Advanced QFT №1

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1 Symmetry & Noether's Theorem

1.1 2D σ-Model

$$\mathcal{L} = -\frac{1}{2}\,\eta_{\alpha\beta}\,\eta_{\mu\nu}\,\partial^\alpha X^\mu\,\partial^\beta X^\nu = -\frac{1}{2}\,\partial^\alpha X_\mu\,\partial_\alpha X^\mu, \quad X^\mu \in \mathbb{R}^{1,D-1} \tag{1}$$

- For $\delta X^\mu = a^\mu + \lambda^\mu{}_\nu X^\nu$, the Lagrangian (density) transforms as follows:

$$\begin{aligned}
\delta\mathcal{L} &= -\partial^\alpha X_\mu\,\partial_\alpha\,\delta X^\mu \\
&= -\partial^\alpha X_\mu\,\partial_\alpha(a^\mu + \lambda^\mu{}_\nu X^\nu) \\
&= -\partial^\alpha X_\mu\,(\partial_\alpha a^\mu + X^\nu\,\partial_\alpha\lambda^\mu{}_\nu + \lambda^\mu{}_\nu\,\partial_\alpha X^\nu) \\
&= -\partial^\alpha X_\mu\,\partial_\alpha a^\mu - \partial^\alpha X^\mu\,\partial_\alpha X^\nu\,\lambda_{\mu\nu} - X^\nu\,\partial^\alpha X^\mu\,\partial_\alpha\lambda_{\mu\nu} \\
&= -\partial^\alpha X_\mu\,\partial_\alpha a^\mu - \partial^\alpha X^\mu\,\partial_\alpha X^\nu\,\lambda_{(\mu\nu)} - X^\nu\,\partial^\alpha X^\mu\,\partial_\alpha\lambda_{\mu\nu}
\end{aligned} \tag{2}$$

Since a^μ and $\lambda^\mu{}_\nu$ are independent, imposing $\delta L = 0$ yields $\partial_\alpha a^\mu = 0$, $a = \text{const}$. Furthermore, if $\delta L = 0$ is to hold for arbitrary X^μ fields, then $\partial_\alpha\lambda_{\mu\nu} = 0$, $\lambda_{(\mu\nu)} = 0$, i.e. $\lambda_{\mu\nu}$ is constant and anti-symmetric over its indices.

- Promote $\delta X \mapsto \epsilon(x)\,\delta X = \epsilon(x)\,(a^\mu + \lambda^\mu{}_\nu X^\nu)$, with $\epsilon(x)$ some localized bump function; using (2) and considering *on-shell* variation, we have:

$$\begin{aligned}
0 = \delta S &= -\int \mathrm{d}^2x\,(\partial^\alpha X_\mu\,a^\mu\,\partial_\alpha\epsilon + X^\nu\,\partial^\alpha X^\mu\,\lambda_{\mu\nu}\,\partial_\alpha\epsilon) \\
&= -\int \mathrm{d}^2x\,\Big(\partial^\alpha X_\mu\,a^\mu + X_{[\nu}\,\partial^\alpha X_{\mu]}\,\lambda^{[\mu\nu]}\Big)\,\partial_\alpha\epsilon
\end{aligned} \tag{3}$$

It is evident (after partial integration) that the following currents are conserved; they are the Noether currents associated with a^μ and $\lambda^{[\mu\nu]}$:

$$j^\alpha_\mu = -\partial^\alpha X_\mu, \quad j^\alpha_{\mu\nu} = -X_{[\nu}\,\partial^\alpha X_{\mu]} = \frac{1}{2}\,(X_\mu\,\partial^\alpha X_\nu - X_\nu\,\partial^\alpha X_\mu) \tag{4}$$

Conserved charge $Q = \int \mathrm{d}^2x\,j^0(x)$, we have:

$$P_\mu = -\int \mathrm{d}x^1\,\partial^0 X_\mu = \int \mathrm{d}x^1\,\partial_0 X_\mu, \quad M_{\mu\nu} = \frac{1}{2}\int \mathrm{d}x^1\,(X_\nu\,\partial_0 X_\mu - X_\mu\,\partial_0 X_\nu) \tag{5}$$

They can be interpreted as spacetime momentum and spacetime angular momentum. ■

1.2 Real Scalar in $(3+1)$ D

$$\mathcal{L} = -\frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}\,m^2\phi^2 \tag{6}$$

- For ϕ: scalar, under $x' = \lambda \circ x$, $\phi(x) \mapsto \phi'(x)$, while:

$$\phi'(x') = \phi(x) \implies \phi'(x) = \phi(\lambda^{-1} \circ x) \tag{7}$$

For $\lambda \sim \lambda^{\mu}{}_{\nu}$: Lorentz transformation, $\eta_{\mu\nu}\lambda^{\mu}{}_{\rho}\lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$, or equivalently, $(\lambda^{-1})^{\mu}{}_{\nu} = \lambda_{\nu}{}^{\mu}$. Therefore,

$$\phi'(x^{\mu}) = \phi(\lambda^{-1} \circ x^{\mu}) = \phi(x^{\nu}\lambda_{\nu}{}^{\mu}) \tag{8}$$

- Under $x'^{\mu} = \lambda^{\mu}{}_{\nu}x^{\nu}$, we have:

$$\begin{aligned}
\mathcal{L}'(x') &= -\frac{1}{2}\,\partial'^{\mu}\phi'(x')\,\partial'_{\mu}\phi'(x') - \frac{1}{2}\,m^2\phi'^2(x') \\
&= -\frac{1}{2}\,\partial'^{\mu}\phi(x)\,\partial'_{\mu}\phi(x) - \frac{1}{2}\,m^2\phi^2(x) \\
&= -\frac{1}{2}\,\eta^{\mu\nu}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\,\partial_{\rho}\phi(x)\,\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\,\partial_{\sigma}\phi(x) - \frac{1}{2}\,m^2\phi^2(x) \\
&= -\frac{1}{2}\,\eta^{\rho\sigma}\partial_{\rho}\phi(x)\,\partial_{\sigma}\phi(x) - \frac{1}{2}\,m^2\phi^2(x) \\
&= \mathcal{L}(x)
\end{aligned} \tag{9}$$

Here we've used $\eta^{\mu\nu}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} = \eta^{\mu\nu}\lambda_{\mu}{}^{\rho}\lambda_{\nu}{}^{\sigma} = \eta^{\rho\sigma}$. Furthermore, $S' = \int \mathrm{d}^4x\,\mathcal{L}'(x) = \int \mathrm{d}^4x'\,\mathcal{L}'(x') = \int \mathrm{d}^4x'\,\mathcal{L}(x) = \int \mathrm{d}^4x\,\mathcal{L}(x) = S$, hence the action is invariant under Lorentz transformation.

- Consider an infinitesimal Lorentz transformation: $\lambda \sim \mathbb{1} + \omega$, then $\eta_{\mu\nu}\lambda^{\mu}{}_{\rho}\lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$ implies that $\omega_{\mu\nu}$ is anti-symmetric: $\omega_{\mu\nu} + \omega_{\nu\nu} = 0$. For $\delta x^{\mu} = \omega^{\mu}{}_{\nu}x^{\nu}$, we have:

$$\delta\phi = -\frac{\partial\phi}{\partial x^{\mu}}\,\delta x^{\mu} = -\omega^{\mu}{}_{\nu}x^{\nu}\,\partial_{\mu}\phi \tag{10}$$

To obtain the corresponding Noether charges, we can simply repeat the operations done in our previous problem; alternatively, we can try to derive a general recipe[1]: for $\mathcal{L} = \mathcal{L}(\phi, \partial_{\mu}\phi)$ and $S = \int \mathrm{d}^4x\,\mathcal{L}$, we have:

$$\begin{aligned}
\delta S &= \int \mathrm{d}^4x\,\delta\mathcal{L} \\
&= \int \mathrm{d}^4x \left(\frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\,\delta\partial_{\mu}\phi\right) \\
&= \int \mathrm{d}^4x \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\right)\delta\phi + \int \mathrm{d}^4x\,\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\,\delta\phi\right)
\end{aligned} \tag{11}$$

If we vary S w.r.t. a symmetry of the system, we will have $\delta\mathcal{L} = \partial_{\mu}K^{\mu}$ some total derivative; when on-shell, such variation gives the conserved current with boundary term K^{μ}:

$$j^{\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\,\delta\phi - K^{\mu} \tag{12}$$

Back to our Lorentz transformation $\delta\phi = -\omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\phi$, we have symmetry variation:

$$\delta\mathcal{L} = -\omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\mathcal{L} = -\partial_{\mu}(\omega^{\mu}{}_{\nu}x^{\nu}\mathcal{L}) \tag{13}$$

We can write this down without explicit calculations, since we know $\mathcal{L}$ itself is a Lorentz scalar, and that's how scalar transforms under Lorentz transformations.

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[1] References: arXiv:1601.03616 and *Tong:* http://damtp.cam.ac.uk/user/tong/qft.html

This gives a boundary term $K^\mu = -\omega^\mu{}_\nu x^\nu \mathcal{L}$, and the Noether current and its corresponding conserved charge can be calculated as follows:

$$j^\mu = -\omega^\sigma{}_\nu x^\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\, \partial_\sigma \phi - \delta^\mu_\sigma \mathcal{L} \right), \tag{14}$$

$$Q = \int \mathrm{d}^3x\, j^0 = -\omega^\sigma{}_\nu \int \mathrm{d}^3x\, x^\nu \left(\partial_0 \phi\, \partial_\sigma \phi - \delta^0_\sigma \mathcal{L} \right), \tag{15}$$

Note that $\omega^\mu{}_\nu$ is arbitrary, therefore Q can be decomposed into independent charges:

$$Q = \frac{1}{2}\,\omega_{\mu\nu} M^{\mu\nu}, \quad M^{\mu\nu} = -\int \mathrm{d}^3x\, 2x^{[[\mu} \left(\partial_0 \phi\, \partial^{\nu]} \phi - \eta^{\nu]0} \mathcal{L} \right), \tag{16}$$

The indices of $M^{\mu\nu}$ are anti-symmetrized to match the degrees of freedom in $\omega_{\mu\nu}$. Note that the $\mathcal{L}$ term only appears when one of the indices is 0.

Note that the canonical momentum:

$$\varpi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} = \partial_0 \phi \tag{17}$$

It is thus natural to re-organize M^{0i} in the following way:

$$\begin{aligned} M^{0i} = -M^{i0} &= -\int \mathrm{d}^3x \left(\varpi \left(x^0 \partial^i - x^i \partial^0 \right) \phi - x^i \mathcal{L} \right) \\ &= -\int \mathrm{d}^3x \left(x^0 \varpi\, \partial^i \phi + x^i \left(\varpi \dot{\phi} - \mathcal{L} \right) \right) \\ &= -\int \mathrm{d}^3x \left(x^0 \varpi\, \partial^i \phi + x^i \mathcal{H} \right) \end{aligned} \tag{18}$$

We've obtained an interesting result: the expression for the boost generator M^{i0} contains the Hamiltonian density $\mathcal{H}$, weighted by the radial distance x^i. This is natural since a boost does indeed contains time evolution for states away from the origin. It's an important result utilized by the so-called *Rindler decomposition*; in fact, M^{i0} becomes the Hamiltonian for an accelerated observer in the Rindler patch[2].

For M^{ij}, we have:

$$M^{ij} = -\int \mathrm{d}^3x\, \dot{\phi} \left(x^i \partial^j - x^j \partial^i \right) \phi \tag{19}$$

This is interpreted as the angular momentum of the field ϕ. Suppose ϕ is a wave packet localized around $\mathbf{x}$ with momentum $\approx \mathbf{p}$, then we have the classical angular momentum up to some factor:

$$M^{ij} \sim \left(x^i p^j - x^j p^i \right) \int \mathrm{d}^3x\, E \phi^2, \quad E = \sqrt{\mathbf{p}^2 + m^2} \tag{20}$$

The $\int \mathrm{d}^3x\, E\phi^2$ factor in the above expression is an $\mathcal{O}(1)$ normalization constant for a particle-like wave packet; to see this, note that $\phi \in \mathbb{R}$ has a phase factor $\phi \sim a\, e^{+ik\cdot x} + a^\dagger e^{-ik\cdot x} \sim \cos(k \cdot x)$,

$$E = \int \mathrm{d}^3x\, \mathcal{H} = \int \mathrm{d}^3x \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 + \cdots \right) \sim \left(E^2 + \mathbf{p}^2 + m^2 + \cdots \right) \int \mathrm{d}^3x\, \frac{1}{2}\,\phi^2, \tag{21}$$

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[2]See the lecture notes of Tom Hartman: `hartmanhep.net/topics2015/gravity-lectures.pdf` or Daniel Harlow `arXiv:1409.1231`.

$$E\int \mathrm{d}^3x\,\phi^2 \sim 1, \tag{22}$$

Indeed, we have: $M^{ij} \sim \left(x^i p^j - x^j p^i\right)$.

Canonical quantization:

$$[\phi(\mathbf{x}), \varpi(\mathbf{y})] = [\phi(\mathbf{x}), \dot{\phi}(\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}) \tag{23}$$

Other equal-time commutators between ϕ, ϖ all just vanish. Operator products are then regularized by normal ordering: $M \mapsto \,:M:$, which can be explicitly implemented by normal ordering of the oscilator modes:

$$\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\left(a_{\mathbf{k}}e^{ik\cdot x} + a^\dagger_{\mathbf{k}'}e^{-ik\cdot x}\right), \quad [a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}') \tag{24}$$

For example, the first term in $M^{0i} = -M^{i0}$ can be expanded as:

$$-x^0\int \mathrm{d}^3x\,\varpi\,\partial^i\phi = x^0\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,k^i a^\dagger_k a_k = x^0 P^i \tag{25}$$

Here P^μ is the momentum operator on the Hilbert space, promoted from the classical $-i\partial^\mu$. M^{0i} is thus further reduced to:

$$M^{0i} = -M^{i0} = x^0P^i - \int \mathrm{d}^3x\,x^i\mathcal{H} \tag{26}$$

We note that this result is almost the classical $x^0P^i - x^iP^0$, but here x^iP^0 is replaced with the integral over energy density $\mathcal{H}$. The result can be nicely re-written with the stress tensor $T^{\mu\nu}$; just run the Noether's procedure with $\delta x^\mu = \epsilon^\mu$, then we shall obtain:

$$\begin{gathered} j'^\mu = \epsilon_\nu T^{\mu\nu}, \quad T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi + \eta^{\mu\nu}\mathcal{L}, \\ Q' = \epsilon_\mu P^\mu, \quad P^\mu = \int \mathrm{d}^3x\,T^{\mu 0} = \int \mathrm{d}^3x\left(\partial^0\phi\,\partial^\mu\phi + \eta^{0\mu}\mathcal{L}\right) \\ M^{\mu\nu} = \int \mathrm{d}^3x\,2x^{[\mu}T^{\nu]0} \end{gathered} \tag{27}$$

The quantization $M \mapsto \,:M:$ is thus reduced to the quantization of $T_{0\nu}$.

TODO: Detailed analysis! HINT: Ward identity!

Notice that $x^{[\mu}\partial^{\nu]} = \frac{1}{2}\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right) = \frac{1}{2}D^{\mu\nu}$ is the Killing vector fields of $\mathbb{R}^{3,1}$, hence they naturally follow the commutation relations of $\mathfrak{so}(3,1)$ (up to a constant coefficient, or an isomorphism)[3]. We have:

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= \int \mathrm{d}^3x\int \mathrm{d}^3y\left[\dot{\phi}D^{\mu\nu}\phi(x), \dot{\phi}D^{\rho\sigma}\phi(y)\right] \\ &= \int \mathrm{d}^3x\,\dot{\phi}\left[D^{\mu\nu}, D^{\rho\sigma}\right]\phi \end{aligned} \tag{28}$$

Similar holds for M^{i0}. Therefore, $M^{\mu\nu}$'s indeed form the Lie algebra $\mathfrak{so}(3,1)$. ■

[3] I would like to thank 林般 for pointing this out.