Logic, Probability and Inductive Reasoning

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Probability, in its modern form, began with the celebrated 17th century correspondence between Blaise Pascal (living in Paris) and Pierre de Fermat (living in Toulouse). This correspondence included a discussion of the "Problem of Points", posed to Pascal by his friend, and gentleman gambler, Antoine Gombaud (see Haigh (2012, Ch. 3) and Diaconis & Skyrms (2018, Ch. 1)). Antoine Gombaud, who used the pen name Chevalier De Méré, was a member of the Mersenne Salon. Other members of this Salon included René Descartes, Thomas Hobbes as well as Pascal himself.

To understand the Problem of Points consider a best of seven series between two equally matched players with stakes Y (i.e., the first player to win four games earns prize Y). Imagine an interruption of the series after the completion of just three games, with the first player having won all three rounds and the second player none. What is a fair division of Y between the two players given the outcomes of the completed games?

The Problem of Points provides an early example of calculating a probability by enumerating *equiprobable cases*; it also represents a very early example of what we now would call *expected utility* calculations. Pascal's solution involved counting the scenarios in which each player would win the series if it were, contrary to fact, allowed to continue. Player one, who has three game victories in hand, only needs to win a single ($r = 1$) additional game to take the series, while player two must win all four remaining games ($s = 4$). Imagine all four remaining games were, contrary to fact, played. Player one will take the series as long as player two wins no more than three of the four remaining games. There are

$$\sum_{q=0}^{s-1} \binom{r+s-1}{q} = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} = 1 + 4 + 6 + 4 = 15 \tag{1}$$

different ways in which player two can win 0, 1, 2 or 3 of the remaining games. There are are $2^4 = 16$ possible sequences of outcomes for these games and hence in 15 out of 16 (hypothetical) scenarios player one will win the series. Since a premise of the problem is that the two players are equally skilled, each of the 16 scenarios are equally likely (i.e., equiprobable), consequently the chance of player one winning the series – were it to continue – is 15/16, while player two's chance of winning is just 1/16.

This particular counting problem is conveniently solved using Pascal's Triangle:

$$\boxed{\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & \diamond & 1 & & & & \\
 & & & 1 & \diamond & 2 & \diamond & 1 & & & \\
 & & 1 & \diamond & 3 & \diamond & 3 & \diamond & 1 & & \\
 & 1 & \diamond & 4 & \diamond & 6 & \diamond & 4 & \diamond & 1 & \\
1 & \diamond & 5 & \diamond & 10 & \diamond & 10 & \diamond & 5 & \diamond & 1
\end{array}}$$

The fourth row, counting from zero, records the binomial coefficients appearing in equation (1) above.

The method of calculating probabilities by counting equiprobable outcomes features widely in early discussions of probability (which not coincidently often considered questions inspired by gambling and other games of chance). In the language of modern probability theory, the distribution of probability mass in such examples is distributed uniformly across all possible cases/scenarios (a *discrete uniform* distribution). The probability of a particular event is then calculated by counting the number of cases under which it occurs (relative to those in which it does not).

Having calculated the win probability for each player given continuation of the series, Pascal reasoned that a fair way to divide the stakes in the event of interruption would be to give $\frac{15}{16} \times Y$ to player one, and the balance to player two. Pascal's suggestion corresponds to giving the *expected payoff* associated with continuing the game to each player.

Pascal's analysis exemplifies the *classical* or *objective* view of probability. In this view there exists a finite list of possible *outcomes* and, via symmetry arguments, each one is known to be *a priori* equally likely to occur. The probability attached to a specific *event* then corresponds to the fraction of outcomes under which it occurs. For example the probability of rolling an "even" with a six-sided die is $\frac{1}{2}$; the probability of drawing an Ace from a well-shuffled deck of cards is $\frac{4}{52}$ and so on.

The objective view of probability seems airtight. Surely we can all agree the role of a die is random, with each side equiprobable? In fact many early researchers in probability agreed with David Hume's view that "there is no such thing as chance in the world" (quoted by Diaconis & Skyrms (2018, p. 200)). Consider the role of a die. If the die is rolled in exactly the same way, its outcome is foreordained to be the same each time. Remember, the development of probability theory occurred during a century that culminated with the publication of Newton's *Principia,* with its associated implication of a clockwork universe.

The apparent randomness of the die role comes from the extreme sensitivity of the outcome to small changes in the mechanics of its role.[1] Whether anything is indeed truly random, or whether apparent randomness stems from chaotic dynamics (in the complex systems sense), is an interesting question. My own view is randomness is, indeed, real (consider radioactive decay), but for most practical applications such nuances do not practically matter: we can proceed "as if" the outcome of a die role (and many other "experiments") is random even it is outcome could be "pre-computed" given sufficient information on initial conditions.[2]

1 The problem of induction

In 1687 Isaac Newton published his Universal Law of Gravitation. Aware of Johannes Kepler's laws of planetary motion, themselves formulated by careful study of Tycho Brahe's astronomical measurements, Newton conjectured that Kepler's laws also applied to the moon. Newton used his theory to construct predictions about the magnitude of the moon's centripetal acceleration, predictions he compared with empirical measurements. The concordance of the moon's measured acceleration with his theoretical predictions was remarkable. Confidence in his theory of gravity increased.

Consider the following two propositions:

Proposition A : "Newton's Universal Law of Gravitation is correct."

Proposition B : "The centripetal acceleration of the moon is accurately predicted by Newton's Universal Law of Gravitation."

Newton and his contemporaries reasoned **inductively** as follows:

> B is true, therefore A is more likely to be true as well.

This is a precarious argument, although it strikes most of us as reasonable. Learning that proposition B is true, does not decisively *prove* the truth of A. Perhaps the concordance

[1] To demonstrate this basic point empirically, Persi Diaconis – a Professor of Statistics at Stanford University – collaborated with the Physics department there on the construction of a coin-flipping machine. This machine was able to repeatedly flip a coin and have it land – repeatedly – heads (or tails). As with die rolling, the apparent randomness of the outcome of a coin flip stems from the sensitivity of the outcome to small changes in the flipping procedure.

[2] John Arbuthnot, in his introduction to his translation of Christiaan Huygen's probability treatise *De Ratiociniis in Ludo Aleae*, asserts that "Chance" is "nothing by but the want of art", meaning that we simply lack the information (the "want of art") required to compute what is objectively a deterministic outcome (as quoted by Diaconis & Skyrms (2018, p. 10)).

between theory and data was due to chance and that, for example, the orbital mechanics of other celestial bodies do not follow Newton's law? We are not claiming certitude. How can we describe and quantify our uncertainty about the "truthiness" of Newton's theory (Proposition A) after witnessing its astounding predictive power (Proposition B)? It turns out that probability helps.

Consider a different type of argument:

1. Premises:
   (a) If A is true, then B is true;
   (b) A is true;
2. Conclusion: therefore B is true.

This is correct example of propositional logic. If the premises of the argument are correct, the stated conclusion logically follows. Newton's law purports to be universal, hence if it is true, then it must explain the orbital mechanics of the moon (Premise 1.a: If A is true, then B is true). If A is true (Premise 1.b), then B must also be true (the Conclusion 2).

Another logically correct argument is:

1. Premises:
   (a) If A is true, then B is true;
   (b) B is false;
2. Conclusion: therefore A is false.

Had Newton discovered a lack of correspondence between the predictions of his theory and the observed orbit of the moon, he could logically conclude that his theory was false (we will assume away the problem of accurate measurement of the moon's orbit in 1688). Logical falsification of Newton's theory is possible (Popper, 1959). We will return to this observation, and its implications for scientific progress, below.

Confirmation, the business Newton was involved in when studying the Moon's orbit, is trickier than falsification. Consider a third argument:

1. Premises:
   (a) If A is true, then B is true;
   (b) B is true;

2. Conclusion: therefore A is true.

This argument is incorrect. It is a fallacy to infer the antecedent A, from the consequent B. In logic this mistake is called "affirming the consequent" or, perhaps more familiarly, confusing necessity and sufficiency (Hacking, 2001).

Newton (and his contemporaries) made the weaker argument

1. Premises:
   (a) If A is true, then B is true;
   (b) B is true;
2. Conclusion: therefore my confidence in the truth of A should increase.

Assessing whether, and by how much, my confidence in A should increase after learning that B is true is an example of the "problem of induction".

Should my confidence in A increase upon learning B? Imagine that I have successfully used Newton's theory of gravitation to successfully predict the orbital paths of many planets, moons, asteroids, dwarf planets and comets in the past. Next imagine a new Kuiper belt dwarf planet is discovered. Am I justified in my confidence that its orbital path will be precisely consonant with Newton's theory? Scottish Enlightenment philosopher, David Hume, in *An Enquiry Concerning Human Understanding*, argued that there is no air-tight argument for reasoning in this way.

> It is impossible, therefore, that any arguments from experience can prove this resemblance of the past to the future, since all these arguments are founded on the supposition of that resemblance (as quoted by Diaconis & Skyrms (2018, p. 192)).

Our confidence that past predictive success is suggestive of future success comes from our *empirical* experience. To justify inductive empirical reasoning by its past empirical success is circular. Many philosophers of science find Hume's argument compelling.

1.1 A brief logic review

Logic is about the truthfulness, or not, of propositions. Probability is about chance and events. Inductive arguments (generally) do not end with certitude. We will use probability to quantify residual uncertainty.

In the balance of this chapter we will try to construct a persuasive system of inductive inference. To so it is helpful to briefly review some basics of logic. Let A and B be two propositions (e.g., A : all econometricians like science fiction, B : Bryan is an econometrican).

Some notation: $\neg A$ denotes negation. That is $\neg A$ denotes "not A" or "A is not true" (e.g., not all econometricians like science fiction). $A \vee B$ denotes disjunction ("OR"); it evaluates to false if, and only if, both A and B are false). $A \wedge B$ denotes conjunction ("AND"); it evaluates to true if, and only if, both A and B are true.

Valid deductive arguments are risk free: if the premises of the argument are true and the rules of logic are followed, the conclusion is true. Consider the following truth table:

A	$\neg A$
1	0
0	1

Here we associate 1 with "true" and 0 with "false". The table indicates that if A is true, then its negation $\neg A$ is false. It is impossible for both A and its negation to be simultaneously true (or simultaneously false). Let $\mathbf{1}(A)$ be an indicator function, equal to 1 if A is true and 0 if A is false. We can represent the information in the truth table above by the equality

$$\mathbf{1}(A) = 1 - \mathbf{1}(\neg A). \tag{2}$$

Next consider the conjunction $A \wedge B$. It evaluates to false whenever either A or B (or both) are false:

A	B	$A \wedge B$
1	1	1
1	0	0
0	1	0
0	0	0

The information in this truth table is contained in the statement

$$\mathbf{1}(A \wedge B) = \mathbf{1}(A)\,\mathbf{1}(B). \tag{3}$$

Finally consider the true value of the disjunction $A \vee B$. It evaluates to true as a long as at least one of A or B evaluates to true:

A	B	$A \vee B$
1	1	1
1	0	1
0	1	1
0	0	0

Agaub using indicator functions we get

$$\begin{aligned}\mathbf{1}(A \vee B) &= 1 - (1 - \mathbf{1}(A))(1 - \mathbf{1}(B)) \\ &= \mathbf{1}(A) + \mathbf{1}(B) - \mathbf{1}(A)\mathbf{1}(B). \end{aligned} \tag{4}$$

1.2 A brief probability review

We will work with the following definition of a probability space and function. See Mitzenmacher & Upfal (2005) for a textbook review.

Definition 1. (Probability Space and Function)

1. Sample space, Ω: the set of all possible outcomes of the random process under consideration;
2. Event space, $\mathcal{A}$: the set of allowable events, with event coinciding with a subset of the outcomes in Ω and
3. Probability function, $\Pr : \mathcal{A} \to \mathbb{R}$ such that
   (a) (Nonnegativity) For any $A \in \mathcal{A}$, $0 \leq \Pr(A) \leq 1$;
   (b) (Normality) $\Pr(\Omega) = 1$and
   (c) (Additivity) for any countable sequence of pairwise disjoint events, $A_1, A_2, \ldots$

$$\Pr\left(\underset{i\geq 1}{\cup} A_i\right) = \sum_{i \geq 1} \Pr(A_i).$$

2 The laws of probability and the coherence of belief

Consider the following two statements:

A : "There is microbial life on Mars."

B : "There is microbial life on Ceres."

We are uncertain about the truth values of these statements. We deal with this uncertainty by assigning probabilities to these statements.

Let $\Pr(A)$ correspond to our personal assessment about the possibility of microbial life on Mars. It is our *ex ante* subjective probability that microbial life exists on Mars. Let $\Pr(B)$ be our personal assessment about whether there is microbial life on Ceres. These probabilities can be interpreted a measures of our "confidences" in A and B. Probability theory provides a useful way to ensure that our confidences are logically consistent or *coherent*.

To develop these arguments will require some basic set notation. Let $\cup$ denote *union*: $A \cup B$ occurs if *either* A *or* B occurs. Let $\cap$ denote *intersection:* $A \cap B$ occurs if both A *and* B occur. Let $A^c = B \setminus A$ be the set of elements in B not in A (i.e,, the set difference or relative complement of A w.r.t B).

With this notation and Definition (1) we can show that

$$\Pr(\neg A) = 1 - \Pr(A). \tag{5}$$

This follows from the equalities $\Pr(\neg A) + \Pr(A) \overset{(3c)}{=} \Pr(\neg A \cup A) \overset{ST}{=} \Pr(\Omega) \overset{3b}{=} 1$. Here the notation "$\overset{(3c)}{=}$" indicates that the equality is an implication of part 3.c of Definition (1) and "$\overset{ST}{=}$" indicates that the equality is an implication of set theory. There is either microbial life on Mars, or there is not. Since one of these two statements is true, the probability of their union is one. Note that (5) is a probability version of our indicator function representation of our negation truth table (i.e., a generalization of equation (2) above).

Coherent beliefs must satisfy (5). One justification for this claim, due to Ramsey (1931) and de Finetti (1992), is that if our beliefs did not satisfy (5), then a bookie could offer us a sequence of fair bets where we would nevertheless lose money with certainty.

For example imagine I simultaneously believed that there is a 50 percent chance that microbial life exists are Mars (i.e., $\Pr(A) = 0.5$) and a 60 percent chance that it does not exist on Mars (i.e., $\Pr(\neg A) = 0.6$). With such beliefs a bookie could sell me a bet that pays \$1 if there is life on Mars for \$0.50. This is fair bet since $\$1 \cdot \Pr(A) = \$1 \cdot 0.50 = \$0.50$: the expected payout from the gamble equals the offered price. The bookie is also able to sell me a bet that pays \$1 if there is no life on either Mars for \$0.60.

If the bookie sells me both bets, then she is *guaranteed* a pay-off of \$0.10:

$$\begin{aligned}
\text{Value-of-Dutch-Book} =& \$0.50 + \$0.60 - \$1 \cdot \Pr(A) + \$1 \cdot \Pr(\neg A) \\
=& \$1.10 - \$1 \cdot [\Pr(A) + \Pr(\neg A)] \\
\overset{(5)}{=}& \$1.10 - \$1 \\
=& \$0.10.
\end{aligned}$$

de Finetti (1992) argued that any beliefs that could be manipulated by a bookie for a certain gain, or equivalently a *sure loss* for the believer, were incoherent. Sure loss bets were termed "Dutch Books" by Ramsey (1931) for unknown reasons. I will generally use the sure loss terminology in what follows. It turns out that belief coherence, and the concordance of those beliefs with axioms of probability (and their implications), coincide. Coherent believers obey the laws of probability. Note coherence of one's beliefs is different from their reasonableness. It is coherent for me to believe their is a 0.9999 chance of life on Mars and only a 0.0001 chance of no life on Mars.

See if you can construct a sure loss bet for these beliefs: $\Pr(A) = 0.5$, $\Pr(\neg A) = 0.4$.

We can also prove a the following *general additivity* property using Definition 1 and some basic set theory:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B). \tag{6}$$

This follows since, for any two events A and B, we have

$$\begin{aligned}
\Pr(A \cup B) &\overset{(ST)}{=} \Pr(A \cup B \setminus A) \\
&\overset{(3c)}{=} \Pr(A) + \Pr(B \setminus A) \\
&\overset{(+0)}{=} \Pr(A) + \Pr(B \setminus A) + \Pr(A \cap B) - \Pr(A \cap B) \\
&\overset{(3c)}{=} \Pr(A) + \Pr((B \setminus A) \cup (A \cap B)) + \Pr(A \cap B) - \Pr(A \cap B) \\
&\overset{(ST)}{=} \Pr(A) + \Pr(B) - \Pr(A \cap B).
\end{aligned}$$

Note that (6) is a probability extension of our indicator function representation of logical disjunction, Equation (4) above. Our (subjective) probability that there is microbial life on either Mars or Ceres, is the sum of the individual probabilities of life on these two celestial bodies minus the probability of life on both of them.

Property (6) extends to more than two events. For example, for any three events A, B and

C, it is easy to show that

$$\Pr(A \cup B \cup C) = \Pr(A) + \Pr(A) + \Pr(C) - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C).$$

In general we have

$$\begin{aligned}\Pr\left(\underset{i \geq 1}{\cup} A_i\right) = & \sum_{i \geq 1} \Pr(A_i) - \sum_{i<j} \Pr(A_i \cap A_j) \\ & + \sum_{i<j<k} \Pr(A_i \cap A_j \cap A_k) - \cdots + (-1)^{L+1} \sum_{i_1 < i_2 < \cdots < i_L} \Pr\left(\overset{L}{\underset{l=1}{\cap}} A_{i_l}\right). \end{aligned} \tag{7}$$

Restriction (6), like negation, also imposes structure on beliefs. Imagine I believe there is a 50 percent chance of life on Mars, a 5 percent chance of life on Ceres and a 70 percent chance that there is life on both Mars and Ceres. That is: $\Pr(A) = 0.5$, $\Pr(B) = 0.05$ and $\Pr(A \cup B) = 0.7$. With such beliefs you, the genius bookie, could offer to:

Buy from me bet that pays \$1 if there is life on Mars for \$0.50.

Buy from a bet that pays \$1 if there is life on Ceres for \$0.05.

Sell me a bet that pays \$1 if there is life on either Mars *or* Ceres for \$0.70.

This series of bets will yield you, the bookie, an expected guaranteed pay-off of $\$0.15 + \$1 \cdot \Pr(A \cap B)$. This is a sure win for you, and a sure loss to me (due to my incoherent beliefs):

$$\begin{aligned}\text{Value-of-Dutch-Book} = & -\$0.50 + \$1 \cdot \Pr(A) - \$0.05 + \$1 \cdot \Pr(B) + \$0.70 - \$1 \cdot \Pr(A \cup B) \\ = & \$0.70 - \$0.50 - \$0.05 + \$1 \cdot [\Pr(A) + \Pr(B) - \Pr(A \cup B)] \\ \overset{(6)}{=} & \$0.15 + \$1 \cdot \Pr(A \cap B).\end{aligned}$$

If my probabilities were coherent, then such a scheme would not be possible.

Probabilism corresponds to the school of thought that a person's beliefs should satisfy the axioms of probability (i.e., Definition 1 and its implications). de Finetti (1992) developed the idea that incoherence of beliefs implies Dutch Book vulnerability; in this sense, incoherent beliefs are not rational. See also Savage (1972). Remember rational beliefs do not need to be reasonable beliefs.

To build some intuition for this approach to subjective belief it is helpful to construct examples of incoherent beliefs, as well as sequences of gambles that involve a sure loss to the incoherent believer (so called Dutch Books).

Some terminology that is helpful for navigating the literature in this area: let E correspond to an event and Y a stake. The *betting quotient* p equals the ratio of the price of the bet to its payout if E occurs. If $p = \Pr(E)$, then the bet is fair to a person who believes event E will occur with probability $\Pr(E)$. A fair bet is one with an expected value of zero. To understand this consider the pay-off matrix:

E	Net Payoff
0 ("False")	$-pY$
1 ("True")	$Y - pY$

The expected payoff associated with bet to a gambler that believes E occurs with probability $\Pr(E)$ is therefore:

$$\begin{aligned}
\text{Expected Payoff} &= [1 - \Pr(E)][-pY] + \Pr(E)[Y - pY] \\
&= \Pr(E)(1-p)Y - [1 - \Pr(E)]pY \\
&= [\Pr(E) - p]Y.
\end{aligned}$$

Thus a bet is:

Fair if its expected value zero (i.e., $p = \Pr(E)$).

Favorable if its expected value positive (i.e., $p < \Pr(E)$).

Disadvantageous if its expected value negative (i.e., $p > \Pr(E)$).

2.1 Conditional probability, Bayes' rule and learning

Definition 2. (Conditional Probability) If $\Pr(B) > 0$, the conditional probability of event A given event B is:

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

The approach we used to assign truth values to conjunctions in logic does not carry over directly to probability. Conjunction in probability requires the probability of one proposition to be evaluated conditional on the other:

$$\Pr(A \cap B) = \Pr(A|B)\Pr(B). \tag{8}$$

If A and B are independent, such that $\Pr(A|B) = \Pr(A)$, then

$$\Pr(A \cap B) = \Pr(A)\Pr(B),$$

which *is* directly analogous to (3) earlier.

When A logically entails B (i.e., if A, then B), then

$$\begin{aligned}\Pr(B) &= \Pr(A \cap B) + \Pr(\neg A \cap B) \\ &= \Pr(A) + \Pr(\neg A \cap B) \\ &\geq \Pr(A).\end{aligned}$$

If Newton's Universal Law of Gravitation is correct (A), then it correctly predicts the orbit of the moon (B). But it is also possible that there is a lucky coincidence and the Newton's law predicts correctly even though the law itself is incorrect.

Conjunction with certainty also has special structure. If $\Pr(A) \in \{0,1\}$ or $\Pr(B) \in \{0,1\}$, then

$$\Pr(A \cap B) = \Pr(A)\Pr(B).$$

Recall the *Law of Total Probability*:

$$\Pr(A) = \Pr(A|B)\Pr(B) + \Pr(A|\neg B)\Pr(\neg B) \tag{9}$$

Using Definition (2) and (9) yields Bayes' rule.

Definition 3. (Bayes' Rule) If $\Pr(A) > 0$, then

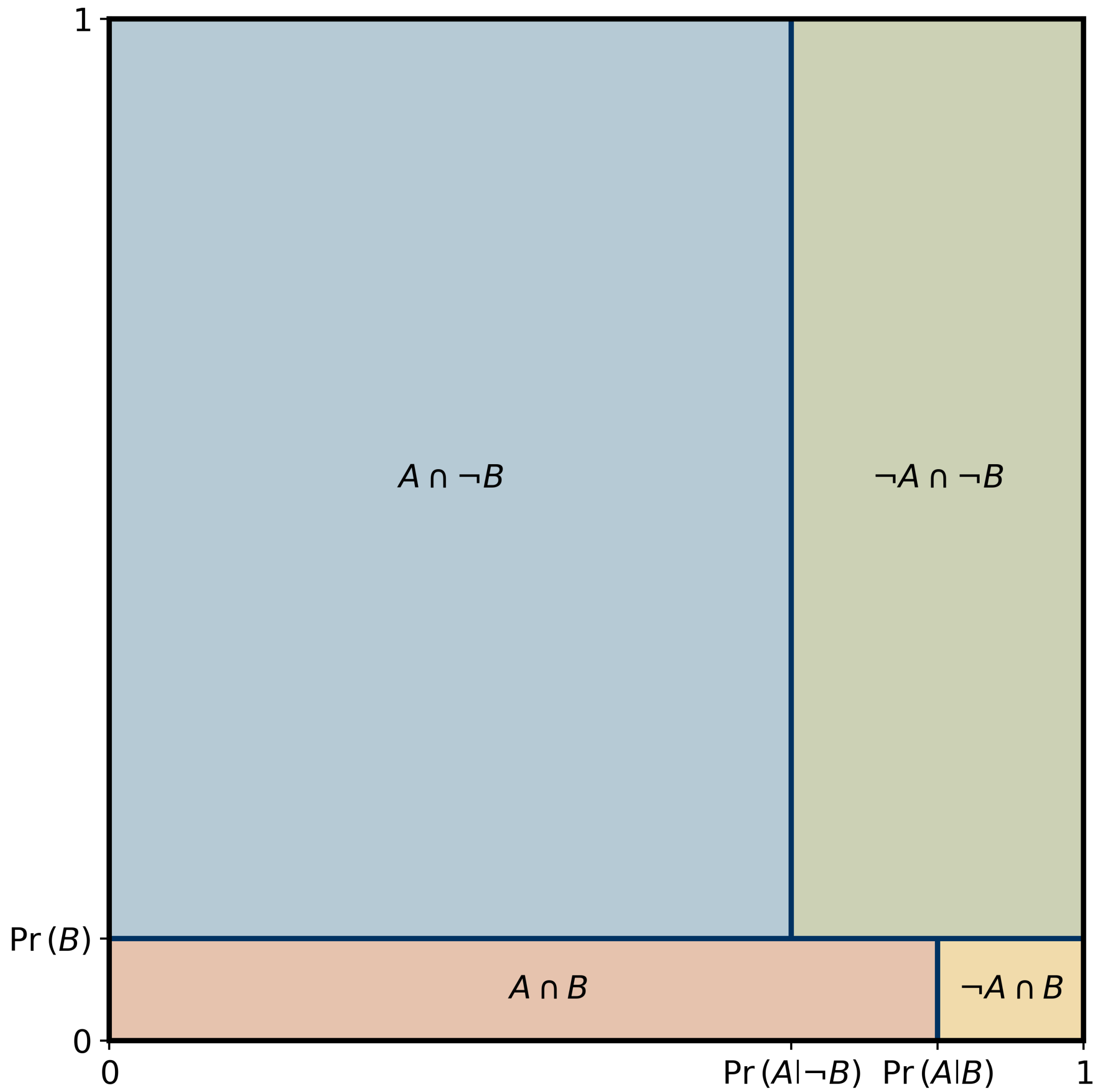
$$\begin{aligned}\Pr(B|A) &= \frac{\Pr(A|B)\Pr(B)}{\Pr(A)} \\ &= \frac{\Pr(A|B)\Pr(B)}{\Pr(A|B)\Pr(B) + \Pr(A|\neg B)\Pr(\neg B)}.\end{aligned}$$

In the problem set you will demonstrate that coherency of beliefs requires the use of Bayes' rule when forming conditional beliefs. This has implications for the conduct of science.

3 Further reading

Savage (1967) discusses the connection between the subjective probability and the problem of induction. Mitzenmacher & Upfal (2005) provides a nice review of the basic of probability theory. Schupbach (2022) discusses Bayes' rule from a philosophy of science perspective (see also the book by Diaconis & Skyrms (2018)). Popper (1959) discusses science without induction.

Figure 1: Simple probability example



<u>Source:</u> Author's calculations.
<u>Notes:</u> Let $\Pr(A \cap \neg B) = 0.70$, $\Pr(\neg A \cap \neg B) = 0.20$, $\Pr(A \cap B) = 0.09$ and $\Pr(\neg A \cap B) = 0.01$. Calculate $\Pr(B|\, A)$.

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