Math 102 • Lab 5 • The Matrix

• Matrices allow us to efficiently solve systems of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_2 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \iff \overbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}^{A \cdot \vec{x} = \vec{b}}$$

$$\underbrace{\left(\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]\right)}_{[A|\vec{b}]}$$

• Instead of writing $A \cdot \vec{x} = \vec{b}$, often we will instead consider the augmented matrix $[A|\vec{b}]$.

• Depending on how the augmented matrix $[A|\vec{b}]$ looks like, we will be able to determine if the associated system of linear equations has i) infinitely many solutions, ii) a unique solution, or iii) no solutions.

Ex:
$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 5 \end{array}\right] \iff \begin{aligned} 1 \cdot x_1 + 0 \cdot x_2 + 2 \cdot x_3 &= 0 \\ 0 \cdot x_1 + 3 \cdot x_2 + 4 \cdot x_3 &= 1 \\ \boxed{0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 5} \end{aligned} \quad \text{Uh Oh}$$

$\therefore$ Inconsistent (No solutions)

Ex:
$$\left[\begin{array}{cccc|c} 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & -3 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \iff \begin{cases} 0x_1 + 1\cdot x_2 + 2\cdot x_3 + 0\cdot x_4 = 5 \\ 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 - 3\cdot x_4 = 6 \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + 0\cdot x_4 = 0 \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + 0\cdot x_4 = 0 \end{cases}$$

$$\iff \begin{cases} x_2 + 2x_3 = 5 \\ x_3 - 3x_4 = 6 \end{cases}$$

Let $x_1 = s =$ whatever
$x_4 = t =$ whatever $\iff$ $x_3 = 6 + 3t$, $t =$ whatever
$x_2 = 5 - 2x_3$
$= 5 - 2(6+3t) = -7 - 6t$

I get a bunch of solutions (infinitely many)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -6 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -7 \\ 6 \\ 0 \end{pmatrix}, \quad s, t \in \mathbb{R}$$

- Faster way to see this:

Pivots

$$\left[\begin{array}{cccc|c} 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & -3 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\begin{array}{cccc} * & & & * \end{array}$$

Free variables

- #Pivots + #Free variables = # columns (of A)
- If #Pivots < # columns, then there are infinitely many solutions (assuming the system is consistent)
- If #Pivots = # columns, then there is a unique solution (assuming the system is consistent)