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## 1 Symmetry & Noether's Theorem

## 1.1 2D $\sigma$ -Model

$$\mathcal{L} = -\frac{1}{2} \eta_{\alpha\beta} \eta_{\mu\nu} \partial^{\alpha} X^{\mu} \partial^{\beta} X^{\nu} = -\frac{1}{2} \partial^{\alpha} X_{\mu} \partial_{\alpha} X^{\mu}, \quad X^{\mu} \in \mathbb{R}^{1,D-1}$$
 (1)

• For  $\delta X^{\mu} = a^{\mu} + \lambda^{\mu}_{\ \nu} X^{\nu}$ , the Lagrangian (density) transforms as follows:

$$\begin{split} \delta \mathcal{L} &= -\partial^{\alpha} X_{\mu} \, \partial_{\alpha} \, \delta X^{\mu} \\ &= -\partial^{\alpha} X_{\mu} \, \partial_{\alpha} (a^{\mu} + \lambda^{\mu}_{\ \nu} X^{\nu}) \\ &= -\partial^{\alpha} X_{\mu} \, (\partial_{\alpha} a^{\mu} + X^{\nu} \, \partial_{\alpha} \lambda^{\mu}_{\ \nu} + \lambda^{\mu}_{\ \nu} \, \partial_{\alpha} X^{\nu}) \\ &= -\partial^{\alpha} X_{\mu} \, \partial_{\alpha} a^{\mu} - \partial^{\alpha} X^{\mu} \, \partial_{\alpha} X^{\nu} \, \lambda_{\mu\nu} - X^{\nu} \, \partial^{\alpha} X^{\mu} \, \partial_{\alpha} \lambda_{\mu\nu} \\ &= -\partial^{\alpha} X_{\mu} \, \partial_{\alpha} a^{\mu} - \partial^{\alpha} X^{\mu} \, \partial_{\alpha} X^{\nu} \, \lambda_{(\mu\nu)} - X^{\nu} \, \partial^{\alpha} X^{\mu} \, \partial_{\alpha} \lambda_{\mu\nu} \end{split}$$

$$(2)$$

Since  $a^{\mu}$  and  $\lambda^{\mu}_{\nu}$  are independent, imposing  $\delta L=0$  yields  $\partial_{\alpha}a^{\mu}=0$ , a= const. Furthermore, if  $\delta L=0$  is to hold for arbitrary  $X^{\mu}$  fields, then  $\partial_{\alpha}\lambda_{\mu\nu}=0$ ,  $\lambda_{(\mu\nu)}=0$ , i.e.  $\lambda_{\mu\nu}$  is constant and anti-symmetric over its indices.

• Promote  $\delta X \mapsto \epsilon(x) \, \delta X = \epsilon(x) \, (a^{\mu} + \lambda^{\mu}_{\ \nu} X^{\nu})$ , with  $\epsilon(x)$  some localized bump function; using (2) and considering *on-shell* variation, we have:

$$0 = \delta S = -\int d^2 x \left( \partial^{\alpha} X_{\mu} a^{\mu} \partial_{\alpha} \epsilon + X^{\nu} \partial^{\alpha} X^{\mu} \lambda_{\mu\nu} \partial_{\alpha} \epsilon \right)$$
$$= -\int d^2 x \left( \partial^{\alpha} X_{\mu} a^{\mu} + X_{[\nu} \partial^{\alpha} X_{\mu]} \lambda^{[\mu\nu]} \right) \partial_{\alpha} \epsilon$$
(3)

It is evident (after partial integration) that the following currents are conserved; they are the Noether currents associated with  $a^{\mu}$  and  $\lambda^{[\mu\nu]}$ :

$$j^{\alpha}_{\mu} = -\partial^{\alpha} X_{\mu}, \quad j^{\alpha}_{\mu\nu} = -X_{[\nu} \,\partial^{\alpha} X_{\mu]} = \frac{1}{2} \left( X_{\mu} \,\partial^{\alpha} X_{\nu} - X_{\nu} \,\partial^{\alpha} X_{\mu} \right) \tag{4}$$

Conserved charge  $Q = \int d^2x \, j^0(x)$ , we have:

$$P_{\mu} = -\int dx^{1} \,\partial^{0} X_{\mu} = \int dx^{1} \,\partial_{0} X_{\mu}, \quad M_{\mu\nu} = \frac{1}{2} \int dx^{1} \,(X_{\nu} \,\partial_{0} X_{\mu} - X_{\mu} \,\partial_{0} X_{\nu})$$
 (5)

They can be interpreted as spacetime momentum and spacetime angular momentum.

## 1.2 Real Scalar in (3+1) D

$$\mathcal{L} = -\frac{1}{2} \partial^{\mu} \phi \, \partial_{\mu} \phi - \frac{1}{2} \, m^2 \phi^2 \tag{6}$$

• For  $\phi$ : scalar, under  $x' = \lambda \circ x$ ,  $\phi(x) \mapsto \phi'(x)$ , while:

$$\phi'(x') = \phi(x) \implies \phi'(x) = \phi(\lambda^{-1} \circ x) \tag{7}$$

For  $\lambda \sim \lambda^{\mu}_{\nu}$ : Lorentz transformation,  $\eta_{\mu\nu}\lambda^{\mu}_{\rho}\lambda^{\nu}_{\sigma} = \eta_{\rho\sigma}$ , or equivalently,  $(\lambda^{-1})^{\mu}_{\nu} = \lambda_{\nu}^{\mu}$ . Therefore,

$$\phi'(x^{\mu}) = \phi(\lambda^{-1} \circ x^{\mu}) = \phi(x^{\nu} \lambda_{\nu}^{\mu}) \tag{8}$$

• Under  $x'^{\mu} = \lambda^{\mu}_{\ \nu} x^{\nu}$ , we have:

$$\mathcal{L}'(x') = -\frac{1}{2} \partial'^{\mu} \phi'(x') \, \partial'_{\mu} \phi'(x') - \frac{1}{2} m^2 \phi'^2(x')$$

$$= -\frac{1}{2} \partial'^{\mu} \phi(x) \, \partial'_{\mu} \phi(x) - \frac{1}{2} m^2 \phi^2(x)$$

$$= -\frac{1}{2} \eta^{\mu\nu} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \, \partial_{\rho} \phi(x) \, \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \, \partial_{\sigma} \phi(x) - \frac{1}{2} m^2 \phi^2(x)$$

$$= -\frac{1}{2} \eta^{\rho\sigma} \partial_{\rho} \phi(x) \, \partial_{\sigma} \phi(x) - \frac{1}{2} m^2 \phi^2(x)$$

$$= \mathcal{L}(x)$$
(9)

Here we've used  $\eta^{\mu\nu} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} = \eta^{\mu\nu} \lambda_{\mu}^{\phantom{\mu}\rho} \lambda_{\nu}^{\phantom{\nu}\sigma} = \eta^{\rho\sigma}$ . Furthermore,  $S' = \int d^4x \, \mathcal{L}'(x) = \int d^4x' \, \mathcal{L}'(x') = \int d^4x' \, \mathcal{L}(x) = \int d^4x' \, \mathcal{L}(x) = \int d^4x' \, \mathcal{L}(x') = \int d$ 

• Consider an infinitesimal Lorentz transformation:  $\lambda \sim \mathbb{1} + \omega$ , then  $\eta_{\mu\nu}\lambda^{\mu}_{\ \rho}\lambda^{\nu}_{\ \sigma} = \eta_{\rho\sigma}$  implies that  $\omega_{\mu\nu}$  is anti-symmetric:  $\omega_{\mu\nu} + \omega_{\nu\nu} = 0$ . For  $\delta x^{\mu} = \omega^{\mu}_{\ \nu} x^{\nu}$ , we have:

$$\delta\phi = -\frac{\partial\phi}{\partial x^{\mu}} \,\delta x^{\mu} = -\omega^{\mu}_{\ \nu} x^{\nu} \,\partial_{\mu}\phi \tag{10}$$

To obtain the corresponding Noether charges, we can simply repeat the operations done in our previous problem; alternatively, we can try to derive a general recipe<sup>1</sup>: for  $\mathcal{L} = \mathcal{L}(\phi, \partial_{\mu}\phi)$  and  $S = \int d^4x \, \mathcal{L}$ , we have:

$$\delta S = \int d^4 x \, \delta \mathcal{L}$$

$$= \int d^4 x \left( \frac{\partial \mathcal{L}}{\partial \phi} \, \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \, \delta \partial_{\mu} \phi \right)$$

$$= \int d^4 x \left( \frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \right) \delta \phi + \int d^4 x \, \partial_{\mu} \left( \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \, \delta \phi \right)$$
(11)

If we vary S w.r.t. a symmetry of the system, we will have  $\delta \mathcal{L} = \partial_{\mu} K^{\mu}$  some total derivative; when on-shell, such variation gives the conserved current with boundary term  $K^{\mu}$ :

$$j^{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \,\delta\phi - K^{\mu} \tag{12}$$

<sup>&</sup>lt;sup>1</sup>References: arXiv:1601.03616 and *Tong:* http://damtp.cam.ac.uk/user/tong/qft.html

Back to our Lorentz transformation  $\delta \phi = -\omega^{\mu}_{\ \nu} x^{\nu} \partial_{\mu} \phi$ , we have symmetry variation:

$$\delta \mathcal{L} = -\omega^{\mu}_{\ \nu} x^{\nu} \partial_{\mu} \mathcal{L} = -\partial_{\mu} (\omega^{\mu}_{\ \nu} x^{\nu} \mathcal{L}) \tag{13}$$

This gives a boundary term  $K^{\mu} = -\omega^{\mu}_{\nu} x^{\nu} \mathcal{L}$ , and the Noether current and its corresponding conserved charge can be calculated as follows:

$$j^{\mu} = -\omega^{\sigma}_{\ \nu} x^{\nu} \left( \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \, \partial_{\sigma} \phi - \delta^{\mu}_{\sigma} \mathcal{L} \right), \tag{14}$$

$$Q = \int d^3x \, j^0 = -\omega^{\sigma}_{\ \nu} \int d^3x \, x^{\nu} \left( \partial_0 \phi \, \partial_\sigma \phi - \delta^0_\sigma \mathcal{L} \right), \tag{15}$$

Note that  $\omega^{\mu}_{\ \nu}$  is arbitrary, therefore Q can be decomposed into independent charges:

$$Q = \omega_{\mu\nu} M^{\mu\nu}, \quad M^{\mu\nu} = -\int d^3x \, x^{[\mu} \Big( \partial_0 \phi \, \partial^{\nu]} \phi - \eta^{\nu]0} \mathcal{L} \Big), \tag{16}$$

The indices of  $M^{\mu\nu}$  are anti-symmetrized to match the degrees of freedom in  $\omega_{\mu\nu}$ . Note that the  $\mathcal{L}$  term only appears when one of the indices is 0.

Canonical quantization:

$$\dot{\phi} = \partial_0 \phi, \quad \Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}, \quad [\phi(\mathbf{x}), \Pi(\mathbf{y})] = [\phi, \dot{\phi}](\mathbf{x}) = i\delta(\mathbf{x} - \mathbf{y})$$
 (17)

Other equal-time commutators between  $\phi$ ,  $\Pi$  all just vanish. We have:

$$M^{0i} = -M^{i0} = -\frac{1}{2} \int d^3x \left( \dot{\phi} \left( x^0 \partial^i - x^i \partial^0 \right) \phi + x^i \mathcal{L} \right), M^{ij} = -\frac{1}{2} \int d^3x \, \dot{\phi} \left( x^i \partial^j - x^j \partial^i \right) \phi$$
 (18)

Notice that  $x^{[\mu}\partial^{\nu]} = \frac{1}{2}(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}) = \frac{1}{2}D^{ij}$  is the Killing vector fields of  $\mathbb{R}^{3,1}$ , hence they naturally follow the commutation relations of  $\mathfrak{so}(3,1)$  (up to a constant coefficient)<sup>2</sup>. We have:

$$[M^{ij}, M^{kl}] = \frac{1}{4} \int d^3x \int d^3y \left[ \dot{\phi} D^{ij} \phi(x), \dot{\phi} D^{kl} \phi(y) \right]$$
$$= \frac{1}{4} \int d^3x \, \dot{\phi} \left[ D^{ij}, D^{kl} \right] \phi$$
(19)

Similar holds for  $M^{i0}$ . Therefore,  $M^{\mu\nu}$ 's indeed form the Lie algebra  $\mathfrak{so}(3,1)$ .

 $<sup>^2</sup>$ I would like to thank 林般 for pointing this out.