Ec241a, Spring 2023

Professor Bryan Graham

Problem Set 1

Due: March 6, 2023

Problem sets are due at 5PM in my mailbox. You may work in groups (indeed I encourage you to all work together), but each student should turn in their own write-up.

1 Mixed Proportional Hazards (MPH)

Helpful reading: Kiefer (1988), Fitzenberger and Wilke (2006), Ridder and Woutersen (2003)

Let $Y \in \mathbb{Y} = [0, \infty)$ denote a random draw from the target population of durations, $X \in \mathbb{X} \subset \mathbb{R}^K$ a corresponding $K \times 1$ vector of observed covariates and A latent heterogeneity (assumed independent of X). Available to the econometrician is a simple random sample with N copies of $Z_i = (X_i', Y_i)'$. We posit a conditional hazard function of

$$\lambda\left(y|\, x, a; \theta\right) = \lambda_0\left(y; \alpha\right) e^{x'\beta + a}. \tag{1}$$

Here $\lambda_0\left(y; \alpha\right)$ is a a known parametric baseline hazard function indexed by the unknown parameter α. The $e^{x'\beta + a}$ term parameterizes the effect of changes in X on the hazard rate. Our goal is to recover consistent and efficient estimates of $\theta = (\alpha, \beta')'$. You may assume that θ equals its population value (i.e., $\theta = \theta_0$) in what follows unless noted otherwise.

[a] Let $\Lambda_0\left(y; \alpha\right) = \int_0^y \lambda_0\left(t; \alpha\right) \mathrm{d}t$ denote the integrated baseline hazard. Show that the conditional survival function equals

$$\Pr\left(Y > y|\, X = x, A = a\right) = S\left(y|\, x, a; \theta\right) = \exp\left(-\Lambda_0\left(y; \alpha\right) e^{x'\beta + a}\right). \tag{2}$$

[b] Let $y_\tau\left(x, a\right) = Q_{Y|X}\left(\tau|\, X = x, A = a\right)$ be the τ^{th} conditional quantile of Y given $X = x$ and $A = a$. Use the Implicit Function Theorem (as applied to $S\left(y_\tau\left(x, a\right)|\, x, a; \theta\right) = (1 - \tau)$) to show that

$$\begin{aligned} \frac{\partial y_\tau\left(x, a\right)}{\partial x} &= -\left\{ \left.\frac{\partial S\left(y|\, x, a\right)}{\partial y}\right|_{y = y_\tau(x,a)} \right\}^{-1} \left.\frac{\partial S\left(y|\, x, a\right)}{\partial x}\right|_{y = y_\tau(x,a)} \\ &= \left\{ \frac{\Lambda_0\left(y_\tau\left(x, a\right); \alpha_0\right)}{\lambda_0\left(y_\tau\left(x, a\right); \alpha_0\right)} \right\} \beta_0. \end{aligned}$$

Use your result to comment on any testable restrictions the MPH places on the conditional distribution of Y given X and A (and/or X alone).

[c] Let $\pi\left(a\right)$ denote the density function of A at $A = a$. The survival function conditional on $X = x$ alone is

$$S\left(y|\, x\right) = \int S\left(y|\, x, a\right) \pi_0\left(a\right) \mathrm{d}a. \tag{3}$$

We will call (3) the observed survival function.

[i] Consider the distribution of latent heterogeneity A in a subpopulation that has survived to y (i.e.,

$Y > y$) and is homogenous in $X = x$. Show that the density of A at $A = a$ in this subpopulation is

$$h(a; x, y) = \frac{S(y|x, a)\,\pi_0(a)}{\int S(y|x, a)\,\pi_0(a)\,\mathrm{d}a}.$$

[ii] Use your answer in part [i] above to show that the observe conditional hazard function is

$$\lambda(y|x; \theta) = \lambda_0(y; \alpha)\, e^{x'\beta}\mathbb{E}\left[e^A \middle| Y \geq y, x\right]. \tag{4}$$

(Hint: use (3) to derive $\lambda(y|x;\theta)$).

Now consider the hazard function relevant for a random draw from the population of units with $X = x$. This *average structural hazard* function is

$$\bar{\lambda}(y|x; \theta) = \lambda_0(y; \alpha)\, e^{x'\beta}\mathbb{E}\left[e^A\right]. \tag{5}$$

Argue that (a) $\mathbb{E}\left[e^A \middle| Y \geq y, x\right]$ is decreasing in y, and hence that (b) the observed hazard function is "rotated to the right" relative to the average structural one (5). Assuming that the MPH specification is correct, argue that a researcher who ignores unobserved heterogeneity might conclude that the baseline hazard implies (more) negative duration dependence than the actual population one.

[c] Define

$$\rho(z; \theta) = \Lambda_0(y; \alpha)\exp(x'\beta). \tag{6}$$

Use (2) and monotonicity of the integrated baseline hazard function to show that

$$\Pr(\rho(Z; \theta) > r|\, X, A) = \exp\left(-re^A\right).$$

Show that this implies that conditional on A, $\rho(Z;\theta)$ is an exponential random variable with rate parameter e^A and independent of X. Show that conditional on X alone $\rho(Z;\theta)$ is a random draw from a mixture of exponentials and remains independent of X.

[d] Recall that for W a exponential random variable with rate λ we have $\mathbb{E}\left[W^l\right] = \frac{l!}{\lambda^l}$ for $l = 1, 2, 3, \ldots$. Use this result, your analysis in part [c] above, and the law of iterated expectations to show that

$$\mathbb{E}[\psi(Z; \theta)] = 0$$

where

$$\psi(Z; \theta) = (X - \mathbb{E}[X]) \otimes \rho(Z; \theta)$$

with $\mathbb{E}[X]$ assumed known.

[e] Assume that $\dim(X) = K = 1$ and further that $\alpha = \alpha_0$ is known to the econometrician. Show that the method-of-moments estimate for β, based on your answer in part [d] above has an asymptotic variance of

$$\frac{2}{\mathbb{V}(X)}\frac{\mathbb{E}\left[e^{-2a}\right]}{\left(\mathbb{E}\left[e^{-a}\right]\right)^2}.$$

[f] Assume that Y is conditionally Weibull such that $\Lambda_0(y; \alpha) = y^\alpha$ and that the econometrician

knowns $\beta = \beta_0$ but seeks to learn α. Show that the method-of-moments estimate for α based upon the moment function in part [d] above has an asymptotic variance of

$$\frac{\alpha^2}{\beta^2}\frac{2}{\mathbb{V}(X)}\frac{\mathbb{E}\left[e^{-2a}\right]}{\left(\mathbb{E}\left[e^{-a}\right]\right)^2}.$$

What happens if $\beta \approx 0$? Comment with reference to Hahn (1994).

2 Semiparametric efficiency bound (SEB)

Helpful reading: Newey (1990), Hahn (1994) and Ridder and Woutersen (2003).

This problem maintains the basic setup and notation established in Problem 1 above.

[a] Observe that $y = \Lambda_0^{-1}\left(\rho(z;\theta)\exp(-x'\beta), \alpha\right)$. Using the change-of-variables formula to show that

$$\begin{aligned} f_{\rho|X}\left(\rho(z;\theta)|\, x\right) &= f\left(y|\, x;\theta,\pi\right)\frac{1}{\lambda(y;\alpha)\exp(x'\beta)} \\ &= f_\rho\left(\rho(z;\theta)\right) \end{aligned}$$

where $f(y|\, x;\theta,\pi) = \int f(y|\, x, a;\theta)\,\pi(a)\,\mathrm{d}a$ and the second equality follows from the independence of $\rho(Z;\theta)$ and X established earlier.

[b] Let $\pi(a;\eta)$ denote a parametric submodel for the nuisance heterogeneity distribution. Using the result in part [a] we have a sub-model log-likelihood function of

$$l(\theta,\eta;Z) = \ln\lambda(Y;\alpha) + X'\beta + \ln f_\rho\left(\rho(Z;\theta);\eta\right).$$

Can your relate your sub-model log-likelihood to the formulation used by Hahn (1994)?

[b] Differentiate with respect to $\theta = (\alpha,\beta')'$ and η to derive the sub-model scores. Conjecture the form of the nuisance tangent set based on your calculations.

[c] Using your results from part [b] show that the efficient score for θ is

$$\psi^{\mathrm{eff}}(Z;\theta) = \left\{ \begin{array}{c} \left(\frac{\frac{\partial\lambda_0(y,\alpha)}{\partial\alpha}}{\lambda_0(y,\alpha)} - \mathbb{E}\left[\left.\frac{\frac{\partial\lambda_0(y,\alpha)}{\partial\alpha}}{\lambda_0(y,\alpha)}\right|\rho(Z;\theta)\right]\right) + \left(\frac{\frac{\partial\Lambda_0(y,\alpha)}{\partial\alpha}}{\Lambda_0(y,\alpha)} - \mathbb{E}\left[\left.\frac{\frac{\partial\Lambda_0(y,\alpha)}{\partial\alpha}}{\Lambda_0(y,\alpha)}\right|\rho(Z;\theta)\right]\right)\frac{f'_\rho(\rho(z;\theta))}{f_\rho(\rho(z;\theta))}\rho(z;\theta) \\ (X-\mathbb{E}[X]) + (X-\mathbb{E}[X])\frac{f'_\rho(\rho(Z;\theta))}{f_\rho(\rho(Z;\theta))}\rho(Z;\theta) \end{array} \right\}.$$

You will need to use independence of X and $\rho(Z;\theta)$ to derive the expression above.

[d] As in Hahn (1994) show that $\theta'\psi^{\mathrm{eff}}(Z;\theta) = 0$ when Y is conditionally Weibull. Discuss the implications of this result for estimation.

[e] Let $h(\rho(z;\theta))$ be a possibly misspecified model for the ratio $\frac{f'_\rho(\rho(Z;\theta))}{f_\rho(\rho(Z;\theta))}$, show that

$$\psi^{\mathrm{dr}}(Z;\theta) = \left\{ \begin{array}{c} \left(\frac{\frac{\partial\lambda_0(y,\alpha)}{\partial\alpha}}{\lambda_0(y,\alpha)} - \mathbb{E}\left[\left.\frac{\frac{\partial\lambda_0(y,\alpha)}{\partial\alpha}}{\lambda_0(y,\alpha)}\right|\rho(Z;\theta)\right]\right) + \left(\frac{\frac{\partial\Lambda_0(y,\alpha)}{\partial\alpha}}{\Lambda_0(y,\alpha)} - \mathbb{E}\left[\left.\frac{\frac{\partial\Lambda_0(y,\alpha)}{\partial\alpha}}{\Lambda_0(y,\alpha)}\right|\rho(Z;\theta)\right]\right)h(\rho(z;\theta))\,\rho(z;\theta) \\ (X-\mathbb{E}[X]) + (X-\mathbb{E}[X])\,h(\rho(z;\theta))\,\rho(Z;\theta) \end{array} \right\},$$

is a valid moment function for θ.

[f] Assume that $e^A \sim \text{Gamma}(\eta_1, \eta_2)$. Derive an expression for $\frac{f'_\rho(\rho(Z;\theta))}{f_\rho(\rho(Z;\theta))}$ in terms of $\rho(Z;\theta)$, η_1 and η_2. Use your expression to construct a locally efficient estimate of θ (assume that the conditional distribution of Y is such that the SEB is non-zero).

Econometrics field exam practice question

Let $\{(W_i, X_i, Y_i)\}_{i=1}^N$ be a simple random sample drawn from a population characterized by (unknown) distribution F_0; Y is a scalar outcome variable of interest, X a vector of policy or treatment variables, and W a vector of additional control variables or confounders. Assume that the conditional distribution of Y_i given W_i and X_i satisfies:

$$Y = X'\beta_0 + h_0(W) + U, \ \mathbb{E}[U|W, X] = 0. \tag{7}$$

Here $\beta_0 \subset \mathbb{R}^K$ is the finite dimensional parameter of interest and $h_0(W)$ an unknown function mapping from a subset of $W \in \mathbb{W} \subset \mathbb{R}^{\dim(W)}$ into $\mathcal{H} \subset \mathbb{R}$. You may assume that all the expressions which appear below are well-defined (i.e., don't worry about regularity conditions in what follows).

[a] Assume that the researcher knows that $h_0(W) = k(W)'\delta_0$ for some known $J \times 1$ vector of basis functions $k(W)$ (which includes a constant) and unknown parameter δ_0. Show that the coefficient on X in the least squares fit of Y onto X and $k(W)$ has an asymptotic limiting distribution of

$$\sqrt{N}\left(\hat{\beta}_{\text{OLS}} - \beta_0\right) \xrightarrow{D} \mathcal{N}\left(0, \sigma^2 \mathbb{E}\left[(X - \Pi_0 k(W))(X - \Pi_0 k(W))'\right]^{-1}\right), \tag{8}$$

with $\Pi_0 = \mathbb{E}\left[Xk(W)'\right]\mathbb{E}\left[k(W)k(W)'\right]^{-1}$ the $K \times J$ matrix of projection coefficients associated with the multivariate regression of X onto $k(W)$. *Note this result maintains the additional assumption of homoscedasticity.* For your reference the partitioned matrix formula is:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B\left(D - CA^{-1}B\right)CA^{-1} & -A^{-1}B\left(D - CA^{-1}B\right) \\ -\left(D - CA^{-1}B\right)CA^{-1} & \left(D - CA^{-1}B\right) \end{bmatrix}.$$

[b] Let

$$e_0(w) = \mathbb{E}[X|W = w]. \tag{9}$$

be a vector of conditional expectation functions (CEFs) of the K policy variables given the confounders. Assume that $e_0(W)$ is known. Show that following unconditional moment identifies β_0:

$$\mathbb{E}[m(Z, \beta_0, e_0(W))] = 0. \tag{10}$$

with $m(Z, \beta_0, e_0(W)) = (Y - X'\beta_0)(X - e_0(W))$. Let $\hat{\beta}_E$ be the method-of-moments estimate associated with this restriction. Is this estimator more or less efficient than the OLS estimator described in part (a)?

[c] Augment the identifying "E-moment" (10) from part (b) with the auxiliary conditional moment

$$\mathbb{E}[(X - e_0(W))|W] = 0. \tag{11}$$

Argue that the efficient estimator based on (10) and (11) coincides with the just-identified moment based upon the unconditional moment restriction

$$\psi\left(Z,\beta_0,e_0\left(W\right),h_0\left(W\right)\right)=m\left(Z,\beta_0,e_0\left(W\right)\right)-\mathbb{E}^*\left[\left.m\left(Z,\beta_0,e_0\left(W\right)\right)\right|\left(X-e_0\left(W\right)\right);W\right].$$

Here $\mathbb{E}^*\left[\left.Y\right|X;W\right]$ denotes the linear regression of Y onto X within a subpopulation homogenous in W (i.e., a conditional linear predictor). You may assume that all "unknowns" except β_0 are known. You may not assume that "unknown unknowns are known" only that "known unknowns are known".

Further show that

$$\psi\left(Z,\beta_0,e_0\left(W\right),h_0\left(W\right)\right)=\left(Y-X'\beta_0-h_0\left(W\right)\right)\left(X-e_0\left(W\right)\right). \tag{12}$$

[d] Replace $h_0\left(w\right)$ in (12) with some arbitrary function of w. Is (12) still mean zero? Now replace $e_0\left(w\right)$ in (12) with some arbitrary function of w. Is (12) still mean zero? Comment on the implications of your answer for estimation.

[e] Write, in the language of your choice, a short Haiku inspired by questions (a) to (d) above.

References

Fitzenberger, B. and Wilke, R. A. (2006). *Modern Econometric Analysis*, chapter Using quantile regression for duration analysis, pages 103 – 118. Springer-Verlag, Berlin.

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Newey, W. K. (1990). Semiparametric efficiency bounds. *Journal of Applied Econometrics*, 5(2):99 – 135.

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