Ec143, Spring 2026

Professor Bryan Graham

Problem Set 2 (Pencil & Paper Part)

Due: February 25th, 2026

Problem sets are due at 5PM in the GSIs mailbox. You may work in groups, but each student should turn in their own problem set write-up. Please list study partners on your homework and reference any AI that was used.

This problem set builds on the material on the iteration and decomposition of linear regression discussed in lecture. You will be asked to connect the empirical part of this problem set to this theoretical material.

[1] [LINEAR REGRESSION] Let $(X', Y) \sim F_{X,Y}$ with Xa $K \times 1$ vectors of regressors (excluding a constant) and Y a scalar outcome variable of interest. Define the linear regression of Y onto a constant and X to equal

$$\begin{aligned}\mathbb{E}^*[Y|1, X] &= \alpha + X'\beta \\ &= R'\eta,\end{aligned}$$

with the second equality following from the definitions $R = (1, X')'$ and $\eta = (\alpha, \beta')'$. The coefficient vector η is given by

$$\eta = \mathbb{E}[RR']^{-1} \times \mathbb{E}[RY].$$

[a] [INTERCEPT-SLOPE] Use the partitioned inverse formula, as well as the definitions of variance, $\mathbb{V}(X) = \mathbb{E}[XX'] - \mathbb{E}[X]\mathbb{E}[X]'$, and covariance, $\mathbb{C}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$, to show that,

$$\eta = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \mathbb{E}[Y] - \mathbb{E}[X]'\beta \\ \mathbb{V}(X)^{-1}\mathbb{C}(X, Y) \end{pmatrix}. \tag{1}$$

"The linear regression always passes through the point $(\mathbb{E}[X], \mathbb{E}[Y])$". True or False? Explain **[2-3 sentences].**

[b] [ORTHOGONAL PROJECTION] Let $U = Y - \alpha - X'\beta$. Using only (1) show that

$$\begin{aligned}\mathbb{E}[U] &= 0 \\ \mathbb{E}[XU] &= 0.\end{aligned} \tag{2}$$

"The linear regression prediction error is orthogonal to any linear function of X". True or False? Explain **[2-3 sentences].**

[2] [LONG, SHORT AND AUXILIARY REGRESSION] Let $(W', X', Y) \sim F_{W,X,Y}$ with W and X respectively $J \times 1$ and $K \times 1$ vectors of regressors (neither including a constant) and Y a scalar outcome variable of interest. The *long regression* equals,

$$\mathbb{E}^*[Y|1, X, W] = \alpha + X'\beta + W'\gamma. \tag{3}$$

Next define two *auxiliary regression* equations

$$\mathbb{E}^*[W|1,X] = \underset{Jx1}{\pi} + \underset{J\times K}{\Pi} X \tag{4}$$

$$\mathbb{E}^*[X|1,W] = \underset{K\times 1}{\lambda} + \underset{K\times J}{\Lambda} W. \tag{5}$$

Finally define the *short regression*

$$\mathbb{E}^*[Y|1,X] = a + X'b. \tag{6}$$

[a] [Law of Iterated Linear Predictors] Show that

$$\begin{aligned} a &= \alpha + \pi'\gamma \\ b &= \beta + \Pi'\gamma, \end{aligned} \tag{7}$$

and further/hence that

$$\mathbb{E}^*[Y|1,X] = \mathbb{E}^*[\mathbb{E}^*[Y|1,X,W]|1,X]. \tag{8}$$

Provide written descriptions of equations (7) and (8). Explain why (7) is sometimes called the "omitted variable bias" formula. **[6-8 sentences]**

[b] [Residual Regression] Let $V = X - \mathbb{E}^*[X|1,W]$. Show that

$$\mathbb{E}^*[Y|1,V] = \mathbb{E}[Y] + \mathbb{E}^*[Y|V].$$

"The residual regression passes through the point $(0, \mathbb{E}[Y])$". True or False? Explain **[2-3 sentences].**

[c] [Hájek Projection] Observe that, by the results of question 1 above, that $\mathbb{C}(V,W) = 0$. A *Hájek Projection* gives the following decomposition

$$\mathbb{E}^*[Y|1,V,W] = \mathbb{E}^*[Y|1,V] + \mathbb{E}^*[Y|1,W] - \mathbb{E}[Y].$$

You may use the expression above without proof for the balance of this problem set. Use the Hájek Projection, as well as your answer from [b] above, to show that

$$\mathbb{E}^*[Y|1,V,W] = \mathbb{E}^*[Y|V] + \mathbb{E}^*[Y|1,W].$$

[d] [Frisch-Waugh, Part 1] Argue, in words **[3-4 sentences]**, that

$$\mathbb{E}^*[Y|1,X,W] = \mathbb{E}^*[Y|1,V,W].$$

[e] [Frisch-Waugh, Part 2] Use (5), as well as your results from parts [a], [c] and [d] above, to show that

$$\begin{aligned} \mathbb{E}^*[Y|1,X,W] &= V'\beta + [\alpha + \lambda'\beta + W'(\gamma + \Lambda'\beta)] \\ &= \mathbb{E}^*[Y|V] + \mathbb{E}^*[Y|1,W]. \end{aligned} \tag{9}$$

[f] Provide a intuitive argument **[3-4 sentences]** for why the linear (residual) regression of Y onto V (w/o a constant) returns the same slope coefficient as that associated with X in the long regression (equation

(3) above).

1 Solution sketches

Equation (1) follows from:

$$
\begin{aligned}
\eta &= \mathbb{E}\left[RR'\right]^{-1} \times \mathbb{E}\left[RY\right] \\
&= \begin{pmatrix} 1 & \mathbb{E}\left[X'\right] \\ \mathbb{E}\left[X\right] & \mathbb{E}\left[XX'\right] \end{pmatrix}^{-1} \times \begin{pmatrix} \mathbb{E}\left[Y\right] \\ \mathbb{E}\left[XY\right] \end{pmatrix} \\
&= \begin{pmatrix} 1 + \frac{1}{1}\mathbb{E}\left[X\right]' \left(\mathbb{E}\left[XX'\right] - \mathbb{E}\left[X\right]\frac{1}{1}\mathbb{E}\left[X\right]'\right)^{-1} \mathbb{E}\left[X\right]\frac{1}{1} & -\frac{1}{1}\mathbb{E}\left[X\right]' \left(\mathbb{E}\left[XX'\right] - \mathbb{E}\left[X\right]\frac{1}{1}\mathbb{E}\left[X\right]'\right)^{-1} \\ -\left(\mathbb{E}\left[XX'\right] - \mathbb{E}\left[X\right]\frac{1}{1}\mathbb{E}\left[X\right]'\right)^{-1} \mathbb{E}\left[X\right]\frac{1}{1} & \left(\mathbb{E}\left[XX'\right] - \mathbb{E}\left[X\right]\frac{1}{1}\mathbb{E}\left[X\right]'\right)^{-1} \end{pmatrix}^{-1} \times \begin{pmatrix} \mathbb{E}\left[Y\right] \\ \mathbb{E}\left[XY\right] \end{pmatrix} \\
&= \begin{pmatrix} 1 + \mathbb{E}\left[X\right]' \mathbb{V}\left(X\right)^{-1} \mathbb{E}\left[X\right] & -\mathbb{E}\left[X'\right] \mathbb{V}\left(X\right)^{-1} \\ -\mathbb{V}\left(X\right)^{-1} \mathbb{E}\left[X\right] & \mathbb{V}\left(X\right)^{-1} \end{pmatrix} \times \begin{pmatrix} \mathbb{E}\left[Y\right] \\ \mathbb{E}\left[XY\right] \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{E}\left[Y\right] + \mathbb{E}\left[X\right]' \mathbb{V}\left(X\right)^{-1} \mathbb{E}\left[X\right] \mathbb{E}\left[Y\right] - \mathbb{E}\left[X'\right] \mathbb{V}\left(X\right)^{-1} \mathbb{E}\left[XY\right] \\ -\mathbb{V}\left(X\right)^{-1} \mathbb{E}\left[X\right] \mathbb{E}\left[Y\right] + \mathbb{V}\left(X\right)^{-1} \mathbb{E}\left[XY\right] \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{E}\left[Y\right] - \mathbb{E}\left[X'\right] \mathbb{V}\left(X\right)^{-1} \left(\mathbb{E}\left[XY\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right]\right) \\ \mathbb{V}\left(X\right)^{-1} \left(\mathbb{E}\left[XY\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right]\right) \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{E}\left[Y\right] - \mathbb{E}\left[X'\right] \mathbb{V}\left(X\right)^{-1} \mathbb{C}\left(X,Y\right) \\ \mathbb{V}\left(X\right)^{-1} \mathbb{C}\left(X,Y\right) \end{pmatrix} \\
&= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{E}\left[Y\right] - \mathbb{E}\left[X\right]' \beta \\ \beta \end{pmatrix}.
\end{aligned}
$$

To show equation (2) start from the observation that

$$U = Y - \mathbb{E}\left[Y\right] - \left(X - \mathbb{E}\left[X\right]\right)' \beta,$$

and hence that

$$
\begin{aligned}
\mathbb{E}\left[XU\right] &= \mathbb{E}\left[XY\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right] - \left(\mathbb{E}\left[XX'\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[X\right]'\right)\beta \\
&= \mathbb{C}\left(X,Y\right) - \mathbb{V}\left(X\right)\beta \\
&= \mathbb{C}\left(X,Y\right) - \mathbb{V}\left(X\right)\mathbb{V}\left(X\right)^{-1}\mathbb{C}\left(X,Y\right) \\
&= \mathbb{C}\left(X,Y\right) - \mathbb{C}\left(X,Y\right) \\
&= 0.
\end{aligned}
$$

To show (7) note that:

$$\begin{aligned}\mathbb{E}^*\left[\mathbb{E}^*\left[Y|\,1,X,W\right]|\,1,X\right] &= \alpha + X'\beta + \mathbb{E}^*\left[W|\,1,X\right]'\gamma \\ &= \alpha + X'\beta + \pi'\gamma + X'\Pi'\gamma \\ &= \alpha + \pi'\gamma + X'\left(\beta + \Pi'\gamma\right) \\ &= a + X'b.\end{aligned}$$

Equation follows from

$$\begin{aligned}\mathbb{E}^*\left[Y|\,1,X,W\right] &= \alpha + X'\beta + W'\gamma \\ &= \alpha + \left(X - \mathbb{E}^*\left[X|\,1,W\right]\right)'\beta + \mathbb{E}^*\left[X|\,1,W\right]'\beta + W'\gamma \\ &= V'\beta + \lambda'\beta + W'\left(\gamma + \Lambda'\beta\right) \\ &= \mathbb{E}^*\left[Y|\,V\right] + \mathbb{E}^*\left[Y|\,1,W\right].\end{aligned}$$