Math 102 · Lab 8 · Orthogonal Matrices

- **Recall:** A matrix $A \in M_{n,n}(\mathbb{R})$ is invertible if there exists a matrix $B \in M_{n,n}(\mathbb{R})$ such that

$$AB = BA = I_n \leftarrow \text{identity matrix}$$

The matrix B is unique: If there is another matrix $\tilde{B} \in M_{n,n}(\mathbb{R})$ such that $A\tilde{B} = \tilde{B}A = I_n$, then

$$\tilde{B} = \tilde{B} I_n = \tilde{B}(AB) = (\tilde{B}A)B = I_n B = B$$

By convention, we denote the inverse B as A^{-1}.

- **Fact:** $A \in M_{n,n}(\mathbb{R})$ is invertible $\iff \det(A) \neq 0$

- **Some properties of determinants:** Let $X, Y \in M_{n,n}(\mathbb{R})$. Then:
  - $\det(XY) = \det(X)\det(Y)$
  - $\det(X) = \det(X^T)$
  - $\det(I_n) = 1$

- **Definition:** A collection of vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are called orthonormal if $\vec{x}_i \cdot \vec{x}_j = 0$ for $1 \leq i \neq j \leq n$ and $\vec{x}_i \cdot \vec{x}_i = 1$ for $1 \leq i \leq n$.

- **Definition:** A matrix $A \in M_{n,n}(\mathbb{R})$ is called orthogonal if its row vectors are orthonormal.

A Hint for the problem set:

Q3. For $A=(a_{ij})_{i,j=1}^{n}, B=(b_{ij})_{i,j=1}^{n} \in M_{n,n}(\mathbb{R})$, their product is given by $AB=(c_{ij})_{i,j=1}^{n}$, where

$$c_{ij} = \underbrace{\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}}_{i^{th} \text{ row of } A} \cdot \left.\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}\right\} j^{th} \text{ column of } B$$