

## Pure QED Partition Function in Covariant Gauge

With the metric convention:  $g \sim (-+++)$ , we have:

$$\mathcal{Z} = \int \mathcal{D}A^\mu e^{-S} \delta[\partial_\mu A^\mu - f] \det[\partial^2 \delta^4(x-y)] \quad (1)$$

Here  $S$  is the Euclidean action:

$$(-S) = \int d^4x \mathcal{L}_{t=-i\tau}, \quad \int d^4x = \int_0^\beta d\tau \int d^3\mathbf{x} \quad (2)$$

For pure QED, we have:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (3)$$

Setting  $t = -i\tau$  is equivalent to carrying out a Wick rotation:  $x^0 \mapsto ix^0$ ,  $A^0 \mapsto iA^0$ , while:

$$g_{\mu\nu} A^\mu A^\nu = g'_{\mu\nu} A'^\mu A'^\nu \implies g_{\mu\nu} \mapsto g'_{\mu\nu} = \delta_{\mu\nu} \quad (4)$$

Under this convention, the Euclidean action is formally unchanged (due to its Lorentz invariance); same applies for the gauge-fixing and the ghost term:

$$\mathcal{L}_{t=-i\tau} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad g_{\mu\nu} \mapsto \delta_{\mu\nu}, \quad (5)$$

$$\delta[\partial_\mu A^\mu - f] \implies \mathcal{L}_{gf} = -\frac{1}{2\rho} (\partial_\mu A^\mu)^2, \quad (6)$$

$$\det[\partial^2 \delta^4(x-y)] \implies \mathcal{L}_{gh} \sim (\partial^2 \bar{\eta}) \eta \sim -\partial_\mu \bar{\eta} \partial^\mu \eta, \quad (7)$$

Here we've dropped some total derivative terms in the ghost Lagrangian. The partition function is then reduced to:

$$\mathcal{Z} = \int \mathcal{D}A^\mu \mathcal{D}\bar{\eta} \mathcal{D}\eta e^{-S'}, \quad (-S') = \int d^4x (\mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{gh})_{t=-i\tau} \quad (8)$$

The action can be further simplified by partial integration and dropping boundary terms:

$$\begin{aligned} (-S') &= \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\rho} (\partial_\mu A^\mu)^2 - \partial_\mu \bar{\eta} \partial^\mu \eta \right) \\ &= \int d^4x \left( -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2\rho} (\partial_\mu A^\mu)^2 - \partial_\mu \bar{\eta} \partial^\mu \eta \right) \\ &= \int d^4x \left( -\frac{1}{2} (\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu) - \frac{1}{2\rho} (\partial_\mu A^\mu \partial_\nu A^\nu) - \partial_\mu \bar{\eta} \partial^\mu \eta \right) \\ &\sim \int d^4x \left( -\frac{1}{2} (-A_\nu \partial^2 A^\nu + A_\mu \partial^\mu \partial_\nu A^\nu) + \frac{1}{2\rho} (A^\mu \partial_\mu \partial_\nu A^\nu) + \bar{\eta} \partial^2 \eta \right) \\ &= \int d^4x \left( -\frac{1}{2} A^\mu \left( -\delta_{\mu\nu} \partial^2 + \partial_\mu \partial_\nu - \frac{1}{\rho} \partial_\mu \partial_\nu \right) A^\nu + \bar{\eta} \partial^2 \eta \right) \\ &= -\frac{1}{2} \int d^4x \left( A^\mu \left( -\delta_{\mu\nu} \partial^2 + \left(1 - \frac{1}{\rho}\right) \partial_\mu \partial_\nu \right) A^\nu - 2\bar{\eta} \partial^2 \eta \right) \end{aligned} \quad (9)$$

With  $\beta = \frac{1}{T}$ , expand  $A^\mu, \eta$  into dimensionless Fourier modes, and we have:

$$A^\mu = \frac{1}{\sqrt{TV}} \sum_k e^{ik_\nu x^\nu} A_k^\mu, \quad \sum_k e^{ik_\nu x^\nu} = V \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} \sum_{n \in \mathbb{Z}} e^{i\omega_n \tau} \quad (10)$$

$$\begin{aligned} \sum_{p,k} \int d^4 x e^{i(p+k) \cdot x} &= \sum_{p,k} (2\pi)^4 \delta^4(p+k) \\ &= V^2 \int \frac{d^3 \mathbf{p} d^3 \mathbf{k}}{(2\pi)^6} \sum_{m,n \in \mathbb{Z}} \beta \delta_{m,-n} \cdot (2\pi)^3 \delta^3(\mathbf{p} + \mathbf{k}) \\ &= \beta V \cdot V \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \sum_{n \in \mathbb{Z}} \int d^3 \mathbf{p} \delta^3(\mathbf{p} + \mathbf{k}) \sum_{n \in \mathbb{Z}} \delta_{m,-n} \\ &= \beta V \sum_k \int d^3 \mathbf{p} \delta^3(\mathbf{p} + \mathbf{k}) \sum_{n \in \mathbb{Z}} \delta_{m,-n}, \end{aligned} \quad (11)$$

$$\begin{aligned} (-S') &= -\frac{1}{2TV} \sum_{p,k} \int d^4 x e^{i(p+k) \cdot x} \left( A_p^\mu \left( -\delta_{\mu\nu}(-k^2) + \left(1 - \frac{1}{\rho}\right) (-k_\mu k_\nu) \right) A_k^\nu - 2\bar{\eta}_p(-k^2) \eta_k \right) \\ &= -\frac{\beta V}{2TV} \sum_k \left( A_{-k}^\mu \left( k^2 \delta_{\mu\nu} - \left(1 - \frac{1}{\rho}\right) k_\mu k_\nu \right) A_k^\nu + 2\bar{\eta}_{-k} k^2 \eta_k \right) \\ &= -\frac{\beta^2}{2} \sum_k \left( A_k^{\mu\dagger} D_{\mu\nu}^{-1}(k) A_k^\nu + 2\bar{\eta}_k^\dagger k^2 \eta_k \right) \end{aligned} \quad (12)$$

Here we've used the reality condition on  $A, \eta$ , namely  $A_k^{\mu\dagger} = A_{-k}^\mu$ , and defined the  $k$ -space inverse propagator  $D_{\mu\nu}^{-1}(k)$ . Similar result applies for  $\eta_k$ , except that we have to be careful about Grassmann variables. Carry out  $\int \mathcal{D}A^\mu \mathcal{D}\bar{\eta} \mathcal{D}\eta$ , and we have:

$$\begin{aligned} \mathcal{Z} &\sim \left( \det_{\mu,\nu,k} \beta^2 D_{\mu\nu}^{-1}(k) \right)^{-1/2} \left( \det_k 2\beta^2 k^2 \right)^{+1} \\ &= \prod_k \left( \beta^{2 \times 4 \times (-1/2)} \cdot \left( \det_{\mu,\nu} D_{\mu\nu}^{-1}(k) \right)^{-1/2} \cdot 2\beta^2 k^2 \right) \\ &= \prod_k \left( \beta^{-4} \left( \det_{\mu,\nu} D_{\mu\nu}^{-1}(k) \right)^{-1/2} \cdot 2\beta^2 k^2 \right) \end{aligned} \quad (13)$$

The determinant is evaluated as follows<sup>1</sup>:

$$\begin{aligned} \det_{\mu,\nu} D_{\mu\nu}^{-1} &= \det_{\mu,\nu} \left( k^2 \delta_{\mu\nu} - \left(1 - \frac{1}{\rho}\right) k_\mu k_\nu \right) \\ &= k^8 \det_{\mu,\nu} \left( \delta_{\mu\nu} - \left(1 - \frac{1}{\rho}\right) \frac{k_\mu k_\nu}{k^2} \right) \\ &= k^8 \left( 1 - \left(1 - \frac{1}{\rho}\right) \frac{k^2}{k^2} \right) \\ &= \frac{1}{\rho} k^8, \end{aligned} \quad (14)$$

---

<sup>1</sup>Reference: Wikipedia: *Determinant # Sylvester's determinant theorem*.

$$\mathcal{Z} \sim \prod_k \frac{2}{\rho} \beta^{-4} k^{-4} \cdot \beta^2 k^2 \sim \prod_k \beta^{-2} k^{-2}, \quad (15)$$

$$\ln \mathcal{Z} \sim - \sum_k \ln (\beta^2 k^2) \quad (16)$$

We see that  $\ln \mathcal{Z}$  is simply twice the result of a neutral scalar field, with mass  $m \rightarrow 0$ , i.e.

$$\ln \mathcal{Z} \sim -2 \times \frac{1}{2} \sum_k \ln (\beta^2 k^2) \sim -2 \sum_{\mathbf{k}} \left( \frac{1}{2} \beta E_k + \ln (1 - e^{-\beta E_k}) \right), \quad (17)$$

$$\mathcal{Z} \sim \prod_{\mathbf{k}} \left\{ \exp \left( -\frac{1}{2} \beta E_k - \ln (1 - e^{-\beta E_k}) \right) \right\}^2, \quad (18)$$

$$\Omega = -T \ln \mathcal{Z} = 2 \sum_{\mathbf{k}} \left( \frac{1}{2} E_k + T \ln (1 - e^{-E_k/T}) \right), \quad (19)$$

$$p = -\frac{\Omega}{V} = -2 \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \left( \frac{1}{2} E_k + T \ln (1 - e^{-E_k/T}) \right), \quad (20)$$

Here  $E_k = \|\mathbf{k}\|$ . Ignore the vacuum contribution to  $p$ , and we have:

$$p = -2T \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \ln (1 - e^{-\|\mathbf{k}\|/T}) = 2 \frac{\pi^2}{90} T^4 \quad (21)$$

## Summary

1. We've completed the calculation of pure QED partition function  $\mathcal{Z}$  and thermodynamic potential  $\Omega$  (energy density  $p$ ) by introducing a “soft” gauge fix  $\mathcal{L}_{gf}$ . Alternatively, we can simply impose a “hard” Lorenz gauge fix  $\delta[\partial_\mu A^\mu]$ ; this can be achieved by taking  $\rho \rightarrow \infty$  in the Gaussian packet  $-\frac{1}{2\rho} (\partial_\mu A^\mu)^2$ , or by integrating out  $A^0$  directly — similar to (12), we have:

$$\mathcal{Z} \sim \int \mathcal{D}A^\mu \mathcal{D}\bar{\eta} \mathcal{D}\eta \delta[\partial_\mu A^\mu] \exp \left( -\frac{\beta^2}{2} \sum_k \left( A_k^{\mu\dagger} (k^2 \delta_{\mu\nu} - k_\mu k_\nu) A_k^\nu + 2\bar{\eta}_k^\dagger k^2 \eta_k \right) \right) \quad (22)$$

$$\sim \int \mathcal{D}A^i \mathcal{D}\bar{\eta} \mathcal{D}\eta \exp \left( -\frac{\beta^2}{2} \sum_k \left( A_k^{\mu\dagger} (k^2 \delta_{\mu\nu} - k_\mu k_\nu) A_k^\nu + 2\bar{\eta}_k^\dagger k^2 \eta_k \right) \right)_{A^0=A^0[A^i]} \quad (23)$$

$$A^0[A^i] = - \int \partial_i A^i d\tau, \quad k_\mu A_k^\mu = 0, \quad (23)$$

Here we've omitted a non-dynamical Jacobian  $\det[\theta(\tau - \tau')] = \det \int^\tau d\tau'' \delta(\tau'' - \tau')$ . The ghost integral gives the same contribution, while the  $A^i$  integral yields:

$$\mathcal{Z}_A = \int \mathcal{D}A^i \exp \left( -\frac{\beta^2}{2} \sum_k \left( A_k^{\mu\dagger} (k^2 \delta_{\mu\nu}) A_k^\nu \right) \right), \quad (24)$$

$$\begin{aligned} A_k^{\mu\dagger} (k^2 \delta_{\mu\nu}) A_k^\nu &= A_k^{i\dagger} (k^2 \delta_{ij}) A_k^j + A_k^{0\dagger} (k^2) A_k^0 = A_k^{i\dagger} (k^2 \delta_{ij}) A_k^j + \frac{k^2}{\omega^2} A_k^{0\dagger} (\omega^2) A_k^0 \\ &\stackrel{(23)}{=} A_k^{i\dagger} (k^2 \delta_{ij}) A_k^j + \frac{k^2}{\omega^2} A_k^{i\dagger} (k_i k_j) A_k^0 = A_k^{i\dagger} k^2 \left( \delta_{ij} + \frac{k_i k_j}{\omega^2} \right) A_k^j \\ &= A_k^{i\dagger} D_{ij}^{-1}(k) A_k^j, \end{aligned} \quad (25)$$

$$\begin{aligned}
\mathcal{Z}_A &\sim \left( \det_{i,j,k} \beta^2 D_{ij}^{-1}(k) \right)^{-1/2} = \prod_k \beta^{-3} k^{-3} \left( 1 + \frac{\mathbf{k}^2}{\omega^2} \right)^{-1/2} = \prod_k (\beta^{-4} k^{-4} \cdot \beta \omega) \\
&= \prod_k \beta^{-4} k^{-4} \prod_{\mathbf{k}} \prod_n 2\pi n \sim \prod_k \beta^{-4} k^{-4}
\end{aligned} \tag{26}$$

We see that the result from a “hard” Lorenz gauge fixing is the same as before, up to a non-dynamical overall coefficient.

2. In our previous calculations, we notice that after functional integration,  $\rho$  is just a non-dynamical overall coefficient in  $\mathcal{Z}$ , hence it can be safely dropped from the final expression; see eq. (15). Therefore, the result is independent of parameter  $\rho$ .