

Bryan

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QCD Partition Function at $\mathcal{O}(g^2)$

$$\ln \mathcal{Z}_{I}^{(2)} = -\frac{1}{2} \underbrace{\begin{array}{c} \\ \\ \\ \\ \end{array}} - \frac{1}{2} \underbrace{\begin{array}{c} \\ \\ \\ \end{array}} + \frac{1}{12} \underbrace{\begin{array}{c} \\ \\ \\ \end{array}} + \frac{1}{8} \underbrace{\begin{array}{c} \\ \\ \\ \end{array}}$$

$$\ln \mathcal{Z}_{I}^{(2)} = \ln \mathcal{Z}^{(a)} + \ln \mathcal{Z}^{(b)} + \ln \mathcal{Z}^{(c)} + \ln \mathcal{Z}^{(d)}$$

$$(1)$$

The contribution of (a) is given by:

$$\ln \mathcal{Z}^{(a)} = \frac{1}{2!} (-1)^1 \frac{T}{V} \sum_k \frac{T}{V} \sum_p \operatorname{Tr} \left(S(k) \left(g \gamma^{\nu} T^b \right) S(p) \left(g \gamma^{\mu} T^a \right) \right) \left(\frac{V}{T} \right) \delta_{ab} \Delta_{\mu\nu} (p - k) \tag{2}$$

Here the trace goes over spinor, color and flavor indices. S(k) is the quark propagator, with suppressed spinor, color and flavor indices, while $\delta_{ab} \Delta_{\mu\nu}$ is the gluon propagator, where a, b are adjoint indices; each vertex contributes a $(g\gamma^{\mu}T^a)$ factor.

In our convention, $k=(\omega_n,\mathbf{k})$ stands for the Euclidean 4-momentum with ω_n : the discrete Matsubara frequency. Each \sum_k comes with a factor $\frac{T}{V}$ while each momentum space delta function comes with an inverse factor: $\frac{V}{T}$; this is due to the fact that:

$$1 = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} (2\pi)^4 \delta^4(k - k_0) \sim \frac{1}{\beta V} \sum_k \beta V \delta_{k,k_0}$$
 (3)

Following the same recipe from QED, we can write down:

$$\ln \mathcal{Z}^{(a)} = -\left(\operatorname{Tr}\left(T^{a}T^{b}\right)\delta_{ab}\right)\frac{1}{2}g^{2}\frac{V}{T}\cdot\frac{T}{V}\sum_{k}\frac{T}{V}\sum_{p}\operatorname{Tr}\left(S(k)\gamma^{\nu}S(p)\gamma^{\mu}\right)\Delta_{\mu\nu}(p-k)$$

$$= -\left(\frac{N_{c}^{2}-1}{2}N_{f}\right)\frac{g^{2}}{288}\frac{V}{T}\left(5T^{4} + \frac{18}{\pi^{2}}T^{2}\mu^{2} + \frac{9}{\pi^{4}}\mu^{4}\right)$$
(4)

The (b) term is structurally similar to the (a) term; now the amplitude can be written down simply by replacing the propagator $S(k) \mapsto W(k)$ of the ghost, while the vertex is $(g\gamma^{\mu}T^{a}) \mapsto (-igk^{\nu}T^{b})$ instead:

$$\ln \mathcal{Z}^{(b)} = \frac{1}{2!} (-1)^1 \frac{T}{V} \sum_k \frac{T}{V} \sum_p \operatorname{Tr} \left(W(k) \left(-igp^{\nu} T^b \right) W(p) \left(-igk^{\mu} T^a \right) \right) \left(\frac{V}{T} \right) \delta_{ab} \Delta_{\mu\nu} (p-k) \quad (5)$$

The trace now goes over suppressed adjoint indices of $W(k)_{ab} = -\delta_{ab}\Delta(k)$ and $(T_a)_{bc} = f_{abc}$, where f_{abc} is the structure constant of $SU(N_c)$. Therefore¹,

$$\ln \mathcal{Z}^{(b)} = -\left(\operatorname{Tr}\left(T^{a}T^{b}\right)\delta_{ab}\right)\frac{1}{2}g^{2}\frac{V}{T}\cdot\frac{T}{V}\sum_{k}\frac{T}{V}\sum_{p}\Delta(k)\,\Delta(p)\left(-k^{\mu}p^{\nu}\right)\Delta_{\mu\nu}(p-k)$$

$$= -\left(\frac{N_{c}^{2}-1}{2}\,2N_{c}\right)\frac{1}{2}g^{2}\frac{V}{T}\cdot\frac{T}{V}\sum_{k}\frac{T}{V}\sum_{p}\Delta(k)\,\Delta(p)\left(-k^{\mu}p^{\nu}\right)\left(-g_{\mu\nu}\right)\Delta(p-k)$$
(6)

 $^{^{1}}$ Tr $(T^{a}T^{b})_{ad}$ in the adjoint representation is precisely the *Killing form* of the $\mathfrak{su}(N_{c})$ algebra, which is $2N_{c}$ times the Tr $(T^{a}T^{b})_{0}$ in the fundamental representation; see Wikipedia: *Killing form*.

Now we compute the remaining $\sum_{k,p}$. We have:

$$\Delta(k)\,\Delta(p)\,(k\cdot p)\,\Delta(p-k) = \frac{k\cdot p}{k^2p^2\,(p-k)^2}\tag{7}$$

The generic method to carry out such summation is by using the mixed representation of the propagator; for some propagator D(k), we have:

$$D(k) = D(w_n, \mathbf{k}) = \int_0^\beta d\tau \, e^{-i\omega_n \tau} \, T \sum_m e^{i\omega_m \tau} D(w_m, \mathbf{k})$$
$$= \int_0^\beta d\tau \, e^{-i\omega_n \tau} \, \tilde{D}(\tau, \mathbf{k}),$$
(8)

$$\tilde{D}(\tau, \mathbf{k}) = T \sum_{m} e^{i\omega_{m}\tau} D(w_{m}, \mathbf{k})$$

$$= T \sum_{m} e^{i\omega_{m}\tau} \int \frac{d\omega}{2\pi} \frac{\rho(\omega, \mathbf{k})}{\omega + i\omega_{m}}$$

$$= \int \frac{d\omega}{2\pi} \rho(\omega, \mathbf{k}) T \sum_{m} \frac{e^{i\omega_{m}\tau}}{\omega + i\omega_{m}}$$

$$= \int \frac{d\omega}{2\pi} \rho(\omega, \mathbf{k}) e^{-\omega\tau} (1 \pm n_{\pm}(\omega)),$$
(9)

$$\rho(\omega, \mathbf{k}) = \frac{1}{i} \left(D(\omega + i\epsilon) - D(\omega - i\epsilon) \right) = 2 \operatorname{Im} D(\omega + i\epsilon, \mathbf{k}), \quad n_{\pm} = \frac{1}{e^{\beta\omega} \mp 1}, \tag{10}$$

Then the Matsubara sum \sum_{ω_n} becomes a sum over exponentials like $e^{-i\omega_n\tau}$, which is easier to deal with. However, for this particular problem, there is a shortcut²; notice that the denominator of (7) is invariant under $p \mapsto k - p$, hence:

$$\sum_{p} \frac{k \cdot p}{k^{2} p^{2} (p - k)^{2}} = \sum_{(k-p)} \frac{k \cdot (k - p)}{k^{2} (k - p)^{2} p^{2}}$$

$$= \sum_{p} \frac{k \cdot (k - p)}{k^{2} p^{2} (p - k)^{2}}$$

$$= \sum_{p} \frac{\frac{1}{2} (k \cdot p + k \cdot (k - p))}{k^{2} p^{2} (p - k)^{2}}$$

$$= \sum_{p} \frac{1}{2 p^{2} (p - k)^{2}},$$
(11)

$$\frac{T}{V} \sum_{k} \frac{T}{V} \sum_{p} \Delta(k) \Delta(p) (k \cdot p) \Delta(p - k) = \frac{1}{2} \frac{T}{V} \sum_{p} \frac{1}{p^{2}} \frac{T}{V} \sum_{k} \frac{1}{(p - k)^{2}}$$

$$= \frac{1}{2} \left(\frac{T}{V} \sum_{p} \frac{1}{p^{2}} \right)^{2} = \frac{1}{2} \left(\frac{T^{2}}{12} \right)^{2}, \tag{12}$$

²Reference: Laine & Vuorinen, Basics of Thermal Field Theory.

$$\ln \mathcal{Z}^{(b)} = -\left(\frac{N_c^2 - 1}{2} \, 2N_c\right) \frac{1}{2} \, g^2 \frac{V}{T} \cdot \frac{1}{2} \left(\frac{T^2}{12}\right)^2 = -\frac{V}{T} \, N_c \left(N_c^2 - 1\right) \frac{1}{4} \, g^2 \frac{T^4}{144} \tag{13}$$

The (c) term is structurally similar to the (b) term, but with a symmetrized 3-gluon vertex:

$$\left(\frac{1}{3!}\right) ig f_{abc} \left(g_{\mu\nu}(k-p)_{\rho} + g_{\nu\rho}(p-q)_{\mu} + g_{\rho\mu}(q-k)_{\nu}\right) = \left(\frac{1}{3!}\right) ig f_{abc} D_{\mu\nu\rho}(k,p,q) \tag{14}$$

To link the legs of two 3-gluon vertices as shown in (c), there are 3! possibilities. Therefore, we have:

$$\ln \mathcal{Z}^{(c)} = \frac{1}{2!} \cdot 3! \cdot \left(\frac{1}{3!}\right)^{2} \frac{T}{V} \sum_{k} \frac{T}{V} \sum_{p} \Delta(k) \Delta(p) \left(\frac{V}{T}\right) \Delta(p-k)$$

$$\times \left(igf_{abc} D_{\mu\nu\rho}(k, -p, p-k)\right) \left(igf^{bac} D^{\nu\mu\rho}(p, -k, k-p)\right)$$

$$= -\frac{1}{12} \frac{V}{T} \cdot \frac{T}{V} \sum_{k} \frac{T}{V} \sum_{p} \Delta(k) \Delta(p) \Delta(p-k)$$

$$\times \left(\frac{N_{c}^{2} - 1}{2} 2N_{c}\right) g^{2} D_{\mu\nu\rho}(k, -p, p-k) D^{\mu\nu\rho}(p, -k, k-p)$$

$$= -\left(\frac{N_{c}^{2} - 1}{2} 2N_{c}\right) \frac{1}{12} g^{2} \frac{V}{T} \cdot \frac{T}{V} \sum_{k} \frac{T}{V} \sum_{p} \Delta(k) \Delta(p) \Delta(p-k)$$

$$\times D_{\mu\nu\rho}(k, -p, p-k) D^{\mu\nu\rho}(p, -k, k-p),$$

$$D_{\mu\nu\rho}(k, -p, p-k) D^{\nu\nu\rho}(p, -k, k-p) = D_{\mu\nu\rho}(k, -p, p-k) D^{\mu\nu\rho}(-k, p, k-p)$$

$$= -D_{\mu\nu\rho}(k, -p, p-k) D^{\mu\nu\rho}(k, -p, p-k) D^{\mu\nu\rho}(k, -p, p-k)$$

$$D_{\mu\nu\rho}(k, -p, p-k) D^{\mu\nu\rho}(p, -k, k-p) = D_{\mu\nu\rho}(k, -p, p-k) D^{\mu\nu\rho}(-k, p, k-p)$$

$$= -D_{\mu\nu\rho}(k, -p, p-k) D^{\mu\nu\rho}(k, -p, p-k)$$

$$= -\left[g_{\mu\nu}g^{\mu\nu}\right] \left((k+p)^2 + (k-2p)^2 + (p-2k)^2\right)$$

$$-2(k+p) \cdot (k-2p)$$

$$-2(k-2p) \cdot (p-2k)$$

$$-2(p-2k) \cdot (k+p)$$

$$= -\left[d+1\right] \cdot 3\left(k^2 + p^2 + (k-p)^2\right)$$

$$-2\left(-\frac{3}{2}\right) \left(k^2 + p^2 + (k-p)^2\right)$$

$$= -3d\left(k^2 + p^2 + (k-p)^2\right),$$
(16)

$$\ln \mathcal{Z}^{(c)} = -\left(\frac{N_c^2 - 1}{2} 2N_c\right) \frac{1}{12} g^2 \frac{V}{T} \cdot \frac{T}{V} \sum_k \frac{T}{V} \sum_p \frac{-3d \left(k^2 + p^2 + (k - p)^2\right)}{k^2 p^2 (p - k)^2}$$

$$= \left(\frac{N_c^2 - 1}{2} 2N_c\right) \frac{d}{4} g^2 \frac{V}{T} \cdot \frac{T}{V} \sum_k \frac{T}{V} \sum_p \frac{k^2 + p^2 + (k - p)^2}{k^2 p^2 (p - k)^2},$$
(17)

$$\frac{T}{V} \sum_{k} \frac{T}{V} \sum_{p} \frac{k^{2} + p^{2} + (k - p)^{2}}{k^{2} p^{2} (p - k)^{2}} = \frac{T}{V} \sum_{k} \frac{T}{V} \sum_{p} \left(\frac{1}{p^{2} (p - k)^{2}} + \frac{1}{k^{2} (p - k)^{2}} + \frac{1}{k^{2} p^{2}} \right) \\
= 3 \left(\frac{T}{V} \sum_{k} \frac{1}{k^{2}} \right)^{2} = 3 \left(\frac{T^{2}}{12} \right)^{2}, \tag{18}$$

$$\ln \mathcal{Z}^{(c)} = \left(\frac{N_c^2 - 1}{2} \, 2N_c\right) \frac{d}{4} \, g^2 \frac{V}{T} \cdot 3 \left(\frac{T^2}{12}\right)^2 = +\frac{V}{T} \, N_c \left(N_c^2 - 1\right) \frac{3d}{4} \, g^2 \frac{T^4}{144} \tag{19}$$

Here we use d to denote spatial dimensions; for d=3 we have $\frac{3d}{4}=\frac{9}{4}$.

The (d) term is built around the symmetrized 4-gluon vertex:

$$\left(\frac{1}{3!}\right)\left(-\frac{g^2}{4}\right)\left(f_{ad} \cdot f_{bc} \cdot (g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma}) + \left[(b,\nu) \leftrightarrow (d,\sigma)\right] + \left[(b,\nu) \leftrightarrow (c,\rho)\right]\right) \tag{20}$$

Here we use "•" to denote a contracted adjoint index, and use $(\cdots) \leftrightarrow (\cdots)$ to mark a switch of indices relative to the *previous* term. By contracting 2 pairs of the indices, we obtain the desired diagram (d). There are 3 ways to do this, therefore:

$$\ln \mathcal{Z}^{(d)} = \frac{1}{1!} \cdot 3 \cdot \left(\frac{1}{3!}\right) \left(-\frac{g^2}{4}\right) \left(-\frac{N_c^2 - 1}{2} 2N_c\right) \frac{T}{V} \sum_k \frac{T}{V} \sum_p \Delta(k) \Delta(p) \left(\frac{V}{T}\right) \delta_{+k-k+p-p} \times \left(-\left((d+1)^2 - (d+1)\right) + 0 + \left((d+1) - (d+1)^2\right)\right)$$
(21)

One of the three terms in the vertex vanishes after contraction, due to the anti-symmetry of f_{abc} . Again, we've used the fact that $g_{\mu\nu}g^{\mu\nu}=d+1$, where d is the spatial dimension.

Note that momentum conservation is automatic at the vertex, which is indicated by the trivial delta function $\delta_{+k-k+p-p} = 1$. In the end, we have:

$$\ln \mathcal{Z}^{(d)} = \left(-\frac{g^2}{8}\right) \left(\frac{N_c^2 - 1}{2} 2N_c\right) \frac{T}{V} \sum_k \frac{T}{V} \sum_p \Delta(k) \Delta(p) \left(\frac{V}{T}\right) 2d (d+1)$$

$$= \left(-\frac{g^2}{8}\right) \left(\frac{N_c^2 - 1}{2} 2N_c\right) \left(\frac{V}{T}\right) 2d (d+1) \left(\frac{T^2}{12}\right)^2$$

$$= -\frac{V}{T} N_c (N_c^2 - 1) \frac{d (d+1)}{4} g^2 \frac{T^4}{144}$$
(22)

For d=3 we have $\frac{d(d+1)}{4}=3$. Combining $(a \sim d)$, we have the full $\mathcal{O}(g^2)$ partition function.