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# 1 Example of limit in <u>Vect</u>:

$$I = \left\{ \begin{array}{c} \bullet \\ \star \end{array} \right\} \xrightarrow{F} \left\{ \begin{array}{c} 0 \\ \mathbb{R} \end{array} \right\}$$

$$\supset \underline{\mathbf{Vect}} \colon \mathbb{R} \text{ vector space },$$

$$\downarrow \\ \Delta(V) \to \left\{ \begin{array}{c} V \\ V \end{array} \right\}$$

$$(1)$$

### (a) By definition, we have:

Where  $g: V \to \lim F$ . More intuitively, for the above  $F: I \to \underline{\mathbf{Vect}}$ , this translates to the following diagram in  $\underline{\mathbf{Vect}}$ :

$$V \xrightarrow{\exists ! g} \lim F \downarrow 0$$

$$\downarrow 0$$

$$\downarrow$$

Now we verify that  $\lim F = \mathbb{R}$ , along with the following choice of  $\sigma$ :

$$\sigma_{\bullet} = 0, \quad \sigma_{\star} = \mathbb{1}_{\mathbb{R}} \tag{4}$$

In fact, for the above diagram (3) to be commutative, we must have  $g = \eta_{\star}$ . Note that such g is unique once  $\sigma$  is chosen; for our choice of  $\sigma$ , if  $g \neq \eta_{\star}$ , then the diagram *cannot* commute. Hence  $\lim F = \mathbb{R}$ , along with the above choice of  $\sigma \colon \Delta(\mathbb{R}) \Rightarrow F$ . In other words, we have:

$$\begin{cases}
\bullet, \star \\
 & \bullet, \star
\end{cases}$$

$$\downarrow^{\Delta(\mathbb{R})} \xrightarrow{\sigma} F$$

$$\begin{cases}
0, \mathbb{R} \\
 & \bullet, \star
\end{cases}$$

(b)(c) From diagram (2) and discussions in (a), we know that:

$$\exists! \ \tau = \Delta(g) \colon \ \Delta(V) \Longrightarrow \Delta(\mathbb{R}) \tag{6}$$

Here  $g: V \to \mathbb{R}$  is fixed uniquely once  $\sigma$  is fixed. However,  $\sigma$  may vary up to isomorphism; therefore, a generic choice of  $\sigma$  is given by:

$$\sigma_{\bullet} = 0, \quad \sigma_{\star} = k \, \mathbb{1}_{\mathbb{R}}, \quad k \in \mathbb{R}$$
 (7)

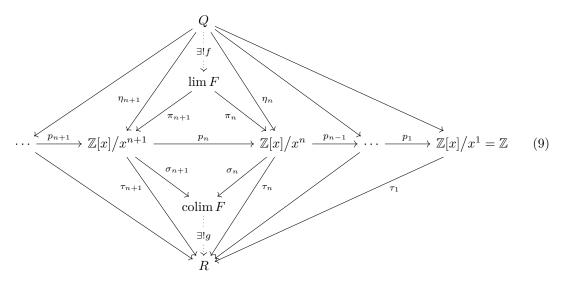
For such g, by the same arguments in (a), we have:

$$\exists ! \ g = \frac{1}{k} \, \eta_{\star} \,, \ \tau = \Delta(g) = \frac{1}{k} \, \Delta(\eta_{\star}), \quad s. \, t. \quad (2), \, (3) \ commutes \tag{8}$$

Note that  $k \in \mathbb{R}$ , for every k there is a different g and  $\tau$ ; hence there are  $\|\{k\}\| = \|\mathbb{R}\|$  many choices of  $\tau$  to make the diagram commute. In particular, for k = 1 we recover  $\tau = \Delta(\eta_*)$ .

## 2 Limit and colimit of polynomial ring:

By definition,



Here  $p_n : \mathbb{Z}[x]/x^{n+1} \to \mathbb{Z}[x]/x^n$  is the natural projection.

Intuitively, if such  $\lim F$  exists, it shall be the "smallest" object that "contains"  $\mathbb{Z}[x]/x^n$  when  $n \to \infty$ . Note that  $\mathbb{Z}[x]/x^n$  is naturally a  $\mathbb{Z}^n$  vector space:

$$\mathbb{Z}[x]/x^n \ni \sum_{m=0}^{n-1} a_m x^m \sim (a_0, a_1, \dots, a_{n-1}) \in \mathbb{Z}^n$$
 (10)

While  $n \to \infty$ , this gives an  $\infty$ -tuple which corresponds to the formal power series<sup>1</sup>:

$$\mathbb{Z}[[x]] \ni \sum_{m=0}^{\infty} a_m x^m \tag{11}$$

The difference between  $\mathbb{Z}[x]$  and  $\mathbb{Z}[[x]]$  is that the latter may contain infinite series while the former may not. Now we confirm that, indeed,  $\lim F = \mathbb{Z}[[x]]$ , along with natural projections  $\pi_n \colon \mathbb{Z}[[x]] \to \mathbb{Z}[x]/x^n$ .

<sup>&</sup>lt;sup>1</sup>See Wikipedia: Formal power series. This is in fact the adic completion of  $\mathbb{Z}[x]$ . I would like to thank 刘逸华 & 谢贤进 for this hint.

In fact, the  $f: R \to \lim F$  in (9) can be explicitly written down as:

$$f = \eta_1 + x \left(\frac{d}{dx} \eta_2\right) + x^2 \left(\frac{1}{2} \frac{d}{dx} \eta_3\right) + \dots = \sum_{m=0}^{\infty} x^m \left\{\frac{1}{m!} \frac{d^m}{dx^m} \eta_{m+1}\right\}$$

$$\sim (\eta_{1,0}, \eta_{2,1}, \dots, \eta_{n,n-1}, \dots)$$
(12)

Here  $\frac{1}{m!} \frac{\mathrm{d}^m}{\mathrm{d}x^m}$  is used to extract the  $a_{n-1}$  coefficient in  $\mathbb{Z}[x]/x^n$ ; this is the last component of  $\eta_n$ , denoted by  $\eta_{n,n-1}$ . Any other choice of f will break communitativity of (9), hence f is fixed uniquely by  $\mathbb{Z}[[x]]$  and  $\pi_n$ 's. Therefore,  $\lim F = \mathbb{Z}[[x]]$ .

On the other hand, colim F is the "largest" object that any map out of  $\mathbb{Z}[x]/x^n$  must "passes through". Also, this should hold for all  $n \in \mathbb{Z}_+$ . Naturally, projections  $\sigma_n \colon \mathbb{Z}[x]/x^n \to \mathbb{Z}$  satisfy the above requirements; we have:

$$\sigma_n \colon x \longmapsto 0, \quad \sum_{m=0}^{n-1} a_m x^m \longmapsto a_0,$$
 (13)

$$\sigma_{n+1} = p_1 \circ p_2 \circ \cdots p_n \tag{14}$$

 $g: \mathbb{Z} \to R$  in (9) is fixed uniquely for such choice of  $\sigma_n$ ; in fact, descend along the  $p_n$  tower in (9), and we have:  $\tau_{n+1} = \tau_n \circ p_n = \tau_{n-1} \circ p_{n-1} \circ p_n = \cdots = \tau_1 \circ p_1 \circ p_2 \circ \cdots \circ p_n = \tau_1 \circ \sigma_{n+1}, \ \forall \ n \in \mathbb{Z}_+,$  hence  $\exists ! \ g = \tau_1$ . Therefore, colim  $F = \mathbb{Z}$ .

## 3 Example of push-out in Groupoid:

Following the same observation as before, the push-out P is the "largest" object that any map out of  $\bullet$  and  $\{0 \leftrightarrow 1\}$  must pass through. By such universal property, P can be no larger than the coproduct:  $\{\bullet\}$   $\{\{0 \leftrightarrow 1\}\}$ . However, we should also consider the equivalence imposed by:

$$\bullet \xleftarrow{f_1} \{0,1\} \xrightarrow{f_2} \{0 \leftrightarrow 1\} \tag{16}$$

Therefore, we simply have  $P = \bullet$ , with  $\tau_{1,2}$  the natural projection. This can be verified with ease: we have  $g = \eta_1$ . It is unique since its image is a single point (with identity map to itself)  $\star \in Q$ , and the point  $\star$  is fixed by commutativity.

# 4 Product and coproduct in Ab:

For  $G_{\alpha} \in \underline{\mathbf{Ab}} \subset \mathbf{Group}$ , note that we have:

$$Free : \mathbf{Set} \rightleftharpoons \mathbf{Group} : Forget$$
 (17)

Therefore, for F: some diagram in **Group**,  $[\lim (\text{Forget} \circ F) = \text{Forget} \circ \lim F]$  if  $\lim F$  exists.

By definition, the product  $\prod_{\alpha} G_{\alpha} \in \underline{\mathbf{Group}}$  is a limit, hence it is idential (as in  $\underline{\mathbf{Set}}$ ) to the *direct product*, with additional entry-wise group multiplication. Same applies for the full subcategory: abelian group  $\underline{\mathbf{Ab}} \subset \mathbf{Group}$ .

On the other hand, the disjoint union of  $G_{\alpha}$ 's as sets will not necessary be a group, the identities  $\mathbb{1}_{\alpha} \in G_{\alpha}$  must be glued together to produce a group structure. Furthermore, free-forgetful adjunction (17) implies that for F': some diagram in **Set**,

$$\operatorname{colim}\left(\operatorname{Free}\circ F'\right) = \operatorname{Free}\circ\operatorname{colim} F',\tag{18}$$

Whenever colim F' exists; in our case, colim F' is the disjoint union of sets:  $\coprod_{\alpha} \text{Forget}(G_{\alpha})$ . Therefore, we should construct a free object in  $\underline{\mathbf{Ab}}$ .

Here we restrict our discussion to  $\underline{\mathbf{Ab}}$ , since the coproduct in  $\underline{\mathbf{Ab}}$  is *not* the same as in  $\underline{\mathbf{Group}}$  — the free product of abelian group is not necessary abelian. Hence, the coproduct in  $\underline{\mathbf{Ab}}$  shall be:

$$\coprod_{\alpha} G_{\alpha} = \bigoplus_{\alpha} G_{\alpha}, \quad i_{\alpha} \colon G_{\alpha} \longleftrightarrow \bigoplus_{\alpha} G_{\alpha}$$
 (19)

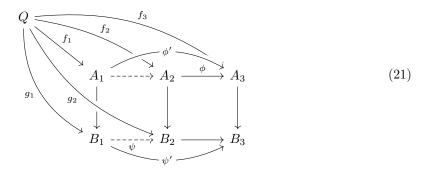
As a set, this is precisely the disjoint union with identities  $0_{\alpha} \in G_{\alpha} \subset \underline{\mathbf{Ab}}$  glued together.

It is then straight-forward to verify its universal property: for  $f_{\alpha} \colon G_{\alpha} \to H$ ,

$$\exists ! f \colon \bigoplus_{\alpha} G_{\alpha} \longrightarrow H, \quad (g_{\alpha})_{\alpha} \longmapsto \sum_{\alpha} f_{\alpha}(g_{\alpha})$$
 (20)

This is compatible with the abelian group multiplication. Note that for the summation to be well-defined, the coproduct must only contain finitely many components; otherwise it is identical to the product in  $\mathbf{Ab}$ .

### 5 Composition of pull-backs:



- (a) If  $A_1$  is the pull-back of  $A_2$  and  $A_2$  is the pull-back of  $A_3$ , then given Q with  $f_3, g_1$ , we have  $g_2 = \psi \circ g_1$ , and  $f_2$  is fixed uniquely by universal property of  $A_2$ , while  $f_1$  is fixed uniquely by universal property of  $A_1$ . Hence,  $A_1$  is the pull-back of  $A_3$ .
- (b) If  $A_1, A_2$  are pull-backs of  $A_3$ , then given Q with  $f_2, g_1$ , we have  $f_3 = \phi \circ f_2$ , and  $f_2$  is fixed uniquely by universal property of  $A_2$ , and  $f_1$  is fixed uniquely by universal property of  $A_1$ . Hence,  $A_1$  is the pull-back of  $A_3$ .

#### → PAST WORK, AS TEMPLATE →

1 For  $F_i \to E_i \xrightarrow{p_i} B$ : coverings in  $Cov_0(B)$  with  $E_i$ : connected and B: path connected and locally path connected, the following diagram commutes:

$$E_1 \xrightarrow{f} E_2$$
  $e_2 = f(e_1),$   $b = p_1(e_1) = p_2(e_2),$ 

To show that f is itself a covering, we need only verify that f is locally trivial with some discrete fiber F. In fact, given any  $e_2 \in E_2$  and  $b = p_2(e_2)$ , there exists some neighborhood  $U \subset B$  that the following diagram holds (by restriction):

$$U \times F_1 \xrightarrow{f} U \times F_2 \qquad e_1 = (b, k_1), \qquad e_2 = (b, k_2(b, k_1)), \quad k_i \in F_i$$

Generally,  $k_2 = k_2(b, k_1)$  depends on the base point  $b \in B$ . However, since B is locally path connected, we can restrict U to be path connected, while  $k_2 \in F_2$ : discrete. Since continuous maps preserve path connectedness,  $k_2$  is in fact independence of b, i.e.  $k_2 = \varphi(k_1)$ .

On the other hand,  $\forall e_2 = (b, k_2) \in U \times \{k_2\} \subset E_2$ , we have its preimage  $f^{-1}(e_2) = \{b\} \times \varphi^{-1}(k_2)$ . Note that  $E_2$  is connected while  $\varphi^{-1}(k_2) \in F_1$  is discrete; for the same reasoning as above,  $\varphi^{-1}(k_2) = F$  is in fact independent of  $k_2$ . This is the discrete fiber F we have been looking for. Hence f is also a covering map<sup>2</sup>.

# | 2 | Cylinder with ends pinched — $\pi_1$ and universal cover:

$$Y = (X \times I)/(X \times \partial I) , \quad I = [0, 1]$$
(22)

Note that Y is homeomorphic to two cones<sup>3</sup>  $CX_1 \coprod CX_2$  with "bases"  $X_i \subset CX_i$  and "vertices"  $v_i$  respectively identified:  $X_1 \sim X_2$ ,  $v_1 \sim v_2 \equiv v$ . X is path connected and so is Y, hence we are free to choose  $\pi_1(Y) = \pi_1(Y, y_0)$ .

First note that paths that do *not* pass through the vertex v are all homotopic, since they are contained in a cone and cones are contractible<sup>4</sup>. Therefore all contributions to  $\pi_1(Y)$  are loop classes that do pass through the vertex v. In other words, morphisms in  $\Pi_1 Y$  are in one-to-one correspondence with morphisms in:

$$\Pi_1([0,1]/_{0\sim 1}) = \Pi_1 S^1$$
 (23)

Therefore, 
$$\pi_1(Y) \cong \pi_1(S^1) = \mathbb{Z}$$
.

<sup>&</sup>lt;sup>2</sup>Reference: math.stackexchange.com/a/109774.

 $<sup>^3{\</sup>rm See}$  discussions from Problem Set  $\mathcal{N}\!\!\!_{2}1.$ 

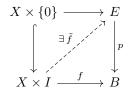
 $<sup>{}^{4}[\</sup>gamma_{1}] = [\gamma_{2} \star \gamma_{2}^{-1} \star \gamma_{1}] = [\gamma_{2}].$ 

The universal cover  $\tilde{Y}$  of Y can be constructed by assigning an induced topology to the space of path classes, same as in the general proof of its existence. Since Y is "degenerate" at its vertex, this is equivalent to "cutting open" Y at its vertex v, and joining  $\mathbb{Z}$  copies them end-to-end. More explicitly, it can be written as:

$$\tilde{Y} = (X \times \mathbb{R}) / \sim, \quad (x, n) \sim (x', n), \ \forall \ x \in X, \ n \in \mathbb{Z}$$
 (24)

While the covering map:  $\tilde{Y} \ni [x,t] \mapsto [x,t-\lfloor t \rfloor] \in Y$ , here  $\lfloor t \rfloor$  is the integer part of  $t \in \mathbb{R}$ .

# $\boxed{\bf 3}$ $\pi_1$ of fiber in fibration:



For  $F \to E \xrightarrow{p} B$ : fibration, by homotopy lifting property (HLP), any homotopy in B can be uniquely lifted to path class in E, provided some "initial condition"  $X \times \{0\}$ . This leads to the following results:

(a) For B: simply-connected, take any loop class  $[\tilde{\gamma}] \in \pi_1(E, e)$  as initial condition; its projection  $[p \circ \tilde{\gamma}] \in \pi_1(B, b) = \{[\mathbb{1}_b]\}$  is trivial, i.e.  $p \circ \tilde{\gamma} \simeq \mathbb{1}_b$ . By HLP, such homotopy can be lifted into E, i.e.

$$p \circ \tilde{\gamma} \simeq \mathbb{1}_b \quad \xrightarrow{\text{lift}} \quad \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}' = \mathbb{1}_b$$
 (25)

In other words,  $\tilde{\gamma} \simeq \tilde{\gamma}' \subset p^{-1}(b)$ , i.e. any loop in E is homotopic to some loop in  $p^{-1}(b) \cong F$ . This implies a surjective group homomorphism  $\pi_1(p^{-1}(b), e) \to \pi_1(E, e)$ , i.e. an epimorphism.

(b) For E: simply-connected, take any loop class  $[\gamma] \in \pi_1(B, b)$  and consider its lifting  $[\tilde{\gamma}]$ . Note that in general  $\tilde{\gamma}$  is not a loop; however, we have  $p \circ \tilde{\gamma} = \gamma$ , hence  $\tilde{\gamma}(0), \tilde{\gamma}(1) \in p^{-1}(b)$ . In general, we have:

$$\gamma \simeq \gamma' \quad \stackrel{\text{lift}}{\Longrightarrow} \quad \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}^{(\prime)} = \gamma^{(\prime)}$$
 (26)

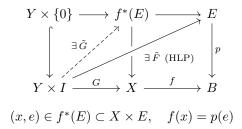
By continuity,  $\tilde{\gamma}(0), \tilde{\gamma}'(0) \in F_0$ : a path component of  $p^{-1}(b)$ ; similarly,  $\tilde{\gamma}(1), \tilde{\gamma}'(1) \in F_1$ . In other words, the start and end points of  $\tilde{\gamma}$  are confined in path components  $F_0$  and  $F_1$ , respectively. Hence a loop class in  $\pi_1(B,b)$  maps to transport between path components:

$$T_{(\cdot)}(e) \colon \pi_1(B,b) \longrightarrow \pi_0(p^{-1}(b))$$

$$[\gamma] \longmapsto T_{[\gamma]}(e)$$
(27)

As a matter of fact,  $T_{(\cdot)}(e)$  is a bijection. For  $T_{[\gamma]} = T_{[\gamma']}$ , they are characterized by two lifted paths  $\tilde{\gamma}, \tilde{\gamma}'$ ; since E is simply connected, they are always homotopic:  $\tilde{\gamma} \simeq \tilde{\gamma}'$ , hence  $[\gamma] = [\gamma']$  by projection p. This means that T is injective. Surjectivity also follows from projection  $\gamma = p \circ \gamma'$ . Therefore,  $T_{(\cdot)}(e)$  gives a bijection between  $\pi_1(B, b)$  and  $\pi_0(p^{-1}(b))$ .

## 4 Pull-back of fibration is fibration:



We need only verify that  $f^*(E) \to X$  also has HLP, i.e. the existence of  $\tilde{F}$  in the above diagram<sup>5</sup>. By HLP of  $E \xrightarrow{p} B$ ,  $\exists \tilde{F} : Y \times I \to E$  as shown above. We can use  $\tilde{F}$  to construct  $\tilde{G}$  explicitly; in fact, first consider:

$$\tilde{G} \colon Y \times I \longrightarrow X \times E$$

$$(y,t) \longmapsto (G(y,t), \tilde{F}(y,t))$$
(28)

Note that  $f \circ G = p \circ \tilde{F}$ ; compared with the definition of  $f^*(E)$ , this implies that the image of  $\tilde{G}$  lies within  $f^*(E) \subset X \times E$ , hence after restriction of its codomain,  $\tilde{G}$  becomes a well-defined lifting of G into  $f^*(E)$ . Therefore,  $f^*(E) \to X$  has HLP, i.e. it is also a fibration.

# 5 More properties of fibration:

- (a) By HLP, given any initial condition  $e \in p^{-1}(b_1)$ , lifting of any path  $b_1 \xrightarrow{\gamma} b_2$  exists. The lifted path with dependence of e can then be written as  $F: p^{-1}(b_1) \times I \to E$ . This is just a generalization of 3 for non-loop paths.
- (b) Similarly, transport  $T_{[\gamma]}$  defined in 3 can be generalized for non-loop paths.  $T_{[\gamma]}$  is well-defined for path class  $[\gamma]$ , since by HLP homotopic paths can be lifted to homotopy in E. Therefore, the transport is fixed up to homotopy, i.e.

$$T: \operatorname{Hom}_{\Pi_1 B}(b_0, b_1) \longrightarrow \operatorname{Hom}_{\underline{\mathbf{hTop}}} \left( p^{-1}(b_0), p^{-1}(b_1) \right)$$

$$[\gamma] \longmapsto T_{[\gamma]}$$
(29)

Note that T defined in this way is also independent of the choice of F, since F simply specifies the starting point of the lifted path; no matter which F we choose, the lifted paths will always be homotopic in E. Hence T is well-defined in the above sense.

(c) T defined above is a functor:  $\Pi_1 B \to \underline{\mathbf{hTop}}$ . To verify this, we need only check that it is compatible with composition and maps identity morphisms to identity morphisms. Indeed,  $T_{[\mathbb{1}_b]} = [\mathbb{1}_{p^{-1}(b)}]$ , and  $T_{[\gamma']\star[\gamma]} = T_{[\gamma'\star\gamma]} = T_{[\gamma']} \circ T_{[\gamma]}$  by joining two lifted paths (up to homotopy).

<sup>&</sup>lt;sup>5</sup>Notice that  $f^*(E)$  is the limit of the diagram, hence this is automatically true by the universal property of  $f^*(E)$ . I would like to thank 刘逸华 for pointing this out. For now, we will stick to a more traditional proof.

(d) For B: path connected, there exists an isomorphism between any two objects in  $\Pi_1 B$  (a path connecting any two points in B), which is mapped to isomorphisms between fibers  $p^{-1}(b)$  in **hTop**. Hence any two fibers of  $E \stackrel{p}{\rightarrow} B$  have the same homotopy type.