

1 Symmetry & Noether's Theorem

1.1 2D σ -Model

$$\mathcal{L} = -\frac{1}{2} \eta_{\alpha\beta} \eta_{\mu\nu} \partial^\alpha X^\mu \partial^\beta X^\nu = -\frac{1}{2} \partial^\alpha X_\mu \partial_\alpha X^\mu, \quad X^\mu \in \mathbb{R}^{1,D-1} \quad (1)$$

- For $\delta X^\mu = a^\mu + \lambda^\mu{}_\nu X^\nu$, the Lagrangian (density) transforms as follows:

$$\begin{aligned} \delta \mathcal{L} &= -\partial^\alpha X_\mu \partial_\alpha \delta X^\mu \\ &= -\partial^\alpha X_\mu \partial_\alpha (a^\mu + \lambda^\mu{}_\nu X^\nu) \\ &= -\partial^\alpha X_\mu (\partial_\alpha a^\mu + X^\nu \partial_\alpha \lambda^\mu{}_\nu + \lambda^\mu{}_\nu \partial_\alpha X^\nu) \\ &= -\partial^\alpha X_\mu \partial_\alpha a^\mu - \partial^\alpha X^\mu \partial_\alpha X^\nu \lambda_{\mu\nu} - X^\nu \partial^\alpha X^\mu \partial_\alpha \lambda_{\mu\nu} \\ &= -\partial^\alpha X_\mu \partial_\alpha a^\mu - \partial^\alpha X^\mu \partial_\alpha X^\nu \lambda_{(\mu\nu)} - X^\nu \partial^\alpha X^\mu \partial_\alpha \lambda_{\mu\nu} \end{aligned} \quad (2)$$

Since a^μ and $\lambda^\mu{}_\nu$ are independent, imposing $\delta L = 0$ yields $\partial_\alpha a^\mu = 0$, $a = \text{const.}$ Furthermore, if $\delta L = 0$ is to hold for arbitrary X^μ fields, then $\partial_\alpha \lambda_{\mu\nu} = 0$, $\lambda_{(\mu\nu)} = 0$, i.e. $\lambda_{\mu\nu}$ is constant and anti-symmetric over its indices.

- Promote $\delta X \mapsto \epsilon(x) \delta X = \epsilon(x) (a^\mu + \lambda^\mu{}_\nu X^\nu)$, with $\epsilon(x)$ some localized bump function; using (2) and considering *on-shell* variation, we have:

$$\begin{aligned} 0 = \delta S &= - \int d^2x (\partial^\alpha X_\mu a^\mu \partial_\alpha \epsilon + X^\nu \partial^\alpha X^\mu \lambda_{\mu\nu} \partial_\alpha \epsilon) \\ &= - \int d^2x \left(\partial^\alpha X_\mu a^\mu + X_{[\nu} \partial^\alpha X_{\mu]} \lambda^{[\mu\nu]} \right) \partial_\alpha \epsilon \end{aligned} \quad (3)$$

It is evident (after partial integration) that the following currents are conserved; they are the Noether currents associated with a^μ and $\lambda^{[\mu\nu]}$:

$$j_\mu^\alpha = -\partial^\alpha X_\mu, \quad j_{\mu\nu}^\alpha = -X_{[\nu} \partial^\alpha X_{\mu]} = \frac{1}{2} (X_\mu \partial^\alpha X_\nu - X_\nu \partial^\alpha X_\mu) \quad (4)$$

Conserved charge $Q = \int d^2x j^0(x)$, we have:

$$P_\mu = - \int dx^1 \partial^0 X_\mu = \int dx^1 \partial_0 X_\mu, \quad M_{\mu\nu} = \frac{1}{2} \int dx^1 (X_\nu \partial_0 X_\mu - X_\mu \partial_0 X_\nu) \quad (5)$$

They can be interpreted as spacetime momentum and spacetime angular momentum. ■

1.2 Real Scalar in $(3+1)$ D

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 \quad (6)$$

- For ϕ : scalar, under $x' = \lambda \circ x$, $\phi(x) \mapsto \phi'(x)$, while:

$$\phi'(x') = \phi(x) \implies \phi'(x) = \phi(\lambda^{-1} \circ x) \quad (7)$$

For $\lambda \sim \lambda^\mu{}_\nu$: Lorentz transformation, $\eta_{\mu\nu} \lambda^\mu{}_\rho \lambda^\nu{}_\sigma = \eta_{\rho\sigma}$, or equivalently, $(\lambda^{-1})^\mu{}_\nu = \lambda_\nu{}^\mu$. Therefore,

$$\phi'(x^\mu) = \phi(\lambda^{-1} \circ x^\mu) = \phi(x^\nu \lambda_\nu{}^\mu) \quad (8)$$

- Under $x'^\mu = \lambda^\mu{}_\nu x^\nu$, we have:

$$\begin{aligned} \mathcal{L}'(x') &= -\frac{1}{2} \partial'^\mu \phi'(x') \partial'_\mu \phi'(x') - \frac{1}{2} m^2 \phi'^2(x') \\ &= -\frac{1}{2} \partial'^\mu \phi(x) \partial'_\mu \phi(x) - \frac{1}{2} m^2 \phi^2(x) \\ &= -\frac{1}{2} \eta^{\mu\nu} \frac{\partial x^\rho}{\partial x'^\mu} \partial_\rho \phi(x) \frac{\partial x^\sigma}{\partial x'^\nu} \partial_\sigma \phi(x) - \frac{1}{2} m^2 \phi^2(x) \\ &= -\frac{1}{2} \eta^{\rho\sigma} \partial_\rho \phi(x) \partial_\sigma \phi(x) - \frac{1}{2} m^2 \phi^2(x) \\ &= \mathcal{L}(x) \end{aligned} \quad (9)$$

Here we've used $\eta^{\mu\nu} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} = \eta^{\mu\nu} \lambda_\mu{}^\rho \lambda_\nu{}^\sigma = \eta^{\rho\sigma}$. Furthermore, $S' = \int d^4x' \mathcal{L}'(x') = \int d^4x \mathcal{L}(x) = S$, hence the action is invariant under Lorentz transformation.

- Consider an infinitesimal Lorentz transformation: $\lambda \sim \mathbb{1} + \omega$, then $\eta_{\mu\nu} \lambda^\mu{}_\rho \lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ implies that $\omega_{\mu\nu}$ is anti-symmetric: $\omega_{\mu\nu} + \omega_{\nu\mu} = 0$. For $\delta x^\mu = \omega^\mu{}_\nu x^\nu$, we have:

$$\delta\phi = -\frac{\partial\phi}{\partial x^\mu} \delta x^\mu = -\omega^\mu{}_\nu x^\nu \partial_\mu \phi \quad (10)$$

To obtain the corresponding Noether charges, we can simply repeat the operations done in our previous problem; alternatively, we can try to derive a general recipe¹: for $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ and $S = \int d^4x \mathcal{L}$, we have:

$$\begin{aligned} \delta S &= \int d^4x \delta \mathcal{L} \\ &= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \partial_\mu \phi \right) \\ &= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \delta \phi + \int d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi \right) \end{aligned} \quad (11)$$

If we vary S w.r.t. a symmetry of the system, we will have $\delta \mathcal{L} = \partial_\mu K^\mu$ some total derivative; when on-shell, such variation gives the conserved current with boundary term K^μ :

$$j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi - K^\mu \quad (12)$$

Back to our Lorentz transformation $\delta\phi = -\omega^\mu{}_\nu x^\nu \partial_\mu \phi$, we have symmetry variation:

$$\delta \mathcal{L} = -\omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = -\partial_\mu (\omega^\mu{}_\nu x^\nu \mathcal{L}) \quad (13)$$

We can write this down without explicit calculations, since we know \mathcal{L} itself is a Lorentz scalar, and that's how scalar transforms under Lorentz transformations.

¹References: [arXiv:1601.03616](https://arxiv.org/abs/1601.03616) and Tong: <http://damtp.cam.ac.uk/user/tong/qft.html>

This gives a boundary term $K^\mu = -\omega^\mu{}_\nu x^\nu \mathcal{L}$, and the Noether current and its corresponding conserved charge can be calculated as follows:

$$j^\mu = -\omega^\sigma{}_\nu x^\nu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\sigma \phi - \delta^\mu_\sigma \mathcal{L} \right), \quad (14)$$

$$Q = \int d^3x j^0 = -\omega^\sigma{}_\nu \int d^3x x^\nu (\partial_0 \phi \partial_\sigma \phi - \delta^\sigma_0 \mathcal{L}), \quad (15)$$

Note that $\omega^\mu{}_\nu$ is arbitrary, therefore Q can be decomposed into independent charges:

$$Q = \frac{1}{2} \omega_{\mu\nu} M^{\mu\nu}, \quad M^{\mu\nu} = - \int d^3x 2x^{[\mu} (\partial_0 \phi \partial^{\nu]} \phi - \eta^{\nu]0} \mathcal{L}), \quad (16)$$

The indices of $M^{\mu\nu}$ are anti-symmetrized to match the degrees of freedom in $\omega_{\mu\nu}$. Note that the \mathcal{L} term only appears when one of the indices is 0.

Note that the canonical momentum:

$$\varpi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} = \partial_0 \phi \quad (17)$$

It is thus natural to re-organize M^{0i} in the following way:

$$\begin{aligned} M^{0i} &= -M^{i0} = - \int d^3x (\varpi (x^0 \partial^i - x^i \partial^0) \phi - x^i \mathcal{L}) \\ &= - \int d^3x (x^0 \varpi \partial^i \phi + x^i (\varpi \dot{\phi} - \mathcal{L})) \\ &= - \int d^3x (x^0 \varpi \partial^i \phi + x^i \mathcal{H}) \end{aligned} \quad (18)$$

We've obtained an interesting result: the expression for the boost generator M^{i0} contains the Hamiltonian density \mathcal{H} , weighted by the radial distance x^i . This is natural since a boost does indeed contains time evolution for states away from the origin. It's an important result utilized by the so-called *Rindler decomposition*; in fact, M^{i0} becomes the Hamiltonian for an accelerated observer in the Rindler patch².

For M^{ij} , we have:

$$M^{ij} = - \int d^3x \dot{\phi} (x^i \partial^j - x^j \partial^i) \phi \quad (19)$$

This is interpreted as the angular momentum of the field ϕ . Suppose ϕ is a wave packet localized around \mathbf{x} with momentum $\approx \mathbf{p}$, then we have the classical angular momentum up to some factor:

$$M^{ij} \sim (x^i p^j - x^j p^i) \int d^3x E \phi^2, \quad E = \sqrt{\mathbf{p}^2 + m^2} \quad (20)$$

The $\int d^3x E \phi^2$ factor in the above expression is an $\mathcal{O}(1)$ normalization constant for a particle-like wave packet; to see this, note that $\phi \in \mathbb{R}$ has a phase factor $\phi \sim a e^{+ik \cdot x} + a^\dagger e^{-ik \cdot x} \sim \cos(k \cdot x)$,

$$E = \int d^3x \mathcal{H} = \int d^3x \left(\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 + \dots \right) \sim (E^2 + \mathbf{p}^2 + m^2 + \dots) \int d^3x \frac{1}{2} \phi^2, \quad (21)$$

²See the lecture notes of Tom Hartman: hartmanhep.net/topics2015/gravity-lectures.pdf or Daniel Harlow [arXiv:1409.1231](https://arxiv.org/abs/1409.1231).

$$E \int d^3x \phi^2 \sim 1, \quad (22)$$

Indeed, we have: $M^{ij} \sim (x^i p^j - x^j p^i)$.

Canonical quantization:

$$[\phi(\mathbf{x}), \varpi(\mathbf{y})] = [\phi(\mathbf{x}), \dot{\phi}(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}) \quad (23)$$

Other equal-time commutators between ϕ, ϖ all just vanish. Operator products are then regularized by normal ordering: $M \mapsto :M:$, which can be explicitly implemented by normal ordering of the oscillator modes:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left(a_{\mathbf{k}} e^{ik \cdot x} + a_{\mathbf{k}}^\dagger e^{-ik \cdot x} \right), \quad [a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \quad (24)$$

For example, the first term in $M^{0i} = -M^{i0}$ can be expanded as:

$$-x^0 \int d^3x \varpi \partial^i \phi = x^0 \int \frac{d^3k}{(2\pi)^3} k^i a_{\mathbf{k}}^\dagger a_{\mathbf{k}} = x^0 P^i \quad (25)$$

Here P^μ is the momentum operator on the Hilbert space, promoted from the classical $-i\partial^\mu$. M^{0i} is thus further reduced to:

$$M^{0i} = -M^{i0} = x^0 P^i - \int d^3x x^i \mathcal{H} \quad (26)$$

We note that this result is almost the classical $x^0 P^i - x^i P^0$, but here $x^i P^0$ is replaced with the integral over energy density \mathcal{H} . The result can be nicely re-written with the stress tensor $T^{\mu\nu}$; just run the Noether's procedure with $\delta x^\mu = \epsilon^\mu$, then we shall obtain:

$$\begin{aligned} j'^\mu &= \epsilon_\nu T^{\mu\nu}, \quad T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi + \eta^{\mu\nu} \mathcal{L}, \\ Q' &= \epsilon_\mu P^\mu, \quad P^\mu = \int d^3x T^{\mu 0} = \int d^3x (\partial^0 \phi \partial^\mu \phi + \eta^{0\mu} \mathcal{L}) \\ M^{\mu\nu} &= \int d^3x 2x^{[\mu} T^{\nu]0} \end{aligned} \quad (27)$$

The quantization $M \mapsto :M:$ is thus reduced to the quantization of $T_{0\nu}$.

TODO: Detailed analysis! HINT: Ward identity!

Notice that $x^{[\mu} \partial^{\nu]} = \frac{1}{2} (x^\mu \partial^\nu - x^\nu \partial^\mu) = \frac{1}{2} D^{\mu\nu}$ is the Killing vector fields of $\mathbb{R}^{3,1}$, hence they naturally follow the commutation relations of $\mathfrak{so}(3,1)$ (up to a constant coefficient, or an isomorphism)³. We have:

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= \int d^3x \int d^3y [\dot{\phi} D^{\mu\nu} \phi(x), \dot{\phi} D^{\rho\sigma} \phi(y)] \\ &= \int d^3x \dot{\phi} [D^{\mu\nu}, D^{\rho\sigma}] \phi \end{aligned} \quad (28)$$

Similar holds for M^{i0} . Therefore, $M^{\mu\nu}$'s indeed form the Lie algebra $\mathfrak{so}(3,1)$. ■

³I would like to thank 林般 for pointing this out.