

Problem sets are due by 5PM EST. You may work in groups (indeed I encourage you to all work together), but each student should turn in their own write-up.

Helpful reading: Newey (1990), Brown and Newey (1998) and Graham (2011). For an alternative, and highly elegant approach, see Chamberlain (1987).

Let $Z = (X', Y)'$ be a vector of modeling variables and $\{Z_i\}_{i=1}^\infty$ an independent and identically distribution random sequence drawn from unknown distribution F_0 . The sole prior restriction on F_0 is that, for some known function $\psi(Z, \beta)$ and unknown parameter $\beta_0 \in \mathbb{B} \subset \mathbb{R}^K$

$$\mathbb{E}[\psi(Z, \beta_0)] = 0.$$

Assume that $\dim(\psi(Z, \beta_0)) = J > \dim(\beta_0) = K$.

Let $a(Z, \beta)$ be some known function of Z and β . The goal is to efficiently estimate the mean

$$\mu_0 = \mathbb{E}[a(Z, \beta_0)].$$

1. Let $X = (\tilde{X}', W)'$ with W a binary treatment indicator (also partition $\beta = (\tilde{\beta}, \alpha)'$) and assume that

$$\Pr(Y = 1 | X) = \Phi(X' \tilde{\beta} + W \alpha).$$

Let $\delta_l = \mathbb{E}[Y | X \in \mathbb{X}_l]$ for $l = 1, \dots, L$. Assume that these conditional means are known to the econometrician (e.g., from register data). The target estimand is

$$\mu_0 = \mathbb{E}[\Phi(X' \tilde{\beta} + \alpha) - \Phi(X' \tilde{\beta})].$$

Outline a plausible empirical setting to which you could adapt the above components. Show how your setting is accommodated by the general setup outlined above.

2. Calculate the semiparametric variance bound, say $\mathcal{I}(\beta_0)^{-1}$, for β_0 . You may use either the approach outlined in Newey (1990) (and also lecture) or that of Chamberlain (1987).
3. Show that the semiparametric variance bound, say $\mathcal{I}(\mu_0)^{-1}$, for μ_0 is

$$\mathcal{I}(\mu_0)^{-1} = \Sigma_{aa} - \Sigma_{a\psi} \Omega_0^{-1} \Sigma'_{a\psi} + \Sigma_{a\psi} \Omega_0^{-1} \Gamma_0 \mathcal{I}(\beta_0)^{-1} \Gamma_0' \Omega_0^{-1} \Sigma'_{a\psi}$$

where $\Sigma_{aa} = \mathbb{E}[(a(Z, \beta_0) - \mu_0)(a(Z, \beta_0) - \mu_0)']$, $\Sigma_{a\psi} = \mathbb{E}[(a(Z, \beta_0) - \mu_0)\psi(Z, \beta_0)']$, $\Omega_0 = \mathbb{E}[\psi(Z, \beta_0)\psi(Z, \beta_0)']$ and $\Gamma_0 = \mathbb{E}\left[\frac{\partial \psi(Z, \beta_0)}{\partial \beta'}\right]$. As before, you may use either the approach of Chamberlain (1987) or Newey (1990) (in this case I also recommend Brown and Newey (1998)). Interpret each of the three components of the inverse information expression above.

4. Assume that $\hat{\beta}$ is an efficient estimate of β_0 .

(a) Is $\hat{\mu} = \frac{1}{N} \sum_{i=1}^N a(Z_i, \hat{\beta})$ efficient? Explain.

- (b) Let \hat{b}_1 and \hat{b}_ψ be the intercept and slope coefficients associated with the least squares fit of $a(Z_i, \hat{\beta})$ onto a constant and $\psi(Z_i, \hat{\beta})$. Let $\hat{\psi}_N = \frac{1}{N} \sum_{i=1}^N \psi(Z_i, \hat{\beta})$.
- Is $\hat{\mu} = \hat{b}_1 + \hat{\psi}_N' \hat{b}_\psi$ consistent for μ_0 ? Is it efficient? Explain.
 - Is $\hat{\mu} = \hat{b}_1$ consistent for μ_0 ? Is it efficient? Explain.
- (c) Assume that β_0 is known (such that the analyst computes the least squares fit of $a(Z_i, \beta_0)$ onto a constant and $\psi(Z_i, \beta_0)$). Show that, in this case,

$$\hat{b}_1 = \sum_{i=1}^N [a(Z_i, \beta_0) - \Sigma_{a\psi} \Omega_0^{-1} \psi(Z_i, \beta_0)] / N + o_p(N^{-1/2})$$

and hence that \hat{b}_1 is asymptotically uncorrelated with $\psi_N = \frac{1}{N} \sum_{i=1}^N \psi(Z_i, \beta_0)$. Connect this observation to your analysis in part (b) above.

References

- Brown, B. W. and Newey, W. K. (1998). Efficient semiparametric estimation of expectations. *Econometrica*, 66(2):453 – 464.
- Chamberlain, G. (1987). Asymptotic efficiency in estimation with conditional moment restrictions. *Journal of Econometrics*, 34(3):305 – 334.
- Graham, B. S. (2011). Efficiency bounds for missing data models with semiparametric restrictions. *Econometrica*, 79(2):437 – 452.
- Newey, W. K. (1990). Efficient instrumental variables estimation of nonlinear models. *Econometrica* 58, 58(4):809 – 837.