Ec143, Spring 2024

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Midterm Review Sheet #1

This review sheet is designed to assist you in your exam preparations. I suggest preparing written answers to each question. You may find it useful to study with your classmates. In the exam you may bring in a single 8.5 x 11 sheet of notes. No calculators or other aides will be permitted. Please bring blue books to the exam. The first midterm exam will occur in class on Thursday, March 5th.

[1] Loan Problem: Beta-Binomial Learning

This problem is adapted from (that late) Gary Chamberlain's undergraduate class at Harvard. Consider a sub-population of borrowers. To make this problem more realistic you may imagine these borrows are homogenous in a vector of observable attributes. Let $Y = 1$ if a borrower repays their loan and $Y = 0$ if they default. We have

$$\Pr(Y = 1|\,\theta) = \theta, \quad \Pr(Y = 0|\,\theta) = 1 - \theta.$$

You work at Proxima Centauri Bank (PCB). PCB is the a largest bank on a generation starship with tens of thousands of passengers traveling to a planetary system 100 light-years from earth. If the borrower pays back the loan the gain to PCB is g, whereas if they default the loss to the bank is l. Expected profits therefore equal

$$g\Pr(Y = 1|\,\theta) - l\Pr(Y = 1|\,\theta) = g\theta - l(1 - \theta).$$

[a] Assume that θ is known. What is the minimal repayment rate (i.e., value of θ) such that is profitable to lend to this group of borrowers?

[b] Let $\mathbf{Y} = (Y_1, \ldots, Y_N)'$ be a vector of past repayment outcomes for a random sample of borrowers. For a given θ what is the ex ante probability of the event $\mathbf{Y} = \mathbf{y}$ (i.e., find an expression for the likelihood $\Pr(\mathbf{Y} = \mathbf{y}|\,\theta) = f(\mathbf{y}|\,\theta)$)?

[c] You have been asked to consider whether it is profitable to continue to lend to this subpopulation (you may base your decision on the dataset introduced in part [b]). You may take one of two actions

$$\mathcal{A} = \{a_1, a_2\}.$$

Action a_1 corresponds to continuing to approve loans for this subpopulation. Whereas action a_2 corresponds to no longer lending to this group (which yields a payoff of zero under all states of nature). Write down the loss function associated these two actions (i.e., an expression for $L(\theta, a_1)$ and $L(\theta, a_2)$). Assume that loss equals the negative of expected profits calculated 'as if' θ were known.

[d] You attended many applied microeconomics seminars as a student. Based on this experience you decide it is best to "let the data speak". Specifically you decide that you will construct a decision rule which maps the data into actions: $d{:}\mathbb{Y} \rightarrow \mathcal{A}$. The data will speak and you, as the ultimate decision-maker, will decide. Here $\mathbb{Y} = \{0,1\}^N$ is the set of possible repayment patterns in your sample. Define *risk* and explain why it equals

$$\begin{aligned} R(\theta, d) &= \mathbb{E}\left[L(\theta, d(\mathbf{Y})) \mid \theta \right] \\ &= \sum_{\mathbf{y} \in \{0,1\}^N} L(\theta, d(\mathbf{y})) f(\mathbf{y}|\theta). \end{aligned}$$

[e] Prior to boarding the starship you worked in a bank in Idaho. From this experience you formed a prior about θ with density $\pi(\theta)$. Using this prior show that average risk equals

$$\begin{aligned} r(\theta, d) &= \int R(\theta, d) \pi(\theta) \, \mathrm{d}\theta \\ &= \sum_{\mathbf{y} \in \{0,1\}^N} \left[\int L(\theta, d(\mathbf{y})) f(\mathbf{y}|\theta) \pi(\theta) \, \mathrm{d}\theta \right], \end{aligned}$$

and argue that you can solve for the average-risk-minimizing decision rule "sample-wise":

$$d_0(\mathbf{y}) = \arg\min_{a \in \mathcal{A}} \int L(\theta, d(\mathbf{y})) \pi(\theta|\mathbf{y}) \, \mathrm{d}\theta,$$

where

$$\pi(\theta|\mathbf{y}) = \frac{f(\mathbf{y}|\theta) \pi(\theta)}{\int f(\mathbf{y}|\theta) \pi(\theta) \, \mathrm{d}t}.$$

[e] Compute the posterior expected loss from making the loan. From your answer deduce the (average risk) optimal decision rule (i.e., the "Bayes' rule").

[f] Show that the likelihood can be written as

$$f(\mathbf{y}|\theta) = \theta^{s_N} (1-\theta)^{N-s_N}$$

with $S_N = \sum_{i=1}^N Y_i$. Assume further a prior on θ of

$$\theta \sim \mathrm{Beta}(\alpha_1, \alpha_2)$$

and hence show that

$$\theta|\,\mathbf{z} \sim \text{Beta}\left(S+\alpha_1, N-S+\alpha_2\right).$$

What is the average risk optimal decision rule under this prior? Why does this decision rule only depend on the data through $S_N = s_N$?

[2] **Incoherence is costly**

Spanky is an avid player of Dungeon's and Dragons. When his character Ademaro, a 325 year old forest elf, is visiting the Yawning Portal he sees the following betting rates posted at the bar for an upcoming unicorn race. There are three unicorns racing: Sparklehoof, Shimmering Sea and Manificent Mountain Mare. The posted rates are pay α to win 1 electrum piece if Sparklehoof wins (event A), pay β to win 1 electrum piece if Shimmering Sea wins (event B) and pay γ to win 1 electrum piece if *either* Sparklehoof or Shimmering Sea win. Assuming that the Yawning Portal will accept any bet consistent with the advertised rate, construct three bets such that regardless of the outcome the Yawning Portal will lose money when $\gamma < \alpha + \beta$. You may also assume that tie races are not possible. Use the follow payoff table to describe the properties of your bets:

Yawning Portal Payoffs

	Payoff (Bet 1)	Payoff (Bet 2)	Payoff (Bet 3)	Total Payoff
$A \wedge \neg B$				
$\neg A \wedge B$				
$\neg A \wedge \neg B$				

What probability rule is the Yawning Portal violating?

[3] **Life is better when you maximize expected utility**

You are deciding whether to incorporate regular aerobic exercise into you life (specifically 45 minutes of vigorous training, three times a week). Let A denote the event "I exercise" and B the event "I die prior to my expected retirement at age 65". Your lifetime utility payoffs are

	B	$\neg B$
A	-10	90
$\neg A$	0	100

[a] Your friend, Not-a-Careful-Thinker says, "since your happier when you don't exercise regardless of how long your live you shouldn't exercise. It is a no-brainer." Your other friend, Blaise, says "dude, maximize expected utility". He says according to the best medical evidence

$$\Pr(B|A) = 0.10, \quad \Pr(B|\neg A) = 0.40.$$

Whose advice do you follow, why?

[b] Whose advice would you follow if, instead,

$$\Pr(B|A) = \Pr(B|\neg A) = 0.25.$$

If you answer to part [a] and [b] differ, then explain why.

[4] **Linear regression**

Let $X \sim \text{Uniform}[-1,1]$ and assume that

$$Y = -\frac{2}{3} + X^2 + V, \quad V|X \sim \mathcal{N}\left(0, \sigma^2\right) \tag{1}$$

[a] Calculate $\mathbb{E}[Y|X]$

[b] Calculate $\mathbb{E}[X^2]$ and $\mathbb{V}(X)$

[c] Calculate $\mathbb{E}[Y]$

[d] Calculate $\mathbb{C}(X,Y)$ and also the coefficient on X in $\mathbb{E}^*[Y|X]$.

[e] Let $U = Y - \mathbb{E}^*[Y|X]$. Show that $\mathbb{C}(U,X) = 0$. Give an intuitive explanation for this result.

[f] Find a function $g(X)$ such that that $\mathbb{C}(U, g(X)) = \mathbb{V}(X^2)$. Give an intuitive explanation for your answer.

[g] Describe how your answers in (d) to (f) would change if (1) held but now $X \sim \text{Uniform}[0,2]$. You may find it helpful to sketch a figure. [5] **More linear regression**

You are given a random sample from South Africa in the late 1980s. Each record in this sample includes, Y, an individual's log income at age 40, X the log permanent income of their parents, and D a binary indicator equaling 1 if the respondent is White and zero if they are Black. Let the best linear predictor of own log income at age forty given parents' log permanent income and own race be

$$\mathbb{E}^*[Y|X,D] = \alpha_0 + \beta_0 X + \gamma_0 D.$$

[a] Let $Q = \Pr(D = 1)$, assume that $\mathbb{V}(X|D = 1) = \mathbb{V}(X|D = 0) = \sigma^2$ and recall the analysis of variance formula $\mathbb{V}(X) = \mathbb{V}(\mathbb{E}[X|D]) + \mathbb{E}[\mathbb{V}(X|D)]$. Show that

$$\mathbb{V}(X) = Q(1-Q)\{\mathbb{E}[X|D=1] - \mathbb{E}[X|D=0]\}^2 + \sigma^2.$$

[b] Let $\mathbb{E}^*[D|X] = \kappa + \lambda X$. Show that

$$\lambda = \frac{Q(1-Q)\{\mathbb{E}[X|D=1] - \mathbb{E}[X|D=0]\}}{Q(1-Q)\{\mathbb{E}[X|D=1] - \mathbb{E}[X|D=0]\}^2 + \sigma^2}.$$

[c] Assume that $\beta_0 = 0$. Show that in this case $\gamma_0 = \mathbb{E}[Y|D=1] - \mathbb{E}[Y|D=0]$.

[d] Let $\mathbb{E}^*[Y|X] = a + bX$. Maintaining the assumption that $\beta_0 = 0$ show that

$$b = \frac{Q(1-Q)\{\mathbb{E}[Y|D=1] - \mathbb{E}[Y|D=0]\}\{\mathbb{E}[X|D=1] - \mathbb{E}[X|D=0]\}}{Q(1-Q)\{\mathbb{E}[X|D=1] - \mathbb{E}[X|D=0]\}^2 + \sigma^2}.$$

[e] Let $Q(1-Q) = 1/10$, $\sigma^2 = 3/10$ and $\mathbb{E}[Y|D=1] - \mathbb{E}[Y|D=0] = \mathbb{E}[X|D=1] - \mathbb{E}[X|D=0] = 3$. Provide a numerical value for $\mathbb{V}(X)$ and b.

[f] On the basis of β_0 a member of the National Party argues that South Africa is a highly mobile society. One the basis of b a member of the African National Congress argues that it is a highly immobile one. Comment on the relative merits of these two assertions.

[6] **Frisch-Waugh Theorem Redux**

Let Y be a scalar random variable, X a K vector of covariates (which includes a constant), and W a vector of additional covariates (which excludes a constant). Consider the long (linear) regression

$$\mathbb{E}^*[Y|W,X] = X'\beta_0 + W'\gamma_0. \tag{2}$$

Next define the short and auxiliary regressions

$$\mathbb{E}^*[Y|X] = X'b_0 \tag{3}$$
$$\mathbb{E}^*[W|X] = \Pi_0 X. \tag{4}$$

[a] Let $V = W - \mathbb{E}^*[W|X]$ be the projection error associated with the auxiliary regression. Show that

$$\begin{aligned}\mathbb{E}^*[Y|V,X] &= \mathbb{E}^*[Y|X] + \mathbb{E}^*[Y|1,V] - \mathbb{E}[Y] \\ &= \mathbb{E}^*[Y|X] + \mathbb{E}^*[Y|V]\end{aligned}$$

where $\mathbb{E}^*[Y|\,1, V]$ denotes the linear regression of Y onto a constant and V, while $\mathbb{E}^*[Y|\,V]$ denotes the corresponding regression without a constant (HINT: Observe that $\mathbb{C}(X, V) = 0$).

[b] Next show that $\mathbb{E}^*[Y|\,V, X] = \mathbb{E}^*[Y|\,W, X]$ and hence that the coefficient on V in $\mathbb{E}^*[Y|\,V, X]$ coincides with that on W in $\mathbb{E}^*[Y|\,W, X]$.

[c] Let $U = Y - \mathbb{E}^*[Y|\,X]$ be the projection error associated with the short regression. Derive the coefficient on V in the linear regression of U onto V (excluding a constant).

[d] Discuss the possible practical value of the results shown in [b] and [c] above.