

QCD Partition Function at $\mathcal{O}(g^2)$

$$\ln \mathcal{Z}_I^{(2)} = -\frac{1}{2} \text{(a)} - \frac{1}{2} \text{(b)} + \frac{1}{12} \text{(c)} + \frac{1}{8} \text{(d)}$$

$$\ln \mathcal{Z}_I^{(2)} = \ln \mathcal{Z}^{(a)} + \ln \mathcal{Z}^{(b)} + \ln \mathcal{Z}^{(c)} + \ln \mathcal{Z}^{(d)} \quad (1)$$

The contribution of (a) is given by:

$$\ln \mathcal{Z}^{(a)} = \frac{1}{2!} (-1)^1 \frac{T}{V} \sum_k \frac{T}{V} \sum_p \text{Tr} \left(S(k) (g\gamma^\nu T^b) S(p) (g\gamma^\mu T^a) \right) \left(\frac{V}{T} \right) \delta_{ab} \Delta_{\mu\nu}(p-k) \quad (2)$$

Here the trace goes over spinor, color and flavor indices. $S(k)$ is the quark propagator, with suppressed spinor, color and flavor indices, while $\delta_{ab} \Delta_{\mu\nu}$ is the gluon propagator, where a, b are adjoint indices; each vertex contributes a $(g\gamma^\mu T^a)$ factor.

In our convention, $k = (\omega_n, \mathbf{k})$ stands for the Euclidean 4-momentum with ω_n : the discrete Matsubara frequency. Each \sum_k comes with a factor $\frac{T}{V}$ while each momentum space delta function comes with an inverse factor: $\frac{V}{T}$; this is due to the fact that:

$$1 = \int \frac{d^4 k}{(2\pi)^4} (2\pi)^4 \delta^4(k - k_0) \sim \frac{1}{\beta V} \sum_k \beta V \delta_{k, k_0} \quad (3)$$

Following the same recipe from QED, we can write down:

$$\begin{aligned} \ln \mathcal{Z}^{(a)} &= - \left(\text{Tr} (T^a T^b) \delta_{ab} \right) \frac{1}{2} g^2 \frac{V}{T} \cdot \frac{T}{V} \sum_k \frac{T}{V} \sum_p \text{Tr} (S(k) \gamma^\nu S(p) \gamma^\mu) \Delta_{\mu\nu}(p-k) \\ &= - \left(\frac{N_c^2 - 1}{2} N_f \right) \frac{g^2}{288} \frac{V}{T} \left(5T^4 + \frac{18}{\pi^2} T^2 \mu^2 + \frac{9}{\pi^4} \mu^4 \right) \end{aligned} \quad (4)$$

The (b) term is structurally similar to the (a) term; now the amplitude can be written down simply by replacing the propagator $S(k) \mapsto W(k)$ of the ghost, while the vertex is $(g\gamma^\mu T^a) \mapsto (-igk^\nu T^b)$ instead:

$$\ln \mathcal{Z}^{(b)} = \frac{1}{2!} (-1)^1 \frac{T}{V} \sum_k \frac{T}{V} \sum_p \text{Tr} \left(W(k) (-igp^\nu T^b) W(p) (-igk^\mu T^a) \right) \left(\frac{V}{T} \right) \delta_{ab} \Delta_{\mu\nu}(p-k) \quad (5)$$

The trace now goes over suppressed *adjoint* indices of $W(k)_{ab} = -\delta_{ab} \Delta(k)$ and $(T_a)_{bc} = f_{abc}$, where f_{abc} is the structure constant of $\text{SU}(N_c)$. Therefore¹,

$$\begin{aligned} \ln \mathcal{Z}^{(b)} &= - \left(\text{Tr} (T^a T^b) \delta_{ab} \right) \frac{1}{2} g^2 \frac{V}{T} \cdot \frac{T}{V} \sum_k \frac{T}{V} \sum_p \Delta(k) \Delta(p) (-k^\mu p^\nu) \Delta_{\mu\nu}(p-k) \\ &= - \left(\frac{N_c^2 - 1}{2} 2N_c \right) \frac{1}{2} g^2 \frac{V}{T} \cdot \frac{T}{V} \sum_k \frac{T}{V} \sum_p \Delta(k) \Delta(p) (-k^\mu p^\nu) (-g_{\mu\nu}) \Delta(p-k) \end{aligned} \quad (6)$$

¹ $\text{Tr} (T^a T^b)_{\text{ad}}$ in the adjoint representation is precisely the *Killing form* of the $\mathfrak{su}(N_c)$ algebra, which is $2N_c$ times the $\text{Tr} (T^a T^b)_0$ in the fundamental representation; see Wikipedia: [Killing form](#).

Now we compute the remaining $\sum_{k,p}$. We have:

$$\Delta(k) \Delta(p) (k \cdot p) \Delta(p - k) = \frac{k \cdot p}{k^2 p^2 (p - k)^2} \quad (7)$$

The generic method to carry out such summation is by using the mixed representation of the propagator; for some propagator $D(k)$, we have:

$$\begin{aligned} D(k) = D(w_n, \mathbf{k}) &= \int_0^\beta d\tau e^{-i\omega_n \tau} T \sum_m e^{i\omega_m \tau} D(w_m, \mathbf{k}) \\ &= \int_0^\beta d\tau e^{-i\omega_n \tau} \tilde{D}(\tau, \mathbf{k}), \end{aligned} \quad (8)$$

$$\begin{aligned} \tilde{D}(\tau, \mathbf{k}) &= T \sum_m e^{i\omega_m \tau} D(w_m, \mathbf{k}) \\ &= T \sum_m e^{i\omega_m \tau} \int \frac{d\omega}{2\pi} \frac{\rho(\omega, \mathbf{k})}{\omega + i\omega_m} \\ &= \int \frac{d\omega}{2\pi} \rho(\omega, \mathbf{k}) T \sum_m \frac{e^{i\omega_m \tau}}{\omega + i\omega_m} \\ &= \int \frac{d\omega}{2\pi} \rho(\omega, \mathbf{k}) e^{-\omega \tau} (1 \pm n_\pm(\omega)), \end{aligned} \quad (9)$$

$$\rho(\omega, \mathbf{k}) = \frac{1}{i} (D(\omega + i\epsilon) - D(\omega - i\epsilon)) = 2 \operatorname{Im} D(\omega + i\epsilon, \mathbf{k}), \quad n_\pm = \frac{1}{e^{\beta\omega} \mp 1}, \quad (10)$$

Then the Matsubara sum \sum_{ω_n} becomes a sum over exponentials like $e^{-i\omega_n \tau}$, which is easier to deal with. However, for this particular problem, there is a shortcut²; notice that the denominator of (7) is invariant under $p \mapsto k - p$, hence:

$$\begin{aligned} \sum_p \frac{k \cdot p}{k^2 p^2 (p - k)^2} &= \sum_{(k-p)} \frac{k \cdot (k - p)}{k^2 (k - p)^2 p^2} \\ &= \sum_p \frac{k \cdot (k - p)}{k^2 p^2 (p - k)^2} \\ &= \sum_p \frac{\frac{1}{2}(k \cdot p + k \cdot (k - p))}{k^2 p^2 (p - k)^2} \\ &= \sum_p \frac{1}{2p^2 (p - k)^2}, \end{aligned} \quad (11)$$

$$\begin{aligned} \frac{T}{V} \sum_k \frac{T}{V} \sum_p \Delta(k) \Delta(p) (k \cdot p) \Delta(p - k) &= \frac{1}{2} \frac{T}{V} \sum_p \frac{1}{p^2} \frac{T}{V} \sum_k \frac{1}{(p - k)^2} \\ &= \frac{1}{2} \left(\frac{T}{V} \sum_p \frac{1}{p^2} \right)^2 = \frac{1}{2} \left(\frac{T^2}{12} \right)^2, \end{aligned} \quad (12)$$

²Reference: Laine & Vuorinen, *Basics of Thermal Field Theory*.

$$\ln \mathcal{Z}^{(b)} = - \left(\frac{N_c^2 - 1}{2} 2N_c \right) \frac{1}{2} g^2 \frac{V}{T} \cdot \frac{1}{2} \left(\frac{T^2}{12} \right)^2 = - \frac{V}{T} N_c (N_c^2 - 1) \frac{1}{4} g^2 \frac{T^4}{144} \quad (13)$$

The (c) term is structurally similar to the (b) term, but with a symmetrized 3-gluon vertex:

$$\left(\frac{1}{3!} \right) igf_{abc} (g_{\mu\nu}(k-p)_\rho + g_{\nu\rho}(p-q)_\mu + g_{\rho\mu}(q-k)_\nu) = \left(\frac{1}{3!} \right) igf_{abc} D_{\mu\nu\rho}(k, p, q) \quad (14)$$

To link the legs of two 3-gluon vertices as shown in (c), there are $3!$ possibilities. Therefore, we have:

$$\begin{aligned} \ln \mathcal{Z}^{(c)} &= \frac{1}{2!} \cdot 3! \cdot \left(\frac{1}{3!} \right)^2 \frac{T}{V} \sum_k \frac{T}{V} \sum_p \Delta(k) \Delta(p) \left(\frac{V}{T} \right) \Delta(p-k) \\ &\quad \times (igf_{abc} D_{\mu\nu\rho}(k, -p, p-k)) (igf^{bac} D^{\nu\mu\rho}(p, -k, k-p)) \\ &= - \frac{1}{12} \frac{V}{T} \cdot \frac{T}{V} \sum_k \frac{T}{V} \sum_p \Delta(k) \Delta(p) \Delta(p-k) \\ &\quad \times \left(\frac{N_c^2 - 1}{2} 2N_c \right) g^2 D_{\mu\nu\rho}(k, -p, p-k) D^{\mu\nu\rho}(p, -k, k-p) \\ &= - \left(\frac{N_c^2 - 1}{2} 2N_c \right) \frac{1}{12} g^2 \frac{V}{T} \cdot \frac{T}{V} \sum_k \frac{T}{V} \sum_p \Delta(k) \Delta(p) \Delta(p-k) \\ &\quad \times D_{\mu\nu\rho}(k, -p, p-k) D^{\mu\nu\rho}(p, -k, k-p), \end{aligned} \quad (15)$$

$$\begin{aligned} D_{\mu\nu\rho}(k, -p, p-k) D^{\nu\mu\rho}(p, -k, k-p) &= D_{\mu\nu\rho}(k, -p, p-k) D^{\mu\nu\rho}(-k, p, k-p) \\ &= -D_{\mu\nu\rho}(k, -p, p-k) D^{\mu\nu\rho}(k, -p, p-k) \\ &= - \boxed{g_{\mu\nu} g^{\mu\nu}} ((k+p)^2 + (k-2p)^2 + (p-2k)^2) \\ &\quad - 2(k+p) \cdot (k-2p) \\ &\quad - 2(k-2p) \cdot (p-2k) \\ &\quad - 2(p-2k) \cdot (k+p) \\ &= - \boxed{d+1} \cdot 3(k^2 + p^2 + (k-p)^2) \\ &\quad - 2 \left(-\frac{3}{2} \right) (k^2 + p^2 + (k-p)^2) \\ &= -3d(k^2 + p^2 + (k-p)^2), \end{aligned} \quad (16)$$

$$\begin{aligned} \ln \mathcal{Z}^{(c)} &= - \left(\frac{N_c^2 - 1}{2} 2N_c \right) \frac{1}{12} g^2 \frac{V}{T} \cdot \frac{T}{V} \sum_k \frac{T}{V} \sum_p \frac{-3d(k^2 + p^2 + (k-p)^2)}{k^2 p^2 (p-k)^2} \\ &= \left(\frac{N_c^2 - 1}{2} 2N_c \right) \frac{d}{4} g^2 \frac{V}{T} \cdot \frac{T}{V} \sum_k \frac{T}{V} \sum_p \frac{k^2 + p^2 + (k-p)^2}{k^2 p^2 (p-k)^2}, \end{aligned} \quad (17)$$

$$\begin{aligned} \frac{T}{V} \sum_k \frac{T}{V} \sum_p \frac{k^2 + p^2 + (k-p)^2}{k^2 p^2 (p-k)^2} &= \frac{T}{V} \sum_k \frac{T}{V} \sum_p \left(\frac{1}{p^2 (p-k)^2} + \frac{1}{k^2 (p-k)^2} + \frac{1}{k^2 p^2} \right) \\ &= 3 \left(\frac{T}{V} \sum_k \frac{1}{k^2} \right)^2 = 3 \left(\frac{T^2}{12} \right)^2, \end{aligned} \quad (18)$$

$$\ln \mathcal{Z}^{(c)} = \left(\frac{N_c^2 - 1}{2} 2N_c \right) \frac{d}{4} g^2 \frac{V}{T} \cdot 3 \left(\frac{T^2}{12} \right)^2 = + \frac{V}{T} N_c (N_c^2 - 1) \frac{3d}{4} g^2 \frac{T^4}{144} \quad (19)$$

Here we use d to denote spatial dimensions; for $d = 3$ we have $\frac{3d}{4} = \frac{9}{4}$.

The (d) term is built around the symmetrized 4-gluon vertex:

$$\left(\frac{1}{3!} \right) \left(-\frac{g^2}{4} \right) \left(f_{ad} \bullet f_{bc} (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma}) + \boxed{(b, \nu) \leftrightarrow (d, \sigma)} + \boxed{(b, \nu) \leftrightarrow (c, \rho)} \right) \quad (20)$$

Here we use “ \bullet ” to denote a contracted adjoint index, and use $\boxed{(\dots) \leftrightarrow (\dots)}$ to mark a switch of indices relative to the *previous* term. By contracting 2 pairs of the indices, we obtain the desired diagram (d). There are 3 ways to do this, therefore:

$$\begin{aligned} \ln \mathcal{Z}^{(d)} &= \frac{1}{1!} \cdot 3 \cdot \left(\frac{1}{3!} \right) \left(-\frac{g^2}{4} \right) \left(-\frac{N_c^2 - 1}{2} 2N_c \right) \frac{T}{V} \sum_k \frac{T}{V} \sum_p \Delta(k) \Delta(p) \left(\frac{V}{T} \right) \delta_{+k-k+p-p} \\ &\quad \times \left(-((d+1)^2 - (d+1)) + 0 + ((d+1) - (d+1)^2) \right) \end{aligned} \quad (21)$$

One of the three terms in the vertex vanishes after contraction, due to the anti-symmetry of f_{abc} . Again, we’ve used the fact that $g_{\mu\nu} g^{\mu\nu} = d+1$, where d is the spatial dimension.

Note that momentum conservation is automatic at the vertex, which is indicated by the trivial delta function $\delta_{+k-k+p-p} = 1$. In the end, we have:

$$\begin{aligned} \ln \mathcal{Z}^{(d)} &= \left(-\frac{g^2}{8} \right) \left(\frac{N_c^2 - 1}{2} 2N_c \right) \frac{T}{V} \sum_k \frac{T}{V} \sum_p \Delta(k) \Delta(p) \left(\frac{V}{T} \right) 2d(d+1) \\ &= \left(-\frac{g^2}{8} \right) \left(\frac{N_c^2 - 1}{2} 2N_c \right) \left(\frac{V}{T} \right) 2d(d+1) \left(\frac{T^2}{12} \right)^2 \\ &= -\frac{V}{T} N_c (N_c^2 - 1) \frac{d(d+1)}{4} g^2 \frac{T^4}{144} \end{aligned} \quad (22)$$

For $d = 3$ we have $\frac{d(d+1)}{4} = 3$. Combining (a~d), we have the full $\mathcal{O}(g^2)$ partition function. ■