

1 Strings on Curved Space:

$$S = \frac{1}{4\pi\alpha'} \int_M d^2\sigma \sqrt{g} \left(i\epsilon^{ab} B_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu + \dots \right), \quad (1)$$

$$T^a_a = -\frac{1}{2\alpha'} \beta_{\mu\nu}^G g^{ab} \partial_a X^\mu \partial_b X^\nu + \dots, \quad (2)$$

$$\beta_{\mu\nu}^G = \alpha' R_{\mu\nu} - \frac{1}{4} \alpha' H_{\mu\lambda\omega} H_\nu^{\lambda\omega} + \dots + \mathcal{O}(\alpha'^2) \quad (3)$$

We want to verify the coefficient of $\alpha' H^2$ term in $\beta_{\mu\nu}^G$; for convenience we've omitted non-related terms in the above expressions.

Note that at $\mathcal{O}(\alpha')$ such term does not depend on the metric $G_{\mu\nu}$, and it depends only on the field strength $H = dB$, not the potential B , hence it's safe to assume:

$$G_{\mu\nu} = \eta_{\mu\nu}, \quad B_{\mu\nu} = \frac{1}{3} H_{\mu\nu\rho} X^\rho, \quad H = \text{const}, \quad (4)$$

$$i\epsilon^{ab} B_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu = \frac{i}{3} H_{\mu\nu\rho} X^\rho \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu, \quad (5)$$

We consider small perturbation away from the classical saddle: $X = X_0 + \xi$, then the 1-loop effective action is obtained by integrating over $\mathcal{O}(\xi^2)$ terms in the perturbed action¹:

$$\Gamma^{(1)}[X_0] = -\ln \int \mathcal{D}\xi \, e^{-S^{(2)}[X_0, \xi]}, \quad (6)$$

$$\begin{aligned} \mathcal{L}^{(2)} &= \frac{i}{3} H_{\mu\nu\rho} \epsilon^{ab} \left(\xi^\rho \partial_a X_0^\mu \partial_b \xi^\nu + \xi^\rho \partial_a \xi^\mu \partial_b X_0^\nu + X_0^\rho \partial_a \xi^\mu \partial_b \xi^\nu \right) \\ &\sim \frac{i}{3} H_{\mu\nu\rho} \epsilon^{ab} \left(\xi^\rho \partial_a X_0^\mu \partial_b \xi^\nu - \xi^\rho \partial_a X_0^\nu \partial_b \xi^\mu - \xi^\mu \partial_a X_0^\rho \partial_b \xi^\nu \right) \\ &= \frac{i}{3} H_{\mu\nu\rho} \epsilon^{ab} \cdot 3\xi^\rho \partial_a X_0^\mu \partial_b \xi^\nu \\ &= i H_{\mu\nu\rho} \epsilon^{ab} \partial_a X_0^\mu (\xi^\rho \partial_b \xi^\nu) \end{aligned} \quad (7)$$

Here we've used the anti-symmetric properties of $H_{\mu\nu\rho}, \epsilon^{ab}$, and ignored any total derivative after integration by parts. This term introduces a cubic interaction vertex in the free background; therefore, $\Gamma^{(1)}$ can be expressed in the following diagram²:



$$\sim \frac{1}{2!} \left(\frac{1}{\alpha'} \right)^2 \int d^2p \left(i H_{\mu\nu\rho} \epsilon^{ab} \partial_a X_0^\mu i p_b \right) \frac{2}{p^4} \left(-\frac{\alpha'}{2} \right)^2 \left(i H_{\mu'\nu\rho} \epsilon^{a'b'} \partial_{a'} X_0^{\mu'} i p_{b'} \right) \quad (8)$$

¹Reference: Prof. Xi Yin's String Notes, see also [arXiv:0812.4408](https://arxiv.org/abs/0812.4408).

²References:

- David Tong, *String Theory*;
- Callan & Thorlacius, *Sigma Models and String Theory*;
- Timo Weigand, *Introduction to String Theory*.

$$= \frac{2}{2!} \left(\frac{1}{\alpha'} \right)^2 \left(-\frac{\alpha'}{2} \right)^2 H_{\mu\lambda\omega} H_{\nu}{}^{\lambda\omega} \partial_a X_0^\mu \partial_b X_0^\nu \int d^2p \frac{p^2 g^{ab} - p^a p^b}{p^4} \quad (9)$$

$$= \frac{2}{2!} \left(-\frac{1}{2} \right)^2 H_{\mu\lambda\omega} H_{\nu}{}^{\lambda\omega} \partial_a X_0^\mu \partial_b X_0^\nu \left(\frac{1}{2} g^{ab} \right) \int d^2p \frac{1}{p^2} \quad (10)$$

$$= \frac{2}{2!} \left(-\frac{1}{2} \right)^2 \left(\frac{1}{2} \right) H_{\mu\lambda\omega} H_{\nu}{}^{\lambda\omega} \partial_a X_0^\mu \partial_b X_0^\nu g^{ab} \int d^2p \frac{1}{p^2} \quad (11)$$

$$= \frac{1}{8} H_{\mu\lambda\omega} H_{\nu}{}^{\lambda\omega} g^{ab} \partial_a X_0^\mu \partial_b X_0^\nu \int d^2p \frac{1}{p^2} \quad (12)$$

Here the $\left(\frac{1}{\alpha'}\right)^2$ coefficient comes from the vertices, while $\left(-\frac{\alpha'}{2}\right)^2$ comes from the propagators. The $p^a p^b$ integral provides an additional $\left(\frac{1}{2}\right)$ factor. The overall normalization is chosen to match the $\alpha' R_{\mu\nu}$ coefficient in $\beta_{\mu\nu}^G \subset T_a^a$, which is $\frac{1}{1!} \times \left(-\frac{1}{2}\right) \times 1 = -\frac{1}{2}$. Therefore, we have:

$$T_a^a \supset \frac{1}{8} H_{\mu\lambda\omega} H_{\nu}{}^{\lambda\omega} g^{ab} \partial_a X_0^\mu \partial_b X_0^\nu, \quad (13)$$

$$\beta_{\mu\nu}^G \supset -\frac{1}{4} \alpha' H_{\mu\lambda\omega} H_{\nu}{}^{\lambda\omega} \quad (14)$$

■

[2] Classical Solutions of 11D SUGRA: Following the convention of *Polchinski*, we have bosonic action:

$$S = \frac{1}{2\kappa^2} \int \left(d^{11}x \sqrt{-g} \mathcal{R} - \frac{1}{2} G \wedge * G - \frac{1}{6} C \wedge G \wedge G \right), \quad (15)$$

Here $G = dC$: a 4-form field. In components, the numerical coefficients would be $\frac{1}{2} \mapsto \frac{1}{2 \times 4!} = \frac{1}{48}$, and $\frac{1}{6} \mapsto \frac{1}{6 \times 3! \times 4! \times 4!} = \frac{1}{20736}$.

Variation of the action yields the EOMs of our theory³; Note that:

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \quad (16)$$

$\frac{\delta S}{\delta g^{\mu\nu}}$ is easier to compute in components; note that the $C \wedge G \wedge G$ term does not depend on $g^{\mu\nu}$, therefore it does not contribute to the EOM. We have the usual Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2} \mathcal{R} g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \quad (17)$$

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{\kappa^2} \left(\frac{4}{48} G_{\mu\sigma_1\sigma_2\sigma_3} G_{\nu}{}^{\sigma_1\sigma_2\sigma_3} - \frac{1}{2} g_{\mu\nu} \cdot \frac{1}{48} G^{\sigma_1\sigma_2\sigma_3\sigma_4} G_{\sigma_1\sigma_2\sigma_3\sigma_4} \right) \\ &= \frac{1}{12\kappa^2} \left(G_{\mu\sigma_1\sigma_2\sigma_3} G_{\nu}{}^{\sigma_1\sigma_2\sigma_3} - \frac{1}{8} g_{\mu\nu} G^{\sigma_1\sigma_2\sigma_3\sigma_4} G_{\sigma_1\sigma_2\sigma_3\sigma_4} \right) \end{aligned} \quad (18)$$

³Reference: [arXiv:hep-th/9912164](https://arxiv.org/abs/hep-th/9912164). I would like to thank *Lucy Smith* for many helpful discussions.

On the other hand, $\frac{\delta S}{\delta C}$ is best carried out using differential forms:

$$\begin{aligned}
0 = \delta_C S &= -\frac{1}{2\kappa^2} \int \left(\delta G \wedge *G + \frac{1}{6} (\delta C \wedge G \wedge G - 2C \wedge \delta G \wedge G) \right) \\
&= -\frac{1}{2\kappa^2} \int \left(\delta(dC) \wedge *G + \frac{1}{6} (\delta C \wedge G \wedge G + 2\delta(dC) \wedge C \wedge G) \right) \\
&= -\frac{1}{2\kappa^2} \int \left(-(-1)^3 \delta C \wedge d *G + \frac{1}{6} (\delta C \wedge G \wedge G - 2(-1)^3 \delta C \wedge d(C \wedge G)) \right) \quad (19) \\
&= -\frac{1}{2\kappa^2} \int \delta C \wedge \left(d *G + \frac{1}{6} (G \wedge G + 2(G \wedge G - C \wedge d^2 C)) \right) \\
&= -\frac{1}{2\kappa^2} \int \delta C \wedge \left(d *G + \frac{1}{2} G \wedge G \right),
\end{aligned}$$

$$d *G + \frac{1}{2} G \wedge G = 0 \quad (20)$$

(a) We hope to find a spacetime solution which is *maximally symmetric* in *some* directions; assume that these directions form a d -dimensional sub-manifold \mathcal{M}_d with:

$$\begin{aligned}
\text{Coordinates: } & x^{\mu'}, \mu' \in \Delta \subset \{0, 1, \dots, 11\}, \\
\text{Induced metric: } & g' = g|_{\mathcal{M}_d}
\end{aligned} \quad (21)$$

The entire spacetime is then a direct product: $\mathcal{M}_d \times \widetilde{\mathcal{M}}_{11-d}$. For \mathcal{M}_d to be maximally symmetric, we expect that $\kappa^2 T_{\mu'\nu'} = -\Lambda g'_{\mu'\nu'}$, i.e. the G -field serves as a cosmological constant Λ . By staring at (18) we find that this can be achieved with⁴:

$$d = 4, \quad G_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} = \alpha \sqrt{|g'|} \epsilon_{\sigma_1 \sigma_2 \sigma_3 \sigma_4}, \quad G^{\sigma_1 \sigma_2 \sigma_3 \sigma_4} = \alpha \frac{\text{sgn } g'}{\sqrt{|g'|}} \epsilon^{\sigma_1 \sigma_2 \sigma_3 \sigma_4}, \quad \{\sigma_i\} \subset \Delta, \quad (22)$$

$$G_{\dots \sigma \dots} = 0, \quad \sigma \notin \Delta, \quad (23)$$

$$T_{\mu\nu} = (\text{sgn } g') \frac{\alpha^2}{12\kappa^2} \left(3! g'_{\mu\nu} - \frac{4!}{8} g_{\mu\nu} \right) = (\text{sgn } g') \frac{\alpha^2}{2\kappa^2} \left(g'_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \right), \quad (24)$$

$$\Lambda g_{\mu\nu} = \mp (\text{sgn } g') \frac{\alpha^2}{4\kappa^2} g_{\mu\nu}, \quad \begin{cases} -: \mu = \mu', \nu = \nu' \in \Delta, & \sim \mathcal{M}_4 \\ +: \mu, \nu \notin \Delta, & \sim \widetilde{\mathcal{M}}_7 \end{cases} \quad (25)$$

Matter EOM is trivially satisfied due to anti-symmetry. We see that the other component $\widetilde{\mathcal{M}}_7$ is also maximally symmetric, but with an opposite sign in its cosmological constant.

The field equations in \mathcal{M}_4 and $\widetilde{\mathcal{M}}_7$ are both of the form $R_{\mu\nu} \propto g_{\mu\nu}$. For $\text{sgn } g' = -1$ i.e. Lorentzian signature, the solution is flat, AdS or dS, depending on the sign of Λ ; for $\text{sgn } g' = +1$, the solution is flat, spherical or hyperbolic. Therefore, we have:

$$\begin{aligned}
\text{sgn } g' = -1, \quad \Lambda_{4,7} &= \pm \frac{\alpha^2}{4\kappa^2}, \quad \mathcal{M}_4 = \text{AdS}_{3,1}, \quad \widetilde{\mathcal{M}}_7 = S^7 \\
\text{sgn } g' = +1, \quad \Lambda_{4,7} &= \mp \frac{\alpha^2}{4\kappa^2}, \quad \mathcal{M}_4 = S^4, \quad \widetilde{\mathcal{M}}_7 = \text{AdS}_{6,1}
\end{aligned} \quad (26)$$

⁴This is in fact the famous *Freund–Robin ansatz*; see Wikipedia: *Freund–Rubin compactification*, and also the original paper: Freund & Robin, *Dynamics of Dimensional Reduction*, 1980.

(b) Global supersymmetries of a theory with the above $\text{AdS}_{4/7} \times S^{4/7}$ background are given by the solutions of:

$$0 = \delta_\eta \psi^\mu \equiv D^\mu \eta(x), \quad \eta: \text{spinor}, \quad (27)$$

$$\begin{aligned} D^\mu &= \nabla^\mu + \frac{1}{288} G_{\nu\rho\sigma\lambda} (\Gamma^{\mu\nu\rho\sigma\lambda} - 8g^{\mu\nu}\Gamma^{\rho\sigma\lambda}) \\ &= \nabla^\mu + \frac{1}{288} G_{\nu'\rho'\sigma'\lambda'} (\Gamma^{\mu\nu'\rho'\sigma'\lambda'} - 8g^{\mu\nu'}\Gamma^{\rho'\sigma'\lambda'}) \\ &= \nabla^\mu + \alpha \begin{cases} \frac{-8 \times 3!}{288} (-\Gamma^\mu \gamma_5) = \frac{1}{6} \Gamma^\mu \gamma_5, & \mu = \mu' \in \Delta, \quad \sim \mathcal{M}_4 \\ \frac{4!}{288} (-\Gamma^\mu) = -\frac{1}{12} \Gamma^\mu, & \mu \notin \Delta, \quad \sim \widetilde{\mathcal{M}}_7 \end{cases} \end{aligned} \quad (28)$$

Note that we've replaced the G indices with \mathcal{M}_4 indices, since G vanish in $\widetilde{\mathcal{M}}_7$ directions; due to anti-symmetry, the G -term can be reduced to simple Γ^μ multiplications according to the μ -direction⁵. Furthermore, the spin connection in ∇^μ is also block diagonalized, same as $g_{\mu\nu}$; hence there is a natural separation of variable⁶:

$$\eta = \eta'(x') \eta''(x''), \quad D_{\mu'} \eta' = 0, \quad D_{\mu''} \eta'' = 0, \quad (29)$$

$$\mu', \eta', x' \sim \mathcal{M}_4, \quad \mu'', \eta'', x'' \sim \widetilde{\mathcal{M}}_7, \quad (30)$$

Due to the presence of an additional Γ , $D_{\mu'} \eta' = 0$ has only 4 linearly independent solutions labeled by μ' , while $D_{\mu''} \eta'' = 0$ is $\text{Spin}(8)$ (or $\text{Spin}(7, 1)$, depending on the signature) invariant, and has $\frac{8 \times 7}{2} = 28$ linearly independent solutions⁷. Hence the total number of SUSYs is $4 + 28 = 32$, for $\text{AdS}_{4/7} \times S^{4/7}$ background.

3 SUSY Sigma Models via Superspace:

$$\begin{aligned} D_{\bar{\theta}} \mathbf{X}^\nu &= (\partial_{\bar{\theta}} + \bar{\theta} \partial_{\bar{z}}) (X^\nu + i\theta \psi^\nu + i\bar{\theta} \tilde{\psi}^\nu + \theta \bar{\theta} F^\nu) \\ &= i\tilde{\psi}^\nu - \theta F^\nu + \bar{\theta} \bar{\partial} X^\nu - i\theta \bar{\theta} \bar{\partial} \psi^\nu, \\ D_{\theta} \mathbf{X}^\mu &= i\psi^\mu + \bar{\theta} F^\mu + \theta \partial X^\mu + i\theta \bar{\theta} \partial \tilde{\psi}^\mu, \end{aligned} \quad (31)$$

$$\begin{aligned} D_{\bar{\theta}} \mathbf{X}^\nu D_{\theta} \mathbf{X}^\mu &= (i\tilde{\psi}^\nu - \theta F^\nu + \bar{\theta} \bar{\partial} X^\nu - i\theta \bar{\theta} \bar{\partial} \psi^\nu) (i\psi^\mu + \bar{\theta} F^\mu + \theta \partial X^\mu + i\theta \bar{\theta} \partial \tilde{\psi}^\mu) \\ &= -\tilde{\psi}^\nu \psi^\mu - i\theta (\tilde{\psi}^\nu \partial X^\mu + \psi^\mu F^\nu) + i\bar{\theta} (\psi^\mu \bar{\partial} X^\nu - \tilde{\psi}^\nu F^\mu) \\ &\quad - \theta \bar{\theta} (\bar{\partial} X^\nu \partial X^\mu + \tilde{\psi}^\nu \partial \tilde{\psi}^\mu - (\bar{\partial} \psi^\nu) \psi^\mu + F^\nu F^\mu), \end{aligned} \quad (32)$$

$$\begin{aligned} G_{\mu\nu}(\mathbf{X}) &= G_{\mu\nu} + (i\theta \psi^\lambda + i\bar{\theta} \tilde{\psi}^\lambda + \theta \bar{\theta} F^\lambda) \partial_\lambda G_{\mu\nu} + \frac{1}{2} \{ i\theta \psi^\rho \partial_\rho, i\bar{\theta} \tilde{\psi}^\sigma \partial_\sigma \} G_{\mu\nu} \\ &= G_{\mu\nu} + (i\theta \psi^\lambda + i\bar{\theta} \tilde{\psi}^\lambda) G_{\mu\nu, \lambda} + \theta \bar{\theta} (F^\lambda G_{\mu\nu, \lambda} + \psi^\rho \tilde{\psi}^\sigma G_{\mu\nu, \rho\sigma}), \end{aligned} \quad (33)$$

⁵Reference for Γ -matrices and spinors: *Polchinski* Vol. II, Appendix B. I'm a bit confused about all the complicated conventions, therefore the coefficients might be off by some factors...

⁶See [arXiv:hep-th/9912164](https://arxiv.org/abs/hep-th/9912164) for more detailed discussions.

⁷Reference: Achilleas Passias, *Aspects of Supergravity in Eleven Dimensions*.

Note that $\int d^2\theta = \partial_\theta \bar{\partial}_{\bar{\theta}}$, hence we need only focus on the $\theta\bar{\theta}$ term in the Lagrangian:

$$\begin{aligned}
4\pi S_G &= \int d^2z d^2\theta G_{\mu\nu}(\mathbf{X}) D_{\bar{\theta}} \mathbf{X}^\mu D_{\theta} \mathbf{X}^\nu = \int d^2z d^2\theta (-\theta\bar{\theta}) \left(G_{\mu\nu} (\partial X^\mu \bar{\partial} X^\nu + \dots) + \dots \right) \\
&= \int d^2z \left(G_{\mu\nu} \left(\partial X^\mu \bar{\partial} X^\nu + \tilde{\psi}^\nu \partial \tilde{\psi}^\mu - (\bar{\partial} \psi^\nu) \psi^\mu + F^\nu F^\mu \right) \right. \\
&\quad \left. + \tilde{\psi}^\nu \psi^\mu \left(F^\lambda G_{\mu\nu,\lambda} + \psi^\rho \tilde{\psi}^\sigma G_{\mu\nu,\rho\sigma} \right) \right. \\
&\quad \left. - G_{\mu\nu,\lambda} \left(\psi^\lambda (\psi^\mu \bar{\partial} X^\nu - \tilde{\psi}^\nu F^\mu) + \tilde{\psi}^\lambda (\tilde{\psi}^\nu \partial X^\mu + \psi^\mu F^\nu) \right) \right)
\end{aligned} \tag{34}$$

Similar result holds for the B contribution S_B . We see that there is no ∂F term in the action, hence F is not dynamical and can be integrated out; we have:

$$0 = \delta_F S = \delta_F (S_G + S_B), \tag{35}$$

$$\begin{aligned}
4\pi \delta S_G &= \int d^2z \left(2G_{\mu\nu} F^\mu \delta F^\nu + G_{\mu\nu,\lambda} (\tilde{\psi}^\nu \psi^\mu \delta F^\lambda - \tilde{\psi}^\nu \psi^\lambda \delta F^\mu - \tilde{\psi}^\lambda \psi^\mu \delta F^\nu) \right) \\
&= \int d^2z \left(2F_\lambda + (G_{\mu\nu,\lambda} - G_{\lambda\mu,\nu} - G_{\lambda\nu,\mu}) \tilde{\psi}^\nu \psi^\mu \right) \delta F^\lambda \\
&= \int d^2z \left(2F_\lambda - 2\Gamma_{\lambda\mu\nu} \tilde{\psi}^\nu \psi^\mu \right) \delta F^\lambda,
\end{aligned} \tag{36}$$

$$\begin{aligned}
4\pi \delta S_B &= \int d^2z \left(0 + (B_{\mu\nu,\lambda} + B_{\lambda\mu,\nu} + B_{\nu\lambda,\mu}) \tilde{\psi}^\nu \psi^\mu \right) \delta F^\lambda = \int d^2z H_{\lambda\mu\nu} \tilde{\psi}^\nu \psi^\mu \delta F^\lambda, \\
F_\lambda &= \left(\Gamma_{\lambda\mu\nu} - \frac{1}{2} H_{\lambda\mu\nu} \right) \tilde{\psi}^\nu \psi^\mu,
\end{aligned} \tag{37}$$

$$F^\lambda = \left(\Gamma_{\mu\nu}^\lambda - \frac{1}{2} H_{\mu\nu}^\lambda \right) \tilde{\psi}^\nu \psi^\mu, \tag{38}$$

Here we've used the (anti-)symmetry of $G_{\mu\nu}$ and $B_{\mu\nu}$, and we adopt the convention that the Levi-Civita connection $\Gamma_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda = G^{\lambda\lambda'} \Gamma_{\lambda'\mu\nu}$; similar holds for $B_{\mu\nu}$ and $H_{\mu\nu}^\lambda$.

Substitute F_λ into S , collect the $\psi^0, \psi^2, \tilde{\psi}^2$ and $\psi^2 \tilde{\psi}^2$ terms respectively, and we have:

$$\begin{aligned}
4\pi S &= \int d^2z \left((G_{\mu\nu} + B_{\mu\nu}) \partial X^\mu \bar{\partial} X^\nu \right. \\
&\quad \left. + (G_{\mu\nu} + B_{\mu\nu}) \left(\tilde{\psi}^\mu \partial \tilde{\psi}^\nu - (\bar{\partial} \psi^\mu) \psi^\nu \right) \right. \\
&\quad \left. - (G_{\mu\nu,\lambda} + B_{\mu\nu,\lambda}) \left(\psi^\lambda \psi^\mu \bar{\partial} X^\nu + \tilde{\psi}^\lambda \tilde{\psi}^\nu \partial X^\mu \right) \right. \\
&\quad \left. + G_{\mu\nu} F^\mu F^\nu - 2 \left(\Gamma_{\lambda\mu\nu} - \frac{1}{2} H_{\lambda\mu\nu} \right) \tilde{\psi}^\nu \psi^\mu F^\lambda \right. \\
&\quad \left. + (G_{\mu\nu,\rho\sigma} + B_{\mu\nu,\rho\sigma}) \tilde{\psi}^\nu \psi^\mu \psi^\rho \tilde{\psi}^\sigma \right) \\
&= \int d^2z \left((G_{\mu\nu} + B_{\mu\nu}) \partial X^\mu \bar{\partial} X^\nu \right. \\
&\quad \left. + G_{\mu\nu} \left(\tilde{\psi}^\mu \partial \tilde{\psi}^\nu + \psi^\mu \bar{\partial} \psi^\nu \right) - (G_{\mu\nu,\lambda} + B_{\mu\nu,\lambda}) \left(\psi^\lambda \psi^\mu \bar{\partial} X^\nu + \tilde{\psi}^\lambda \tilde{\psi}^\nu \partial X^\mu \right) \right. \\
&\quad \left. - F_\lambda F^\lambda + (G_{\mu\nu,\rho\sigma} + B_{\mu\nu,\rho\sigma}) \psi^\mu \psi^\rho \tilde{\psi}^\nu \tilde{\psi}^\sigma \right)
\end{aligned} \tag{39}$$

Here we've performed some integration by parts to clean up the result. Note that some terms involving $B_{\mu\nu}$ vanish conveniently (up to integration by parts) due to anti-symmetry.

The $\psi^2, \tilde{\psi}^2$ terms in the integrand can be further simplified as follows:

$$\begin{aligned}
\mathcal{L}_{\psi^2} &= G_{\mu\nu} \psi^\mu \bar{\partial} \psi^\nu - (G_{\mu\nu, \lambda} + B_{\mu\nu, \lambda}) \psi^\lambda \psi^\mu \bar{\partial} X^\nu \\
&= G_{\mu\nu} \psi^\mu \bar{\partial} \psi^\nu - (G_{\mu[\nu, \lambda]} + B_{\mu[\nu, \lambda]}) \psi^\lambda \psi^\mu \bar{\partial} X^\nu \\
&= G_{\mu\nu} \psi^\mu \bar{\partial} \psi^\nu - \left(-\Gamma_{\lambda\mu\nu} + \frac{1}{2} H_{\lambda\mu\nu} \right) \psi^\lambda \psi^\mu \bar{\partial} X^\nu \\
&= G_{\mu\nu} \psi^\mu \left(\bar{\partial} \psi^\nu + \left(\Gamma_{\rho\sigma}^\nu - \frac{1}{2} H_{\rho\sigma}^\nu \right) \psi^\rho \bar{\partial} X^\sigma \right) \\
&= G_{\mu\nu} \psi^\mu \left(\bar{\partial} \psi^\nu + \left(\Gamma_{\rho\sigma}^\nu + \frac{1}{2} H_{\rho\sigma}^\nu \right) \psi^\sigma \bar{\partial} X^\rho \right) = G_{\mu\nu} \psi^\mu \bar{\mathcal{D}} \psi^\nu, \\
\mathcal{L}_{\tilde{\psi}^2} &= G_{\mu\nu} \tilde{\psi}^\mu \partial \tilde{\psi}^\nu - (G_{\mu\nu, \lambda} + B_{\mu\nu, \lambda}) \tilde{\psi}^\lambda \tilde{\psi}^\mu \partial X^\nu \\
&= G_{\mu\nu} \tilde{\psi}^\mu \left(\partial \tilde{\psi}^\nu + \left(\Gamma_{\rho\sigma}^\nu - \frac{1}{2} H_{\rho\sigma}^\nu \right) \tilde{\psi}^\sigma \partial X^\rho \right) = G_{\mu\nu} \tilde{\psi}^\mu \mathcal{D} \tilde{\psi}^\nu,
\end{aligned} \tag{40}$$

For the $\psi^2 \tilde{\psi}^2$ term, recall that $R_{\mu\nu\rho\sigma} = e_\mu [\nabla_\rho, \nabla_\sigma] e_\nu$, $\nabla_\sigma e_\nu = e_\lambda \Gamma_{\sigma\nu}^\lambda$, and we have:

$$\begin{aligned}
\mathcal{L}_{\psi^2 \tilde{\psi}^2} &= \psi^\mu \psi^\nu \tilde{\psi}^\rho \tilde{\psi}^\sigma \left(G_{\mu\rho, \nu\sigma} + B_{\mu\rho, \nu\sigma} + \left(\Gamma_{\lambda\mu\rho} - \frac{1}{2} H_{\lambda\mu\rho} \right) \left(\Gamma_{\nu\sigma}^\lambda - \frac{1}{2} H_{\nu\sigma}^\lambda \right) \right) \\
&= \psi^\mu \psi^\nu \tilde{\psi}^\rho \tilde{\psi}^\sigma \left(G_{\mu\rho, \nu\sigma} + \Gamma_{\lambda\mu\rho} \Gamma_{\nu\sigma}^\lambda + B_{\mu\rho, \nu\sigma} - \frac{1}{2} \left(\Gamma_{\mu\rho}^\lambda H_{\lambda\nu\sigma} + \Gamma_{\nu\sigma}^\lambda H_{\lambda\mu\rho} \right) + \frac{1}{4} H_{\mu\rho}^\lambda H_{\lambda\nu\sigma} \right) \\
&= \mathcal{L}_G + \mathcal{L}_B + \frac{1}{4} H_{\mu\rho}^\lambda H_{\lambda\nu\sigma} \psi^\mu \psi^\nu \tilde{\psi}^\rho \tilde{\psi}^\sigma,
\end{aligned} \tag{41}$$

$$\begin{aligned}
\mathcal{L}_G &= \psi^\mu \psi^\nu \tilde{\psi}^\rho \tilde{\psi}^\sigma (G_{\mu\rho, \nu\sigma} + \Gamma_{\lambda\mu\rho} \Gamma_{\nu\sigma}^\lambda) \\
&= \psi^{[\mu} \psi^{\nu]} \tilde{\psi}^{[\rho} \tilde{\psi}^{\sigma]} (G_{\mu\rho, \nu\sigma} + \Gamma_{\lambda\mu\rho} \Gamma_{\nu\sigma}^\lambda) \\
&= \frac{1}{2} \psi^\mu \psi^\nu \tilde{\psi}^\rho \tilde{\psi}^\sigma \left\{ \left(\frac{1}{2} (G_{\mu\rho, \nu\sigma} - G_{\mu\sigma, \nu\rho}) + \Gamma_{\lambda\mu\rho} \Gamma_{\nu\sigma}^\lambda \right) - (\dots)_{\rho \leftrightarrow \sigma} \right\} \\
&= \frac{1}{2} R_{\mu\nu\rho\sigma} \psi^\mu \psi^\nu \tilde{\psi}^\rho \tilde{\psi}^\sigma,
\end{aligned} \tag{42}$$

$$\mathcal{L}_B = \frac{1}{2} \nabla_\rho H_{\mu\nu\sigma} \psi^\mu \psi^\nu \tilde{\psi}^\rho \tilde{\psi}^\sigma,$$

Therefore, the total action is:

$$\begin{aligned}
S &= \frac{1}{4\pi} \int d^2 z \left((G_{\mu\nu} + B_{\mu\nu}) \partial X^\mu \bar{\partial} X^\nu \right. \\
&\quad \left. + G_{\mu\nu} (\tilde{\psi}^\mu \mathcal{D} \tilde{\psi}^\nu + \psi^\mu \bar{\mathcal{D}} \psi^\nu) \right. \\
&\quad \left. + \left(\frac{1}{2} R_{\mu\nu\rho\sigma} + \frac{1}{2} \nabla_\rho H_{\mu\nu\sigma} + \frac{1}{4} H_{\mu\rho}^\lambda H_{\lambda\nu\sigma} \right) \psi^\mu \psi^\nu \tilde{\psi}^\rho \tilde{\psi}^\sigma \right)
\end{aligned} \tag{43}$$

4 Mixed Anomaly Between Diffeomorphism and Axial $U(1)$ Symmetry:

(a) Calculations of such anomaly is (schematically) similar to the usual axial anomaly; instead of the A_μ legs, we now have two $h_{\mu\nu}$ legs in the triangular diagram.

Again we chose the Pauli–Villars regularization with a regulator field ψ' of mass $M \rightarrow \infty$. The $\partial^\mu J_\mu^A$ insertion is then reduced to:

$$\partial^\mu J_\mu^A = \partial_\mu (i \bar{\psi}' \gamma^\mu \gamma^5 \psi') = i \bar{\psi}' (2M \gamma^5) \psi' \tag{44}$$

The fermion–fermion–graviton vertex is given by $h_{\mu\nu}T^{\mu\nu}$, and (up to integration by parts) we have:

$$T^{\mu\nu} = \frac{i}{2}\bar{\psi}\gamma^{(\mu}\overleftrightarrow{\partial}^{\nu)}\psi \sim \frac{i}{2}\bar{\psi}\gamma^{(\mu}(-2\partial^{\nu)}\psi = -i\bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi, \quad (45)$$

$$h_{\mu\nu}T^{\mu\nu} = \bar{\psi}\left(-ih_{\mu\nu}\gamma^{(\mu}\partial^{\nu)}\right)\psi, \quad (46)$$

This is very similar to the A_μ coupling, except that there is an extra derivative ∂^ν . Denote the polarization of graviton as $\varepsilon_{\mu\nu}$, then in momentum space the interaction vertex $\sim \varepsilon_{\mu\nu}\gamma^{(\mu}(k_1^\nu + k_2^\nu)$, and we have:

$$\begin{aligned} \langle \partial^\mu J_\mu^A \rangle_h &\sim \frac{1}{2!} \times 2 \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left(2M\gamma_5 \cdot \frac{k+M}{k^2+M^2} \cdot \varepsilon_1(2k+p_1) \cdot \frac{k+p_1+M}{(k+p_1)^2+M^2} \cdot \varepsilon_2(2k+2p_1+p_2) \cdot \frac{k+p_1+p_2+M}{(k+p_1+p_2)^2+M^2} \right) \\ &\sim \int \frac{d^4k}{(2\pi)^4} 2M^2(4\varepsilon_{\mu\nu\rho\sigma})\varepsilon_1^{\mu\mu'}(2k+p_1)_{\mu'}p_1^\nu\varepsilon_2^{\rho\rho'}(2k+2p_1+p_2)_{\rho'}p_2^\sigma \left(\frac{1}{k^2+M^2} \cdots \right) \\ &\sim 8M^2\varepsilon_{\mu\nu\rho\sigma}p_1^\nu p_2^\sigma \varepsilon_1^{\mu\mu'}\varepsilon_2^{\rho\rho'} \int \frac{d^4k}{(2\pi)^4} \frac{(2k+p_1)_{\mu'}(2k+2p_1+p_2)_{\rho'}}{(k^2+M^2)((k+p_1)^2+M^2)((k+p_1+p_2)^2+M^2)} \\ &\sim 8M^2\varepsilon_{\mu\nu\rho\sigma}p_1^\nu p_2^\sigma \varepsilon_1^{\mu\mu'}\varepsilon_2^{\rho\rho'} \int \frac{d^4k}{(2\pi)^4} \frac{4k_{\mu'}k_{\rho'}+p_{1,\mu'}p_{2,\rho'}}{(k^2+M^2)^3} \end{aligned} \quad (47)$$

There are, in fact, 2 diagrams accounting for this amplitude with $1 \leftrightarrow 2$ symmetry; here we simply take one contribution with an additional factor of 2, and imply $1 \leftrightarrow 2$ symmetrization in the above expressions.

Note that due to the additional $k_{\mu'}, k_{\rho'}$ the integral is no longer finite but logarithmic divergent: $\int^\Lambda d^4k \frac{k^2}{k^6} \sim \ln \Lambda$. More specifically⁸, we have:

$$\begin{aligned} \langle \partial^\mu J_\mu^A \rangle_h &\sim 8M^2\varepsilon_{\mu\nu\rho\sigma}p_1^\nu p_2^\sigma \varepsilon_1^{\mu\mu'}\varepsilon_2^{\rho\rho'} \frac{\text{Vol } S^3}{(2\pi)^4} \int \left(\frac{4k_{\mu'}k_{\rho'}k^3 dk}{(k^2+M^2)^3} + p_{1,\mu'}p_{2,\rho'} \frac{k^3 dk}{(k^2+M^2)^3} \right) \\ &\sim 8M^2\varepsilon_{\mu\nu\rho\sigma}p_1^\nu p_2^\sigma \varepsilon_1^{\mu\mu'}\varepsilon_2^{\rho\rho'} \frac{2\pi^2}{(2\pi)^4} \int \left(\delta_{\mu'\rho'} \frac{k^5 dk}{(k^2+M^2)^3} + p_{1,\mu'}p_{2,\rho'} \frac{k^3 dk}{(k^2+M^2)^3} \right) \\ &\sim 8M^2\varepsilon_{\mu\nu\rho\sigma}p_1^\nu p_2^\sigma \varepsilon_1^{\mu\mu'}\varepsilon_2^{\rho\rho'} \frac{1}{8\pi^2} \left(\delta_{\mu'\rho'} \frac{1}{2} \ln \frac{\Lambda^2}{M^2} + p_{1,\mu'}p_{2,\rho'} \frac{1}{4M^2} \right) \\ &\sim \frac{1}{4\pi^2} \varepsilon_{\mu\nu\rho\sigma}p_1^\nu p_2^\sigma \varepsilon_1^{\mu\mu'}\varepsilon_2^{\rho\rho'} \left(2\delta_{\mu'\rho'} M^2 \ln \frac{\Lambda^2}{M^2} + p_{1,\mu'}p_{2,\rho'} \right) \end{aligned} \quad (48)$$

The second term is very much similar to the axial anomaly result, while the first term diverges.

However, we believe that the divergent term must be canceled by other diagrams; otherwise, it will contribute a $p^\nu p^\sigma \delta_{\mu'\rho'} \varepsilon_1^{\mu\mu'} \varepsilon_2^{\rho\rho'} = p^\nu p^\sigma (\varepsilon_1)^\mu{}_\alpha (\varepsilon_2)^{\rho\alpha} \sim (\partial h)^2$ term in the final result, which is not diff-invariant. The second term, on the other hand, is diff-invariant:

$$R_{\mu\nu\alpha\beta} = p_\beta p_{[\nu} \varepsilon_{\mu]\alpha} - p_\alpha p_{[\nu} \varepsilon_{\mu]\beta}, \quad (49)$$

⁸References:

- David Tong, *Gauge Theory*;
- A. Zee, *QFT in a Nutshell*;
- [arXiv:0802.0634](#);
- Wikipedia: *Common integrals in quantum field theory*.

$$\begin{aligned}
\langle \partial^\mu J_\mu^A \rangle_h &\sim \frac{1}{4\pi^2} \epsilon_{\mu\nu\rho\sigma} (\varepsilon^{\mu\mu'} p_{1,\mu'} p_1^{\nu'}) (\varepsilon^{\rho\rho'} p_{2,\rho'} p_2^\sigma) \\
&\sim \frac{1}{4\pi^2} \epsilon_{\mu\nu\rho\sigma} \frac{1}{4! \times 2 \times 2} \times \frac{1}{2} R_{\mu\nu\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta} \\
&\sim \frac{1}{768\pi^2} \epsilon_{\mu\nu\rho\sigma} R_{\mu\nu\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta}
\end{aligned} \tag{50}$$

(b) The next order contribution would come from the covariant derivative⁹:

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{2} \omega_\mu{}^{ab} \sigma_{ab} \psi \tag{51}$$

Where $\omega_\mu{}^{ab}$ is the spin connections, and $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$; when linearized this contributes to the following interaction vertex:

$$\mathcal{L}' = -\frac{i}{4} h_\lambda{}^\alpha \partial_\mu h_{\nu\alpha} \bar{\psi} \Gamma^{\mu\lambda\nu} \psi, \quad \Gamma^{\mu\lambda\nu} = \gamma^{[\mu} \gamma^\lambda \gamma^{\nu]}, \tag{52}$$

$$\text{Feynman rule:} \quad -\frac{i}{4} \Gamma^{\mu\lambda\nu} (p_1 - p_2)_\mu (\varepsilon_1)_\lambda{}^\alpha (\varepsilon_2)_{\nu\alpha}, \tag{53}$$

We see a $(\varepsilon_1)_\lambda{}^\alpha (\varepsilon_2)_{\nu\alpha}$ factor, much similar to the factor in the divergent term in (a). Note that this vertex already contains 3 γ -matrices; by joining it with the anomalous vertex $\partial_\mu J_A^\mu$, we obtain a simple 1-loop “seagull” diagram (with graviton wings) :



$$\begin{aligned}
\langle \partial^\mu J_\mu^A \rangle'_h &\sim 2 \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left(2M \gamma_5 \cdot \frac{\not{k} + M}{k^2 + M^2} \cdot \left(-\frac{1}{4} \right) \cancel{\varepsilon_1 \varepsilon_2 (p_1 - p_2)} \cdot \frac{\not{k} + \not{p}_1 + \not{p}_2 + M}{(k + p_1 + p_2)^2 + M^2} \right) \\
&\sim - \int \frac{d^4 k}{(2\pi)^4} M^2 (4\epsilon_{\mu\nu\rho\sigma}) \delta_{\mu'\rho'} \varepsilon_1^{\mu\mu'} \varepsilon_2^{\rho\rho'} (p_1 - p_2)^\nu (p_1 + p_2)^\sigma \left(\frac{1}{k^2 + M^2} \cdots \right) \\
&\sim -4M^2 \epsilon_{\mu\nu\rho\sigma} (2p_1^\nu p_2^\sigma) \varepsilon_1^{\mu\mu'} \varepsilon_2^{\rho\rho'} \int \frac{d^4 k}{(2\pi)^4} \frac{\delta_{\mu'\rho'}}{(k^2 + M^2)^2} \\
&\sim -8M^2 \epsilon_{\mu\nu\rho\sigma} p_1^\nu p_2^\sigma \varepsilon_1^{\mu\mu'} \varepsilon_2^{\rho\rho'} \frac{1}{8\pi^2} \left(\delta_{\mu'\rho'} \frac{1}{2} \ln \frac{\Lambda^2}{M^2} \right)
\end{aligned} \tag{54}$$

Compare with the result in (a), and we see that the divergences cancel each other out precisely.

(c) For an anomalous vertex with hypercharge Y , there will be an additional Y factor in the front of $\langle \partial_\mu J_A^\mu \rangle$; summing over a family of matter gives the total anomaly¹⁰:

$$\langle \partial_\mu J_A^\mu \rangle \propto \sum \text{Tr } T_a T_b Y \propto \delta_{ab} \sum Y \tag{55}$$

When the summation goes over all states in a complete generation, we have $\sum Y = 0$, i.e. the anomaly cancels.

⁹Reference: Alvarez-Gaume & Witten, *Gravitational Anomalies*.

¹⁰Reference: Tong, and Wikipedia: *Anomaly (physics) # Anomaly cancellation*.