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## 1 BRST Symmetry

The BRST transformation of  $c^a$  ghost is:

$$\delta c^a = \frac{1}{2} f^a_{bc} c^b c^c \Lambda, \quad \delta c = \delta c^a T_a = \frac{1}{2} [c, c\Lambda]$$
 (1)

 $D_{\mu}=\partial_{\mu}+A^a_{\mu}\,T_a,\,T_a$  acts on  $c^a$  by adjoint representation:  $(T_a)^c_{\ b}\,c^b=f^c_{\ ab}\,c^b,$  i.e.

$$T_a \cdot c = (T_a)^c_b c^b T_c = f^c_{ab} T_c c^b = [T_a, T_b] c^b = [T_a, c],$$
 (2)

$$D_{\mu}c = \partial_{\mu}c + [A_{\mu}, c] = [D_{\mu}, c], \tag{3}$$

$$D_{\mu} \, \delta c = \partial_{\mu} \, \delta c + \left[ A_{\mu}, \delta c \right], \tag{4}$$

$$(D_{\mu} \delta c)^{a} = \partial_{\mu} \delta c^{a} + \left[ A_{\mu}, \delta c \right]^{a}$$

$$= \partial_{\mu} \delta c^{a} + A_{\mu}^{c} \left[ T_{c}, T_{b} \right]^{a} \delta c^{b}$$

$$= \partial_{\mu} \delta c^{a} + A_{\mu}^{c} f_{cb}^{a} \delta c^{b}$$

$$= \partial_{\mu} \delta c^{a} + A_{\mu}^{c} \left( T_{c} \right)^{a}{}_{b} \delta c^{b}$$

$$= D_{\mu} (\delta c^{a}),$$

$$(5)$$

i.e. 
$$(D_{\mu} \delta c)^{a} - \frac{1}{2} D_{\mu} (f^{a}_{bc} c^{b} c^{c} \Lambda) = (D_{\mu} \delta c)^{a} + \frac{1}{2} D_{\mu} (f^{a}_{bc} c^{b} \Lambda c^{c}) = 0$$
 (6)

## 2 Relativistic Particle

$$L_q = L + L_{gf} + L_{gh}, (7)$$

$$L = \frac{1}{2e} \left( \frac{1}{c_0} \frac{dX}{dt} \right)^2 - \frac{e}{2} m^2 c_0^4, \tag{8}$$

$$L_{ab} = -e\dot{b}c\tag{9}$$

• For  $t \mapsto t' = t - \xi(t)$ , we have gauge transformation:  $\delta X^{\mu} = \xi \dot{X}^{\mu}$ ,  $\delta e = \frac{\mathrm{d}}{\mathrm{d}t} \left( e \xi \right)$ ,  $\delta L = \frac{\mathrm{d}}{\mathrm{d}t} \left( \xi L \right)$ , replace  $\xi \mapsto c\Lambda$ , and we have BRST transformation:

$$\delta X^{\mu} = c\Lambda \dot{X}^{\mu} = c\dot{X}^{\mu}\Lambda, \quad \delta e = \frac{\mathrm{d}}{\mathrm{d}t}(ec\Lambda) = \frac{\mathrm{d}}{\mathrm{d}t}(ec)\Lambda$$
 (10)

• Assume nilpotency, and we have:

$$0 = \delta_{\Lambda} \delta_{\Lambda'} X^{\mu} = \left( (\delta_{\Lambda} c) \dot{X}^{\mu} + c \, \delta_{\Lambda} \dot{X}^{\mu} \right) \Lambda'$$

$$= \left( (\delta_{\Lambda} c) \dot{X}^{\mu} + c \, \left( \dot{c} \dot{X}^{\mu} + e \ddot{X}^{\mu} \right) \Lambda \right) \Lambda'$$

$$= \left( \delta_{\Lambda} c + c \dot{c} \Lambda \right) \dot{X}^{\mu} \Lambda',$$
(11)

$$\delta_{\Lambda}c = -c\dot{c}\Lambda \tag{12}$$

$$\delta_{\Lambda}\delta_{\Lambda'}e = \frac{\mathrm{d}}{\mathrm{d}t} ((\delta_{\Lambda}e)c + e\delta_{\Lambda}c)\Lambda'$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathrm{d}}{\mathrm{d}t}(ec)\Lambda c - ec\dot{c}\Lambda\right)\Lambda'$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} (e\dot{c}\Lambda c - ec\dot{c}\Lambda)\Lambda' = 0$$
(13)

• The BRST transformation for c is also nilpotent:

$$\delta_{\Lambda}\delta_{\Lambda'}c = -\left(\left(\delta_{\Lambda}c\right)\dot{c} + c\,\delta_{\Lambda}\dot{c}\right)\Lambda'$$

$$= -\left(-c\dot{c}\Lambda\,\dot{c} - c\,c\dot{c}\Lambda\right)\Lambda' = 0$$
(14)

• Gauge fixing f = e(t) - 1 = 0 can be imposed by:

$$\delta[f] \sim \int \mathcal{D}d(t) \exp\left(i \int dt \, d(t) \, f(t)\right),$$
 (15)

$$L_{gf} = d(t)f(t) = d(t)(e(t) - 1)$$
 (16)

The quantum action is  $S_q = \int dt L_q$ ,  $L_q = L[X, e] + L_{gf}[e, d] + L_{gh}[e, b, c]$ . We want  $S_q$  to be BRST invariant, which will help determine transformation rules for b, d; consider:

$$\delta(L_{gf} + L_{gh}) = (e - 1) \, \delta d + d \, \delta e - \delta \left( e\dot{b}c \right)$$

$$= (e - 1) \, \delta d + d(t) \, \frac{\mathrm{d}}{\mathrm{d}t} \left( ec \right) \Lambda + \left( ec \right) \delta \dot{b}$$

$$= (e - 1) \, \delta d + \frac{\mathrm{d}}{\mathrm{d}t} \left( ec \right) \left( d(t) \Lambda - \delta b \right) + \frac{\mathrm{d}}{\mathrm{d}t} \left( ec \, \delta b \right)$$
(17)

We find a natural choice of  $\delta b$ ,  $\delta d$ :

$$\delta d = 0, \quad \delta b = d(t) \Lambda, \quad \delta(L_{gf} + L_{gh}) = \frac{\mathrm{d}}{\mathrm{d}t} (ec \,\delta b)$$
 (18)

• The complete quantum action is BRST invariant, since:

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}t} (\xi L)_{\xi \mapsto c\Lambda} = \frac{\mathrm{d}}{\mathrm{d}t} (cL) \Lambda, \quad \delta(L_{gf} + L_{gh}) = \frac{\mathrm{d}}{\mathrm{d}t} (ec \,\delta b), \tag{19}$$

$$\delta S_q = \int dt \, \delta L_q = \int dt \left( \delta L + \delta (L_{gf} + L_{gh}) \right) = 0$$
 (20)

• Note that  $\frac{\delta S_q}{\delta d} = 0 \implies f = e - 1 = 0$ . Moreover,

$$\frac{\delta S_q}{\delta e} = 0 \implies -\frac{1}{2} \left( \frac{1}{e^2 c_0^2} \dot{X}^2 + m^2 c_0^4 \right) + d - \dot{b}c = 0, \tag{21}$$

$$d = d[X, b, c] = \frac{1}{2} \left( \frac{1}{c_0^2} \dot{X}^2 + m^2 c_0^4 \right) + \dot{b}c$$
 (22)

Therefore, it is convenient to consider the reduced Lagrangian  $L_q[X, b, c] = (L_q)_{e=1, d=d[X]}$ , where e, d are integrated out<sup>1</sup>. The symmetries are thus reduced to:

$$\delta X^{\mu} = c\dot{X}^{\mu}\Lambda, \quad \delta b = d[X, b, c]\Lambda, \quad \delta c = -c\dot{c}\Lambda,$$
 (23)

<sup>&</sup>lt;sup>1</sup>Reference: Polchinski.

$$\delta L_q = \frac{\mathrm{d}}{\mathrm{d}t} \left( cL\Lambda + ec \ \delta b \right)_{e=1} = \frac{\mathrm{d}}{\mathrm{d}t} \left( cL_{e=1} + \frac{c}{2} \left( \frac{1}{c_0^2} \dot{X}^2 + m^2 c_0^4 \right) \right) \Lambda = \frac{\mathrm{d}}{\mathrm{d}t} \left( c \left( \frac{1}{c_0} \dot{X} \right)^2 \right) \Lambda, \tag{24}$$

$$L_q = \frac{1}{2} \left( \frac{1}{c_0} \dot{X} \right)^2 - \frac{1}{2} m^2 c_0^4 - \dot{b}c$$
 (25)

On the other hand, the *on-shell* variation is given by:

$$\delta_0 L_q = \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\partial L_q}{\partial \dot{X}^{\mu}} \, \delta X^{\mu} + \frac{\partial L_q}{\partial \dot{b}} \, \delta b \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left\{ c \left( \left( \frac{1}{c_0} \dot{X} \right)^2 + d[X, b, c] \right) \right\} \Lambda, \tag{26}$$

The  $\dot{b}c$  term in d[X, b, c] is killed by the c multiplication: c d[X, b, c] = c d[X]. Therefore, the canonical BRST charge Q is given by:

$$0 = \delta_0 L_q - \delta L_q = \frac{\mathrm{d}Q}{\mathrm{d}t} \Lambda = \frac{\mathrm{d}}{\mathrm{d}t} (c \, d[X]) \Lambda, \tag{27}$$

$$Q = c d[X] = \frac{c}{2} \left( \frac{1}{c_0^2} \dot{X}^2 + m^2 c_0^4 \right)$$
 (28)

• Note that  $L_{gh} = -\dot{b}c = b\dot{c} - \frac{\mathrm{d}}{\mathrm{d}t}(bc) \sim b\dot{c}$ ; for future convenience, let's replace  $(-\dot{b}c) \mapsto b\dot{c}$  in the Lagrangian, and we have:

$$p_{\mu} = \frac{\partial L}{\partial \dot{X}^{\mu}} = \frac{1}{c_0^2} \dot{X}_{\mu}, \quad p_c = \frac{\partial L}{\partial \dot{c}} \equiv \left(b\dot{c}\right) \frac{\overleftarrow{\partial}}{\partial \dot{c}} = b \tag{29}$$

Here we adopt the "right" derivative convention, in this case the Hamiltonian:

$$H = p_{\mu}\dot{X}^{\mu} + p_{c}\dot{c} - L_{q} = \frac{c_{0}^{2}}{2}p_{\mu}p^{\mu} + \frac{1}{2}m^{2}c_{0}^{4} = \frac{1}{2}\left(p^{2}c_{0}^{2} + m^{2}c_{0}^{4}\right)$$
(30)

• We have:

$$Q = \frac{c}{2} \left( \frac{1}{c_0^2} \dot{X}^2 + m^2 c_0^4 \right) = \frac{c}{2} \left( p^2 c_0^2 + m^2 c_0^4 \right) = cH$$
 (31)

After canonical quantization,  $p_{\mu}, p_c$  and H are promoted to Hermitian operators, and:

$$Q^2 = cH \cdot cH = 0 \tag{32}$$

• Note that:

$$[p_{\mu}, X^{\nu}] = -i\delta^{\nu}_{\mu}, \quad [p_{\mu}, \mathcal{F}(X)] = -i\partial_{\mu}\mathcal{F}(X), \tag{33}$$

i.e.  $p_{\mu}$  acts on  $\mathcal{F}(X)$  by X-derivative; from the path integral perspective, we have:

$$\langle p_{\mu} \mathcal{F}(X) \rangle = \int \mathcal{D}p \, \mathcal{D}X \, \mathcal{D}b \, \mathcal{D}c \, e^{iS[p,X,b,c]} \, p_{\mu} \mathcal{F}(X),$$
 (34)

$$S[p, X, b, c] = \int dt \left( p_{\mu} \dot{X}^{\mu} + \dot{b}c - H[p] \right), \tag{35}$$

$$\int dt' \frac{\delta S}{\delta X^{\mu}(t')} = \int dt' \int dt \, p_{\mu} \, \partial_t \, \delta(t - t') \sim -\int dt' \, \dot{p}_{\mu}(t') = -p_{\mu}, \tag{36}$$

$$\langle p_{\mu} \mathcal{F}(X) \rangle = \int dt' \int \mathcal{D}p \, \mathcal{D}X \, \mathcal{D}b \, \mathcal{D}c \, \left( i \, \frac{\delta}{\delta X^{\mu}(t')} \, e^{iS[p,X,b,c]} \right) \mathcal{F}(X)$$

$$= \int \mathcal{D}p \, \mathcal{D}X \, \mathcal{D}b \, \mathcal{D}c \, e^{iS[p,X,b,c]} \int dt' \, \left( -i \, \frac{\delta}{\delta X^{\mu}(t')} \, \right) \mathcal{F}(X), \tag{37}$$

$$p_{\mu} \mathcal{F}(X) \sim \int dt' \left( -i \frac{\delta}{\delta X^{\mu}(t')} \right) \mathcal{F}(X) = -i \frac{\partial}{\partial X^{\mu}} \mathcal{F}(X)$$
 (38)

## 2 RELATIVISTIC PARTICLE

For  $e^{ik_{\mu}X^{\mu}}|0\rangle$ ,  $\mathcal{F}(X)=e^{ik_{\mu}X^{\mu}}$ , it is Q-closed iff.

$$0 = Q e^{ik_{\mu}X^{\mu}} |0\rangle = \frac{c}{2} \left( p^{2} c_{0}^{2} + m^{2} c_{0}^{4} \right) e^{ik_{\mu}X^{\mu}} |0\rangle$$
$$= \frac{c}{2} \left( k^{2} c_{0}^{2} + m^{2} c_{0}^{4} \right) \left( e^{ik_{\mu}X^{\mu}} |0\rangle \right),$$
(39)

$$k^2 c_0^2 = k_\mu k^\mu c_0^2 = -E^2 + \mathbf{k}^2 c_0^2 = -m^2 c_0^4, \tag{40}$$

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Or  $E^2={f k}^2c_0^2+m^2c_0^4.$  This is the dispersion relation of a relativistic particle.