Advanced QFT №2

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1 Local Transformation

$$\delta A_\mu^a = \partial_\mu \lambda^a(x) + f^a{}_{bc} A_\mu^b \lambda^c(x), \tag{1}$$

Here f_{abc} is the totally anti-symmetric structure constant for a semi-simple Lie algebra $\mathfrak{g}$, with generators $\{T_a\}_a$ and normalized Killing form δ_{ab}.

- The field strength is defined as follows:

$$\begin{aligned} F_{\mu\nu} \equiv F_{\mu\nu}^a T_a = [D_\mu, D_\nu] &= \big[\partial_\mu + A_\mu, \partial_\nu + A_\nu\big] \\ &= \mathrm{d}A + A \wedge A \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + f^a{}_{bc} A_\mu^b A_\nu^c \, T_a \end{aligned} \tag{2}$$

Adjoint indices $a, b, \cdots$ are sometimes suppressed by contracting with T_a's. By exploiting the anti-symmetric property of $f^a{}_{bc}$, along with the Jacobi identity, we get the infinitesimal transformation:

$$\begin{aligned} \delta F_{\mu\nu}^a &= \partial_\mu \, \delta A_\nu^a - \partial_\nu \, \delta A_\mu^a + f^a{}_{bc} \, \delta(A_\mu^b A_\nu^c) \\ &= f^a{}_{bc} \Big(\lambda^c \big(\partial_\mu A_\nu^b - \partial_\nu A_\mu^b\big) + \big(A_\nu^b \, \partial_\mu \lambda^c - A_\mu^b \, \partial_\nu \lambda^c\big) + \delta(A_\mu^b A_\nu^c) \Big) \\ &= f^a{}_{bc} \Big(\lambda^c \big(F_{\mu\nu}^b - f^b{}_{de} A_\mu^d A_\nu^e\big) + \big(A_\nu^b \, \big(\underline{\delta A_\mu^c} - f^c{}_{de} A_\mu^d \, \lambda^e\big) - A_\mu^b \, \big(\underline{\delta A_\nu^c} - f^c{}_{de} A_\nu^d \, \lambda^e\big)\big) + \underline{\delta(A_\mu^b A_\nu^c)} \Big) \\ &= f^a{}_{bc} \Big(\lambda^c \big(F_{\mu\nu}^b - f^b{}_{de} A_\mu^d A_\nu^e\big) - \big(f^c{}_{de} A_\nu^b A_\mu^d \, \lambda^e - f^c{}_{de} A_\mu^b A_\nu^d \, \lambda^e\big) \Big) \\ &= \underline{f^a{}_{bc}} \Big(\lambda^c \Big(F_{\mu\nu}^b - \underline{f^b{}_{de}} A_\mu^d A_\nu^e \Big) - \Big(\underline{f^c{}_{de}} A_\nu^b A_\mu^d \, \lambda^e - \underline{f^c{}_{de}} A_\mu^b A_\nu^d \, \lambda^e \Big) \Big) \\ &= f^a{}_{bc} \, \lambda^c F_{\mu\nu}^b \end{aligned} \tag{3}$$

When contracted with T_a, this yields:

$$\delta F_{\mu\nu} = \lambda^c F_{\mu\nu}^b \, f^a{}_{bc} T_a = \lambda^c F_{\mu\nu}^b \, [T_b, T_c] = F_{\mu\nu} \cdot \lambda - \lambda \cdot F_{\mu\nu}, \tag{4}$$

$$\lambda = \lambda^c(x) \, T_c, \quad F_{\mu\nu} = F_{\mu\nu}^b \, T_b, \tag{5}$$

$$F_{\mu\nu} \;\longmapsto\; e^{-\Lambda^a(x) \, T_a} \, F_{\mu\nu} \, e^{\Lambda^a(x) \, T_a} \tag{6}$$

The exponentiation is valid even for local $\lambda = \lambda(x)$, since it is produced by integrating along the fiber direction $\lambda \to \Lambda$, not the spacetime direction x. This is the finite transformation w.r.t. $\Lambda(x)$.

- For any matter field ψ furnishing a representation of $\mathfrak{g}$, we have:

$$T_a \psi = (T_a)^i{}_j \, \psi^j, \quad \delta\psi = -\lambda^a(x) \, T_a \psi, \tag{7}$$

$$\psi \;\longmapsto\; e^{-\Lambda^a(x) \, T_a} \, \psi, \tag{8}$$

$$D_\mu \psi \;\longmapsto\; e^{-\Lambda^a(x) \, T_a} \, D_\mu \psi, \tag{9}$$

In fact, (1) is chosen to ensure that $D_\mu\psi$ transforms gauge covariantly just like ψ. Therefore,

$$\begin{aligned} D_\mu = \partial_\mu + A_\mu \;\longmapsto\; & e^{-\Lambda^a(x)\,T_a} \circ D_\mu \circ e^{\Lambda^a(x)\,T_a} \\ &= e^{-\Lambda} \circ (\partial_\mu + A_\mu) \circ e^{\Lambda} \\ &= e^{-\Lambda} \circ \partial_\mu \circ e^{\Lambda} + e^{-\Lambda} A_\mu\, e^{\Lambda}, \quad \Lambda = \Lambda^a(x)\,T_a, \end{aligned} \tag{10}$$

$$A_\mu \;\longmapsto\; e^{-\Lambda}\,(\partial_\mu e^{\Lambda}) + e^{-\Lambda} A_\mu\, e^{\Lambda} = T_a\,\partial_\mu \Lambda^a(x) + e^{-\Lambda} A_\mu\, e^{\Lambda} \tag{11}$$

- $F^2 \equiv F \wedge F$, we have:

$$\begin{aligned} F^2 &= (\mathrm{d}A + A \wedge A) \wedge (\mathrm{d}A + A \wedge A) \\ &= \mathrm{d}A \wedge \mathrm{d}A + \mathrm{d}A \wedge A \wedge A + A \wedge A \wedge \mathrm{d}A + A \wedge A \wedge A \wedge A \end{aligned} \tag{12}$$

The last term is proportional to $\epsilon_{abcd}\, T^a\, T^b\, T^c\, T^d$, hence its trace will vanish; therefore,

$$\begin{aligned} \mathrm{tr}\, F^2 &= \mathrm{tr}\,(\mathrm{d}A \wedge \mathrm{d}A + \mathrm{d}A \wedge A \wedge A + A \wedge A \wedge \mathrm{d}A) \\ &= \mathrm{tr}\left(\mathrm{d}(\mathrm{d}A \wedge A) + \frac{2}{3}\,\mathrm{d}(A \wedge A \wedge A)\right) \\ &= \mathrm{d}\,\mathrm{tr}\left(\mathrm{d}A \wedge A + \frac{2}{3}\,A \wedge A \wedge A\right) = \mathrm{d}\omega\,, \end{aligned} \tag{13}$$

$$\omega = \mathrm{tr}\left(\mathrm{d}A \wedge A + \frac{2}{3}\,A \wedge A \wedge A\right) \tag{14}$$

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2 Relativistic Particle

$$L = \frac{1}{2e}\left(\frac{1}{c}\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2 - \frac{e}{2}\,m^2 c^4 \tag{15}$$

- For $t \mapsto t' = t - \xi(t)$, we have $X'(t') = X(t)$, therefore:

$$\delta X^\mu = -\,\delta t\,\frac{\mathrm{d}X^\mu}{\mathrm{d}t} = \xi(t)\,\dot{X}^\mu, \tag{16}$$

Or more explicitly, $X^\mu(t) \mapsto X^\mu(t) + \xi(t)\,\dot{X}^\mu$.

- We have:

$$\begin{aligned} \delta L &= \frac{1}{ec^2}\,\dot{X}_\mu\,\delta\dot{X}^\mu - \frac{\delta e}{2}\,\frac{1}{e^2c^2}\dot{X}^2 - \frac{\delta e}{2}\,m^2c^4 \\ &= \frac{1}{ec^2}\,\xi\dot{X}_\mu\ddot{X}^\mu + \frac{1}{ec^2}\,\dot{\xi}\dot{X}^2 - \frac{\delta e}{2}\,\frac{1}{e^2c^2}\dot{X}^2 - \frac{\delta e}{2}\,m^2c^4 \end{aligned} \tag{17}$$

For $S = \int \mathrm{d}t\, L$ to be invariant, δL should be reduced to a total derivative, which can then be reduced to some vanishing boundary terms.

Consider $\delta e = \frac{\mathrm{d}}{\mathrm{d}t}(e\xi) = \dot{e}\xi + e\dot{\xi}$, and we have:

$$\begin{aligned} \delta L &= \frac{1}{ec^2}\,\xi\dot{X}_\mu\ddot{X}^\mu + \frac{1}{2ec^2}\,\dot{\xi}\dot{X}^2 - \frac{\dot{e}}{2e^2c^2}\,\xi\dot{X}^2 - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\,e\xi\,m^2c^4\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(\frac{1}{2ec^2}\dot{X}^2 - \frac{e}{2}\,m^2c^4\right)\xi\right\} = \frac{\mathrm{d}}{\mathrm{d}t}(\xi L) \end{aligned} \tag{18}$$

Indeed we get a total derivative; therefore,

$$\delta e = \frac{\mathrm{d}}{\mathrm{d}t}\big(e\xi\big), \quad \delta S = \int \delta L = \int \mathrm{d}\big(\xi L\big) = 0 \tag{19}$$

- $e(t)$ can be seen as a gauge field coupled to X, which captures the t–reparametrization redundancy through the gauge transformation parameter $\xi(t)$. A natural gauge choice is fixing $f = e(t) - 1 \equiv 0$, which is equivalent to setting $t = \tau$: the proper time, or affine parametrization for the massless case.

The gauge invariant path integral is constructed as follows:

$$\begin{aligned}
\mathcal{Z} &= \frac{1}{\int \mathcal{D}\xi} \int \mathcal{D}X\, \mathcal{D}e\, e^{iS} \\
&= \frac{1}{\int \mathcal{D}\xi} \int \mathcal{D}X\, \mathcal{D}e\, e^{iS} \int \mathcal{D}f\, \delta\big[f\big] \\
&= \frac{1}{\int \mathcal{D}\xi} \int \mathcal{D}X\, \mathcal{D}e\, e^{iS} \int \mathcal{D}\xi\, \delta\big[f_\xi\big] \det \frac{\delta f_\xi}{\delta \xi} \\
&= \frac{1}{\int \mathcal{D}\xi} \int \mathcal{D}\xi \int \mathcal{D}X\, \mathcal{D}e\, e^{iS}\, \delta\big[f_\xi\big] \det \frac{\delta f_\xi}{\delta \xi} \\
&= \frac{1}{\int \mathcal{D}\xi} \int \mathcal{D}\xi \int \mathcal{D}X_\xi\, \mathcal{D}e_\xi\, e^{iS_\xi}\, \delta\big[f_\xi\big] \det \frac{\delta f_\xi}{\delta \xi}\bigg|_{e_\xi} \\
&= \frac{1}{\int \mathcal{D}\xi} \int \mathcal{D}\xi \int \mathcal{D}X\, \mathcal{D}e\, e^{iS}\, \delta\big[f\big] \det \frac{\delta f_\xi}{\delta \xi}\bigg|_{\xi=0} \\
&= \int \mathcal{D}X\, \mathcal{D}e\, e^{iS}\, \delta\big[f\big] \det \frac{\delta f_\xi}{\delta \xi}\bigg|_{\xi=0}
\end{aligned} \tag{20}$$

Here the gauge-transformed quantities are marked with a ξ subscript. Note that in the final expression, f is *not* integrated out and can be *any* possible gauge-fixing function, i.e. $\Big(\delta\big[f\big] \det \frac{\delta f_\xi}{\delta \xi}\Big)_{\xi=0}$ is in fact f-independent.

These are the first steps of the Faddeev–Popov (FP) procedure; it achieves several things at once: first it imposes a gauge-fixing $f = 0$, and then it removes the gauge redundancy with the help of FP determinant $\big(\det \frac{\delta f_\xi}{\delta \xi}\big)_{\xi=0}$, while implicitly imposing the constraints resulted from the gauge-fixing process. The constraints implemented by $\big(\det \frac{\delta f_\xi}{\delta \xi}\big)_{\xi=0}$ can be made explicit with the help of BRST formalism.

The gauge-fixing term $\delta\big[f\big]$ can be replaced by a Gaussian packet with width parameter ζ. More rigorously, up to an overall constant coefficient, we have the following equivalence:

$$\delta\big[f\big] \sim \delta\big[f - f_0\big] \sim \int \mathcal{D}f_0\, \exp\bigg(-\frac{i}{2\zeta}\int \mathrm{d}t\, f_0^2\bigg)\, \delta\big[f - f_0\big] = \exp\bigg(i\int \mathrm{d}t\, L_{gf}\bigg), \tag{21}$$

$$L_{gf} = -\frac{1}{2\zeta} f^2 = -\frac{1}{2\zeta}\,(e-1)^2, \tag{22}$$

Here f_0 is some gauge-invariant shift of f, namely $(f_0)_\xi = f_0$. f_0 can be seen as a non-dynamical auxiliary field that enforce the gauge fixing, much similar to a Lagrange multiplier. On the other hand, the determinant can be evaluated using Faddeev–Popov (FP) ghosts b, c:

$$\det P \sim \int \mathscr{D}b\, \mathscr{D}c \exp\Big(i \int \mathrm{d}t \int \mathrm{d}t'\, b(t) \cdot P(t,t') \cdot c(t')\Big), \tag{23}$$

$$\frac{\delta f_\xi(t)}{\delta \xi(t')}\bigg|_{\xi=0} = \frac{\delta}{\delta \xi(t')}\bigg|_{\xi=0} \Big(e + \frac{\mathrm{d}}{\mathrm{d}t}\big(e\xi\big) - 1\Big)_{(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(e(t)\,\delta(t-t')\Big), \tag{24}$$

$$\begin{aligned} \det \frac{\delta f_\xi(t)}{\delta \xi(t')}\bigg|_{\xi=0} &\sim \int \mathscr{D}b\, \mathscr{D}c \exp\Big(i \int \mathrm{d}t \int \mathrm{d}t'\, b(t) \Big(e(t)\,\delta(t-t')\Big)\, c(t')\Big) \\ &\sim \int \mathscr{D}b\, \mathscr{D}c \exp\Big(-i \int \mathrm{d}t\, e\dot{b}c\Big), \end{aligned} \tag{25}$$

$$L_{gh} = -e\dot{b}c \tag{26}$$

In summary, we have:

$$\mathcal{Z} = \int \mathscr{D}X\, \mathscr{D}e\, \mathscr{D}b\, \mathscr{D}c\, e^{iS_q}, \quad S_q = \int \mathrm{d}t\, L_q, \tag{27}$$

$$L_q = L + L_{gf} + L_{gh} = L - \frac{1}{2\zeta}\,(e-1)^2 - e\dot{b}c, \tag{28}$$

S_q is the quantum action under the gauge-fixing condition $f = e(t) - 1 = 0$. ■

3 2D σ-Model

$$\mathcal{L} = -\frac{1}{2}\,\partial_\alpha X^\mu\,\partial_\beta X_\mu \sqrt{-h}\, h^{\alpha\beta}, \quad X\colon \Sigma^{1,1} \to \mathbb{R}^{D-1,1} \tag{29}$$

- The action is diff-invariant; under $\sigma^\alpha \mapsto \sigma^\alpha + \xi^\alpha$, we have:

$$\delta X^\alpha = \mathcal{L}_\xi X^\alpha, \quad \delta h^{\alpha\beta} = \mathcal{L}_\xi h^{\alpha\beta} \tag{30}$$

$\mathcal{L}_\xi$ is the Lie derivative along ξ^α. Note that $0 = \delta\big(h_{\alpha\beta}h^{\beta\gamma}\big)$, hence we have:

$$\delta h_{\alpha\beta} = -h_{\alpha\alpha'}h_{\beta\beta'}\,\delta h^{\alpha'\beta'} = \xi^\gamma\partial_\gamma h_{\alpha\beta} + (\partial_\alpha\xi^\gamma)\,h_{\gamma\beta} + (\partial_\beta\xi^\gamma)\,h_{\alpha\gamma} = \mathcal{L}_\xi h_{\alpha\beta}, \tag{31}$$

$$\delta\sqrt{-h} = \frac{1}{2}\sqrt{-h}\,h^{\alpha\beta}\,\delta h_{\alpha\beta}\,, \tag{32}$$

Furthermore, we have $\mathcal{L}_\xi \mathrm{d}X = \mathrm{d}(\mathcal{L}_\xi X)$, i.e. $\partial_\alpha\,\delta X = \partial_\alpha\,\mathcal{L}_\xi X = \partial_\alpha\big(\xi^\gamma\partial_\gamma X\big) = \mathcal{L}_\xi\big(\partial_\alpha X\big)$. Note that due to the $\sqrt{-h}$ factor, $\mathcal{L}$ is not a scalar but a *scalar density.* For convenience, define $\mathcal{L} = \widetilde{\mathcal{L}}\sqrt{-h}$, then $\widetilde{\mathcal{L}} = -\frac{1}{2}\,h^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X_\mu$ is a scalar; using chain rule, we obtain:

$$\begin{aligned}
\delta\mathcal{L} &= \sqrt{-h}\,\delta\widetilde{\mathcal{L}} + \widetilde{\mathcal{L}}\,\delta\sqrt{-h} \\
&= \sqrt{-h}\,\mathcal{L}_\xi\widetilde{\mathcal{L}} + \widetilde{\mathcal{L}}\,\delta\sqrt{-h} \\
&= \sqrt{-h}\,\xi^\gamma\partial_\gamma\widetilde{\mathcal{L}} + \widetilde{\mathcal{L}}\,\delta\sqrt{-h} \\
&= \partial_\gamma\left(\xi^\gamma\widetilde{\mathcal{L}}\sqrt{-h}\right) - \widetilde{\mathcal{L}}\left(\sqrt{-h}\,(\partial_\gamma\xi^\gamma) + \xi^\gamma\,(\partial_\gamma\sqrt{-h})\right) + \widetilde{\mathcal{L}}\,\delta\sqrt{-h} \\
&= \partial_\gamma\left(\xi^\gamma\mathcal{L}\right) - \widetilde{\mathcal{L}}\left(\sqrt{-h}\,(\partial_\gamma\xi^\gamma) + \xi^\gamma\,(\partial_\gamma\sqrt{-h}) - \delta\sqrt{-h}\right) \\
&= \partial_\gamma\left(\xi^\gamma\mathcal{L}\right) - \widetilde{\mathcal{L}}\sqrt{-h}\left(\partial_\gamma\xi^\gamma + \frac{1}{2}\,\xi^\gamma\,h^{\alpha\beta}\partial_\gamma h_{\alpha\beta} - \frac{1}{2}\,h^{\alpha\beta}\,\delta h_{\alpha\beta}\right) \\
&= \partial_\gamma\left(\xi^\gamma\mathcal{L}\right) - \widetilde{\mathcal{L}}\sqrt{-h}\left(\partial_\gamma\xi^\gamma - \frac{1}{2}\,h^{\alpha\beta}\left((\partial_\alpha\xi^\gamma)\,h_{\gamma\beta} + (\partial_\beta\xi^\gamma)\,h_{\alpha\gamma}\right)\right) \\
&= \partial_\gamma\left(\xi^\gamma\mathcal{L}\right) - \widetilde{\mathcal{L}}\sqrt{-h}\left(\partial_\gamma\xi^\gamma - \partial_\gamma\xi^\gamma\right) \\
&= \partial_\gamma\left(\xi^\gamma\mathcal{L}\right)
\end{aligned} \tag{33}$$

We see that $\delta\mathcal{L}$ is a total derivative, hence $\delta S = \int \mathrm{d}^2\sigma\,\delta\mathcal{L} = 0$, i.e. the action is diff-invariant.

- The action is Weyl invariant; with $\delta h^{\alpha\beta} = -\lambda(\sigma)\,h^{\alpha\beta}$, we have:

$$\begin{aligned}
\delta\left(\sqrt{-h}\,h^{\alpha\beta}\right) &= \sqrt{-h}\,\delta h^{\alpha\beta} + h^{\alpha\beta}\,\delta\sqrt{-h} \\
&= \sqrt{-h}\,h^{\alpha\beta}\left(-\lambda - \frac{1}{2}\,h_{\alpha'\beta'}\,\delta h^{\alpha'\beta'}\right) \\
&= \sqrt{-h}\,h^{\alpha\beta}\left(-\lambda + \frac{1}{2}\,\lambda\,h_{\alpha'\beta'}h^{\alpha'\beta'}\right) \\
&= \sqrt{-h}\,h^{\alpha\beta}\left(-\lambda + \frac{2}{2}\,\lambda\right) \\
&= 0
\end{aligned} \tag{34}$$

Here we've used the fact that $h_{\alpha\beta}h^{\alpha\beta} = \delta^\alpha_\alpha = 2$. Therefore, $\delta\mathcal{L} = -\frac{1}{2}\,\partial_\alpha X^\mu\,\partial_\beta X_\mu\delta\big(\sqrt{-h}\,h^{\alpha\beta}\big) = 0$, i.e. the action is Weyl invariant.

- FP quantization of this system follows the same recipe as the point particle case above:

$$\mathcal{Z} = \int \mathcal{D}X\ \mathcal{D}h\ \mathcal{D}b\ \mathcal{D}c\ e^{iS_q}, \quad S_q = \int \mathrm{d}^2\sigma\,\mathcal{L}_q, \tag{35}$$

$$\mathcal{L}_q = \mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{gh} \tag{36}$$

Given gauge fixing: $f^{\alpha\beta} = h^{\alpha\beta} - h^{\alpha\beta}_{(0)}$, we have:

$$\mathcal{L}_{gf} = -\frac{1}{2\zeta}\,f^{\alpha\beta}f_{\alpha\beta}\sqrt{-h} = -\frac{1}{2\zeta}\left(h^{\alpha\beta} - h^{\alpha\beta}_{(0)}\right)\left(h_{\alpha\beta} - h^{(0)}_{\alpha\beta}\right)\sqrt{-h} \tag{37}$$

The FP ghost term $\mathscr{L}_{gh}$ is given by functional determinant; we have:

$$\left.\frac{\delta f_\xi^{\alpha\beta}(\sigma)}{\delta\xi^\gamma(\sigma')}\right|_0 = \left.\frac{\delta}{\delta\xi^\gamma(\sigma')}\right|_0 \left(\mathcal{L}_\xi h^{\alpha\beta} - \lambda h^{\alpha\beta}\right)_{(\sigma)} \tag{38}$$

$$= \delta(\sigma-\sigma')\,\partial_\gamma h^{\alpha\beta} - \delta_\gamma^\alpha\,\partial^\beta\delta(\sigma-\sigma') - \delta_\gamma^\beta\,\partial^\alpha\delta(\sigma-\sigma') \tag{39}$$

$$= -\delta_\gamma^\alpha\,\nabla^\beta\delta(\sigma-\sigma') - \delta_\gamma^\beta\,\nabla^\alpha\delta(\sigma-\sigma'), \tag{40}$$

$$\left.\frac{\delta f_\xi^{\alpha\beta}(\sigma)}{\delta\lambda(\sigma')}\right|_0 = -\delta(\sigma-\sigma')\,h^{\alpha\beta}, \tag{41}$$

Here we've replaced ∂ with ∇ which commutes with the metric $h_{\alpha\beta}$. Define $\Xi^\Gamma = (\xi^\gamma, \lambda)$ to combine all gauge parameters, and use fermionic FP ghosts: $b_{\alpha\beta}$, $c^\Gamma = (c^\gamma, c')$ to contract the indices; after some integration by parts, we have:

$$\det\left.\frac{\delta f_\xi^{\alpha\beta}(\sigma)}{\delta\Xi^\Gamma(\sigma')}\right|_0 \sim \int \mathscr{D}b_{\alpha\beta}\,\mathscr{D}c^\gamma\,\mathscr{D}c'\,\exp\left(i\int \mathrm{d}^2\sigma\,\sqrt{-h}\,b_{\alpha\beta}\left(-\nabla^\beta c^\alpha - \nabla^\alpha c^\beta - h^{\alpha\beta}c'\right)\right) \tag{42}$$

To simplify the action, it is common[1] to integrate c' out, which constrains $b_{\alpha\beta}$ to be symmetric traceless: $b_{\alpha\beta}h^{\alpha\beta} = b_\alpha^\alpha = 0$. The resulting $b_{\alpha\beta}$ has 2 degrees of freedom, same as c^γ.

In the end, we have:

$$\det\left.\frac{\delta f_\xi^{\alpha\beta}(\sigma)}{\delta\Xi^\Gamma(\sigma')}\right|_0 \sim \int \mathscr{D}b_{\alpha\beta}\,\mathscr{D}c^\gamma\,\exp\left(-2i\int \mathrm{d}^2\sigma\,\sqrt{-h}\,b_{\alpha\beta}\,\nabla^\alpha c^\beta\right) \tag{43}$$

Therefore, FP quantization with $f^{\alpha\beta} = h^{\alpha\beta} - h_{(0)}^{\alpha\beta}$ yields:

$$\mathcal{Z} = \int \mathscr{D}X\,\mathscr{D}h^{\alpha\beta}\,\mathscr{D}b_{\alpha\beta}\,\mathscr{D}c^\gamma\,e^{iS_q}, \quad S_q = \int \mathrm{d}^2\sigma\,\mathscr{L}_q, \quad \mathscr{L}_q = \mathscr{L} + \mathscr{L}_{gf} + \mathscr{L}_{gh}, \tag{44}$$

$$\mathscr{L}_{gf} = -\frac{1}{2\zeta}\,f^{\alpha\beta}f_{\alpha\beta}\sqrt{-h} = -\frac{1}{2\zeta}\left(h^{\alpha\beta} - h_{(0)}^{\alpha\beta}\right)\left(h_{\alpha\beta} - h_{\alpha\beta}^{(0)}\right)\sqrt{-h}, \tag{45}$$

$$\mathscr{L}_{gh} = -2b_{\alpha\beta}\,\nabla^\alpha c^\beta\sqrt{-h} \tag{46}$$

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[1] References: *Tong:* http://damtp.cam.ac.uk/user/tong/string.html, and also *Polchinski*.