2D Yang–Mills & Cohomological Field Theory

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Contents

Main References

This review is almost entirely based on the following references:

[1] *Cordes, Moore, Ramgoolam*, arXiv:hep-th/9411210

[2] Witten, *On Quantum Gauge Theories in Two Dimensions*, 1991

It is basically a miniaturized, reorganized, pedagocial subset of [1], focusing on its section 3, with brief summary of its section 9, 10, 14 and 15. Many basic facts in this review are therefore uncited to avoid repeated citations of [1]. All uncited claims, unless otherwise specified, can be traced back to [1]. However, it is highly possible that I, in my infinite stupidity, misunderstood some ideas from [1]; so please feel free to point out my mistakes.

1 Introduction

We start by writing down the usual Yang–Mills action in 2D (YM_2), in Euclidean signature:

$$I_{\mathrm{YM}_2} = +\frac{1}{4e^2}\int_{\Sigma_T} \mathrm{d}^2x\,\sqrt{G}\,\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}), \qquad \sqrt{G} = \sqrt{\det G_{\mu\nu}} \tag{1}$$

Here we will try follow the convention of [1], despite the fact that it is, unfortunately, not quite self-consistent. Σ_T stands for the 2D *target*; in the large N limit, it is possible to realize YM_2 as a string theory with worldsheet Σ_W, as is proposed by D. Gross and W. Taylor, among others [3–5].

Note that in 2D, the $\mathfrak{g}$-valued curvature form $F = F^a_{\mu\nu}\,T_a\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$ is a *top form*; here T_a is the generator of Lie algebra $\mathfrak{g} = \mathrm{Lie}\,G$, and G is the compact gauge group, e.g. $G = \mathrm{SU}(N)$. This means that in 2D, we have:

$$F = f\mu, \quad f = \star F, \tag{2}$$

$$\mu = \sqrt{G}\,\mathrm{d}^2x\,, \quad F^a_{\mu\nu} = \sqrt{G}\,\epsilon_{\mu\nu} f^a \tag{3}$$

Here μ is the volume form on Σ_T, and f is some $\mathfrak{g}$-valued 0-form. The original YM_2 action can thus be rewritten as:

$$I_{\mathrm{YM}_2} = \frac{1}{2e^2}\int_{\Sigma_T} \mathrm{d}^2x\,\sqrt{G}\,\operatorname{Tr}(f^2) = \frac{1}{2e^2}\int_{\Sigma_T} \mu \operatorname{Tr}(f^2) \tag{4}$$

First we would like to examine the $e^2 \to 0$ limit of this theory. This can be achieved by a *Hubbard–Stratonovich transformation*[1]; namely, we introduce an additional $\mathfrak{g}$-valued field ϕ that serves as a Lagrangian multiplier; consider:

$$I[\phi, A] = \int \left(i \operatorname{Tr}(\phi F) + \frac{1}{2}\, e^2 \mu \operatorname{Tr}(\phi^2) \right) \tag{5}$$

Using the functional version of the integral identity: $\int \mathrm{d}x\, e^{-\frac{e^2}{2}x^2 - ixy} \sim e^{-y^2/(2e^2)}$, it is straightforward to verify that [2], up to an overall coefficient,

$$\int \mathscr{D}\phi\, e^{-I[\phi,A]} \sim e^{-I_{\mathrm{YM}_2}[A]} \tag{6}$$

The advantage of this formulation is that the $e^2 \to 0$ limit becomes non-singular; in fact, now we can simply set $e^2 = 0$, and get:

$$I[\phi, A] \longrightarrow I_0[\phi, A] = \int i \operatorname{Tr}(\phi F) \tag{7}$$

This action is in fact *topological*; there is no explicit metric dependence in the action. The measure μ comes with e^2 in the $I[\phi, A]$ action; setting $e^2 \to 0$ eliminates the metric dependence. Integrating out ϕ fixes $F = 0$, i.e. we need only sum over the moduli of *flat connections*. For a principal G bundle $P \to \Sigma_T$, this is given by:

$$\mathcal{M}_0 = \mathcal{M}(F = 0, P \to \Sigma_T) = \Big\{ A \in \mathcal{A}(P) \,\Big|\, F(A) = 0 \Big\} \Big/ \mathcal{G}(P) \ \subset\ \mathcal{A}(P) \Big/ \mathcal{G}(P) \tag{8}$$

$$\mathcal{A}(P) = \Big\{ \text{all possible connections } A \text{ on } P \Big\}$$
$$\mathcal{G}(P) = \Big\{ \text{all possible gauge transformations on } P \Big\}$$

The moduli space $\mathcal{M}_0$ is far from trivial. Flatness implies that all contractible loops correspond to trivial holonomy; only non-trivial circles, i.e. elements of the homotopy group $\pi_1(\Sigma_T)$, may have non-trivial holonomy. Furthermore, holonomies that differ by a global gauge transformation are by definition, equivalent. In fact, we have [6]:

$$\mathcal{M}_0 = \operatorname{Hom}\big(\pi_1(\Sigma_T), G\big) \big/ G \tag{9}$$

Note that this only identifies the topology of $\mathcal{M}_0$; to compute the path integral, we need to derive the measure on $\mathcal{M}_0$ following the Faddeev–Popov procedure, which is implemented in [2].

[1] See `Wikipedia`: *Hubbard–Stratonovich transformation*.

For $e^2 \neq 0$, the action $I[\phi, A]$ is metric dependent. Somewhat surprisingly, the path integral still contains information about the topology of $\mathcal{M}$. This is an example of a so-called *cohomological field theory*. The idea of [1] is to start from 2D Yang–Mills as a concrete example, and then use its results to motivate a thorough study of cohomological field theory.

As is summarized in [1], *topological field theories* (TFT's), largely introduced by E. Witten, may be grouped into two classes: *Schwarz type* and *cohomological type*[2]. Cohomological field theories, including 2D Yang–Mills with coupling $e^2 \neq 0$, are *not* manifestly metric independent; however, they have a Grassmann-odd nilpotent BRST operator Q, and physical observables are Q-cohomology classes; amplitudes involving these observables are metric independent, thus they are indeed *topological*. Generally speaking, a TFT need *not* be metric independent; the important thing is that it computes *topological invariants*.

On the other hand, *Schwarz type* theories have Lagrangians which are metric independent and hence, formally, the quantum theory is expected to be topological. Examples of such theories include the $e^2 = 0$ YM_2 described above, and also the Chern–Simons theory in 3D. Also, there is a 4D analog of the action $I[\phi, A]$, given by:

$$\int \Big(\mathrm{Tr}\,(BF) + e^2\,\mathrm{Tr}\,(B \wedge \star B) \Big) \tag{10}$$

The first term with BF is also manifestly topological, similar to $e^2 = 0$ YM_2; therefore Schwarz type theories are also called BF type theories.

Following [1], we will first review the exact solution of YM_2, and then try to generalize some aspects for a generic cohomological field theory.

2 Exact solution of 2D Yang–Mills

2.1 Canonical quantization on the cylinder

One can perform the usual canonical quantization with I_{YM_2} on the cylinder, with coordinates $(x^0, x^1) = (t, x) \in \mathbb{R}^1 \times S^1$. We shall make full use of the gauge redundancies in 2D; recall that a generic gauge transformation can be written as:

$$\begin{aligned} A'_\mu &= g A_\mu g^{-1} + g\,\partial_\mu (g^{-1}) \\ &\simeq A_\mu + D_\mu \circ \lambda, \end{aligned} \qquad A_\mu = A^a_\mu(t, x), \quad g = e^{-\lambda^a(t,x)\,T_a}, \tag{11}$$

$$D_\mu = \partial_\mu + A^a_\mu\,T_a, \tag{12}$$

$$(D_\mu \circ \lambda)^a = \partial_\mu \lambda^a + A^b_\mu\,f^a{}_{bc}\,\lambda^c, \quad (T_b)^a{}_c = f^a{}_{bc} \tag{13}$$

Here D_μ is the $\mathfrak{g}$-valued covariant derivative; it acts on the $\mathfrak{g}$-valued gauge parameters $\lambda^a(t, x)$ by the adjoint representation $(T_b)^a{}_c = f^a{}_{bc}$. It is thus possible to choose the *temporal gauge* $A_0 = 0$, by simply solving a first order ODE of $\lambda^a(t, x)$, with respect to the variable $x^0 = t$.

[2] Cohomological type TFT's are also called *Witten type* TFT's, e.g. in [7]. However, [1] chooses to call them *cohomological*, probably to avoid confusion, since Witten has done wonderful work on both types of the theories.

We can further reduce $A_1(t, x)$ with remaining gauge redundancies; with some t-independent, but x-dependent $g = g(x)$, we can preserve $A_0 = 0$, while reducing $A_1(t, x) = A_1(t)$. This is basically the *Coulomb gauge* in 2D, i.e. we have $\partial_1 A_1 = 0$.

Further simplifications can be achieved by working in the *Schrödinger picture*, or *Schrödinger representation*. A nice treatment of 4D Yang–Mills from this "novel" perspective can be found in [8]. In conventional formulations of QFT, we are used to work in the *Heisenberg* or *interactive picture*, where the fields evolve in time: $A_1 = A_1(t)$ and satisfy some operator equations of motion (EOM's), which for free theories look identical to the classical EOM's. Alternatively, we can take the quantum mechanical approach, and decompose the fields at each time slice $t = t_0$ to a set of time-independent energy eigenstates; in the case of YM_2, we have $A_1 = \text{const}$. The time evolution is then tracked by the *wave functional* $\Psi_t[A_1]$. Since the gauge-fixed A_1 has no spacetime dependence, we've actually obtained a equivalent 0-dimensional field theory, i.e. a quantum mechanical system.

There are still remaining gauge redundancies; with another spacetime independent, *global* gauge transformation, we can rotate $A_1 = A_1^a T_a \in \mathfrak{g}$ to the *Cartan subalgebra*, i.e. the maximal abelian subalgebra of $\mathfrak{g}$. Finally, we demand that A_1 is invariant under the *Weyl group*, which is the symmetry of the Cartan subalgebra. Therefore, the *physical* Hilbert space of YM_2 consists of states given by:

$$\Psi[A_1], \quad A_1 = A_1^a T_a \in \mathit{Cartan} \Big/ \mathit{Weyl} \tag{14}$$

Alternatively, we can also work with a partial gauge fixing, e.g. we only impose the temporal gauge $A_0 = 0$, and try to solve for $\Psi_t[A_1(x)]$ by looking at the "Maxwell's equation" in 2D. The time evolution is taken care of by the Schrödinger equation for Ψ_t; for now we need only look at the spatial constraints. We have:

$$D_1 F_{10} = 0, \quad F_{\mu\nu} = [D_\mu, D_\nu] \tag{15}$$

This is simply the YM_2 version of the *Gauss's law* $\vec{\nabla} \cdot \vec{E} = 0$. In YM_2, we have only one $\mathfrak{g}$-valued component of the field strength:

$$E = E^a T_a = F_{10} \tag{16}$$

One can think of the Gauss's law constraint as the result of integrating out A_0, which imposes it's EOM $\frac{\delta I_{\mathrm{YM}_2}}{\delta A_0} = 0$, which is precisely (15). However, for a gauged system, there are subtleties that we need to look out for. One should account for the gauge volume, which can be treated properly with Faddeev–Popov path integral, and the proper way to implement the constraints is through BRST quantization.

Fortunately, the naïve Gauss's law constraint *does* work in this example. In fact, if we solve the Gauss's law constraint as an operator equation of A_1, we will get further gauge-fixing [8]. Alternatively, if we ignore the constraint and proceed with canonical quantization, which might be more convenient in some cases, we would expect *unphysical* degrees of freedom like *null states* to show up, due to the unfixed gauge redundancies. The constraint can then be utilized to identify *physical* degrees of freedom, by demanding that it annilates physical states:

$$D_1 \circ E\,|\Psi\rangle = 0, \quad E = F_{10} \tag{17}$$

Note that this idea is very much similar to *old covariant quantization* and the *Virasoro constraint* in string theory [9]. Again, for a more rigorous treatment, we should turn to the BRST cohomology, but for now this is sufficient.

In fact, we can actually proved (17) by demanding the wave functional $\Psi[A_1(x)]$ to be gauge-invariant [8]; we shall work in the $A_1(x)$ basis, with $\Psi[A_1(x)] = \langle A_1(x)|\Psi\rangle$. The canonical momentum operator is then given by:

$$E = \frac{\delta}{\delta A_1}, \quad [E(x), A_1(x')] = \delta(x - x') \tag{18}$$

Just like the usual quantum mechanical $P = -i\frac{\partial}{\partial X}$; the $(-i)$ factor is gone since we are working in Euclidean signature.

We demand that $\Psi[A_1(x)]$ is gauge-invariant under the *remaining* t-independent gauge transformations, $\lambda = \lambda(x)$:

$$\begin{aligned} 0 = \delta\Psi[A_1(x)] &= \int \mathrm{d}x \, \frac{\delta\Psi}{\delta A_1(x)} \, \delta A_1(x) \,, \quad \delta A_1(x) = D_1 \circ \lambda(x) \\ &= -\int \mathrm{d}x \, \lambda(x) \left(D_1 \circ \frac{\delta}{\delta A_1(x)} \right) \Psi[A_1(x)], \quad E = \frac{\delta}{\delta A_1(x)} \\ &= -\int \mathrm{d}x \, \lambda(x) \, (D_1 \circ E) \, \Psi[A_1(x)], \quad \forall \, \lambda(x) \end{aligned} \tag{19}$$

Indeed this is (17) in the $A_1(x)$ basis. The formal solution of (17) is quite similar to the Schrödinger equation, but with path-ordering instead of time-ordering [10]:

$$\Psi = \Psi[W], \quad W = \mathcal{P} \exp \int_0^L A_1(x) \tag{20}$$

From (20) we see that in fact there is no direct dependence of $A_1(x)$ in $\Psi[A_1(x)]$; it only depends on the holonomy W around the S^1 circle. Note that W is similar to the *Wilson loop* operator, but not quite, due to the lack of a trace Tr. Such W is not exactly gauge-invariant, but in fact gauge-covariant with respect to the base point:

$$W' = g_0 W g_0^{-1}, \quad g_0 = g(x = 0) \tag{21}$$

Hence further demanding invariance under global gauge transformations requires that Ψ only depends on the *conjugacy class* of W. Again we recover (14), namely,

$$\boxed{\textbf{The Hilbert space of } \mathrm{YM}_2 \textbf{ is the space of } \textbf{\textit{class functions}} \textbf{ on } G} \tag{22}$$

By the Peter–Weyl theorem for G compact we can decompose the Hilbert space by the unitary irreducible representations (*irreps*) $\{R\}$ of G. Consequently, a natural basis for the Hilbert space is provided by the *characters* $\{\chi_R\}$ of the irreps; thus we have, relative to the W basis,

$$\langle W|R\rangle = \chi_R(W) = \mathrm{Tr}_R(W) \tag{23}$$

This is the wave function of the R basis. Here Tr_R denotes trace with respect to the irrep R; recall that W is $\mathfrak{g}$-valued and do not have a well-defined trace until we specify a irrep, in this case R. This is the gauge-invariant Wilson loop operator. Note that we are considering pure YM_2 without matter; the irrep R is not some *a priori* matter representation, but arises naturally as we consider class functions on G. This is kind of mysterious; does it has a physical interpretation, e.g. like sum over possible matter representations?

2.2 Hamiltonian and time evolution

Let us now find the Hamiltonian for this theory. We are familiar with the the Hamiltonian density in 4D: $\frac{e^2}{2}\left(\vec{E}^2+\vec{B}^2\right)$; here in 2D, we do not have the magnetic field $\vec{B}$, just a one component $\mathfrak{g}$-valued $E = E^a T_a = F_{10}$. Therefore,

$$H = \frac{e^2}{2}\int \mathrm{d}x\,\mathrm{Tr}\,E^2 = \frac{e^2}{2}\int \mathrm{d}x\,\delta^{ab}\frac{\delta}{\delta A_1^a(x)}\frac{\delta}{\delta A_1^b(x)} \to \frac{1}{2}\,e^2 L\,\mathrm{Tr}\left(W\frac{\partial}{\partial W}\right)^2, \tag{24}$$

$$E_a = \frac{\delta}{\delta A_1^a(x)} \to T_a W\frac{\partial}{\partial W}, \quad \mathrm{Tr}\,(T_a T_b) = \delta_{ab} \tag{25}$$

Note that we've followed a more convenient trace convention. In the last step we've restricted to the physical Hilbert space with $\Psi = \Psi(W)$. Consider H action on the basis $\langle W|R\rangle = \chi_R(W) = \mathrm{Tr}_R(W)$, we have:

$$E_a\,\chi_R(W) = E_a\,\mathrm{Tr}_R(W) = \mathrm{Tr}_R(T_a W), \tag{26}$$

$$\begin{aligned} H\,\chi_R(W) &= \frac{1}{2}\,e^2 L\,\mathrm{Tr}_R\left(\delta^{ab}T_aT_bW\right) \\ &= \frac{1}{2}\,e^2 L\left(\delta^{ab}T_aT_b\right)_R \mathrm{Tr}_R(W) \\ &= \frac{1}{2}\,e^2 L C_2(R)\,\chi_R(W), \end{aligned} \tag{27}$$

$$C_2(R) = \left(\delta^{ab}T_aT_b\right)_R \tag{28}$$

Here $C_2(R)$ is the *quadratic Casimir* of the irrep R; it is proportional to $\mathbb{1}_R$, or can be treated as a c-number when restricted to the irrep. We find out that:

$$\boxed{\textbf{The Hamiltonian } H \textbf{ is diagonalized in the } R \textbf{ basis and } \propto C_2} \tag{29}$$

$$H = \frac{1}{2}\,e^2 L C_2 \tag{30}$$

2.3 Basic amplitudes

Having diagonalized the Hamiltonian, we can immediately write down the W basis propagator by summing over the R basis; we have:

$$\begin{aligned} \langle W_1|e^{-HT}|W_2\rangle &= \sum_R \langle W_1|R\rangle\, e^{-H(R)\,T}\,\langle R|W_2\rangle \\ &= \sum_R \chi_R(W_1)\,\chi_R(W_2^\dagger)\,e^{-\frac{1}{2}\,e^2 a\,C_2(R)} = Z(W_1,W_2;a), \end{aligned} \tag{31}$$

$$a = LT, \quad \chi_R^\dagger(W) = (\mathrm{Tr}_R\,W)^\dagger = \mathrm{Tr}_R(W^\dagger) = \chi_R(W^\dagger) \tag{32}$$

This is the **cylinder amplitude**. Note that the combination $\frac{1}{2}e^2 a$ enters together, where $a = LT$ is the area of the cylinder; this is evident from the action $I[\phi, A]$ in (5), where $e^2\mu$ comes together as a parameter. The area dependence is also evident from the scalar action (4), where we that the theory is invariant under *area preserving diffeomorphism*, SDiff(Σ_T). From now on we will absorb $\frac{e^2}{2}$ into a, and it will be the only dimensionful parameter of the theory.

By the orthogonality relations of characters, it is straightforward to verify that the propagator satisfy the *gluing property*:

$$\int \mathrm{d}W\, Z(W_1, W; T_1)\, Z(W, W_2; T_2) = Z(W_1, W_2; T_1 + T_2) \tag{33}$$

We see that YM_2 has very similar properties as those from the categorical approach of CFT and TFT, given by Segal and Atiyah [11, 12].

With the gluing property, we can see what happens when one end of the cylinder shrinks to zero size; this is achieved by gluing one cylinder with finite size a, with another one that has $a' \to 0$. As $a' \to 0$, the corresponding amplitude should be well approximated by the topological action $I_0[\phi, A]$ given in (7); as we've mentioned before, integrating out ϕ sets $F = 0$, which in turn forces the holonomy $W \equiv \mathbb{1}$. The wave function is then given by a Dirac delta function:

$$\Psi(W) = \delta(W - \mathbb{1}) \tag{34}$$

This is, up to an overall coefficient, the delta function with respect to the bi-invariant *Haar measure* on the compact G. Here *bi-invariance* means that the measure is invariant under left and right group multiplication, hence also invariant under conjugacy; therefore it is also a well-defined measure on G classes, where W actually belongs. Gluing the cylinder a with the infinitesimal cylinder $a' \to 0$, we find the **disk (cap) amplitude**:

$$Z(W, \mathbb{1}; a) = \sum_R \chi_R(W)\, (\mathrm{Tr}_R\, \mathbb{1})\, e^{-aC_2(R)} = \sum_R (\dim R)\, \chi_R(W)\, e^{-aC_2(R)} \tag{35}$$

By now it is clear that the theory is indeed *topological*; there are no x-dependent, propagating degrees of freedom. To see degrees of freedom, we must investigate the theory on spacetimes of nontrivial topology[3]. In fact, we can compute the amplitudes for more complicated surfaces by standard gluing techniques familiar from axiomatic TFT.

3 From YM_2 to general cohomological theory

We've learned quite a few things from our study of YM_2; for example, we see that a topological theory need not be manifestly metric independent. Also, it is possible, and sometimes more elegant, to solve the theory without a complete gauge fixing. These ideas can all be utilized for the study of a general cohomological field theory.

From a mathematical point of view, topological field theory is the study of *intersection theory* on moduli spaces using physical methods [1, 13]. In the physical framework these moduli spaces are presented in the general form:

$$\mathcal{M} = \left\{ A \in \mathcal{A} \,\middle|\, DA = 0 \right\} \Big/ \mathcal{G} \tag{36}$$

We've seen an example of this, from (8).

Note that the usual Faddeev–Popov procedure and coventional BRST cohomology requires gauge-fixing. Now we would like to develop an alternative method which does not rely on a specific gauge

[3] Then isn't this, in some sense, a theory of Euclidean "gravity", or at least a "string theory" by itself? As it includes a sum over topologies.

choice. In YM_2, postponing the gauge fixing leads us to the class functions on G, which is more convenient to work with than the gauge-fixed *Cartan / Weyl*. More generally, we can directly work on the space of fields $\mathcal{A}$ before gauge fixing, and consider $\mathcal{G}$*-equivariant* quantities on $\mathcal{A}$. Here $\mathcal{G}$ is generally an infinite-dimensional Lie group, the group of all possible gauge transformations. This is the idea of *equivariant cohomology*.

3.1 Equivariant cohomology from "supersymmetry"

Here we will briefly illustrate the construction of a differential complex, which will be a model for equivariant cohomology. We want something that is "BRST-like", without going through the Faddeev–Popov gauge-fixing process. The idea is to mimic the BRST field contents by introducing *ghosts*, which can be systematically constructed using differential graded algebra (DGA); the grading is the *ghost number*.

Note that "physical" fields such as A^a_μ and matter fields are form a $\mathfrak{g}$-module, where $\mathfrak{g} = \mathrm{Lie}\,\mathcal{G}$; to include ghost fields, we basically add an additional grading and work with the $\mathfrak{g}[\epsilon]$-module. Here ϵ is a Grassmann-odd parameter with $\deg \epsilon = -1$, and:

$$\mathfrak{g}[\epsilon] = (\mathfrak{g} \otimes 1) \oplus (\mathfrak{g} \otimes \epsilon) \tag{37}$$

It turns out that the *usual* Lie algebra cohomology of this *supersymmetrized* Lie algebra $\mathfrak{g}[\epsilon]$ is the same as the equivariant cohomology of the original Lie algebra $\mathfrak{g}$ [1].

In fact, the Lie (super)algebra cohomology for $\mathfrak{g}[\epsilon]$ is precisely the BRST cohomology with b, c, β, γ ghosts, known from superstring theory [1, 14]. Hence it's possible to relate equivariant cohomology with familiar Lie superalgebra cohomology. An example of this is $\mathcal{G} = \mathrm{U}(1)$, whose equivariant cohomology is given by the Lie algebra cohomology of the *twisted* $\mathcal{N} = 2$ supersymmetry algebra in 2D [1]. The twisted algebra is obtained from the original algebra by a *topological twist*, which redefines the energy-momentum tensor and makes it Q-exact, thus rendering the theory topological [15, 16]. From the level of the algebra, we have:

$$L_0 = \{G_0, Q_0\} \tag{38}$$

Where L_0 is the zero mode of the energy momentum. There is a beautiful generalization of this by considering $\mathcal{G} = \mathrm{Diff}(S^1)$, whose corresponding Lie algebra is given by the *Witt algebra*, or *Virasoro algebra* $\mathfrak{Vir}_c$ with central charge $c = 0$. The $\mathcal{G} = \mathrm{U}(1)$ we've considered is just the *global* part of the local symmetry group $\mathcal{G} = \mathrm{Diff}(S^1)$. Similarly, the $\mathrm{Diff}(S^1)$-equivariant cohomology is precisely the Lie algebra cohomology of the twisted $\mathcal{N} = 2$ *superconformal algebra*.

For a more concrete example, let's go back to Yang–Mills, and consider the $\mathcal{G}$-equivariant cohomology of the space of connections $\mathcal{A}$. Again, fields are simply $\mathfrak{g}[\epsilon]$-modules; to compute equivariant cohomology we must construct a differential complex, which is achieved by adding ghosts. A choice of differential complex amounts to a choice of ghosts. One of the simpliest choice is given by the *Cartan model* [1, 13]; we have:

- Degree 0: $A^a_\mu(x)$, Yang–Mills
- Degree 1: $\psi^a_\mu(x)$, 1-form ghosts
- Degree 2: $\phi^a(x)$, commuting 0-form

The BRST-like symmetry $\delta = [Q, -]_{\pm}$ is given by:

$$\delta A_\mu = \psi_\mu, \quad \delta\psi_\mu = -D_\mu\phi, \quad \delta\phi = 0 \tag{39}$$

On the other hand, the conventional BRST symmetry from Faddeev–Popov is given by:

$$\tilde{\delta} A_\mu = -D_\mu c, \quad \tilde{\delta} c = \frac{1}{2}[c, c] \tag{40}$$

3.2 Localization

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nonabelian localization theorem

Mathematically, the ultimate objects of study are intersection numbers.

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