Advanced QFT №2

1   Local Transformation

$$\delta A_\mu^a = \partial_\mu \lambda^a(x) + f^a{}_{bc} A_\mu^b \lambda^c(x), \tag{1}$$

Here f_{abc} is the totally anti-symmetric structure constant for a semi-simple Lie algebra $\mathfrak{g}$, with generators $\{T_a\}_a$ and normalized Killing form δ_{ab}.

- The field strength is defined as follows:

$$
\begin{aligned}
F_{\mu\nu} \equiv F_{\mu\nu}^a T_a = [D_\mu, D_\nu] &= \big[\partial_\mu + A_\mu, \partial_\nu + A_\nu\big] \\
&= \mathrm{d}A + A \wedge A \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu + f^a{}_{bc} A_\mu^b A_\nu^c \, T_a
\end{aligned}
\tag{2}
$$

Adjoint indices $a, b, \cdots$ are sometimes suppressed by contracting with T_a's. By exploiting the anti-symmetric property of $f^a{}_{bc}$, along with the Jacobi identity, we get the infinitesimal transformation:

$$
\begin{aligned}
\delta F_{\mu\nu}^a &= \partial_\mu \, \delta A_\nu^a - \partial_\nu \, \delta A_\mu^a + f^a{}_{bc} \, \delta(A_\mu^b A_\nu^c) \\
&= f^a{}_{bc} \Big(\lambda^c \big(\partial_\mu A_\nu^b - \partial_\nu A_\mu^b \big) + \big(A_\nu^b \, \partial_\mu \lambda^c - A_\mu^b \, \partial_\nu \lambda^c \big) + \delta(A_\mu^b A_\nu^c) \Big) \\
&= f^a{}_{bc} \Big(\lambda^c \big(F_{\mu\nu}^b - f^b{}_{de} A_\mu^d A_\nu^e \big) + \big(A_\nu^b \big(\underline{\delta A_\mu^c} - f^c{}_{de} A_\mu^d \lambda^e \big) - A_\mu^b \big(\underline{\delta A_\nu^c} - f^c{}_{de} A_\nu^d \lambda^e \big) \big) + \underline{\delta(A_\mu^b A_\nu^c)} \Big) \\
&= f^a{}_{bc} \Big(\lambda^c \big(F_{\mu\nu}^b - f^b{}_{de} A_\mu^d A_\nu^e \big) - \big(f^c{}_{de} A_\nu^b A_\mu^d \lambda^e - f^c{}_{de} A_\mu^b A_\nu^d \lambda^e \big) \Big) \\
&= \underline{f^a{}_{bc}} \Big(\lambda^c \big(F_{\mu\nu}^b - \underline{f^b{}_{de}} A_\mu^d A_\nu^e \big) - \big(\underline{f^c{}_{de}} A_\nu^b A_\mu^d \lambda^e - \underline{f^c{}_{de}} A_\mu^b A_\nu^d \lambda^e \big) \Big) \\
&= f^a{}_{bc} \lambda^c F_{\mu\nu}^b
\end{aligned}
\tag{3}
$$

When contracted with T_a, this yields:

$$\delta F_{\mu\nu} = \lambda^c F_{\mu\nu}^b \, f^a{}_{bc} T_a = \lambda^c F_{\mu\nu}^b \, [T_b, T_c] = F_{\mu\nu} \cdot \lambda - \lambda \cdot F_{\mu\nu}, \tag{4}$$

$$\lambda = \lambda^c(x) \, T_c, \quad F_{\mu\nu} = F_{\mu\nu}^b \, T_b, \tag{5}$$

$$F_{\mu\nu} \;\longmapsto\; e^{-\Lambda^a(x) \, T_a} \, F_{\mu\nu} \, e^{\Lambda^a(x) \, T_a} \tag{6}$$

The exponentiation is valid even for local $\lambda = \lambda(x)$, since it is produced by integrating along the fiber direction $\lambda \to \Lambda$, not the spacetime direction x. This is the finite transformation w.r.t. $\Lambda(x)$.

- For any matter field ψ furnishing a representation of $\mathfrak{g}$, we have:

$$T_a \psi = (T_a)^i{}_j \, \psi^j, \quad \delta \psi = -\lambda^a(x) \, T_a \psi, \tag{7}$$

$$\psi \;\longmapsto\; e^{-\Lambda^a(x) \, T_a} \, \psi, \tag{8}$$

$$D_\mu \psi \;\longmapsto\; e^{-\Lambda^a(x) \, T_a} \, D_\mu \psi, \tag{9}$$

In fact, (1) is chosen to ensure that $D_\mu \psi$ transforms gauge covariantly just like ψ. Therefore,

$$
\begin{aligned}
D_\mu = \partial_\mu + A_\mu \;\longmapsto\; & e^{-\Lambda^a(x)\,T_a} \circ D_\mu \circ e^{\Lambda^a(x)\,T_a} \\
= & e^{-\Lambda} \circ (\partial_\mu + A_\mu) \circ e^{\Lambda} \\
= & e^{-\Lambda} \circ \partial_\mu \circ e^{\Lambda} + e^{-\Lambda} A_\mu\, e^{\Lambda}, \quad \Lambda = \Lambda^a(x)\,T_a,
\end{aligned}
\tag{10}
$$

$$
A_\mu \;\longmapsto\; e^{-\Lambda}\left(\partial_\mu e^{\Lambda}\right) + e^{-\Lambda}A_\mu\,e^{\Lambda} = T_a\,\partial_\mu \Lambda^a(x) + e^{-\Lambda}A_\mu\,e^{\Lambda}
\tag{11}
$$

- $F^2 \equiv F \wedge F$, we have:

$$
\begin{aligned}
F^2 &= (\mathrm{d}A + A \wedge A) \wedge (\mathrm{d}A + A \wedge A) \\
&= \mathrm{d}A \wedge \mathrm{d}A + \mathrm{d}A \wedge A \wedge A + A \wedge A \wedge \mathrm{d}A + A \wedge A \wedge A \wedge A
\end{aligned}
\tag{12}
$$

The last term is proportional to $\epsilon_{abcd}\,T^a\,T^b\,T^c\,T^d$, hence its trace will vanish; therefore,

$$
\begin{aligned}
\operatorname{tr} F^2 &= \operatorname{tr}\left(\mathrm{d}A \wedge \mathrm{d}A + \mathrm{d}A \wedge A \wedge A + A \wedge A \wedge \mathrm{d}A\right) \\
&= \operatorname{tr}\left(\mathrm{d}(\mathrm{d}A \wedge A) + \frac{2}{3}\,\mathrm{d}(A \wedge A \wedge A)\right) \\
&= \mathrm{d}\operatorname{tr}\left(\mathrm{d}A \wedge A + \frac{2}{3}\,A \wedge A \wedge A\right) = \mathrm{d}\omega\,,
\end{aligned}
\tag{13}
$$

$$
\omega = \operatorname{tr}\left(\mathrm{d}A \wedge A + \frac{2}{3}\,A \wedge A \wedge A\right)
\tag{14}
$$

∎

2 Relativistic Particle

$$
L = \frac{1}{2e}\left(\frac{1}{c}\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2 - \frac{e}{2}\,m^2 c^4
\tag{15}
$$

- For $t \mapsto t' = t - \xi(t)$, we have $X'(t') = X(t)$, therefore:

$$
\delta X^\mu = -\,\delta t\,\frac{\mathrm{d}X^\mu}{\mathrm{d}t} = \xi(t)\,\dot{X}^\mu,
\tag{16}
$$

Or more explicitly, $X^\mu(t) \mapsto X^\mu(t) + \xi(t)\,\dot{X}^\mu$.

- We have:

$$
\begin{aligned}
\delta L &= \frac{1}{ec^2}\,\dot{X}_\mu\,\delta\dot{X}^\mu - \frac{\delta e}{2}\frac{1}{e^2 c^2}\dot{X}^2 - \frac{\delta e}{2}\,m^2 c^4 \\
&= \frac{1}{ec^2}\,\xi\dot{X}_\mu\ddot{X}^\mu + \frac{1}{ec^2}\,\dot{\xi}\dot{X}^2 - \frac{\delta e}{2}\frac{1}{e^2 c^2}\dot{X}^2 - \frac{\delta e}{2}\,m^2 c^4
\end{aligned}
\tag{17}
$$

For $S = \int \mathrm{d}t\,L$ to be invariant, δL should be reduced to a total derivative, which can then be reduced to some vanishing boundary terms.

Consider $\delta e = \frac{\mathrm{d}}{\mathrm{d}t}(e\xi) = \dot{e}\xi + e\dot{\xi}$, and we have:

$$
\begin{aligned}
\delta L &= \frac{1}{ec^2}\,\xi\dot{X}_\mu\ddot{X}^\mu + \frac{1}{2ec^2}\,\dot{\xi}\dot{X}^2 - \frac{\dot{e}}{2e^2 c^2}\,\xi\dot{X}^2 - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\,e\xi\,m^2 c^4\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(\frac{1}{2ec^2}\dot{X}^2 - \frac{e}{2}\,m^2 c^4\right)\xi\right\} = \frac{\mathrm{d}}{\mathrm{d}t}(\xi L)
\end{aligned}
\tag{18}
$$

Indeed we get a total derivative; therefore,

$$\delta e = \frac{\mathrm{d}}{\mathrm{d}t}(e\xi), \quad \delta S = \int \delta L = \int \mathrm{d}(\xi L) = 0 \tag{19}$$

• $e(t)$ can be seen as a gauge field coupled to X, which captures the t–reparametrization redundancy through the gauge transformation parameter $\xi(t)$. A natural gauge choice is fixing $f = e(t) - 1 \equiv 0$, which is equivalent to setting $t = \tau$: the proper time, or affine parametrization for the massless case.

The gauge invariant path integral is constructed as follows:

$$
\begin{aligned}
\mathcal{Z} &= \frac{1}{\int \mathscr{D}\xi} \int \mathscr{D}X \ \mathscr{D}e \ e^{iS} \\
&= \frac{1}{\int \mathscr{D}\xi} \int \mathscr{D}X \ \mathscr{D}e \ e^{iS} \int \mathscr{D}f \ \delta[f] \\
&= \frac{1}{\int \mathscr{D}\xi} \int \mathscr{D}X \ \mathscr{D}e \ e^{iS} \int \mathscr{D}\xi \ \delta[f_\xi] \det \frac{\delta f_\xi}{\delta \xi} \\
&= \frac{1}{\int \mathscr{D}\xi} \int \mathscr{D}\xi \int \mathscr{D}X \ \mathscr{D}e \ e^{iS} \ \delta[f_\xi] \det \frac{\delta f_\xi}{\delta \xi} \\
&= \frac{1}{\int \mathscr{D}\xi} \int \mathscr{D}\xi \int \mathscr{D}X_\xi \ \mathscr{D}e_\xi \ e^{iS_\xi} \ \delta[f_\xi] \det \frac{\delta f_\xi}{\delta \xi}\bigg|_{e_\xi} \\
&= \frac{1}{\int \mathscr{D}\xi} \int \mathscr{D}\xi \int \mathscr{D}X \ \mathscr{D}e \ e^{iS} \ \delta[f] \det \frac{\delta f_\xi}{\delta \xi}\bigg|_{\xi=0} \\
&= \int \mathscr{D}X \ \mathscr{D}e \ e^{iS} \ \delta[f] \det \frac{\delta f_\xi}{\delta \xi}\bigg|_{\xi=0}
\end{aligned}
\tag{20}
$$

Here the gauge-transformed quantities are marked with a ξ subscript. Note that in the final expression, f is *not* integrated out and can be *any* possible gauge-fixing function, i.e. $\left(\delta[f] \det \frac{\delta f_\xi}{\delta \xi}\right)_{\xi=0}$ is in fact f-independent.

The gauge-fixing term $\delta[f]$ can be replaced by a Gaussian packet with width parameter ζ. More rigorously, up to an overall constant coefficient, we have the following equivalence:

$$\delta[f] \sim \delta[f - f_0] \sim \int \mathscr{D}f_0 \exp\left(-\frac{i}{2\zeta}\int \mathrm{d}t \ f_0^2\right) \delta[f - f_0] = \exp\left(i \int \mathrm{d}t \ L_{gf}\right), \tag{21}$$

$$L_{gf} = -\frac{1}{2\zeta}f^2 = -\frac{1}{2\zeta}(e-1)^2, \tag{22}$$

Here f_0 is some gauge-invariant shift of f, namely $(f_0)_\xi = f_0$. f_0 can be seen as a non-dynamical auxiliary field that enforce the gauge fixing, much similar to a Lagrange multiplier. On the other hand, the determinant can be evaluated using Faddeev–Popov (FP) ghosts b, c:

$$\det P \sim \int \mathscr{D}b \ \mathscr{D}c \exp\left(i \int \mathrm{d}t \int \mathrm{d}t' \ b(t) \cdot P(t,t') \cdot c(t')\right), \tag{23}$$

$$\frac{\delta f_\xi(t)}{\delta \xi(t')}\bigg|_{\xi=0} = \frac{\delta}{\delta \xi(t')}\bigg|_{\xi=0}\left(e + \frac{\mathrm{d}}{\mathrm{d}t}(e\xi) - 1\right)_{(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\left(e(t) \ \delta(t-t')\right), \tag{24}$$

$$\det \frac{\delta f_\xi(t)}{\delta \xi(t')} \bigg|_{\xi=0} \sim \int \mathcal{D}b\, \mathcal{D}c \, \exp\left(i \int \mathrm{d}t \int \mathrm{d}t'\, b(t) \left(e(t)\, \delta(t-t') \right) c(t') \right)$$

$$\sim \int \mathcal{D}b\, \mathcal{D}c \, \exp\left(-i \int \mathrm{d}t \, e\dot{b}c \right), \tag{25}$$

$$L_{gh} = -e\dot{b}c \tag{26}$$

In summary, we have:

$$\mathcal{Z} = \int \mathcal{D}X\, \mathcal{D}e\, \mathcal{D}b\, \mathcal{D}c \, e^{iS_q}, \quad S_q = \int \mathrm{d}t\, L_q, \tag{27}$$

$$L_q = L + L_{gf} + L_{gh} = L - \frac{1}{2\zeta}\,(e-1)^2 - e\dot{b}c, \tag{28}$$

S_q is the quantum action under the gauge-fixing condition $f = e(t) - 1 = 0$. ∎

3   2D σ-Model

$$\mathcal{L} = -\frac{1}{2}\,\partial_\alpha X^\mu\, \partial_\beta X_\mu \sqrt{-h}\, h^{\alpha\beta}, \quad X : \Sigma^{1,1} \to \mathbb{R}^{D-1,1} \tag{29}$$

- The action is diff-invariant; under $\sigma^\alpha \mapsto \sigma^\alpha + \xi^\alpha$, we have:

$$\delta X^\alpha = \mathcal{L}_\xi X^\alpha, \quad \delta h^{\alpha\beta} = \mathcal{L}_\xi h^{\alpha\beta} \tag{30}$$

$\mathcal{L}_\xi$ is the Lie derivative along ξ^α. Note that $0 = \delta\!\left(h_{\alpha\beta} h^{\beta\gamma}\right)$, hence we have:

$$\delta h_{\alpha\beta} = -h_{\alpha\alpha'} h_{\beta\beta'}\, \delta h^{\alpha'\beta'} = \xi^\gamma \partial_\gamma h_{\alpha\beta} + (\partial_\alpha \xi^\gamma)\, h_{\gamma\beta} + (\partial_\beta \xi^\gamma)\, h_{\alpha\gamma} = \mathcal{L}_\xi h_{\alpha\beta}, \tag{31}$$

$$\delta\sqrt{-h} = \frac{1}{2}\sqrt{-h}\, h^{\alpha\beta}\, \delta h_{\alpha\beta}\,, \tag{32}$$

Furthermore, we have $\mathcal{L}_\xi \mathrm{d}X = \mathrm{d}(\mathcal{L}_\xi X)$, i.e. $\partial_\alpha\, \delta X = \partial_\alpha\, \mathcal{L}_\xi X = \partial_\alpha\!\left(\xi^\gamma \partial_\gamma X\right) = \mathcal{L}_\xi\!\left(\partial_\alpha X\right)$. Note that due to the $\sqrt{-h}$ factor, $\mathcal{L}$ is not a scalar but a *scalar density*. For convenience, define $\mathcal{L} = \widetilde{\mathcal{L}} \sqrt{-h}$, then $\widetilde{\mathcal{L}} = -\frac{1}{2}\, h^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X_\mu$ is a scalar; using chain rule, we obtain:

$$\begin{aligned}
\delta\mathcal{L} &= \sqrt{-h}\,\delta\widetilde{\mathcal{L}} + \widetilde{\mathcal{L}}\,\delta\sqrt{-h} \\
&= \sqrt{-h}\,\mathcal{L}_\xi \widetilde{\mathcal{L}} + \widetilde{\mathcal{L}}\,\delta\sqrt{-h} \\
&= \sqrt{-h}\,\xi^\gamma \partial_\gamma \widetilde{\mathcal{L}} + \widetilde{\mathcal{L}}\,\delta\sqrt{-h} \\
&= \partial_\gamma\!\left(\xi^\gamma \widetilde{\mathcal{L}}\sqrt{-h}\right) - \widetilde{\mathcal{L}}\left(\sqrt{-h}\,(\partial_\gamma \xi^\gamma) + \xi^\gamma (\partial_\gamma \sqrt{-h})\right) + \widetilde{\mathcal{L}}\,\delta\sqrt{-h} \\
&= \partial_\gamma\!\left(\xi^\gamma \mathcal{L}\right) - \widetilde{\mathcal{L}}\left(\sqrt{-h}\,(\partial_\gamma \xi^\gamma) + \xi^\gamma (\partial_\gamma \sqrt{-h}) - \delta\sqrt{-h}\right) \\
&= \partial_\gamma\!\left(\xi^\gamma \mathcal{L}\right) - \widetilde{\mathcal{L}}\sqrt{-h}\left(\partial_\gamma \xi^\gamma + \frac{1}{2}\,\xi^\gamma h^{\alpha\beta}\partial_\gamma h_{\alpha\beta} - \frac{1}{2}\,h^{\alpha\beta}\,\delta h_{\alpha\beta}\right) \\
&= \partial_\gamma\!\left(\xi^\gamma \mathcal{L}\right) - \widetilde{\mathcal{L}}\sqrt{-h}\left(\partial_\gamma \xi^\gamma - \frac{1}{2}\,h^{\alpha\beta}\left((\partial_\alpha \xi^\gamma)\,h_{\gamma\beta} + (\partial_\beta \xi^\gamma)\,h_{\alpha\gamma}\right)\right) \\
&= \partial_\gamma\!\left(\xi^\gamma \mathcal{L}\right) - \widetilde{\mathcal{L}}\sqrt{-h}\left(\partial_\gamma \xi^\gamma - \partial_\gamma \xi^\gamma\right) \\
&= \partial_\gamma\!\left(\xi^\gamma \mathcal{L}\right)
\end{aligned} \tag{33}$$

We see that $\delta\mathcal{L}$ is a total derivative, hence $\delta S = \int \mathrm{d}^2\sigma\, \delta\mathcal{L} = 0$, i.e. the action is diff-invariant.

- The action is Weyl invariant; with $\delta h^{\alpha\beta} = -\lambda(\sigma)\, h^{\alpha\beta}$, we have:

$$\begin{aligned}
\delta\left(\sqrt{-h}\, h^{\alpha\beta}\right) &= \sqrt{-h}\, \delta h^{\alpha\beta} + h^{\alpha\beta}\, \delta\sqrt{-h} \\
&= \sqrt{-h}\, h^{\alpha\beta}\left(-\lambda - \frac{1}{2}\, h_{\alpha'\beta'}\, \delta h^{\alpha'\beta'}\right) \\
&= \sqrt{-h}\, h^{\alpha\beta}\left(-\lambda + \frac{1}{2}\, \lambda\, h_{\alpha'\beta'} h^{\alpha'\beta'}\right) \\
&= \sqrt{-h}\, h^{\alpha\beta}\left(-\lambda + \frac{2}{2}\, \lambda\right) \\
&= 0
\end{aligned} \tag{34}$$

Here we've used the fact that $h_{\alpha\beta} h^{\alpha\beta} = \delta^\alpha_\alpha = 2$. Therefore, $\delta\mathcal{L} = -\frac{1}{2}\, \partial_\alpha X^\mu\, \partial_\beta X_\mu \delta\left(\sqrt{-h}\, h^{\alpha\beta}\right) = 0$, i.e. the action is Weyl invariant.

- FP quantization of this system follows the same recipe as the point particle case above:

$$\mathcal{Z} = \int \mathcal{D}X\, \mathcal{D}h\, \mathcal{D}b\, \mathcal{D}c\, e^{iS_q}, \quad S_q = \int \mathrm{d}^2\sigma\, \mathcal{L}_q, \tag{35}$$

$$\mathcal{L}_q = \mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{gh} \tag{36}$$

Given gauge fixing: $f^{\alpha\beta} = h^{\alpha\beta} - h^{\alpha\beta}_{(0)}$, we have:

$$\mathcal{L}_{gf} = -\frac{1}{2\zeta}\, f^{\alpha\beta} f_{\alpha\beta} \sqrt{-h} = -\frac{1}{2\zeta}\left(h^{\alpha\beta} - h^{\alpha\beta}_{(0)}\right)\left(h_{\alpha\beta} - h^{(0)}_{\alpha\beta}\right)\sqrt{-h} \tag{37}$$

The FP ghost term $\mathcal{L}_{gh}$ is given by functional determinant; we have:

$$\left.\frac{\delta f^{\alpha\beta}_\xi(\sigma)}{\delta \xi^\gamma(\sigma')}\right|_0 = \left.\frac{\delta}{\delta \xi^\gamma(\sigma')}\right|_0 \left(\mathcal{L}_\xi h^{\alpha\beta} - \lambda h^{\alpha\beta}\right)_{(\sigma)} \tag{38}$$

$$= \delta(\sigma - \sigma')\, \partial_\gamma h^{\alpha\beta} - \delta^\alpha_\gamma\, \partial^\beta \delta(\sigma - \sigma') - \delta^\beta_\gamma\, \partial^\alpha \delta(\sigma - \sigma') \tag{39}$$

$$= -\delta^\alpha_\gamma\, \nabla^\beta \delta(\sigma - \sigma') - \delta^\beta_\gamma\, \nabla^\alpha \delta(\sigma - \sigma'), \tag{40}$$

$$\left.\frac{\delta f^{\alpha\beta}_\xi(\sigma)}{\delta \lambda(\sigma')}\right|_0 = -\delta(\sigma - \sigma')\, h^{\alpha\beta}, \tag{41}$$

Here we've replaced ∂ with ∇ which commutes with the metric $h_{\alpha\beta}$. Define $\Xi^\Gamma = (\xi^\gamma, \lambda)$ to combine all gauge parameters, and use fermionic FP ghosts: $b_{\alpha\beta}$, $c^\Gamma = (c^\gamma, c')$ to contract the indices; after some integration by parts, we have:

$$\det \left.\frac{\delta f^{\alpha\beta}_\xi(\sigma)}{\delta \Xi^\Gamma(\sigma')}\right|_0 \sim \int \mathcal{D}b_{\alpha\beta}\, \mathcal{D}c^\gamma\, \mathcal{D}c'\, \exp\left(i \int \mathrm{d}^2\sigma\, \sqrt{-h}\, b_{\alpha\beta}\left(-\nabla^\beta c^\alpha - \nabla^\alpha c^\beta - h^{\alpha\beta} c'\right)\right) \tag{42}$$

To simplify the action, it is common[1] to integrate c' out, which constrains $b_{\alpha\beta}$ to be symmetric traceless: $b_{\alpha\beta} h^{\alpha\beta} = b^\alpha_\alpha = 0$. The resulting $b_{\alpha\beta}$ has 2 degrees of freedom, same as c^γ.

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[1] References: *Tong:* http://damtp.cam.ac.uk/user/tong/string.html, and also *Polchinski.*

In the end, we have:

$$\det \left.\frac{\delta f_\xi^{\alpha\beta}(\sigma)}{\delta \Xi^\Gamma(\sigma')}\right|_0 \sim \int \mathcal{D}b_{\alpha\beta} \ \mathcal{D}c^\gamma \ \exp\left(-2i \int \mathrm{d}^2\sigma \sqrt{-h}\, b_{\alpha\beta} \, \nabla^\alpha c^\beta\right) \tag{43}$$

Therefore, FP quantization with $f^{\alpha\beta} = h^{\alpha\beta} - h_{(0)}^{\alpha\beta}$ yields:

$$\mathcal{Z} = \int \mathcal{D}X \ \mathcal{D}h^{\alpha\beta} \ \mathcal{D}b_{\alpha\beta} \ \mathcal{D}c^\gamma \ e^{iS_q}, \quad S_q = \int \mathrm{d}^2\sigma \, \mathcal{L}_q, \quad \mathcal{L}_q = \mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{gh}, \tag{44}$$

$$\mathcal{L}_{gf} = -\frac{1}{2\zeta} \, f^{\alpha\beta} f_{\alpha\beta} \sqrt{-h} = -\frac{1}{2\zeta} \left(h^{\alpha\beta} - h_{(0)}^{\alpha\beta}\right)\left(h_{\alpha\beta} - h_{\alpha\beta}^{(0)}\right)\sqrt{-h}, \tag{45}$$

$$\mathcal{L}_{gh} = -2b_{\alpha\beta} \, \nabla^\alpha c^\beta \sqrt{-h} \tag{46}$$

$$\blacksquare$$