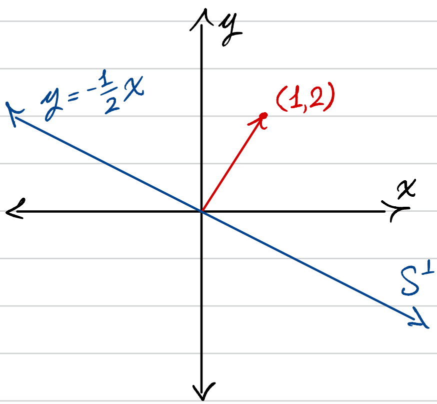
Math 102 · Lab 9 · Hyperplanes

Definition: Consider the vector space $\mathbb{R}^n$ and let $S \subset \mathbb{R}^n$ be some ~~Proposition~~ subset. The orthogonal complement of S, denoted $S^\perp$, is a subspace of $\mathbb{R}^n$ given by:

$$S^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{s} = 0 \text{ for all } \vec{s} \in S\}$$

Example: Consider the vector space $\mathbb{R}^2$ and the subset $S \subset \mathbb{R}^2$ given by $S = \{(1,2)\}$.



$$\begin{aligned} S^\perp &= \{\vec{x} \in \mathbb{R}^2 \mid \vec{x} \cdot \vec{s} = 0 \text{ for all } \vec{s} \in S\} \\ &= \{\vec{x} \in \mathbb{R}^2 \mid \vec{x} \cdot (1,2) = 0\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid x + 2y = 0\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid y = -\tfrac{1}{2}x\} \end{aligned}$$

So: $S^\perp$ is a line. More specifically, $S^\perp$ is a 1-dimensional subspace of $\mathbb{R}^2$.

Example: Consider the vector space $\mathbb{R}^3$ and the subset $S \subset \mathbb{R}^3$ given by $S = \{(5,-3,4)\}$

$$\begin{aligned} S^\perp &= \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} \cdot \vec{s} = 0 \text{ for all } \vec{s} \in S\} \\ &= \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} \cdot (5,-3,4) = 0\} \\ &= \{(x,y,z) \in \mathbb{R}^3 \mid 5x - 3y + 4z = 0\} \end{aligned}$$

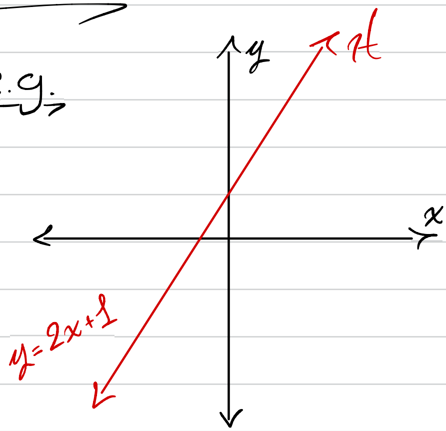
So: $S^\perp$ is a plane with normal vector $\vec{n} = (5,-3,4)$. More specifically, $S^\perp$ is a 2-dimensional subspace of $\mathbb{R}^3$

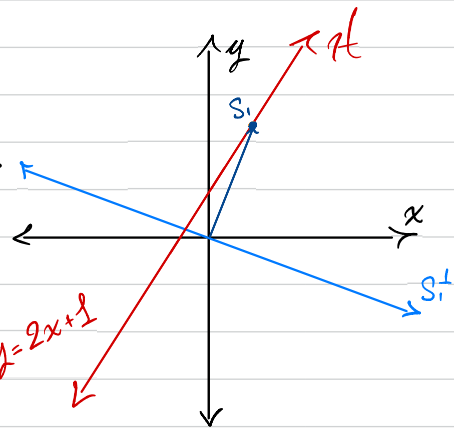
Definition: A hyperplane of $\mathbb{R}^2$ is a line in $\mathbb{R}^2$. A hyperplane of $\mathbb{R}^3$ is a plane in $\mathbb{R}^3$.

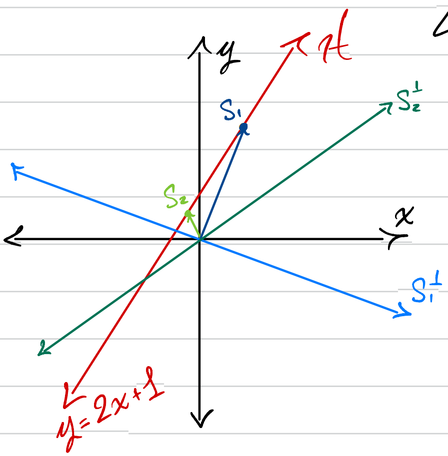
Problem: Does every hyperplane $\mathcal{H}$ in $\mathbb{R}^2$ satisfy $(\mathcal{H}^\perp)^\perp = \mathcal{H}$?

Answer: NO

e.g.:







So: $\mathcal{H}^\perp = \{(0,0)\}$

$\Rightarrow (\mathcal{H}^\perp)^\perp = \mathbb{R}^2$

But: If $\mathcal{H}$ is a subspace, then $(\mathcal{H}^\perp)^\perp = \mathcal{H}$