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Math 102 — Lab 11

Let $T_1(\theta_1)$ be the linear transformation that rotates any vector in $\mathbb{R}^3$ through an angle θ_1 in a right-handed sense about the x_1-axis (so if your right thumb points along the positive x_1-axis, the rotation is in the direction that your fingers curl). It will have matrix

$$T_1(\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}.$$

An important property of this transformation is that it leaves any vector parallel to the x_1-axis unchanged, as you would expect. You can easily check this by confirming that

$$[T_1(\theta_1)] \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}.$$

But a vector in the x_2x_3-plane gets rotated:

$$[T_1(\theta_1)] \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2\cos\theta_1 - x_3\sin\theta_1 \\ x_2\sin\theta_1 + x_3\cos\theta_1 \end{bmatrix}.$$

The length, however, is unchanged since

$$(x_2\cos\theta_1 - x_3\sin\theta_1)^2 + (x_2\sin\theta_1 + x_3\cos\theta_1)^2 = x_2^2 + x_3^2.$$

1. Write the matrix for rotation through angle θ_2 in the right-handed sense about the x_2-axis. Write the matrix for rotation through angle θ_3 in the right-handed sense about the x_3-axis. [2]

Answer:

$$T_2(\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$

$$T_3(\theta_3) = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Find the matrix A which corresponds to the linear transformation T obtained by first rotating a vector in $\mathbb{R}^3$ through angle $\pi/3$ about the x_3-axis and then through angle $\pi/4$ about the x_1-axis. [3]

(T is the matrix you found in question 2)

3. Now we'll find the axis of rotation. That is, we find the vectors $\mathbf{v}$ that are unchanged under rotation by T. These vectors obey [3]

$$T\mathbf{v} = \mathbf{v} \qquad ...(*)$$

so they are *eigenvectors of T with eigenvalue* 1. Using your answer from problem 2, write equation $(*)$ as $M\mathbf{v} = \mathbf{0}$ where M is a matrix all of whose entries are known numbers, and $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Check that the solution $\mathbf{v}$ that satisfies $(*)$ is

$$v = t\left(\sqrt{3}, -1, 1+\sqrt{2}\right), \quad t \in \mathbb{R}.$$

This is the axis of this rotation.

4. Discuss how you might try to find the angle of rotation T about the axis parallel to $\mathbf{v}$. [2]

Hint: T is a product of rotation matrices (Q2) and it turns out that T will also be a rotation matrix but the axis of rotation is more complicated (Q3).

Suppose $\vec{\omega}$ is perpendicular to the axis $\vec{v}$ in Q3. Is there a way to compute the angle between $\vec{\omega}$ and $T(\vec{\omega})$?