Math 102 · Lab 6 · Logic

- A statement of the form "If A, then B" or "A implies B", where A and B are either TRUE or FALSE, is called a conditional statement.

- We often write "$A \Rightarrow B$" instead of "If A, then B" or "A implies B"

  - $A \Rightarrow B$ is TRUE if both A and B are TRUE.

  - $A \Rightarrow B$ is FALSE if A is TRUE but B is FALSE.

Example: Is the following TRUE or FALSE:

S: If $X \in M_{2,2}(\mathbb{R})$ such that $X^2 = 0$ and $P \in M_{2,2}(\mathbb{R})$ is invertible, then $(PXP^{-1})^2 = 0$.

- This statement is of the form $A \Rightarrow B$, where
A = "$X \in M_{2,2}(\mathbb{R})$ such that $X^2 = 0$ and $P \in M_{2,2}(\mathbb{R})$ is invertible"
and B = "$(PXP^{-1})^2 = 0$"

  - A is TRUE but is B TRUE or FALSE?

Well,
$$
\begin{aligned}
(PXP^{-1})^2 &= (PXP^{-1})(PXP^{-1}) \\
&= PX(P^{-1}P)XP^{-1} \\
&= PXIXP^{-1} \quad (P^{-1}P = I) \\
&= PX^2P^{-1} \\
&= P0P^{-1} \quad (X^2 = 0) \\
&= 0
\end{aligned}
$$

(We call this a proof)

So: B is TRUE and therefore, $A \Rightarrow B$ is TRUE

Example: Is the following statement TRUE or FALSE:

S: If $X \in M_{2,2}(\mathbb{R})$, then X^{-1} exists.

- This is $A \Rightarrow B$, where $A =$ "$X \in M_{2,2}(\mathbb{R})$" and $B =$ "X^{-1} exists".

  - A is TRUE, but is B TRUE or FALSE?

Well, if $X = O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then X^{-1} does not exist (there is no matrix X^{-1} such that $OX^{-1} = I = X^{-1}O$).

(There is no $a,b,c,d \in \mathbb{R}$ so that $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$)

$\boxed{\text{So: } B \text{ is FALSE, and therefore } A \Rightarrow B \text{ is FALSE.}}$

In this case, we say that $X = O$ is a counterexample to the statement $A \Rightarrow B$.

For the problem set:

- If the statement is TRUE, then prove it (examples alone will not get full marks)
- If the statement is FALSE, then find a counterexample.