

1 Gravity

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2 \quad (1)$$

$$f(r) = 1 - \frac{GM}{r} + \frac{Q^2}{r^2} = \frac{(r - r_+)(r - r_-)}{r^2} \quad (2)$$

1. Event horizon(s): $f(r) = 0$, we have:

- (a) $M > |Q|$, $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$, 2 event horizons;
- (b) $M = |Q|$, $r_{\pm} = M$, 1 event horizon;
- (c) $M < |Q|$, no event horizon! “Naked” singularity.

2. New coordinate: $v = t + r^*$,

$$r^* = r + \frac{1}{2k_+} \ln \frac{|r - r_+|}{r_+} + \frac{1}{2k_-} \ln \frac{|r - r_-|}{r_-}, \quad k_{\pm} = \frac{r_{\pm} - r_{\mp}}{2r_{\pm}^2} \quad (3)$$

We have:

$$dt = dv - dr^* = dv - \left(1 + \frac{1}{2k_+} \frac{1}{r - r_+} + \frac{1}{2k_-} \frac{1}{r - r_-} \right) dr \quad (4)$$

$$\begin{aligned} &= dv - \left(1 + \frac{1}{r^2 f(r)} \frac{r_+^2 (r - r_-) - r_-^2 (r - r_+)}{r_+ - r_-} \right) dr \\ &= dv - \left(1 + \frac{1}{r^2 f(r)} \left((r_+ + r_-) r - r_+ r_- \right) \right) dr \\ &= dv - \frac{1}{f(r)} dr \end{aligned} \quad (5)$$

Therefore,

(a) The new metric:

$$\begin{aligned} ds^2 &= -f(r) \left(dv - \frac{1}{f(r)} dr \right)^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2 \\ &= -f(r) dv^2 + 2 dv dr + r^2 d\Omega^2 \end{aligned} \quad (6)$$

It is only singular at $r = 0$.

Note: during the exam I panicked when I saw (3), and I made a very stupid mistake in step (4). However, I knew what this new coordinate is trying to achieve — it’s aiming to eliminate the coordinate singularities in $\frac{1}{f} dr^2$ by absorbing it into dv^2 , so I guessed the result (5) correctly and carried on. I hope they gave me some points for getting the right answer, despite with some wrong process ($>_<$).

(b) $\frac{\partial}{\partial v}$ is a Killing vector field, for the metric components are all v -independent. More precisely, since $\frac{\partial}{\partial v}$ itself is a coordinate basis, we have the Lie derivative:


$$\mathcal{L}_{\frac{\partial}{\partial v}} g_{\mu\nu} = \partial_v g_{\mu\nu} = 0 \quad (7)$$

(c) $\left\| \frac{\partial}{\partial v} \right\|^2 = g_{\mu\nu} \delta_v^\mu \delta_v^\nu = g_{vv} = -f(r)$, therefore, for $M > |Q|$ we have:

- $\frac{\partial}{\partial v}$ timelike: $r > r_+$ and $r < r_-$
- spacelike: $r_- < r < r_+$
- null: $r = r_+$ and $r = r_-$

2 QFT

We shall restore the reasonable convention: $\eta_{\mu\nu} \sim (-, +, +, +)$.

1. 1PI: diagrammatic correction to the (1-particle) propagator that cannot be split into 2 disconnected parts by cutting one line; e.g. 

2. Consider the following Lagrangian:

$$\mathcal{L} = -\frac{1}{2}Z(\partial\phi_r)^2 - \frac{1}{2}m^2Z\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 - \frac{1}{2}\delta_Z(\partial\phi_r)^2 - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4 \quad (8)$$

The convention here is rather bizarre; normally we write down the UV Lagrangian \mathcal{L}_{UV} and split it into 2 parts, one is the effective IR Lagrangian \mathcal{L}_{IR} and the other one is the counterterm:


$$\begin{aligned} \mathcal{L}_{\text{UV}} &= -\frac{1}{2}Z(\partial\phi_r)^2 - \frac{1}{2}m^2Z\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 \\ &= \left(-\frac{1}{2}(\partial\phi_r)^2 - \frac{1}{2}m_p^2\phi_r^2 - \frac{\lambda_p}{4!}\phi_r^4 \right) - \left(-\frac{1}{2}\delta_Z(\partial\phi_r)^2 - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4 \right) \\ &= \mathcal{L}_{\text{IR}} + \mathcal{L}_{\text{ct}} \end{aligned} \quad (9)$$

Normally, we use \mathcal{L} to denote the UV Lagrangian \mathcal{L}_{UV} ; this is the convention adopted by numerous standard textbooks, incl. *Peskin & Schroeder* [1], *Weinberg*, and also *Srednicki*. However, the Lagrangian in (8) seems to be \mathcal{L}_{IR} instead of \mathcal{L}_{UV} . Anyway, we have:

$$Z + \delta_Z = 1, \quad m^2Z + \delta_m = m_p^2, \quad \lambda + \delta_\lambda = \lambda_p \quad (10)$$

Where m_p, λ_p is the physical IR couplings, fixed by the renormalization scheme. The convention here is really confusing and somewhat inconsistent; e.g. if we choose to write the UV mass term as $-\frac{1}{2}m^2Z\phi_r^2$, then the corresponding UV interaction term should look like $-\frac{\lambda}{4!}Z^2\phi_r^4$, but here we do not have the Z^2 factor. Also, we usually use m_0, λ_0 to denote bare couplings, but here it seems that they are denoted by m, λ .

We can write down the renormalized Feynman rules nonetheless, despite some sign issues due to the conventions; to avoid further confusion, we will adopt the usual notation: m_0, λ_0 for bare couplings, and $m = m_p, \lambda = \lambda_p$ for physical couplings. We have:

- Renormalized propagator: $\frac{-i}{p^2 + m^2 - i\epsilon}$
- Renormalized vertex: $-i\lambda$
- Counterterm ϕ^2 vertex: $+i(\delta_Z(-p^2) + \delta_m)$, 
- Counterterm ϕ^4 vertex: $+i\delta_\lambda$

The t -integral is related to Beta functions; consider $t \mapsto \frac{t^2}{1+t^2}$, and we have:

$$\int_0^\infty \frac{t^d dt}{1+t^2} = \frac{1}{2} \int_0^1 t^{\frac{D}{2}-1} (1-t)^{-\frac{D}{2}} dt = \frac{\Gamma(\frac{D}{2}) \Gamma(1-\frac{D}{2})}{2\Gamma(1)} = \frac{1}{2} \Gamma(\frac{D}{2}) \Gamma(1-\frac{D}{2}) = \frac{\pi}{2 \sin \frac{\pi D}{2}} \quad (19)$$

The last line is *Euler's reflection formula*, but here we actually don't need that since the $\Gamma(\frac{D}{2})$ factor is canceled by $A(S^d)$. In the end we have:

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m^2} = \frac{\pi^{D/2}}{(2\pi)^D} \Gamma(1-\frac{D}{2}) m^{D-2} = \frac{1}{(4\pi)^{D/2}} \Gamma(1-\frac{D}{2}) m^{D-2}, \quad (20)$$

$$\text{---}\bullet\text{---} \quad = \quad \frac{-i\lambda}{2} \frac{1}{(4\pi)^{D/2}} \Gamma(1-\frac{D}{2}) m^{D-2} \quad (21)$$

We then have to include counterterm contributions so that the renormalization condition (15) is satisfied; we have:

$$\begin{aligned} -iM(p^2) &\sim \text{---}\bullet\text{---} + \text{---}\otimes\text{---} = \frac{-i\lambda}{2} \frac{1}{(4\pi)^{D/2}} \Gamma(1-\frac{D}{2}) m^{D-2} + i(\delta_Z(-p^2) + \delta_m) \\ &\sim 0 + 0 \cdot (p^2 + m^2) + \mathcal{O}(p^4) \end{aligned} \quad (22)$$

Therefore,

$$\delta_Z = 0, \quad \delta_m = \frac{\lambda}{2} \frac{1}{(4\pi)^{D/2}} \Gamma(1-\frac{D}{2}) m^{D-2} \quad (23)$$

Alternatively, if we are working in $D = 4 = 3 + 1$ dimensions, it's easier to impose a naïve cutoff Λ , which gives:

$$\begin{aligned} \int^\Lambda \frac{k^d dk}{k^2 + m^2} &\sim \int^\Lambda k^{d-2} dk + \int^\Lambda k^d dk \left(\frac{1}{k^2 + m^2} - \frac{1}{k^2} \right) \\ &= \int^\Lambda k^{d-2} dk - m^2 \int^\Lambda \frac{k^{d-2} dk}{k^2 + m^2}, \quad d = D - 1 = 3 \\ &= \frac{\Lambda^2}{2} - \frac{m^2}{2} \ln \left(1 + \frac{\Lambda^2}{m^2} \right), \end{aligned} \quad (24)$$

Similarly, with $A(S^3) = 2\pi^2$, we have:

$$\begin{aligned} \delta_Z &= 0, \quad \delta_m = \frac{\lambda}{2} \frac{2\pi^2}{(2\pi)^4} \left\{ \frac{\Lambda^2}{2} - \frac{m^2}{2} \ln \left(1 + \frac{\Lambda^2}{m^2} \right) \right\} \\ &= \frac{\lambda}{32\pi^2} \left\{ \Lambda^2 - m^2 \ln \left(1 + \frac{\Lambda^2}{m^2} \right) \right\} \end{aligned} \quad (25)$$

References

- [1] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to Quantum Field Theory*. Addison-Wesley, Reading, USA, **1995**. ISBN: 978-0-201-50397-5.