Advanced QFT №1

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1 Symmetry & Noether's Theorem

1.1 2D σ-Model

$$\mathcal{L} = -\frac{1}{2}\,\eta_{\alpha\beta}\,\eta_{\mu\nu}\,\partial^\alpha X^\mu\,\partial^\beta X^\nu = -\frac{1}{2}\,\partial^\alpha X_\mu\,\partial_\alpha X^\mu, \quad X^\mu \in \mathbb{R}^{1,D-1} \tag{1}$$

- For $\delta X^\mu = a^\mu + \lambda^\mu{}_\nu X^\nu$, the Lagrangian (density) transforms as follows:

$$\begin{aligned}
\delta\mathcal{L} &= -\partial^\alpha X_\mu\,\partial_\alpha\,\delta X^\mu \\
&= -\partial^\alpha X_\mu\,\partial_\alpha(a^\mu + \lambda^\mu{}_\nu X^\nu) \\
&= -\partial^\alpha X_\mu\,(\partial_\alpha a^\mu + X^\nu\,\partial_\alpha\lambda^\mu{}_\nu + \lambda^\mu{}_\nu\,\partial_\alpha X^\nu) \\
&= -\partial^\alpha X_\mu\,\partial_\alpha a^\mu - \partial^\alpha X^\mu\,\partial_\alpha X^\nu\,\lambda_{\mu\nu} - X^\nu\,\partial^\alpha X^\mu\,\partial_\alpha\lambda_{\mu\nu} \\
&= -\partial^\alpha X_\mu\,\partial_\alpha a^\mu - \partial^\alpha X^\mu\,\partial_\alpha X^\nu\,\lambda_{(\mu\nu)} - X^\nu\,\partial^\alpha X^\mu\,\partial_\alpha\lambda_{\mu\nu}
\end{aligned} \tag{2}$$

Since a^μ and $\lambda^\mu{}_\nu$ are independent, imposing $\delta L = 0$ yields $\partial_\alpha a^\mu = 0$, $a = \text{const}$. Furthermore, if $\delta L = 0$ is to hold for arbitrary X^μ fields, then $\partial_\alpha\lambda_{\mu\nu} = 0$, $\lambda_{(\mu\nu)} = 0$, i.e. $\lambda_{\mu\nu}$ is constant and anti-symmetric over its indices.

- Promote $\delta X \mapsto \epsilon(x)\,\delta X = \epsilon(x)\,(a^\mu + \lambda^\mu{}_\nu X^\nu)$, with $\epsilon(x)$ some localized bump function; using (2) and considering *on-shell* variation, we have:

$$\begin{aligned}
0 = \delta S &= -\int \mathrm{d}^2x\,(\partial^\alpha X_\mu\,a^\mu\,\partial_\alpha\epsilon + X^\nu\,\partial^\alpha X^\mu\,\lambda_{\mu\nu}\,\partial_\alpha\epsilon) \\
&= -\int \mathrm{d}^2x\,\Big(\partial^\alpha X_\mu\,a^\mu\ + X_{[\nu}\,\partial^\alpha X_{\mu]}\,\lambda^{[\mu\nu]}\Big)\,\partial_\alpha\epsilon
\end{aligned} \tag{3}$$

It is evident (after partial integration) that the following currents are conserved; they are the Noether currents associated with a^μ and $\lambda^{[\mu\nu]}$:

$$j^\alpha_\mu = -\partial^\alpha X_\mu, \quad j^\alpha_{\mu\nu} = -X_{[\nu}\,\partial^\alpha X_{\mu]} = \frac{1}{2}\,(X_\mu\,\partial^\alpha X_\nu - X_\nu\,\partial^\alpha X_\mu) \tag{4}$$

Conserved charge $Q = \int \mathrm{d}^2x\,j^0(x)$, we have:

$$P_\mu = -\int \mathrm{d}x^1\,\partial^0 X_\mu = \int \mathrm{d}x^1\,\partial_0 X_\mu, \quad M_{\mu\nu} = \frac{1}{2}\int \mathrm{d}x^1\,(X_\nu\,\partial_0 X_\mu - X_\mu\,\partial_0 X_\nu) \tag{5}$$

They can be interpreted as spacetime momentum and spacetime angular momentum. ■

1.2 Real Scalar in $(3+1)$ D

$$\mathcal{L} = -\frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}\,m^2\phi^2 \tag{6}$$

- For ϕ: scalar, under $x' = \lambda \circ x$, $\phi(x) \mapsto \phi'(x)$, while:

$$\phi'(x') = \phi(x) \implies \phi'(x) = \phi(\lambda^{-1} \circ x) \tag{7}$$

For $\lambda \sim \lambda^\mu{}_\nu$: Lorentz transformation, $\eta_{\mu\nu}\lambda^\mu{}_\rho\lambda^\nu{}_\sigma = \eta_{\rho\sigma}$, or equivalently, $(\lambda^{-1})^\mu{}_\nu = \lambda_\nu{}^\mu$. Therefore,

$$\phi'(x^\mu) = \phi(\lambda^{-1} \circ x^\mu) = \phi(x^\nu \lambda_\nu{}^\mu) \tag{8}$$

- Under $x'^\mu = \lambda^\mu{}_\nu x^\nu$, we have:

$$\begin{aligned}
\mathcal{L}'(x') &= -\frac{1}{2}\,\partial'^\mu\phi'(x')\,\partial'_\mu\phi'(x') - \frac{1}{2}\,m^2\phi'^2(x') \\
&= -\frac{1}{2}\,\partial'^\mu\phi(x)\,\partial'_\mu\phi(x) - \frac{1}{2}\,m^2\phi^2(x) \\
&= -\frac{1}{2}\,\eta^{\mu\nu}\frac{\partial x^\rho}{\partial x'^\mu}\,\partial_\rho\phi(x)\,\frac{\partial x^\sigma}{\partial x'^\nu}\,\partial_\sigma\phi(x) - \frac{1}{2}\,m^2\phi^2(x) \\
&= -\frac{1}{2}\,\eta^{\rho\sigma}\partial_\rho\phi(x)\,\partial_\sigma\phi(x) - \frac{1}{2}\,m^2\phi^2(x) \\
&= \mathcal{L}(x)
\end{aligned} \tag{9}$$

Here we've used $\eta^{\mu\nu}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} = \eta^{\mu\nu}\lambda_\mu{}^\rho\lambda_\nu{}^\sigma = \eta^{\rho\sigma}$. Furthermore, $S' = \int \mathrm{d}^4x\,\mathcal{L}'(x) = \int \mathrm{d}^4x'\,\mathcal{L}'(x') = \int \mathrm{d}^4x'\,\mathcal{L}(x) = \int \mathrm{d}^4x\,\mathcal{L}(x) = S$, hence the action is invariant under Lorentz transformation.

- Consider an infinitesimal Lorentz transformation: $\lambda \sim \mathbb{1} + \omega$, then $\eta_{\mu\nu}\lambda^\mu{}_\rho\lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ implies that $\omega_{\mu\nu}$ is anti-symmetric: $\omega_{\mu\nu} + \omega_{\nu\nu} = 0$. For $\delta x^\mu = \omega^\mu{}_\nu x^\nu$, we have:

$$\delta\phi = -\frac{\partial\phi}{\partial x^\mu}\,\delta x^\mu = -\omega^\mu{}_\nu x^\nu\,\partial_\mu\phi \tag{10}$$

To obtain the corresponding Noether charges, we can simply repeat the operations done in our previous problem; alternatively, we can try to derive a general recipe[1]: for $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$ and $S = \int \mathrm{d}^4x\,\mathcal{L}$, we have:

$$\begin{aligned}
\delta S &= \int \mathrm{d}^4x\,\delta\mathcal{L} \\
&= \int \mathrm{d}^4x\left(\frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\partial_\mu\phi\right) \\
&= \int \mathrm{d}^4x\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \int \mathrm{d}^4x\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi\right)
\end{aligned} \tag{11}$$

If we vary S w.r.t. a symmetry of the system, we will have $\delta\mathcal{L} = \partial_\mu K^\mu$ some total derivative; when on-shell, such variation gives the conserved current with boundary term K^μ:

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi - K^\mu \tag{12}$$

[1] References: arXiv:1601.03616 and *Tong:* http://damtp.cam.ac.uk/user/tong/qft.html

Back to our Lorentz transformation $\delta\phi = -\omega^\mu{}_\nu x^\nu \partial_\mu \phi$, we have symmetry variation:

$$\delta\mathcal{L} = -\omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = -\partial_\mu(\omega^\mu{}_\nu x^\nu \mathcal{L}) \tag{13}$$

This gives a boundary term $K^\mu = -\omega^\mu{}_\nu x^\nu \mathcal{L}$, and the Noether current and its corresponding conserved charge can be calculated as follows:

$$j^\mu = -\omega^\sigma{}_\nu x^\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\sigma \phi - \delta^\mu_\sigma \mathcal{L} \right), \tag{14}$$

$$Q = \int \mathrm{d}^3x\, j^0 = -\omega^\sigma{}_\nu \int \mathrm{d}^3x\, x^\nu \left(\partial_0\phi\, \partial_\sigma \phi - \delta^0_\sigma \mathcal{L} \right), \tag{15}$$

Note that $\omega^\mu{}_\nu$ is arbitrary, therefore Q can be decomposed into independent charges:

$$Q = \omega_{\mu\nu} M^{\mu\nu}, \quad M^{\mu\nu} = -\int \mathrm{d}^3x\, x^{[\mu} \left(\partial_0\phi\, \partial^{\nu]}\phi - \eta^{\nu]0}\mathcal{L} \right), \tag{16}$$

The indices of $M^{\mu\nu}$ are anti-symmetrized to match the degrees of freedom in $\omega_{\mu\nu}$. Note that the $\mathcal{L}$ term only appears when one of the indices is 0.

Canonical quantization:

$$\dot{\phi} = \partial_0 \phi, \quad \Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}, \quad [\phi(\mathbf{x}), \Pi(\mathbf{y})] = [\phi, \dot{\phi}](\mathbf{x}) = i\delta(\mathbf{x} - \mathbf{y}) \tag{17}$$

Other equal-time commutators between ϕ, Π all just vanish. We have:

$$M^{0i} = -M^{i0} = -\frac{1}{2} \int \mathrm{d}^3x \left(\dot{\phi} \left(x^0 \partial^i - x^i \partial^0 \right) \phi + x^i \mathcal{L} \right), M^{ij} = -\frac{1}{2} \int \mathrm{d}^3x\, \dot{\phi} \left(x^i \partial^j - x^j \partial^i \right) \phi \tag{18}$$

Notice that $x^{[\mu}\partial^{\nu]} = \frac{1}{2}\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) = \frac{1}{2} D^{ij}$ is the Killing vector fields of $\mathbb{R}^{3,1}$, hence they naturally follow the commutation relations of $\mathfrak{so}(3,1)$ (up to a constant coefficient)[2]. We have:

$$\begin{aligned} [M^{ij}, M^{kl}] &= \frac{1}{4} \int \mathrm{d}^3x \int \mathrm{d}^3y \left[\dot{\phi} D^{ij} \phi(x), \dot{\phi} D^{kl} \phi(y) \right] \\ &= \frac{1}{4} \int \mathrm{d}^3x\, \dot{\phi} \left[D^{ij}, D^{kl} \right] \phi \end{aligned} \tag{19}$$

Similar holds for M^{i0}. Therefore, $M^{\mu\nu}$'s indeed form the Lie algebra $\mathfrak{so}(3,1)$. ■

[2] I would like to thank 林般 for pointing this out.