String Theory №4 [WIP]

Bryan Compiled @ 2020/04/28

1 Stringy Physics!

$$T(z) = -\frac{1}{\alpha'}:\partial X^\mu \partial X_\mu:, \quad \tilde{T}(\bar{z}) = -\frac{1}{\alpha'}:\bar\partial X^\mu \bar\partial X_\mu:, \tag{1}$$

$$V_k = :e^{ik\cdot X(z,\bar z)}:, \quad G_{e,k} = e_{\mu\nu}:\partial X^\mu_z\,\bar\partial X^\nu_{\bar z}\,e^{ik\cdot X(z,\bar z)}:, \tag{2}$$

We sometimes use subscripts like ∂X^μ_z to denote variable dependence to avoid clutter.

(a) The weight of a primary operator is given by its OPE with T and $\tilde T$. For exponential operators, there is a neat formula for cross contractions[1]:

$$\begin{aligned}
T(z)\,V_k(w,\bar w) &= \exp\left\{\int \mathrm{d}^2z' \int \mathrm{d}^2w'\, \overbracket{X^\mu_{z'} X^\nu_{w'}}\, \frac{\delta}{\delta X^\mu_{z'}}\frac{\delta}{\delta X^\nu_{w'}}\right\}:T_z\, e^{ik\cdot X_w}: \\
&= \exp\left\{\int \mathrm{d}^2z'\, \overbracket{X^\mu_{z'} X^\nu_{w}}\, \frac{\delta}{\delta X^\mu_{z'}}\, ik_\nu\right\}:T_z\, e^{ik\cdot X_w}: \\
&= :\left\{\exp\left(ik_\nu \int \mathrm{d}^2z'\, \overbracket{X^\mu_{z'} X^\nu_{w}}\, \frac{\delta}{\delta X^\mu_{z'}}\right) T_z\right\} e^{ik\cdot X_w}: \\
&\sim -\frac{1}{\alpha'}:\left\{2\partial_z\left(ik_\sigma \overbracket{X^\mu_z X^\sigma_w}\right)\partial_z X_\mu + \partial_z\left(ik_\rho \overbracket{X^\mu_z X^\rho_w}\right)\partial_z\left(ik_\sigma \overbracket{X_{z,\mu} X^\sigma_w}\right)\right\} e^{ik\cdot X_w}: \\
&\sim -\frac{1}{\alpha'}:\left\{2\left(-\frac{\alpha'}{2}\frac{ik^\mu}{z-w}\right)\partial_z X_\mu + \left(-\frac{\alpha'}{2}\frac{ik^\mu}{z-w}\right)\left(-\frac{\alpha'}{2}\frac{ik_\mu}{z-w}\right)\right\} e^{ik\cdot X_w}: \\
&\sim \frac{\alpha' k^2}{4}\frac{V_k(w,\bar w)}{(z-w)^2} + \frac{\partial V_k(w,\bar w)}{z-w}
\end{aligned} \tag{3}$$

Here we've used the result that $ik_\sigma \overbracket{X^\mu_z X^\sigma_w} = ik^\mu(-\frac{\alpha'}{2})\ln|z-w|^2$. We see that V_k is a primary of weight $(1,1)$ iff. $\frac{\alpha' k^2}{4} = 1$, or $m^2 = -k^2 = -\frac{4}{\alpha'}$. This is the mass shell condition for the closed string tachyon (at level 0). On the other hand,

$$G_{e,k} = e_{\mu\nu} G^{\mu\nu}_k, \tag{4}$$

$$\begin{aligned}
T(z)\,G^{\mu\nu}_k(0) \sim\ & :\overbracket{T_z\,\partial X^\mu_0}\,\bar\partial X^\nu_0\, e^{ik\cdot X_0}: + \cancel{:\overbracket{T_z\,\partial X^\mu_0\,\bar\partial X^\nu_0}\, e^{ik\cdot X_0}:} + :\overbracket{T_z\,\partial X^\mu_0\,\bar\partial X^\nu_0\, e^{ik\cdot X_0}}: \\
& + \cancel{:\underbracket{\overbracket{T_z\,\partial X^\mu_0}\,\bar\partial X^\nu_0}\, e^{ik\cdot X_0}:} + :\underbracket{\overbracket{T_z\,\partial X^\mu_0\,\bar\partial X^\nu_0\, e^{ik\cdot X_0}}}: \\
\sim\ & \left(\frac{1}{z^2}G^{\mu\nu}_k(0) + \frac{1}{z}:\partial^2 X^\mu_0\,\bar\partial X^\nu_0\, e^{ik\cdot X_0}:\right) + \left(\frac{\alpha' k^2}{4}\frac{1}{z^2}G^{\mu\nu}_k(0) + \frac{1}{z}:\partial X^\mu_0\,\bar\partial X^\nu_0\,\partial\, e^{ik\cdot X_0}:\right) \\
& - \frac{2}{\alpha'}\left(-\frac{\alpha'}{2}\eta^{\sigma\mu}\frac{1}{z^2}\right)\left(-\frac{\alpha'}{2}\frac{ik_\sigma}{z}\right):\bar\partial X^\nu_0\, e^{ik\cdot X_0}: \\
\sim\ & ik^\mu:\bar\partial X^\nu_0\, e^{ik\cdot X_0}:\left(-\frac{\alpha'}{2}\right)\frac{1}{z^3} + \left(1+\frac{\alpha' k^2}{4}\right)\frac{G^{\mu\nu}_k(0)}{z^2} + \frac{\partial G^{\mu\nu}_k(0)}{z},
\end{aligned} \tag{5}$$

$$\tilde T(\bar z)\,G^{\mu\nu}_k(0) \sim ik^\nu:\partial X^\mu_0\, e^{ik\cdot X_0}:\left(-\frac{\alpha'}{2}\right)\frac{1}{\bar z^3} + \left(1+\frac{\alpha' k^2}{4}\right)\frac{G^{\mu\nu}_k(0)}{\bar z^2} + \frac{\partial G^{\mu\nu}_k(0)}{\bar z}, \tag{6}$$

[1] Reference: *Polchinski*, and physics.stackexchange.com/a/389193.

Therefore, $G_{e,k}$ is a primary of weight $(1,1)$ iff. $1+\frac{\alpha' k^2}{4} = 1$ and $k^\mu e_{\mu\nu} = 0 = k^\nu e_{\mu\nu}$. The first equation gives the mass shell condition $m^2 = -k^2 = 0$ for a massless boson, while the second equation constrains the polarization to be transverse. These are the physical constraints for a massless gauge boson, which is the level 1 excitation for a bosonic closed string.

(b) The form of any primary 3-point function is completely fixed by $\mathrm{PSL}(2,\mathbb{C})$ invariance[2]. In fact, for any holomorphic $\phi_i(z_i)$ with weight h_i, by translational invariance, we have:

$$\langle \phi_1(z_1)\,\phi_2(z_2)\,\phi_3(z_3)\rangle = f(z_{12}, z_{23}, z_{31}), \quad z_{ij} = z_i - z_j, \tag{7}$$

Furthermore, scaling invariance requires that f is quasi-homogeneous:

$$\begin{aligned} z \mapsto z' = \lambda^{-1} z, \quad f \mapsto & \left\langle \lambda^{h_1}\phi_1(\lambda z_1)\,\lambda^{h_2}\phi_2(\lambda z_2)\,\lambda^{h_3}\phi_3(\lambda z_3)\right\rangle \\ &= \lambda^{h_1+h_2+h_3} f(\lambda z_{12}, \lambda z_{23}, \lambda z_{31}) \\ &= f(z_{12}, z_{23}, z_{31}), \end{aligned} \tag{8}$$

$$f = \sum_{a+b+c=\sum_i h_i} f_{abc} = \sum_{a+b+c=\sum_i h_i} \frac{C_{abc}}{z_{12}^a z_{23}^b z_{31}^c} \tag{9}$$

On the other hand, for special conformal transformation[3] $\frac{1}{\bar z} \mapsto \frac{1}{\bar z'} = \frac{1}{\bar z} + a$, we have:

$$z \longmapsto z' = \frac{1}{\frac{1}{z}+\bar a} = \frac{z}{1+z\bar a} = w(z), \quad \frac{\partial z}{\partial z'} = \frac{1}{(1-z\bar a)^2} = \frac{1}{\kappa^2}, \quad z_{ij} = \frac{z'_{ij}}{\kappa_i \kappa_j}, \tag{10}$$

$$f \longmapsto f\Big(w^{-1}(z_{12}), w^{-1}(z_{23}), w^{-1}(z_{31})\Big)\frac{1}{\kappa_1^{2h_1}\kappa_2^{2h_2}\kappa_3^{2h_3}} = f(z_{12}, z_{23}, z_{31}), \tag{11}$$

$$f_{abc}\Big(w^{-1}(z_{12}), w^{-1}(z_{23}), w^{-1}(z_{31})\Big) = f_{abc}(z_{12}, z_{23}, z_{31})\,\kappa_1^{c+a}\kappa_2^{a+b}\kappa_3^{b+c}, \tag{12}$$

We see that f is invariant under special conformal transformation iff. $f = f_{abc}$ where:

$$c+a = 2h_1, \quad a+b = 2h_2, \quad b+c = 2h_3, \tag{13}$$

$$\text{i.e.} \quad a = h_1+h_2-h_3, \quad b = h_2+h_3-h_1, \quad c = h_3+h_1-h_2, \tag{14}$$

In the above discussions we've restricted ϕ_i to be holomorphic; for *spin-less* $\phi_i = \phi_i(z,\bar z)$, $h_i = \tilde h_i$, $\Delta_i = h_i + \tilde h_i$, the holomorphic and anti-holomorphic contributions can be nicely combined, and we have:

$$f = \frac{C}{|z_{12}|^{2a}|z_{23}|^{2b}|z_{31}|^{2c}}, \tag{15}$$

$$2a = \Delta_1+\Delta_2-\Delta_3, \quad 2b = \Delta_2+\Delta_3-\Delta_1, \quad 2c = \Delta_3+\Delta_1-\Delta_2, \tag{16}$$

$$\langle V_{k_1}(z_1,\bar z_1)\,V_{k_2}(z_2,\bar z_2)\,G_{e,k_3}(z_3,\bar z_3)\rangle = \frac{A(k_1,k_2,e)}{|z_{12}|^2|z_{23}|^2|z_{31}|^2} \tag{17}$$

[2]Reference: Blumenhagen, *Introduction to CFT*, and also *Di Francesco et al.*

[3]See *Di Francesco et al*, and also github.com/davidsd/ph229.

(c) Following the recipe in (a), we have:

$$
\begin{aligned}
V_{k_1}(z_1,\bar{z}_1)\,V_{k_2}(z_2,\bar{z}_2) &= :\exp\left(ik_{1,\mu} ik_{2,\nu} X_1^\mu X_2^\nu\right) e^{ik_1\cdot X_1}\, e^{ik_2\cdot X_2}: \\
&= \exp\left(\frac{\alpha'}{2}\, k_1\cdot k_2 \ln|z_{12}|^2\right) :e^{ik_1\cdot X_1}\, e^{ik_2\cdot X_2}: \\
&= |z_{12}|^{\alpha' k_1\cdot k_2} :e^{ik_1\cdot X_1}\, e^{ik_2\cdot X_2}:
\end{aligned}
\tag{18}
$$

Apply the on-shell conditions, and we find that:

$$
\alpha' k_1\cdot k_2 = \frac{\alpha'}{2}(k_1+k_2)^2 - \frac{\alpha'}{2}k_1^2 - \frac{\alpha'}{2}k_2^2 = \frac{\alpha'}{2}(-k_3)^2 - \frac{\alpha'}{2}k_1^2 - \frac{\alpha'}{2}k_2^2 = 0-2-2 = -4 \tag{19}
$$

We are interested in the $|z_{12}|^{-2}$ term in the OPE around z_2; it will contribute to the 3-point function discussed in (b). Note that:

$$
\begin{aligned}
:e^{ik_1\cdot X_1}\, e^{ik_2\cdot X_2}: &= :\left(\cdots + \frac{1}{2}(ik_1\cdot X_1)^2 + \cdots\right) e^{ik_2\cdot X_2}: \\
&= :\left(\cdots - \frac{1}{2}\, k_1^\mu k_1^\nu \big(X_2 + z_{12}\,\partial X_2 + \bar{z}_{12}\,\bar{\partial} X_2 + \cdots\big)_\mu (\cdots)_\nu + \cdot\right) e^{ik_2\cdot X_2}: \\
&= :\left(\cdots - k_{1,\mu}k_{1,\nu}\left(z_{12}\bar{z}_{12}\,\partial X_2^\mu\,\bar{\partial} X_2^\nu\right) + \cdots\right) e^{ik_2\cdot X_2}: \\
&= \cdots - |z_{12}|^2\, k_{1,\mu}k_{1,\nu}\, G_{k_2}^{\mu\nu}(z_2,\bar{z}_2) + \cdots,
\end{aligned}
\tag{20}
$$

$$
V_{k_1}(z_1,\bar{z}_1)\,V_{k_2}(z_2,\bar{z}_2) = \cdots + \frac{O_{k_1,k_2}(z_2,\bar{z}_2)}{|z_{12}|^2} + \cdots, \tag{21}
$$

$$
O_{k_1,k_2}(z_2,\bar{z}_2) = -k_{1,\rho}k_{1,\sigma}\, G_{k_2}^{\rho\sigma}(z_2,\bar{z}_2), \tag{22}
$$

Consider the same limit: $z_1\to z_2$ of the 3-point function, and we find that:

$$
\begin{aligned}
z_1\to z_2,\quad \langle V_{k_1}(z_1,\bar{z}_1)\,V_{k_2}(z_2,\bar{z}_2)\,G_{e,k_3}(z_3,\bar{z}_3)\rangle &\to \frac{1}{|z_{12}|^2}\frac{A(k_1,k_2,e)}{|z_{23}|^4} \\
&\sim \frac{1}{|z_{12}|^2}\,\big\langle O_{k_1,k_2}(z_2,\bar{z}_2)\,G_{e,k_3}(z_3,\bar{z}_3)\big\rangle,
\end{aligned}
\tag{23}
$$

We see that in the $z_2\to z_3$ limit, we should obtain:

$$
O_{k_1,k_2}(z_2,\bar{z}_2)\,G_{e,k_3}(z_3,\bar{z}_3) = \cdots + \frac{A(k_1,k_2,e)}{|z_{23}|^4} + \cdots, \tag{24}
$$

Note that $A(k_1,k_2,e) = A(k_1,k_2,e)\,\mathbb{1}$ is simply a number; therefore, when finding $A(k_1,k_2,e)$, it is safe to ignore all (non-identity) operator contributions, as they should cancel each other. Similar to (a), we have:

$$
\begin{aligned}
G_{k_2}^{\rho\sigma}(z_2,\bar{z}_2)\,G_{k_3}^{\mu\nu}(z_3,\bar{z}_3) = \cdots &+ :\partial X_2^\rho\,\bar{\partial} X_2^\sigma\, e^{ik_2\cdot X_2}\,\partial X_3^\mu\,\bar{\partial} X_3^\nu\, e^{ik_3\cdot X_3}: \\
&+ :\partial X_2^\rho\,\bar{\partial} X_2^\sigma\, e^{ik_2\cdot X_2}\,\partial X_3^\mu\,\bar{\partial} X_3^\nu\, e^{ik_3\cdot X_3}: \\
&+ :\partial X_2^\rho\,\bar{\partial} X_2^\sigma\, e^{ik_2\cdot X_2}\,\partial X_3^\mu\,\bar{\partial} X_3^\nu\, e^{ik_3\cdot X_3}: \\
&+ :\partial X_2^\rho\,\bar{\partial} X_2^\sigma\, e^{ik_2\cdot X_2}\,\partial X_3^\mu\,\bar{\partial} X_3^\nu\, e^{ik_3\cdot X_3}: + \cdots
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
G^{\rho\sigma}_{k_2}(z_2,\bar{z}_2)\,G^{\mu\nu}_{k_3}(z_3,\bar{z}_3) \sim \cdots &+ \left(-\frac{\alpha'}{2}\eta^{\rho\mu}\frac{1}{z_{23}^2}\right)\left(-\frac{\alpha'}{2}\eta^{\sigma\nu}\frac{1}{\bar{z}_{23}^2}\right)\times 1 \\
&+ \left(-\frac{\alpha'}{2}\eta^{\rho\mu}\frac{1}{z_{23}^2}\right)\left(-\frac{\alpha'}{2}\frac{ik_3^\sigma}{\bar{z}_{23}}\right)\left(-\frac{\alpha'}{2}\frac{ik_2^\nu}{\bar{z}_{32}}\right) + (z\leftrightarrow\bar{z},\ \rho\leftrightarrow\sigma,\ \mu\leftrightarrow\nu) \\
&+ \left(-\frac{\alpha'}{2}\frac{ik_2^\rho}{z_{23}}\right)\left(-\frac{\alpha'}{2}\frac{ik_2^\sigma}{\bar{z}_{23}}\right)\left(-\frac{\alpha'}{2}\frac{ik_3^\mu}{z_{32}}\right)\left(-\frac{\alpha'}{2}\frac{ik_3^\nu}{\bar{z}_{32}}\right) + \cdots
\end{aligned}
$$

$$
\begin{aligned}
O_{k_1,k_2}(z_2,\bar{z}_2)\,G^{\mu\nu}_{k_3}(z_3,\bar{z}_3) \sim \cdots &- k_1^\mu k_1^\nu\left(\frac{\alpha'^2}{4}\right)\frac{1}{|z_{23}|^4} \\
&- i^2(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu)\left(\frac{\alpha'}{2}(k_1\cdot k_3)\right)\left(\frac{\alpha'^2}{4}\right)\frac{1}{|z_{23}|^4} \\
&- i^4 k_3^\mu k_3^\nu\left(\frac{\alpha'}{2}(k_1\cdot k_2)\right)^2\left(\frac{\alpha'^2}{4}\right)\frac{1}{|z_{23}|^4} + \cdots
\end{aligned}
\tag{26}
$$

Again, apply the on-shell conditions, and we find that:

$$
\frac{\alpha'}{2}k_1\cdot k_2 = -2,\quad \frac{\alpha'}{2}k_1\cdot k_3 = -\frac{\alpha'}{2}k_1\cdot(k_1+k_2) = -\frac{\alpha'}{2}k_1^2 - \frac{\alpha'}{2}k_1\cdot k_2 = -2-(-2) = 0, \tag{27}
$$

$$
\begin{aligned}
A(k_1,k_2,e) &= -\frac{\alpha'^2}{4}\left(4\cancel{e_{\mu\nu}k_3^\mu k_3^\nu} + e_{\mu\nu}k_1^\mu k_1^\nu\right) = -\frac{\alpha'^2}{4}e_{\mu\nu}k_1^\mu k_1^\nu \\
&= -\frac{\alpha'^2}{4}e_{\mu\nu}(k_2+k_3)^\mu(k_2+k_3)^\nu = -\frac{\alpha'^2}{4}e_{\mu\nu}k_2^\mu k_2^\nu \\
&= -\frac{\alpha'^2}{8}e_{\mu\nu}\left(k_1^\mu k_1^\nu + k_2^\mu k_2^\nu\right) \\
&= -\frac{\alpha'^2}{8}e_{\mu\nu}\Big(k_{12}^\mu k_{12}^\nu + (k_1^\mu k_2^\nu + k_1^\nu k_2^\mu)\Big),
\end{aligned}
\tag{28}
$$

On the other hand,

$$
0 = e_{\mu\nu}k_3^\mu k_3^\nu = e_{\mu\nu}(k_1+k_2)^\mu(k_1+k_2)^\nu = e_{\mu\nu}\Big(k_{12}^\mu k_{12}^\nu + 2(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu)\Big) \tag{29}
$$

$$
A(k_1,k_2,e) = -\frac{\alpha'^2}{8}e_{\mu\nu}k_{12}^\mu k_{12}^\nu\left(1-\frac{1}{2}\right) = -\frac{\alpha'^2}{16}e_{\mu\nu}\,k_{12}^\mu k_{12}^\nu \tag{30}
$$

$\boxed{2}$ Strings Scattering Off a Heavy Particle:

A heavy particle can be modeled by some D0-brane with Neumann boundary condition in the X_0 direction[4]. The scattering of a closed string tachyon off the heavy particle can then be computed via a disc diagram with two insertions.

(a) The conformal Killing group (CKG) of the disc is $\mathrm{PSL}(2,\mathbb{R})$. It is a 3 dimensional $\mathbb{R}$ Lie group, generated by 3 conformal Killing vectors (CKV's); therefore, it is possible to partially fix the positions of the two insertions V_1, V_2. On the upper half plane, this can be implemented by putting z_1, z_2 on the imaginary axis, with z_2 fixed and z_1 integrated[5]:

$$
\mathcal{A} = g_c^2 e^{-\lambda}\int_0^{z_2}\mathrm{d}z_1\left\langle :c_1^x e^{ik_1\cdot X_1}::c_2\tilde{c}_2\,e^{ik_2\cdot X_2}:\right\rangle,\quad z_2 = i,\quad z_1 = iy,\quad y\in[0,1] \tag{31}
$$

[4]Reference: arXiv:hep-th/9611214, arXiv:hep-th/9605168, and *Polchinski*.

[5]Reference: arXiv:0812.4408. I would like to thank Lucy Smith for pointing this out.

Here c^x comes from the CKV that brings $z_1 \to iy$. On the disc this can be taken to be a rotation around z_2; when mapped to the upper half plane and at around the imaginary axis, this is simply a translation along the $x = \frac{1}{2}(z+\bar{z})$ direction[6], i.e.

$$
\text{CKV}:\ \partial_x = \delta_x^a\,\partial_a \quad \Longrightarrow \quad \text{Ghost}:\ c^x, \tag{32}
$$

$$
c^x\partial_x + c^y\partial_y = c^z\partial_z + c^{\bar{z}}\partial_{\bar{z}}, \quad c^x = \frac{1}{2}\big(c^z + c^{\bar{z}}\big) = \frac{1}{2}\big(c(z) + \tilde{c}(\bar{z})\big), \tag{33}
$$

The ghost contribution is then:

$$
\begin{aligned}
\big\langle c_1^x c_2 \tilde{c}_2 \big\rangle = \big\langle c^x(z_1)\, c(z_2)\, \tilde{c}(\bar{z}_2)\big\rangle &= \frac{1}{2}\Big(\big\langle c(z_1)\, c(z_2)\, \tilde{c}(\bar{z}_2)\big\rangle + \big\langle \tilde{c}(z_1)\, c(z_2)\, \tilde{c}(\bar{z}_2)\big\rangle\Big) \\
&= \frac{1}{2}\Big(\big\langle c(z_1)\, c(z_2)\, c(z_2')\big\rangle + \big\langle c(z_1')\, c(z_2)\, c(z_2')\big\rangle\Big), \quad z' = \bar{z}, \\
&= \frac{C^g_{D^2}}{2}\,(z_{12} z_{12'} z_{22'} + z_{1'2} z_{1'2'} z_{22'}), \quad z_1, z_2 \in i\mathbb{R}, \\
&= 2C^g_{D^2}\big(z_1^2 - z_2^2\big) z_2
\end{aligned} \tag{34}
$$

On the other hand, the $e^{ik_j\cdot X_j}$ contribution is very similar to what we compute in [1], except that now we should be careful about the boundary conditions of X^μ on the upper half plane, which affect the XX contraction in the formulae. For Neumann boundary condition: $\partial_y X^0 = 0$, the half-plane propagator from z' can be constructed with an image at $\bar{z}'$ with *the same charge*, i.e. we have:

$$
\overbracket{X_1^0 X_2^0} = -\frac{\alpha'}{2}\,\eta^{00}\ln|z_1 - z_2|^2 - \frac{\alpha'}{2}\,\eta^{00}\ln|z_1 - \bar{z}_2|^2 \tag{35}
$$

While for Dirichlet boundary $X^i = \text{const}$, we can always select the origin so that $X^i = 0$, and in this case the image should have *the opposite charge*, i.e.

$$
\overbracket{X_1^i X_2^j} = -\frac{\alpha'}{2}\,\delta^{ij}\ln|z_1 - z_2|^2 + \frac{\alpha'}{2}\,\delta^{ij}\ln|z_1 - \bar{z}_2|^2, \tag{36}
$$

$$
\begin{aligned}
\Longrightarrow \quad :e^{ik_1\cdot X_1}::e^{ik_2\cdot X_2}: &= \exp\left(ik_{1,\mu} ik_{2,\nu}\overbracket{X_1^\mu X_2^\nu}\right) :e^{ik_1\cdot X_1}\, e^{ik_2\cdot X_2}: \\
&= |z_{12}|^{\alpha' k_1\cdot k_2}\,|z_{1\bar{2}}|^{\alpha'\left(-k_1^0 k_2^0 - \delta_{ij} k_1^i k_2^j\right)} :e^{ik_1\cdot X_1}\, e^{ik_2\cdot X_2}:
\end{aligned} \tag{37}
$$

Before further calculations, we note that the normal ordering defined here on D^2 differs from that on the usual $\mathbb{C}^2$; in fact, there are also self-contractions with image charge[7]:

$$
\overbracket{X^\mu(z,\bar{z})\, X^\nu}(\bar{z}, z) = G_r^{\mu\nu}(z,\bar{z}) = \mp\frac{\alpha'}{2}\,\eta^{\mu\nu}\ln|z - \bar{z}|^2, \tag{38}
$$

$$
\Longrightarrow \quad \Big\langle :e^{ik_1\cdot X_1}\, e^{ik_2\cdot X_2}: \Big\rangle_{D^2} = \Big\langle :e^{ik_1\cdot X_1}\, e^{ik_2\cdot X_2}: \Big\rangle_{\mathbb{C}^2} \exp\left(\frac{1}{2}\sum_n ik_{n,\mu} ik_{n,\nu}\overbracket{X_n^\mu X_n^\nu}\right), \quad n = 1, 2 \tag{39}
$$

The "$\mp$" sign choice depends on the boundary condition.

---

[6]Reference: *Polchinski*, Chapter 5 & 6.

[7]This is very much similar to the torus situation, where we also have to consider self-contractions with image charges. More rigorous discussion of G^r is given in *Polchinski*.

Therefore,

$$
\begin{aligned}
\Big\langle :e^{ik_1\cdot X_1}::e^{ik_2\cdot X_2}:\Big\rangle_{D^2} &= \Big\langle :e^{ik_1\cdot X_1}\,e^{ik_2\cdot X_2}:\Big\rangle_{\mathbb{C}^2} \exp\left(ik_{1,\mu}ik_{2,\nu}\overbracket{X_1^\mu X_2^\nu}\right)\exp\left(\frac{1}{2}\sum_n ik_{n,\mu}ik_{n,\nu}\overbracket{X_n^\mu X_n^\nu}\right)\\
&= \Big\langle :e^{ik_1\cdot X_1}\,e^{ik_2\cdot X_2}:\Big\rangle_{\mathbb{C}^2} \exp\left(\frac{1}{2}\sum_{m,n} ik_{m,\mu}ik_{n,\nu}\overbracket{X_m^\mu X_n^\nu}\right)\\
&= \Big\langle :e^{ik_1\cdot X_1}\,e^{ik_2\cdot X_2}:\Big\rangle_{\mathbb{C}^2} |z_{12}|^{\alpha' k_1\cdot k_2}|z_{1\bar{2}}|^{\alpha'(-k_1^0k_2^0-\mathbf{k}_1\cdot\mathbf{k}_2)}\prod_n |z_{n\bar{n}}|^{\frac{\alpha'}{2}(-(k_n^0)^2-\mathbf{k}_n^2)}
\end{aligned}
\tag{40}
$$

Note that X^i has no zero mode due to the Dirichlet boundary, hence $\int \mathcal{D}X$ gives a delta function in only the Neumann direction: $\delta\big(k_1^0+k_2^0\big)$. Physically, this means that only the energy is conversed; the momentum k^i is not conserved since the heavy D0-brane does not recoil. It is therefore convenient to define these on shell variables:

$$
s=\omega^2=(k_1^0)^2=(k_2^0)^2,\quad t=-(\mathbf{k}_1+\mathbf{k}_2)^2=-\mathbf{k}_1^2-\mathbf{k}_2^2-2\mathbf{k}_1\cdot\mathbf{k}_2=2\left(-\omega^2-\mathbf{k}_1\cdot\mathbf{k}_2-\frac{4}{\alpha'}\right),
\tag{41}
$$

$$
\mathbf{k_1}\cdot\mathbf{k_2}=-\frac{t}{2}-\omega^2-\frac{4}{\alpha'},\quad k_1\cdot k_2=-\omega(-\omega)+\mathbf{k_1}\cdot\mathbf{k_2}
\tag{42}
$$

Here we've used the on-shell condition: $m^2=-k^2=\omega^2-\mathbf{k}^2=-\frac{4}{\alpha'}$ for tachyons. The previous expressions can then be simplified to:

$$
\begin{aligned}
\Big\langle :e^{ik_1\cdot X_1}::e^{ik_2\cdot X_2}:\Big\rangle_{D^2} &= \Big\langle :e^{ik_1\cdot X_1}\,e^{ik_2\cdot X_2}:\Big\rangle_{\mathbb{C}^2}|z_{12}|^{-\frac{\alpha' t}{2}-4}|z_{1\bar{2}}|^{+\frac{\alpha' t}{2}+4+2\alpha'\omega^2}\prod_n|2z_n|^{-\alpha'\omega^2-2}\\
&= iC_{D^2}^X\,2\pi\,\delta\big(k_1^0+k_2^0\big)\,|z_{12}|^{-\frac{\alpha' t}{2}-4}|z_{1\bar{2}}|^{+\frac{\alpha' t}{2}+4+2\alpha'\omega^2}\prod_n|2z_n|^{-\alpha'\omega^2-2}\\
&= iC_{D^2}^X\,2\pi\,\delta\big(k_1^0+k_2^0\big)\,f\big(|z_{12}|,|z_{1\bar{2}}|,|z_1|,|z_2|\big),
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
\mathcal{A} &= g_c^2\underline{e^{-\lambda}}\cdot i\underline{C_{D^2}^X}\,2\pi\,\delta\big(k_1^0+k_2^0\big)\cdot 2\underline{C_{D^2}^g}\int_0^{z_2}\mathrm{d}z_1\,\left(z_1^2-z_2^2\right)z_2\,f\big(|z_{12}|,|z_{1\bar{2}}|,|z_1|,|z_2|\big)\\
&= g_c^2\underline{C_{D^2}}\,2\pi\,\delta\big(k_1^0+k_2^0\big)\cdot 2i\int_0^1 i\,\mathrm{d}y\,\left((iy)^2-i^2\right)i\cdot f\big(1-y,1+y,2y,2\big)\\
&= -ig_c^2C_{D^2}\,2\pi\,\delta\big(k_1^0+k_2^0\big)\cdot 2\cdot 2^{-2\alpha'\omega^2-4}\int_0^1\mathrm{d}y\,(1-y^2)\,f\big(1-y,1+y,y,1\big),
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
\int_0^1\mathrm{d}y\,(1-y^2)\,f\big(1-y,1+y,y,1\big) &= \int_0^1\mathrm{d}y\,(1-y)^{-\frac{\alpha' t}{2}-4+1}(1+y)^{+\frac{\alpha' t}{2}+4+2\alpha'\omega^2+1}y^{-\alpha'\omega^2-2}\\
&= \int_0^1\mathrm{d}y\,y^{a-1}(1-y)^{2b-1}(1+y)^{-2a-2b+1},\quad t=\frac{1-y}{1+y},\\
&= -2^{1-2a}\int_0^1\mathrm{d}t\,(-t)^{2b-1}(1-t^2)^{a-1}\\
&= 2^{-2a}\int_0^1\mathrm{d}(t^2)\,(t^2)^{b-1}(1-t^2)^{a-1}\\
&= 2^{-2a}B\left(a=-\alpha'\omega^2-1,\,b=-\frac{\alpha' t}{4}-1\right)
\end{aligned}
\tag{45}
$$

Here $B(a,b)=\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$ is the Euler Beta function.

Putting everything together, we obatin:

$$\begin{aligned}\mathcal{A} &= -ig_c^2 C_{D^2}\, 2\pi\, \delta\big(k_1^0 + k_2^0\big) \cdot \frac{1}{2}\, B\left(-\alpha'\omega^2 - 1,\, -\frac{\alpha' t}{4} - 1\right) \\ &= -ig_c^2 C_{D^2}\, \pi\, \delta\big(k_1^0 + k_2^0\big)\, B\left(-\alpha'\omega^2 - 1,\, -\frac{\alpha' t}{4} - 1\right)\end{aligned} \tag{46}$$

In fact C_{D^2} can be further computed by path integral or by comparing physical results. Here we settle for this generic coefficient since it's already enough for our following discussions[8].

(b) The Regge limit is found by taking the high energy limit while keeping the momentum transfer fixed; in this case it is achieved by:

$$\text{Regge:} \quad s = \omega^2 \to \infty, \quad t = -(\mathbf{k}_1 + \mathbf{k}_2)^2 \text{ fixed,} \tag{47}$$

$$\begin{aligned}\mathcal{A} \propto B\left(a = -\alpha' s - 1,\, b = -\frac{\alpha' t}{4} - 1\right) &= \frac{\Gamma(-\alpha' s - 1)}{\Gamma\big(-\alpha' s - \frac{\alpha' t}{4} - 2\big)}\, \Gamma\left(-\frac{\alpha' t}{4} - 1\right) \\ &\sim \left\{e\left(\alpha' s + \frac{\alpha' t}{4} + 3\right)\right\}^{\frac{\alpha' t}{4}+1} \Gamma\left(-\frac{\alpha' t}{4} - 1\right) \\ &\sim \left(e\alpha'\omega^2\right)^{\frac{\alpha' t}{4}+1} \Gamma\left(-\frac{\alpha' t}{4} - 1\right) \\ &\sim \left(\omega^2\right)^{\frac{\alpha' t}{4}+1} \Gamma\left(-\frac{\alpha' t}{4} - 1\right)\end{aligned} \tag{48}$$

Here we've used the Stirling's approximation[9]: $\ln\Gamma(z+1) = \ln z! \sim z\ln z - z$. On the other hand, the hard scattering limit is found by keeping the scattering angle fixed, i.e.

$$\text{Hard scattering:} \quad s = \omega^2 \to \infty, \quad (t/s) \equiv \lambda \text{ fixed,} \tag{49}$$

$$\mathcal{A} \propto B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} \sim \exp\left\{-\alpha'\left(s\ln(\alpha' s) + \tfrac{t}{4}\ln\tfrac{\alpha' t}{4} + \tfrac{u}{4}\ln\tfrac{\alpha' u}{4}\right)\right\}, \tag{50}$$

$$s = \omega^2 = (k_1^0)^2 = (k_2^0)^2, \quad t = -(\mathbf{k}_1 + \mathbf{k}_2)^2, \quad u = -(\mathbf{k}_1 - \mathbf{k}_2)^2, \tag{51}$$

$$s + \frac{t}{4} + \frac{u}{4} = -\frac{4}{\alpha'}, \tag{52}$$

Here we've introduced an additional u variable, and we see that the result is symmetric under $t \leftrightarrow u$. We find that the amplititude exhibits similar limits as the Veneziano amplititude.

(c)

[8] And I have run out of time and energy.

[9] For the validity of Stirling's approximation when $z \in \mathbb{C}$ and $|z| \to \infty$, see `Wikipedia:` *Stirling's formula for the gamma function*.

✒ PAST WORK, AS TEMPLATE ✒

$\boxed{3}$ Strings on Curved Space:

$$S = \frac{1}{4\pi\alpha'}\int_M \mathrm{d}^2\sigma\,\sqrt{g}\,\Big(i\epsilon^{ab}B_{\mu\nu}(X)\,\partial_a X^\mu \partial_b X^\nu + \cdots\Big), \tag{53}$$

$$T^a{}_a = -\frac{1}{2\alpha'}\,\beta^G_{\mu\nu}\,g^{ab}\partial_a X^\mu \partial_b X^\nu + \cdots\,, \tag{54}$$

$$\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} - \frac{1}{4}\,\alpha' H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega} + \cdots + \mathcal{O}\big(\alpha'^2\big) \tag{55}$$

We want to verify the coefficient of $\alpha' H^2$ term in $\beta^G_{\mu\nu}$; for convenience we've omitted non-related terms in the above expressions.

Note that at $\mathcal{O}(\alpha')$ such term does not depend on the metric $G_{\mu\nu}$, and it depends only on the field strength $H = \mathrm{d}B$, not the potential B, hence it's safe to assume:

$$G_{\mu\nu} = \eta_{\mu\nu}, \quad B_{\mu\nu} = \frac{1}{3}H_{\mu\nu\rho}X^\rho, \quad H = \text{const}, \tag{56}$$

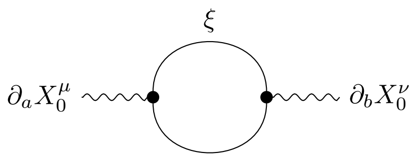
$$i\epsilon^{ab}B_{\mu\nu}(X)\,\partial_a X^\mu \partial_b X^\nu = \frac{i}{3}H_{\mu\nu\rho}\,X^\rho \epsilon^{ab}\partial_a X^\mu \partial_b X^\nu, \tag{57}$$

We consider small perturbation away from the classical saddle: $X = X_0 + \xi$, then the 1-loop effective action is obtained by integrating over $\mathcal{O}\big(\xi^2\big)$ terms in the perturbed action[10]:

$$\Gamma^{(1)}[X_0] = -\ln\int \mathscr{D}\xi\, e^{-S^{(2)}[X_0,\xi]}, \tag{58}$$

$$\begin{aligned}
\mathcal{L}^{(2)} &= \frac{i}{3}H_{\mu\nu\rho}\,\epsilon^{ab}\Big(\xi^\rho\,\partial_a X_0^\mu\,\partial_b\xi^\nu + \xi^\rho\,\partial_a\xi^\mu\,\partial_b X_0^\nu + X_0^\rho\,\partial_a\xi^\mu\,\partial_b\xi^\nu\Big)\\
&\sim \frac{i}{3}H_{\mu\nu\rho}\,\epsilon^{ab}\Big(\xi^\rho\,\partial_a X_0^\mu\,\partial_b\xi^\nu - \xi^\rho\,\partial_a X_0^\nu\,\partial_b\xi^\mu - \xi^\mu\,\partial_a X_0^\rho\,\partial_b\xi^\nu\Big)\\
&= \frac{i}{3}H_{\mu\nu\rho}\,\epsilon^{ab}\cdot 3\xi^\rho\,\partial_a X_0^\mu\,\partial_b\xi^\nu\\
&= iH_{\mu\nu\rho}\,\epsilon^{ab}\,\partial_a X_0^\mu\,(\xi^\rho\partial_b\xi^\nu)
\end{aligned} \tag{59}$$

Here we've used the anti-symmetric properties of $H_{\mu\nu\rho}, \epsilon^{ab}$, and ignored any total derivative after integration by parts. This term introduces a cubic interaction vertex in the free background; therefore, $\Gamma^{(1)}$ can be expressed in the following diagram[11]:



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[10]Reference: Prof. Xi Yin's String Notes, see also `arXiv:0812.4408`.

[11]References:
- David Tong, *String Theory*;
- Callan & Thorlacius, *Sigma Models and String Theory*;
- Timo Weigand, *Introduction to String Theory*.

$$\sim \frac{1}{2!}\left(\frac{1}{\alpha'}\right)^2 \int \mathrm{d}^2p \left(iH_{\mu\nu\rho}\,\epsilon^{ab}\,\partial_a X_0^\mu\, ip_b\right)\frac{2}{p^4}\left(-\frac{\alpha'}{2}\right)^2\left(iH_{\mu'}{}^{\nu\rho}\,\epsilon^{a'b'}\,\partial_{a'}X_0^{\mu'}\, ip_{b'}\right) \tag{60}$$

$$= \frac{2}{2!}\left(\frac{1}{\alpha'}\right)^2\left(-\frac{\alpha'}{2}\right)^2 H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega}\partial_a X_0^\mu\,\partial_b X_0^\nu \int \mathrm{d}^2p\,\frac{p^2 g^{ab} - p^a p^b}{p^4} \tag{61}$$

$$= \frac{2}{2!}\left(-\frac{1}{2}\right)^2 H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega}\partial_a X_0^\mu\,\partial_b X_0^\nu\left(\frac{1}{2}\,g^{ab}\right)\int \mathrm{d}^2p\,\frac{1}{p^2} \tag{62}$$

$$= \frac{2}{2!}\left(-\frac{1}{2}\right)^2\left(\frac{1}{2}\right) H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega}\partial_a X_0^\mu\,\partial_b X_0^\nu\, g^{ab}\int \mathrm{d}^2p\,\frac{1}{p^2} \tag{63}$$

$$= \frac{1}{8}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega}g^{ab}\partial_a X_0^\mu\,\partial_b X_0^\nu \int \mathrm{d}^2p\,\frac{1}{p^2} \tag{64}$$

Here the $\left(\frac{1}{\alpha'}\right)^2$ coefficient comes from the vertices, while $\left(-\frac{\alpha'}{2}\right)^2$ comes from the propagators. The $p^a p^b$ integral provides an additional $(\frac{1}{2})$ factor. The overall normalization is chosen to match the $\alpha' R_{\mu\nu}$ coefficient in $\beta^G_{\mu\nu} \subset T^a{}_a$, which is $\frac{1}{1!}\times(-\frac{1}{2})\times 1 = -\frac{1}{2}$. Therefore, we have:

$$T^a{}_a \supset \frac{1}{8}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega}g^{ab}\partial_a X_0^\mu\,\partial_b X_0^\nu, \tag{65}$$

$$\beta^G_{\mu\nu} \supset -\frac{1}{4}\,\alpha' H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega} \tag{66}$$

■

$\boxed{4}$ **Classical Solutions of 11D SUGRA:** Following the convention of *Polchinski*, we have bosonic action:

$$S = \frac{1}{2\kappa^2}\int\left(\mathrm{d}^{11}x\,\sqrt{-g}\,\mathcal{R} - \frac{1}{2}\,G\wedge *G - \frac{1}{6}\,C\wedge G\wedge G\right), \tag{67}$$

Here $G = \mathrm{d}C$: a 4-form field. In components, the numerical coefficients would be $\frac{1}{2}\mapsto\frac{1}{2\times 4!} = \frac{1}{48}$, and $\frac{1}{6}\mapsto\frac{1}{6\times 3!\times 4!\times 4!} = \frac{1}{20736}$.

Variation of the action yields the EOMs of our theory[12]; Note that:

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\,\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu} \tag{68}$$

$\frac{\delta S}{\delta g^{\mu\nu}}$ is easier to compute in components; note that the $C\wedge G\wedge G$ term does not depend on $g^{\mu\nu}$, therefore it does not contribute to the EOM. We have the usual Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{69}$$

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{\kappa^2}\left(\frac{4}{48}G_{\mu\sigma_1\sigma_2\sigma_3}G_\nu{}^{\sigma_1\sigma_2\sigma_3} - \frac{1}{2}\,g_{\mu\nu}\cdot\frac{1}{48}\,G^{\sigma_1\sigma_2\sigma_3\sigma_4}G_{\sigma_1\sigma_2\sigma_3\sigma_4}\right) \\ &= \frac{1}{12\kappa^2}\left(G_{\mu\sigma_1\sigma_2\sigma_3}G_\nu{}^{\sigma_1\sigma_2\sigma_3} - \frac{1}{8}\,g_{\mu\nu}\,G^{\sigma_1\sigma_2\sigma_3\sigma_4}G_{\sigma_1\sigma_2\sigma_3\sigma_4}\right) \end{aligned} \tag{70}$$

[12]Reference: arXiv:hep-th/9912164. I would like to thank *Lucy Smith* for many helpful discussions.

On the other hand, $\frac{\delta S}{\delta C}$ is best carried out using differential forms:

$$\begin{aligned}
0 = \delta_C S &= -\frac{1}{2\kappa^2}\int \left(\delta G \wedge * G + \frac{1}{6}\Big(\delta C \wedge G \wedge G - 2C \wedge \delta G \wedge G\Big)\right) \\
&= -\frac{1}{2\kappa^2}\int \left(\delta(\mathrm{d}C) \wedge * G + \frac{1}{6}\big(\delta C \wedge G \wedge G + 2\,\delta(\mathrm{d}C) \wedge C \wedge G\big)\right) \\
&= -\frac{1}{2\kappa^2}\int \left(-(-1)^3\,\delta C \wedge \mathrm{d} * G + \frac{1}{6}\big(\delta C \wedge G \wedge G - 2\,(-1)^3\,\delta C \wedge \mathrm{d}(C \wedge G)\big)\right) \\
&= -\frac{1}{2\kappa^2}\int \delta C \wedge \left(\mathrm{d} * G + \frac{1}{6}\Big(G \wedge G + 2\big(G \wedge G - C \wedge \cancel{\mathrm{d}^2 C}\big)\Big)\right) \\
&= -\frac{1}{2\kappa^2}\int \delta C \wedge \left(\mathrm{d} * G + \frac{1}{2}\,G \wedge G\right),
\end{aligned} \tag{71}$$

$$\mathrm{d} * G + \frac{1}{2}\,G \wedge G = 0 \tag{72}$$

(a) We hope to find a spacetime solution which is *maximally symmetric* in *some* directions; assume that these directions form a d-dimensional sub-manifold $\mathcal{M}_d$ with:

$$\begin{aligned}
\text{Coordinates:}\quad & x^{\mu'},\ \mu' \in \Delta \subset \{0, 1, \cdots, 11\}, \\
\text{Induced metric:}\quad & g' = g|_{\mathcal{M}_d}
\end{aligned} \tag{73}$$

The entire spacetime is then a direct product: $\mathcal{M}_d \times \widetilde{\mathcal{M}}_{11-d}$. For $\mathcal{M}_d$ to be maximally symmetric, we expect that $\kappa^2 T_{\mu'\nu'} = -\Lambda g'_{\mu'\nu'}$, i.e. the G-field serves as a cosmological constant Λ. By staring at (70) we find that this can be achieved with[13]:

$$d = 4, \quad G_{\sigma_1\sigma_2\sigma_3\sigma_4} = \alpha\sqrt{|g'|}\,\epsilon_{\sigma_1\sigma_2\sigma_3\sigma_4}, \quad G^{\sigma_1\sigma_2\sigma_3\sigma_4} = \alpha\,\frac{\operatorname{sgn} g'}{\sqrt{|g'|}}\,\epsilon^{\sigma_1\sigma_2\sigma_3\sigma_4}, \quad \{\sigma_i\} \subset \Delta, \tag{74}$$

$$G_{\cdots\,\sigma\,\cdots} = 0, \quad \sigma \notin \Delta, \tag{75}$$

$$T_{\mu\nu} = (\operatorname{sgn} g')\,\frac{\alpha^2}{12\kappa^2}\left(3!\,g'_{\mu\nu} - \frac{4!}{8}\,g_{\mu\nu}\right) = (\operatorname{sgn} g')\,\frac{\alpha^2}{2\kappa^2}\left(g'_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\right), \tag{76}$$

$$\Lambda g_{\mu\nu} = \mp(\operatorname{sgn} g')\,\frac{\alpha^2}{4\kappa^2}\,g_{\mu\nu}, \quad \begin{cases} -:\ \mu = \mu', \nu = \nu' \in \Delta, & \sim \mathcal{M}_4 \\ +:\ \mu, \nu \notin \Delta, & \sim \widetilde{\mathcal{M}}_7 \end{cases} \tag{77}$$

Matter EOM is trivially satisfied due to anti-symmetricity. We see that the other component $\widetilde{\mathcal{M}}_7$ is also maximally symmetric, but with an opposite sign in its cosmological constant.

The field equations in $\mathcal{M}_4$ and $\widetilde{\mathcal{M}}_7$ are both of the form $R_{\mu\nu} \propto g_{\mu\nu}$. For $\operatorname{sgn} g' = -1$ i.e. Lorentzian signature, the solution is flat, AdS or dS, depending on the sign of Λ; for $\operatorname{sgn} g' = -1$, the solution is flat, spherical or hyperbolic. Therefore, we have:

$$\begin{aligned}
\operatorname{sgn} g' = -1, \quad \Lambda_{4,7} = \pm\frac{\alpha^2}{4\kappa^2}, \quad \mathcal{M}_4 = \mathrm{AdS}_{3,1}, \quad \widetilde{\mathcal{M}}_7 = S^7 \\
\operatorname{sgn} g' = +1, \quad \Lambda_{4,7} = \mp\frac{\alpha^2}{4\kappa^2}, \quad \mathcal{M}_4 = S^4, \quad \widetilde{\mathcal{M}}_7 = \mathrm{AdS}_{6,1}
\end{aligned} \tag{78}$$

[13] This is in fact the famous *Freund–Robin ansatz*; see `Wikipedia:` *Freund– Rubin compactification*, and also the original paper: Freund & Robin, *Dynamics of Dimensional Reduction*, 1980.

(b) Global supersymmetries of a theory with the above $\mathrm{AdS}_{4/7} \times S^{4/7}$ background are given by the solutions of:

$$0 = \delta_\eta \psi^\mu \equiv D^\mu \eta(x), \quad \eta\text{: spinor}, \tag{79}$$

$$\begin{aligned} D^\mu &= \nabla^\mu + \frac{1}{288}\, G_{\nu\rho\sigma\lambda}\big(\Gamma^{\mu\nu\rho\sigma\lambda} - 8g^{\mu\nu}\Gamma^{\rho\sigma\lambda}\big) \\ &= \nabla^\mu + \frac{1}{288}\, G_{\nu'\rho'\sigma'\lambda'}\big(\Gamma^{\mu\nu'\rho'\sigma'\lambda'} - 8g^{\mu\nu'}\Gamma^{\rho'\sigma'\lambda'}\big) \\ &= \nabla^\mu + \alpha \begin{cases} \dfrac{-8\times 3!}{288}\,(-\Gamma^\mu\gamma_5) = \dfrac{1}{6}\,\Gamma^\mu\gamma_5, \quad \mu = \mu' \in \Delta, & \sim \mathcal{M}_4 \\ \dfrac{4!}{288}\,(-\Gamma^\mu) = -\dfrac{1}{12}\,\Gamma^\mu, \quad \mu \notin \Delta, & \sim \widetilde{\mathcal{M}}_7 \end{cases} \end{aligned} \tag{80}$$

Note that we've replaced the G indices with $\mathcal{M}_4$ indices, since G vanish in $\widetilde{\mathcal{M}}_7$ directions; due to anti-symmetricity, the G-term can be reduced to simple Γ^μ multiplications according to the μ-direction[14]. Furthermore, the spin connection in ∇^μ is also block diagonalized, same as $g_{\mu\nu}$; hence there is a natural separation of variable[15]:

$$\eta = \eta'(x')\,\eta''(x''), \quad D_{\mu'}\eta' = 0, \quad D_{\mu''}\eta'' = 0, \tag{81}$$

$$\mu', \eta', x' \sim \mathcal{M}_4, \quad \mu'', \eta'', x'' \sim \widetilde{\mathcal{M}}_7, \tag{82}$$

Due to the presence of an additional Γ, $D_{\mu'}\eta' = 0$ has only 4 linearly independent solutions labeled by μ', while $D_{\mu''}\eta'' = 0$ is Spin(8) (or Spin(7, 1), depending on the signature) invariant, and has $\frac{8\times 7}{2} = 28$ linearly independent solutions[16]. Hence the total number of SUSYs is $4 + 28 = 32$, for $\mathrm{AdS}_{4/7} \times S^{4/7}$ background.

5 SUSY Sigma Models via Superspace:

$$\begin{aligned} D_{\bar\theta}\mathbf{X}^\nu &= \big(\partial_{\bar\theta} + \bar\theta\partial_{\bar z}\big)\big(X^\nu + i\theta\psi^\nu + i\bar\theta\tilde\psi^\nu + \theta\bar\theta F^\nu\big) \\ &= i\tilde\psi^\nu - \theta F^\nu + \bar\theta\,\bar\partial X^\nu - i\theta\bar\theta\,\bar\partial\psi^\nu, \\ D_\theta \mathbf{X}^\mu &= i\psi^\mu + \bar\theta F^\mu + \theta\,\partial X^\mu + i\theta\bar\theta\,\partial\tilde\psi^\mu, \end{aligned} \tag{83}$$

$$\begin{aligned} D_{\bar\theta}\mathbf{X}^\nu D_\theta\mathbf{X}^\mu &= \Big(i\tilde\psi^\nu - \theta F^\nu + \bar\theta\,\bar\partial X^\nu - i\theta\bar\theta\,\bar\partial\psi^\nu\Big)\Big(i\psi^\mu + \bar\theta F^\mu + \theta\,\partial X^\mu + i\theta\bar\theta\,\partial\tilde\psi^\mu\Big) \\ &= -\tilde\psi^\nu\psi^\mu - i\theta\Big(\tilde\psi^\nu\partial X^\mu + \psi^\mu F^\nu\Big) + i\bar\theta\Big(\psi^\mu\bar\partial X^\nu - \tilde\psi^\nu F^\mu\Big) \\ &\quad - \theta\bar\theta\Big(\bar\partial X^\nu\partial X^\mu + \tilde\psi^\nu\partial\tilde\psi^\mu - (\bar\partial\psi^\nu)\psi^\mu + F^\nu F^\mu\Big), \end{aligned} \tag{84}$$

$$\begin{aligned} G_{\mu\nu}(\mathbf{X}) &= G_{\mu\nu} + \Big(i\theta\psi^\lambda + i\bar\theta\tilde\psi^\lambda + \theta\bar\theta F^\lambda\Big)\,\partial_\lambda G_{\mu\nu} + \frac{1}{2}\Big\{i\theta\psi^\rho\partial_\rho,\, i\bar\theta\tilde\psi^\sigma\partial_\sigma\Big\}\,G_{\mu\nu} \\ &= G_{\mu\nu} + \Big(i\theta\psi^\lambda + i\bar\theta\tilde\psi^\lambda\Big)\,G_{\mu\nu,\lambda} + \theta\bar\theta\Big(F^\lambda G_{\mu\nu,\lambda} + \psi^\rho\tilde\psi^\sigma G_{\mu\nu,\rho\sigma}\Big), \end{aligned} \tag{85}$$

---

[14] Reference for Γ-matrices and spinors: *Polchinski* Vol. II, Appendix B. I'm a bit confused about all the complicated conventions, therefore the coefficients might be off by some factors...

[15] See arXiv:hep-th/9912164 for more detailed discussions.

[16] Reference: Achilleas Passias, *Aspects of Supergravity in Eleven Dimensions.*

Note that $\int \mathrm{d}^2\theta = \partial_\theta \partial_{\bar\theta}$, hence we need only focus on the $\theta\bar\theta$ term in the Lagrangian:

$$
\begin{aligned}
4\pi S_G &= \int \mathrm{d}^2 z\, \mathrm{d}^2\theta\, G_{\mu\nu}(\mathbf{X})\, D_{\bar\theta}\mathbf{X}^\mu D_\theta \mathbf{X}^\nu = \int \mathrm{d}^2 z\, \mathrm{d}^2\theta\, (-\theta\bar\theta)\Big(G_{\mu\nu}\big(\partial X^\mu \bar\partial X^\nu + \cdots\big) + \cdots\Big) \\
&= \int \mathrm{d}^2 z \,\bigg(G_{\mu\nu}\Big(\partial X^\mu \bar\partial X^\nu + \tilde\psi^\nu \partial \tilde\psi^\mu - (\bar\partial \psi^\nu)\psi^\mu + F^\nu F^\mu\Big) \\
&\qquad + \tilde\psi^\nu \psi^\mu \Big(F^\lambda G_{\mu\nu,\lambda} + \psi^\rho \tilde\psi^\sigma G_{\mu\nu,\rho\sigma}\Big) \\
&\qquad - G_{\mu\nu,\lambda}\Big(\psi^\lambda\big(\psi^\mu \bar\partial X^\nu - \tilde\psi^\nu F^\mu\big) + \tilde\psi^\lambda\big(\tilde\psi^\nu \partial X^\mu + \psi^\mu F^\nu\big)\Big)\bigg)
\end{aligned} \tag{86}
$$

Similar result holds for the B contribution S_B. We see that there is no ∂F term in the action, hence F is not dynamical and can be integrated out; we have:

$$
0 = \delta_F S = \delta_F (S_G + S_B), \tag{87}
$$

$$
\begin{aligned}
4\pi\, \delta S_G &= \int \mathrm{d}^2 z \,\Big(2G_{\mu\nu} F^\mu\, \delta F^\nu + G_{\mu\nu,\lambda}\big(\tilde\psi^\nu \psi^\mu\, \delta F^\lambda - \tilde\psi^\nu \psi^\lambda\, \delta F^\mu - \tilde\psi^\lambda \psi^\mu\, \delta F^\nu\big)\Big) \\
&= \int \mathrm{d}^2 z \,\Big(2F_\lambda + \big(G_{\mu\nu,\lambda} - G_{\lambda\mu,\nu} - G_{\lambda\nu,\mu}\big)\, \tilde\psi^\nu \psi^\mu\Big)\, \delta F^\lambda \\
&= \int \mathrm{d}^2 z \,\Big(2F_\lambda - 2\Gamma_{\lambda\mu\nu} \tilde\psi^\nu \psi^\mu\Big)\, \delta F^\lambda \,,
\end{aligned} \tag{88}
$$

$$
4\pi\, \delta S_B = \int \mathrm{d}^2 z \,\Big(0 + \big(B_{\mu\nu,\lambda} + B_{\lambda\mu,\nu} + B_{\nu\lambda,\mu}\big)\, \tilde\psi^\nu \psi^\mu\Big)\, \delta F^\lambda = \int \mathrm{d}^2 z\, H_{\lambda\mu\nu} \tilde\psi^\nu \psi^\mu\, \delta F^\lambda \,,
$$

$$
F_\lambda = \bigg(\Gamma_{\lambda\mu\nu} - \frac{1}{2} H_{\lambda\mu\nu}\bigg) \tilde\psi^\nu \psi^\mu, \tag{89}
$$

$$
F^\lambda = \bigg(\Gamma^\lambda_{\mu\nu} - \frac{1}{2} H^\lambda_{\mu\nu}\bigg) \tilde\psi^\nu \psi^\mu, \tag{90}
$$

Here we've used the (anti-)symmetry of $G_{\mu\nu}$ and $B_{\mu\nu}$, and we adopt the convention that the Levi-Civita connection $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} = G^{\lambda\lambda'} \Gamma_{\lambda'\mu\nu}$; similar holds for $B_{\mu\nu}$ and $H^\lambda_{\mu\nu}$.

Substitute F_λ into S, collect the $\psi^0, \psi^2, \tilde\psi^2$ and $\psi^2\tilde\psi^2$ terms respectively, and we have:

$$
\begin{aligned}
4\pi S &= \int \mathrm{d}^2 z \,\bigg(\left(G_{\mu\nu} + B_{\mu\nu}\right) \partial X^\mu \bar\partial X^\nu \\
&\qquad + \left(G_{\mu\nu} + \cancel{B_{\mu\nu}}\right)\Big(\tilde\psi^\mu \partial \tilde\psi^\nu - (\bar\partial \psi^\mu)\psi^\nu\Big) \\
&\qquad - \left(G_{\mu\nu,\lambda} + B_{\mu\nu,\lambda}\right)\Big(\psi^\lambda \psi^\mu \bar\partial X^\nu + \tilde\psi^\lambda \tilde\psi^\nu \partial X^\mu\Big) \\
&\qquad + G_{\mu\nu} F^\mu F^\nu - 2\bigg(\Gamma_{\lambda\mu\nu} - \frac{1}{2} H_{\lambda\mu\nu}\bigg) \tilde\psi^\nu \psi^\mu F^\lambda \\
&\qquad + \left(G_{\mu\nu,\rho\sigma} + B_{\mu\nu,\rho\sigma}\right) \tilde\psi^\nu \psi^\mu \psi^\rho \tilde\psi^\sigma \bigg) \\
&= \int \mathrm{d}^2 z \,\bigg(\left(G_{\mu\nu} + B_{\mu\nu}\right) \partial X^\mu \bar\partial X^\nu \\
&\qquad + G_{\mu\nu}\Big(\tilde\psi^\mu \partial \tilde\psi^\nu + \psi^\mu \bar\partial \psi^\nu\Big) - \left(G_{\mu\nu,\lambda} + B_{\mu\nu,\lambda}\right)\Big(\psi^\lambda \psi^\mu \bar\partial X^\nu + \tilde\psi^\lambda \tilde\psi^\nu \partial X^\mu\Big) \\
&\qquad - F_\lambda F^\lambda + \left(G_{\mu\nu,\rho\sigma} + B_{\mu\nu,\rho\sigma}\right) \psi^\mu \psi^\rho \tilde\psi^\nu \tilde\psi^\sigma \bigg)
\end{aligned} \tag{91}
$$

Here we've performed some integration by parts to clean up the result. Note that some terms involving $B_{\mu\nu}$ vanish conveniently (up to integration by parts) due to anti-symmetricity.

The $\psi^2, \tilde{\psi}^2$ terms in the integrand can be further simplified as follows:

$$\begin{aligned}
\mathcal{L}_{\psi^2} &= G_{\mu\nu}\psi^\mu\bar{\partial}\psi^\nu - (G_{\mu\nu,\lambda} + B_{\mu\nu,\lambda})\,\psi^\lambda\psi^\mu\bar{\partial}X^\nu \\
&= G_{\mu\nu}\psi^\mu\bar{\partial}\psi^\nu - \big(G_{\mu[\nu,\lambda]} + B_{\mu[\nu,\lambda]}\big)\,\psi^\lambda\psi^\mu\bar{\partial}X^\nu \\
&= G_{\mu\nu}\psi^\mu\bar{\partial}\psi^\nu - \left(-\Gamma_{\lambda\mu\nu} + \frac{1}{2}H_{\lambda\mu\nu}\right)\psi^\lambda\psi^\mu\bar{\partial}X^\nu \\
&= G_{\mu\nu}\psi^\mu\left(\bar{\partial}\psi^\nu + \left(\Gamma^\nu_{\rho\sigma} - \frac{1}{2}H^\nu_{\rho\sigma}\right)\psi^\rho\bar{\partial}X^\sigma\right) \\
&= G_{\mu\nu}\psi^\mu\left(\bar{\partial}\psi^\nu + \left(\Gamma^\nu_{\rho\sigma} + \frac{1}{2}H^\nu_{\rho\sigma}\right)\psi^\sigma\bar{\partial}X^\rho\right) = G_{\mu\nu}\psi^\mu\bar{\mathcal{D}}\psi^\nu,
\end{aligned} \tag{92}$$

$$\begin{aligned}
\mathcal{L}_{\tilde{\psi}^2} &= G_{\mu\nu}\tilde{\psi}^\mu\partial\tilde{\psi}^\nu - (G_{\mu\nu,\lambda} + B_{\mu\nu,\lambda})\,\tilde{\psi}^\lambda\tilde{\psi}^\nu\partial X^\mu \\
&= G_{\mu\nu}\tilde{\psi}^\mu\left(\partial\tilde{\psi}^\nu + \left(\Gamma^\nu_{\rho\sigma} - \frac{1}{2}H^\nu_{\rho\sigma}\right)\tilde{\psi}^\sigma\partial X^\rho\right) = G_{\mu\nu}\tilde{\psi}^\mu\mathcal{D}\tilde{\psi}^\nu,
\end{aligned}$$

For the $\psi^2\tilde{\psi}^2$ term, recall that $R_{\mu\nu\rho\sigma} = e_\mu[\nabla_\rho, \nabla_\sigma]\,e_\nu$, $\nabla_\sigma e_\nu = e_\lambda\Gamma^\lambda_{\sigma\nu}$, and we have:

$$\begin{aligned}
\mathcal{L}_{\psi^2\tilde{\psi}^2} &= \psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma\left(G_{\mu\rho,\nu\sigma} + B_{\mu\rho,\nu\sigma} + \left(\Gamma_{\lambda\mu\rho} - \frac{1}{2}H_{\lambda\mu\rho}\right)\left(\Gamma^\lambda_{\nu\sigma} - \frac{1}{2}H^\lambda_{\nu\sigma}\right)\right) \\
&= \psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma\left(G_{\mu\rho,\nu\sigma} + \Gamma_{\lambda\mu\rho}\Gamma^\lambda_{\nu\sigma} + B_{\mu\rho,\nu\sigma} - \frac{1}{2}\left(\Gamma^\lambda_{\mu\rho}H_{\lambda\nu\sigma} + \Gamma^\lambda_{\nu\sigma}H_{\lambda\mu\rho}\right) + \frac{1}{4}H^\lambda_{\mu\rho}H_{\lambda\nu\sigma}\right) \\
&= \mathcal{L}_G + \mathcal{L}_B + \frac{1}{4}H^\lambda_{\mu\rho}H_{\lambda\nu\sigma}\,\psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma,
\end{aligned} \tag{93}$$

$$\begin{aligned}
\mathcal{L}_G &= \psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma\left(G_{\mu\rho,\nu\sigma} + \Gamma_{\lambda\mu\rho}\Gamma^\lambda_{\nu\sigma}\right) \\
&= \psi^{[\mu}\psi^{\nu]}\tilde{\psi}^{[\rho}\tilde{\psi}^{\sigma]}\left(G_{\mu\rho,\nu\sigma} + \Gamma_{\lambda\mu\rho}\Gamma^\lambda_{\nu\sigma}\right) \\
&= \frac{1}{2}\,\psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma\left\{\left(\frac{1}{2}\left(G_{\mu\rho,\nu\sigma} - G_{\mu\sigma,\nu\rho}\right) + \Gamma_{\lambda\mu\rho}\Gamma^\lambda_{\nu\sigma}\right) - \left(\cdots\right)_{\rho\leftrightarrow\sigma}\right\} \\
&= \frac{1}{2}R_{\mu\nu\rho\sigma}\,\psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma,
\end{aligned} \tag{94}$$

$$\mathcal{L}_B = \frac{1}{2}\,\nabla_\rho H_{\mu\nu\sigma}\,\psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma,$$

Therefore, the total action is:

$$\begin{aligned}
S = \frac{1}{4\pi}\int \mathrm{d}^2 z\,\Bigg(&(G_{\mu\nu} + B_{\mu\nu})\,\partial X^\mu\bar{\partial}X^\nu \\
&+ G_{\mu\nu}\left(\tilde{\psi}^\mu\mathcal{D}\tilde{\psi}^\nu + \psi^\mu\bar{\mathcal{D}}\psi^\nu\right) \\
&+ \left(\frac{1}{2}R_{\mu\nu\rho\sigma} + \frac{1}{2}\,\nabla_\rho H_{\mu\nu\sigma} + \frac{1}{4}H^\lambda_{\mu\rho}H_{\lambda\nu\sigma}\right)\psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma\Bigg)
\end{aligned} \tag{95}$$

$\boxed{6}$ Mixed Anomaly Between Diffeomorphism and Axial $U(1)$ Symmetry:

(a) Calculations of such anomaly is (schematically) similar to the usual axial anomaly; instead of the A_μ legs, we now have two $h_{\mu\nu}$ legs in the triangular diagram.

Again we chose the Pauli–Villars regularization with a regulator field ψ' of mass $M \to \infty$. The $\partial^\mu J^A_\mu$ insertion is then reduced to:

$$\partial^\mu J^A_\mu = \partial_\mu\big(i\bar{\psi}'\gamma^\mu\gamma^5\psi'\big) = i\bar{\psi}'(2M\gamma^5)\psi' \tag{96}$$

The fermion–fermion–graviton vertex is given by $h_{\mu\nu}T^{\mu\nu}$, and (up to integration by parts) we have:

$$T^{\mu\nu} = \frac{i}{2}\bar{\psi}\gamma^{(\mu}\overleftrightarrow{\partial}^{\nu)}\psi \sim \frac{i}{2}\bar{\psi}\gamma^{(\mu}\big(-2\partial^{\nu)}\big)\psi = -i\bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi, \tag{97}$$

$$h_{\mu\nu}T^{\mu\nu} = \bar{\psi}\left(-ih_{\mu\nu}\gamma^{(\mu}\partial^{\nu)}\right)\psi, \tag{98}$$

This is very similar to the A_μ coupling, except that there is an extra derivative ∂^ν. Denote the polarization of graviton as $\varepsilon_{\mu\nu}$, then in momentum space the interaction vertex $\sim \epsilon_{\mu\nu}\gamma^\mu(k_1^\nu + k_2^\nu)$, and we have:

$$\begin{aligned}
\langle\partial^\mu J^A_\mu\rangle_h &\sim \frac{1}{2!}\times 2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{Tr}\left(2M\gamma_5\cdot\frac{\not{k}+M}{k^2+M^2}\cdot\varepsilon_1\cancel{(2k+p_1)}\cdot\frac{\not{k}+\not{p}_1+M}{(k+p_1)^2+M^2}\cdot\varepsilon_2\cancel{(2k+2p_1+p_2)}\cdot\frac{\not{k}+\not{p}_1+\not{p}_2+M}{(k+p_1+p_2)^2+M^2}\right)\\
&\sim \int\frac{\mathrm{d}^4k}{(2\pi)^4}\,2M^2(4\epsilon_{\mu\nu\rho\sigma})\,\varepsilon_1^{\mu\mu'}(2k+p_1)_{\mu'}\,p_1^\nu\,\varepsilon_2^{\rho\rho'}(2k+2p_1+p_2)_{\rho'}\,p_2^\sigma\left(\frac{1}{k^2+M^2}\cdots\right)\\
&\sim 8M^2\epsilon_{\mu\nu\rho\sigma}\,p_1^\nu p_2^\sigma\,\varepsilon_1^{\mu\mu'}\,\varepsilon_2^{\rho\rho'}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(2k+p_1)_{\mu'}(2k+2p_1+p_2)_{\rho'}}{(k^2+M^2)\big((k+p_1)^2+M^2\big)\big((k+p_1+p_2)^2+M^2\big)}\\
&\sim 8M^2\epsilon_{\mu\nu\rho\sigma}\,p_1^\nu p_2^\sigma\,\varepsilon_1^{\mu\mu'}\,\varepsilon_2^{\rho\rho'}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{4k_{\mu'}k_{\rho'}+p_{1,\mu'}p_{2,\rho'}}{(k^2+M^2)^3}
\end{aligned}\tag{99}$$

There are, in fact, 2 diagrams accounting for this amplitude with $1\leftrightarrow 2$ symmetry; here we simply take one contribution with an additional factor of 2, and imply $1\leftrightarrow 2$ symmetrization in the above expressions.

Note that due to the additional $k_{\mu'}k_{\rho'}$ the integral is no longer finite but logarithmic divergent: $\int^\Lambda \mathrm{d}^4k\,\frac{k^2}{k^6}\sim\ln\Lambda$. More specifically[17], we have:

$$\begin{aligned}
\langle\partial^\mu J^A_\mu\rangle_h &\sim 8M^2\epsilon_{\mu\nu\rho\sigma}\,p_1^\nu p_2^\sigma\,\varepsilon_1^{\mu\mu'}\,\varepsilon_2^{\rho\rho'}\frac{\mathrm{Vol}\,S^3}{(2\pi)^4}\int\left(\frac{4k_{\mu'}k_{\rho'}k^3\,\mathrm{d}k}{(k^2+M^2)^3}+p_{1,\mu'}p_{2,\rho'}\frac{k^3\,\mathrm{d}k}{(k^2+M^2)^3}\right)\\
&\sim 8M^2\epsilon_{\mu\nu\rho\sigma}\,p_1^\nu p_2^\sigma\,\varepsilon_1^{\mu\mu'}\,\varepsilon_2^{\rho\rho'}\frac{2\pi^2}{(2\pi)^4}\int\left(\delta_{\mu'\rho'}\frac{k^5\,\mathrm{d}k}{(k^2+M^2)^3}+p_{1,\mu'}p_{2,\rho'}\frac{k^3\,\mathrm{d}k}{(k^2+M^2)^3}\right)\\
&\sim 8M^2\epsilon_{\mu\nu\rho\sigma}\,p_1^\nu p_2^\sigma\,\varepsilon_1^{\mu\mu'}\,\varepsilon_2^{\rho\rho'}\frac{1}{8\pi^2}\left(\delta_{\mu'\rho'}\frac{1}{2}\ln\frac{\Lambda^2}{M^2}+p_{1,\mu'}p_{2,\rho'}\frac{1}{4M^2}\right)\\
&\sim \frac{1}{4\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\,p_1^\nu p_2^\sigma\,\varepsilon_1^{\mu\mu'}\,\varepsilon_2^{\rho\rho'}\left(2\delta_{\mu'\rho'}M^2\ln\frac{\Lambda^2}{M^2}+p_{1,\mu'}p_{2,\rho'}\right)
\end{aligned}\tag{100}$$

The second term is very much similar to the axial anomaly result, while the first term diverges.

However, we believe that the divergent term must be canceled by other diagrams; otherwise, it will contribute a $p^\nu p^\sigma\,\delta_{\mu'\rho'}\varepsilon_1^{\mu\mu'}\varepsilon_2^{\rho\rho'} = p^\nu p^\sigma(\varepsilon_1)^\mu{}_\alpha(\varepsilon_2)^{\rho\alpha}\sim(\partial h)^2$ term in the final result, which is not diff-invariant. The second term, on the other hand, is diff-invariant:

$$R_{\mu\nu\alpha\beta} = p_\beta\,p_{[\nu}\,\varepsilon_{\mu]\alpha} - p_\alpha\,p_{[\nu}\,\varepsilon_{\mu]\beta}, \tag{101}$$

[17] References:

- David Tong, *Gauge Theory*;
- A. Zee, *QFT in a Nutshellz*;
- `arXiv:0802.0634`;
- `Wikipedia`: *Common integrals in quantum field theory*.

$$
\begin{aligned}
\langle \partial^\mu J^A_\mu \rangle_h &\sim \frac{1}{4\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\,(\varepsilon^{\mu\mu'} p_{1,\mu'} p_1^\nu)(\varepsilon^{\rho\rho'} p_{2,\rho'} p_2^\sigma) \\
&\sim \frac{1}{4\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\frac{1}{4!\times 2\times 2}\times\frac{1}{2}R_{\mu\nu\alpha\beta}R_{\rho\sigma}{}^{\alpha\beta} \\
&\sim \frac{1}{768\pi^2}\,\epsilon_{\mu\nu\rho\sigma}R_{\mu\nu\alpha\beta}R_{\rho\sigma}{}^{\alpha\beta}
\end{aligned}
\tag{102}
$$

(b) The next order contribution would come from the covariant derivative[18]:

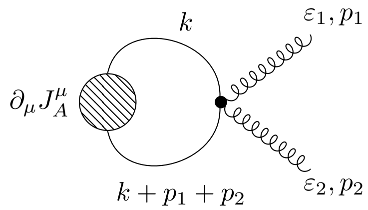
$$
\nabla_\mu\psi = \partial_\mu\psi + \frac{1}{2}\,\omega_\mu{}^{ab}\sigma_{ab}\psi \tag{103}
$$

Where $\omega_\mu{}^{ab}$ is the spin connections, and $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$; when linearized this contributes to the following interaction vertex:

$$
\mathcal{L}' = -\frac{i}{4}\,h_\lambda{}^\alpha\partial_\mu h_{\nu\alpha}\,\bar\psi\,\Gamma^{\mu\lambda\nu}\psi, \quad \Gamma^{\mu\lambda\nu} = \gamma^{[\mu}\gamma^\lambda\gamma^{\nu]}, \tag{104}
$$

$$
\text{Feynman rule:} \quad -\frac{i}{4}\,\Gamma^{\mu\lambda\nu}(p_1-p_2)_\mu\,(\varepsilon_1)_\lambda{}^\alpha(\varepsilon_2)_{\nu\alpha}, \tag{105}
$$

We see a $(\varepsilon_1)_\lambda{}^\alpha(\varepsilon_2)_{\nu\alpha}$ factor, much similar to the factor in the divergent term in (a). Note that this vertex already contains 3 γ-matrices; by joining it with the anomalous vertex $\partial_\mu j^\mu_A$, we obtain a simple 1-loop "seagull" diagram (with graviton wings) :



$$
\begin{aligned}
\langle \partial^\mu J^A_\mu \rangle'_h &\sim 2\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{Tr}\left(2M\gamma_5\cdot\frac{\not k + M}{k^2+M^2}\cdot\left(-\frac{1}{4}\right)\cancel{\varepsilon_1\varepsilon_2(p_1-p_2)}\cdot\frac{\not k + \not p_1 + \not p_2 + M}{(k+p_1+p_2)^2+M^2}\right) \\
&\sim -\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,M^2(4\epsilon_{\mu\nu\rho\sigma})\,\delta_{\mu'\rho'}\varepsilon_1^{\mu\mu'}\,\varepsilon_2^{\rho\rho'}\,(p_1-p_2)^\nu\,(p_1+p_2)^\sigma\left(\frac{1}{k^2+M^2}\cdots\right) \\
&\sim -4M^2\epsilon_{\mu\nu\rho\sigma}\,(2p_1^\nu p_2^\sigma)\,\varepsilon_1^{\mu\mu'}\,\varepsilon_2^{\rho\rho'}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\delta_{\mu'\rho'}}{(k^2+M^2)^2} \\
&\sim -8M^2\epsilon_{\mu\nu\rho\sigma}\,p_1^\nu p_2^\sigma\,\varepsilon_1^{\mu\mu'}\,\varepsilon_2^{\rho\rho'}\,\frac{1}{8\pi^2}\left(\delta_{\mu'\rho'}\frac{1}{2}\ln\frac{\Lambda^2}{M^2}\right)
\end{aligned}
\tag{106}
$$

Compare with the result in (a), and we see that the divergences cancel each other out precisely.

(c) For an anomalous vertex with hypercharge Y, there will be an additional Y factor in the front of $\langle\partial_\mu J^\mu_A\rangle$; summing over a family of matter gives the total anomaly[19]:

$$
\langle\partial_\mu J^\mu_A\rangle \propto \sum \mathrm{Tr}\,T_aT_bY \propto \delta_{ab}\sum Y \tag{107}
$$

When the summation goes over all states in a complete generation, we have $\sum Y = 0$, i.e. the anomaly cancels.

[18] Reference: Alvarez-Gaume & Witten, *Gravitational Anomalies*.

[19] Reference: *Tong*, and `Wikipedia`: *Anomaly (physics) # Anomaly cancellation*.