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Compiled @ 2021/02/22

## Ward Identity with Hard Thermal Loop

The hard thermal loop (HTL) approximation of the photon self-energy is given by:

$$\Pi_{\mu\nu}(Q) = m_E^2 \left( \frac{q^0}{q} \int \frac{\mathrm{d}^2 \Omega(\hat{\mathbf{k}})}{4\pi} \frac{\hat{K}_{\mu} \hat{K}_{\nu}}{\hat{\mathbf{k}} \cdot \hat{\mathbf{q}} + q^0/q} - g_{\mu 0} g_{\nu 0} \right)$$
(1)

Where  $m_E^2 = \frac{e^2 T^2}{3}$ . Here we've chosen the Lorentzian signature  $g \sim (-+++)$ , and  $Q^\mu \sim (q^0, \mathbf{q})$ , where  $q^0 = -i\omega$ ,  $q = \|\mathbf{q}\|$ , while the normalized  $\hat{K}_\mu \sim (1, \hat{\mathbf{k}})$ ,  $\hat{K}^\mu \sim (-1, \hat{\mathbf{k}})$ . We have:

$$Q^{\mu}\hat{K}_{\mu} = q^{0} + \hat{\mathbf{k}} \cdot \mathbf{q} = q \left( \hat{\mathbf{k}} \cdot \hat{\mathbf{q}} + q^{0}/q \right), \quad Q^{\mu}g_{\mu 0} = q_{0} = -q^{0},$$
 (2)

$$Q^{\mu}\Pi_{\mu\nu} = m_E^2 q^0 \left( \int \frac{\mathrm{d}^2 \Omega(\hat{\mathbf{k}})}{4\pi} \hat{K}_{\nu} + g_{\nu 0} \right) = 0$$
 (3)

More specifically, we have  $Q^{\mu}\Pi_{\mu 0} \propto \left(\int \frac{\mathrm{d}^2\Omega(\hat{\mathbf{k}})}{4\pi} \mathbf{1}\right) - 1 = 0$ , and also  $Q^{\mu}\Pi_{\mu 0} \propto \int \frac{\mathrm{d}^2\Omega(\hat{\mathbf{k}})}{4\pi} \hat{K}_i = 0$ . Therefore, the self-energy is transverse.

## 2 Decomposition of $\Pi_{\mu\nu}$

As we know,  $Q^{\mu}\Pi_{\mu\nu}=0$ , hence it can be nicely decomposed by various projections related to  $Q^{\mu}$ ; recall that:

$$E_{\mu\nu} = \frac{Q_{\mu}Q_{\nu}}{Q^2} = \hat{Q}_{\mu}\hat{Q}_{\nu}, \quad P_{\mu\nu} = g_{\mu\nu} - E_{\mu\nu}, \tag{4}$$

$$N^{\mu} = P^{\mu\nu}(0, \mathbf{q})_{\nu} = P^{\mu j}q_{j} = -P^{\mu 0}q_{0} = P^{\mu 0}q^{0}, \quad Q_{\mu}N^{\mu} = 0,$$

$$P_{L}^{\mu\nu} = \hat{N}^{\mu}\hat{N}^{\nu}, \quad P_{T}^{\mu\nu} = P^{\mu\nu} - P_{L}^{\mu\nu},$$
(5)

$$P_L^{\mu\nu} = \hat{N}^{\mu} \hat{N}^{\nu}, \quad P_T^{\mu\nu} = P^{\mu\nu} - P_L^{\mu\nu},$$
 (6)

Note that  $P_T^{\mu\nu}$  annihilates both  $Q^{\mu}$  and  $N^{\mu}$ , by linearity it also annihilates both  $(1,\mathbf{0})$  and  $(0,\mathbf{q})$ , i.e.

$$P_T^{\mu 0} q_0 = 0, \quad P_T^{\mu 0} = 0, \tag{7}$$

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$$P_T^{ij} = \delta^{ij} - \hat{Q}^i \hat{Q}^j - \hat{N}^i \hat{N}^j = \delta^{ij} - \hat{q}^i \hat{q}^j$$
(8)

The transverse self-energy can then be decomposed as:

$$\Pi^{\mu\nu} = \Pi_L P_L^{\mu\nu} + \Pi_T P_T^{\mu\nu},\tag{9}$$

$$\Pi^{\mu}_{\ \mu} = \Pi_L + 2\Pi_T, \quad \Pi^{00} = \Pi_L P_L^{00},$$
(10)

$$N^i = P^{ij}q_j = \left(1 - \frac{q^2}{Q^2}\right)q^i = -\frac{q_0^2}{Q^2}q^i, \quad N^0 = \left(0 - \frac{q^2}{Q^2}\right)q^0 = -\frac{q^2}{Q^2}q^0, \quad N^2 = -\frac{q_0^2q^2}{Q^2}$$
 (11)

$$P_L^{00} = \frac{N_0^2}{N^2} = -\frac{q^2}{Q^2} \tag{12}$$

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We see that  $\Pi_{\mu\nu}$  is entirely determined by  $\Pi^{00}$  and  $\Pi^{\mu}_{\mu}$ . More explicitly, we have<sup>1</sup>:

$$\Pi_{L} = -\frac{Q^{2}}{q^{2}} \Pi^{00} 
\Pi_{T} = \frac{1}{2} \left( \Pi^{\mu}_{\mu} + \frac{Q^{2}}{q^{2}} \Pi^{00} \right) = \frac{1}{2} \left( -\frac{q_{0}^{2}}{q^{2}} \Pi^{00} + \Pi^{i}_{i} \right)$$
(13)

 $\Pi^{00}, \Pi^{i}_{i}$  is computed explicitly as follows:

$$\Pi_{ij} = m_E^2 \frac{q^0}{q} \int \frac{\mathrm{d}^2 \Omega\left(\hat{\mathbf{k}}\right)}{4\pi} \frac{\hat{k}_i \hat{k}_j}{\hat{\mathbf{k}} \cdot \hat{\mathbf{q}} + q^0/q}, \quad \Pi^{00} = m_E^2 \left(\frac{q^0}{q} \int \frac{\mathrm{d}^2 \Omega\left(\hat{\mathbf{k}}\right)}{4\pi} \frac{1}{\hat{\mathbf{k}} \cdot \hat{\mathbf{q}} + q^0/q} - 1\right)$$
(14)

$$\Pi^{i}_{i} = m_{E}^{2} \frac{q^{0}}{q} \int \frac{\mathrm{d}^{2}\Omega\left(\hat{\mathbf{k}}\right)}{4\pi} \frac{\hat{k}^{i}\hat{k}_{i}}{\hat{\mathbf{k}} \cdot \hat{\mathbf{q}} + q^{0}/q}, \quad \hat{k}^{i}\hat{k}_{i} = 1,$$

$$= m_{E}^{2} \frac{q^{0}}{q} \int_{0}^{\pi} \frac{2\pi \sin\theta \,\mathrm{d}\theta}{4\pi} \frac{1}{\cos\theta + q^{0}/q}$$

$$= m_{E}^{2} \frac{q^{0}}{q} \int_{-1}^{1} \frac{\mathrm{d}z}{2} \frac{1}{z + q^{0}/q}$$

$$= m_{E}^{2} \frac{q^{0}}{q} H\left(\frac{q^{0}}{q}\right), \quad H(x) = \frac{1}{2} \ln \frac{x + 1}{x - 1},$$
(15)

$$\Pi^{00} = \Pi^i_{\ i} - m_E^2,\tag{16}$$

$$\Pi_{L} = -m_{E}^{2} \frac{Q^{2}}{q^{2}} \left( \frac{q^{0}}{q} H \left( \frac{q^{0}}{q} \right) - 1 \right), \quad \frac{Q^{2}}{q^{2}} = 1 - \frac{q_{0}^{2}}{q^{2}},$$

$$\Pi_{T} = \frac{1}{2} \left( -\frac{q_{0}^{2}}{q^{2}} \Pi^{00} + \Pi^{i}_{i} \right) = \frac{1}{2} \left( \frac{Q^{2}}{q^{2}} \Pi^{i}_{i} + \frac{q_{0}^{2}}{q^{2}} m_{E}^{2} \right)$$

$$= \frac{1}{2} m_{E}^{2} \frac{q^{0}}{q} \left( \frac{Q^{2}}{q^{2}} H \left( \frac{q^{0}}{q} \right) + \frac{q^{0}}{q} \right)$$
(17)

## 3 HTL in QCD

The gluon self-energy at 1-loop is similar to the QED situation, except that now we should include the extra degrees of freedom in the summation.

The quark 1-loop correction is structurally identical to the QED fermion 1-loop correction, but enhanced  $N_f$  fold due to the flavor degrees of freedom. Also,  $\text{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$  gives an extra  $\frac{1}{2}$  factor. In the end, the quark loop contribution is given by the overall factor:

$$m_E^2 \longmapsto m_{\text{quark}}^2 \, \delta^{ab} = \frac{1}{2} N_f \, \delta^{ab}$$
 (18)

The rest is identical to the QED case.

<sup>&</sup>lt;sup>1</sup>Note that the metric convention here might result in signs that may differ from the textbook.

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The ghost loop is also similar, except now we have a factor of  $N_c$  instead of  $\frac{1}{2}N_f$ , and with a frequency summation given by:

$$\frac{T}{V} \sum_{K} \Delta(K) \left( -i (K - Q)^{\nu} \right) \Delta(K - Q) \left( -i K^{\mu} \right), \quad \Delta(K) = \frac{1}{K^{2}}$$

$$= -\frac{T}{V} \sum_{K} \frac{(K - Q)^{\nu} K^{\mu}}{K^{2} (K - Q)^{2}} \simeq -\frac{T}{V} \sum_{K} \frac{K^{\mu} K^{\nu}}{K^{2} (K - Q)^{2}} \tag{19}$$

We've computed a similar Matsubara summation before, while treating the QED fermionic loop; the method still applies here but with bosonic frequencies. This produces an additional factor of  $(-\frac{1}{8})$ , relative to the fermionic case<sup>2</sup>. In the end, we have:

$$m_E^2 \longmapsto m_{\text{ghost}}^2 \, \delta^{ab} = -\frac{1}{8} N_c \, \delta^{ab}$$
 (20)

 $<sup>^2 {\</sup>rm Reference} :$  Laine & Vuorinen, Basics of Thermal Field Theory.