

## 1 Local Transformation

$$\delta A_\mu^a = \partial_\mu \lambda^a(x) + f_{bc}^a A_\mu^b \lambda^c(x), \quad (1)$$

Here  $f_{abc}$  is the totally anti-symmetric structure constant for a semi-simple Lie algebra  $\mathfrak{g}$ , with generators  $\{T_a\}_a$  and normalized Killing form  $\delta_{ab}$ .

- The field strength is defined as follows:

$$\begin{aligned} F_{\mu\nu} &\equiv F_{\mu\nu}^a T_a = [D_\mu, D_\nu] = [\partial_\mu + A_\mu, \partial_\nu + A_\nu] \\ &= dA + A \wedge A \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + f_{bc}^a A_\mu^b A_\nu^c T_a \end{aligned} \quad (2)$$

Adjoint indices  $a, b, \dots$  are sometimes suppressed by contracting with  $T_a$ 's. By exploiting the anti-symmetric property of  $f_{bc}^a$ , along with the Jacobi identity, we get the infinitesimal transformation:

$$\begin{aligned} \delta F_{\mu\nu}^a &= \partial_\mu \delta A_\nu^a - \partial_\nu \delta A_\mu^a + f_{bc}^a \delta(A_\mu^b A_\nu^c) \\ &= f_{bc}^a \left( \lambda^c (\partial_\mu A_\nu^b - \partial_\nu A_\mu^b) + (A_\nu^b \partial_\mu \lambda^c - A_\mu^b \partial_\nu \lambda^c) + \delta(A_\mu^b A_\nu^c) \right) \\ &= f_{bc}^a \left( \lambda^c (F_{\mu\nu}^b - f_{de}^b A_\mu^d A_\nu^e) + (A_\nu^b (\delta A_\mu^c - f_{de}^c A_\mu^d \lambda^e) - A_\mu^b (\delta A_\nu^c - f_{de}^c A_\nu^d \lambda^e)) + \delta(A_\mu^b A_\nu^c) \right) \\ &= f_{bc}^a \left( \lambda^c (F_{\mu\nu}^b - f_{de}^b A_\mu^d A_\nu^e) - (f_{de}^c A_\nu^b A_\mu^d \lambda^e - f_{de}^c A_\mu^b A_\nu^d \lambda^e) \right) \\ &= f_{bc}^a \left( \lambda^c (F_{\mu\nu}^b - f_{de}^b A_\mu^d A_\nu^e) - (f_{de}^c A_\nu^b A_\mu^d \lambda^e - f_{de}^c A_\mu^b A_\nu^d \lambda^e) \right) \\ &= f_{bc}^a \lambda^c F_{\mu\nu}^b \end{aligned} \quad (3)$$

When contracted with  $T_a$ , this yields:

$$\delta F_{\mu\nu} = \lambda^c F_{\mu\nu}^b f_{bc}^a T_a = \lambda^c F_{\mu\nu}^b [T_b, T_c] = F_{\mu\nu} \cdot \lambda - \lambda \cdot F_{\mu\nu}, \quad (4)$$

$$\lambda = \lambda^c(x) T_c, \quad F_{\mu\nu} = F_{\mu\nu}^b T_b, \quad (5)$$

$$F_{\mu\nu} \longmapsto e^{-\Lambda^a(x) T_a} F_{\mu\nu} e^{\Lambda^a(x) T_a} \quad (6)$$

The exponentiation is valid even for local  $\lambda = \lambda(x)$ , since it is produced by integrating along the fiber direction  $\lambda \rightarrow \Lambda$ , not the spacetime direction  $x$ . This is the finite transformation w.r.t.  $\Lambda(x)$ .

- For any matter field  $\psi$  furnishing a representation of  $\mathfrak{g}$ , we have:

$$T_a \psi = (T_a)^i_j \psi^j, \quad \delta \psi = -\lambda^a(x) T_a \psi, \quad (7)$$

$$\psi \longmapsto e^{-\Lambda^a(x) T_a} \psi, \quad (8)$$

$$D_\mu \psi \longmapsto e^{-\Lambda^a(x) T_a} D_\mu \psi, \quad (9)$$

In fact, (1) is chosen to ensure that  $D_\mu \psi$  transforms gauge covariantly just like  $\psi$ . Therefore,

$$\begin{aligned} D_\mu = \partial_\mu + A_\mu &\longmapsto e^{-\Lambda^a(x) T_a} \circ D_\mu \circ e^{\Lambda^a(x) T_a} \\ &= e^{-\Lambda} \circ (\partial_\mu + A_\mu) \circ e^\Lambda \\ &= e^{-\Lambda} \circ \partial_\mu \circ e^\Lambda + e^{-\Lambda} A_\mu e^\Lambda, \quad \Lambda = \Lambda^a(x) T_a, \end{aligned} \quad (10)$$

$$A_\mu \longmapsto e^{-\Lambda} (\partial_\mu e^\Lambda) + e^{-\Lambda} A_\mu e^\Lambda = T_a \partial_\mu \Lambda^a(x) + e^{-\Lambda} A_\mu e^\Lambda \quad (11)$$

- $F^2 \equiv F \wedge F$ , we have:

$$\begin{aligned} F^2 &= (dA + A \wedge A) \wedge (dA + A \wedge A) \\ &= dA \wedge dA + dA \wedge A \wedge A + A \wedge A \wedge dA + A \wedge A \wedge A \wedge A \end{aligned} \quad (12)$$

The last term is proportional to  $\epsilon_{abcd} T^a T^b T^c T^d$ , hence its trace will vanish; therefore,

$$\begin{aligned} \text{tr } F^2 &= \text{tr} (dA \wedge dA + dA \wedge A \wedge A + A \wedge A \wedge dA) \\ &= \text{tr} \left( d(dA \wedge A) + \frac{2}{3} d(A \wedge A \wedge A) \right) \\ &= d \text{tr} \left( dA \wedge A + \frac{2}{3} A \wedge A \wedge A \right) = d\omega, \end{aligned} \quad (13)$$

$$\omega = \text{tr} \left( dA \wedge A + \frac{2}{3} A \wedge A \wedge A \right) \quad (14)$$

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## 2 Relativistic Particle

$$L = \frac{1}{2e} \left( \frac{1}{c} \frac{dX}{dt} \right)^2 - \frac{e}{2} m^2 c^4 \quad (15)$$

- For  $t \mapsto t' = t - \xi(t)$ , we have  $X'(t') = X(t)$ , therefore:

$$\delta X^\mu = -\delta t \frac{dX^\mu}{dt} = \xi(t) \dot{X}^\mu, \quad (16)$$

Or more explicitly,  $X^\mu(t) \mapsto X^\mu(t) + \xi(t) \dot{X}^\mu$ .

- We have:

$$\begin{aligned} \delta L &= \frac{1}{ec^2} \dot{X}_\mu \delta \dot{X}^\mu - \frac{\delta e}{2} \frac{1}{e^2 c^2} \dot{X}^2 - \frac{\delta e}{2} m^2 c^4 \\ &= \frac{1}{ec^2} \xi \dot{X}_\mu \ddot{X}^\mu + \frac{1}{ec^2} \dot{\xi} \dot{X}^2 - \frac{\delta e}{2} \frac{1}{e^2 c^2} \dot{X}^2 - \frac{\delta e}{2} m^2 c^4 \end{aligned} \quad (17)$$

For  $S = \int dt L$  to be invariant,  $\delta L$  should be reduced to a total derivative, which can then be reduced to some vanishing boundary terms.

Consider  $\delta e = \frac{d}{dt}(e\xi) = \dot{e}\xi + e\dot{\xi}$ , and we have:

$$\begin{aligned} \delta L &= \frac{1}{ec^2} \xi \dot{X}_\mu \ddot{X}^\mu + \frac{1}{2ec^2} \dot{\xi} \dot{X}^2 - \frac{\dot{e}}{2e^2 c^2} \xi \dot{X}^2 - \frac{d}{dt} \left( \frac{1}{2} e \xi m^2 c^4 \right) \\ &= \frac{d}{dt} \left\{ \left( \frac{1}{2ec^2} \dot{X}^2 - \frac{e}{2} m^2 c^4 \right) \xi \right\} = \frac{d}{dt} (\xi L) \end{aligned} \quad (18)$$

Indeed we get a total derivative; therefore,

$$\delta e = \frac{d}{dt}(e\xi), \quad \delta S = \int \delta L = \int d(\xi L) = 0 \quad (19)$$

•  $e(t)$  can be seen as a gauge field coupled to  $X$ , which captures the  $t$ -reparametrization redundancy through the gauge transformation parameter  $\xi(t)$ . A natural gauge choice is fixing  $f = e(t) - 1 \equiv 0$ , which is equivalent to setting  $t = \tau$ : the proper time, or affine parametrization for the massless case.

The gauge invariant path integral is constructed as follows:

$$\begin{aligned} \mathcal{Z} &= \frac{1}{\int \mathcal{D}\xi} \int \mathcal{D}X \mathcal{D}e e^{iS} \\ &= \frac{1}{\int \mathcal{D}\xi} \int \mathcal{D}X \mathcal{D}e e^{iS} \int \mathcal{D}f \delta[f] \\ &= \frac{1}{\int \mathcal{D}\xi} \int \mathcal{D}X \mathcal{D}e e^{iS} \int \mathcal{D}\xi \delta[f_\xi] \det \frac{\delta f_\xi}{\delta \xi} \\ &= \frac{1}{\int \mathcal{D}\xi} \int \mathcal{D}\xi \int \mathcal{D}X \mathcal{D}e e^{iS} \delta[f_\xi] \det \frac{\delta f_\xi}{\delta \xi} \\ &= \frac{1}{\int \mathcal{D}\xi} \int \mathcal{D}\xi \int \mathcal{D}X_\xi \mathcal{D}e_\xi e^{iS_\xi} \delta[f_\xi] \det \frac{\delta f_\xi}{\delta \xi} \\ &= \frac{1}{\int \mathcal{D}\xi} \int \mathcal{D}\xi \int \mathcal{D}X_\xi \mathcal{D}e_\xi e^{iS_\xi} \delta[f_\xi] \det \frac{\delta f_\xi}{\delta \xi} \\ &= \frac{1}{\int \mathcal{D}\xi} \int \mathcal{D}\xi \int \mathcal{D}X \mathcal{D}e e^{iS} \delta[f] \det \frac{\delta f_\xi}{\delta \xi} \Big|_{\xi=0} \\ &= \int \mathcal{D}X \mathcal{D}e e^{iS} \delta[f] \det \frac{\delta f_\xi}{\delta \xi} \Big|_{\xi=0} \end{aligned} \quad (20)$$

Here the gauge-transformed quantities are marked with a  $\xi$  subscript. Note that in the final expression,  $f$  is *not* integrated out and can be *any* possible gauge-fixing function, i.e.  $\left(\delta[f] \det \frac{\delta f_\xi}{\delta \xi}\right)_{\xi=0}$  is in fact  $f$ -independent.

The gauge-fixing term  $\delta[f]$  can be replaced by a Gaussian packet with width parameter  $\zeta$ . More rigorously, up to an overall constant coefficient, we have the following equivalence:

$$\delta[f] \sim \delta[f - f_0] \sim \int \mathcal{D}f_0 \exp\left(-\frac{i}{2\zeta} \int dt f_0^2\right) \delta[f - f_0] = \exp\left(i \int dt L_{gf}\right), \quad (21)$$

$$L_{gf} = -\frac{1}{2\zeta} f^2 = -\frac{1}{2\zeta} (e - 1)^2, \quad (22)$$

Here  $f_0$  is some gauge-invariant shift of  $f$ , namely  $(f_0)_\xi = f_0$ .  $f_0$  can be seen as a non-dynamical auxiliary field that enforce the gauge fixing, much similar to a Lagrange multiplier. On the other hand, the determinant can be evaluated using Faddeev–Popov (FP) ghosts  $b, c$ :

$$\det P \sim \int \mathcal{D}b \mathcal{D}c \exp\left(i \int dt \int dt' b(t) \cdot P(t, t') \cdot c(t')\right), \quad (23)$$

$$\left. \frac{\delta f_\xi(t)}{\delta \xi(t')} \right|_{\xi=0} = \left. \frac{\delta}{\delta \xi(t')} \right|_{\xi=0} \left( e + \frac{d}{dt}(e\xi) - 1 \right)_{(t)} = \frac{d}{dt}(e(t) \delta(t - t')), \quad (24)$$

$$\begin{aligned} \det \frac{\delta f_\xi(t)}{\delta \xi(t')} \Big|_{\xi=0} &\sim \int \mathcal{D}b \mathcal{D}c \exp \left( i \int dt \int dt' b(t) \left( e(t) \delta(t-t') \right) c(t') \right) \\ &\sim \int \mathcal{D}b \mathcal{D}c \exp \left( -i \int dt e \dot{b} c \right), \end{aligned} \quad (25)$$

$$L_{gh} = -e \dot{b} c \quad (26)$$

In summary, we have:

$$\mathcal{Z} = \int \mathcal{D}X \mathcal{D}e \mathcal{D}b \mathcal{D}c e^{iS_q}, \quad S_q = \int dt L_q, \quad (27)$$

$$L_q = L + L_{gf} + L_{gh} = L - \frac{1}{2\zeta} (e-1)^2 - e \dot{b} c, \quad (28)$$

$S_q$  is the quantum action under the gauge-fixing condition  $f = e(t) - 1 = 0$ .  $\blacksquare$

### 3 2D $\sigma$ -Model

$$\mathcal{L} = -\frac{1}{2} \partial_\alpha X^\mu \partial_\beta X_\mu \sqrt{-h} h^{\alpha\beta}, \quad X: \Sigma^{1,1} \rightarrow \mathbb{R}^{D-1,1} \quad (29)$$

- The action is diff-invariant; under  $\sigma^\alpha \mapsto \sigma^\alpha + \xi^\alpha$ , we have:

$$\delta X^\alpha = \mathcal{L}_\xi X^\alpha, \quad \delta h^{\alpha\beta} = \mathcal{L}_\xi h^{\alpha\beta} \quad (30)$$

$\mathcal{L}_\xi$  is the Lie derivative along  $\xi^\alpha$ . Note that  $0 = \delta(h_{\alpha\beta} h^{\beta\gamma})$ , hence we have:

$$\delta h_{\alpha\beta} = -h_{\alpha\alpha'} h_{\beta\beta'} \delta h^{\alpha'\beta'} = \xi^\gamma \partial_\gamma h_{\alpha\beta} + (\partial_\alpha \xi^\gamma) h_{\gamma\beta} + (\partial_\beta \xi^\gamma) h_{\alpha\gamma} = \mathcal{L}_\xi h_{\alpha\beta}, \quad (31)$$

$$\delta \sqrt{-h} = \frac{1}{2} \sqrt{-h} h^{\alpha\beta} \delta h_{\alpha\beta}, \quad (32)$$

Furthermore, we have  $\mathcal{L}_\xi dX = d(\mathcal{L}_\xi X)$ , i.e.  $\partial_\alpha \delta X = \partial_\alpha \mathcal{L}_\xi X = \partial_\alpha (\xi^\gamma \partial_\gamma X) = \mathcal{L}_\xi (\partial_\alpha X)$ . Note that due to the  $\sqrt{-h}$  factor,  $\mathcal{L}$  is not a scalar but a *scalar density*. For convenience, define  $\tilde{\mathcal{L}} = \mathcal{L} \sqrt{-h}$ , then  $\tilde{\mathcal{L}} = -\frac{1}{2} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu$  is a scalar; using chain rule, we obtain:

$$\begin{aligned} \delta \mathcal{L} &= \sqrt{-h} \delta \tilde{\mathcal{L}} + \tilde{\mathcal{L}} \delta \sqrt{-h} \\ &= \sqrt{-h} \mathcal{L}_\xi \tilde{\mathcal{L}} + \tilde{\mathcal{L}} \delta \sqrt{-h} \\ &= \sqrt{-h} \xi^\gamma \partial_\gamma \tilde{\mathcal{L}} + \tilde{\mathcal{L}} \delta \sqrt{-h} \\ &= \partial_\gamma (\xi^\gamma \tilde{\mathcal{L}} \sqrt{-h}) - \tilde{\mathcal{L}} \left( \sqrt{-h} (\partial_\gamma \xi^\gamma) + \xi^\gamma (\partial_\gamma \sqrt{-h}) \right) + \tilde{\mathcal{L}} \delta \sqrt{-h} \\ &= \partial_\gamma (\xi^\gamma \mathcal{L}) - \tilde{\mathcal{L}} \left( \sqrt{-h} (\partial_\gamma \xi^\gamma) + \xi^\gamma (\partial_\gamma \sqrt{-h}) - \delta \sqrt{-h} \right) \\ &= \partial_\gamma (\xi^\gamma \mathcal{L}) - \tilde{\mathcal{L}} \sqrt{-h} \left( \partial_\gamma \xi^\gamma + \frac{1}{2} \xi^\gamma h^{\alpha\beta} \partial_\gamma h_{\alpha\beta} - \frac{1}{2} h^{\alpha\beta} \delta h_{\alpha\beta} \right) \\ &= \partial_\gamma (\xi^\gamma \mathcal{L}) - \tilde{\mathcal{L}} \sqrt{-h} \left( \partial_\gamma \xi^\gamma - \frac{1}{2} h^{\alpha\beta} \left( (\partial_\alpha \xi^\gamma) h_{\gamma\beta} + (\partial_\beta \xi^\gamma) h_{\alpha\gamma} \right) \right) \\ &= \partial_\gamma (\xi^\gamma \mathcal{L}) - \tilde{\mathcal{L}} \sqrt{-h} \left( \partial_\gamma \xi^\gamma - \partial_\gamma \xi^\gamma \right) \\ &= \partial_\gamma (\xi^\gamma \mathcal{L}) \end{aligned} \quad (33)$$

We see that  $\delta \mathcal{L}$  is a total derivative, hence  $\delta S = \int d^2\sigma \delta \mathcal{L} = 0$ , i.e. the action is diff-invariant.

- The action is Weyl invariant; with  $\delta h^{\alpha\beta} = -\lambda(\sigma) h^{\alpha\beta}$ , we have:

$$\begin{aligned}
\delta(\sqrt{-h} h^{\alpha\beta}) &= \sqrt{-h} \delta h^{\alpha\beta} + h^{\alpha\beta} \delta\sqrt{-h} \\
&= \sqrt{-h} h^{\alpha\beta} \left( -\lambda - \frac{1}{2} h_{\alpha'\beta'} \delta h^{\alpha'\beta'} \right) \\
&= \sqrt{-h} h^{\alpha\beta} \left( -\lambda + \frac{1}{2} \lambda h_{\alpha'\beta'} h^{\alpha'\beta'} \right) \\
&= \sqrt{-h} h^{\alpha\beta} \left( -\lambda + \frac{2}{2} \lambda \right) \\
&= 0
\end{aligned} \tag{34}$$

Here we've used the fact that  $h_{\alpha\beta} h^{\alpha\beta} = \delta_\alpha^\alpha = 2$ . Therefore,  $\delta \mathcal{L} = -\frac{1}{2} \partial_\alpha X^\mu \partial_\beta X_\mu \delta(\sqrt{-h} h^{\alpha\beta}) = 0$ , i.e. the action is Weyl invariant.

- FP quantization of this system follows the same recipe as the point particle case above:

$$\mathcal{Z} = \int \mathcal{D}X \mathcal{D}h \mathcal{D}b \mathcal{D}c e^{iS_q}, \quad S_q = \int d^2\sigma \mathcal{L}_q, \tag{35}$$

$$\mathcal{L}_q = \mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{gh} \tag{36}$$

Given gauge fixing:  $f^{\alpha\beta} = h^{\alpha\beta} - h_{(0)}^{\alpha\beta}$ , we have:

$$\mathcal{L}_{gf} = -\frac{1}{2\zeta} f^{\alpha\beta} f_{\alpha\beta} \sqrt{-h} = -\frac{1}{2\zeta} \left( h^{\alpha\beta} - h_{(0)}^{\alpha\beta} \right) \left( h_{\alpha\beta} - h_{\alpha\beta}^{(0)} \right) \sqrt{-h} \tag{37}$$

The FP ghost term  $\mathcal{L}_{gh}$  is given by functional determinant; we have:

$$\left. \frac{\delta f_\xi^{\alpha\beta}(\sigma)}{\delta \xi^\gamma(\sigma')} \right|_0 = \left. \frac{\delta}{\delta \xi^\gamma(\sigma')} \right|_0 (\mathcal{L}_\xi h^{\alpha\beta} - \lambda h^{\alpha\beta})_{(\sigma)} \tag{38}$$

$$= \delta(\sigma - \sigma') \partial_\gamma h^{\alpha\beta} - \delta_\gamma^\alpha \partial^\beta \delta(\sigma - \sigma') - \delta_\gamma^\beta \partial^\alpha \delta(\sigma - \sigma') \tag{39}$$

$$= -\delta_\gamma^\alpha \nabla^\beta \delta(\sigma - \sigma') - \delta_\gamma^\beta \nabla^\alpha \delta(\sigma - \sigma'), \tag{40}$$

$$\left. \frac{\delta f_\xi^{\alpha\beta}(\sigma)}{\delta \lambda(\sigma')} \right|_0 = -\delta(\sigma - \sigma') h^{\alpha\beta}, \tag{41}$$

Here we've replaced  $\partial$  with  $\nabla$  which commutes with the metric  $h_{\alpha\beta}$ . Define  $\Xi^\Gamma = (\xi^\gamma, \lambda)$  to combine all gauge parameters, and use fermionic FP ghosts:  $b_{\alpha\beta}$ ,  $c^\Gamma = (c^\gamma, c')$  to contract the indices; after some integration by parts, we have:

$$\det \left. \frac{\delta f_\xi^{\alpha\beta}(\sigma)}{\delta \Xi^\Gamma(\sigma')} \right|_0 \sim \int \mathcal{D}b_{\alpha\beta} \mathcal{D}c^\gamma \mathcal{D}c' \exp \left( i \int d^2\sigma \sqrt{-h} b_{\alpha\beta} \left( -\nabla^\beta c^\alpha - \nabla^\alpha c^\beta - h^{\alpha\beta} c' \right) \right) \tag{42}$$

To simplify the action, it is common<sup>1</sup> to integrate  $c'$  out, which constrains  $b_{\alpha\beta}$  to be symmetric traceless:  $b_{\alpha\beta} h^{\alpha\beta} = b_\alpha^\alpha = 0$ . The resulting  $b_{\alpha\beta}$  has 2 degrees of freedom, same as  $c^\gamma$ .

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<sup>1</sup>References: Tong: <http://damtp.cam.ac.uk/user/tong/string.html>, and also Polchinski.

In the end, we have:

$$\det \frac{\delta f_{\xi}^{\alpha\beta}(\sigma)}{\delta \Xi^{\Gamma}(\sigma')} \Big|_0 \sim \int \mathcal{D}b_{\alpha\beta} \mathcal{D}c^{\gamma} \exp \left( -2i \int d^2\sigma \sqrt{-h} b_{\alpha\beta} \nabla^{\alpha} c^{\beta} \right) \quad (43)$$

Therefore, FP quantization with  $f^{\alpha\beta} = h^{\alpha\beta} - h_{(0)}^{\alpha\beta}$  yields:

$$\mathcal{Z} = \int \mathcal{D}X \mathcal{D}h^{\alpha\beta} \mathcal{D}b_{\alpha\beta} \mathcal{D}c^{\gamma} e^{iS_q}, \quad S_q = \int d^2\sigma \mathcal{L}_q, \quad \mathcal{L}_q = \mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{gh}, \quad (44)$$

$$\mathcal{L}_{gf} = -\frac{1}{2\zeta} f^{\alpha\beta} f_{\alpha\beta} \sqrt{-h} = -\frac{1}{2\zeta} \left( h^{\alpha\beta} - h_{(0)}^{\alpha\beta} \right) \left( h_{\alpha\beta} - h_{\alpha\beta}^{(0)} \right) \sqrt{-h}, \quad (45)$$

$$\mathcal{L}_{gh} = -2b_{\alpha\beta} \nabla^{\alpha} c^{\beta} \sqrt{-h} \quad (46)$$

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