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1 T-duality of Heterotic Strings¹

We use $d \leq 10$ to denote the number of noncompact dimensions; the remaining $m \geq d$ dimensions are compactified. For heterotic strings, the $I \geq 10$ dimensions of the left-moving sector are already compactified on a lattice Γ_{16} or $\Gamma_8 \times \Gamma_8$. Here we use the label I to index the 16 internal dimensions.

(a) Generally, if we compactify an open string on the x^m direction: $x^m \cong x^m + 2\pi R$ with constant backgrounds A_m , then its zero mode spectrum, with winding w = 0, can be obtained from canonical quantization of the effective point particle action, with an additional gauge action term in the form of a Wilson line²:

$$-W_q = -iq \int dx^m A_m \sim -iq \int d\tau A_m \dot{X}^m$$
 (1)

By imposing that the canonical momentum to be periodic along x^m , we find that:

$$k_m = \frac{n_m}{R} - qA_m \tag{2}$$

To obtain the winding states, we have to reproduce the above action from the world-sheet description. For heterotic strings with $m < 10 \le I \le 26$, this can be achieved by adding the following term to the usual world-sheet action³:

$$S_A \propto \int \mathrm{d}^2 \sigma \, \epsilon^{ab} A^I_\mu \, \partial_a X^\mu \, \partial_b X_I$$
 (3)

With proper normalization to match the result in (2).

Canonical quantization then produces⁴:

$$k_{m} = \frac{n_{m}}{R} \pm \frac{w_{m}R}{\alpha'} - q_{I}A_{m}^{I} - \frac{w_{n}R}{2}A_{I}^{n}A_{m}^{I}, \tag{4}$$

$$k_L^I = \sqrt{\frac{2}{\alpha'}} \left(q^I + w^m R A_m^I \right), \tag{5}$$

The " \pm " signs in k_m correspond to the left and right-moving sector, respectively. Only the left-moving sector has an additional 16 dimensional internal torus, therefore k^I is labeled with an "L".

Note that the charge q^I now takes value on the Γ_{16} or $\Gamma_8 \times \Gamma_8$ lattice, and:

$$l \circ l' = \frac{\alpha'}{2} \left(k_L^I k_{L,I}' + k_L^m k_{L,m}' - k_R^m k_{R,m}' \right) = q^I q_I' + 2nw$$
 (6)

We can then see that the new "extended" lattice indeed satisfies the even and self-dual conditions, which follows from the even and self-dual properties of Γ_{16} or $\Gamma_8 \times \Gamma_8$.

 $^{^1\}mathrm{I}$ would like to thank Lucy Smith for help with this problem.

²Reference: *Polchinski*, Chapter 8.

³Reference: Blumenhagen et al., Basic Concepts of String Theory.

 $^{^4 {\}it Reference: Polchinski},$ Chapter 11.

(b) With m = 9 and $G_{dd} = 1$, we have:

$$W_q = \exp\left(-iq_I\theta^I\right), \quad A_9^I = -\frac{\theta^I}{2\pi R}$$
 (7)

Note that W_q captures the phase change of the paths that wind around x^9 ; the extra phase from a non-trivial Wilson line might affect the boundary condition of some states while leaving others intact, thus breaking the original gauge symmetry. Our discussions here closely follow Polchinski, Chapter 11.

For the SO(32) theory with:

$$RA_9^I = \operatorname{diag}\left(\left(\frac{1}{2}\right)^8, 0^8\right)$$
 (8)

Adjoint states are labeled by a pair of indices valued in $1, \dots, 32$; those with one index from $1 \le A \le 16$ and one from $17 \le A \le 32$ are anti-periodic due to the additional phase $e^{i\pi} = -1$ from the Wilson line, so the gauge symmetry is reduced to $SO(16) \times SO(16)$.

Similarly, for the $E_8 \times E_8$ theory with:

$$R'A_9^I = \operatorname{diag}(1, 0^7, 1, 0^7) \tag{9}$$

Note that Γ_8 , the root lattice of E_8 , is basically the root lattice union an additional spinor weight lattice of SO(16). With the above Wilson line, the integer-charged states from the SO(16) root lattice in each E_8 remain periodic, while the half-integer charged states from the SO(16) spinor lattices become anti-periodic, due to the additional phase $e^{i\frac{1}{2}\cdot 2\pi}=-1$. Again the gauge symmetry is broken down to SO(16) \times SO(16).

In summary, with the above Wilson line, the SO(32) and $E_8 \times E_8$ theory shares an unbroken gauge of SO(16) \times SO(16). Consider the spectrum of the SO(16) \times SO(16) neutral states, i.e. those with internal momentum:

$$k_L^I = \sqrt{\frac{2}{\alpha'}} \left(q^I + wRA_9^I \right) = 0 \tag{10}$$

For the SO(32) theory, since $q^I \in \Gamma_{16}$ while $RA_9^I = \text{diag}\left((\frac{1}{2})^8, 0^8\right)$, we must have w = 2m for this to hold. The same goes for the $E_8 \times E_8$ theory; therefore, we have:

$$k_{L,R} = \frac{\tilde{n}}{R} \pm \frac{2mR}{\alpha'}, \quad k'_{L,R} = \frac{\tilde{n}'}{R'} \pm \frac{2m'R'}{\alpha'}, \tag{11}$$

$$\tilde{n} = n + 2m, \quad \tilde{n}' = n' + 2m'$$
 (12)

(c) If the two theories are related by T-duality, then we should expect:

$$(k_L, k_R) \longleftrightarrow (k_L', -k_R'), \tag{13}$$

Under suitable mapping of parameters. Indeed, it is straightforward to verify that $(\tilde{n}, m) \leftrightarrow (m', \tilde{n}')$ realizes this, along with $RR' = \alpha'/2$. The above arguments can then be generalized to higher levels, by acting on fermionized left-moving fields λ^A and carefully organizing representations. We see that the two heterotic string theories are equivalent under T-duality.

2 String Junction⁵

For a string junction to be mechanically stable, the tension force exerted on the junction must cancel each other; this is a Newtonian mechanics problem, but with (p,q)-string tension given by the BPS bound:

$$\tau_{(p,q)} = \frac{\sqrt{p^2 + q^2/g^2}}{2\pi\alpha'} \tag{14}$$

Stability of the system implies that the BPS bound should be saturated.

From Newtonian mechanics, we know that three forces cannot cancel each other unless they are co-planar. Therefore, a 3-string junction must be co-planar in order to be stable. Suppose they lie in the (X^1, X^2) plane, then the tension exerted on the junction can expressed as:

$$\vec{T}_i = \tau_{(p_i, q_i)} \left(\cos \theta_i, \sin \theta_i\right), \quad i = 1, 2, 3, \tag{15}$$

 $\sum_{i} \vec{T}_{i} = 0$ gives two equations, and we have two independent unknowns (the angle between two pairs of strings); therefore if a solution exists, it should be unique up to rotations and reflections.

In fact, a solution can be found by simple observations:

$$\cos \theta_i = \frac{p_i}{\sqrt{p^2 + q^2/g^2}}, \quad \sin \theta_i = \frac{q_i/g}{\sqrt{p^2 + q^2/g^2}},$$
 (16)

It satisfies $\sum_{i} \vec{T}_{i} = 0$ since that total (p,q) vanishes at each junction.

To find the remaining supersymmetries of this system, we start from the original supersymmetries of a (p,q) string (which saturates the BPS bound) extended along the $\hat{X} = (\cos \theta, \sin \theta)$ direction:

$$\frac{1}{2L\tau_{(p,q)}} \left\{ \begin{bmatrix} Q_{\alpha} \\ \tilde{Q}_{\alpha} \end{bmatrix}, \begin{bmatrix} Q_{\beta}^{\dagger} \ \tilde{Q}_{\beta}^{\dagger} \end{bmatrix} \right\} = \delta_{\alpha\beta} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (\Gamma^{0}\Gamma^{\theta})_{\alpha\beta} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}, \tag{17}$$

$$\Gamma^{\theta} = \hat{X} \cdot \vec{\Gamma} = \Gamma^{1} \cos \theta + \Gamma^{2} \sin \theta \tag{18}$$

We see that the algebra depends on θ , i.e. it is different for strings in different directions. However, if we can find a (maximal) subalgebra that is independent of θ , then we would have found the remaining supersymmetries of the full system⁶.

We begin with diagonalizing the matrix on the RHS with:

$$U(\frac{\theta}{2}) = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix},\tag{19}$$

$$\frac{1}{2L\tau_{(p,q)}} \left\{ U^{\mathrm{T}} \begin{bmatrix} Q_{\alpha} \\ \tilde{Q}_{\alpha} \end{bmatrix}, \left[Q_{\beta}^{\dagger} \ \tilde{Q}_{\beta}^{\dagger} \right] U \right\} = \begin{bmatrix} (\mathbb{1} + \Gamma^{0}\Gamma^{\theta})_{\alpha\beta} & 0 \\ 0 & (\mathbb{1} - \Gamma^{0}\Gamma^{\theta})_{\alpha\beta} \end{bmatrix}, \tag{20}$$

Note that $(\mathbb{1} + \Gamma^0 \Gamma^\theta)(\mathbb{1} - \Gamma^0 \Gamma^\theta) = 0$ and $(\mathbb{1} + \Gamma^0 \Gamma^\theta) + (\mathbb{1} - \Gamma^0 \Gamma^\theta) = 2\mathbb{1}$, i.e. they are orthogonal to each other; acting $(\mathbb{1} \pm \Gamma^0 \Gamma^\theta)$ on both sides, we find the following combinations, which gives the 16 SUSYs of a (p,q) string:

$$(\mathbb{1} - \Gamma^0 \Gamma^\theta) \left(\cos \frac{\theta}{2} Q + \sin \frac{\theta}{2} \tilde{Q} \right)_{\alpha} = 0 = (\mathbb{1} + \Gamma^0 \Gamma^\theta) \left(-\sin \frac{\theta}{2} Q + \cos \frac{\theta}{2} \tilde{Q} \right)_{\beta}$$
 (21)

⁵Reference: arXiv:0812.4408.

⁶The $\tau_{(p,q)}$ factor can be absorbed by rescaling generators, hence does not matter in our discussions.

For further simplification, we can isolate the θ dependence in Γ^{θ} by working in a specific representation of the Clifford algebra, e.g. the Dirac representation given by Polchinski. Then α is given by 10 D spinor components: $\alpha = (s_0, s_1, s_2, s_3, s_4), \ s_i = \pm$, with additional chirality constraints from both Q and \tilde{Q} : $\prod_i s_i = +$. In the end, we have 16 independent components as expected.

Details of the expansion are given in arXiv:0812.4408. When the dust settles, we find that the 16 SUSYs in (21) is given by:

$$\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}Q + \sin\frac{\theta}{2}\tilde{Q}\right)_{(++s)} + \cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}Q + \sin\frac{\theta}{2}\tilde{Q}\right)_{(--s)},\tag{22a}$$

$$\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}Q + \sin\frac{\theta}{2}\tilde{Q}\right)_{(+-s)} - \cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}Q + \sin\frac{\theta}{2}\tilde{Q}\right)_{(-+s)},\tag{22b}$$

$$\cos\frac{\theta}{2}\left(-\sin\frac{\theta}{2}Q + \cos\frac{\theta}{2}\tilde{Q}\right)_{(++s)} - \sin\frac{\theta}{2}\left(-\sin\frac{\theta}{2}Q + \cos\frac{\theta}{2}\tilde{Q}\right)_{(--s)},\tag{22c}$$

$$\cos\frac{\theta}{2}\left(-\sin\frac{\theta}{2}Q + \cos\frac{\theta}{2}\tilde{Q}\right)_{(+-s)} + \sin\frac{\theta}{2}\left(-\sin\frac{\theta}{2}Q + \cos\frac{\theta}{2}\tilde{Q}\right)_{(-+s)},\tag{22d}$$

$$s = (s_2 s_3 s_4), \quad \prod_i s_i = +$$
 (23)

By trial and error, we can find the 8 linear combinations that are independent of θ ; they are:

$$(a) + (c) \implies \tilde{Q}_{(++s)} + Q_{(--s)}, \tag{24}$$

$$(b) + (d) \implies \tilde{Q}_{(+-s)} - Q_{(-+s)}, \tag{25}$$

Therefore, the string junction is $\frac{8}{32} = \frac{1}{4}$ BPS.

$|\,3\,|$ Two and Three-Point Functions in AdS/CFT

Consider a scalar field $\phi(x,z)$ in Poincaré AdS₅ (with radius R=1) satisfying:

$$(\nabla^2 - m^2) \phi(x, z) = 0, \quad \phi(x, z) \to \begin{cases} z^{\delta} \phi_0(x), & z \to 0, \\ \text{regular}, & z \to \infty, \end{cases} \quad \delta = 2 - \sqrt{m^2 + 4}$$
 (26)

It can be constructed via the boundary-to-bulk propagator K_{Δ} :

$$\phi(x,z) = \int d^4x' K_{\Delta}(x,z;x') \phi(x'), \qquad (27)$$

$$K_{\Delta}(x,z;x') = \frac{(\Delta - 1)(\Delta - 2)}{\pi^2} \left(\frac{z}{z^2 + \|x - x'\|^2}\right)^{\Delta}, \quad \Delta = 2 + \sqrt{m^2 + 4}$$
 (28)

(a) To verify this, we first check that the boundary conditions are indeed satisfied by K_{Δ} ; note that $\Delta \geq 2 > 0$, and we have:

$$z \to 0, \quad l \neq 0, \quad \left(\frac{z}{z^2 + l^2}\right)^{\Delta} \to 0,$$
 (29)

i.e. the only contribution comes from the $l \to 0$ case, where we have:

$$l = ||x - x'|| \to 0, \quad \int d^4 x' \, K_{\Delta}(x, z; x') = \frac{(\Delta - 1)(\Delta - 2)}{\pi^2} \int_0^{\infty} 2\pi^2 l^3 \, dl \left(\frac{z}{z^2 + l^2}\right)^{\Delta}$$
$$= \frac{(\Delta - 1)(\Delta - 2)}{\pi^2} \cdot \frac{2\pi^2}{2(\Delta - 1)(\Delta - 2)} z^{4-\Delta}$$
(30)
$$= z^{\delta}$$

Therefore, we have:

$$z \to 0$$
, $K_{\Delta}(x, z; x') \to z^{\delta} \delta^4(x - x')$, $\phi(x, z) \to z^{\delta} \phi_0(x)$, (31)

The other boundary condition is convenient to check; we have:

$$z \to \infty$$
, $K_{\Delta}(x, z; x') \propto z^{-\Delta} \to 0$, $\phi(x, z)$ regular. (32)

Now we need only check that K_{Δ} satisfies the equation of motion; in Poincaré AdS₅ we have:

$$\nabla^2 = z^2 \left(\partial_z^2 - \frac{3}{z} \, \partial_z + \partial_x^2 \right) \tag{33}$$

With the help of MathematicaTM, it is straightforward to check that $(\nabla^2 - m^2) \left(\frac{z}{z^2 + l^2}\right)^{\Delta} = 0$, therefore $(\nabla^2 - m^2) \phi(x, z) = 0$.

The AdS/CFT dictionary is given by:

$$\left\langle e^{\int d^4 x \, C_O \, \phi_0(x) \, O(x)} \right\rangle_{\text{CFT}} = e^{-S[\phi_0]}$$
 (34)

Where $S[\phi_0]$ the bulk effective action evaluated on the solution to the equation of motion:

$$S[\phi_0] = \int d^4x \, dz \, \sqrt{-G} \left\{ \frac{1}{2} \left(\partial_\mu \phi \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{3} g \phi^3 + \dots \right\}$$
 (35)

(b) The CFT 2-point function $\langle O(x) O(y) \rangle$ can be computed with the above dictionary, using the usual effective formalism, but with the bulk action instead of the boundary action:

$$\langle O(x) O(y) \rangle = \frac{1}{C_O^2} \left. \frac{\delta^2}{\delta \phi_0(x) \delta \phi_0(y)} e^{-S[\phi_0]} \right|_{\phi_0 = 0}$$
(36)

For 2-point function, we only need terms $\sim \mathcal{O}(\phi^2)$; note that:

$$\frac{\delta\phi(x,z)}{\delta\phi_0(x')} = K_{\Delta}(x,z;x'),\tag{37}$$

$$\delta S \left[\phi_{0}\right] \sim \int d^{4}x \, dz \, \sqrt{-G} \left(-\nabla^{2} + m^{2}\right) \phi \, \delta\phi + \int_{z \to 0} d^{4}x \, \sqrt{-G} \, \partial^{z}\phi \, \delta\phi$$

$$= 0 - \int_{z \to 0} d^{4}x \, z^{-5}z^{2} \partial_{z}\phi \, \delta\phi = -\int_{z \to 0} d^{4}x \, z^{-3} \partial_{z}\phi \, \delta\phi \,,$$

$$\therefore \langle O(x) \, O(y) \rangle = + \frac{1}{C_{O}^{2}} \frac{\delta}{\delta \phi_{0}(x)} e^{-S[\phi_{0}]} \int_{z \to 0} d^{4}x' \, z^{-3} \partial_{z}\phi(x', z) \, K_{\Delta}(x', z; y) \Big|_{\phi_{0} = 0}$$

$$= \frac{1}{C_{O}^{2}} \int_{z \to 0} d^{4}x' \, z^{-3} \partial_{z} K_{\Delta}(x', z; x) \, K_{\Delta}(x', z; y)$$

$$= \frac{1}{C_{O}^{2}} \int_{z \to 0} d^{4}x' \, z^{-3} \partial_{z} K_{\Delta}(x', z; x) \, z^{\delta} \delta^{4}(x' - y)$$

$$= \frac{z^{\delta - 3}}{C_{O}^{2}} \, \partial_{z} K_{\Delta}(x, z; y)$$

$$\sim \frac{z^{\delta - 3}}{C_{O}^{2}} \frac{(\Delta - 1)(\Delta - 2)}{\pi^{2}} \frac{\Delta z^{\Delta - 1}}{\|x - y\|^{2\Delta}}$$

$$= \frac{1}{C_{O}^{2}} \frac{\Delta(\Delta - 1)(\Delta - 2)}{\pi^{2}} \frac{1}{\|x - y\|^{2\Delta}}$$

Therefore, if we want $\langle O(x) O(y) \rangle = \frac{1}{\|x-y\|^{2\Delta}}$, then we have⁷:

$$C_O = \frac{1}{\pi} \sqrt{\Delta(\Delta - 1)(\Delta - 2)} \tag{40}$$

Here $z \to 0$ is a cutoff parameter.

(c) Similarly, we can use the dictionary to compute 3-point functions; we have⁸:

$$\langle O(x_1) \, O(x_2) \, O(x_3) \rangle = \frac{1}{C_O^3} \left(-\frac{g}{3} \right) \int d^4 x \, dz \, \sqrt{-G} \, K_\Delta(x, z; x_1) \, K_\Delta(x, z; x_2) \, K_\Delta(x, z; x_3) \quad (41)$$

This is a difficult integral; as is suggested by arXiv:hep-th/9804058, we can use an important symmetry of AdS/CFT — the inversion $\vec{x} \mapsto \frac{\vec{x}}{x^2}$, to complete the integration.

By conformal symmetry, we know that the 3-point function is of the form:

$$\langle O(x_1) O(x_2) O(x_3) \rangle = A(x_1, x_2, x_3) = \frac{C_{OOO}}{|x_{12}|^{\Delta} |x_{23}|^{\Delta} |x_{31}|^{\Delta}}, \quad x_{ij} = x_i - x_j$$
 (42)

First set $x_3 = 0$, then perform inversion on all other points:

$$x_i = \frac{x_i'}{x_i'^2}, \quad (x, z) = \frac{(x', z')}{r'^2}, \quad r^2 = x^2 + z^2, \quad r^2 r'^2 = 1 = x_i^2 x_i'^2,$$
 (43)

$$\frac{d^4 x \, dz}{z^5} = \frac{d^4 x' \, dz'}{z'^5},\tag{44}$$

$$\frac{z}{z^{2} + \|x - x_{i}\|^{2}} = \frac{z}{r^{2} + x_{i}^{2} - 2x \cdot x_{i}} = \frac{z'/r'^{2}}{1/r'^{2} + 1/x_{i}'^{2} - 2x' \cdot x_{i}'/(r'^{2}x_{i}'^{2})}$$

$$= \frac{z'}{r'^{2} + x_{i}'^{2} - 2x' \cdot x_{i}'} x_{i}'^{2} = \frac{z'}{z'^{2} + \|x' - x_{i}'\|^{2}} x_{i}'^{2}, \tag{45}$$

$$K_{\Delta}(x, z; x_i) = K_{\Delta}(x', z'; x_i') |x_i'|^{2\Delta} = \frac{1}{|x_i|^{2\Delta}} K_{\Delta}(x', z'; x_i'), \tag{46}$$

With these in mind, we find that:

$$A(x_1, x_2, 0) = -\frac{g}{3C_O^3} \frac{1}{|x_1|^{2\Delta}} \frac{1}{|x_2|^{2\Delta}} \frac{(\Delta - 1)(\Delta - 2)}{\pi^2} \int \frac{\mathrm{d}^4 x' \,\mathrm{d} z'}{z'^5} K_{\Delta}(x', z'; x_1') K_{\Delta}(x', z'; x_2') z'^{\Delta}$$

$$(47)$$

The integral can then be completed using Feynman parameters; in the end we obtain:

$$A(x_1, x_2, 0) \propto \frac{1}{|x_1|^{2\Delta}} \frac{1}{|x_2|^{2\Delta}} \frac{1}{|x_1' - x_2'|^{2\Delta}} = \frac{1}{|x_1|^{\Delta} |x_2|^{\Delta} |x_1 - x_2|^{\Delta}},$$
 (48)

$$C_{OOO} = -\frac{g}{3C_O^3} \frac{1}{2\pi^4} \left(\frac{\Gamma(\frac{\Delta}{2})}{\Gamma(\Delta - 2)} \right)^3 \Gamma\left(\frac{3\Delta - 4}{2}\right) \tag{49}$$

⁷Reference: arXiv:hep-th/9804058. Again I would like to thank Lucy Smith for helpful hints.

⁸Reference: arXiv:hep-th/9905111, and arXiv:hep-th/9804058.