Math 102 · Lab 11 · Linear Transformations

Alert!! Don't forget about USRI's! 😮

• **Definition:** Let V and W be vector spaces (over $\mathbb{R}$). A map $\varphi: V \to W$ is said to be linear (or a linear transformation) if

$$\varphi(\lambda_1 \cdot \vec{v_1} + \lambda_2 \cdot \vec{v_2}) = \lambda_1 \cdot \varphi(\vec{v_1}) + \lambda_2 \cdot \varphi(\vec{v_2}) \quad \text{for all } \lambda_1, \lambda_2 \in \mathbb{R},\ \vec{v_1}, \vec{v_2} \in V$$

• **Fact:** Every linear transformation $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ is represented by a matrix ($\varphi(\vec{v}) = M_\varphi \cdot \vec{v}$, where M_φ is an $m \times n$ matrix), where

$$M_\varphi = \left[\varphi(\vec{e_1}) \,\vdots\, \varphi(\vec{e_2}) \,\vdots\, \cdots \,\vdots\, \varphi(\vec{e_n})\right]$$

• **Fact:** Every linear transformation $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ is determined by $\varphi(\vec{e_1}), \varphi(\vec{e_2}), \ldots, \varphi(\vec{e_n})$

Proof: Let $\vec{v} \in \mathbb{R}^n$. Then $\vec{v} = \lambda_1 \cdot \vec{e_1} + \lambda_2 \cdot \vec{e_2} + \cdots + \lambda_n \cdot \vec{e_n}$ for some $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ (because $\{\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}\}$ is a basis for $\mathbb{R}^n$).
Hence,

$$\begin{aligned}
\varphi(\vec{v}) &= \varphi(\lambda_1 \cdot \vec{e_1} + \lambda_2 \cdot \vec{e_2} + \cdots + \lambda_n \cdot \vec{e_n}) \\
&= \varphi(\lambda_1 \cdot \vec{e_1}) + \varphi(\lambda_2 \cdot \vec{e_2}) + \cdots + \varphi(\lambda_n \cdot \vec{e_n}) \\
&= \lambda_1 \cdot \varphi(\vec{e_1}) + \lambda_2 \cdot \varphi(\vec{e_2}) + \cdots + \lambda_n \cdot \varphi(\vec{e_n})
\end{aligned}$$

$\square$

Example: Consider the linear transformation $\varphi: \mathbb{R}^3 \to \mathbb{R}^2$ given by $\varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x+13z \\ 2x-5y-7z \end{bmatrix}$

Here, $\varphi(\vec{e_1}) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\varphi(\vec{e_2}) = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$, and $\varphi(\vec{e_3}) = \begin{bmatrix} 13 \\ -7 \end{bmatrix}$.

Then $M_\varphi = \left[\varphi(\vec{e_1}) \,\vdots\, \varphi(\vec{e_2}) \,\vdots\, \varphi(\vec{e_3})\right] = \begin{bmatrix} 3 & 0 & 13 \\ 2 & -5 & -7 \end{bmatrix}$

(Notice that $M_\varphi \cdot \vec{e_1} = \begin{bmatrix} 3 & 0 & 13 \\ 2 & -5 & -7 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \varphi(\vec{e_1})$

Similarly, $M_\varphi \cdot \vec{e_i} = \varphi(\vec{e_i})$ for $i = 2, 3$)

We should then have $\varphi(\vec{u}) = M_\varphi \cdot \vec{u}$ for any $\vec{u} \in \mathbb{R}^3$

e.g., Consider $\vec{u} = 2 \cdot \vec{e_1} + \vec{e_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^3$

Then $\varphi\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3\cdot 2 + 13 \cdot 0 \\ 2 \cdot 2 - 5 \cdot 1 - 7 \cdot 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

and $M_\varphi \cdot \vec{u} = \begin{bmatrix} 3 & 0 & 13 \\ 2 & -5 & -7 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4-5 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix} = \varphi\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right)$