precisely the  $x^i$  operator in “momentum-space”, and one can make sense of it by considering a generic  $n$ -particle state:

$$|\Psi\rangle = \int d^3k_1 \cdots d^3k_n \Psi(\mathbf{k}_1, \dots, \mathbf{k}_n) a_{k_1}^\dagger \cdots a_{k_n}^\dagger |0\rangle \quad (31)$$

Where  $\Psi(\mathbf{k}_1, \dots, \mathbf{k}_n)$  is the  $n$ -particle wave function; using  $a_k a_{k'}^\dagger = a_{k'}^\dagger a_k + (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}')$  recursively, we get in the end that:

$$\mathcal{O}_i |\Psi\rangle = \int d^3k_1 \cdots d^3k_n \left\{ \sum_{m=1}^n E_{k_m} \left( +i \frac{\partial}{\partial k_m^i} \right) \Psi(\mathbf{k}_1, \dots, \mathbf{k}_n) \right\} a_{k_1}^\dagger \cdots a_{k_n}^\dagger |0\rangle \quad (32)$$

We see that indeed  $\mathcal{O}_i$  acts as  $E_k(+i \frac{\partial}{\partial k^i})$  on the momentum-space  $n$ -particle wave function, consistent with the result of ordinary quantum mechanics; ...

**TODO: Detailed analysis! HINT: Ward identity!**

Notice that  $x^{[\mu} \partial^{\nu]} = \frac{1}{2} (x^\mu \partial^\nu - x^\nu \partial^\mu) = \frac{1}{2} D^{\mu\nu}$  is the Killing vector fields of  $\mathbb{R}^{3,1}$ , hence they naturally follow the commutation relations of  $\mathfrak{so}(3,1)$  (up to a constant coefficient, or an isomorphism)<sup>4</sup>. We have:

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= \int d^3x \int d^3y \left[ \dot{\phi} D^{\mu\nu} \phi(x), \dot{\phi} D^{\rho\sigma} \phi(y) \right] \\ &= \int d^3x \dot{\phi} [D^{\mu\nu}, D^{\rho\sigma}] \phi \end{aligned} \quad (33)$$

Similar holds for  $M^{i0}$ . Therefore,  $M^{\mu\nu}$ 's indeed form the Lie algebra  $\mathfrak{so}(3,1)$ . ■

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<sup>4</sup>I would like to thank 林般 for pointing this out.