String Theory №4

Bryan Compiled @ 2020/10/20

1 Stringy Physics!

$$T(z) = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu: , \quad \tilde{T}(\bar{z}) = -\frac{1}{\alpha'} :\bar{\partial} X^\mu \bar{\partial} X_\mu: , \tag{1}$$

$$V_k = :e^{ik\cdot X(z,\bar{z})}: , \quad G_{e,k} = e_{\mu\nu} :\partial X_z^\mu \, \bar{\partial} X_{\bar{z}}^\nu \, e^{ik\cdot X(z,\bar{z})}: , \tag{2}$$

We sometimes use subscripts like ∂X_z^μ to denote variable dependence to avoid clutter.

(a) The weight of a primary operator is given by its OPE with T and $\tilde{T}$. For exponential operators, there is a neat formula for cross contractions[1]:

$$\begin{aligned}
T(z)\, V_k(w,\bar{w}) &= \exp\left\{ \int \mathrm{d}^2 z' \int \mathrm{d}^2 w' \, \overbracket{X_{z'}^\mu X_{w'}^\nu} \frac{\delta}{\delta X_{z'}^\mu} \frac{\delta}{\delta X_{w'}^\nu} \right\} :T_z \, e^{ik\cdot X_w}: \\
&= \exp\left\{ \int \mathrm{d}^2 z' \, \overbracket{X_{z'}^\mu X_w^\nu} \frac{\delta}{\delta X_{z'}^\mu} \, ik_\nu \right\} :T_z \, e^{ik\cdot X_w}: \\
&= :\left\{ \exp\left(ik_\nu \int \mathrm{d}^2 z' \, \overbracket{X_{z'}^\mu X_w^\nu} \frac{\delta}{\delta X_{z'}^\mu} \right) T_z \right\} e^{ik\cdot X_w}: \\
&\sim -\frac{1}{\alpha'} : \left\{ 2\partial_z \left(ik_\sigma \overbracket{X_z^\mu X_w^\sigma} \right) \partial_z X_\mu + \partial_z \left(ik_\rho \overbracket{X_z^\mu X_w^\rho} \right) \partial_z \left(ik_\sigma \overbracket{X_{z,\mu} X_w^\sigma} \right) \right\} e^{ik\cdot X_w}: \\
&\sim -\frac{1}{\alpha'} : \left\{ 2\left(-\frac{\alpha'}{2} \frac{ik^\mu}{z-w} \right) \partial_z X_\mu + \left(-\frac{\alpha'}{2} \frac{ik^\mu}{z-w} \right) \left(-\frac{\alpha'}{2} \frac{ik_\mu}{z-w} \right) \right\} e^{ik\cdot X_w}: \\
&\sim \frac{\alpha' k^2}{4} \frac{V_k(w,\bar{w})}{(z-w)^2} + \frac{\partial V_k(w,\bar{w})}{z-w}
\end{aligned} \tag{3}$$

Here we've used the result that $ik_\sigma \overbracket{X_z^\mu X_w^\sigma} = ik^\mu (-\frac{\alpha'}{2}) \ln |z-w|^2$. We see that V_k is a primary of weight $(1,1)$ iff. $\frac{\alpha' k^2}{4} = 1$, or $m^2 = -k^2 = -\frac{4}{\alpha'}$. This is the mass shell condition for the closed string tachyon (at level 0). On the other hand,

$$G_{e,k} = e_{\mu\nu} G_k^{\mu\nu}, \tag{4}$$

$$\begin{aligned}
T(z)\, G_k^{\mu\nu}(0) \sim{}& :\overbracket{T_z \, \partial X_0^\mu} \, \bar{\partial} X_0^\nu \, e^{ik\cdot X_0}: + \cancel{:\overbracket{T_z \, \partial X_0^\mu \, \bar{\partial} X_0^\nu} \, e^{ik\cdot X_0}:} + :\overbracket{T_z \, \partial X_0^\mu \, \bar{\partial} X_0^\nu \, e^{ik\cdot X_0}}: \\
&+ \cancel{:\overbracket{\underbracket{T_z \, \partial X_0^\mu} \, \bar{\partial} X_0^\nu} \, e^{ik\cdot X_0}:} + :\overbracket{\underbracket{T_z \, \partial X_0^\mu} \, \bar{\partial} X_0^\nu \, e^{ik\cdot X_0}}: \\
\sim{}& \left(\frac{1}{z^2} G_k^{\mu\nu}(0) + \frac{1}{z} :\partial^2 X_0^\mu \, \bar{\partial} X_0^\nu \, e^{ik\cdot X_0}: \right) + \left(\frac{\alpha' k^2}{4} \frac{1}{z^2} G_k^{\mu\nu}(0) + \frac{1}{z} :\partial X_0^\mu \, \bar{\partial} X_0^\nu \, \partial \, e^{ik\cdot X_0}: \right) \\
&- \frac{2}{\alpha'} \left(-\frac{\alpha'}{2} \eta^{\sigma\mu} \frac{1}{z^2} \right) \left(-\frac{\alpha'}{2} \frac{ik_\sigma}{z} \right) :\bar{\partial} X_0^\nu \, e^{ik\cdot X_0}: \\
\sim{}& ik^\mu :\bar{\partial} X_0^\nu \, e^{ik\cdot X_0}: \left(-\frac{\alpha'}{2} \right) \frac{1}{z^3} + \left(1 + \frac{\alpha' k^2}{4} \right) \frac{G_k^{\mu\nu}(0)}{z^2} + \frac{\partial G_k^{\mu\nu}(0)}{z},
\end{aligned} \tag{5}$$

$$\tilde{T}(\bar{z})\, G_k^{\mu\nu}(0) \sim ik^\nu :\partial X_0^\mu \, e^{ik\cdot X_0}: \left(-\frac{\alpha'}{2} \right) \frac{1}{\bar{z}^3} + \left(1 + \frac{\alpha' k^2}{4} \right) \frac{G_k^{\mu\nu}(0)}{\bar{z}^2} + \frac{\partial G_k^{\mu\nu}(0)}{\bar{z}}, \tag{6}$$

[1] Reference: *Polchinski*, and physics.stackexchange.com/a/389193.

Therefore, $G_{e,k}$ is a primary of weight $(1,1)$ iff. $1+\frac{\alpha' k^2}{4}=1$ and $k^\mu e_{\mu\nu}=0=k^\nu e_{\mu\nu}$. The first equation gives the mass shell condition $m^2=-k^2=0$ for a massless boson, while the second equation constrains the polarization to be transverse. These are the physical constraints for a massless gauge boson, which is the level 1 excitation for a bosonic closed string.

(b) The form of any primary 3-point function is completely fixed by $\mathrm{PSL}(2,\mathbb{C})$ invariance[2]. In fact, for any holomorphic $\phi_i(z_i)$ with weight h_i, by translational invariance, we have:

$$\langle \phi_1(z_1)\,\phi_2(z_2)\,\phi_3(z_3)\rangle = f(z_{12},z_{23},z_{31}), \quad z_{ij}=z_i-z_j, \tag{7}$$

Furthermore, scaling invariance requires that f is quasi-homogeneous:

$$\begin{aligned} z\mapsto z'=\lambda^{-1}z, \quad f\mapsto &\left\langle \lambda^{h_1}\phi_1(\lambda z_1)\,\lambda^{h_2}\phi_2(\lambda z_2)\,\lambda^{h_3}\phi_3(\lambda z_3)\right\rangle \\ &=\lambda^{h_1+h_2+h_3} f(\lambda z_{12},\lambda z_{23},\lambda z_{31}) \\ &= f(z_{12},z_{23},z_{31}), \end{aligned} \tag{8}$$

$$f=\sum_{a+b+c=\sum_i h_i} f_{abc} = \sum_{a+b+c=\sum_i h_i} \frac{C_{abc}}{z_{12}^a z_{23}^b z_{31}^c} \tag{9}$$

On the other hand, for special conformal transformation[3] $\frac{1}{z}\mapsto\frac{1}{z'}=\frac{1}{z}+a$, we have:

$$z \longmapsto z'=\frac{1}{\frac{1}{z}+\bar a}=\frac{z}{1+z\bar a}=w(z), \quad \frac{\partial z}{\partial z'}=\frac{1}{(1-z\bar a)^2}=\frac{1}{\kappa^2}, \quad z_{ij}=\frac{z'_{ij}}{\kappa_i\kappa_j}, \tag{10}$$

$$f \longmapsto f\Big(w^{-1}(z_{12}),w^{-1}(z_{23}),w^{-1}(z_{31})\Big)\frac{1}{\kappa_1^{2h_1}\kappa_2^{2h_2}\kappa_3^{2h_3}} = f(z_{12},z_{23},z_{31}), \tag{11}$$

$$f_{abc}\Big(w^{-1}(z_{12}),w^{-1}(z_{23}),w^{-1}(z_{31})\Big) = f_{abc}(z_{12},z_{23},z_{31})\,\kappa_1^{c+a}\kappa_2^{a+b}\kappa_3^{b+c}, \tag{12}$$

We see that f is invariant under special conformal transformation iff. $f=f_{abc}$ where:

$$c+a=2h_1, \quad a+b=2h_2, \quad b+c=2h_3, \tag{13}$$

$$\text{i.e.}\quad a=h_1+h_2-h_3, \quad b=h_2+h_3-h_1, \quad c=h_3+h_1-h_2, \tag{14}$$

In the above discussions we've restricted ϕ_i to be holomorphic; for *spin-less* $\phi_i=\phi_i(z,\bar z)$, $h_i=\tilde h_i$, $\Delta_i=h_i+\tilde h_i$, the holomorphic and anti-holomorphic contributions can be nicely combined, and we have:

$$f=\frac{C}{|z_{12}|^{2a}|z_{23}|^{2b}|z_{31}|^{2c}}, \tag{15}$$

$$2a=\Delta_1+\Delta_2-\Delta_3, \quad 2b=\Delta_2+\Delta_3-\Delta_1, \quad 2c=\Delta_3+\Delta_1-\Delta_2, \tag{16}$$

$$\langle V_{k_1}(z_1,\bar z_1)\,V_{k_2}(z_2,\bar z_2)\,G_{e,k_3}(z_3,\bar z_3)\rangle = \frac{A(k_1,k_2,e)}{|z_{12}|^2|z_{23}|^2|z_{31}|^2} \tag{17}$$

[2]Reference: Blumenhagen, *Introduction to CFT*, and also *Di Francesco et al.*

[3]See *Di Francesco et al*, and also github.com/davidsd/ph229.

(c) Following the recipe in (a), we have:

$$\begin{aligned} V_{k_1}(z_1,\bar z_1)\,V_{k_2}(z_2,\bar z_2) &= :\exp\left(ik_{1,\mu}ik_{2,\nu}X_1^\mu X_2^\nu\right)e^{ik_1\cdot X_1}\,e^{ik_2\cdot X_2}: \\ &= \exp\left(\frac{\alpha'}{2}\,k_1\cdot k_2\ln|z_{12}|^2\right):e^{ik_1\cdot X_1}\,e^{ik_2\cdot X_2}: \\ &= |z_{12}|^{\alpha' k_1\cdot k_2}:e^{ik_1\cdot X_1}\,e^{ik_2\cdot X_2}: \end{aligned} \tag{18}$$

Apply the on-shell conditions, and we find that:

$$\alpha' k_1\cdot k_2 = \frac{\alpha'}{2}(k_1+k_2)^2-\frac{\alpha'}{2}k_1^2-\frac{\alpha'}{2}k_2^2=\frac{\alpha'}{2}(-k_3)^2-\frac{\alpha'}{2}k_1^2-\frac{\alpha'}{2}k_2^2=0-2-2=-4 \tag{19}$$

We are interested in the $|z_{12}|^{-2}$ term in the OPE around z_2; it will contribute to the 3-point function discussed in (b). Note that:

$$\begin{aligned} :e^{ik_1\cdot X_1}\,e^{ik_2\cdot X_2}: &= :\left(\cdots+\frac{1}{2}(ik_1\cdot X_1)^2+\cdots\right)e^{ik_2\cdot X_2}: \\ &= :\left(\cdots-\frac{1}{2}\,k_1^\mu k_1^\nu\left(X_2+z_{12}\,\partial X_2+\bar z_{12}\,\bar\partial X_2+\cdots\right)_\mu\left(\cdots\right)_\nu+\cdot\right)e^{ik_2\cdot X_2}: \\ &= :\left(\cdots-k_{1,\mu}k_{1,\nu}\left(z_{12}\bar z_{12}\,\partial X_2^\mu\,\bar\partial X_2^\nu\right)+\cdots\right)e^{ik_2\cdot X_2}: \\ &= \cdots-|z_{12}|^2\,k_{1,\mu}k_{1,\nu}\,G_{k_2}^{\mu\nu}(z_2,\bar z_2)+\cdots, \end{aligned} \tag{20}$$

$$V_{k_1}(z_1,\bar z_1)\,V_{k_2}(z_2,\bar z_2)=\cdots+\frac{O_{k_1,k_2}(z_2,\bar z_2)}{|z_{12}|^2}+\cdots, \tag{21}$$

$$O_{k_1,k_2}(z_2,\bar z_2)=-k_{1,\rho}k_{1,\sigma}\,G_{k_2}^{\rho\sigma}(z_2,\bar z_2), \tag{22}$$

Consider the same limit: $z_1\to z_2$ of the 3-point function, and we find that:

$$\begin{aligned} z_1\to z_2,\quad \langle V_{k_1}(z_1,\bar z_1)\,V_{k_2}(z_2,\bar z_2)\,G_{e,k_3}(z_3,\bar z_3)\rangle &\to \frac{1}{|z_{12}|^2}\frac{A(k_1,k_2,e)}{|z_{23}|^4} \\ &\sim \frac{1}{|z_{12}|^2}\left\langle O_{k_1,k_2}(z_2,\bar z_2)\,G_{e,k_3}(z_3,\bar z_3)\right\rangle, \end{aligned} \tag{23}$$

We see that in the $z_2\to z_3$ limit, we should obtain:

$$O_{k_1,k_2}(z_2,\bar z_2)\,G_{e,k_3}(z_3,\bar z_3)=\cdots+\frac{A(k_1,k_2,e)}{|z_{23}|^4}+\cdots, \tag{24}$$

Note that $A(k_1,k_2,e)=A(k_1,k_2,e)\,\mathbb{1}$ is simply a number; therefore, when finding $A(k_1,k_2,e)$, it is safe to ignore all (non-identity) operator contributions, as they should cancel each other. Similar to (a), we have:

$$\begin{aligned} G_{k_2}^{\rho\sigma}(z_2,\bar z_2)\,G_{k_3}^{\mu\nu}(z_3,\bar z_3) = \cdots &+ :\partial X_2^\rho\,\bar\partial X_2^\sigma\,e^{ik_2\cdot X_2}\,\partial X_3^\mu\,\bar\partial X_3^\nu\,e^{ik_3\cdot X_3}: \\ &+ :\partial X_2^\rho\,\bar\partial X_2^\sigma\,e^{ik_2\cdot X_2}\,\partial X_3^\mu\,\bar\partial X_3^\nu\,e^{ik_3\cdot X_3}: \\ &+ :\partial X_2^\rho\,\bar\partial X_2^\sigma\,e^{ik_2\cdot X_2}\,\partial X_3^\mu\,\bar\partial X_3^\nu\,e^{ik_3\cdot X_3}: \\ &+ :\partial X_2^\rho\,\bar\partial X_2^\sigma\,e^{ik_2\cdot X_2}\,\partial X_3^\mu\,\bar\partial X_3^\nu\,e^{ik_3\cdot X_3}:+\cdots \end{aligned} \tag{25}$$

$$
\begin{aligned}
G^{\rho\sigma}_{k_2}(z_2,\bar z_2)\,G^{\mu\nu}_{k_3}(z_3,\bar z_3) \sim \cdots &+ \left(-\frac{\alpha'}{2}\eta^{\rho\mu}\frac{1}{z_{23}^2}\right)\left(-\frac{\alpha'}{2}\eta^{\sigma\nu}\frac{1}{\bar z_{23}^2}\right)\times 1 \\
&+ \left(-\frac{\alpha'}{2}\eta^{\rho\mu}\frac{1}{z_{23}^2}\right)\left(-\frac{\alpha'}{2}\frac{ik_3^\sigma}{\bar z_{23}}\right)\left(-\frac{\alpha'}{2}\frac{ik_2^\nu}{\bar z_{32}}\right) + \left(z\leftrightarrow\bar z,\ \rho\leftrightarrow\sigma,\ \mu\leftrightarrow\nu\right) \\
&+ \left(-\frac{\alpha'}{2}\frac{ik_2^\rho}{z_{23}}\right)\left(-\frac{\alpha'}{2}\frac{ik_2^\sigma}{\bar z_{23}}\right)\left(-\frac{\alpha'}{2}\frac{ik_3^\mu}{z_{32}}\right)\left(-\frac{\alpha'}{2}\frac{ik_3^\nu}{\bar z_{32}}\right) + \cdots
\end{aligned}
$$

$$
\begin{aligned}
O_{k_1,k_2}(z_2,\bar z_2)\,G^{\mu\nu}_{k_3}(z_3,\bar z_3) \sim \cdots &- k_1^\mu k_1^\nu\left(\frac{\alpha'^2}{4}\right)\frac{1}{|z_{23}|^4} \\
&- i^2(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu)\left(\frac{\alpha'}{2}(k_1\cdot k_3)\right)\left(\frac{\alpha'^2}{4}\right)\frac{1}{|z_{23}|^4} \\
&- i^4 k_3^\mu k_3^\nu\left(\frac{\alpha'}{2}(k_1\cdot k_2)\right)^2\left(\frac{\alpha'^2}{4}\right)\frac{1}{|z_{23}|^4} + \cdots
\end{aligned}
\tag{26}
$$

Again, apply the on-shell conditions, and we find that:

$$
\frac{\alpha'}{2}k_1\cdot k_2 = -2,\quad \frac{\alpha'}{2}k_1\cdot k_3 = -\frac{\alpha'}{2}k_1\cdot(k_1+k_2) = -\frac{\alpha'}{2}k_1^2 - \frac{\alpha'}{2}k_1\cdot k_2 = -2-(-2) = 0, \tag{27}
$$

$$
\begin{aligned}
A(k_1,k_2,e) &= -\frac{\alpha'^2}{4}\left(\cancel{4e_{\mu\nu}k_3^\mu k_3^\nu} + e_{\mu\nu}k_1^\mu k_1^\nu\right) = -\frac{\alpha'^2}{4}e_{\mu\nu}k_1^\mu k_1^\nu \\
&= -\frac{\alpha'^2}{4}e_{\mu\nu}(k_2+k_3)^\mu(k_2+k_3)^\nu = -\frac{\alpha'^2}{4}e_{\mu\nu}k_2^\mu k_2^\nu \\
&= -\frac{\alpha'^2}{8}e_{\mu\nu}\left(k_1^\mu k_1^\nu + k_2^\mu k_2^\nu\right) \\
&= -\frac{\alpha'^2}{8}e_{\mu\nu}\left(k_{12}^\mu k_{12}^\nu + (k_1^\mu k_2^\nu + k_1^\nu k_2^\mu)\right),
\end{aligned}
\tag{28}
$$

On the other hand,

$$
0 = e_{\mu\nu}k_3^\mu k_3^\nu = e_{\mu\nu}(k_1+k_2)^\mu(k_1+k_2)^\nu = e_{\mu\nu}\left(k_{12}^\mu k_{12}^\nu + 2(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu)\right) \tag{29}
$$

$$
A(k_1,k_2,e) = -\frac{\alpha'^2}{8}e_{\mu\nu}k_{12}^\mu k_{12}^\nu\left(1-\frac{1}{2}\right) = -\frac{\alpha'^2}{16}e_{\mu\nu}\,k_{12}^\mu k_{12}^\nu \tag{30}
$$

$\boxed{2}$ Strings Scattering Off a Heavy Particle:

A heavy particle can be modeled by some D0-brane with Neumann boundary condition in the X_0 direction[4]. The scattering of a closed string tachyon off the heavy particle can then be computed via a disc diagram with two insertions.

(a) The conformal Killing group (CKG) of the disc is $\mathrm{PSL}(2,\mathbb{R})$. It is a 3 dimensional $\mathbb{R}$ Lie group, generated by 3 conformal Killing vectors (CKV's); therefore, it is possible to partially fix the positions of the two insertions V_1, V_2. On the upper half plane, this can be implemented by putting z_1, z_2 on the imaginary axis, with z_2 fixed and z_1 integrated[5]:

$$
\mathcal{A} = g_c^2 e^{-\lambda}\int_0^{z_2}\mathrm{d}z_1\left\langle :c_1^x e^{ik_1\cdot X_1}::c_2\tilde c_2\, e^{ik_2\cdot X_2}:\right\rangle,\quad z_2 = i,\quad z_1 = iy,\quad y\in[0,1] \tag{31}
$$

[4]Reference: arXiv:hep-th/9611214, arXiv:hep-th/9605168, and *Polchinski*.

[5]Reference: arXiv:0812.4408. I would like to thank Lucy Smith for pointing this out.

Here c^x comes from the CKV that brings $z_1 \to iy$. On the disc this can be taken to be a rotation around z_2; when mapped to the upper half plane and at around the imaginary axis, this is simply a translation along the $x = \frac{1}{2}(z + \bar{z})$ direction[6], i.e.

$$\text{CKV}: \ \partial_x = \delta_x^a \partial_a \quad \Longrightarrow \quad \text{Ghost}: \ c^x, \tag{32}$$

$$c^x \partial_x + c^y \partial_y = c^z \partial_z + c^{\bar{z}} \partial_{\bar{z}}, \quad c^x = \frac{1}{2}\big(c^z + c^{\bar{z}}\big) = \frac{1}{2}\big(c(z) + \tilde{c}(\bar{z})\big), \tag{33}$$

The ghost contribution is then:

$$\begin{aligned}\big\langle c_1^x c_2 \tilde{c}_2 \big\rangle = \big\langle c^x(z_1)\, c(z_2)\, \tilde{c}(\bar{z}_2) \big\rangle &= \frac{1}{2}\Big(\big\langle c(z_1)\, c(z_2)\, \tilde{c}(\bar{z}_2)\big\rangle + \big\langle \tilde{c}(z_1)\, c(z_2)\, \tilde{c}(\bar{z}_2)\big\rangle \Big) \\ &= \frac{1}{2}\Big(\big\langle c(z_1)\, c(z_2)\, c(z_2')\big\rangle + \big\langle c(z_1')\, c(z_2)\, c(z_2')\big\rangle \Big), \quad z' = \bar{z}, \\ &= \frac{C^g_{D^2}}{2}\big(z_{12} z_{12'} z_{22'} + z_{1'2} z_{1'2'} z_{22'}\big), \quad z_1, z_2 \in i\mathbb{R}, \\ &= 2C^g_{D^2}\big(z_1^2 - z_2^2\big) z_2 \end{aligned} \tag{34}$$

On the other hand, the $e^{ik_j \cdot X_j}$ contributions is similar to what we've computed in $\boxed{1}$, except that now we should be careful about the boundary conditions of X^μ, which affect the XX contraction in the formulae. For Neumann boundary condition: $\partial_y X^0 = 0$, the half-plane propagator from z' can be constructed with an image at $\bar{z}'$ with *the same charge*, i.e. we have:

$$\overbracket{X_1^0 X_2^0} = -\frac{\alpha'}{2}\, \eta^{00} \ln|z_1 - z_2|^2 - \frac{\alpha'}{2}\, \eta^{00} \ln|z_1 - \bar{z}_2|^2 \tag{35}$$

While for Dirichlet boundary $X^i = \text{const}$, we can always select the origin so that $X^i = 0$, and in this case the image should have *the opposite charge*, i.e.

$$\overbracket{X_1^i X_2^j} = -\frac{\alpha'}{2}\, \delta^{ij} \ln|z_1 - z_2|^2 + \frac{\alpha'}{2}\, \delta^{ij} \ln|z_1 - \bar{z}_2|^2, \tag{36}$$

$$\begin{aligned}\Longrightarrow \quad :e^{ik_1\cdot X_1}::e^{ik_2 \cdot X_2}: &= \exp\left(ik_{1,\mu} ik_{2,\nu} \overbracket{X_1^\mu X_2^\nu} \right) :e^{ik_1\cdot X_1}\, e^{ik_2\cdot X_2}: \\ &= |z_{12}|^{\alpha' k_1 \cdot k_2} |z_{1\bar{2}}|^{\alpha'\left(-k_1^0 k_2^0 - \mathbf{k}_1 \cdot \mathbf{k}_2\right)} :e^{ik_1\cdot X_1}\, e^{ik_2\cdot X_2}: \end{aligned} \tag{37}$$

Before further calculations, we note that the normal ordering defined here on D^2 differs from that on the usual $\mathbb{C}^2$; in fact, there are also self-contractions with image charge[7]:

$$\overbracket{X^\mu(z,\bar{z})\, X^\nu(\bar{z}, z)} = G_r^{\mu\nu}(z, \bar{z}) = \mp \frac{\alpha'}{2}\, \eta^{\mu\nu} \ln|z - \bar{z}|^2, \tag{38}$$

$$\Longrightarrow \quad \Big\langle :e^{ik_1\cdot X_1}\, e^{ik_2\cdot X_2}: \Big\rangle_{D^2} = \Big\langle :e^{ik_1\cdot X_1}\, e^{ik_2\cdot X_2}: \Big\rangle_{\mathbb{C}^2} \exp\left(\frac{1}{2}\sum_n ik_{n,\mu} ik_{n,\nu} \overbracket{X_n^\mu X_n^\nu} \right), \quad n = 1, 2 \tag{39}$$

The "$\mp$" sign choice depends on the boundary condition.

---

[6]Reference: *Polchinski*, Chapter 5 & 6.

[7]This is very much similar to the torus situation, where we also have to consider self-contractions with image charges. More rigorous discussion of G^r is given in *Polchinski*.

Therefore,

$$
\begin{aligned}
\Big\langle :e^{ik_1\cdot X_1}::e^{ik_2\cdot X_2}:\Big\rangle_{D^2} &= \Big\langle :e^{ik_1\cdot X_1}\,e^{ik_2\cdot X_2}:\Big\rangle_{\mathbb{C}^2} \exp\left(ik_{1,\mu}ik_{2,\nu}\overbracket{X_1^\mu X_2^\nu}\right)\exp\left(\frac{1}{2}\sum_n ik_{n,\mu}ik_{n,\nu}\overbracket{X_n^\mu X_n^\nu}\right)\\
&= \Big\langle :e^{ik_1\cdot X_1}\,e^{ik_2\cdot X_2}:\Big\rangle_{\mathbb{C}^2} \exp\left(\frac{1}{2}\sum_{m,n} ik_{m,\mu}ik_{n,\nu}\overbracket{X_m^\mu X_n^\nu}\right)\\
&= \Big\langle :e^{ik_1\cdot X_1}\,e^{ik_2\cdot X_2}:\Big\rangle_{\mathbb{C}^2} |z_{12}|^{\alpha' k_1\cdot k_2}|z_{1\bar{2}}|^{\alpha'\left(-k_1^0k_2^0-\mathbf{k}_1\cdot\mathbf{k}_2\right)}\prod_n |z_{n\bar{n}}|^{\frac{\alpha'}{2}\left(-(k_n^0)^2-\mathbf{k}_n^2\right)}
\end{aligned}
\tag{40}
$$

Note that X^i has no zero mode due to the Dirichlet boundary, hence $\int \mathscr{D}X$ gives a delta function in only the Neumann direction: $\delta\big(k_1^0+k_2^0\big)$. Physically, this means that only the energy is conversed; the momentum k^i is not conserved since the heavy D0-brane does not recoil. It is therefore convenient to define these on shell variables:

$$
s=\omega^2=(k_1^0)^2=(k_2^0)^2,\quad t=-(\mathbf{k}_1+\mathbf{k}_2)^2=-\mathbf{k}_1^2-\mathbf{k}_2^2-2\mathbf{k}_1\cdot\mathbf{k}_2=2\left(-\omega^2-\mathbf{k}_1\cdot\mathbf{k}_2-\frac{4}{\alpha'}\right),
\tag{41}
$$

$$
\mathbf{k_1}\cdot\mathbf{k_2}=-\frac{t}{2}-\omega^2-\frac{4}{\alpha'},\quad k_1\cdot k_2=-\omega(-\omega)+\mathbf{k_1}\cdot\mathbf{k_2}
\tag{42}
$$

Here we've used the on-shell condition: $m^2=-k^2=\omega^2-\mathbf{k}^2=-\frac{4}{\alpha'}$ for tachyons. The previous expressions can then be simplified to:

$$
\begin{aligned}
\Big\langle :e^{ik_1\cdot X_1}::e^{ik_2\cdot X_2}:\Big\rangle_{D^2} &= \Big\langle :e^{ik_1\cdot X_1}\,e^{ik_2\cdot X_2}:\Big\rangle_{\mathbb{C}^2}|z_{12}|^{-\frac{\alpha' t}{2}-4}|z_{1\bar{2}}|^{+\frac{\alpha' t}{2}+4+2\alpha'\omega^2}\prod_n |2z_n|^{-\alpha'\omega^2-2}\\
&= iC_{D^2}^X\,2\pi\,\delta\big(k_1^0+k_2^0\big)\,|z_{12}|^{-\frac{\alpha' t}{2}-4}|z_{1\bar{2}}|^{+\frac{\alpha' t}{2}+4+2\alpha'\omega^2}\prod_n |2z_n|^{-\alpha'\omega^2-2}\\
&= iC_{D^2}^X\,2\pi\,\delta\big(k_1^0+k_2^0\big)\,f\big(|z_{12}|,|z_{1\bar{2}}|,|z_1|,|z_2|\big),
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
\mathcal{A} &= g_c^2\underline{e^{-\lambda}}\cdot i\underline{C_{D^2}^X}\,2\pi\,\delta\big(k_1^0+k_2^0\big)\cdot 2\underline{C_{D^2}^g}\int_0^{z_2}\mathrm{d}z_1\,\big(z_1^2-z_2^2\big)z_2\,f\big(|z_{12}|,|z_{1\bar{2}}|,|z_1|,|z_2|\big)\\
&= g_c^2\underline{C_{D^2}}\,2\pi\,\delta\big(k_1^0+k_2^0\big)\cdot 2i\int_0^1 i\,\mathrm{d}y\,\big((iy)^2-i^2\big)\,i\cdot f\big(1-y,1+y,2y,2\big)\\
&= -ig_c^2C_{D^2}\,2\pi\,\delta\big(k_1^0+k_2^0\big)\cdot 2\cdot 2^{-2\alpha'\omega^2-4}\int_0^1\mathrm{d}y\,(1-y^2)\,f\big(1-y,1+y,y,1\big),
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
\int_0^1 \mathrm{d}y\,(1-y^2)\,f\big(1-y,1+y,y,1\big) &= \int_0^1 \mathrm{d}y\,(1-y)^{-\frac{\alpha' t}{2}-4+1}(1+y)^{+\frac{\alpha' t}{2}+4+2\alpha'\omega^2+1}y^{-\alpha'\omega^2-2} \\
&= \int_0^1 \mathrm{d}y\,y^{a-1}(1-y)^{2b-1}(1+y)^{-2a-2b+1}, \quad y' = \frac{1-y}{1+y}, \\
&= -2^{1-2a}\int_0^1 \mathrm{d}y'\,(-y')^{2b-1}(1-y'^2)^{a-1} \\
&= 2^{-2a}\int_0^1 \mathrm{d}(y'^2)\,(y'^2)^{b-1}(1-y'^2)^{a-1} \\
&= 2^{-2a}B\bigg(a=-\alpha'\omega^2-1,\, b=-\frac{\alpha' t}{4}-1\bigg)
\end{aligned}
\tag{45}
$$

Here $B(a,b)=\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$ is the Euler Beta function.

Putting everything together, we obtain:

$$
\begin{aligned}
\mathcal{A} &= -ig_c^2 C_{D^2}\,2\pi\,\delta\big(k_1^0+k_2^0\big)\cdot\frac{1}{2}\,B\bigg(-\alpha'\omega^2-1,\,-\frac{\alpha' t}{4}-1\bigg) \\
&= -ig_c^2 C_{D^2}\,\pi\,\delta\big(k_1^0+k_2^0\big)\,B\bigg(-\alpha'\omega^2-1,\,-\frac{\alpha' t}{4}-1\bigg)
\end{aligned}
\tag{46}
$$

In fact $g_c^2 C_{D^2}$ can be further computed by path integral or by comparing physical results. Here we settle for this generic coefficient since it's already enough for our following discussions[8].

(b) The Regge limit is found by taking the high energy limit while keeping the momentum transfer fixed; in this case it is achieved by:

$$
\text{Regge:}\quad s=\omega^2\to\infty,\quad t=-(\mathbf{k}_1+\mathbf{k}_2)^2\ \text{ fixed,} \tag{47}
$$

$$
\begin{aligned}
\mathcal{A}\propto B\bigg(a=-\alpha' s-1,\, b=-\frac{\alpha' t}{4}-1\bigg) &= \frac{\Gamma(-\alpha' s-1)}{\Gamma\big(-\alpha' s-\frac{\alpha' t}{4}-2\big)}\,\Gamma\bigg(-\frac{\alpha' t}{4}-1\bigg) \\
&\sim \bigg\{e\bigg(\alpha' s+\frac{\alpha' t}{4}+3\bigg)\bigg\}^{\frac{\alpha' t}{4}+1}\Gamma\bigg(-\frac{\alpha' t}{4}-1\bigg) \\
&\sim \big(e\alpha'\omega^2\big)^{\frac{\alpha' t}{4}+1}\,\Gamma\bigg(-\frac{\alpha' t}{4}-1\bigg) \\
&\sim \big(\omega^2\big)^{\frac{\alpha' t}{4}+1}\,\Gamma\bigg(-\frac{\alpha' t}{4}-1\bigg)
\end{aligned}
\tag{48}
$$

Here we've used the Stirling's approximation[9]: $\ln\Gamma(z+1)=\ln z!\sim z\ln z-z$. On the other hand, the hard scattering limit is found by keeping the scattering angle fixed, i.e.

$$
\text{Hard scattering:}\quad s=\omega^2\to\infty,\quad (t/s)\equiv\lambda\ \text{ fixed,} \tag{49}
$$

$$
\mathcal{A}\propto B(a,b)=\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\sim\exp\Big\{-\alpha'\Big(s\ln(\alpha' s)+\tfrac{t}{4}\ln\tfrac{\alpha' t}{4}+\tfrac{u}{4}\ln\tfrac{\alpha' u}{4}\Big)\Big\}, \tag{50}
$$

[8] And I have run out of time and energy.

[9] For the validity of Stirling's approximation when $z\in\mathbb{C}$, $\arg z=\pi-\epsilon$ and $|z|\to\infty$, see `Wikipedia:` *Stirling's formula for the gamma function*.

$$s=\omega^2=(k_1^0)^2=(k_2^0)^2, \quad t=-(\mathbf{k}_1+\mathbf{k}_2)^2, \quad u=-(\mathbf{k}_1-\mathbf{k}_2)^2, \tag{51}$$

$$s+\frac{t}{4}+\frac{u}{4}=-\frac{4}{\alpha'}, \tag{52}$$

Here we've introduced an additional u variable, and we see that the result is symmetric under $t \leftrightarrow u$. We find that under the above limits, the amplitude exhibits similar behaviors as the Veneziano amplitude.

(c) Note that $\Gamma(z)$ has no zeros on $\mathbb{C}^2$, therefore the poles of $B(a,b)=\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$ is given by the poles of $\Gamma(a)$ and $\Gamma(b)$:

$$a=-\alpha's-1=0,-1,-2,\cdots, \quad b=-\frac{\alpha't}{4}-1=0,-1,-2,\cdots, \tag{53}$$

The first set of poles gives:

$$\omega^2=-\frac{1}{\alpha'},0,\frac{1}{\alpha'},\frac{2}{\alpha'},\cdots \tag{54}$$

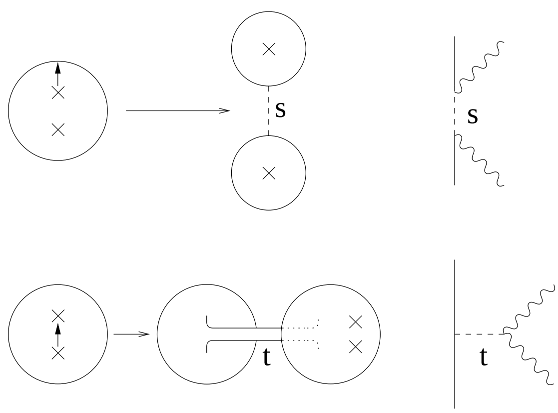
This is precisely the open string spectrum in $D=26$. Going back to (43) ~ (45), we see that they come from the singularities as $y=|z_1|\to 0$ or $|z_{12}|\to 1$, i.e. the two insertions are far apart.

This is the s-channel contribution; physically, it means that the closed string is first absorbed by the D0-brane, then it propagates as an intermediate open string on the brane, and is re-emitted in the end.

The other set of poles gives:

$$-t=(\mathbf{k}_1+\mathbf{k}_2)^2=\frac{4}{\alpha'},0,\cancel{-\frac{4}{\alpha'},-\frac{8}{\alpha'}},\cdots \tag{55}$$

Only two poles are realistic due to $\mathbf{k}^2\geq 0$. This is the t-channel contribution, which corresponds to a closed string exchange between the D0-brane and the incoming particle. The above factorization channels are nicely illustrated in the diagram below, borrowed from `arXiv:hep-th/9611214`.



$\boxed{3}$ Compton Scattering Between $U(N)$ Gluons and Adjoint Tachyons:

Such scattering is captured by a disc diagram with $AATT$ insertions at the boundary; following the same recipe as before, we write down[10]:

$$\begin{aligned}\mathcal{A} = g_o'^2 g_o^2 e^{-\lambda} \int \mathrm{d}y \Big\langle :e_\mu \partial_y X^\mu\, e^{ik\cdot X}(y): :c_1^y\, e_{1,\nu}\partial_y X_1^\nu\, e^{ik_1\cdot X_1}: :c_2^y e^{ik_2\cdot X_2}: :c_3^y e^{ik_2\cdot X_2}: \Big\rangle \\ \times \operatorname{Tr}\left(\lambda^a \lambda^{a_1}\lambda^{a_2}\lambda^{a_3} : \text{in } (y, y_1, y_2, y_3) \text{ order}\right) \\ + (2 \leftrightarrow 3)\end{aligned} \tag{56}$$

Where $y_i = y_{1,2,3}$ are fixed while $y_0 = y$ is integrated. Note that for $\boxed{2}$ we use (x, y) to label the points on the upper half plane so that the boundary coordinate is $x = (x, 0)$; here in order to match with *Polchinski*, we label the boundary points with $y = (y, 0)$ instead.

We see that $\mathcal{A}$ is structurally similar to the 4-point Veneziano amplitude $\mathcal{A}_{TTTT}$ and also the 3-point amplitude $\mathcal{A}_{TTA}$; by analogy we can evaluate $\mathcal{A}$ in a similar fashion[11]:

$$\begin{aligned}\mathcal{A} = g_o'^2 g_o^2 e^{-\lambda} \prod_{i<j} |y_{ij}|^{2\alpha' k_i\cdot k_j + 1}\, e_\mu e_{1,\nu} \\ \times \int \mathrm{d}y \prod_j |y_{0j}|^{2\alpha' k\cdot k_j} \left\{ -2\alpha' \frac{\eta^{\mu\nu}}{y_{01}^2} + (-2\alpha')\left(\frac{ik_1^\mu}{y_{01}} + \frac{ik_2^\mu}{y_{02}} + \frac{ik_3^\mu}{y_{03}}\right)(-2\alpha')\left(\frac{ik^\nu}{y_{10}} + \frac{ik_2^\nu}{y_{12}} + \frac{ik_3^\nu}{y_{13}}\right)\right\} \\ \times \operatorname{Tr}(\cdots)\, iC^X_{D_2} C^g_{D_2}\, (2\pi)^{26}\delta^{26}\left(\textstyle\sum_i k_i\right) \\ + (2 \leftrightarrow 3)\end{aligned} \tag{57}$$

Notice the doubling due to the boundary contraction $\overbrace{X_1^\mu X_2^\nu} = 2\times\left(-\frac{\alpha'}{2}\right)\ln y_{12}^2 = -2\alpha' \ln y_{12}$. Further simplifications can be achieved by using the on-shell and physical conditions for open string states:

$$k^2 = k_1^2 = 0, \quad e\cdot k = e_1 \cdot k_1 = 0, \quad k_{2,3}^2 = \frac{1}{\alpha'} = -m^2, \tag{58}$$

$$\begin{aligned} s &= -\alpha'(k+k_1)^2 = -2\alpha' k\cdot k_1, \\ t &= -\alpha'(k+k_2)^2 = -2\alpha' k\cdot k_2 - 1, \\ u &= -\alpha'(k+k_3)^2 = -2\alpha' k\cdot k_3 - 1,\end{aligned} \tag{59}$$

$$s + t + u = -2, \tag{60}$$

$$2\alpha' k_1\cdot k_2 = \alpha'(k_1+k_2)^2 - \alpha' k_1^2 - \alpha' k_2^2 = \alpha'(k+k_3)^2 - 1 = -u - 1 = 2\alpha' k\cdot k_3, \tag{61}$$

$$2\alpha' k_1\cdot k_3 = -t - 1 = 2\alpha' k\cdot k_2, \tag{62}$$

$$2\alpha' k_2\cdot k_3 = \alpha'(k+k_1)^2 - 2 = -s - 2, \tag{63}$$

[10] I would like to thank 谷夏 for help with this problem.

[11] For the ∂X contribution, see *Polchinski*, arXiv:1403.4553 and arXiv:1507.02172. Note that there are contractions between two ∂X fields.

$$\begin{aligned}\mathcal{A} = ig_o'^2 g_o^2 C_{D^2}\, |y_{12}|^{-u}\, |y_{13}|^{-t}\, |y_{23}|^{-s-2}\, e_\mu e_{1,\nu} \\ \times \int \mathrm{d}y\, |y_{01}|^{-s} |y_{02}|^{-t-1} |y_{03}|^{-u-1} \left\{ \frac{\eta^{\mu\nu}}{y_{01}^2} - 2\alpha' \left(\frac{ik_1^\mu}{y_{01}} + \frac{ik_2^\mu}{y_{02}} + \frac{ik_3^\mu}{y_{03}} \right) \left(-\frac{ik^\nu}{y_{01}} + \frac{ik_2^\nu}{y_{12}} + \frac{ik_3^\nu}{y_{13}} \right) \right\} \\ \times \mathrm{Tr}\,(\cdots)\,(-2\alpha')(2\pi)^{26}\delta^{26}\left(\textstyle\sum_i k_i\right) \\ + \left(2 \leftrightarrow 3\right)\end{aligned} \tag{64}$$

Fix $y_1 \to \infty, y_2 = 0, y_3 = 1$, and we get:

$$\begin{aligned}\mathcal{A} = ig_o'^2 g_o^2 C_{D^2}\, |y_1|^{-u-t}\, e_\mu e_{1,\nu} \\ \times \int \mathrm{d}y\, |y_{01}|^{-s} |y_{02}|^{-t-1} |y_{03}|^{-u-1} \left\{ \frac{\eta^{\mu\nu}}{y_{01}^2} - 2\alpha' \left(\frac{ik_1^\mu}{y_{01}} + \frac{ik_2^\mu}{y_{02}} + \frac{ik_3^\mu}{y_{03}} \right) \left(-\frac{ik^\nu}{y_{01}} + \frac{ik_2^\nu}{y_{12}} + \frac{ik_3^\nu}{y_{13}} \right) \right\} \\ \times \mathrm{Tr}\,(\cdots)\,\cdots + \cdots\end{aligned} \tag{65}$$

$$\begin{aligned}\mathcal{A} &= ig_o'^2 g_o^2 C_{D^2} \\ &\quad\times \int \mathrm{d}y\, |y_{02}|^{-t-1} |y_{03}|^{-u-1} \left| \frac{y_1^{s+2}}{y_{01}^s} \right| \left\{ \frac{e \cdot e_1}{y_{01}^2} - 2\alpha' e_\mu \left(\frac{ik_1^\mu}{y_{01}} + \frac{ik_2^\mu}{y_{02}} + \frac{ik_3^\mu}{y_{03}} \right) e_{1,\nu} \left(-\frac{ik^\nu}{y_{01}} + \frac{ik_2^\nu}{y_{12}} + \frac{ik_3^\nu}{y_{13}} \right) \right\} \\ &\quad\times \mathrm{Tr}\,(\cdots)\,\cdots + \cdots \\ &= ig_o'^2 g_o^2 C_{D^2} \int \mathrm{d}y \left\{ (e \cdot e_1)\, f(t,u) - 2\alpha' g(t,u) \right\} \times \cdots + \cdots\end{aligned} \tag{66}$$

Note that the definition of $f(t,u)$ and $g(t,u)$ contains the $\mathrm{Tr}\,(\cdots)$ factor.

The limit $y_1 \to \infty$ should be treated with care. In fact, the integral along the boundary splits into three ranges[12]:

$$\int \mathrm{d}y = \left\{ \int_{y_1 \to -\infty}^{y_3=0} + \int_{y_2=0}^{y_3=1} + \int_{y_3=1}^{y_1 \to +\infty} \right\} \mathrm{d}y \tag{67}$$

Notice the difference of $y_1 \to \pm\infty$; this is due to the S^1 topology of the boundary. Therefore,

$$\int \mathrm{d}y\, f(t,u) = \int \mathrm{d}y\, |y_{02}|^{-t-1} |y_{03}|^{-u-1} \left| \frac{y_1}{y_{01}} \right|^{s+2} \mathrm{Tr}\,(\cdots), \tag{68}$$

$$\mathrm{Tr}\left(\lambda^a \lambda^{a_1} \lambda^{a_2} \lambda^{a_3}\right) \equiv T_{0123}, \tag{69}$$

$$\begin{aligned}\int_{y_1 \to -\infty}^{y_2=0} \mathrm{d}y\, f(t,u) &= \int_{y_1 \to -\infty}^{0} \mathrm{d}y\, (1-y)^{-t-1} (-y)^{-u-1} \left(\frac{-y_1}{y - y_1} \right)^{s+2} T_{1023} = T_{1023} B(-u, -s-1), \\ \int_{y_3=1}^{y_1 \to +\infty} \mathrm{d}y\, f(t,u) &= \int_{1}^{y_1 \to +\infty} \mathrm{d}y\, (y-1)^{-t-1} y^{-u-1} \left(\frac{y_1}{y_1 - y} \right)^{s+2} T_{1230} = T_{1230} B(-t, -s-1), \\ \int_{y_3=0}^{y_2=1} \mathrm{d}y\, f(t,u) &= \int_0^1 \mathrm{d}y\, (1-y)^{-t-1} y^{-u-1} t_{1203} = T_{1203} B(-t, -u),\end{aligned} \tag{70}$$

[12] Reference: *Polchinski*'s discussion of $\mathcal{A}_{TTTT}$.

$$\int \mathrm{d}y\, f(t,u) = T_{1203}B(-t,-u) + T_{1230}B(-t,-s-1) + T_{1023}B(-u,-s-1) \tag{71}$$

Under $2 \leftrightarrow 3$, we have $t \leftrightarrow u$; therefore,

$$\begin{aligned}\int \mathrm{d}y\, f(t,u) + \big(2 \leftrightarrow 3\big) &= (T_{1203} + T_{1302})\, B(-t,-u) \\ &\quad + (T_{1230} + T_{1032})\, B(-t,-s-1) \\ &\quad + (T_{1023} + T_{1320})\, B(-u,-s-1) \\ &\equiv \alpha B(-t,-u) + \beta B(-t,-s-1) + \gamma B(-u,-s-1)\end{aligned} \tag{72}$$

The $g(t,u)$ factor can be computed in a similar manner. In the end we have[13]:

$$\begin{aligned}\mathcal{A} &= i g_o'^2 g_o^2 C_{D^2}\,(2\pi)^{26}\delta^{26}\big(\textstyle\sum_i k_i\big) \\ &\quad \times (-2\alpha')\Big\{(e_0\cdot e_1)\,\big(\alpha B(-t,-u) + \beta B(-t,-s-1) + \gamma B(-u,-s-1)\big) \\ &\qquad + (-2\alpha')(e_0\cdot k_2)(e_1\cdot k_3)\,\big(\alpha B(-t,-u) - \beta B(-t,-s-1) - \gamma B(-u,-s-1)\big) + \big(2 \leftrightarrow 3\big)\Big\}\end{aligned} \tag{73}$$

[13] Reference: arXiv:0801.3358.