Finite Temperature Field Theory №5

Correlations of Pauli Spinor Fields

The Lagrangian of a nonrelativistic free particle is given by:

$$\mathcal{L} = \bar{\psi}\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m}\right)\psi \tag{1}$$

The Euclidean action is then given by:

$$(-S) = \int \mathrm{d}^4x \, \mathcal{L}_{t=-i\tau} = -\int \mathrm{d}^4x \, \bar{\psi}\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m}\right)\psi = -\beta H \tag{2}$$

Note that we have $N = \int \mathrm{d}^3\mathbf{x}\,\bar{\psi}\psi = \int \mathrm{d}^3\mathbf{x}\,n$ as a conserved charge, therefore it is natural to include N in the partition function:

$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \, \exp\int \mathrm{d}^4x \left\{-\bar{\psi}\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m}\right)\psi + \mu n\right\} \tag{3}$$

For a spin-$\frac{1}{2}$ free fermion, ψ is given by a 2-component Pauli spinor: $\psi \sim (\psi_+, \psi_-)$, and $\bar{\psi} = \psi^\dagger$ is the conjugate transpose of ψ. Relevant observables in the Heisenberg picture are then given by:

$$n = \psi^\dagger\psi, \quad \tilde{n} = n - \langle n \rangle, \quad s_i = \psi^\dagger \sigma_i \psi, \tag{4}$$

$$D_n = \left\langle \mathcal{T}_\tau \, \tilde{n}(x)\,\tilde{n}(x')\right\rangle, \quad D_{ij} = \left\langle \mathcal{T}_\tau\, s_i(x)\,s_j(x')\right\rangle \tag{5}$$

To compute the **density correlation** D_n, define the generating functional:

$$Z[j] = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \, \exp\int \mathrm{d}^4x \left\{-\bar{\psi}\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m}\right)\psi + \mu n(x) + j(x)\,n(x)\right\} \tag{6}$$

To remove the non-zero $\langle n \rangle$, consider:

$$W[j] = \ln Z[j], \quad D_n = \left\langle \mathcal{T}_\tau \, \tilde{n}(x)\,\tilde{n}(x')\right\rangle = \left.\frac{\delta^2 W[j]}{\delta j(x)\,\delta j(x')}\right|_{j=0} = \left.\frac{\delta^2 W^{(2)}[j]}{\delta j(x)\,\delta j(x')}\right|_{j=0} \tag{7}$$

$$W^{(2)} \sim \mathcal{O}(j^2) \tag{8}$$

For free theory, $W^{(2)}$ can be computed explicitly by mode expansions:

$$\psi = \frac{1}{\sqrt{V}}\sum_k e^{ik\cdot x}\psi_k, \quad j = \sum_q e^{-iq\cdot x}j_q, \quad \sum_k e^{ik\cdot x} = V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}}\sum_{n\in\mathbb{Z}}e^{i\omega_n\tau} \tag{9}$$

$$
\begin{aligned}
Z[j] &= \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \, \exp\int \mathrm{d}^4x \left\{-\bar{\psi}\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu - j(x)\right)\psi\right\} \\
&= \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \, \exp\left\{-\beta\sum_{k,k'}\bar{\psi}_k\left(\left(i\omega + \frac{\mathbf{k}^2}{2m} - \mu\right)\delta_{k,k'} - j_{k-k'}\right)\psi_{k'}\right\} \\
&\sim \left\{\det_{k,k',\bullet}\left(\beta D_0^{-1}(k)\,\delta_{k,k'} - \beta j_{k-k'}\right)\right\}^{+1}, \quad D_0^{-1}(k) = i\omega + \frac{\mathbf{k}^2}{2m} - \mu, \\
&= \left\{\det_{k,k'}\left(\beta D_0^{-1}(k)\,\delta_{k,k'} - \beta j_{k-k'}\right)\right\}^{+2}
\end{aligned}
\tag{10}
$$

Note that the determinant should go over the implicit spinor indices as well, which results in a power of 2 in the final expression.

We can then compute $W[j]$ using Jacobi's formula:

$$
\begin{aligned}
W[j] = \ln Z[j] &= 2 \ln \det_{k,k'} \left(\beta D_0^{-1}(k)\, \delta_{k,k'} - \beta j_{k-k'} \right) \\
&= 2 \operatorname*{Tr}_{k,k'} \ln \left(\beta D_0^{-1}(k)\, \delta_{k,k'} - \beta j_{k-k'} \right) \\
&= 2 \operatorname*{Tr}_{k,k'} \left\{ \delta_{k,k'} \ln \left(\beta D_0^{-1}(k) \right) + \ln \left(\mathbb{1}_{k,k'} - j_{k-k'} D_0(k) \right) \right\}
\end{aligned}
\tag{11}
$$

The first term is the vacuum contribution with no dependence of j, hence it is irrelevant in $W^{(2)} \sim \mathcal{O}(j^2)$. Expansion of the matrix log in the second term reveals that:

$$
\ln \left(\mathbb{1}_{k,k'} - j_{k-k'} D_0(k) \right) = -\sum_{n=1}^{\infty} \frac{1}{n} \left(j_{k-k'} D_0(k) \right)^n_{k,k'}
\tag{12}
$$

$$
\begin{aligned}
W^{(2)} &= 2 \operatorname*{Tr}_{k,k'} \frac{-1}{2} \left(j_{k-k'} D_0(k) \right)^2_{k,k'} \\
&= -\operatorname*{Tr}_{k,k'} \sum_{q} \left(j_{k-q} D_0(k) \right) \left(j_{q-k'} D_0(q) \right) \\
&= -\sum_{k,q} j_{k-q} D_0(k)\, j_{q-k} D_0(q) \\
&= -\sum_{k} \sum_{q-k} j_{k-q} D_0(k)\, j_{q-k} D_0(q-k+k) \\
&= -\sum_{q} \sum_{k} j_{-q} \Big(D_0(k)\, D_0(q+k) \Big) j_q \\
&= \frac{\beta V}{2} \sum_{k} j_{-k} D_n(k)\, j_k,
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
D_n(k) &= -\frac{2}{\beta V} \sum_{q} D_0(q)\, D_0(k+q) \\
&= -\frac{2}{\beta V} \sum_{q} \frac{1}{i\omega_q + E_q} \frac{1}{i(\omega_q + \omega_k) + E_{q+k}}, \quad E_q = \frac{\mathbf{q}^2}{2m} - \mu, \\
&= -\frac{2}{\beta V} \sum_{q} \left(\frac{1}{i\omega_q + E_q} - \frac{1}{i\omega_q + i\omega_k + E_{q+k}} \right) \frac{1}{i\omega_k + E_{q+k} - E_k} \\
&= -2 \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{i\omega_k + E_{q+k} - E_q}\, T \sum_{\omega_q} \left(\frac{1}{i\omega_q + E_q} - \frac{1}{i\omega_q + i\omega_k + E_{q+k}} \right) \\
&= -2 \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{i\omega_k + E_{q+k} - E_q} \Big(\big(1 - n(E_q)\big) - \big(1 - n(E_{q+k} + i\omega_k)\big) \Big) \\
&= -2 \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{i\omega_k + E_{q+k} - E_q} \left(\frac{1}{-e^{\beta E_{q+k}} + 1} - \frac{1}{e^{\beta E_q} + 1} \right)
\end{aligned}
\tag{14}
$$

Here we've completed the Matsubara sum of the fermionic frequencies, using:

$$
T \sum_{\omega_q} \frac{1}{i\omega_q + E_q} = 1 - n(E_q), \quad n(E_q) = \frac{1}{e^{\beta E_q} + 1}, \quad n(E_q + i\omega_k) = \frac{1}{-e^{\beta E_q} + 1}
\tag{15}
$$

The retarded propagator and the spectral density is obtained by analytic continuation:

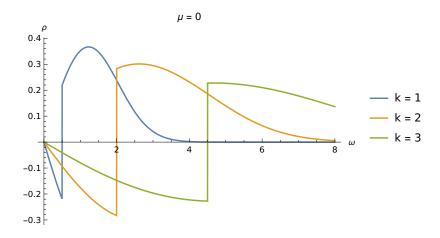$$D_n^R(\omega, \mathbf{k}) = D_n(\omega \to i\omega - \epsilon, \mathbf{k})$$

$$= -2 \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{\omega + i\epsilon - (E_{q+k} - E_q)} \left(\frac{1}{e^{\beta E_{q+k}} - 1} + \frac{1}{e^{\beta E_q} + 1} \right), \tag{16}$$
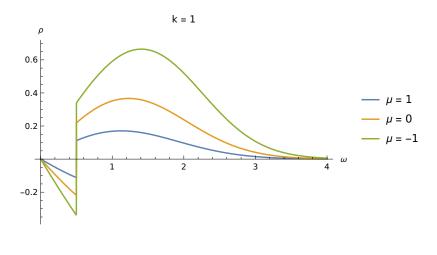
$$\rho_n = 2 \operatorname{Im} D_n^R(\omega, \mathbf{k}), \quad \frac{1}{x \pm i\epsilon} = \mathcal{P}\frac{1}{x} \mp i\pi\delta(x)$$

$$= 4\pi \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \delta\left(\omega - (E_{q+k} - E_q)\right) \left(\frac{1}{e^{\beta E_{q+k}} - 1} + \frac{1}{e^{\beta E_q} + 1} \right)$$

$$= 4\pi \cdot \frac{1}{(2\pi)^3} \cdot 2\pi \int_0^\pi \sin\theta\, \mathrm{d}\theta \int_0^\infty q^2\, \mathrm{d}q\, \delta\left(\omega - (E_{q+k} - E_q)\right) \left(\frac{1}{e^{\beta(E_q+\omega)} - 1} + \frac{1}{e^{\beta E_q} + 1} \right) \tag{17}$$

$$= \frac{1}{\pi} \int_0^\pi \sin\theta\, \mathrm{d}\theta\, \frac{m}{k\cos\theta} \int_0^\infty q^2\, \mathrm{d}q\, \delta\left(q - \frac{2m\omega - k^2}{2k\cos\theta}\right) \left(\frac{1}{e^{\beta(E_q+\omega)} - 1} + \frac{1}{e^{\beta E_q} + 1} \right)$$

$$= \frac{1}{\pi} \int_{-1}^1 \mathrm{d}x\, \frac{m}{kx} \left(\frac{q_0}{x}\right)^2 \left(\frac{1}{e^{\beta(E_0/x^2 - \mu + \omega)} - 1} + \frac{1}{e^{\beta(E_0/x^2 - \mu)} + 1} \right) \theta\left(\frac{q_0}{x}\right)$$

We have $k = \|\mathbf{k}\|$, $x = \cos\theta$, $q_0 = \frac{2m\omega - k^2}{2k}$, $E_0 = \frac{q_0^2}{2m}$, and the Heaviside θ-function $\theta\left(\frac{q_0}{x}\right)$ which enforces that $q_0/x > 0$. With $\mathrm{d}\left(\frac{1}{x^2}\right) = -\frac{2}{x^3}\,\mathrm{d}x$, we can rewrite the above integral into:

$$\rho_n = \frac{1}{\pi} \left(-\operatorname{sgn} q_0\right)\left(-\frac{mq_0^2}{2k}\right) \int_1^\infty \mathrm{d}y \left(\frac{1}{e^{\beta(E_0 y - \mu + \omega)} - 1} + \frac{1}{e^{\beta(E_0 y - \mu)} + 1} \right), \quad y = \frac{1}{x^2},$$

$$= \left(\operatorname{sgn} q_0\right) \frac{mq_0^2}{2\pi k} \cdot \frac{1}{\beta E_0} \left\{ -\ln\left(1 - e^{-\beta(E_0 - \mu + \omega)}\right) + \ln\left(1 + e^{-\beta(E_0 - \mu)}\right) \right\} \tag{18}$$

$$= \left(\operatorname{sgn} q_0\right) \frac{m^2 T}{k\pi} \ln \frac{1 + e^{-\beta(E_0 - \mu)}}{1 - e^{-\beta(E_0 - \mu + \omega)}}, \quad k = \|\mathbf{k}\|, \quad q_0 = \frac{2m\omega - k^2}{2k}, \quad E_0 = \frac{q_0^2}{2m}$$

We see that the result grows linearly in T, and flips sign at $q_0 = 0$ or $\omega = \frac{k^2}{2m}$. Note that this is precisely the dispersion relation of ψ. We can interpret this result as particle pairs being created at such frequencies. For $m = 1$, $T = 1$, plots of ρ_n as a function of ω with various μ, k is shown below.

For the **spin correlations** D_{ij}, the above analysis still holds, and we need only change the source term in (6) from $j(x)\,n(x)$ to $J^i(x)\,s_i(x)$. We have:

$$Z[j] \sim \left\{ \det_{k,k',\bullet} \left(\beta D_0^{-1}(k)\,\delta_{k,k'} - \beta J^i_{k-k'}\sigma_i \right) \right\}^{+1}, \quad D_0^{-1}(k) = i\omega + \frac{\mathbf{k}^2}{2m} - \mu \tag{19}$$

There are now no-trivial dependence on the spinor indices. After completing the trace of Pauli matrices, we find that the result is basically the same as before, but with an additional δ_{ij} factor:

$$\begin{aligned}
W^{(2)} &= -\frac{1}{2} \operatorname*{Tr}_{k,k',\bullet} \sum_q \left(J^i_{k-q}\sigma_i D_0(k) \right) \left(J^j_{q-k'}\sigma_j D_0(q) \right) \\
&= -\frac{1}{2} \sum_q \sum_k J^i_{-q} \left(\operatorname{tr}(\sigma_i\sigma_j)\, D_0(k)\, D_0(q+k) \right) J^j_q \\
&= \frac{\beta V}{2} \sum_k J^i_{-k} D_{ij}(k)\, J^j_k,
\end{aligned} \tag{20}$$

$$D_{ij}(k) = \frac{1}{2}\operatorname{tr}(\sigma_i\sigma_j)D_n(k) = \delta_{ij}D_n(k), \quad \rho_{ij} = \delta_{ij}D_{ij}(k) \tag{21}$$

This means that the spin correlation along a same direction is identical with the density correlation, while there is no correlation between spins in different directions.