Finite Temperature Field Theory №5

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Correlations of Pauli Spinor Fields

The Lagrangian of a nonrelativistic free particle is given by:

$$\mathcal{L} = \bar{\psi}\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m}\right)\psi \tag{1}$$

The Euclidean action is then given by:

$$(-S) = \int \mathrm{d}^4x\, \mathcal{L}_{t=-i\tau} = -\int \mathrm{d}^4x\, \bar{\psi}\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m}\right)\psi = -\beta H \tag{2}$$

Note that we have $N = \int \mathrm{d}^3\mathbf{x}\, \bar{\psi}\psi = \int \mathrm{d}^3\mathbf{x}\, n$ as a conserved charge, therefore it is natural to include N in the partition function:

$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \int \mathscr{D}\bar{\psi}\,\mathscr{D}\psi\, \exp \int \mathrm{d}^4x \left\{-\bar{\psi}\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m}\right)\psi + \mu n\right\} \tag{3}$$

For a spin-$\frac{1}{2}$ free fermion, ψ is given by a 2-component Pauli spinor: $\psi \sim (\psi_+, \psi_-)$, and $\bar{\psi} = \psi^\dagger$ is the conjugate transpose of ψ. Relevant observables in the Heisenberg picture are then given by:

$$n = \psi^\dagger\psi, \quad \tilde{n} = n - \langle n\rangle, \quad s_i = \psi^\dagger\sigma_i\psi, \tag{4}$$

$$D_n = \left\langle \mathcal{T}_\tau\, \tilde{n}(x)\, \tilde{n}(x')\right\rangle, \quad D_{ij} = \left\langle \mathcal{T}_\tau\, s_i(x)\, s_j(x')\right\rangle \tag{5}$$

To compute the **density correlation** D_n, define the generating functional:

$$Z[j] = \int \mathscr{D}\bar{\psi}\,\mathscr{D}\psi\, \exp \int \mathrm{d}^4x \left\{-\bar{\psi}\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m}\right)\psi + \mu n(x) + j(x)\, n(x)\right\} \tag{6}$$

To remove the non-zero $\langle n\rangle$, consider:

$$W[j] = \ln Z[j], \quad D_n = \left\langle \mathcal{T}_\tau\, \tilde{n}(x)\, \tilde{n}(x')\right\rangle = \left.\frac{\delta^2 W[j]}{\delta j(x)\, \delta j(x')}\right|_{j=0} = \left.\frac{\delta^2 W^{(2)}[j]}{\delta j(x)\, \delta j(x')}\right|_{j=0} \tag{7}$$

$$W^{(2)} \sim \mathcal{O}(j^2) \tag{8}$$

For free theory, $W^{(2)}$ can be computed explicitly by mode expansions:

$$\psi = \frac{1}{\sqrt{V}}\sum_k e^{ik\cdot x}\psi_k, \quad j = \sum_q e^{-iq\cdot x} j_q, \quad \sum_k e^{ik\cdot x} = V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{n\in\mathbb{Z}} e^{i\omega_n\tau} \tag{9}$$

$$\begin{aligned} Z[j] &= \int \mathscr{D}\bar{\psi}\,\mathscr{D}\psi\, \exp \int \mathrm{d}^4x \left\{-\bar{\psi}\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu - j(x)\right)\psi\right\} \\ &= \int \mathscr{D}\bar{\psi}\,\mathscr{D}\psi\, \exp \left\{-\beta \sum_{k,k'} \bar{\psi}_k \left(\left(i\omega + \tfrac{\mathbf{k}^2}{2m} - \mu\right)\delta_{k,k'} - j_{k-k'}\right)\psi_{k'}\right\} \\ &\sim \left\{\det_{k,k',\bullet}\left(\beta D_0^{-1}(k)\,\delta_{k,k'} - \beta j_{k-k'}\right)\right\}^{+1}, \quad D_0^{-1}(k) = i\omega + \tfrac{\mathbf{k}^2}{2m} - \mu, \\ &= \left\{\det_{k,k'}\left(\beta D_0^{-1}(k)\,\delta_{k,k'} - \beta j_{k-k'}\right)\right\}^{+2} \end{aligned} \tag{10}$$

Note that the determinant should go over the implicit spinor indices as well, which results in a power of 2 in the final expression.

We can then compute $W[j]$ using Jacobi's formula:

$$
\begin{aligned}
W[j] = \ln Z[j] &= 2 \ln \det_{k,k'} \left(\beta D_0^{-1}(k)\, \delta_{k,k'} - \beta j_{k-k'}\right) \\
&= 2 \operatorname*{Tr}_{k,k'} \ln \left(\beta D_0^{-1}(k)\, \delta_{k,k'} - \beta j_{k-k'}\right) \\
&= 2 \operatorname*{Tr}_{k,k'} \left\{\delta_{k,k'} \ln \left(\beta D_0^{-1}(k)\right) + \ln \left(\mathbb{1}_{k,k'} - j_{k-k'} D_0(k)\right)\right\}
\end{aligned}
\tag{11}
$$

The first term is the vacuum contribution with no dependence of j, hence it is irrelevant in $W^{(2)} \sim \mathcal{O}\big(j^2\big)$. Expansion of the matrix log in the second term reveals that:

$$
\ln \left(\mathbb{1}_{k,k'} - j_{k-k'} D_0(k)\right) = -\sum_{n=1}^{\infty} \frac{1}{n} \Big(j_{k-k'} D_0(k)\Big)^n_{k,k'}
\tag{12}
$$

$$
\begin{aligned}
W^{(2)} &= 2 \operatorname*{Tr}_{k,k'} \frac{-1}{2} \Big(j_{k-k'} D_0(k)\Big)^2_{k,k'} \\
&= -\operatorname*{Tr}_{k,k'} \sum_q \Big(j_{k-q} D_0(k)\Big) \Big(j_{q-k'} D_0(q)\Big) \\
&= -\sum_{k,q} j_{k-q} D_0(k)\, j_{q-k} D_0(q) \\
&= -\sum_k \sum_{q-k} j_{k-q} D_0(k)\, j_{q-k} D_0(q-k+k) \\
&= -\sum_q \sum_k j_{-q} \Big(D_0(k)\, D_0(q+k)\Big) j_q \\
&= \frac{\beta V}{2} \sum_k j_{-k} D_n(k)\, j_k,
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
D_n(k) &= -\frac{2}{\beta V} \sum_q D_0(q)\, D_0(k+q) \\
&= -\frac{2}{\beta V} \sum_q \frac{1}{i\omega_q + E_q} \frac{1}{i\left(\omega_q + \omega_k\right) + E_{q+k}}, \quad E_q = \frac{\mathbf{q}^2}{2m} - \mu, \\
&= -\frac{2}{\beta V} \sum_q \left(\frac{1}{i\omega_q + E_q} - \frac{1}{i\omega_q + i\omega_k + E_{q+k}}\right) \frac{1}{i\omega_k + E_{q+k} - E_k} \\
&= -2 \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{i\omega_k + E_{q+k} - E_q}\, T \sum_{\omega_q} \left(\frac{1}{i\omega_q + E_q} - \frac{1}{i\omega_q + i\omega_k + E_{q+k}}\right) \\
&= -2 \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{i\omega_k + E_{q+k} - E_q} \Big(\big(1 - n(E_q)\big) - \big(1 - n(E_{q+k} + i\omega_k)\big)\Big) \\
&= -2 \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{i\omega_k + E_{q+k} - E_q} \left(\frac{1}{-e^{\beta E_{q+k}} + 1} - \frac{1}{e^{\beta E_q} + 1}\right)
\end{aligned}
\tag{14}
$$

Here we've completed the Matsubara sum of the fermionic frequencies, using:

$$
T \sum_{\omega_q} \frac{1}{i\omega_q + E_q} = 1 - n(E_q), \quad n(E_q) = \frac{1}{e^{\beta E_q} + 1}, \quad n(E_q + i\omega_k) = \frac{1}{-e^{\beta E_q} + 1}
\tag{15}
$$

The retarded propagator and the spectral density is obtained by analytic continuation:

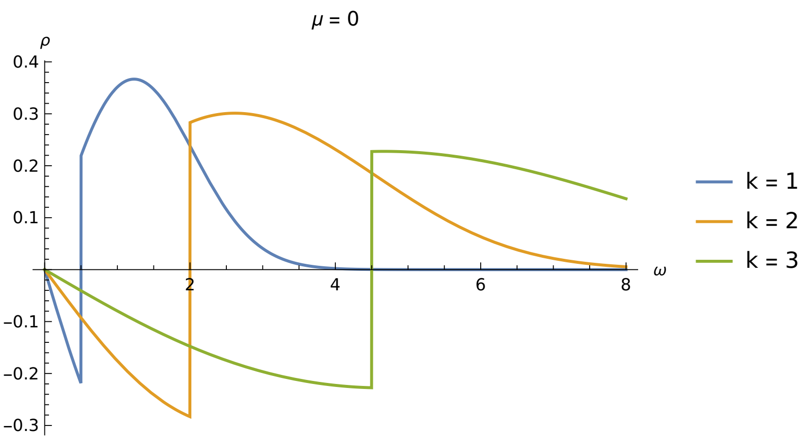
$$\begin{aligned} D_n^R(\omega,\mathbf{k}) &= D_n(\omega\to i\omega-\epsilon,\mathbf{k}) \\ &= -2\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\,\frac{1}{\omega+i\epsilon-(E_{q+k}-E_q)}\left(\frac{1}{e^{\beta E_{q+k}}-1}+\frac{1}{e^{\beta E_q}+1}\right), \end{aligned} \tag{16}$$

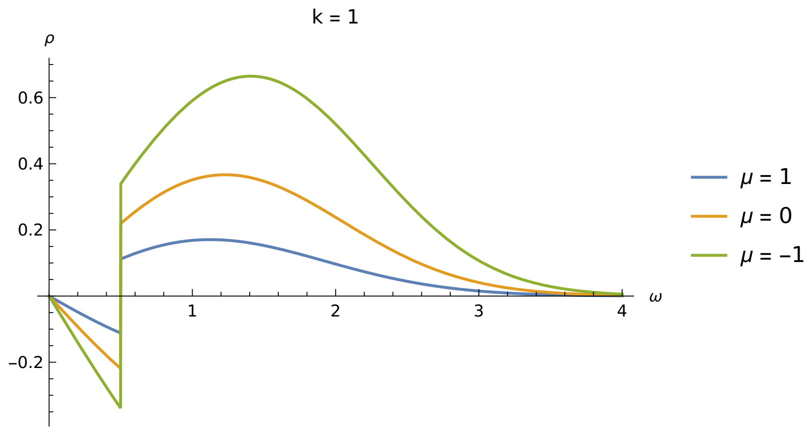
$$\begin{aligned} \rho_n &= 2\,\mathrm{Im}\,D_n^R(\omega,\mathbf{k}),\qquad \frac{1}{x\pm i\epsilon}=\mathcal{P}\frac{1}{x}\mp i\pi\delta(x) \\ &= 4\pi\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\,\delta\Big(\omega-(E_{q+k}-E_q)\Big)\left(\frac{1}{e^{\beta E_{q+k}}-1}+\frac{1}{e^{\beta E_q}+1}\right) \\ &= 4\pi\cdot\frac{1}{(2\pi)^3}\cdot 2\pi\int_0^\pi\sin\theta\,\mathrm{d}\theta\int_0^\infty q^2\,\mathrm{d}q\;\delta\Big(\omega-(E_{q+k}-E_q)\Big)\left(\frac{1}{e^{\beta(E_q+\omega)}-1}+\frac{1}{e^{\beta E_q}+1}\right) \\ &= \frac{1}{\pi}\int_0^\pi\sin\theta\,\mathrm{d}\theta\,\frac{m}{k\cos\theta}\int_0^\infty q^2\,\mathrm{d}q\;\delta\left(q-\frac{2m\omega-k^2}{2k\cos\theta}\right)\left(\frac{1}{e^{\beta(E_q+\omega)}-1}+\frac{1}{e^{\beta E_q}+1}\right) \\ &= \frac{1}{\pi}\int_{-1}^1\mathrm{d}x\,\frac{m}{kx}\left(\frac{q_0}{x}\right)^2\left(\frac{1}{e^{\beta(E_0/x^2-\mu+\omega)}-1}+\frac{1}{e^{\beta(E_0/x^2-\mu)}+1}\right)\theta\Big(\frac{q_0}{x}\Big) \end{aligned} \tag{17}$$

We have $k=\|\mathbf{k}\|,\ x=\cos\theta,\ q_0=\frac{2m\omega-k^2}{2k},\ E_0=\frac{q_0^2}{2m}$, and the Heaviside θ-function $\theta\big(\frac{q_0}{x}\big)$ which enforces that $q_0/x>0$. With $\mathrm{d}\big(\frac{1}{x^2}\big)=-\frac{2}{x^3}\,\mathrm{d}x$, we can rewrite the above integral into:

$$\begin{aligned} \rho_n &= \frac{1}{\pi}\left(-\operatorname{sgn}q_0\right)\left(-\frac{mq_0^2}{2k}\right)\int_1^\infty\mathrm{d}y\left(\frac{1}{e^{\beta(E_0y-\mu+\omega)}-1}+\frac{1}{e^{\beta(E_0y-\mu)}+1}\right),\quad y=\frac{1}{x^2}, \\ &= \left(\operatorname{sgn}q_0\right)\frac{mq_0^2}{2\pi k}\cdot\frac{1}{\beta E_0}\left\{-\ln\left(1-e^{-\beta(E_0-\mu+\omega)}\right)+\ln\left(1+e^{-\beta(E_0-\mu)}\right)\right\} \\ &= \left(\operatorname{sgn}q_0\right)\frac{m^2T}{k\pi}\ln\frac{1+e^{-\beta(E_0-\mu)}}{1-e^{-\beta(E_0-\mu+\omega)}},\quad k=\|\mathbf{k}\|,\quad q_0=\frac{2m\omega-k^2}{2k},\quad E_0=\frac{q_0^2}{2m} \end{aligned} \tag{18}$$

We see that the result grows linearly in T, and flips sign at $q_0=0$ or $\omega=\frac{k^2}{2m}$. Note that this is precisely the dispersion relation of ψ. We can interpret this result as particle pairs being created at such frequencies. For $m=1,\ T=1$, plots of ρ_n as a function of ω with various μ,k is shown below.





For the **spin correlations** D_{ij}, the above analysis still holds, and we need only change the source term in (6) from $j(x)\, n(x)$ to $J^i(x)\, s_i(x)$. We have:

$$Z[j] \sim \left\{ \det_{k,k',\bullet} \left(\beta D_0^{-1}(k)\, \delta_{k,k'} - \beta J^i_{k-k'} \sigma_i\right) \right\}^{+1}, \quad D_0^{-1}(k) = i\omega + \tfrac{\mathbf{k}^2}{2m} - \mu \tag{19}$$

There are now no-trivial dependence on the spinor indices. After completing the trace of Pauli matrices, we find that the result is basically the same as before, but with an additional δ_{ij} factor:

$$\begin{aligned} W^{(2)} &= -\frac{1}{2} \operatorname*{Tr}_{k,k',\bullet} \sum_q \left(J^i_{k-q} \sigma_i D_0(k)\right) \left(J^j_{q-k'} \sigma_j D_0(q)\right) \\ &= -\frac{1}{2} \sum_q \sum_k J^i_{-q} \Big(\mathrm{tr}(\sigma_i \sigma_j)\, D_0(k)\, D_0(q+k)\Big) J^j_q \\ &= \frac{\beta V}{2} \sum_k J^i_{-k} D_{ij}(k)\, J^j_k, \end{aligned} \tag{20}$$

$$D_{ij}(k) = \frac{1}{2}\, \mathrm{tr}(\sigma_i \sigma_j) D_n(k) = \delta_{ij} D_n(k), \quad \rho_{ij} = \delta_{ij} D_{ij}(k) \tag{21}$$

This means that the spin correlation along a same direction is identical with the density correlation, while there is no correlation between spins in different directions.