

1 Symmetry & Noether's Theorem

1.1 2D σ -Model

$$\mathcal{L} = -\frac{1}{2} \eta_{\alpha\beta} \eta_{\mu\nu} \partial^\alpha X^\mu \partial^\beta X^\nu = -\frac{1}{2} \partial^\alpha X_\mu \partial_\alpha X^\mu, \quad X^\mu \in \mathbb{R}^{1,D-1} \quad (1)$$

- For $\delta X^\mu = a^\mu + \lambda^\mu{}_\nu X^\nu$, the Lagrangian (density) transforms as follows:

$$\begin{aligned} \delta \mathcal{L} &= -\partial^\alpha X_\mu \partial_\alpha \delta X^\mu \\ &= -\partial^\alpha X_\mu \partial_\alpha (a^\mu + \lambda^\mu{}_\nu X^\nu) \\ &= -\partial^\alpha X_\mu (\partial_\alpha a^\mu + X^\nu \partial_\alpha \lambda^\mu{}_\nu + \lambda^\mu{}_\nu \partial_\alpha X^\nu) \\ &= -\partial^\alpha X_\mu \partial_\alpha a^\mu - \partial^\alpha X^\mu \partial_\alpha X^\nu \lambda_{\mu\nu} - X^\nu \partial^\alpha X^\mu \partial_\alpha \lambda_{\mu\nu} \\ &= -\partial^\alpha X_\mu \partial_\alpha a^\mu - \partial^\alpha X^\mu \partial_\alpha X^\nu \lambda_{(\mu\nu)} - X^\nu \partial^\alpha X^\mu \partial_\alpha \lambda_{\mu\nu} \end{aligned} \quad (2)$$

Since a^μ and $\lambda^\mu{}_\nu$ are independent, imposing $\delta L = 0$ yields $\partial_\alpha a^\mu = 0$, $a = \text{const.}$ Furthermore, if $\delta L = 0$ is to hold for arbitrary X^μ fields, then $\partial_\alpha \lambda_{\mu\nu} = 0$, $\lambda_{(\mu\nu)} = 0$, i.e. $\lambda_{\mu\nu}$ is constant and anti-symmetric over its indices.

- Promote $\delta X \mapsto \epsilon(x) \delta X = \epsilon(x) (a^\mu + \lambda^\mu{}_\nu X^\nu)$, with $\epsilon(x)$ some localized bump function; using (2) and considering *on-shell* variation, we have:

$$\begin{aligned} 0 = \delta S &= - \int d^2 x (\partial^\alpha X_\mu a^\mu \partial_\alpha \epsilon + X^\nu \partial^\alpha X^\mu \lambda_{\mu\nu} \partial_\alpha \epsilon) \\ &= - \int d^2 x \left(\partial^\alpha X_\mu a^\mu + X_{[\nu} \partial^\alpha X_{\mu]} \lambda^{[\mu\nu]} \right) \partial_\alpha \epsilon \end{aligned} \quad (3)$$

It is evident (after partial integration) that the following currents are conserved; they are the Noether currents associated with a^μ and $\lambda^{[\mu\nu]}$:

$$j_\mu^\alpha = -\partial^\alpha X_\mu, \quad j_{\mu\nu}^\alpha = -X_{[\nu} \partial^\alpha X_{\mu]} = \frac{1}{2} (X_\mu \partial^\alpha X_\nu - X_\nu \partial^\alpha X_\mu) \quad (4)$$

Conserved charge $Q = \int d^2 x j^0(x)$, we have:

$$P_\mu = - \int dx^1 \partial^0 X_\mu = \int dx^1 \partial_0 X_\mu, \quad M_{\mu\nu} = \frac{1}{2} \int dx^1 (X_\nu \partial_0 X_\mu - X_\mu \partial_0 X_\nu) \quad (5)$$

They can be interpreted as spacetime momentum and spacetime angular momentum. ■

1.2 Real Scalar in $(3+1)$ D

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 \quad (6)$$

- For ϕ : scalar, under $x' = \lambda \circ x$, $\phi(x) \mapsto \phi'(x)$, while:

$$\phi'(x') = \phi(x) \implies \phi'(x) = \phi(\lambda^{-1} \circ x) \quad (7)$$

For $\lambda \sim \lambda^\mu_\nu$: Lorentz transformation, $\eta_{\mu\nu} \lambda^\mu_\rho \lambda^\nu_\sigma = \eta_{\rho\sigma}$, or equivalently, $(\lambda^{-1})^\mu_\nu = \lambda^\mu_\nu$. Therefore,

$$\phi'(x^\mu) = \phi(\lambda^{-1} \circ x^\mu) = \phi(x^\nu \lambda_\nu^\mu) \quad (8)$$

- Under $x'^\mu = \lambda^\mu_\nu x^\nu$, we have:

$$\begin{aligned} \mathcal{L}'(x') &= -\frac{1}{2} \partial'^\mu \phi'(x') \partial'_\mu \phi'(x') - \frac{1}{2} m^2 \phi'^2(x') \\ &= -\frac{1}{2} \partial'^\mu \phi(x) \partial'_\mu \phi(x) - \frac{1}{2} m^2 \phi^2(x) \\ &= -\frac{1}{2} \eta^{\mu\nu} \frac{\partial x^\rho}{\partial x'^\mu} \partial_\rho \phi(x) \frac{\partial x^\sigma}{\partial x'^\nu} \partial_\sigma \phi(x) - \frac{1}{2} m^2 \phi^2(x) \\ &= -\frac{1}{2} \eta^{\rho\sigma} \partial_\rho \phi(x) \partial_\sigma \phi(x) - \frac{1}{2} m^2 \phi^2(x) \\ &= \mathcal{L}(x) \end{aligned} \quad (9)$$

Here we've used $\eta^{\mu\nu} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} = \eta^{\mu\nu} \lambda_\mu^\rho \lambda_\nu^\sigma = \eta^{\rho\sigma}$. Furthermore, $S' = \int d^4x \mathcal{L}'(x) = \int d^4x' \mathcal{L}'(x') = \int d^4x' \mathcal{L}(x) = \int d^4x \mathcal{L}(x) = S$, hence the action is invariant under Lorentz transformation.

- Consider an infinitesimal Lorentz transformation: $\lambda \sim \mathbb{1} + \omega$, then $\eta_{\mu\nu} \lambda^\mu_\rho \lambda^\nu_\sigma = \eta_{\rho\sigma}$ implies that $\omega_{\mu\nu}$ is anti-symmetric: $\omega_{\mu\nu} + \omega_{\nu\mu} = 0$. For $\delta x^\mu = \omega^\mu_\nu x^\nu$, we have:

$$\delta\phi = -\frac{\partial\phi}{\partial x^\mu} \delta x^\mu = -\omega^\mu_\nu x^\nu \partial_\mu \phi \quad (10)$$

To obtain the corresponding Noether charges, we can simply repeat the operations done in our previous problem; alternatively, we can try to derive a general recipe¹: for $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ and $S = \int d^4x \mathcal{L}$, we have:

$$\begin{aligned} \delta S &= \int d^4x \delta \mathcal{L} \\ &= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\partial_\mu \phi \right) \\ &= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \int d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) \end{aligned} \quad (11)$$

If we vary S w.r.t. a symmetry of the system, we will have $\delta\mathcal{L} = \partial_\mu K^\mu$ some total derivative; when on-shell, such variation gives the conserved current with boundary term K^μ :

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - K^\mu \quad (12)$$

¹References: [arXiv:1601.03616](https://arxiv.org/abs/1601.03616) and Tong: <http://damtp.cam.ac.uk/user/tong/qft.html>

Back to our Lorentz transformation $\delta\phi = -\omega^\mu{}_\nu x^\nu \partial_\mu \phi$, we have symmetry variation:

$$\delta\mathcal{L} = -\omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = -\partial_\mu (\omega^\mu{}_\nu x^\nu \mathcal{L}) \quad (13)$$

This gives a boundary term $K^\mu = -\omega^\mu{}_\nu x^\nu \mathcal{L}$, and the Noether current and its corresponding conserved charge can be calculated as follows:

$$j^\mu = -\omega^\sigma{}_\nu x^\nu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\sigma \phi - \delta^\mu_\sigma \mathcal{L} \right), \quad (14)$$

$$Q = \int d^3x j^0 = -\omega^\sigma{}_\nu \int d^3x x^\nu (\partial_0 \phi \partial_\sigma \phi - \delta^\sigma_0 \mathcal{L}), \quad (15)$$

Note that $\omega^\mu{}_\nu$ is arbitrary, therefore Q can be decomposed into independent charges:

$$Q = \omega_{\mu\nu} M^{\mu\nu}, \quad M^{\mu\nu} = - \int d^3x x^{[\mu} (\partial_0 \phi \partial^{\nu]} \phi - \eta^{\nu]0} \mathcal{L}), \quad (16)$$

The indices of $M^{\mu\nu}$ are anti-symmetrized to match the degrees of freedom in $\omega_{\mu\nu}$. Note that the \mathcal{L} term only appears when one of the indices is 0.

Canonical quantization:

$$\dot{\phi} = \partial_0 \phi, \quad \Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}, \quad [\phi(\mathbf{x}), \Pi(\mathbf{y})] = [\phi, \dot{\phi}](\mathbf{x}) = i\delta(\mathbf{x} - \mathbf{y}) \quad (17)$$

Other equal-time commutators between ϕ, Π all just vanish. We have:

$$M^{0i} = -M^{i0} = -\frac{1}{2} \int d^3x \left(\dot{\phi} (x^0 \partial^i - x^i \partial^0) \phi + x^i \mathcal{L} \right), \quad M^{ij} = -\frac{1}{2} \int d^3x \dot{\phi} (x^i \partial^j - x^j \partial^i) \phi \quad (18)$$

Notice that $x^{[\mu} \partial^{\nu]} = \frac{1}{2} (x^\mu \partial^\nu - x^\nu \partial^\mu) = \frac{1}{2} D^{ij}$ is the Killing vector fields of $\mathbb{R}^{3,1}$, hence they naturally follow the commutation relations of $\mathfrak{so}(3,1)$ (up to a constant coefficient)². We have:

$$\begin{aligned} [M^{ij}, M^{kl}] &= \frac{1}{4} \int d^3x \int d^3y \left[\dot{\phi} D^{ij} \phi(x), \dot{\phi} D^{kl} \phi(y) \right] \\ &= \frac{1}{4} \int d^3x \dot{\phi} [D^{ij}, D^{kl}] \phi \end{aligned} \quad (19)$$

Similar holds for M^{i0} . Therefore, $M^{\mu\nu}$'s indeed form the Lie algebra $\mathfrak{so}(3,1)$. ■

²I would like to thank 林般 for pointing this out.