Orthosymplectic, Periplectic, and Twisted Super Yangians
(Draft)

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Chapter 1

Introduction

Under Development

Lie groups serve as the most well-developed theory of continuous symmetries of mathematical structures, which consequently makes them essential tools for many parts of modern mathematics and theoretical physics. The theory of Lie groups is just a part of a larger theory known as Lie theory, which describes the correspondence between Lie groups and Lie algebras: Lie algebras, being linear approximations of Lie groups, are simpler objects that yet still encode many of the properties of Lie groups via the Lie bracket operation.

The study of Lie algebras leads to the original examples of *quantum groups*, which first appeared in the theory of quantum integrable systems and then later formalized by Vladimir Drinfel'd. In particular, quantum groups arose as a tool from theoretical physics to produce non-trivial solutions of the *quantum Yang-Baxter equation* (QYBE), a famous equation first appearing in statistical mechanics and that is named after physicists Chen-Ning Yang and Rodney Baxter. Ever since their discovery, quantum groups have evolved into a mature theory in their own right and are now an integral branch of mathematics and theoretical physics which has captivated the curiosity of many researchers in both communities.

One of the two families of quantum groups of affine type are called *Yangians*. In short, any complex simple Lie algebra $\mathfrak{g}$ gives rise to a particular associative (Hopf) algebra $\mathbf{Y}(\mathfrak{g})$, which we call the Yangian of $\mathfrak{g}$. Yangians come equipped with the foundational property that they produce rational solutions to the aforementioned QYBE via their *representations*. In essence, representation theory allows one to translate between abstract algebra problems and linear algebra problems. In our case, this powerful idea allows us to use inherent properties of a quantum group to yield sought-after linear functions: solutions to the QYBE.

Chapter 2

Orthosymplectic Super Yangians

2.1 The Super Yangian $\mathbf{Y}(\mathfrak{osp}_{M|N})$

2.1.1 Preliminaries

The Gradation Index and Orthosymplectic Lie Superalgebra

By convention, $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ denotes the set of natural numbers, $\mathbb{Z}$ is the set of all integers, $\mathbb{Z}^+$ denotes the set of positive integers, $\mathbb{C}$ is the field of complex numbers, $\mathbb{Q}$ is the field of rational numbers, and $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ denotes the field of two elements. Let us fix our ground field to be $\mathbb{C}$. Unless otherwise stated, all linear algebraic notions are formulated with respect to this fixed base field $\mathbb{C}$ (i.e., vector space = $\mathbb{C}$-vector space, algebra = $\mathbb{C}$-algebra, linear map = $\mathbb{C}$-linear map, $\otimes = \otimes_{\mathbb{C}}$, etc...).

Definition 2.1.1. Fix two integers $d \in \mathbb{N}$, $D \in \mathbb{Z}^+$ such that $d \leq D$. For a subset $\mathbf{d} \subseteq \{1, 2, \dots, D\}$ of cardinality d, we introduce the *gradation index*

$$[\,\cdot\,]_{\mathbf{d}} \colon \{1, 2, \dots, D\} \to \mathbb{Z}_2 \tag{2.1}$$

given by $(-1)^{[i]_{\mathbf{d}}} = 1$ for $i \in \mathbf{d}$ and $(-1)^{[i]_{\mathbf{d}}} = -1$ for $i \in \mathbf{d}^c = \{1, 2, \ldots, D\} \setminus \mathbf{d}$. When $\mathbf{d} = \{1, 2, \ldots, d\}$, we set $[\,\cdot\,] = [\,\cdot\,]_{\mathbf{d}}$.

We will often be working with some super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ that is graded with respect to the gradation index (2.1), but we shall also denote the gradation of homogeneous elements in V with the similar notation: $[\,\cdot\,]\colon V_{\bar{0}} \sqcup V_{\bar{1}} \to \mathbb{Z}_2, v \mapsto [v]$, where $[v] = \gamma \in \mathbb{Z}_2$ if $v \in V_\gamma$. Note that in a super vector space V, elements in $V_{\bar{0}}$ are said to be *even* and elements in $V_{\bar{1}}$ are said to be *odd*.

For instance, if d and D are integers as in Definition (2.1.1) and V is a vector space with basis $\{x_i\}_{1=1}^{D}$, then V may be equipped with the $\mathbb{Z}_2$-grading given by $[x_i] := [i]_{\mathbf{d}}$ for $i = 1, 2, \ldots, D$. That is, we have the decomposition $V = V_{\mathbf{d}} \oplus V_{\mathbf{d}^c}$, where $V_{\mathbf{d}} = \bigoplus_{i \in \mathbf{d}} \mathbb{C}x_i$ is the even subspace and $V_{\mathbf{d}^c} = \bigoplus_{i \in \mathbf{d}^c} \mathbb{C}x_i$ is the odd subspace.

The space of $\mathbb{C}$-linear maps $V \to V$, denoted $\operatorname{End}_{\mathbb{C}} V$ (or $\operatorname{End} V$ for short), carries a natural $\mathbb{Z}_2$-grading via the assignment $(\operatorname{End} V)_\gamma = \{\varphi \in \operatorname{End} V \mid \varphi(V_\eta) \subseteq V_{\eta+\gamma},\ \eta \in \mathbb{Z}_2\}$. That is, such grading is provided by $[E_{ij}] := [i]_{\mathbf{d}} + [j]_{\mathbf{d}}$, where $\{E_{ij}\}_{i,j=1}^{D}$ is the collection of the matrix units of $\operatorname{End} V$ with respect to the fixed basis $\{x_i\}_{1=1}^{D}$.

For natural numbers $M, N \in \mathbb{N}$, the super vector space graded with respect to the gradation index (2.1) we will often utilize is the vector space $\mathbb{C}^{M+N}$ whose standard basis is given by $\mathcal{B} = \{e_i\}_{i=1}^{M+N}$. We denote $\mathbb{C}_{\mathbf{d}}^{M|N}$ to be the vector space $\mathbb{C}^{M+N}$ equipped with the $\mathbb{Z}_2$-grading given by $[e_i] := [i]_{\mathbf{d}}$ for $i = 1, 2, \ldots, M+N$, where $d = M$ and $D = M+N$ as in Definition (2.1.1). The space $\operatorname{End} \mathbb{C}_{\mathbf{d}}^{M|N}$ is $\mathbb{Z}_2$-graded and will now let the set of standard matrix units of $\operatorname{End} \mathbb{C}_{\mathbf{d}}^{M|N}$ with respect to $\mathcal{B}$ be denoted $\{E_{ij}\}_{i,j=1}^{M+N}$. Furthermore, when $\mathbf{d} = \{1, 2, \ldots, M\}$ we set $\mathbb{C}^{M|N} = \mathbb{C}_{\mathbf{d}}^{M|N}$.

The space $\operatorname{End} \mathbb{C}^{M|N}$ is a unital associative superalgebra and therefore carries the structure of a Lie superalgebra with the Lie superbracket given by the super-commutator

$$[E_{ij}, E_{kl}] := \delta_{jk} E_{il} - (-1)^{([i]+[j])([k]+[l])} \delta_{li} E_{kj},$$

where δ_{ij} is the Kronecker delta. We shall denote the space $\operatorname{End}\mathbb{C}^{M|N}$ as $\mathfrak{gl}_{M|N} = \mathfrak{gl}(\mathbb{C}^{M|N})$ when it is equipped with the above Lie superalgebra structure and call it the *general Lie superalgebra.*

Definition 2.1.2. Let $M, N \in \mathbb{N}$ such that N is even and let $d = M$ and $D = M+N$ as in Definition (2.1.1). Further, consider $\mathbf{d}$ and $\mathbf{d}^c$ as ordered sets with respect to the canonical ordering of natural numbers, so that $\mathbf{d}[j]$ and $\mathbf{d}^c[j]$ denotes the j^{th} elements in these sets and $\mathbf{d}^c_{N/2}$ denotes the first $\frac{N}{2}$ integers in $\mathbf{d}^c$. For each integer $1 \leq i \leq M+N$, define the sign

$$\theta_i^{\mathbf{d}} := \begin{cases} 1 & \text{if} \quad i \in \mathbf{d} \cup \mathbf{d}^c_{N/2} \\ -1 & \text{if} \quad i \in \mathbf{d}^c \setminus \mathbf{d}^c_{N/2} \end{cases} \tag{2.2}$$

and define the conjugate index $\bar{\imath}^{\mathbf{d}}$ as

$$\bar{\imath}^{\mathbf{d}} := \begin{cases} \mathbf{d}[M+1-j] & \text{if} \quad i = \mathbf{d}[j] \;\text{ for some }\; 1 \leq j \leq M \\ \mathbf{d}^c[N+1-j] & \text{if} \quad i = \mathbf{d}^c[j] \;\text{ for some }\; 1 \leq j \leq N. \end{cases} \tag{2.3}$$

When $\mathbf{d} = \{1, 2, \dots, M\}$, we set $\theta_i = \theta_i^{\mathbf{d}}$ and $\bar{\imath} = \bar{\imath}^{\mathbf{d}}$. In this case,

$$\theta_i := \begin{cases} 1 & \text{if} \;\; 1 \leq i \leq M+\frac{N}{2} \\ -1 & \text{if} \;\; M+\frac{N}{2}+1 \leq i \leq M+N \end{cases} \tag{2.4}$$

and

$$\bar{\imath} := \begin{cases} M+1-i & \text{if} \quad 1 \leq i \leq M \\ 2M+N+1-i & \text{if} \quad M+1 \leq i \leq M+N. \end{cases} \tag{2.5}$$

The *super-transpose* is the $\mathbb{C}$-linear map defined by

$$\begin{aligned} (-)^{st}\colon \operatorname{End}\mathbb{C}^{M|N} &\to \operatorname{End}\mathbb{C}^{M|N} \\ E_{ij} &\mapsto E_{ij}^{st} := (-1)^{[i][j]+[i]} E_{ji} \end{aligned} \tag{2.6}$$

and is furthermore a superalgebra antimorphism; that is, $(A_1A_2)^{st} = (-1)^{[A_1][A_2]} A_2^{st} A_1^{st}$ for any homogeneous elements $A_1, A_2 \in \operatorname{End}\mathbb{C}^{M|N}$. Under the superalgebra identification $\operatorname{End}\mathbb{C}^{M|N} \cong \operatorname{Mat}_{M|N}(\mathbb{C}) = \operatorname{Mat}_{M+N}(\mathbb{C})$, an element $A \in \operatorname{End}\mathbb{C}^{M|N}$ may be regarded as $(M+N) \times (M+N)$ matrix

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \tag{2.7}$$

where $A_{00} \in \operatorname{Mat}_M(\mathbb{C})$, $A_{01} \in \operatorname{Mat}_{M\times N}(\mathbb{C})$, $A_{10} \in \operatorname{Mat}_{N\times M}(\mathbb{C})$, and $A_{11} \in \operatorname{Mat}_N(\mathbb{C})$. If $(-)'$ denotes the conventional matrix transpose, then A^{st} may be regarded as the matrix

$$\begin{pmatrix} A'_{00} & A'_{10} \\ -A'_{01} & A'_{11} \end{pmatrix}. \tag{2.8}$$

Assume that $b\colon \mathbb{C}^{M|N} \times \mathbb{C}^{M|N} \to \mathbb{C}$ is an even, super-symmetric, non-degenerate $\mathbb{C}$-bilinear form; hence, N is necessarily even. The *ortho-symplectic Lie superalgebra* $\mathfrak{osp}_{M|N} = \mathfrak{osp}_{M|N}(\mathbb{C}^{M|N}, b)$ is defined as the Lie sub-superalgebra of $\mathfrak{gl}_{M|N}$ preserving such bilinear form b. That is, $\mathfrak{osp}_{M|N}$ is the Lie superalgebra generated by homogeneous elements $\varphi \in \mathfrak{gl}_{M|N}$ satisfying the relation $b(\varphi(v), w) + (-1)^{[\varphi][v]} b(v, \varphi(w)) = 0$ for all homogeneous vectors $v, w \in \mathbb{C}^{M|N}$. The associated matrix of $b(x, y)$ with respect to the standard basis $\mathcal{B}$ is given by $B = \big(b(e_i, e_j)\big)_{i,j=1}^{M+N}$, which necessarily has the form

$$B = \begin{pmatrix} G & 0 \\ 0 & J \end{pmatrix}$$

where $G \in \operatorname{Mat}_M(\mathbb{C})$ and $J \in \operatorname{Mat}_N(\mathbb{C})$ are invertible matrices satisfying $G' = G$ and $J' = -J$. In this way, if $[v]_{\mathcal{B}}, [w]_{\mathcal{B}} \in \operatorname{Mat}_{(M|1)\times(N|0)}(\mathbb{C})$ represent the coordinate vectors of $v, w \in \mathbb{C}^{M|N}$ with respect to the basis $\mathcal{B}$, then $b(v, w) = [v]_{\mathcal{B}}^{st} B [w]_{\mathcal{B}}$. The definition of $\mathfrak{osp}_{M|N}$ is independent of the selection of such a bilinear form, so we may assume

$$G = \big(\delta_{i\bar{\jmath}}\big)_{i,j=1}^{M} \quad \text{and} \quad J = \big(\theta_j \delta_{i\bar{\jmath}}\big)_{i,j=1}^{N}, \quad \text{so} \quad B = \big(\theta_j \delta_{i\bar{\jmath}}\big)_{i,j=1}^{M+N} \tag{2.9}$$

Regarding $A \in \mathfrak{gl}_{M|N}$ in its matrix form (2.7), then we have $A \in \mathfrak{osp}_{M|N}$ if and only if $A^{st}B + BA = 0$. That is, if we have $A_{00} \in \mathfrak{so}_M$, $A_{11} \in \mathfrak{sp}_N$, and $(A_{10})'J + GA_{01} = 0$.

Definition 2.1.3. The *super-transposition* is the $\mathbb{C}$-linear map defined by

$$\begin{aligned} (-)^t\colon \operatorname{End}\mathbb{C}^{M|N} &\to \operatorname{End}\mathbb{C}^{M|N} \\ E_{ij} &\mapsto E_{ij}^t := (-1)^{[i][j]+[i]}\theta_i\theta_j E_{\bar{j}\bar{\imath}} \end{aligned} \tag{2.10}$$

We note that the super-transposition is in fact an involution, unlike the super-transpose which is of order 4, and satisfies $(A_1A_2)^t = (-1)^{[A_1][A_2]}A_2^tA_1^t$ for homogeneous maps $A_1, A_2 \in \operatorname{End}\mathbb{C}^{M|N}$. Moreover, the super-transposition and the super-transpose commute: $(-)^t \circ (-)^{st} = (-)^{st} \circ (-)^t$. Finally, given an index $1 \le k \le m$, we shall denote by $(-)^{t_k}$ the map

$$\mathrm{id}^{\otimes(k-1)} \otimes (-)^t \otimes \mathrm{id}^{\otimes(m-k)} \in \operatorname{End}\left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes m}.$$

Proposition 2.1.4. *For $1 \le i,j \le M+N$, define the operators*

$$F_{ij} := E_{ij} - E_{ij}^t = E_{ij} - (-1)^{[i][j]+[i]}\theta_i\theta_j E_{\bar{j}\bar{\imath}} \in \operatorname{End}\mathbb{C}^{M|N}. \tag{2.11}$$

As a Lie superalgebra, $\mathfrak{osp}_{M|N}$ is generated by the operators $\{F_{ij} \,|\, 1 \le i,j \le M+N\}$ satisfying the relations

$$\begin{aligned} \left[F_{ij}, F_{kl}\right] = \delta_{jk}F_{il} - \delta_{il}(-1)^{([i]+[j])([k]+[l])}F_{kj} \\ - \delta_{\bar{\imath}k}(-1)^{[i][j]+[i]}\theta_i\theta_j F_{\bar{j}l} + \delta_{\bar{j}l}(-1)^{([i]+[l])[k]}\theta_{\bar{\imath}}\theta_{\bar{j}}F_{k\bar{\imath}} \end{aligned} \tag{2.12}$$

and

$$F_{ij} + (-1)^{[i][j]+[i]}\theta_i\theta_j F_{\bar{j}\bar{\imath}} = 0 \tag{2.13}$$

The R-Matrix and SQYBE

Every space in this work will be regarded as an object in the symmetric monoidal category of super vector spaces over $\mathbb{C}$, denoted $\mathrm{sVect}_{\mathbb{C}}$, which is equipped with the *super-braiding* σ. As such, given any two objects X and Y in $\mathrm{sVect}_{\mathbb{C}}$ we have the isomorphism $\sigma_{X,Y}\colon X \otimes Y \to Y \otimes X$, $x \otimes y \mapsto (-1)^{[x][y]}y \otimes x$ on homogeneous elements

$x \in X$ and $y \in Y$.

If we are given homogeneous linear maps $\varphi \in \operatorname{End} X$, $\psi \in \operatorname{End} Y$, then their (super) tensor product is the homogeneous linear map in $\operatorname{End}(X \otimes Y)$, denoted $\varphi \otimes \psi$, given by

$$\begin{aligned} \varphi \otimes \psi \colon X \otimes Y &\to X \otimes Y \\ x \otimes y &\mapsto (-1)^{[\psi][x]} \varphi(x) \otimes \psi(y) \end{aligned}$$

Note that when φ and ψ are even (or just ψ), then their (super) tensor product is just the traditional tensor product of linear maps. For instance, the operator $E_{ij} \otimes E_{kl}$ in $\left(\operatorname{End} \mathbb{C}^{M|N}\right)^{\otimes 2} \cong \operatorname{End}(\mathbb{C}^{M|N} \otimes \mathbb{C}^{M|N})$ acts on basis elements $e_a \otimes e_b$ via the formula

$$(E_{ij} \otimes E_{kl})\, e_a \otimes e_b = \delta_{ja} \delta_{lb} (-1)^{([k]+[l])[a]} e_i \otimes e_k$$

If we further suppose that X and Y are superalgebras in $\mathrm{sVect}_{\mathbb{C}}$ with multiplication maps $\mu_X \colon X \otimes X \to X$ and $\mu_Y \colon Y \otimes Y \to Y$, then multiplication in $X \otimes Y$ is defined by the composition $(\mu_X \otimes \mu_Y) \circ (\mathrm{id}_X \otimes \sigma \otimes \mathrm{id}_Y)$. Explicitly, this multiplication is given by $(x_1 \otimes y_1)(x_2 \otimes y_2) = (-1)^{[y_1][x_2]} x_1 x_2 \otimes y_1 y_2$ on homogeneous elements.

Let the *super permutation operator* in $\operatorname{End} \mathbb{C}^{M|N} \otimes \operatorname{End} \mathbb{C}^{M|N}$ be given by

$$P := \sum_{i,j=1}^{M+N} (-1)^{[j]} E_{ij} \otimes E_{ji} \tag{2.14}$$

and further define the Q operator

$$Q := P^{t_1} = P^{t_2} = \sum_{i,j=1}^{M+N} (-1)^{[i][j]} \theta_i \theta_j E_{ij} \otimes E_{\bar{\imath}\bar{\jmath}}, \tag{2.15}$$

noting the relations $P = Q^{t_1} = Q^{t_2}$. Acting on basis elements $e_a \otimes e_b$, these operators yield

$$P(e_a \otimes e_b) = (-1)^{[a][b]} e_b \otimes e_a \quad \text{and}$$

$$Q(e_a \otimes e_b) = \delta_{\bar{a}b}\theta_a(-1)^{[a]} \sum_{i=1}^{M+N} \theta_i e_i \otimes e_{\bar{\imath}}.$$

Now, we define the *R-matrix* $R(u) \in \left(\operatorname{End}\mathbb{C}^{M|N} \otimes \operatorname{End}\mathbb{C}^{M|N}\right)(u)$ to be the rational function in the formal parameter u with coefficients in $\operatorname{End}\mathbb{C}^{M|N} \otimes \operatorname{End}\mathbb{C}^{M|N}$ given by

$$R(u) := \mathrm{id}^{\otimes 2} - \frac{P}{u} + \frac{Q}{u-\kappa}, \tag{2.16}$$

where $\kappa = \kappa_{M,N} := (M-N-2)/2$.

These newly defined operators (4.2) and (2.15) satisfy the relations

$$P^2 = \mathrm{id}^{\otimes 2}, \quad PQ = QP = Q, \quad \text{and} \quad Q^2 = (M-N)Q. \tag{2.17}$$

Using these relations, one may prove the equations

$$R^{t_1}(-u+\kappa) = R(u), \tag{2.18}$$

$$R(u)R(-u) = \left(1 - \frac{1}{u^2}\right)\mathrm{id}^{\otimes 2}, \tag{2.19}$$

known as *crossing symmetry* and *unitarity*, respectively.

Now, given an element $S \in \left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2}$ and indices $1 \leq i < j \leq 3$, we define a new element S_{ij} in $\left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 3}$ by

$$S_{12} := S \otimes \mathrm{id}, \quad S_{23} := \mathrm{id} \otimes S, \quad \text{and} \quad S_{13} := (\mathrm{id} \otimes \tau)(S \otimes \mathrm{id})(\mathrm{id} \otimes \tau),$$

where $\tau \colon \mathbb{C}^{M|N} \otimes \mathbb{C}^{M|N} \to \mathbb{C}^{M|N} \otimes \mathbb{C}^{M|N}, v_1 \otimes v_2 \mapsto v_2 \otimes v_1$ is the twist map. When $S = S(u)$ depends on some formal parameter u, then we write $S_{ij}(u)$ for $S(u)_{ij}$.

As was shown in [**?**, Section 2] the R-matrix (4.3) satisfies the *super quantum Yang-Baxter equation* (SQYBE)

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u). \tag{2.20}$$

Equation (2.20) may be regarded as an equality in the space $\left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 3}[\![u^{\pm}, v^{\pm}]\!]$ since any rational function may be interpreted as an element in a power series algebra. For instance, the R-matrix (4.3) may be regarded as an element in $\left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 3}[\![u^{-1}]\!]$ since $1/(u-\kappa) = \sum_{n=0}^{\infty} \kappa^n u^{-n-1}$.

2.1.2 Notation on Tensor Spaces, Supermatrices, and Maps of Power Series

Embedding Elements into Higher Order Tensor Spaces

Let W be an arbitrary super vector space and let V be a finite-dimensional super vector space of dimension d with a fixed basis $\{b_1, \dots, b_d\}$, where $\{E_{ij}\}_{i,j=1}^{d}$ denotes the matrix units of $\operatorname{End} V$ with respect to this basis. We will often wish to represent objects in $(\operatorname{End} V) \otimes W$ in a larger space $(\operatorname{End} V)^{\otimes m} \otimes W$ for some integer $m \geq 2$. For this, we note that for the index $1 \leq k \leq m$, we have an injective? morphism of super vector spaces

$$\begin{aligned} \varphi_k \colon\ (\operatorname{End} V) \otimes W &\to (\operatorname{End} V)^{\otimes m} \otimes W \\ \psi \otimes w &\mapsto \mathrm{id}^{\otimes(k-1)} \otimes \psi \otimes \mathrm{id}^{\otimes(m-k)} \otimes w, \end{aligned}$$

and set $X_k = \varphi_k(X)$ for an element $X \in (\operatorname{End} V) \otimes W$. Explicitly, if $X = \sum_{i,j=1}^{d} E_{ij} \otimes w_{ij}$, then

$$X_k = \sum_{i,j=1}^{d} \mathrm{id}^{\otimes(k-1)} \otimes E_{ij} \otimes \mathrm{id}^{\otimes(m-k)} \otimes w_{ij} \in (\operatorname{End} V)^{\otimes m} \otimes W.$$

If W is a formal power series superalgebra or if $X = X(u)$ depends on some formal parameter u, we shall write $X_k(u)$ instead of $X(u)_k$ for the element $\varphi_k(X(u))$.

As a generalization of the above when W is a superalgebra $\mathcal{A}$ with unit $\mathbf{1}$, we may desire to represent objects in $(\operatorname{End} V) \otimes \mathcal{A}$ in a larger space $(\operatorname{End} V)^{\otimes m} \otimes \mathcal{A}^{\otimes n}$ for some integers $m, n \in \mathbb{Z}^{+}$. For this, we note that for the indices $1 \leq k \leq m$ and $1 \leq l \leq n$, we have an injective? morphism of superalgebras

$$\begin{aligned}\varphi_{k[l]}\colon\ (\operatorname{End} V)\otimes\mathcal{A} &\to (\operatorname{End} V)^{\otimes m}\otimes\mathcal{A}^{\otimes n}\\ \psi\otimes a &\mapsto \mathrm{id}^{\otimes(k-1)}\otimes\psi\otimes\mathrm{id}^{\otimes(m-k)}\otimes\mathbf{1}^{\otimes(l-1)}\otimes a\otimes\mathbf{1}^{\otimes(n-l)},\end{aligned}$$

and set $X_{k[l]}=\varphi_{k[l]}(X)$ for an element $X\in(\operatorname{End} V)\otimes\mathcal{A}$. Explicitly, if we express X as the sum $\sum_{i,j=1}^{d}E_{ij}\otimes a_{ij}$, then

$$X_{k[l]}=\sum_{i,j=1}^{d}\mathrm{id}^{\otimes(k-1)}\otimes E_{ij}\otimes\mathrm{id}^{\otimes(m-k)}\otimes\mathbf{1}^{\otimes(l-1)}\otimes a_{ij}\otimes\mathbf{1}^{\otimes(n-l)}\in(\operatorname{End} V)^{\otimes m}\otimes\mathcal{A}^{\otimes n}.$$

If $k=1$ we shall abbreviate $X_{1[l]}$ by $X_{[l]}$ and if $l=1$ we shall abbreviate $X_{k[1]}$ by X_k just as above. If $\mathcal{A}$ is a formal power series superalgebra or if $X=X(u)$ depends on some formal parameter u, we shall write $X_{k[l]}(u)$ instead of $X(u)_{k[l]}$ for the element $\varphi_{k[l]}(X(u))$.

Analogously, we will like to express elements of $\mathcal{A}\otimes\mathcal{A}$ in $\mathcal{A}^{\otimes m}$ for some integer $m\geq 3$. For indices $1\leq k<l\leq m$, we also have the injective? morphism of super algebras

$$\begin{aligned}\varphi_{kl}\colon \mathcal{A}\otimes\mathcal{A} &\to \mathcal{A}^{\otimes m}\\ a\otimes b &\mapsto \mathbf{1}^{\otimes(k-1)}\otimes a\otimes\mathbf{1}^{\otimes(l-k-1)}\otimes b\otimes\mathbf{1}^{\otimes(m-l)}\end{aligned}$$

and set $X_{kl}=\varphi_{kl}(X)$ for an element $X\in\mathcal{A}\otimes\mathcal{A}$. Explicitly, if $X=\sum_{i=1}^{r}a_i\otimes b_i$, then

$$X_{kl}=\sum_{i=1}^{r}\mathbf{1}^{\otimes(k-1)}\otimes a_i\otimes\mathbf{1}^{\otimes(l-k-1)}\otimes b_i\otimes\mathbf{1}^{\otimes(m-l)}.$$

Again, if $\mathcal{A}$ is a formal power series superalgebra or $X=X(u)$ depends on some formal parameter u, then we write $X_{kl}(u)$ instead of $X(u)_{kl}$ for the element $\varphi_{kl}(X(u))$.

Supermatrices and Supermatrix Multiplication

For a superalgebra $\mathcal{A}$ and integers $K, L, M, N \in \mathbb{N}$ such that $K+L+M+N \geq 1$, we let $\mathrm{Mat}_{(K|L)\times(M|N)}(\mathcal{A})$ denote the collection of all supermatices over $\mathcal{A}$ of dimension $(K|L)\times(M|N)$. That is, each element $A \in \mathrm{Mat}_{(K|L)\times(M|N)}(\mathcal{A})$ is a 2×2 block matrix

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

where $A_{00} \in \mathrm{Mat}_{K\times M}(\mathcal{A})$, $A_{01} \in \mathrm{Mat}_{K\times N}(\mathcal{A})$, $A_{10} \in \mathrm{Mat}_{L\times M}(\mathcal{A})$, and $A_{11} \in \mathrm{Mat}_{L\times N}(\mathcal{A})$. The collection of supermatrices $\mathrm{Mat}_{(K|L)\times(M|N)}(\mathcal{A})$ comes equipped with a $\mathbb{Z}_2$-grading $\bigoplus_{\gamma\in\mathbb{Z}_2} \mathrm{Mat}_{(K|L)\times(M|N)}(\mathcal{A})_\gamma$ given in the following way: $A \in \mathrm{Mat}_{(K|L)\times(M|N)}(\mathcal{A})_{\bar{0}}$ if the diagonal matrices A_{00} and A_{11} consist of only even elements in $\mathcal{A}$ and the off-diagonal matrices A_{01} and A_{10} consist of only odd elements in $\mathcal{A}$; conversely, $A \in \mathrm{Mat}_{(K|L)\times(M|N)}(\mathcal{A})_{\bar{1}}$ is odd if the diagonal matrices A_{00} and A_{11} consist of only odd elements in $\mathcal{A}$ and the off-diagonal matrices A_{01} and A_{10} consist of only even elements in $\mathcal{A}$.

One may define a scalar multiplication action of $\mathcal{A}$ on $\mathrm{Mat}_{(K|L)\times(M|N)}(\mathcal{A})$ as in the traditional case, and when dimensions permit, supermatrices may be added and multiplied just as conventional matrices. Under such scalar multiplication and matrix addition, $\mathrm{Mat}_{(K|L)\times(M|N)}(\mathcal{A})$ forms an $\mathcal{A}$-supermodule. Furthermore, when $K = M$ and $L = N$, we shall write $\mathrm{Mat}_{M|N}(\mathcal{A})$ as short for $\mathrm{Mat}_{(K|L)\times(M|N)}(\mathcal{A})$. In this case, the collection $\mathrm{Mat}_{M|N}(\mathcal{A})$ also is a superalgebra over $\mathcal{A}$ via matrix multiplication.

An important operation on supermatrices is the *super-transpose*

$$(-)^{st}\colon\ \mathrm{Mat}_{(K|L)\times(M|N)}(\mathcal{A}) \to \mathrm{Mat}_{(M|N)\times(K|L)}(\mathcal{A})$$

described by the assignment $A \mapsto A^{st}$ on homogeneous matrices, where if $(-)'$ denotes the conventional matrix transpose, then the supermatrix A^{st} is given by

$$\begin{pmatrix} A'_{00} & -(-1)^{[A]}A'_{10} \\ (-1)^{[A]}A'_{01} & A'_{11} \end{pmatrix}.$$

One can readily verify that the super-transpose is $\mathcal{A}$-linear and satisfies the property $(AB)^{st} = (-1)^{[A][B]}B^{st}A^{st}$ for any homogeneous supermatrices A, B with compatible dimensions for matrix multiplication. Furthermore, when $\mathcal{A} = \mathbb{C} = \mathbb{C}^{1|0}$ with the trivial $\mathbb{Z}_2$-gradation, notice that the above form of the super-transpose coincides with the formula (2.8).

In the non-super case, when $\mathcal{A}$ is assumed to only have an algebra structure, one identifies the algebra $\mathrm{End}(\mathbb{C}^{M+N}) \otimes \mathcal{A}$ with $\mathrm{Mat}_{M+N}(\mathcal{A})$ so that multiplication in $\mathrm{End}(\mathbb{C}^{M+N}) \otimes \mathcal{A}$ may be regarded as simply matrix multiplication. When $\mathcal{A}$ is super, one invariably encounters signs occurring with multiplication in $\mathrm{End}(\mathbb{C}^{M|N}) \otimes \mathcal{A}$ that does not occur with (super)matrix multiplication, so we have the following:

Proposition 2.1.5. *When $\mathcal{A}$ is a superalgebra, there is an algebra isomorphism*

$$\begin{gathered} \left(\mathrm{End}(\mathbb{C}^{M|N}) \otimes \mathcal{A}\right)_{\bar{0}} \to \mathrm{Mat}_{M|N}(\mathcal{A})_{\bar{0}} \\ \sum_{i,j=1}^{M+N} (-1)^{[i][j]+[j]} E_{ij} \otimes A_{ij} \mapsto (A_{ij})_{i,j=1}^{M+N}, \end{gathered} \tag{2.21}$$

where the elements A_{ij} are homogeneous of degree $[A_{ij}] = [E_{ij}] = [i]+[j]$.

Proof. The map is bijective and linear, so all left to do is to show that it is multiplicative. Considering even supermatrices $(A_{ij})_{i,j=1}^{M+N}, (B_{ij})_{i,j=1}^{M+N} \in \mathrm{Mat}_{M|N}(\mathcal{A})_{\bar{0}}$, note that

$$\begin{aligned} &\left(\sum_{i,j=1}^{M+N} (-1)^{[i][j]+[j]} E_{ij} \otimes A_{ij}\right)\left(\sum_{k,l=1}^{M+N} (-1)^{[k][l]+[l]} E_{kl} \otimes B_{kl}\right) \\ &\qquad = \sum_{i,j,k,l=1}^{M+N} (-1)^{[i][j]+[j]+[k][l]+[l]+([i]+[j])([k]+[l])} \delta_{jk} E_{il} \otimes A_{ij}B_{kl} \end{aligned}$$

$$= \sum_{i,l=1}^{M+N} (-1)^{[i][l]+[l]} E_{il} \otimes \sum_{k=1}^{M+N} A_{ik} B_{kl}$$

and the above element is sent to $\left(\sum_{k=1}^{M+N} A_{ik} B_{kl}\right)_{i,l=1}^{M+N} = (A_{ij})_{i,j=1}^{M+N} (B_{kl})_{k,l=1}^{M+N}$. □

Maps of Power Series and Module Actions

We will often use power series to define maps between spaces. In particular, this will be useful when our spaces are (countably) infinite-dimensional. To see how this works, suppose that W_1 and W_2 are super vector spaces or superalgebras such that $\{A_n\}_{n=0}^{\infty}$ generates W_1. Given $a(u) = \sum_{n=0}^{\infty} A_n u^{-n} \in W_1[\![u^{-1}]\!]$ and some element $b(u) = \sum_{n=0}^{\infty} B_n u^{-n} \in W_2[\![u^{-1}]\!]$, we shall write

$$\varphi \colon a(u) \to b(u)$$

to mean the morphism $\varphi \colon W_1 \to W_2$ defined by $\varphi(A_n) = B_n$ for all $n \in \mathbb{N}$.

Suppose that V is a finite-dimensional super vector space of dimension D graded with respect to some gradation index (2.1). Given $A(u) = \sum_{i,j=1}^{D} (-1)^{[i][j]+[j]} E_{ij} \otimes A_{ij}(u)$ in $\operatorname{End} V \otimes W_1[\![u^{-1}]\!]$ and $B(u) = \sum_{i,j=1}^{D} (-1)^{[i][j]+[j]} E_{ij} \otimes B_{ij}(u)$ in $\operatorname{End} V \otimes W_2[\![u^{-1}]\!]$, where $A_{ij}(u) = \sum_{n=0}^{\infty} A_{ij}^{(n)} u^{-n}$ and $B_{ij}(u) = \sum_{n=0}^{\infty} B_{ij}^{(n)} u^{-n}$ such that $\{A_{ij}^{(n)} \mid 1 \le i, j \le D,\ n \in \mathbb{N}\}$ is a generating set for W_1, we shall similarly write

$$\varphi \colon A(u) \to B(u)$$

to mean the morphism $\varphi \colon W_1 \to W_2$ defined by $\varphi(A_{ij}^{(n)}) = B_{ij}^{(n)}$ for all $1 \le i, j \le D$, and $n \in \mathbb{N}$.

Suppose now that $\mathcal{A}$ is a superalgebra generated by $\{A_{ij}^{(n)} \mid 1 \le i, j \le D,\ n \in \mathbb{N}\}$. When studying a representation of $\mathcal{A}$, say V, we would like an efficient way to describe how the generating set of $\mathcal{A}$ acts on such representation. To this extent, if we set

$A_{ij}(u) = \sum_{n=0}^{\infty} A_{ij}^{(n)} u^{-n}$ and $v_{ij}(u) = \sum_{n=0}^{\infty} v_{ij}^{(n)} u^{-n} \in V[\![u^{-1}]\!]$, then when we say that the action of $\mathcal{A}$ on $v \in V$ is given by

$$A_{ij}(u)v = v_{ij}(u),$$

we mean the action $A_{ij}^{(n)} v = v_{ij}^{(n)}$ for all $1 \leq i, j \leq D$ and $n \in \mathbb{N}$.

2.1.3 Orthosymplectic Super Yangians

The first definition of the *super Yangian* $\mathbf{Y}(\mathfrak{osp}_{M|N})$ of $\mathfrak{osp}_{M|N}$ was stated in [?, Section 3]. In this section, we shall define the super Yangian as some quotient of the *extended super Yangian* $\mathbf{X}(\mathfrak{osp}_{M|N})$, which was also introduced in [?, Section 3] but via different terminology. The following construction also yields isomorphic representations of the Yangians of $\mathfrak{so}_M$ when $N = 0$, and $\mathfrak{sp}_N$ when $M = 0$, whose definitions were first provided in [?, Section 2]. To start, we shall recall such definitions of $\mathbf{Y}(\mathfrak{a})$ when $\mathfrak{a} = \mathfrak{so}_M$ and $\mathfrak{a} = \mathfrak{sp}_N$.

The Orthogonal and Symplectic Yangians $\mathbf{Y}(\mathfrak{so}_M)$ and $\mathbf{Y}(\mathfrak{sp}_N)$

In this section, we shall not suppose that any space is equipped with a $\mathbb{Z}_2$-grading; however, we shall keep the sign conventions introduced in Definition (2.1.2) and assume $M, N \in \mathbb{N}$ such that $M + N \geq 1$. To start, we define the *transposition* as the $\mathbb{C}$-linear map defined by

$$\begin{aligned} (-)^{\mathsf{t}} \colon \operatorname{End} \mathbb{C}^{M+N} &\to \operatorname{End} \mathbb{C}^{M+N} \\ E_{ij} &\mapsto E_{ij}^{\mathsf{t}} := \theta_i \theta_j E_{\bar{j}\bar{\imath}} \end{aligned} \tag{2.22}$$

Let the *permutation operator* in $\operatorname{End}\mathbb{C}^{M+N}\otimes\operatorname{End}\mathbb{C}^{M+N}$ be given by

$$\mathsf{P} := \sum_{i,j=1}^{M+N} E_{ij}\otimes E_{ji} \tag{2.23}$$

and further define the Q operator

$$\mathsf{Q} := \mathsf{P}^{\mathsf{t}_1} = \mathsf{P}^{\mathsf{t}_2} = \sum_{i,j=1}^{M+N} \theta_i\theta_j E_{ij}\otimes E_{\bar{\imath}\bar{\jmath}}, \tag{2.24}$$

noting the relations $\mathsf{P} = \mathsf{Q}^{\mathsf{t}_1} = \mathsf{Q}^{\mathsf{t}_2}$. The R-*matrix* $\mathsf{R}(u)\in\left(\operatorname{End}\mathbb{C}^{M+N}\otimes\operatorname{End}\mathbb{C}^{M+N}\right)(u)$ is the rational function in the formal parameter u with coefficients in $\operatorname{End}\mathbb{C}^{M+N}\otimes\operatorname{End}\mathbb{C}^{M+N}$ given by

$$\mathsf{R}(u) := \mathrm{id}^{\otimes 2} - \frac{\mathsf{P}}{u} + \frac{\mathsf{Q}}{u-\mathsf{k}}, \tag{2.25}$$

where $\mathsf{k} = \mathsf{k}_{M,N} := (M+N-2\delta_{0N}+2\delta_{0M})/2$. When $M=0$ or $N=0$, it is known that the R-matrix (2.25) satisfies the *quantum Yang-Baxter equation* (QYBE)

$$\mathsf{R}_{12}(u)\mathsf{R}_{13}(u+v)\mathsf{R}_{23}(v) = \mathsf{R}_{23}(v)\mathsf{R}_{13}(u+v)\mathsf{R}_{12}(u). \tag{2.26}$$

Definition 2.1.6. The *extended Yangian* $\mathbf{X}(\mathfrak{osp}_{M+N})$ is the unital associative $\mathbb{C}$-algebra on generators $\{\mathsf{t}_{ij}^{(n)} \mid 1\le i,j\le M+N,\ n\in\mathbb{Z}^+\}$ subject to the defining RTT-*relation*

$$\begin{gathered}\mathsf{R}(u-v)\mathsf{T}_1(u)\mathsf{T}_2(v) = \mathsf{T}_2(v)\mathsf{T}_1(u)\mathsf{R}(u-v) \\ \text{in}\quad \left(\operatorname{End}\mathbb{C}^{M+N}\right)^{\otimes 2}\otimes\mathbf{X}(\mathfrak{osp}_{M+N})[\![u^{\pm 1},v^{\pm 1}]\!],\end{gathered} \tag{2.27}$$

where $\mathsf{R}(u-v)$ is identified with $\mathsf{R}(u-v)\otimes\mathbf{1}$ and

$$\mathsf{T}(u) := \sum_{i,j=1}^{M+N} E_{ij}\otimes\mathsf{t}_{ij}(u)\in\left(\operatorname{End}\mathbb{C}^{M+N}\right)\otimes\mathbf{X}(\mathfrak{osp}_{M+N})[\![u^{-1}]\!], \tag{2.28}$$

is the *generating matrix* consisting of the *generating series*

$$\mathsf{t}_{ij}(u) := \sum_{n=0}^{\infty}\mathsf{t}_{ij}^{(n)}u^{-n}\in\mathbf{X}(\mathfrak{osp}_{M+N})[\![u^{-1}]\!],\quad\text{where}\quad \mathsf{t}_{ij}^{(0)} = \delta_{ij}\mathbf{1} \tag{2.29}$$

When $N = 0$, the algebra $\mathbf{X}(\mathfrak{osp}_{M+0})$ is called the *extended Yangian* $\mathbf{X}(\mathfrak{so}_M)$ of $\mathfrak{so}_M$. Accordingly, when $M = 0$, the algebra $\mathbf{X}(\mathfrak{osp}_{0+N})$ is called the *extended Yangian* $\mathbf{X}(\mathfrak{sp}_N)$ of $\mathfrak{sp}_N$.

Setting $\mathsf{T}^{\mathsf{t}}(u+\mathsf{k}) := ((-)^{\mathsf{t}} \otimes \mathbf{1})\mathsf{T}(u+\mathsf{k})$, we consider the series $\mathsf{Z}(u) := \mathsf{T}^{\mathsf{t}}(u+\mathsf{k})\mathsf{T}(u)$ and let $(\mathsf{Z}(u) - \mathbf{1})$ denote the two-sided ideal of $\mathbf{X}(\mathfrak{osp}_{M+N})$ generated by the coefficients of $(\mathsf{Z}(u) - \mathbf{1})$. We therefore, get the desired definition

Definition 2.1.7. The *Yangian* $\mathbf{Y}(\mathfrak{osp}_{M+N})$ is the quotient of $\mathbf{X}(\mathfrak{osp}_{M+N})$ by the two-sided ideal $(\mathsf{Z}(u) - \mathbf{1})$, i.e.,

$$\mathbf{Y}(\mathfrak{osp}_{M+N}) := \mathbf{X}(\mathfrak{osp}_{M+N})/(\mathsf{Z}(u) - \mathbf{1}). \tag{2.30}$$

When $N = 0$, the quotient $\mathbf{Y}(\mathfrak{osp}_{M+0})$ is called the *Yangian* $\mathbf{Y}(\mathfrak{so}_M)$ of $\mathfrak{so}_M$. Accordingly, when $M = 0$, the quotient $\mathbf{Y}(\mathfrak{osp}_{0+N})$ is called the *Yangian* $\mathbf{Y}(\mathfrak{sp}_N)$ of $\mathfrak{sp}_N$.

We are now set to return to the super-setting in the next section to introduce the construction of super Yangians of $\mathfrak{osp}_{M|N}$ that translate these algebras to their superalgebra analogues.

The Extended Orthosymplectic Super Yangian $\mathbf{X}(\mathfrak{osp}_{M|N})$

Definition 2.1.8. The *extended super Yangian* $\mathbf{X}(\mathfrak{osp}_{M|N})$ of $\mathfrak{osp}_{M|N}$ is the unital associative $\mathbb{C}$-superalgebra on generators $\{T_{ij}^{(n)} \mid 1 \leq i, j \leq M+N,\ n \in \mathbb{Z}^+\}$, with $\mathbb{Z}_2$-grade $\left[T_{ij}^{(n)}\right] := [i]+[j]$ for all $n \in \mathbb{Z}^+$, subject to the defining *RTT-relation*

$$\begin{gathered} R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v) \\ \text{in} \quad \left(\operatorname{End} \mathbb{C}^{M|N}\right)^{\otimes 2} \otimes \mathbf{X}(\mathfrak{osp}_{M|N})[\![u^{\pm 1}, v^{\pm 1}]\!], \end{gathered} \tag{2.31}$$

where $R(u-v)$ is identified with $R(u-v)\otimes\mathbf{1}$ and

$$T(u) := \sum_{i,j=1}^{M+N} (-1)^{[i][j]+[j]} E_{ij}\otimes T_{ij}(u) \in \big(\operatorname{End}\mathbb{C}^{M|N}\big)\otimes \mathbf{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!], \tag{2.32}$$

is the *generating matrix* consisting of the *generating series*

$$T_{ij}(u) := \sum_{n=0}^{\infty} T_{ij}^{(n)} u^{-n} \in \mathbf{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!], \quad \text{where} \quad T_{ij}^{(0)} = \delta_{ij}\mathbf{1} \tag{2.33}$$

cf. [**?**, Section 3].

Written in terms of power series, the RTT-relation (4.7) has the form

$$\begin{aligned}\big[T_{ij}(u), T_{kl}(v)\big] &= \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\Big(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)\Big)\\ &\quad - \frac{1}{u-v-\kappa}\Bigg(\delta_{\bar{\imath}k}\sum_{p=1}^{M+N}(-1)^{[i][j]+[i]+[j][p]}\theta_i\theta_p T_{pj}(u)T_{\bar{p}l}(v)\\ &\qquad - \delta_{\bar{\jmath}l}\sum_{p=1}^{M+N}(-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]}\theta_j\theta_p T_{k\bar{p}}(v)T_{ip}(u)\Bigg).\end{aligned} \tag{2.34}$$

where $[\,\cdot\,,\cdot\,]$ is understood as the super-bracket

$$\big[T_{ij}(u), T_{kl}(v)\big] = T_{ij}(u)T_{kl}(v) - (-1)^{([i]+[j])([k]+[l])}T_{kl}(v)T_{ij}(u).$$

If we want relations in terms of the explicit generators of $\mathbf{X}(\mathfrak{osp}_{M|N})$, we can multiply equation (4.10) by $(u-v)(u-v-\kappa)$ and equate the coefficients of $u^{-a}v^{-b}$, so the RTT-relation has the explicit form

$$\begin{aligned}&\big[T_{ij}^{(a+2)}, T_{kl}^{(b)}\big] + \big[T_{ij}^{(a)}, T_{kl}^{(b+2)}\big] - 2\big[T_{ij}^{(a+1)}, T_{kl}^{(b+1)}\big]\\ &\quad = \kappa\Big(\big[T_{ij}^{(a+1)}, T_{kl}^{(b)}\big] - \big[T_{ij}^{(a)}, T_{kl}^{(b+1)}\big] - (-1)^{[i][j]+[i][k]+[j][k]}\Big(T_{kj}^{(a)}T_{il}^{(b)} - T_{kj}^{(b)}T_{il}^{(a)}\Big)\Big)\\ &\qquad + (-1)^{[i][k]+[i][l]+[k][l]}\Big(T_{kj}^{(a+1)}T_{il}^{(b)} - T_{kj}^{(b)}T_{il}^{(a+1)} - T_{kj}^{(a)}T_{il}^{(b+1)} + T_{kj}^{(b+1)}T_{il}^{(a)}\Big)\end{aligned}$$

$$
\begin{aligned}
&-\Bigg(\delta_{\bar{\imath}k}\sum_{p=1}^{M+N}(-1)^{[i][j]+[i]+[j][p]}\theta_i\theta_p\Big(T_{pj}^{(a+1)}T_{\bar{p}l}^{(b)}-T_{pj}^{(a)}T_{\bar{p}l}^{(b+1)}\Big)\\
&\qquad-\delta_{\bar{\jmath}l}\sum_{p=1}^{M+N}(-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]}\theta_j\theta_p\Big(T_{k\bar{p}}^{(b)}T_{ip}^{(a+1)}-T_{k\bar{p}}^{(b+1)}T_{ip}^{(a)}\Big)\Bigg),
\end{aligned}
\tag{2.35}
$$

where $a, b \geq -2$ and by convention we have $T_{ij}^{(n)} = 0$ for $n < 0$.

Remark 2.1.9. Definition (2.1.8) of $\mathbf{X}(\mathfrak{osp}_{M|N})$ inherently relies on the selection of the set $\mathbf{d}$ for the gradation index (2.1). Suppose that $\mathbf{X}^{\mathbf{d}_1}(\mathfrak{osp}_{M|N})$ and $\mathbf{X}^{\mathbf{d}_2}(\mathfrak{osp}_{M|N})$ denote two definitions of the extended super Yangian in terms of two different sets $\mathbf{d}_1$ and $\mathbf{d}_2$ as in Definition (2.1.1); accordingly, we denote the generating series for each of these definitions as $T_{ij}^{\mathbf{d}_1}(u)$ and $T_{ij}^{\mathbf{d}_2}(u)$, respectively. If the bijection $\sigma \in \mathfrak{S}_{M+N}$ satisfies $[i]_{\mathbf{d}_1} = [\sigma(i)]_{\mathbf{d}_2}$, $\theta_i^{\mathbf{d}_1} = \theta_{\sigma(i)}^{\mathbf{d}_2}$, and $\sigma(\bar{\imath}^{\mathbf{d}_1}) = \overline{\sigma(i)}^{\mathbf{d}_2}$, then

$$
\begin{aligned}
\mathbf{X}^{\mathbf{d}_1}(\mathfrak{osp}_{M|N}) &\xrightarrow{\sim} \mathbf{X}^{\mathbf{d}_2}(\mathfrak{osp}_{M|N})\\
T_{ij}^{\mathbf{d}_1}(u) &\mapsto T_{\sigma(i)\sigma(j)}^{\mathbf{d}_2}(u)
\end{aligned}
$$

is an isomorphism of superalgebras.

Remark 2.1.10. Now, when $N = 0$ the super permutation operator (4.2) and non-super permutation operator (2.23) coincide: $\mathsf{P} = P$. One can also readily verify in this case that $\mathsf{Q} = Q$ and $\mathsf{k}_{M,0} = \kappa_{M,0}$, so the matrices (4.3) and (2.25) are equal: $\mathsf{R}(u) = R(u)$. Hence, the assignment $\mathsf{T}(u) \mapsto T(u)$ yields an algebra isomorphism $\mathbf{X}(\mathfrak{so}_M) \xrightarrow{\sim} \mathbf{X}(\mathfrak{osp}_{M|0})$. Alternatively, when $M = 0$ we have $\mathsf{P} = -P$, $\mathsf{Q} = -Q$, and $\mathsf{k}_{0,N} = -\kappa_{0,N}$; hence, $\mathsf{R}(u) = R(-u)$. Exchanging $(u, v) \mapsto (-u, -v)$ in the *RTT*-relation (4.7) therefore shows that $\mathsf{T}(u) \mapsto T(-u)$ induces an algebra isomorphism $\mathbf{X}(\mathfrak{so}_N) \xrightarrow{\sim} \mathbf{X}(\mathfrak{osp}_{0|N})$.

Automorphisms of $\mathbf{X}(\mathfrak{osp}_{M|N})$

Proposition 2.1.11. *For any formal series* $f = f(u) = 1 + \sum_{n=1}^{\infty} f_n u^{-n} \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ *and* $a \in \mathbb{C}$*, the maps*

$$\mu_f\colon T(u) \mapsto f(u)T(u) \tag{2.36}$$

$$\tau_a\colon T(u) \mapsto T(u-a) \tag{2.37}$$

induce superalgebra automorphisms of $\mathbf{X}(\mathfrak{osp}_{M|N})$*, cf.* [?, Section 1], [?, Section 1]. *More explicitly, the maps above take the form*

$$\mu_f\colon T_{ij}^{(n)} \mapsto \sum_{a+b=n} f_a T_{ij}^{(b)} \quad \text{and} \quad \tau_a\colon T_{ij}^{(n)} \mapsto \sum_{k=1}^{n} \binom{n-1}{n-k} a^{n-k} T_{ij}^{(k)} \quad \text{for} \quad n \in \mathbb{Z}^+.$$

Proof. We first note that both induced maps are grade-preserving. Furthermore, each are invertible: the inverse of (3.20) is $\mu_{f^{-1}}$ since $f(u)$ has constant term 1 and the inverse of (2.37) is τ_{-a}.

To show that the induced maps are superalgebra morphisms, it suffices to show that images of (3.20) and (2.37) satisfy the RTT-relation (4.7). To show the first, one multiplies the RTT-relation by $f(u)f(v)$, and the second one follows since the RTT-relation is invariant under shifts of the parameter u.

□

Anti-Automorphisms of $\mathbf{X}(\mathfrak{osp}_{M|N})$

We may regard $T(u)$ defined by (4.8) as a formal power series in u^{-1} whose coefficients lie in $\operatorname{End}\mathbb{C}^{M|N} \otimes \mathbf{X}(\mathfrak{osp}_{M|N})$. Since the constant term of the power series $T(u)$ is the unit object $\mathbb{1} = \mathrm{id} \otimes \mathbf{1}$, we conclude that $T(u)$ has an inverse, denoted $T(u)^{-1}$. Further, we shall understand $T^t(u)$ as $((-)^t \otimes \mathbf{1})T(u)$ and therefore have

$$T^t(u) = \sum_{i,j=1}^{M+N} (-1)^{[i]+[j]} \theta_i \theta_j E_{\bar{j}\bar{i}} \otimes T_{ij}(u) = \sum_{i,j=1}^{M+N} (-1)^{[i]+[j]} \theta_{\bar{i}} \theta_{\bar{j}} E_{ij} \otimes T_{\bar{j}\bar{i}}(u)$$
$$= \sum_{i,j=1}^{M+N} (-1)^{[i][j]+[j]} E_{ij} \otimes (-1)^{[i][j]+[j]} \theta_i \theta_j T_{\bar{j}\bar{i}}(u),$$

Hence, by interpreting $T(u) = \big(T_{ij}(u)\big)_{i,j=1}^{M+N}$ as a matrix in $\mathrm{Mat}_{M+N}\big(\mathbf{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]\big)$, then

$$T^t(u) = \big((-1)^{[i][j]+[j]} \theta_i \theta_j T_{\bar{j}\bar{i}}(u)\big)_{i,j=1}^{M+N}, \tag{2.38}$$

so we accordingly define

$$T^t_{ij}(u) = (-)^{[i][j]+[j]} \theta_i \theta_j T_{\bar{j}\bar{i}}(u) \quad \text{and} \quad \big(T^{(n)}_{ij}\big)^t = (-)^{[i][j]+[j]} \theta_i \theta_j T^{(n)}_{\bar{j}\bar{i}}. \tag{2.39}$$

Similarly, we interpret $T^t_1(u)$ as $((-)^t \otimes \mathrm{id} \otimes \mathbf{1})T_1(u)$ and $T^t_2(u)$ as $(\mathrm{id} \otimes (-)^t \otimes \mathbf{1})T_2(u)$.

Let $(-)^\circ \in \mathrm{End}\, \mathbf{X}(\mathfrak{osp}_{M|N})$ be a superalgebra anti-morphism in the category $\mathrm{sVect}_{\mathbb{C}}$. For generating series $T_{ij}(u), T_{kl}(v) \in \mathbf{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$, we then have the set of equalities $(T_{ij}(u)T_{kl}(v))^\circ = T^\circ_{ij}(u) \cdot_{\mathrm{op}} T^\circ_{kl}(v) = (-1)^{([i]+[j])([k]+[l])} T^\circ_{kl}(v) T^\circ_{ij}(u)$, where $\cdot_{\mathrm{op}}$ stands for multiplication for algebras in the opposite category $\mathrm{sVect}^{\mathrm{op}}_{\mathbb{C}}$. The map $(-)^\circ$ must therefore satisfy the relations

$$\begin{aligned} T^\circ_{ij}(u) \cdot_{\mathrm{op}} T^\circ_{kl}(v) &- (-1)^{([i]+[j])([k]+[l])} T^\circ_{kl}(v) \cdot_{\mathrm{op}} T^\circ_{ij}(u) \\ &= \frac{1}{u-v} (-1)^{[i][j]+[i][k]+[j][k]} \Big(T^\circ_{kj}(u) \cdot_{\mathrm{op}} T^\circ_{il}(v) - T^\circ_{kj}(v) \cdot_{\mathrm{op}} T^\circ_{il}(u) \Big) \\ &\quad - \frac{1}{u-v-\kappa} \Bigg(\delta_{\bar{\imath}k} \sum_{p=1}^{M+N} (-1)^{[i][j]+[i]+[j][p]} \theta_i \theta_p T^\circ_{pj}(u) \cdot_{\mathrm{op}} T^\circ_{\bar{p}l}(v) \\ &\qquad - \delta_{\bar{\jmath}l} \sum_{p=1}^{M+N} (-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]} \theta_j \theta_p T^\circ_{k\bar{p}}(v) \cdot_{\mathrm{op}} T^\circ_{ip}(u) \Bigg). \end{aligned}$$

Rewritten, these relations have the form

$$\big[T^\circ_{ij}(u), T^\circ_{kl}(v)\big] = \frac{1}{u-v} (-1)^{[j][k]+[j][l]+[k][l]} \Big(T^\circ_{il}(u) T^\circ_{kj}(v) - T^\circ_{il}(v) T^\circ_{kj}(u) \Big)$$

$$+\frac{1}{u-v-\kappa}\Bigg(\delta_{\bar{\imath}k}\sum_{p=1}^{M+N}(-1)^{[i][j]+[i]+[j][l]+[l][p]+[p]}\theta_i\theta_p T^{\circ}_{\bar{p}l}(v)T^{\circ}_{pj}(u)$$

$$-\delta_{\bar{\jmath}l}\sum_{p=1}^{M+N}(-1)^{[j][k]+[j]+[k][p]}\theta_j\theta_p T^{\circ}_{ip}(u)T^{\circ}_{k\bar{p}}(v)\Bigg).$$

Defining $T^{\circ}(u)=\sum_{i,j=1}^{M+N}(-1)^{[i][j]+[j]}E_{ij}\otimes T^{\circ}_{ij}(u)$ and $T^{\circ}_i(u)$, $i=1,2$ in the suitable ways, these relations may be equivalently written as a variant of the RTT-relation:

$$R(u-v)T^{\circ}_2(v)T^{\circ}_1(u)=T^{\circ}_1(u)T^{\circ}_2(v)R(u-v). \tag{2.40}$$

Proposition 2.1.12. *The maps*

$$\sigma\colon T(u)\mapsto T(-u) \tag{2.41}$$

$$(-)^t\colon T(u)\mapsto T^t(u) \tag{2.42}$$

$$S\colon T(u)\mapsto T(u)^{-1} \tag{2.43}$$

induce superalgebra anti-automorphisms of $\mathbf{X}(\mathfrak{osp}_{M|N})$*, cf. [?, Section 1] and [?, Section 1]*

Proof. We first observe that the maps defined by (2.41) and (2.42) are both grade-preserving and involutive. To show that these induced maps are anti-morphisms, it suffices that show that the images of (2.41) and (2.42) satisfy the variant RTT-relation (5.1). In order to obtain the relation

$$R(u-v)T_2(-v)T_1(-u)=T_1(-u)T_2(-v)R(u-v),$$

first perform the exchange $(u,v)\mapsto(-v,-u)$ in the RTT-relation (4.7). One then multiplies both sides by $R(u-v)$ and utilizes equation (2.19) to imply the result.

Let us now show that (2.42) is an anti-morphism. Applying the map $(-)^{t_1}\otimes\mathbf{1}$, which we will identify as $(-)^{t_1}$, to the RTT-relation (4.7) yields the equation

$$(R(u-v)T_1(u))^{t_1}\, T_2(v) = T_2(v)\,(T_1(u)R(u-v))^{t_1}\,,$$

which may be expressed as

$$T_1^{t_1}(u)R^{t_1}(u-v)T_2(v) = T_2(v)R^{t_1}(u-v)T_1^{t_1}(u) \tag{2.44}$$

by using the equations $(R(u-v)T_1(u))^{t_1} = T_1^{t_1}(u)R^{t_1}(u-v)$ and $(T_1(u)R(u-v))^{t_1} = R^{t_1}(u-v)T_1^{t_1}(u)$. Next, by applying the map $(-)^{t_2}\otimes \mathbf{1}$, identified as $(-)^{t_2}$, to equation (2.44), one obtains

$$T_1^{t_1}(u)\left(R^{t_1}(u-v)T_2(v)\right)^{t_2} = \left(T_2(v)R^{t_1}(u-v)\right)^{t_2} T_1^{t_1}(u),$$

which yields the desired relation

$$T_1^{t_1}(u)T_2^{t_2}(v)R(u-v) = R(u-v)T_2^{t_2}(v)T_1^{t_1}(u),$$

since $\left(R^{t_1}(u-v)T_2(v)\right)^{t_2} = T_2^{t_2}(v)R(u-v)$ and $\left(T_2(v)R^{t_1}(u-v)\right)^{t_2} = R(u-v)T_2^{t_2}(v)$.

Since $T(u)^{-1} = \sum_{i,j=1}^{M+N}(-1)^{[i][j]+[j]}E_{ij}\otimes T_{ij}^{\bullet}(u)$, where $T_{ij}^{\bullet}(u) = \mathbf{1} + \sum_{n=1}^{\infty} T_{ij}^{\bullet(n)}u^{-n}$ and

$$T_{ij}^{\bullet(n)} = -T_{ij}^{(n)} + \sum_{s=2}^{n}(-1)^s \sum_{\sum_{j=1}^{s} k_j = n} \left(\sum_{a_1,\dots,a_{s-1}=1}^{M+N} T_{ia_1}^{(k_1)}T_{a_1a_2}^{(k_2)}\cdots T_{a_{s-1}j}^{(k_s)} \right)$$

with $k_j\in\mathbb{Z}^+$ for each k_j in the sum $\sum_{j=1}^{s} k_j = n$, the map (2.43) is grade-preserving.

To prove that (2.43) is an algebra anti-morphism amounts to observing that the RTT-relation (4.7) is equivalent to the equation

$$R(u-v)T_2(v)^{-1}T_1(u)^{-1} = T_1(u)^{-1}T_2(v)^{-1}R(u-v).$$

To see that the map is invertible, first consider the composition

$$\omega := S \circ \sigma \colon T(u) \mapsto T(-u)^{-1}. \tag{2.45}$$

Consequently, ω is a morphism of superalgebras and is in fact involutive. Indeed, ω acting on the identity $\omega(T(u))T(-u) = \mathbb{1}$ yields

$$\omega^2(T(u))T(u)^{-1} = \mathbb{1},$$

proving that ω^2 is the identity map. Hence, S is bijective as well. □

We emphasize the the map (2.43) is *not* involutive. Moreover, such map will be the antipode of a Hopf superstructure on $\mathbf{X}(\mathfrak{osp}_{M|N})$.

Hopf Superalgebra Structure of $\mathbf{X}(\mathfrak{osp}_{M|N})$

Proposition 2.1.13. *The extended super Yangian $\mathbf{X}(\mathfrak{osp}_{M|N})$ has a Hopf superalgebra structure given by the comultiplication*

$$\begin{aligned} \Delta \colon \mathbf{X}(\mathfrak{osp}_{M|N}) &\to \mathbf{X}(\mathfrak{osp}_{M|N}) \otimes \mathbf{X}(\mathfrak{osp}_{M|N}) \\ T(u) &\mapsto T_{[1]}(u)T_{[2]}(u) \end{aligned} \tag{2.46}$$

the counit

$$\begin{aligned} \varepsilon \colon \mathbf{X}(\mathfrak{osp}_{M|N}) &\to \mathbb{C} \\ T(u) &\mapsto \mathbb{1} \end{aligned} \tag{2.47}$$

and the antipode

$$\begin{aligned} S \colon \mathbf{X}(\mathfrak{osp}_{M|N}) &\to \mathbf{X}(\mathfrak{osp}_{M|N}) \\ T(u) &\mapsto T(u)^{-1} \end{aligned} \tag{2.48}$$

previously given as (2.43), cf. [?, Section 3]. We observe that the comultiplication is the map

$$\Delta\colon T_{ij}(u) \mapsto \sum_{k=1}^{M+N} T_{ik}(u) \otimes T_{kj}(u),$$

which is more explicitly given by $\Delta(T_{ij}^{(n)}) = \sum_{k=1}^{M+N} \sum_{a+b=n} T_{ik}^{(a)} \otimes T_{kj}^{(b)}$*. The counit map is explicitly given by* $\varepsilon(T_{ij}^{(n)}) = \delta_{n0}\delta_{ij}$*.*

The Central Series $\mathcal{Z}(u)$

Let us define $Z(u) := T^t(u+\kappa)T(u)$ and further consider the series $\mathcal{Z}(u)$ lying in $\mathbf{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$ such that $\mathrm{id} \otimes \mathcal{Z}(u) = Z(u)$. Multiplying both sides of the *RTT*-relation by $u - v - \kappa$, setting $u = v + \kappa$, and replacing v by u yields the equation

$$QT_1(u+\kappa)T_2(u) = T_2(u)T_1(u+\kappa)Q. \tag{2.49}$$

Using that $QT_1(u) = QT_2^t(u)$ and $T_1(u)Q = T_2^t(u)Q$ and transposing the first tensor factor of (3.26), we deduce

$$P \otimes \mathcal{Z}(u) = PT_2^t(u+\kappa)T_2(u) = T_2(u)T_2^t(u+\kappa)P.$$

Multiplying the above on left by P, we yield $\mathrm{id}^{\otimes 2} \otimes \mathcal{Z}(u) = T_2^t(u+\kappa)T_2(u)$. Similarly, if instead we multiply on the right by P, we obtain $\mathrm{id}^{\otimes 2} \otimes \mathcal{Z}(u) = T_2(u)T_2^t(u+\kappa)$. Therefore,

$$Z(u) = T^t(u+\kappa)T(u) = T(u)T^t(u+\kappa), \tag{2.50}$$

or rather put, since

$$\begin{aligned} T^t(u+\kappa)T(u) &= \left(\sum_{i,k=1}^{M+N} (-1)^{[i][k]+[k]} E_{ik} \otimes T_{ik}^t(u+\kappa)\right)\left(\sum_{l,j=1}^{M+N} (-1)^{[l][j]+[j]} E_{lj} \otimes T_{lj}(u)\right) \\ &= \sum_{i,j,k=1}^{M+N} (-1)^{[i][k]+[k]+[k][j]+[j]+([i]+[k])([k]+[j])} E_{ij} \otimes T_{ik}^t(u+\kappa)T_{kj}(u) \end{aligned}$$

$$= \sum_{i,j,k=1}^{M+N} (-1)^{[i][j]+[j]} E_{ij} \otimes T^t_{ik}(u+\kappa) T_{kj}(u)$$

and

$$\begin{aligned} T(u)T^t(u+\kappa) &= \left(\sum_{i,k=1}^{M+N} (-1)^{[i][k]+[k]} E_{ik} \otimes T_{ik}(u) \right) \left(\sum_{l,j=1}^{M+N} (-1)^{[l][j]+[j]} E_{lj} \otimes T^t_{lj}(u+\kappa) \right) \\ &= \sum_{i,j,k=1}^{M+N} (-1)^{[i][k]+[k]+[k][j]+[j]+([i]+[k])([k]+[j])} E_{ij} \otimes T_{ik}(u) T^t_{kj}(u+\kappa) \\ &= \sum_{i,j,k=1}^{M+N} (-1)^{[i][j]+[j]} E_{ij} \otimes T_{ik}(u) T^t_{kj}(u+\kappa) \end{aligned}$$

are both equal to $\mathrm{id} \otimes \mathcal{Z}(u)$, then

$$\sum_{k=1}^{M+N} T^t_{ik}(u+\kappa) T_{kj}(u) = \sum_{k=1}^{M+N} T_{ik}(u) T^t_{kj}(u+\kappa) = \delta_{ij} \mathcal{Z}(u), \tag{2.51}$$

where

$$\mathcal{Z}(u) = \mathbf{1} + \sum_{n=1}^{\infty} \mathcal{Z}_n u^{-n} \in \mathbf{1} + u^{-1} \mathbf{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!] \tag{2.52}$$

We note that the coefficients of $\mathcal{Z}(u)$ are homogeneous of even degree, so $\mathcal{Z}(u)$ lies in the even subalgebra of $\mathbf{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$. Let us furthermore denote $(\mathcal{Z}(u) - \mathbf{1})$ to mean the two-sided ideal of $\mathbf{X}(\mathfrak{osp}_{M|N})$ generated by the coefficients of $\mathcal{Z}(u) - \mathbf{1}$. We note that this ideal is in fact graded since it is generated by homogeneous elements.

Explicitly, the coefficients of $\mathcal{Z}(u)$ are given by

$$\delta_{ij} \mathcal{Z}_n = \sum_{k=1}^{M+N} (-1)^{[i][k]+[k]} \theta_i \theta_k \sum_{a+b=n} \sum_{p=1}^{a} \binom{a-1}{a-p} (-\kappa)^{a-p} T^{(p)}_{\bar{k}\bar{\imath}} T^{(b)}_{kj} \tag{2.53}$$

Definition 2.1.14. Let $\mathbf{ZX}(\mathfrak{osp}_{M|N})$ denote the subalgebra of $\mathbf{X}(\mathfrak{osp}_{M|N})$ generated by the coefficients of the series $\mathcal{Z}(u)$ as in (3.29), cf. [**?**, Section 3].

Proposition 2.1.15. *The coefficients of the series* $\mathcal{Z}(u) \in \mathbf{1} + u^{-1}\mathbf{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$ *given by the equation* $T^t(u+\kappa)T(u) = T(u)T^t(u+\kappa) = \mathrm{id} \otimes \mathcal{Z}(u)$ *lie in the center of*

$\mathbf{X}(\mathfrak{osp}_{M|N})$. *Furthermore,*

$$\Delta\colon \mathcal{Z}(u) \mapsto \mathcal{Z}(u) \otimes \mathcal{Z}(u) \tag{2.54}$$

where Δ *is the comultiplication map* (**??**), *and* $(\mathcal{Z}(u) - \mathbf{1})$ *is a graded bi-ideal.*

Proof. This proof was provided in [**?**, Theorem 3.1], but let us reproduce the argument here. We observe

$$Z(u)T_2(v) = T_1^t(u+\kappa)T_1(u)T_2(v) = T_1^t(u+\kappa)R(u-v)^{-1}T_2(v)T_1(u)R(u-v), \tag{2.55}$$

where we used the RTT-relation. Now, by transposing the first tensor factor of the RTT-relation (4.7) and using properties (2.18) and (2.19), we get

$$T_1^t(u+\kappa)R(u-v)^{-1}T_2(v) = T_2(v)R(u-v)^{-1}T_1^t(u+\kappa). \tag{2.56}$$

Therefore,

$$\begin{aligned} Z(u)T_2(v) &= T_2(v)R(u-v)^{-1}T_1^t(u+\kappa)T_1(u)R(u-v) \\ &= T_2(v)R(u-v)^{-1}Z(u)R(u-v) = T_2(v)Z(u), \end{aligned}$$

since $Z(u)$ commutes with $R(u-v)$. Furthermore, $\Delta\colon \mathcal{Z}(u) \mapsto \mathcal{Z}(u) \otimes \mathcal{Z}(u)$ is readily verified, since

$$\begin{aligned} \Delta(\mathcal{Z}(u)) &= \sum_{k=1}^{M+N} (-1)^{[i][k]+[k]}\theta_i\theta_k \Delta\big(T_{\bar{k}\bar{\imath}}(u+\kappa)T_{ki}(u)\big) \\ &= \sum_{a,b,k=1}^{M+N} (-1)^{[i][k]+[k]}\theta_i\theta_k \big(T_{\bar{k}a}(u+\kappa) \otimes T_{a\bar{\imath}}(u+\kappa)\big)\big(T_{kb}(u) \otimes T_{bi}(u)\big) \\ &= \sum_{a,b,k=1}^{M+N} (-1)^{([a]+[i])[b]+[a][k]+[k]}\theta_i\theta_k T_{\bar{k}a}(u+\kappa)T_{kb}(u) \otimes T_{a\bar{\imath}}(u+\kappa)T_{bi}(u) \\ &= \sum_{a,b,k=1}^{M+N} (-1)^{([a]+[i])([a]+[b])} T^t_{\bar{a}k}(u+\kappa)T_{kb}(u) \otimes T^t_{i\bar{a}}(u+\kappa)T_{bi}(u) \\ &= \sum_{a,b=1}^{M+N} (-1)^{([a]+[i])([a]+[b])} \delta_{\bar{a}b}\mathcal{Z}(u) \otimes T^t_{i\bar{a}}(u+\kappa)T_{bi}(u) = \mathcal{Z}(u) \otimes \mathcal{Z}(u). \end{aligned}$$

Let us set $\mathcal{I} = (\mathcal{Z}(u) - \mathbf{1})$. One may verify that $\varepsilon\colon \mathcal{Z}(u) \mapsto 1$ and so $\varepsilon(\mathcal{I}) = 0$. Moreover, since $\Delta(\mathcal{Z}_n) = \sum_{a+b=n} \mathcal{Z}_a \otimes \mathcal{Z}_b$ (where $\mathcal{Z}_0 = \mathbf{1}$), then for $X \in \mathbf{X}(\mathfrak{osp}_{M|N})$ we have $\Delta(X\mathcal{Z}_n), \Delta(\mathcal{Z}_n X) \in \mathcal{I} \otimes \mathbf{X}(\mathfrak{osp}_{M|N}) + \mathbf{X}(\mathfrak{osp}_{M|N}) \otimes \mathcal{I}$, so $\mathcal{I}$ is a coideal. □

Now, under the axioms of the Hopf superalgebra structure, the image of $\mathcal{Z}(u)$ under the antipode is given by

$$S\colon \mathcal{Z}(u) \mapsto \mathcal{Z}(u)^{-1}. \tag{2.57}$$

We therefore have the immediate corollary:

Corollary 2.1.16. $\mathbf{ZX}(\mathfrak{osp}_{M|N})$ *is a sub-Hopf superalgebra and* $(\mathcal{Z}(u) - \mathbf{1})$ *is a graded Hopf ideal of of* $\mathbf{X}(\mathfrak{osp}_{M|N})$*.*

By identifying $\mathcal{Z}(u)$ with $Z(u)$, equation (2.50) shows that the inverse of $T(u)$ is given by

$$T(u)^{-1} = \mathcal{Z}(u)^{-1} T^t(u + \kappa), \tag{2.58}$$

so the antipode S is the mapping

$$S\colon T(u) \mapsto \mathcal{Z}(u)^{-1} T^t(u + \kappa). \tag{2.59}$$

In particular, the square of the antipode is given by

$$S^2\colon T(u) \mapsto \frac{\mathcal{Z}(u)}{\mathcal{Z}(u + \kappa)} T(u + 2\kappa), \tag{2.60}$$

cf. [**?**, Section 2].

The Associated Graded Superalgebra $\operatorname{gr} \mathbf{X}(\mathfrak{osp}_{M|N})$

We shall now consider two (ascending algebra) filtrations on $\mathbf{X}(\mathfrak{osp}_{M|N})$, denoted $\mathbf{E}^{\mathrm{Ab}}\big(\mathbf{X}(\mathfrak{osp}_{M|N})\big) = \mathbf{E}^{\mathrm{Ab}} = \{\mathbf{E}_n^{\mathrm{Ab}}\}_{n \in \mathbb{N}}$ and $\mathbf{E}\big(\mathbf{X}(\mathfrak{osp}_{M|N})\big) = \mathbf{E} = \{\mathbf{E}_n\}_{n \in \mathbb{N}}$, given via the respective filtration degree assignments

$$\deg_{\mathrm{Ab}} T_{ij}^{(n)} = n \quad \text{and} \quad \deg_{\mathbf{E}} T_{ij}^{(n)} = n-1. \tag{2.61}$$

Note that the second filtration is given explicitly by

$$\begin{aligned}\mathbf{E}_n &= \mathrm{span}_{\mathbb{C}} \big\{ \textstyle\prod_{a=1}^{\gamma} T_{i_a j_a}^{(k_a)} \mid \gamma \in \mathbb{Z}^+,\ 1 \le i_a, j_a \le M+N,\ \sum_{a=1}^{\gamma} \deg_{\mathbf{E}} T_{i_a j_a}^{(k_a)} \le n \big\} \\ &= \mathrm{span}_{\mathbb{C}} \big\{ \textstyle\prod_{a=1}^{\gamma} T_{i_a j_a}^{(k_a)} \mid \gamma \in \mathbb{Z}^+,\ 1 \le i_a, j_a \le M+N,\ \sum_{a=1}^{\gamma} k_a \le n+\gamma \big\}\end{aligned}$$

for $n \in \mathbb{Z}^+$ and

$$\mathbf{E}_0 = \mathrm{span}_{\mathbb{C}} \big\{ \textstyle\prod_{a=1}^{\gamma} T_{i_a j_a}^{(k_a)} \mid \gamma \in \mathbb{Z}^+,\ 1 \le i_a, j_a \le M+N,\ k_a \in \{0,1\} \big\}.$$

From the defining relations (4.10), one can deduce that the associated graded superalgebra $\mathrm{gr}_{\mathrm{Ab}}\, \mathbf{X}(\mathfrak{osp}_{M|N}) = \bigoplus_{n\in\mathbb{N}} \mathbf{E}_n^{\mathrm{Ab}}/\mathbf{E}_{n-1}^{\mathrm{Ab}}$ corresponding to the first filtration $\mathbf{E}^{\mathrm{Ab}}$ is supercommutative. Our focus will divert to the second filtration $\mathbf{E}$ which will induce a more interesting associated graded superalgebra:

$$\mathrm{gr}\, \mathbf{X}(\mathfrak{osp}_{M|N}) = \mathrm{gr}_{\mathbf{E}}\, \mathbf{X}(\mathfrak{osp}_{M|N}) = \bigoplus_{n\in\mathbb{N}} \mathbf{E}_n/\mathbf{E}_{n-1}.$$

We note that $\mathrm{gr}\, \mathbf{X}(\mathfrak{osp}_{M|N})$ inherits a $\mathbb{Z}_2$-graded structure from $\mathbf{X}(\mathfrak{osp}_{M|N})$ by assigning $\mathbb{Z}_2$-grade $[i]+[j]$ to the image $\overline{T}_{ij}^{(n)}$ of $T_{ij}^{(n)}$ in $\mathbf{E}_n/\mathbf{E}_{n-1}$. Furthermore, by endowing $\mathbf{X}(\mathfrak{osp}_{M|N})^{\otimes 2}$ with the filtration $\mathbf{E}^2 = \{\mathbf{E}_n^2\}_{n\in\mathbb{N}}$ induced by $\mathbf{E}$, i.e., $\mathbf{E}_n^2 = \bigoplus_{i+j=n} \mathbf{E}_i \otimes \mathbf{E}_j$, and assigning $\mathbb{C}$ with the trivial ascending algebra filtration $\mathbf{C} = \{\mathbf{C}_n\}_{n\in\mathbb{N}}$ where $\mathbf{C}_n = \mathbb{C}$ for all $n \in \mathbb{N}$, we observe that each of the Hopf superalgebra structure maps on $\mathbf{X}(\mathfrak{osp}_{M|N})$ preserve their relative filtrations. Therefore, the structure maps Δ and ε descend to the respective superalgebra morphisms

$$\begin{aligned}\mathrm{gr}\,\Delta\colon\ \mathrm{gr}\, \mathbf{X}(\mathfrak{osp}_{M|N}) &\to \mathrm{gr}\big(\mathbf{X}(\mathfrak{osp}_{M|N})^{\otimes 2}\big) \cong \big(\mathrm{gr}\, \mathbf{X}(\mathfrak{osp}_{M|N})\big)^{\otimes 2} \\ \overline{T}_{ij}^{(n)} &\mapsto \overline{T}_{ij}^{(n)} \otimes \mathbf{1} + \mathbf{1} \otimes \overline{T}_{ij}^{(n)},\ n \in \mathbb{Z}^+,\end{aligned} \tag{2.62}$$

where the above isomorphism is induced by the map $\mathbf{E}_n^2 \to \bigoplus_{i+j=n} \mathbf{E}_i/\mathbf{E}_{i-1} \otimes \mathbf{E}_j/\mathbf{E}_{j-1}$, and

$$\begin{aligned} \operatorname{gr}\varepsilon\colon\ \operatorname{gr}\mathbf{X}(\mathfrak{osp}_{M|N}) &\to \mathbb{C} \\ \overline{T}_{ij}^{(n)} &\mapsto \delta_{0n}\delta_{ij}. \end{aligned} \tag{2.63}$$

Of course, the antipode S will also descend to the superalgebra antimorphism

$$\begin{aligned} \operatorname{gr} S\colon\ \operatorname{gr}\mathbf{X}(\mathfrak{osp}_{M|N}) &\to \operatorname{gr}\mathbf{X}(\mathfrak{osp}_{M|N}) \\ \overline{T}_{ij}^{(n)} &\mapsto \overline{T}_{ij}^{\bullet(n)} = -\overline{T}_{ij}^{(n)} \end{aligned} \tag{2.64}$$

The fact that the Hopf algebra structure maps on $\mathbf{X}(\mathfrak{osp}_{M|N})$ are filtration preserving is equivalent to saying that $\mathbf{E}$ is a Hopf filtration on $\mathbf{X}(\mathfrak{osp}_{M|N})$. As such, $\operatorname{gr}\mathbf{X}(\mathfrak{osp}_{M|N})$ is therefore equipped with an $\mathbb{N}$-graded Hopf superalgebra structure given by the morphisms (2.62), (2.63), and (2.64) [?, Proposition 7.9.2]; however, one could also verify the Hopf superalgebra axioms directly.

Given a Lie superalgebra $\mathfrak{g}$, allow $\mathfrak{g}[z]$ to denote the *polynomial current Lie superalgebra* associated to $\mathfrak{g}$; that is, $\mathfrak{g}[z]$ is equal to $\mathfrak{g} \otimes \mathbb{C}[z]$, as a super vector space (where $\mathbb{C}[z] = \mathbb{C}^{1|0}[z]$), and is equipped with the Lie superbracket

$$[X \otimes f(z), Y \otimes g(z)] := [X, Y] \otimes f(z)g(z) \quad \text{for } X, Y \in \mathfrak{g}, \text{ and } f(z), g(z) \in \mathbb{C}[z].$$

Equivalently, $\mathfrak{g}[z]$ may be regarded as the Lie superalgebra of polynomial maps $f\colon \mathbb{C} \to \mathfrak{g}$ with Lie superbracket given point-wise, and $\mathfrak{g}[z]$ is an $\mathbb{N}$-graded Lie superalgebra $\bigoplus_{n\in\mathbb{N}} \mathfrak{g}[z]_n$, where $\mathfrak{g}[z]_n = \mathfrak{g} \otimes \mathbb{C}z^n$. Furthermore, we shall use the identification $Xz^n = X \otimes z^n$ for elements in $\mathfrak{g}[z]$.

Given such Lie superalgebra $\mathfrak{g}$, we let $\mathfrak{U}(\mathfrak{g})$ denote its universal enveloping superalgebra. That is, $\mathfrak{U}(\mathfrak{g}) = T(\mathfrak{g})/I(\mathfrak{g})$, where $T(\mathfrak{g})$ is the tensor superalgebra of $\mathfrak{g}$ and $I(\mathfrak{g})$ is the two-sided ideal generated by elements of the form $X \otimes Y - (-1)^{[X][Y]} Y \otimes X - [X, Y]$, where $X, Y \in \mathfrak{g}$ are homogeneous. Furthermore, $\mathfrak{U}(\mathfrak{g})$ is endowed with a Hopf superalgebra structure given by the comultiplication $\Delta\colon \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$, $X \mapsto X \otimes 1 + 1 \otimes X$, counit $\varepsilon\colon \mathfrak{U}(\mathfrak{g}) \to \mathbb{C}$, $X \mapsto 0$, and antipode $S\colon \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$, $X \mapsto -X$ for $X \in \mathfrak{g}$ and where the $\mathbb{Z}_2$-gradation on $\mathfrak{U}(\mathfrak{g})$ is induced by the one on $\mathfrak{g}$. In the case when $\mathfrak{g} = \mathfrak{g}[z]$ is a polynomial current Lie superalgebra, we see that $\mathfrak{U}(\mathfrak{g}[z])$ is an $\mathbb{N}$-graded superalgebra $\bigoplus_{n\in\mathbb{N}} \mathfrak{U}^n(\mathfrak{g}[z])$, where

$$\mathfrak{U}^n(\mathfrak{g}[z]) = \operatorname{span}_{\mathbb{C}}\big\{ \textstyle\prod_{a=1}^{\gamma} X_a z^{k_a} \mid \gamma \in \mathbb{Z}^+,\ X_a \in \mathfrak{g},\ \sum_{a=1}^{\gamma} k_a = n \big\}.$$

Let us now consider a central extension $\mathfrak{osp}_{M|N} \oplus \mathfrak{z}_{\mathsf{c}}$ of $\mathfrak{osp}_{M|N}$ by a 1-dimensional abelian Lie superalgebra $\mathfrak{z}_{\mathsf{c}} := \mathbb{C}^{1|0}\mathsf{c}$. As a Lie superalgebra, $(\mathfrak{osp}_{M|N} \oplus \mathfrak{z}_{\mathsf{c}})[z]$ is generated by the elements $\{F_{ij}z^m, \mathsf{c}z^n \mid 1 \le i,j \le M+N,\ m,n \in \mathbb{N}\}$ satisfying the relations

$$\begin{aligned}[] [F_{ij}z^m, F_{kl}z^n] = \delta_{jk}F_{il}z^{m+n} &- \delta_{il}(-1)^{([i]+[j])([k]+[l])}F_{kj}z^{m+n} \\ &- \delta_{\bar{i}k}(-1)^{[i][j]+[i]}\theta_i\theta_j F_{\bar{j}l}z^{m+n} + \delta_{\bar{j}l}(-1)^{([i]+[j])[k]}\theta_{\bar{i}}\theta_{\bar{j}}F_{k\bar{i}}z^{m+n}, \end{aligned} \tag{2.65}$$

$$F_{ij}z^n + (-1)^{[i][j]+[i]}\theta_i\theta_j F_{\bar{j}\bar{i}}z^n = 0, \quad \text{and} \quad \big[F_{ij}z^m, \mathsf{c}z^n\big] = 0. \tag{2.66}$$

Note that by the defining equation for the central series coefficients $\mathcal{Z}_n$, we have

$$\delta_{ij}\mathcal{Z}_n \equiv T_{ij}^{(n)} + (-1)^{[i][j]+[j]}\theta_i\theta_j T_{\bar{j}\bar{i}}^{(n)} \mod \mathbf{E}_{n-2} \tag{2.67}$$

In particular, $\mathcal{Z}_n$ has filtration degree $n-1$, so we shall let $\overline{\mathcal{Z}}_n$ denote the image of $\mathcal{Z}_n$ in $\mathbf{E}_{n-1}/\mathbf{E}_{n-2}$.

Proposition 2.1.17. *There is a surjective $\mathbb{N}$-graded Hopf superalgebra morphism*

$$\begin{aligned} \Psi\colon \mathfrak{U}\big((\mathfrak{osp}_{M|N} \oplus \mathfrak{z}_{\mathsf{c}})[z]\big) &\to \operatorname{gr}\mathbf{X}(\mathfrak{osp}_{M|N}) \\ F_{ij}z^{n-1} &\mapsto (-1)^{[i]}\big(\overline{T}_{ij}^{(n)} - \tfrac{1}{2}\delta_{ij}\overline{\mathcal{Z}}_n\big) \\ \mathsf{c}z^{n-1} &\mapsto \overline{\mathcal{Z}}_n \end{aligned} \tag{2.68}$$

Proof. To find the expression for the commutator $\left[T_{ij}^{(m)}, T_{kl}^{(n)}\right]$, we can use the expansions

$$\frac{1}{u-v} = \sum_{r=0}^{\infty} u^{-r-1}v^{r} \quad \text{and} \quad \frac{1}{u-v-\kappa} = \sum_{s=0}^{\infty} u^{-s-1}(v+\kappa)^{s} = \sum_{s=0}^{\infty}\sum_{b+c=s}\binom{s}{b}\kappa^{b}u^{-s-1}v^{c}$$

in the defining relations (4.10) and equate the coefficients of $u^{-m}v^{-n}$ to yield

$$\begin{aligned}\left[T_{ij}^{(m)}, T_{kl}^{(n)}\right] &= (-1)^{[i][j]+[i][k]+[j][k]}\sum_{a=1}^{\min(m,n)}\left(T_{kj}^{(a-1)}T_{il}^{(m+n-a)} - T_{kj}^{(m+n-a)}T_{il}^{(a-1)}\right)\\ &\quad + \delta_{\bar{\jmath}l}\sum_{p=1}^{M+N}\sum_{a=1}^{m}\sum_{b=0}^{m-a}\binom{m-a}{b}\kappa^{b}(-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]}\theta_{j}\theta_{p}T_{k\bar{p}}^{(m+n-a-b)}T_{ip}^{(a-1)}\\ &\quad - \delta_{\bar{\imath}k}\sum_{p=1}^{M+N}\sum_{a=1}^{m}\sum_{b=0}^{m-a}\binom{m-a}{b}\kappa^{b}(-1)^{[i][j]+[i]+[p]}\theta_{i}\theta_{p}T_{pj}^{(a-1)}T_{\bar{p}l}^{(m+n-a-b)}.\end{aligned} \tag{2.69}$$

Passing the relations (2.69) to the associated graded superalgebra yields

$$\begin{aligned}\left[\overline{T}_{ij}^{(m)}, \overline{T}_{kl}^{(n)}\right] &= \delta_{jk}(-1)^{[k]}\overline{T}_{il}^{(m+n-1)} - \delta_{il}(-1)^{([i]+[j])[k]+[j][l]}\overline{T}_{kj}^{(m+n-1)}\\ &\quad - \delta_{\bar{\imath}k}(-1)^{[i][j]+[i]+[j]}\theta_{i}\theta_{j}\overline{T}_{\bar{\jmath}l}^{(m+n-1)} + \delta_{\bar{\jmath}l}(-1)^{([i]+[j])[k]+[j]}\theta_{i}\theta_{j}\overline{T}_{k\bar{\imath}}^{(m+n-1)}.\end{aligned}$$

Hence,

$$\begin{aligned}&\left[(-1)^{[i]}\left(\overline{T}_{ij}^{(m)} - \tfrac{1}{2}\delta_{ij}\overline{\mathcal{Z}}_{n}\right), (-1)^{[k]}\left(\overline{T}_{kl}^{(n)} - \tfrac{1}{2}\delta_{kl}\overline{\mathcal{Z}}_{n}\right)\right]\\ &\quad = \delta_{jk}(-1)^{[i]}\left(\overline{T}_{il}^{(m+n-1)} - \tfrac{1}{2}\delta_{il}\overline{\mathcal{Z}}_{n}\right) - \delta_{il}(-1)^{([i]+[j])([k]+[l])}(-1)^{[k]}\left(\overline{T}_{kj}^{(m+n-1)} - \tfrac{1}{2}\delta_{kj}\overline{\mathcal{Z}}_{n}\right)\\ &\qquad - \delta_{\bar{\imath}k}(-1)^{[i][j]+[i]}\theta_{i}\theta_{j}(-1)^{[j]}\left(\overline{T}_{\bar{\jmath}l}^{(m+n-1)} - \tfrac{1}{2}\delta_{\bar{\jmath}l}\overline{\mathcal{Z}}_{n}\right)\\ &\qquad + \delta_{\bar{\jmath}l}(-1)^{([i]+[j])[k]}\theta_{\bar{\imath}}\theta_{\bar{\jmath}}(-1)^{[k]}\left(\overline{T}_{k\bar{\imath}}^{(m+n-1)} - \tfrac{1}{2}\delta_{k\bar{\imath}}\overline{\mathcal{Z}}_{n}\right).\end{aligned}$$

Furthermore, we know

$$\begin{aligned}&(-1)^{[i]}\left(\overline{T}_{ij}^{(n)} - \tfrac{1}{2}\delta_{ij}\overline{\mathcal{Z}}_{n}\right) + (-1)^{[i][j]+[i]}\theta_{i}\theta_{j}(-1)^{[j]}\left(\overline{T}_{\bar{\jmath}\bar{\imath}}^{(n)} - \tfrac{1}{2}\delta_{\bar{\jmath}\bar{\imath}}\overline{\mathcal{Z}}_{n}\right)\\ &\quad = (-1)^{[i]}\left(\overline{T}_{ij}^{(n)} + (-1)^{[i][j]+[j]}\theta_{i}\theta_{j}\overline{T}_{\bar{\jmath}\bar{\imath}}^{(n)}\right) - (-1)^{[i]}\delta_{ij}\overline{\mathcal{Z}}_{n} = \bar{0},\end{aligned}$$

and $\left[(-1)^{[i]}\left(\overline{T}_{ij}^{(m)} - \tfrac{1}{2}\delta_{ij}\overline{\mathcal{Z}}_{m}\right), \overline{\mathcal{Z}}_{n}\right] = \bar{0}$ is always true.

Therefore, the Lie superalgebra morphism $\Psi\colon (\mathfrak{osp}_{M|N} \oplus \mathfrak{z}_{\mathsf{c}})[z] \to \mathrm{Lie}(\mathrm{gr}\,\mathbf{X}(\mathfrak{osp}_{M|N}))$ extends to a morphism of superalgebras $\Psi\colon \mathfrak{U}\big((\mathfrak{osp}_{M|N} \oplus \mathfrak{z}_{\mathsf{c}})[z]\big) \to \mathrm{gr}\,\mathbf{X}(\mathfrak{osp}_{M|N})$. The map Ψ preserves the $\mathbb{Z}_2$-gradation by definition and can also be seen to preserve the $\mathbb{N}$-gradings. Furthermore, the morphism is surjective since $\mathrm{gr}\,\mathbf{X}(\mathfrak{osp}_{M|N})$ is generated by the elements $\overline{T}_{ij}^{(n)}$, whereas Ψ maps $(-1)^{[i]}F_{ij}z^{n-1}$ to $\overline{T}_{ij}^{(n)}$ for $i \neq j$ and maps $((-1)^{[k]}F_{kk} + \frac{1}{2}\mathsf{c})z^{n-1}$ to $\overline{T}_{kk}^{(n)}$.

Lastly, it can be seen that the mapping is a morphism of Hopf superalgebras, from the description of the Hopf superstructures on $\mathfrak{U}\big((\mathfrak{osp}_{M|N} \oplus \mathfrak{z}_{\mathsf{c}})[z]\big)$ and $\mathrm{gr}\,\mathbf{X}(\mathfrak{osp}_{M|N})$ as before. $\square$

The Orthosymplectic Super Yangian $\mathbf{Y}(\mathfrak{osp}_{M|N})$

Definition 2.1.18. The *super Yangian* $\mathbf{Y}(\mathfrak{osp}_{M|N})$ of $\mathfrak{osp}_{M|N}$ is the quotient of $\mathbf{X}(\mathfrak{osp}_{M|N})$ by the two-sided ideal $(\mathcal{Z}(u) - \mathbf{1})$, i.e.,

$$\mathbf{Y}(\mathfrak{osp}_{M|N}) := \mathbf{X}(\mathfrak{osp}_{M|N})/(\mathcal{Z}(u) - \mathbf{1}). \tag{2.70}$$

Equivalently, $\mathbf{Y}(\mathfrak{osp}_{M|N})$ is the unital associative $\mathbb{C}$-superalgebra on the generators $\{\mathcal{T}_{ij}^{(n)} \mid 1 \leq i,j \leq M+N,\ n \in \mathbb{Z}^+\}$, with $\mathbb{Z}_2$-grade $\big[\mathcal{T}_{ij}^{(n)}\big] := [i]+[j]$ for all $n \in \mathbb{Z}^+$, subject to the defining *RTT-relation*

$$\begin{aligned} R(u-v)\mathcal{T}_1(u)\mathcal{T}_2(v) &= \mathcal{T}_2(v)\mathcal{T}_1(u)R(u-v) \\ \text{in} \quad \big(\mathrm{End}\,\mathbb{C}^{M|N}\big)^{\otimes 2} &\otimes \mathbf{Y}(\mathfrak{osp}_{M|N})[\![u^{\pm 1}, v^{\pm 1}]\!], \end{aligned} \tag{2.71}$$

such that $R(u-v)$ is identified with $R(u-v) \otimes \mathbf{1}$, and

$$\mathcal{T}^t(u+\kappa)\mathcal{T}(u) = \mathbb{1} \quad \text{in} \quad \mathrm{End}\,\mathbb{C}^{M|N} \otimes \mathbf{Y}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!], \tag{2.72}$$

where

$$\mathcal{T}(u) := \sum_{i,j=1}^{M+N} (-1)^{[i][j]+[j]} E_{ij} \otimes \mathcal{T}_{ij}(u) \in \left(\operatorname{End} \mathbb{C}^{M|N}\right) \otimes \mathbf{Y}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!], \tag{2.73}$$

is the *generating matrix* consisting of the *generating series*

$$\mathcal{T}_{ij}(u) := \sum_{n=0}^{\infty} \mathcal{T}_{ij}^{(n)} u^{-n} \in \mathbf{Y}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!], \quad \text{where} \quad \mathcal{T}_{ij}^{(0)} = \delta_{ij}\mathbf{1}. \tag{2.74}$$

Note that the defining relations for the super Yangian in terms of power series takes the form

$$\begin{aligned}\left[\mathcal{T}_{ij}(u), \mathcal{T}_{kl}(v)\right] = {} & \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\Big(\mathcal{T}_{kj}(u)\mathcal{T}_{il}(v) - \mathcal{T}_{kj}(v)\mathcal{T}_{il}(u)\Big) \\ & - \frac{1}{u-v-\kappa}\Bigg(\delta_{\bar{\imath}k} \sum_{p=1}^{M+N} (-1)^{[i][j]+[i]+[j][p]} \theta_i \theta_p \mathcal{T}_{pj}(u)\mathcal{T}_{\bar{p}l}(v) \\ & \qquad - \delta_{\bar{\jmath}l} \sum_{p=1}^{M+N} (-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]} \theta_j \theta_p \mathcal{T}_{k\bar{p}}(v)\mathcal{T}_{ip}(u)\Bigg),\end{aligned} \tag{2.75}$$

and

$$\sum_{k=1}^{M+N} \mathcal{T}_{ik}^{t}(u+\kappa)\mathcal{T}_{kj}(u) = \delta_{ij}\mathbf{1}. \tag{2.76}$$

Note that since $(\mathcal{Z}(u) - \mathbf{1})$ is a graded Hopf ideal, the quotient of $\mathbf{X}(\mathfrak{osp}_{M|N})$ by $(\mathcal{Z}(u) - \mathbf{1})$ comes equipped with a unique Hopf superstructure such that the canonical projection on the quotient $\mathbf{X}(\mathfrak{osp}_{M|N}) \twoheadrightarrow \mathbf{X}(\mathfrak{osp}_{M|N})/(\mathcal{Z}(u) - \mathbf{1})$ is a morphism of Hopf superalgebras. Hence, there is Hopf superalgebra structure on $\mathbf{Y}(\mathfrak{osp}_{M|N})$ given by the comultiplication

$$\begin{aligned}\Delta \colon \mathbf{Y}(\mathfrak{osp}_{M|N}) &\to \mathbf{Y}(\mathfrak{osp}_{M|N}) \otimes \mathbf{Y}(\mathfrak{osp}_{M|N}) \\ \mathcal{T}(u) &\mapsto \mathcal{T}_{[1]}(u)\mathcal{T}_{[2]}(u),\end{aligned} \tag{2.77}$$

the counit

$$\begin{aligned} \varepsilon\colon \mathbf{Y}(\mathfrak{osp}_{M|N}) &\to \mathbb{C} \\ \mathcal{T}(u) &\mapsto \mathbb{1}, \end{aligned} \tag{2.78}$$

and the antipode

$$\begin{aligned} S\colon \mathbf{Y}(\mathfrak{osp}_{M|N}) &\to \mathbf{Y}(\mathfrak{osp}_{M|N}) \\ \mathcal{T}(u) &\mapsto \mathcal{T}(u)^{-1}. \end{aligned} \tag{2.79}$$

In particular, we see that the antipode of $\mathbf{Y}(\mathfrak{osp}_{M|N})$ is given by

$$S\colon \mathcal{T}(u) \mapsto \mathcal{T}^{t}(u+\kappa) \tag{2.80}$$

The Associated Graded Superalgebra $\operatorname{gr}\mathbf{Y}(\mathfrak{osp}_{M|N})$

The filtrations $\mathbf{E}^{\mathrm{Ab}}$ and $\mathbf{E}$ on $\mathbf{X}(\mathfrak{osp}_{M|N})$ will endow filtrations $\overline{\mathbf{E}}^{\mathrm{Ab}} = \{\overline{\mathbf{E}}^{\mathrm{Ab}}_n\}_{n\in\mathbb{N}}$ and $\overline{\mathbf{E}} = \{\overline{\mathbf{E}}_n\}_{n\in\mathbb{N}}$ on the quotient $\mathbf{Y}(\mathfrak{osp}_{M|N}) = \mathbf{X}(\mathfrak{osp}_{M|N})/(\mathcal{Z}(u)-1)$ such that $\overline{\mathbf{E}}^{\mathrm{Ab}}_n = \mathbf{E}^{\mathrm{Ab}}_n/\big(\mathbf{E}^{\mathrm{Ab}}_n \cap (\mathcal{Z}(u)-1)\big)$ and $\overline{\mathbf{E}}_n = \mathbf{E}_n/\big(\mathbf{E}_n \cap (\mathcal{Z}(u)-1)\big)$. For simplicity, we shall set $\mathbf{F}^{\mathrm{Ab}} = \{\mathbf{F}^{\mathrm{Ab}}_n\}_{n\in\mathbb{N}} := \overline{\mathbf{E}}^{\mathrm{Ab}}$ and $\mathbf{F} = \{\mathbf{F}_n\}_{n\in\mathbb{N}} := \overline{\mathbf{E}}$. Equivalently, these filtrations are given by the respective filtration degree assignments

$$\deg_{\mathrm{Ab}} \mathcal{T}^{(n)}_{ij} = n \quad \text{and} \quad \deg_{\mathbf{F}} \mathcal{T}^{(n)}_{ij} = n-1. \tag{2.81}$$

The second filtration is given explicitly by

$$\begin{aligned} \mathbf{F}_n &= \operatorname{span}_{\mathbb{C}}\big\{ \textstyle\prod_{a=1}^{\gamma} \mathcal{T}^{(k_a)}_{i_a j_a} \mid \gamma \in \mathbb{Z}^+,\ 1 \le i_a, j_a \le M+N,\ \sum_{a=1}^{\gamma} \deg_{\mathbf{F}} \mathcal{T}^{(k_a)}_{i_a j_a} \le n \big\} \\ &= \operatorname{span}_{\mathbb{C}}\big\{ \textstyle\prod_{a=1}^{\gamma} \mathcal{T}^{(k_a)}_{i_a j_a} \mid \gamma \in \mathbb{Z}^+,\ 1 \le i_a, j_a \le M+N,\ \sum_{a=1}^{\gamma} k_a \le n+\gamma \big\} \end{aligned}$$

for $n \in \mathbb{Z}^+$ and

$$\mathbf{F}_0 = \operatorname{span}_{\mathbb{C}}\big\{ \textstyle\prod_{a=1}^{\gamma} T^{(k_a)}_{i_a j_a} \mid \gamma \in \mathbb{Z}^+,\ 1 \le i_a, j_a \le M+N,\ k_a \in \{0,1\} \big\}.$$

From the defining relations (2.75), one can deduce that the associated graded superalgebra $\mathrm{gr}_{\mathrm{Ab}}\,\mathbf{Y}(\mathfrak{osp}_{M|N}) = \bigoplus_{n\in\mathbb{N}} \mathbf{F}^{\mathrm{Ab}}_n/\mathbf{F}^{\mathrm{Ab}}_{n-1}$ corresponding to the first filtration $\mathbf{F}^{\mathrm{Ab}}$ is supercommutative. For now, we will direct our attention to the second filtration $\mathbf{F}$ which will induce a more interesting associated graded superalgebra:

$$\mathrm{gr}\,\mathbf{Y}(\mathfrak{osp}_{M|N}) = \mathrm{gr}_{\mathbf{F}}\,\mathbf{Y}(\mathfrak{osp}_{M|N}) = \bigoplus_{n\in\mathbb{N}} \mathbf{F}_n/\mathbf{F}_{n-1},$$

We note that $\mathrm{gr}\,\mathbf{Y}(\mathfrak{osp}_{M|N})$ inherits a $\mathbb{Z}_2$-graded structure from $\mathbf{Y}(\mathfrak{osp}_{M|N})$ by assigning $\mathbb{Z}_2$-grade $[i]+[j]$ to the image $\overline{\mathcal{T}}^{(n)}_{ij}$ of $\mathcal{T}^{(n)}_{ij}$ in $\mathbf{F}_n/\mathbf{F}_{n-1}$. Furthermore, by endowing $\mathbf{Y}(\mathfrak{osp}_{M|N})^{\otimes 2}$ with the filtration $\mathbf{F}^2 = \{\mathbf{F}^2_n\}_{n\in\mathbb{N}}$ induced by $\mathbf{F}$, i.e., $\mathbf{F}^2_n = \bigoplus_{i+j=n} \mathbf{F}_i \otimes \mathbf{F}_j$, and assigning $\mathbb{C}$ with the trivial ascending algebra filtration $\mathbf{C} = \{\mathbf{C}_n\}_{n\in\mathbb{N}}$ where $\mathbf{C}_n = \mathbb{C}$ for all $n \in \mathbb{N}$, we observe that each of the Hopf superalgebra structure maps on $\mathbf{Y}(\mathfrak{osp}_{M|N})$ preserve their relative filtrations. Therefore, the structure maps Δ and ε descend to the superalgebra morphisms

$$\begin{aligned} \mathrm{gr}\,\Delta\colon\ \mathrm{gr}\,\mathbf{Y}(\mathfrak{osp}_{M|N}) &\to \mathrm{gr}\left(\mathbf{Y}(\mathfrak{osp}_{M|N})^{\otimes 2}\right) \cong \left(\mathrm{gr}\,\mathbf{Y}(\mathfrak{osp}_{M|N})\right)^{\otimes 2} \\ \overline{\mathcal{T}}^{(n)}_{ij} &\mapsto \overline{\mathcal{T}}^{(n)}_{ij} \otimes \mathbf{1} + \mathbf{1} \otimes \overline{\mathcal{T}}^{(n)}_{ij},\ n \in \mathbb{Z}^+ \end{aligned} \tag{2.82}$$

where the above isomorphism is induced by the map $\mathbf{F}^2_n \to \bigoplus_{i+j=n} \mathbf{F}_i/\mathbf{F}_{i-1} \otimes \mathbf{F}_j/\mathbf{F}_{j-1}$, and

$$\begin{aligned} \mathrm{gr}\,\varepsilon\colon\ \mathrm{gr}\,\mathbf{Y}(\mathfrak{osp}_{M|N}) &\to \mathbb{C} \\ \overline{\mathcal{T}}^{(n)}_{ij} &\mapsto \delta_{0n}\delta_{ij} \end{aligned} \tag{2.83}$$

The antipode S will also descend to the superalgebra antimorphism

$$\begin{aligned} \mathrm{gr}\,S\colon\ \mathrm{gr}\,\mathbf{Y}(\mathfrak{osp}_{M|N}) &\to \mathrm{gr}\,\mathbf{Y}(\mathfrak{osp}_{M|N}) \\ \overline{\mathcal{T}}^{(n)}_{ij} &\mapsto (-1)^{[i][j]+[j]}\theta_i\theta_j \overline{\mathcal{T}}^{(n)}_{\bar{j}\bar{i}} = -\overline{\mathcal{T}}^{(n)}_{ij}. \end{aligned} \tag{2.84}$$

The fact that the Hopf algebra structure maps on $\mathbf{Y}(\mathfrak{osp}_{M|N})$ are filtration preserving is equivalent to saying that $\mathbf{F}$ is a Hopf filtration on $\mathbf{Y}(\mathfrak{osp}_{M|N})$. As such, $\mathrm{gr}\,\mathbf{Y}(\mathfrak{osp}_{M|N})$ is therefore equipped with an $\mathbb{N}$-graded Hopf superalgebra structure given by the morphisms (2.82), (2.83), and (2.84) [**?**, Proposition 7.9.2]; however, one could also verify the Hopf superalgebra axioms directly.

Let us now consider the polynomial current superalgebra $\mathfrak{osp}_{M|N}[z]$. As a Lie superalgebra, it is generated by the elements $\{F_{ij}z^n \mid 1 \leq i,j \leq M+N,\ n \in \mathbb{N}\}$ satisfying the relations

$$\begin{aligned}[F_{ij}z^m, F_{kl}z^n] &= \delta_{jk}F_{il}z^{m+n} - \delta_{il}(-1)^{([i]+[j])([k]+[l])}F_{kj}z^{m+n} \\ &\quad - \delta_{\bar{\imath}k}(-1)^{[i][j]+[i]}\theta_i\theta_j F_{\bar{\jmath}l}z^{m+n} + \delta_{\bar{\jmath}l}(-1)^{([i]+[j])[k]}\theta_{\bar{\imath}}\theta_{\bar{\jmath}}F_{k\bar{\imath}}z^{m+n},\end{aligned} \tag{2.85}$$

$$\text{and} \quad F_{ij}z^n + (-1)^{[i][j]+[i]}\theta_i\theta_j F_{\bar{\jmath}\bar{\imath}}z^n = 0. \tag{2.86}$$

We now have an analogue of Proposition (2.1.17) for the super Yangian $\mathbf{Y}(\mathfrak{osp}_{M|N})$:

Proposition 2.1.19. *There is a surjective* $\mathbb{N}$*-graded Hopf superalgebra morphism*

$$\begin{aligned}\Phi\colon \mathfrak{U}(\mathfrak{osp}_{M|N}[z]) &\to \operatorname{gr}\mathbf{Y}(\mathfrak{osp}_{M|N}) \\ F_{ij}z^{n-1} &\mapsto (-1)^{[i]}\overline{\mathcal{T}}_{ij}^{(n)}\end{aligned} \tag{2.87}$$

Proof. The proof is similar to the proof of Proposition (2.1.17). Passing the relations (2.75) to the associated graded superalgebra yields

$$\begin{aligned}\left[\overline{\mathcal{T}}_{ij}^{(m)}, \overline{\mathcal{T}}_{kl}^{(n)}\right] &= \delta_{jk}(-1)^{[k]}\overline{\mathcal{T}}_{il}^{(m+n-1)} - \delta_{il}(-1)^{([i]+[j])[k]+[j][l]}\overline{\mathcal{T}}_{kj}^{(m+n-1)} \\ &\quad - \delta_{\bar{\imath}k}(-1)^{[i][j]+[i]+[j]}\theta_i\theta_j\overline{\mathcal{T}}_{\bar{\jmath}l}^{(m+n-1)} + \delta_{\bar{\jmath}l}(-1)^{([i]+[j])[k]+[j]}\theta_i\theta_j\overline{\mathcal{T}}_{k\bar{\imath}}^{(m+n-1)}.\end{aligned}$$

Hence,

$$\begin{aligned}\left[(-1)^{[i]}\overline{\mathcal{T}}_{ij}^{(m)}, (-1)^{[k]}\overline{\mathcal{T}}_{kl}^{(n)}\right] &= \delta_{jk}(-1)^{[i]}\overline{\mathcal{T}}_{il}^{(m+n-1)} - \delta_{il}(-1)^{([i]+[j])([k]+[l])}(-1)^{[k]}\overline{\mathcal{T}}_{kj}^{(m+n-1)} \\ &\quad - \delta_{\bar{\imath}k}(-1)^{[i][j]+[i]}\theta_i\theta_j(-1)^{[j]}\overline{\mathcal{T}}_{\bar{\jmath}l}^{(m+n-1)} \\ &\quad + \delta_{\bar{\jmath}l}(-1)^{([i]+[j])[k]}\theta_{\bar{\imath}}\theta_{\bar{\jmath}}(-1)^{[k]}\overline{\mathcal{T}}_{k\bar{\imath}}^{(m+n-1)}.\end{aligned}$$

Lastly, by using the expansion

$$(u+\kappa)^{-m} = u^{-m}(1-(-\kappa u^{-1}))^{-m} = \sum_{p=m}^{\infty}\binom{p-1}{p-m}(-\kappa)^{p-m}u^{-p},$$

the relations (2.76) are explicitly given by

$$\sum_{k=1}^{M+N}\left(\delta_{ik}\mathcal{T}_{kj}^{(n)}+\sum_{m=1}^{n}\sum_{p=1}^{m}\binom{m-1}{m-p}(-\kappa)^{m-p}(-1)^{[i][k]+[k]}\theta_i\theta_k\mathcal{T}_{\bar{k}\bar{\imath}}^{(p)}\mathcal{T}_{kj}^{(n-m)}\right)=\delta_{0n}\delta_{ij}\mathbf{1}$$

Passing these relations to the associated graded superalgebra, we yield the equation

$$\overline{\mathcal{T}}_{ij}^{(n)}+(-1)^{[i][j]+[j]}\theta_i\theta_j\overline{\mathcal{T}}_{\bar{\jmath}\bar{\imath}}^{(n)}=0\quad\text{for}\ \ n\in\mathbb{Z}^+,$$

hence, $(-1)^{[i]}\overline{\mathcal{T}}_{ij}^{(n)}+(-1)^{[i][j]+[i]}\theta_i\theta_j(-1)^{[j]}\overline{\mathcal{T}}_{\bar{\jmath}\bar{\imath}}^{(n)}=0.$

Therefore, the Lie superalgebra morphism $\Phi\colon\mathfrak{osp}_{M|N}[z]\to\mathrm{Lie}(\mathrm{gr}\,\mathbf{Y}(\mathfrak{osp}_{M|N}))$ extends to a morphism of superalgebras $\Phi\colon\mathfrak{U}(\mathfrak{osp}_{M|N}[z])\to\mathrm{gr}\,\mathbf{Y}(\mathfrak{osp}_{M|N})$. This map preserves the $\mathbb{Z}_2$-gradation by definition and can also be seen to preserve the $\mathbb{N}$-gradings. This morphism is surjective as well since $\mathrm{gr}\,\mathbf{Y}(\mathfrak{osp}_{M|N})$ is generated by the elements $\overline{\mathcal{T}}_{ij}^{(n)}$.

Lastly, it can be seen that the mapping is a morphism of Hopf superalgebras, from the description of the Hopf superstructures on $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ and $\mathrm{gr}\,\mathbf{Y}(\mathfrak{osp}_{M|N})$ as before. □

2.2 Poincaré-Birkhoff-Witt Theorem for Orthosymplectic Super Yangians

Consider now the vector representation of $\mathfrak{osp}_{M|N}$ on the super vector space $\mathbb{C}^{M|N}$: $\rho\colon\mathfrak{osp}_{M|N}\to\mathfrak{gl}(\mathbb{C}^{M|N})$, where

$$\rho(F_{ij})\colon e_k\mapsto\delta_{jk}e_i-\delta_{\bar{\imath}k}(-1)^{[i][j]+[i]}\theta_i\theta_j e_{\bar{\jmath}}.$$

For any $a\in\mathbb{C}$, there is a Lie superalgebra morphism $\mathrm{ev}_a\colon\mathfrak{osp}_{M|N}[z]\to\mathfrak{osp}_{M|N}$ induced by the assignment $z\mapsto a$. Pre-composing ev_a with the vector representation, we yield the *evaluation representation* of $\mathfrak{osp}_{M|N}[z]$ at a given by

$$\begin{aligned} \rho\circ \mathrm{ev}_a\colon \mathfrak{osp}_{M|N}[u] &\to \mathfrak{gl}(\mathbb{C}^{M|N}) \\ F_{ij}z^n &\mapsto a^n\rho(F_{ij}). \end{aligned}$$

The evaluation representation extends uniquely to a representation of $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ on $\mathbb{C}^{M|N}$, denoted ρ_a, and such map will be called the *evaluation representation* of $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ at a. For any complex numbers $a_1,\ldots,a_n\in\mathbb{C}$, we may therefore consider the tensor product of such evaluation representations of $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$,

$$\rho_{a_1\rightharpoondown a_n} := \left(\textstyle\bigotimes_{i=1}^n \rho_{a_i}\right)\circ\Delta_{n-1}, \tag{2.1}$$

where $\Delta_{n-1}\colon \mathfrak{U}(\mathfrak{osp}_{M|N}[z])\to\big(\mathfrak{U}(\mathfrak{osp}_{M|N}[z])\big)^{\otimes n}$ is the unique $(n-1)$-fold comultiplication sending $X\in\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ to the element $\sum_X X_{(1)}\otimes X_{(2)}\otimes\cdots\otimes X_{(n)}$ in Sweedler notation.

Lemma 2.2.1. *The intersection* $\bigcap_{n\in\mathbb{Z}^+}\bigcap_{(a_1,\ldots,a_n)\in\mathbb{C}^n}\ker(\rho_{a_1\rightharpoondown a_n})$ *lying in* $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ *is trivial.*

Proof. Step 1. Let $\{X_i\}_{i=1}^d$ be a homogeneous basis of $\mathfrak{osp}_{M|N}$ and write $\chi_i=\rho(X_i)$ for indices $i=1,2,\ldots,d$. By endowing a total ordering '$\preceq$' on the collection of basis elements $\{X_bz^m \mid 1\le b\le d,\ m\in\mathbb{N}\}$ of $\mathfrak{osp}_{M|N}[z]$, the PBW Theorem for Lie superalgebras establishes that $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ has a PBW basis of ordered monomials of the form $\prod_{j=1}^n X_{b_j}z^{m_j}$, where $n\in\mathbb{Z}^+$, $X_{b_j}z^{m_j}\preceq X_{b_{j+1}}z^{m_{j+1}}$ for indices $j=1,\ldots,n-1$, and $X_{b_j}z^{m_j}\neq X_{b_{j+1}}z^{m_{j+1}}$ if X_{b_j} is an odd element. Given a non-zero element $\mathbf{A}$ in $\mathfrak{U}(\mathfrak{osp}_{M|N})$, we can therefore express $\mathbf{A}$ as a unique linear combination of the PBW basis elements of $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ and we let $\{M_i=\prod_{j=1}^n X_{b_{ij}}z^{m_{ij}}\}_{i=1}^p$ denote the collection of maximal length ordered monomials occurring in such expression of $\mathbf{A}$. For every such maximal length monomial, we consider their corresponding supersymmetrized object

$$M_i^\sigma := \sum_{\sigma\in\mathfrak{S}_n}\epsilon(\sigma,M_i)\bigotimes_{j=1}^n X_{b_{i\sigma(j)}}z^{m_{i\sigma(j)}}\in(\mathfrak{osp}_{M|N}[z])^{\otimes n}, \tag{2.2}$$

where $\epsilon\colon \mathfrak{S}_n \times (\mathfrak{osp}_{M|N}[z])^{\otimes n} \to \{\pm 1\}$ is the Koszul sign defined by the assignment $\epsilon(\sigma, x) = \prod_{(k,l)\in \mathrm{Inv}(\sigma)} (-1)^{[x_{\sigma(k)}][x_{\sigma(l)}]}$ given that $x = x_1 \otimes x_2 \otimes \cdots \otimes x_n \in (\mathfrak{osp}_{M|N}[z])^{\otimes n}$ is homogeneous and $\mathrm{Inv}(\sigma) = \{(k,l) \mid k < l,\, \sigma(k) > \sigma(l)\}$.

Step 2. We now show that the p supersymmetrized elements (2.2) are linearly independent; accordingly, it suffices to prove that their images under the projection $T(\mathfrak{osp}_{M|N}[z]) \twoheadrightarrow \mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ are so. To this end, we first express each monomial in the sum $\sum_{\sigma\in\mathfrak{S}_n} \epsilon(\sigma, M_i) \prod_{j=1}^n X_{b_{i\sigma(j)}} z^{m_{i\sigma(j)}}$ in terms of the PBW basis for $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ with respect to the total order '$\preceq$'. For instance, if $\sigma(k) > k$, then using the defining relations of the universal enveloping algebra allow us to write the monomial $\prod_{j=1}^n X_{b_{i\sigma(j)}} z^{m_{i\sigma(j)}}$ as

$$(\pm 1)\, X_{b_{i\sigma(1)}} z^{m_{i\sigma(1)}} \cdots X_{b_{i\sigma(k)}} z^{m_{i\sigma(k)}} X_{b_{i\sigma(k-1)}} z^{m_{i\sigma(k-1)}} \cdots X_{b_{i\sigma(n)}} z^{m_{i\sigma(n)}} \\ + X_{b_{i\sigma(1)}} z^{m_{i\sigma(1)}} \cdots \big[X_{b_{i\sigma(k-1)}}, X_{b_{i\sigma(k)}}\big] z^{m_{i\sigma(k-1)}+m_{i\sigma(k)}} \cdots X_{b_{i\sigma(n)}} z^{m_{i\sigma(n)}},$$

where $\pm 1 = (-1)^{[X_{b_{i\sigma(k-1)}}][X_{b_{i\sigma(k)}}]}$. By repeated use of the defining relations of the universal enveloping algebra, one yields that

$$\prod_{j=1}^n X_{b_{i\sigma(j)}} z^{m_{i\sigma(j)}} = \epsilon(\sigma, M_i) \prod_{j=1}^n X_{b_{ij}} z^{m_{ij}} \mod \mathfrak{U}_{n-1}(\mathfrak{osp}_{M|N}[z]), \tag{2.3}$$

where $\{\mathfrak{U}_n(\mathfrak{osp}_{M|N}[z])\}_{n\in\mathbb{N}}$ is the ascending algebra filtration on $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ determined by lengths of PBW basis elements. Therefore, the linear independence of the supersymmetrized elements (2.2) amounts to whether or not there is a non-trivial solution to

$$\sum_{i=1}^p \lambda_i n! \prod_{j=1}^n X_{b_{ij}} z^{m_{ij}} \equiv 0 \mod \mathfrak{U}_{n-1}(\mathfrak{osp}_{M|N}[z]),$$

but this is not possible unless $\lambda_i = 0$ for all $i = 1, 2, \ldots, p$.

Step 3. Since

$$\rho_{a_1 \to a_n} \colon X_b z^m \mapsto \sum_{k=1}^n a_k^m \chi_b^{[k]}, \qquad \chi_b^{[k]} := \mathrm{id}^{\otimes(k-1)} \otimes \chi_b \otimes \mathrm{id}^{\otimes(n-k)} \in \mathrm{End}(\mathbb{C}^{M|N})^{\otimes n},$$

then the image of the an ordered monomial $\prod_{j=1}^{r} X_{b_j} z^{m_j}$ under $\rho_{a_1 \to a_n}$ is given by

$$\sum_{k_1,\ldots,k_r=1}^{n} a_{k_1}^{m_1} \cdots a_{k_r}^{m_r} \chi_{b_1}^{[k_1]} \cdots \chi_{b_r}^{[k_r]} \in \mathrm{End}(\mathbb{C}^{M|N})^{\otimes n}. \tag{2.4}$$

By completing the collection $\{\chi_i\}_{i=1}^{d}$ to a homogeneous basis $\{\chi_i\}_{i=1}^{(M+N)^2}$ of $\mathrm{End}(\mathbb{C}^{M|N})$ such that $\chi_j = \mathrm{id}$ for some $j = d+1, \ldots, (M+N)^2$, we consider the subspace of $\mathrm{End}(\mathbb{C}^{M|N})^{\otimes n}$ given by

$$W_n := \mathrm{span}_{\mathbb{C}} \left\{ \chi_{i_1} \otimes \cdots \otimes \chi_{i_n} \mid 1 \leq i_j \leq (M+N)^2,\ \chi_{i_j} = \mathrm{id} \ \text{ for some } \ 1 \leq j \leq n \right\}.$$

Since the identity id must occur in at least one tensor factor of the elements spanning W_n, the image of any ordered monomial in $\mathfrak{U}_{n-1}(\mathfrak{osp}_{M|N}[z])$ under $\rho_{a_1 \to a_n}$ will be contained in W_n. Moreover, from (2.4), the image of the monomial M_i under $\rho_{a_1 \to a_n}$ may be written as

$$\sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma, M_i) a_1^{m_{i\sigma(1)}} \cdots a_n^{m_{i\sigma(n)}} \bigotimes_{j=1}^{n} \chi_{b_{i\sigma(j)}} \quad \mathrm{mod}\ W_n. \tag{2.5}$$

Under the identification $\phi\colon (\mathfrak{osp}_{M|N}[z])^{\otimes n} \xrightarrow{\sim} (\mathfrak{osp}_{M|N})^{\otimes n}[z_1, \ldots, z_n]$, the images of the supersymmetrized elements M_i^{σ} under ϕ are given by

$$\sum_{\sigma \in \mathfrak{S}_n} \Big(\epsilon(\sigma, M_i) \bigotimes_{j=1}^{n} X_{b_{i\sigma(j)}} \Big) z_1^{m_{i\sigma(1)}} \cdots z_n^{m_{i\sigma(n)}}. \tag{2.6}$$

Now, since ρ is a faithful representation, then so is $\rho^{\otimes n}\colon \mathfrak{U}(\mathfrak{osp}_{M|N})^{\otimes n} \to \mathrm{End}(\mathbb{C}^{M|N})^{\otimes n}$ and its extension $\rho^{\otimes n}[z_1, \ldots, z_n]\colon \mathfrak{U}(\mathfrak{osp}_{M|N})^{\otimes n}[z_1, \ldots, z_n] \to \mathrm{End}(\mathbb{C}^{M|N})^{\otimes n}[z_1, \ldots, z_n]$. By the linear independence of the elements (2.2), and equivalently the elements (2.6), the images of these latter elements under $\rho^{\otimes n}[z_1, \ldots, z_n]$ are linearly independent. That is, a non-zero linear combination $\sum_{i=1}^{p} \lambda_i \phi(M_i^{\sigma})$ implies that the sum of polynomials

$$\sum_{i=1}^{p} \lambda_i \sum_{\sigma \in \mathfrak{S}_n} \Big(\epsilon(\sigma, M_i) \bigotimes_{j=1}^{n} \chi_{b_{i\sigma(j)}} \Big) z_1^{m_{i\sigma(1)}} \cdots z_n^{m_{i\sigma(n)}}$$

is non-zero. Hence, there exists numbers $a_1, \dots, a_n \in \mathbb{C}$ such that

$$\sum_{i=1}^{p} \lambda_i \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma, M_i) a_1^{m_{i\sigma(1)}} \cdots a_n^{m_{i\sigma(n)}} \bigotimes_{j=1}^{n} \chi_{b_{i\sigma(j)}} \neq 0$$

Comparing the above with (2.5), we conclude that that image of $\rho_{a_1 \to a_n}(A)$ in the quotient $\operatorname{End}(\mathbb{C}^{M|N})^{\otimes n}/W_n$ is non-zero and therefore $\rho_{a_1 \to a_n}(A) \neq 0$, proving the lemma. $\square$

By substituting $u \mapsto u - v$ in the SQYBE (2.20), one can readily verify that the assignment

$$\begin{aligned} R\colon \mathbf{X}(\mathfrak{osp}_{M|N}) &\to \operatorname{End} \mathbb{C}^{M|N} \\ T(u) &\mapsto R(u) \end{aligned} \tag{2.7}$$

defines a representation of $\mathbf{X}(\mathfrak{osp}_{M|N})$, which we call the *R-matrix representation* of the extended super Yangian. On the level of power series, the R-matrix representation takes the form

$$R\colon T_{ij}(u) \mapsto \delta_{ij} \operatorname{id} - \frac{(-1)^{[i][j]} E_{ji}}{u} + \frac{(-1)^{[j]} \theta_i \theta_j E_{\bar{i}\bar{j}}}{u - \kappa} \tag{2.8}$$

In various sections of this work, we will want to use a variant of the R-matrix representation which we will construct here. Similar to before, replace $u \mapsto u - v$ in the SQBYE (2.20) but now also use the unitarity relation (2.19) to yield

$$R_{12}(u-v) R_{23}(-v) R_{13}(-u) = R_{13}(-u) R_{23}(-v) R_{12}(u-v). \tag{2.9}$$

Therefore, by applying the super-transpose (2.6) to the third tensor factor of equation (2.9), we get

$$R_{12}(u-v) R_{13}^{st_3}(-u) R_{23}^{st_3}(-v) = R_{23}^{st_3}(-v) R_{13}^{st_3}(-u) R_{12}(u-v),$$

where

$$R^{st_2}(u) = \mathrm{id}^{\otimes 2} - \frac{P^{st_2}}{u} + \frac{Q^{st_2}}{u-\kappa}$$

and $(-)^{st_2} = \mathrm{id} \otimes (-)^{st}$. Hence, the assignment $\varphi\colon T(u) \mapsto R^{st_2}(-u)$ defines a representation $\varphi\colon \mathbf{X}(\mathfrak{osp}_{M|N}) \to \mathrm{End}\,\mathbb{C}^{M|N}$. Finally, postcomposing φ with the automorphism τ_{-a} as in (2.37) will result in a representation of $\mathbf{X}(\mathfrak{osp}_{M|N})$ given by

$$\begin{aligned} \varrho_a := \varphi \circ \tau_{-a}\colon \mathbf{X}(\mathfrak{osp}_{M|N}) &\to \mathrm{End}\,\mathbb{C}^{M|N} \\ T(u) &\mapsto R^{st_2}(a-u) \end{aligned} \tag{2.10}$$

for any $a \in \mathbb{C}$. On the level of power series, such representation takes the form

$$\varrho_a\colon T_{ij}(u) \mapsto \delta_{ij}\,\mathrm{id} + \frac{(-1)^{[j]}E_{ij}}{u-a} - \frac{(-1)^{[i][j]}\theta_{\bar{\imath}}\theta_{\bar{\jmath}}E_{\bar{\jmath}\bar{\imath}}}{u+\kappa-a}. \tag{2.11}$$

Theorem 2.2.2. *There is an* $\mathbb{N}$*-graded Hopf superalgebra isomorphism*

$$\begin{aligned} \Phi\colon \mathfrak{U}(\mathfrak{osp}_{M|N}[z]) &\xrightarrow{\sim} \mathrm{gr}\,\mathbf{Y}(\mathfrak{osp}_{M|N}) \\ F_{ij}z^{n-1} &\mapsto (-1)^{[i]}\overline{\mathcal{T}}_{ij}^{(n)}. \end{aligned} \tag{2.12}$$

Proof. By Proposition (2.1.19), we only need to show that this map is injective. To this end, let $A \in U(\mathfrak{osp}_{M|N}[z])$ be a nonzero element that is homogeneous of degree d with respect to the $\mathbb{N}$-gradation. That is,

$$A = \sum A^{k_1;\dots;k_m}_{i_1j_1;\dots;i_mj_m} F_{i_1j_1}z^{k_1-1}\cdots F_{i_mj_m}z^{k_m-1}, \qquad A^{k_1;\dots;k_m}_{i_1j_1;\dots;i_mj_m} \in \mathbb{C}, \tag{2.13}$$

summed over the indices i_b, j_b, k_b, $1 \le b \le m$ such that $1 \le i_b, j_b \le M+N$ and $\sum_{b=1}^m k_b = d+m$. Considering the element

$$\widetilde{A} = \sum A^{k_1;\dots;k_m}_{i_1j_1;\dots;i_mj_m} \mathcal{T}^{(k_1)}_{i_1j_1}\cdots \mathcal{T}^{(k_m)}_{i_mj_m} \in \mathbf{Y}(\mathfrak{osp}_{M|N})$$

whose summation is given over the same indices as in (3.11), then $\Psi(A)$ coincides with the image of $\widetilde{A}$ in $\mathrm{gr}_d\,\mathbf{Y}(\mathfrak{osp}_{M|N})$, so it suffices to prove that the degree of $\widetilde{A}$ is d.

Now, by replacing $u \mapsto u - v$ in the SQBYE (2.20) and using the relation (2.19), we obtain

$$R_{12}(u-v)R_{23}(-v)R_{13}(-u) = R_{13}(-u)R_{23}(-v)R_{12}(u-v). \tag{2.14}$$

Therefore, by applying the super transpose to the third tensor factor of equation (2.14), we get

$$R_{12}(u-v)R_{13}^{st_3}(-u)R_{23}^{st_3}(-v) = R_{23}^{st_3}(-v)R_{13}^{st_3}(-u)R_{12}(u-v), \tag{2.15}$$

where

$$R^{st_2}(u) = \mathrm{id}^{\otimes 2} - \frac{P^{st_2}}{u} + \frac{Q^{st_2}}{u-\kappa}$$

and $(-)^{st_2} = \mathrm{id} \otimes (-)^{st}$. Hence, the assignment $\varphi\colon T(u) \mapsto R^{st_2}(-u)$ defines a representation $\varphi\colon \mathbf{X}(\mathfrak{osp}_{M|N}) \to \operatorname{End}\mathbb{C}^{M|N}$. Postcomposing φ with the automorphism (2.37) will result in a representation of $\mathbf{X}(\mathfrak{osp}_{M|N})$ given by $\varrho_a := \varphi \circ \tau_a\colon T(u) \mapsto R^{st_2}(a-u)$ for any $a \in \mathbb{C}$. Explicitly, this is given by

$$\varrho_a\colon T_{ij}(u) \mapsto \delta_{ij}\,\mathrm{id} + \frac{E_{ij}}{u-a} - \frac{(-1)^{[i][j]+[i]}\theta_i\theta_j E_{\bar{j}\bar{\imath}}}{u+\kappa-a}. \tag{2.16}$$

Now, by equation (2.51) and the fact that the transposition and super-transpose commute, we have that $\varrho_a\colon Z(u) \mapsto R^{st_2}(a-u)(R^t(a-u-\kappa))^{st_2}$. Using the relations

$$(P^{st_2})^2 = (M-N)\theta_0 P^{st_2}, \quad P^{st_2}Q^{st_2} = Q^{st_2}P^{st_2} = \theta_0 P^{st_2}, \quad \text{and} \quad (Q^{st_2})^2 = \mathrm{id}^{\otimes 2},$$

we observe that

$$\varrho_a\colon \mathcal{Z}(u) \mapsto \mathrm{id} - \frac{\mathrm{id}}{(u+\kappa-a)^2}. \tag{2.17}$$

By Proposition (2.3.2) there exists a unique series $f_a(u) \in \mathrm{id} + u^{-1}\operatorname{End}\mathbb{C}^{M|N}[\![u^{-1}]\!]$ such that

$$f_a(u)f_a(u+\kappa) = \varrho_a(\mathcal{Z}(u)^{-1}) = \frac{(u+\kappa-a)^2}{(u+\kappa-a)^2-1}\,\mathrm{id},$$

and since $\varrho_a(\mathcal{Y}(u)^{-1})\varrho_a(\mathcal{Y}(u+\kappa)^{-1}) = \varrho_a(\mathcal{Z}(u)^{-1})$, then by uniqueness we yield the equality $\varrho_a(\mathcal{Y}(u)^{-1}) = f_a(u)$. Now, since $\mathcal{T}_{ij}(u) = \mathcal{Y}(u)^{-1}T_{ij}(u)$, then the image of

$\mathcal{T}_{ij}(u)$ under ϱ_a is given by

$$\varrho_a \colon \mathcal{T}_{ij}(u) \mapsto f_a(u)\left(\delta_{ij}\,\mathrm{id} + \frac{E_{ij}}{u-a} - \frac{(-1)^{[i][j]+[i]}\theta_i\theta_j E_{\bar{j}\bar{\imath}}}{u+\kappa-a}\right). \tag{2.18}$$

Using the expansion

$$\begin{aligned}\frac{(u+\kappa-a)^2}{(u+\kappa-a)^2-1} = \sum_{m=0}^{\infty}(u+\kappa-a)^{-2m} &= \sum_{m=0}^{\infty} u^{-2m}\left(\sum_{k=0}^{\infty} u^{-k}(a-\kappa)^k\right)^{2m} \\ &= 1 + \sum_{m=1}^{\infty}\sum_{p=0}^{\infty} \vartheta_{a-\kappa}(2m,p)u^{-p-2m},\end{aligned} \tag{2.19}$$

the coefficient of u^{-n} for $n \geq 1$ in the series $\varrho_a(\mathcal{Z}(u)^{-1})$ is $c_n = \sum_{2m+p=n} \vartheta_{a-\kappa}(2m,p)\,\mathrm{id}$ for $m \in \mathbb{Z}^+$, $p \in \mathbb{N}$. If we write $f_a(u) = \sum_{n=0}^{\infty} f_a^{(n)} u^{-n}$, $f_a^{(0)} = \mathrm{id}$, we therefore have the relation $c_n = \sum_{r+s+t=n} \vartheta_{-\kappa}(r,s) f_a^{(s)} f_a^{(t)}$ for $r,s,t \in \mathbb{N}$. Now, by regarding a as a formal variable, we observe that the coefficients $f_a^{(n)}$ of $f_a(u)$ are polynomials in a with coefficients in $\mathrm{End}\,\mathbb{C}^{M|N}$ since the coefficients c_n are so.

In particular, by regarding $f_a^{(n)} \in \mathrm{End}\,\mathbb{C}^{M|N}[a]$, then $\deg f_a^{(1)} = 0$ and $\deg f_a^{(n)} = n-2$ for $n \geq 2$. This follows from the fact that $c_1 = 0$ and then inducting on the formula $c_n = \sum_{r+s+t=n} \vartheta_{-\kappa}(r,s) f_a^{(s)} f_a^{(t)}$.

Using the expansions $\frac{1}{u-a} = \sum_{n=0}^{\infty} a^n u^{-n-1}$ and $\frac{1}{u+\kappa-a} = \sum_{n=0}^{\infty}(a-\kappa)^n u^{-n-1}$, we observe that the coefficient of u^{-n} in $\varrho_a(\mathcal{T}_{ij}(u))$ is

$$\delta_{ij} f_a^{(n)} + \sum_{r+s=n} f_a^{(r)} E_{ij} a^{s-1} - \sum_{r+s=n} f_a^{(r)} (-1)^{[i][j]+[i]}\theta_i\theta_j E_{\bar{j}\bar{\imath}}(a-\kappa)^{s-1},$$

where $r \in \mathbb{N}$ and $s \in \mathbb{Z}^+$. In particular, the above is a polynomial in a of degree $n-1$ and the coefficient of $a^{n-1}u^{-n}$ in $\varrho_a(\mathcal{T}_{ij}(u))$ is precisely $\rho(F_{ij})$.

Now, since $\iota \colon \mathbf{Y}(\mathfrak{osp}_{M|N}) \hookrightarrow \mathbf{X}(\mathfrak{osp}_{M|N})$ is a sub-Hopf superalgebra, we may then construct a representation $\varrho_{a_1 \to a_n} := (\bigotimes_{i=1}^{n} \varrho_{a_i}) \circ \Delta_{n-1} \colon \mathbf{Y}(\mathfrak{osp}_{M|N}) \to \left(\mathrm{End}\,\mathbb{C}^{M|N}\right)^{\otimes n}$

for any $(a_1, \ldots, a_n) \in \mathbb{C}^n$. Now since $\Delta_{n-1}(\mathcal{T}_{i_1j_1}^{(k_1)} \cdots \mathcal{T}_{i_mj_m}^{(k_m)})$ is given up to sign by

$$\sum_{\sum_{s=1}^{m} \alpha_{r,s}=k_r} \sum_{b_{r,s}=1}^{M+N} \mathcal{T}_{i_1b_{1,1}}^{(\alpha_{1,1})} \cdots \mathcal{T}_{i_mb_{m,1}}^{(\alpha_{m,1})} \otimes \mathcal{T}_{b_{1,1}b_{1,2}}^{(\alpha_{1,2})} \cdots \mathcal{T}_{b_{m,1}b_{m,2}}^{(\alpha_{m,2})} \otimes \cdots \otimes \mathcal{T}_{b_{1,n-1}j_1}^{(\alpha_{1,n})} \cdots \mathcal{T}_{b_{m,n-1}j_m}^{(\alpha_{m,n})},$$

or rather,

$$\sum_{q_1,\ldots,q_m=1}^{n} (\mathcal{T}_{i_1j_1}^{k_1})^{[q_1]} \cdots (\mathcal{T}_{i_mj_m}^{k_m})^{[q_m]} \ + \ \text{lower degree terms}$$

where $(\mathcal{T}_{i_bj_b}^{k_b})^{[q_b]} = \mathbf{1}^{\otimes(q_b-1)} \otimes \mathcal{T}_{i_bj_b}^{k_b} \otimes \mathbf{1}^{\otimes(n-q_b)}$ and $\deg \mathcal{T}_{i_1j_1}^{\alpha_1} \cdots \mathcal{T}_{i_mj_m}^{\alpha_m} = \sum_{b=1}^{m} \deg \mathcal{T}_{i_bj_b}^{(\alpha_b)}$, then the image of the monomial $\mathcal{T}_{i_1j_1}^{k_1} \cdots \mathcal{T}_{i_mj_m}^{k_m}$ under the representation $\varrho_{\star_{i=1}^{n} a_i}$ will a polynomial in $a_1, \ldots, a_n$ of degree $\leq -m + \sum_{i=1}^{m} k_i$. Modulo terms of polynomial degree strictly less than $-m + \sum_{i=1}^{m} k_i$, this image may be written as

$$\sum_{q_1,\ldots,q_m=1}^{n} a_{q_1}^{k_1-1} \cdots a_{q_m}^{k_m-1} \rho(F_{i_1j_1})^{[q_1]} \cdots \rho(F_{i_mj_m})^{[q_m]},$$

where $\rho(F_{i_bj_b})^{[q_b]} = \mathrm{id}^{\otimes(q_b-1)} \otimes \rho(F_{i_bj_b}) \otimes \mathrm{id}^{\otimes(n-q_b)}$. In particular, the image of the element $\widetilde{A}$ under $\varrho_{\star_{i=1}^{n} a_i}$ will be a polynomial in $a_1, \ldots, a_n$ of degree $\leq d$. This image may be displayed as

$$\sum A_{i_1,j_1;\ldots;i_m,j_m}^{k_1;\ldots;k_m} \sum_{q_1,\ldots,q_m=1}^{n} a_{q_1}^{k_1-1} \cdots a_{q_m}^{k_m-1} \rho(F_{i_1j_1})^{[q_1]} \cdots \rho(F_{i_mj_m})^{[q_m]},$$

modulo polynomial terms of degree $< d$, however this is simply $\rho_{\star_{i=1}^{n} a_i}(A)$. By the previous lemma we know that there exists $a_1, \ldots, a_n \in \mathbb{C}$ such that $\rho_{\star_{i=1}^{n} a_i}(A) \neq 0$, thus $\widetilde{A}$ under $\varrho_{\star_{i=1}^{n} a_i}$ will be a polynomial in $a_1, \ldots, a_n$ of degree d and therefore the gradation degree of $\widetilde{A}$ is d. Otherwise, if the gradation degree of $\widetilde{A}$ is $b < d$, then $\widetilde{A}$ under $\varrho_{\star_{i=1}^{n} a_i}$ will be a polynomial in $a_1, \ldots, a_n$ of degree $b < d$. $\square$

By setting

$$\mathcal{B}^M = \left\{(i,j) \in (\mathbb{Z}_{M+N}^{+})^2 \mid M+1 < i+j\right\}$$

$$\text{and} \quad \mathcal{B}_N = \left\{(i,j) \in (\mathbb{Z}^+_{M+N})^2 \mid 2M+N < i+j\right\},$$

then the collections $\left\{F_{ij} \mid (i,j) \in \mathcal{B}^M\right\}$ and $\{F_{ij} \mid (i,j) \in \mathcal{B}_N\}$ form bases for $\mathfrak{so}_M$ and $\mathfrak{sp}_N$, respectively. If we further define

$$\mathcal{C} = \left\{(i,j) \in (\mathbb{Z}^+_{M+N})^2 \mid 1 \le i \le M,\ M+1 \le j \le M+N\right\},$$

then the collection

$$\mathcal{B}_{M|N} = \mathcal{B}^M \cup \mathcal{B}_N \cup \mathcal{C} \tag{2.20}$$

indexes a basis $\left\{F_{ij} \mid (i,j) \in \mathcal{B}_{M|N}\right\}$ for $\mathfrak{osp}_{M|N}$.

Corollary 2.2.3 (PBW Theorem for $\mathbf{Y}(\mathfrak{osp}_{M|N})$). *Given any total ordering '$\preceq$' on the set*

$$\mathsf{Y} = \left\{\mathcal{T}^{(n)}_{ij} \mid (i,j,n) \in \mathcal{B}_{M|N} \times \mathbb{Z}^+\right\}, \tag{2.21}$$

then the collection of all ordered monomials of the form

$$\mathcal{T}^{(n_1)}_{i_1 j_1}\mathcal{T}^{(n_2)}_{i_2 j_2}\cdots\mathcal{T}^{(n_k)}_{i_k j_k} \tag{2.22}$$

where $\mathcal{T}^{(n_a)}_{i_a j_a} \in \mathsf{Y}$, $\mathcal{T}^{(n_a)}_{i_a j_a} \preceq \mathcal{T}^{(n_{a+1})}_{i_{a+1} j_{a+1}}$, and $\mathcal{T}^{(n_a)}_{i_a j_a} \neq \mathcal{T}^{(n_{a+1})}_{i_{a+1} j_{a+1}}$ if $(i_a, j_a) \in \mathcal{C}$, forms a basis for the super Yangian $\mathbf{Y}(\mathfrak{osp}_{M|N})$.

Proof. This follows from Theorem (2.2.2) and the PBW Theorem for Lie superalgebras. □

Corollary 2.2.4. *(i) The supercenter of $\mathbf{Y}(\mathfrak{osp}_{M|N})$ is trivial,*

(ii) the supercenter of $\mathbf{X}(\mathfrak{osp}_{M|N})$ is $\mathbf{ZX}(\mathfrak{osp}_{M|N})$,

(iii) the coefficients of the series $\mathcal{Z}(u) - \mathbf{1}$ are algebraically independent over $\mathbb{C}$. Therefore, $\mathbf{ZX}(\mathfrak{osp}_{M|N}) \cong \mathbb{C}[\xi_i \mid i \in \mathbb{Z}^+]$.

Corollary 2.2.5 (PBW Theorem for $\mathbf{X}(\mathfrak{osp}_{M|N})$). *Given any total ordering '$\preceq$' on the* *set*

$$\mathsf{X} = \{T_{ij}^{(n)}, \mathcal{Z}_r \mid (i,j,n) \in \mathcal{B}_{M|N} \times \mathbb{Z}^+,\ r \in \mathbb{Z}^+\}, \tag{2.23}$$

then the collection of all ordered monomials of the form

$$X_{m_1} X_{m_2} \cdots X_{m_k}$$

where $X_{m_a} \in \mathsf{X}$*,* $X_{m_a} \preceq X_{m_{a+1}}$*, and* $X_{m_a} \neq X_{m_{a+1}}$ *if* $X_a = T_{i_a j_a}^{(n_a)}$ *with* $(i_a, j_a) \in \mathcal{C}$*, forms a basis for the extended super Yangian* $\mathbf{X}(\mathfrak{osp}_{M|N})$*.*

Proposition 2.2.6. *There is a superalgebra embedding*

$$\begin{aligned} \iota\colon \mathfrak{U}(\mathfrak{osp}_{M|N}) &\hookrightarrow \mathbf{Y}(\mathfrak{osp}_{M|N}) \\ F_{ij} &\mapsto (-1)^{[i]} \mathcal{T}_{ij}^{(1)} \end{aligned} \tag{2.24}$$

Consequently, there is also a superalgebra emedding

$$\begin{aligned} \mathfrak{U}(\mathfrak{osp}_{M|N}) &\hookrightarrow \mathbf{X}(\mathfrak{osp}_{M|N}) \\ F_{ij} &\mapsto \frac{1}{2}(-1)^{[i]} \big(T_{ij}^{(1)} - (-1)^{[i][j]+[j]} \theta_i \theta_j T_{\bar{j}\bar{i}}^{(1)}\big) \end{aligned} \tag{2.25}$$

Proof. Via the defining relations of the Yangian, we have

$$\begin{aligned} \big[\mathcal{T}_{ij}^{(1)}, \mathcal{T}_{kl}(v)\big] &= \delta_{jk}(-1)^{[j]} \mathcal{T}_{il}(v) - \delta_{il}(-1)^{[i]+([i]+[j])([k]+[l])} \mathcal{T}_{kj}(v) \\ &\quad - \delta_{\bar{i}k}(-1)^{[j]+[i][j]+[i]} \theta_i \theta_j \mathcal{T}_{\bar{j}l}(v) + \delta_{\bar{j}l}(-1)^{[j]+([i]+[j])[k]} \theta_i \theta_j \mathcal{T}_{k\bar{i}}(v). \end{aligned}$$

so the coefficient of $(-1)^{[i]+[k]} v^{-1}$ in the above yields a desired relation. Furthermore, we have

$$\mathcal{T}_{ij}^{(1)} + (-1)^{[i][j]+[j]} \theta_i \theta_j \mathcal{T}_{\bar{j}\bar{i}}^{(1)} = 0$$

so multiply the above expression by $(-1)^{[i]}$. □

2.3 Structure of the Extended Orthosymplectic Super Yangian

The Yangian as a subalgebra of the Extended Yangian

RE-WRITE SECTION

Definition 2.3.1. The *super Yangian* $\mathbf{Y}(\mathfrak{osp}_{M|N})$ of $\mathfrak{osp}_{M|N}$ is the subalgebra of $\mathbf{X}(\mathfrak{osp}_{M|N})$ stable under superalgebra automorphisms of the form $\mu_f \colon T(u) \mapsto f(u)T(u)$ for all formal series $f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$. That is,

$$\mathbf{Y}(\mathfrak{osp}_{M|N}) := \left\{\mathcal{Y} \in \mathbf{X}(\mathfrak{osp}_{M|N}) \mid \mu_f(\mathcal{Y}) = \mathcal{Y} \text{ for all } f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]\right\},$$

cf. [?, Section 2].

Proposition 2.3.2. *If $\mathcal{A}$ is a commutative unital associative $\mathbb{K}$-algebra, then for any formal series*

$$a(u) = \mathbf{1} + \sum_{n=1}^{\infty} a_n u^{-n} \in \mathbf{1} + u^{-1}\mathcal{A}[\![u^{-1}]\!]$$

and any $\kappa \in \mathbb{K}$, there exists a unique formal series

$$y(u) = \mathbf{1} + \sum_{n=1}^{\infty} y_n u^{-n} \in \mathbf{1} + u^{-1}\mathcal{A}[\![u^{-1}]\!]$$

such that

$$a(u) = y(u)y(u+\kappa). \tag{2.1}$$

Proof. We follow the argument as in [?, Section 2.15], [?, Theorem 3.1]. If we write the equality (2.1) in regards to the coefficients of $a(u)$, then we yield the relations

$$a_n = 2y_n + B_n(y_1, \ldots, y_{n-1}), \qquad n \geq 1, \tag{2.2}$$

where B_n is a quadratic polynomial in $n-1$ indeterminates over $\mathbb{K}$. One may then inductively solve for the coefficients of $y(u)$ since the above relation implies that y_n will be a quadratic polynomial in $a_1, ..., a_n$. By construction, such a series $y(u)$ is unique. $\square$

Theorem 2.3.3. *The extended Yangian $\mathbf{X}(\mathfrak{osp}_{M|N})$ is isomorphic as algebras to the tensor product of its subalgebras $\mathbf{ZX}(\mathfrak{osp}_{M|N})$ and $\mathbf{Y}(\mathfrak{osp}_{M|N})$:*

$$\mathbf{X}(\mathfrak{osp}_{M|N}) \cong \mathbf{ZX}(\mathfrak{osp}_{M|N}) \otimes \mathbf{Y}(\mathfrak{osp}_{M|N}).$$

Proof. We follow the argument as in [?, Section 2.16], [?, Theorem 3.1]. Firstly, by Proposition (2.3.2) there exists a unique series $\mathcal{Y}(u) = \mathbf{1} + \sum_{n=1}^{\infty} \mathcal{Y}_n u^{-n}$ in $\mathbf{ZX}(\mathfrak{osp}_{M|N})$ such that

$$\mathcal{Z}(u) = \mathcal{Y}(u)\mathcal{Y}(u+\kappa). \tag{2.3}$$

Now, from the relations (2.51) the image of $\mathcal{Z}(u)$ under the automorphism (3.20) is

$$\mu_f \colon \mathcal{Z}(u) \mapsto f(u)f(u+\kappa)\mathcal{Z}(u). \tag{2.4}$$

In particular, we note that $\mathbf{ZX}(\mathfrak{osp}_{M|N})$ is stable under the automorphism μ_f. From equation (2.3) we deduce that

$$\mathcal{Z}(u) = f(u)^{-1}\mu_f(\mathcal{Y}(u))f(u+\kappa)^{-1}\mu_f(\mathcal{Y}(u+\kappa))$$

and therefore by uniqueness we yield

$$\mu_f \colon \mathcal{Y}(u) \mapsto f(u)\mathcal{Y}(u). \tag{2.5}$$

If we define the series

$$\mathcal{T}_{ij}(u) := \mathcal{Y}(u)^{-1}T_{ij}(u) = \delta_{ij}\mathbf{1} + \sum_{n=1}^{\infty} \mathcal{T}_{ij}^{(n)} u^{-n} \in \mathbf{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!], \tag{2.6}$$

then $\mathcal{T}_{ij}(u)$ is stable under μ_f, showing that $\mathcal{T}_{ij}(u) \in \mathbf{Y}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$. Moreover, since $T_{ij}(u) = \mathcal{Y}(u)\mathcal{T}_{ij}(u)$, then we have the product $\mathbf{X}(\mathfrak{osp}_{M|N}) = \mathbf{ZX}(\mathfrak{osp}_{M|N}) \cdot \mathbf{Y}(\mathfrak{osp}_{M|N})$.

We now aim to show that the coefficients of $\mathcal{Z}(u) - \mathbf{1}$ are algebraically independent over $\mathbf{Y}(\mathfrak{osp}_{M|N})$. In fact, by the relations (2.2) it suffices to show that the coefficients of $\mathcal{Y}(u)$ are algebraically independent over $\mathbf{Y}(\mathfrak{osp}_{M|N})$. To this extent, suppose to the contrary that for some $n \in \mathbb{Z}^+$ there exists a nonzero polynomial $B(x_1, \ldots, x_n)$ in $\mathbf{Y}(\mathfrak{osp}_{M|N})[x_1, \ldots, x_n]$ such that

$$B(\mathcal{Y}_1, \ldots, \mathcal{Y}_n) = 0$$

and let n be the minimal positive integer with respect to this property. Now, given $a \in \mathbb{C}$ and if we consider the series $f(u) = 1 + au^{-n}$, then by having μ_f act on the above equation one yields $B(\mathcal{Y}_1, \ldots, \mathcal{Y}_n + a\mathbf{1}) = 0$. Explicitly, if

$$B(\mathcal{Y}_1, \ldots, \mathcal{Y}_n) = \sum_{(k_1,\ldots,k_n)\in J} \gamma_{k_1,\ldots,k_n} \mathcal{Y}_1^{k_1} \cdots \mathcal{Y}_n^{k_n}$$

for some finite subset $J \subset (\mathbb{Z}^+)^n$, then

$$\begin{aligned} 0 &= B(\mathcal{Y}_1, \ldots, \mathcal{Y}_n + a_1\mathbf{1}) - B(\mathcal{Y}_1, \ldots, \mathcal{Y}_n) \\ &= \sum_{(k_1,\ldots,k_n)\in J} \gamma Y_{k_1,\ldots,k_n} \mathcal{Y}_1^{k_1} \cdots \mathcal{Y}_{n-1}^{k_{n-1}} \left(\sum_{i=0}^{k_n-1} \binom{k_n}{i} a_1^{k_n-i} \mathcal{Y}_n^i \right). \end{aligned}$$

The polynomial $B_1(x_1, \ldots, x_n) = B(x_1, \ldots, x_n + a_1\mathbf{1}) - B(x_1, \ldots, x_n)$ is nonzero for $a_1 \in \mathbb{C}^*$ and is of one degree less than B. If we inductively define

$$B_j(x_1, \ldots, x_n) = B_{j-1}(x_1, \ldots, x_n + a_j\mathbf{1}) - B_{j-1}(x_1, \ldots, x_n),$$

for some $a_j \in \mathbb{C}^*$, then we will yield a nonzero polynomial $B_m(x_1, \ldots, x_{n-1})$ in $\mathbf{Y}(\mathfrak{osp}_{M|N})[x_1, \ldots, x_{n-1}]$ for some $m \in \mathbb{Z}^+$ such that $B_m(\mathcal{Y}_1, \ldots, \mathcal{Y}_{n-1}) = 0$, contradicting the choice of $n \in \mathbb{Z}^+$.

Let us now consider the $\mathbb{C}$-linear multiplication map

$$\begin{aligned}\mu\colon \mathbf{ZX}(\mathfrak{osp}_{M|N}) \otimes \mathbf{Y}(\mathfrak{osp}_{M|N}) &\to \mathbf{X}(\mathfrak{osp}_{M|N}) \\ \mathcal{Z} \otimes \mathcal{Y} &\mapsto \mathcal{Z}\mathcal{Y}.\end{aligned} \tag{2.7}$$

This map is an algebra morphism since $\mathbf{ZX}(\mathfrak{osp}_{M|N})$ is comprised of central elements. Surjectivity follows from that $\mathbf{X}(\mathfrak{osp}_{M|N}) = \mathbf{ZX}(\mathfrak{osp}_{M|N}) \cdot \mathbf{Y}(\mathfrak{osp}_{M|N})$ and the kernel of the map is trivial by the algebraic independence of $\mathbf{ZX}(\mathfrak{osp}_{M|N})$ over $\mathbf{Y}(\mathfrak{osp}_{M|N})$. □

We note that the coefficients of the series $\mathcal{Y}(u) \in \mathbf{ZX}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$ as in (2.3) are homogeneous of even degree. Therefore, $\mathcal{Y}(u)$ lies in the even subalgebra of $\mathbf{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$.

Corollary 2.3.4. *The super Yangian $\mathbf{Y}(\mathfrak{osp}_{M|N})$ is isomorphic as algebras to the quotient of $\mathbf{X}(\mathfrak{osp}_{M|N})$ by the two-sided ideal $(\mathcal{Z}(u) - \mathbf{1})$, i.e.,*

$$\mathbf{Y}(\mathfrak{osp}_{M|N}) \cong \mathbf{X}(\mathfrak{osp}_{M|N})/(\mathcal{Z}(u) - \mathbf{1}). \tag{2.8}$$

Equivalently, $\mathbf{Y}(\mathfrak{osp}_{M|N})$ is generated by the elements $\mathcal{T}_{ij}^{(n)}$ (2.6), where $1 \le i, j \le M+N$ and $n \in \mathbb{N}$, subject only to the relations

$$\begin{aligned}[\mathcal{T}_{ij}(u), \mathcal{T}_{kl}(v)] = {}& \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\Big(\mathcal{T}_{kj}(u)\mathcal{T}_{il}(v) - \mathcal{T}_{kj}(v)\mathcal{T}_{il}(u)\Big) \\ & - \frac{1}{u-v-\kappa}\Bigg(\delta_{\bar{\imath}k}\sum_{p=1}^{M+N}(-1)^{[i][j]+[i]+[j][p]}\theta_i\theta_p\mathcal{T}_{pj}(u)\mathcal{T}_{\bar{p}l}(v) \\ & \qquad - \delta_{\bar{\jmath}l}\sum_{p=1}^{M+N}(-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]}\theta_j\theta_p\mathcal{T}_{k\bar{p}}(v)\mathcal{T}_{ip}(u)\Bigg),\end{aligned} \tag{2.9}$$

and

$$\sum_{k=1}^{M+N}(-1)^{[i][k]+[k]}\theta_i\theta_k\mathcal{T}_{\bar{k}\bar{\imath}}(u+\kappa)\mathcal{T}_{kj}(u) = \delta_{ij}\mathbf{1}. \tag{2.10}$$

Proof. We follow the argument as in [?, Corollary 3.2]. By Theorem (2.3.3), we know that $\mathbf{X}(\mathfrak{osp}_{M|N}) \cong \big(\mathbb{C}\mathbf{1} \otimes \mathbf{Y}(\mathfrak{osp}_{M|N})\big) \oplus \big(\mathbb{C}\,[\mathcal{Z}_n \mid n \in \mathbb{Z}^+] \otimes \mathbf{Y}(\mathfrak{osp}_{M|N})\big)$, and so by the isomorphism (2.7) we conclude that $\mathbf{X}(\mathfrak{osp}_{M|N}) = (\mathcal{Z}(u) - \mathbf{1}) \oplus \mathbf{Y}(\mathfrak{osp}_{M|N})$. Therefore, the first assertion follows by the second isomorphism theorem of algebras.

To show that $\mathbf{Y}(\mathfrak{osp}_{M|N})$ is generated by the elements $\mathcal{T}_{ij}^{(n)}$, $1 \leq i, j \leq M+N$, $n \in \mathbb{Z}^+$, we first observe that in the proof of Theorem (2.3.3), it was shown that any element $X \in \mathbf{X}(\mathfrak{osp}_{M|N})$ may be uniquely written as $B(\mathcal{Y}_1, \ldots, \mathcal{Y}_n)$ for some nonzero polynomial B in $n \in \mathbb{N}$ indeterminates with coefficients in $\mathbf{Y}(\mathfrak{osp}_{M|N})$. If X lies in $\mathbf{Y}(\mathfrak{osp}_{M|N})$, then $X = \mu_f(X) = B(\mu_f(\mathcal{Y}_1), \ldots, \mu_f(\mathcal{Y}_n))$ for all $f(u) \in \mathbf{1} + u^{-1}\mathbb{C}[\![u^{-1}]\!]$. However, uniqueness would imply that B must then be a constant polynomial.

By using equation (2.6) and the fact that $\mathcal{Y}(u)$ is central in $\mathbf{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$, we see that $[\mathcal{T}_{ij}(u), \mathcal{T}_{kl}(v)] = \mathcal{Y}(u)^{-2}[T_{ij}(u), T_{kl}(v)]$ and so relation (2.9) follows from (4.10). Relation (2.10) is derived from (2.51). Conversely, equations (2.9) and (2.10) are the defining relations for $\mathbf{Y}(\mathfrak{osp}_{M|N})$ since they are equivalent to the relations (4.10) and $\mathcal{Z}(u) = \mathbf{1}$, respectively. $\square$

We note that from equation (2.6) the $\mathbb{Z}_2$-grade of the elements $\mathcal{T}_{ij}^{(n)}$ is $[i]+[j]$ since the coefficients of $\mathcal{Y}(u)$ are homogeneous of even degree. Hence, $\mathbf{Y}(\mathfrak{osp}_{M|N})$ is $\mathbb{Z}_2$-graded by Corollary (2.3.4). Furthermore, by setting $\mathcal{T}(u) = \sum_{i,j=1}^{M+N}(-1)^{[i][j]+[j]}E_{ij} \otimes \mathcal{T}_{ij}(u)$, then we may rewrite the defining relations of $\mathbf{Y}(\mathfrak{osp}_{M|N})$ as

$$R(u-v)\mathcal{T}_1(u)\mathcal{T}_2(v) = \mathcal{T}_2(v)\mathcal{T}_1(u)R(u-v), \tag{2.11}$$

$$\mathcal{T}^t(u+\kappa)\mathcal{T}(u) = \mathbb{1}. \tag{2.12}$$

We observe that the isomorphism (2.7) is grade-preserving. Since the two-sided ideal $(\mathcal{Z}(u) - \mathbf{1})$ is graded, the quotient of $\mathbf{X}(\mathfrak{osp}_{M|N})$ by $(\mathcal{Z}(u) - \mathbf{1})$ is $\mathbb{Z}_2$-graded and so the isomorphism (2.8) may then be verified to be grade-preserving as well.

Furthermore, since $(\mathcal{Z}(u) - \mathbf{1})$ is a Hopf ideal the quotient of $\mathbf{X}(\mathfrak{osp}_{M|N})$ by $(\mathcal{Z}(u) - \mathbf{1})$ has a unique Hopf superstructure such that the canonical projection on the quotient

$\mathbf{X}(\mathfrak{osp}_{M|N}) \twoheadrightarrow \mathbf{X}(\mathfrak{osp}_{M|N})/(\mathcal{Z}(u) - \mathbf{1})$ is a morphism of Hopf superalgebras. The isomorphism (2.8) then equips a Hopf superstructure onto $\mathbf{Y}(\mathfrak{osp}_{M|N})$. In fact, this will just be the sub-Hopf superstructure on $\mathbf{Y}(\mathfrak{osp}_{M|N})$ endowed from $\mathbf{X}(\mathfrak{osp}_{M|N})$.

Proposition 2.3.5. *The super Yangian $\mathbf{Y}(\mathfrak{osp}_{M|N})$ is a sub-Hopf superalgebra of the extended super Yangian $\mathbf{X}(\mathfrak{osp}_{M|N})$, whose Hopf structure maps are given by the restriction of those on $\mathbf{X}(\mathfrak{osp}_{M|N})$.*

Proof. Using (2.54) and (2.3), we observe that

$$\begin{aligned}\Delta(\mathcal{Y}(u))\Delta(\mathcal{Y}(u+\kappa)) = \Delta(\mathcal{Z}(u)) = \mathcal{Z}(u) \otimes \mathcal{Z}(u) \\ = (\mathcal{Y}(u) \otimes \mathcal{Y}(u))(\mathcal{Y}(u+\kappa) \otimes \mathcal{Y}(u+\kappa)).\end{aligned}$$

Therefore, by Proposition (2.3.2) where $a(u) = \mathcal{Z}(u) \otimes \mathcal{Z}(u)$ and $y(u) = \mathcal{Y}(u) \otimes \mathcal{Y}(u)$, we have by uniqueness $\Delta(\mathcal{Y}(u)) = \mathcal{Y}(u) \otimes \mathcal{Y}(u)$. In particular, $\Delta(\mathcal{Y}(u)^{-1}) = \mathcal{Y}(u)^{-1} \otimes \mathcal{Y}(u)^{-1}$ and so

$$\begin{aligned}\Delta(\mathcal{T}_{ij}(u)) = \Delta(\mathcal{Y}(u)^{-1})\Delta(T_{ij}(u)) &= \sum_{k=1}^{M+N} (-1)^{[k]} \mathcal{Y}(u)^{-1} T_{ik}(u) \otimes \mathcal{Y}(u)^{-1} T_{kj}(u) \\ &= \sum_{k=1}^{M+N} (-1)^{[k]} \mathcal{T}_{ik}(u) \otimes \mathcal{T}_{kj}(u),\end{aligned}$$

showing that $\Delta(\mathbf{Y}(\mathfrak{osp}_{M|N})) \subseteq \mathbf{Y}(\mathfrak{osp}_{M|N}) \otimes \mathbf{Y}(\mathfrak{osp}_{M|N})$. Furthermore, by using (2.57) and equation (2.3), we get

$$S(\mathcal{Y}(u))S(\mathcal{Y}(u+\kappa)) = S(\mathcal{Z}(u)) = \mathcal{Z}(u)^{-1} = \mathcal{Y}(u)^{-1}\mathcal{Y}(u+\kappa)^{-1}.$$

By Proposition (2.3.2) where $a(u) = \mathcal{Z}(u)^{-1}$ and $y(u) = \mathcal{Y}(u)^{-1}$, we therefore yield by uniqueness that $S(\mathcal{Y}(u)) = \mathcal{Y}(u)^{-1}$ and so $S(\mathcal{Y}(u)^{-1}) = \mathcal{Y}(u)$. Collecting the generators of $\mathbf{Y}(\mathfrak{osp}_{M|N})$ into the matrix $\mathcal{T}(u) = \mathcal{Y}(u)^{-1}T(u)$, we observe that

$$S\colon \mathcal{T}(u) = \mathcal{Y}(u)^{-1}T(u) \mapsto \mathcal{Y}(u)T(u)^{-1} = \mathcal{T}(u)^{-1}.$$

From relations (2.58), (2.4), and (2.5), one sees that $\mathcal{Y}(u)T(u)^{-1}$ is left invariant under automorphisms of the form (3.20). Therefore, $S(\mathbf{Y}(\mathfrak{osp}_{M|N})) \subseteq \mathbf{Y}(\mathfrak{osp}_{M|N})$. $\square$

Corollary 2.3.6. *There exists Hopf superalgebra isomorphisms*

$$\mathbf{X}(\mathfrak{osp}_{M|N}) \cong \mathbf{ZX}(\mathfrak{osp}_{M|N}) \otimes \mathbf{Y}(\mathfrak{osp}_{M|N}),$$
$$\mathbf{Y}(\mathfrak{osp}_{M|N}) \cong \mathbf{X}(\mathfrak{osp}_{M|N})/(\mathcal{Z}(u) - \mathbf{1}).$$

That is, the Hopf superalgebra structure on $\mathbf{Y}(\mathfrak{osp}_{M|N})$ is given by

$$\begin{aligned} \Delta\colon \mathbf{Y}(\mathfrak{osp}_{M|N}) &\to \mathbf{Y}(\mathfrak{osp}_{M|N}) \otimes \mathbf{Y}(\mathfrak{osp}_{M|N}) \\ \mathcal{T}(u) &\mapsto \mathcal{T}_{[1]}(u)\mathcal{T}_{[2]}(u) \end{aligned} \tag{2.13}$$

the counit

$$\begin{aligned} \varepsilon\colon \mathbf{Y}(\mathfrak{osp}_{M|N}) &\to \mathbb{C} \\ \mathcal{T}(u) &\mapsto \mathbb{1} \end{aligned} \tag{2.14}$$

and the antipode

$$\begin{aligned} S\colon \mathbf{Y}(\mathfrak{osp}_{M|N}) &\to \mathbf{Y}(\mathfrak{osp}_{M|N}) \\ \mathcal{T}(u) &\mapsto \mathcal{T}(u)^{-1} \end{aligned} \tag{2.15}$$

In particular, from (2.11), we see that the antipode of $\mathbf{Y}(\mathfrak{osp}_{M|N})$ is given by

$$S\colon \mathcal{T}(u) \mapsto \mathcal{T}^t(u+\kappa) \tag{2.16}$$

Furthermore, we note that the elements $\mathcal{Z}_n$, $\mathcal{Y}_n$, and $\mathcal{Y}_n^\bullet$ are all of degree $n-1$, where $\mathcal{Y}(u)^{-1} = \mathbf{1} + \sum_{n=1}^{\infty} \mathcal{Y}_n^\bullet u^{-n}$. The first of these can be seen from equation (2.51). For the second, we observe that from equation (2.3) we yield (re-write:

$$\mathcal{Z}_n = \sum_{m+p+q=n} \vartheta_{-\kappa}(m,p)\mathcal{Y}_p\mathcal{Y}_q, \tag{2.17}$$

where $\mathcal{Y}_0 = \mathbf{1}$ and $\vartheta_{-\kappa}(m,p) = \binom{m+p-1}{p-1}(-\kappa)^m$ such that $\vartheta_{-\kappa}(m,0) = 1$), which therefore implies the result by induction on $n \in \mathbb{Z}^+$. To see why $\deg_{\mathbf{E}} \mathcal{Y}_n^{\bullet} = n-1$, we write (re-write: $\widetilde{\mathcal{Y}}_n = \sum_{s=1}^{n}(-1)^s \sum_{\sum_{i=1}^{s} k_i = n} \prod_{i=1}^{s} \mathcal{Y}_{k_i}$, $k_i \in \mathbb{Z}^+$ for $i = 1, \ldots, s$. Then the result follows from the fact that $\prod_{i=1}^{s} \mathcal{Y}_{k_i} \in \prod_{i=1}^{s} \mathbf{F}^X_{k_i-1} \subseteq \mathbf{F}^X_{n-s}$).

By writing (re-write:

$$T^t_{ij}(u+\kappa) = (-1)^{[i][j]+[j]}\theta_i\theta_j \sum_{m,p=0}^{\infty} \vartheta_{-\kappa}(m,p) T^{(p)}_{\bar{j}\bar{i}} u^{-m-p},$$

then by equation (2.51) we observe that

$$\sum_{j=1}^{M+N} (-1)^{[i][j]+[i][l]}\theta_i\theta_j \sum_{m,p,q=0}^{\infty} \vartheta_{-\kappa}(m,p) T^{(p)}_{\bar{j}\bar{i}} T^{(q)}_{jl} u^{-m-p-q} = \delta_{il}\mathcal{Z}(u).$$

Therefore, we yield the relations

$$(-1)^{[i][l]}T^{(n)}_{il} + (-1)^{[l]}\theta_i\theta_l T^{(n)}_{\bar{l}\bar{i}} = \delta_{il}\mathcal{Z}_n \mod \mathbf{F}^X_{n-2}.$$

Multiplying the above equation by $(-1)^{[i][l]+[l]}$, we therefore have

$$(-1)^{[l]}T^{(n)}_{il} + (-1)^{[i][l]}\theta_i\theta_l T^{(n)}_{\bar{l}\bar{i}} = \delta_{il}\mathcal{Z}_n \mod \mathbf{F}^X_{n-2}. \tag{2.18}$$

From equation (2.6), the degree of $\mathcal{T}^{(n)}_{ij}$ is at most $n-1$. Moreover, we obtain

$$\mathcal{T}^{(n)}_{ij} = T^{(n)}_{ij} - \delta_{ij}(-1)^{[j]}\mathcal{Y}_n \mod \mathbf{F}^X_{n-2}$$

from (2.6). In particular, using that $\mathcal{Y}_n = \frac{1}{2}\mathcal{Z}_n \mod \mathbf{F}^X_{n-2}$ from (2.17), we deduce

$$\mathcal{T}^{(n)}_{ij} = \frac{1}{2}\left(T^{(n)}_{ij} - (-1)^{[i][j]+[j]}\theta_i\theta_j T^{(n)}_{\bar{j}\bar{i}}\right) \mod \mathbf{F}^X_{n-2} \tag{2.19}$$

from (2.18).) Now, the filtration $\mathbf{E}$ on $\mathbf{X}(\mathfrak{osp}_{M|N})$ endows a filtration $\overline{\mathbf{E}} = \{\overline{\mathbf{E}}_n\}_{n\in\mathbb{N}}$ on the quotient $\mathbf{X}(\mathfrak{osp}_{M|N})/(\mathcal{Z}(u)-1)$ such that $\overline{\mathbf{E}}_n = \mathbf{E}_n/(\mathbf{E}_n \cap (\mathcal{Z}(u)-1))$. Therefore, the isomorphism (2.8) will induce a filtration $\mathbf{F} = \{\mathbf{F}_n\}_{n\in\mathbb{N}}$ on $\mathbf{Y}(\mathfrak{osp}_{M|N})$ such that

$\deg_{\mathbf{F}} \mathcal{T}_{ij}^{(n)} = n-1$. That is,

$$\mathbf{F}_n = \operatorname{span}_{\mathbb{C}} \left\{ \prod_{a=1}^{\gamma} \mathcal{T}_{i_a j_a}^{(k_a)} \mid \gamma \in \mathbb{Z}^+,\ 1 \le i_a, j_a \le M+N,\ \sum_{a=1}^{\gamma} k_a \le n+\gamma \right\}$$

for $n \in \mathbb{Z}^+$ and

$$\mathbf{F}_0 = \operatorname{span}_{\mathbb{C}} \left\{ \prod_{a=1}^{\gamma} T_{i_a j_a}^{(k_a)} \mid \gamma \in \mathbb{Z}^+,\ 1 \le i_a, j_a \le M+N,\ k_a \in \{0,1\} \right\}.$$

We hence have the associated graded algebra of $\mathbf{Y}(\mathfrak{osp}_{M|N})$:

$$\operatorname{gr}_{\mathbf{F}} \mathbf{Y}(\mathfrak{osp}_{M|N}) = \bigoplus_{n \in \mathbb{N}} \operatorname{gr}_n \mathbf{Y}(\mathfrak{osp}_{M|N}) = \bigoplus_{n \in \mathbb{N}} \mathbf{F}_n / \mathbf{F}_{n-1},$$

The isomorphism (2.8) will induce an isomorphism on the graded superalgebras

$$\operatorname{gr}_{\mathbf{F}} \mathbf{Y}(\mathfrak{osp}_{M|N}) \cong \operatorname{gr}_{\overline{\mathbf{E}}} (\mathbf{X}(\mathfrak{osp}_{M|N})/(\mathcal{Z}(u)-1)) = \bigoplus_{n \in \mathbb{N}} \overline{\mathbf{E}}_n / \overline{\mathbf{E}}_{n-1}$$

and shall now let $\overline{\mathcal{T}}_{ij}^{(n)}$ to denote the image of $\mathcal{T}_{ij}^{(n)}$ in the $\operatorname{gr}_{n-1} \mathbf{Y}(\mathfrak{osp}_{M|N})$. Accordingly, $\operatorname{gr}_{\mathbf{F}} \mathbf{Y}(\mathfrak{osp}_{M|N})$ will be endowed with an $\mathbb{N}$-graded Hopf superalgebra structure given by comultiplication

$$\begin{aligned} \operatorname{gr}_{\mathbf{F}} \Delta \colon\ \operatorname{gr}_{\mathbf{F}} \mathbf{Y}(\mathfrak{osp}_{M|N}) &\to \left(\operatorname{gr}_{\mathbf{F}} \mathbf{Y}(\mathfrak{osp}_{M|N})\right)^{\otimes 2} \\ \overline{\mathcal{T}}_{ij}^{(n)} &\mapsto \overline{\mathcal{T}}_{ij}^{(n)} \otimes \mathbf{1} + \mathbf{1} \otimes \overline{\mathcal{T}}_{ij}^{(n)},\ n \in \mathbb{Z}^+ \end{aligned} \tag{2.20}$$

counit

$$\begin{aligned} \operatorname{gr}_{\mathbf{F}} \varepsilon \colon\ \operatorname{gr}_{\mathbf{F}} \mathbf{Y}(\mathfrak{osp}_{M|N}) &\to \mathbb{C} \\ \overline{\mathcal{T}}_{ij}^{(n)} &\mapsto \delta_{0n} \delta_{ij} \end{aligned} \tag{2.21}$$

and antipode

$$\begin{aligned} \operatorname{gr}_{\mathbf{F}} S \colon\ \operatorname{gr}_{\mathbf{F}} \mathbf{Y}(\mathfrak{osp}_{M|N}) &\to \operatorname{gr}_{\mathbf{F}} \mathbf{Y}(\mathfrak{osp}_{M|N}) \\ \overline{\mathcal{T}}_{ij}^{(n)} &\mapsto (-1)^{[i][j]+[j]} \theta_i \theta_j \overline{\mathcal{T}}_{\bar{j}\bar{\imath}}^{(n)} = -\overline{\mathcal{T}}_{ij}^{(n)}. \end{aligned} \tag{2.22}$$

2.4 Representation Theory of $\mathbf{X}(\mathfrak{osp}_{M|N})$

2.4.1 Root Systems of $\mathfrak{osp}_{M|N}$

Let us first observe that the coefficient of u^{-1} in equation (2.51) yields the equality

$$T_{ij}^{(1)} + (T_{ij}^{(1)})^t = \delta_{ij}\mathcal{Z}_1. \tag{2.1}$$

By identifying the elements $F_{ij} \in \mathfrak{osp}_{M|N}$ with their images in $\mathbf{X}(\mathfrak{osp}_{M|N})$ under the embedding (2.25), equation (2.1) infers $F_{ij} = (-1)^{[i]}T_{ij}^{(1)} - \frac{1}{2}\delta_{ij}(-1)^{[i]}\mathcal{Z}_1$. Furthermore, the embedding (2.25) equips $\mathbf{X}(\mathfrak{osp}_{M|N})$ with an $\mathfrak{osp}_{M|N}$-module structure. To determine the action of the generators F_{ij} on $\mathbf{X}(\mathfrak{osp}_{M|N})$, we first note that the defining relations (4.10) of $\mathbf{X}(\mathfrak{osp}_{M|N})$ imply

$$\begin{aligned}\left[T_{ij}^{(1)}, T_{kl}(v)\right] &= \delta_{jk}(-1)^{[j]}T_{il}(v) - \delta_{il}(-1)^{[i]+([i]+[j])([k]+[l])}T_{kj}(v)\\ &\quad - \delta_{\bar{\imath}k}(-1)^{[j]+[i][j]+[i]}\theta_i\theta_j T_{\bar{\jmath}l}(v) + \delta_{\bar{\jmath}l}(-1)^{[j]+([i]+[j])[k]}\theta_i\theta_j T_{k\bar{\imath}}(v).\end{aligned} \tag{2.2}$$

Since $\mathcal{Z}_1$ lies within the supercenter of $\mathbf{X}(\mathfrak{osp}_{M|N})$, the generators F_{ij} therefore act on $\mathbf{X}(\mathfrak{osp}_{M|N})$ by the formula

$$\begin{aligned}\left[F_{ij}, T_{kl}(u)\right] &= \delta_{jk}(-1)^{[i]+[j]}T_{il}(u) - \delta_{il}(-1)^{([i]+[j])([k]+[l])}T_{kj}(u)\\ &\quad - \delta_{\bar{\imath}k}(-1)^{[i][j]+[j]}\theta_i\theta_j T_{\bar{\jmath}l}(u) + \delta_{\bar{\jmath}l}(-1)^{([i]+[j])([k]+1)}\theta_i\theta_j T_{k\bar{\imath}}(u).\end{aligned} \tag{2.3}$$

Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{osp}_{M|N}$ given by

$$\mathfrak{h} = \operatorname{span}_{\mathbb{C}}\left\{F_{hh} \mid \left\lceil \tfrac{M}{2}\right\rceil + 1 \le h \le M,\ M + \tfrac{N}{2} + 1 \le h \le M+N\right\}, \tag{2.4}$$

where $\lceil\cdot\rceil : \mathbb{Q} \to \mathbb{N}$ denotes the ceiling function. Letting $\lfloor\cdot\rfloor : \mathbb{Q} \to \mathbb{N}$ denote the floor function, then $\mathfrak{osp}_{M|N}$ is of rank $\left\lfloor \frac{M}{2}\right\rfloor + \frac{N}{2}$ and the action of the Cartan subalgebra on

$\mathbf{X}(\mathfrak{osp}_{M|N})$ is given by

$$\begin{aligned}\big[F_{hh}, T_{ij}(u)\big] &= \delta_{hi}T_{hj}(u) - \delta_{hj}T_{ih}(u) - \delta_{\bar{h}i}T_{\bar{h}j}(u) + \delta_{\bar{h}j}T_{i\bar{h}}(u). \\ &= (\delta_{hi} - \delta_{hj} - \delta_{h\bar{\imath}} + \delta_{h\bar{\jmath}})T_{ij}(u)\end{aligned} \tag{2.5}$$

Let $\left\{\varepsilon_i, \delta_j \mid 1 \leq i \leq \lfloor \frac{M}{2} \rfloor,\ 1 \leq j \leq \frac{N}{2}\right\}$ denote the dual basis for $\mathfrak{h}$, where ε_i and δ_j are those $\mathbb{C}$-linear functionals in $\mathfrak{h}^*$ given by

$$\varepsilon_i(F_{hh}) = \begin{cases} \delta_{\lceil \frac{M}{2} \rceil + i, h} & \text{if} \quad \lceil \frac{M}{2} \rceil + 1 \leq h \leq M \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

$$\text{and} \quad \delta_j(F_{hh}) = \begin{cases} \delta_{M + \frac{N}{2} + j, h} & \text{if} \quad M + \frac{N}{2} + 1 \leq h \leq M + N \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

To introduce a notion of positivity for the root system of $\mathfrak{osp}_{M|N}$, we declare the non-zero even generators F_{ij} of $\mathfrak{osp}_{M|N}$ with indices satisfying $i < j$ to be *positive even root vectors*; the collection of their corresponding roots will form a system of positive even roots, which we will denote Φ^+_{even}.

We will complete the set Φ^+_{even} to positive root system of $\mathfrak{osp}_{M|N}$ in two ways by selecting appropriate collections of odd roots to be positive. Before describing these selections, we first establish some notation by denoting the first collection of odd positive roots as $\Phi^+_{\text{odd[I]}}$, and the second by $\Phi^+_{\text{odd[II]}}$. Hence, we will denote the two positive root systems for $\mathfrak{osp}_{M|N}$ as $\Phi^+_{\text{I}} := \Phi^+_{\text{even}} \cup \Phi^+_{\text{odd[I]}}$ and $\Phi^+_{\text{II}} := \Phi^+_{\text{even}} \cup \Phi^+_{\text{odd[II]}}$.

The orthosymplectic Lie superalgebras comprise three infinite families of basic Lie superalgebras: $B(m,n) = \mathfrak{osp}_{(2m+1)|2n}$ for integers $m \geq 0$ and $n \geq 1$; $C(n+1) = \mathfrak{osp}_{2|2n}$ for integers $n \geq 1$; and $D(m,n) = \mathfrak{osp}_{2m|2n}$ for integers $m \geq 2$ and $n \geq 1$. The description the root system of $\mathfrak{osp}_{M|N}$ will vary depending on which family, it is necessary to describe to root systems of each of these three families, as these will all be different.

For the following descriptions of positive root systems Note that the Cartan subalgebra action on $\mathfrak{osp}_{M|N}$ is described by

$$\begin{aligned}\left[F_{hh}, F_{ij}\right] &= \delta_{hi} F_{hj} - \delta_{hj} F_{ih} - \delta_{\bar{h}i} F_{\bar{h}j} + \delta_{\bar{h}j} F_{i\bar{h}}, \\ &= (\delta_{hi} - \delta_{hj} - \delta_{h\bar{\imath}} + \delta_{h\bar{\jmath}}) F_{ij}.\end{aligned}$$

Let us consider the first case of $\mathfrak{osp}_{M|N} = B(m,n) = \mathfrak{osp}_{(2m+1)|2n}$. Here, the Cartan subalgebra is given by $\mathfrak{h} = \mathrm{span}_{\mathbb{C}} \{F_{hh} \mid m{+}2 \leq a \leq M,\ M{+}n{+}1 \leq a \leq M{+}N\}$. As mentioned before, we deem the non-zero even elements F_{ij}, where $i < j$, as positive root vectors. Using the relations

$$\begin{aligned}\left[F_{hh}, F_{ij}\right] &= \delta_{hi} F_{hj} - \delta_{hj} F_{ih} - \delta_{\bar{h}i} F_{\bar{h}j} + \delta_{\bar{h}j} F_{i\bar{h}}, \\ &= (\delta_{hi} - \delta_{hj} - \delta_{h\bar{\imath}} + \delta_{h\bar{\jmath}}) F_{ij}\end{aligned}$$

we find that the collection of their associated positive even roots is given by

$$\Phi^+_{\text{even}} = \{\varepsilon_i {-} \varepsilon_j,\ -\varepsilon_i {-} \varepsilon_j,\ -\varepsilon_q,\ \delta_k {-} \delta_l,\ -\delta_k {-} \delta_l,\ -2\delta_p\}, \tag{2.8}$$

with $1 \leq i < j \leq m$, $1 \leq q \leq m$, $1 \leq k < l \leq n$, and $1 \leq p \leq n$. The first selection of positive odd roots is

$$\Phi^+_{\text{odd[I]}} = \{-\varepsilon_i {+} \delta_k,\ -\varepsilon_i {-} \delta_k,\ -\delta_p\}, \tag{2.9}$$

where $1 \leq i \leq m$ and $1 \leq k, p \leq n$. The second choice shall be

$$\Phi^+_{\text{odd[II]}} = \{\varepsilon_i {-} \delta_k,\ -\varepsilon_i {-} \delta_k,\ -\delta_p\}, \tag{2.10}$$

when $1 \leq i \leq m$ and $1 \leq k, p \leq n$ as well.

In the case $\mathfrak{osp}_{M|N} = C(n{+}1) = \mathfrak{osp}_{2|2n}$, our Cartan subalgebra is given by the span $\mathfrak{h} = \mathrm{span}_{\mathbb{C}} \{F_{hh} \mid h = 2,\ 2{+}n{+}1 \leq h \leq 2{+}N\}$ and the even positive roots are given by

$$\Phi^+_{\text{even}} = \{\delta_k {-} \delta_l,\ -\delta_k {-} \delta_l,\ -2\delta_p\}, \tag{2.11}$$

where $1 \leq k < l \leq n$ and $1 \leq p \leq n$. Accordingly, the first selection of positive odd roots is given by

$$\Phi^{+}_{\text{odd[I]}} = \{-\varepsilon_1 + \delta_k,\ -\varepsilon_1 - \delta_k\}, \tag{2.12}$$

where $1 \leq k \leq n$. The second choice is

$$\Phi^{+}_{\text{odd[II]}} = \{\varepsilon_1 - \delta_k,\ -\varepsilon_1 - \delta_k\}, \tag{2.13}$$

with $1 \leq k \leq n$.

Lastly, when $\mathfrak{osp}_{M|N} = D(m,n) = \mathfrak{osp}_{2m|2n}$, our Cartan subalgebra is given by the span $\mathfrak{h} = \operatorname{span}_{\mathbb{C}}\{F_{hh} \mid m+1 \leq h \leq M,\ M+n+1 \leq h \leq M+N\}$. The even positive roots are

$$\Phi^{+}_{\text{even}} = \{\varepsilon_i - \varepsilon_j,\ -\varepsilon_i - \varepsilon_j,\ \delta_k - \delta_l,\ -\delta_k - \delta_l,\ -2\delta_p\}, \tag{2.14}$$

where $1 \leq i < j \leq m$, $1 \leq k < l \leq n$, and $1 \leq p \leq n$. To complete this to a set of positive roots for $D(m,n)$, the first selection of positive odd roots is

$$\Phi^{+}_{\text{odd[I]}} = \{-\varepsilon_i + \delta_k,\ -\varepsilon_i - \delta_k\}, \tag{2.15}$$

where $1 \leq i \leq m$ and $1 \leq k \leq n$. A second selection is given by

$$\Phi^{+}_{\text{odd[II]}} = \{\varepsilon_i - \delta_k,\ -\varepsilon_i - \delta_k\}, \tag{2.16}$$

with $1 \leq i \leq m$ and $1 \leq k \leq n$.

Setting $\mathbb{Z}^{+}_{M+N} := [1, M+N] \cap \mathbb{Z}^{+}$, we consider the following subsets of $(\mathbb{Z}^{+}_{M+N})^2$:

$$\Gamma_{0,0} = \{(i,j) \mid 1 \leq i < j \leq M\}, \quad \Gamma_{1,1} = \{(i,j) \mid M+1 \leq i < j \leq M+N\},$$

$$\begin{aligned}
\Gamma^{\mathrm{I}}_{0,1a} &= \left\{(i,j) \mid 1 \leq i \leq \left\lfloor \tfrac{M}{2} \right\rfloor,\ M+1 \leq j \leq M+\tfrac{N}{2}\right\},\\
\Gamma^{\mathrm{I}}_{0,1b} &= \left\{(i,j) \mid 1 \leq i \leq \left\lceil \tfrac{M}{2} \right\rceil,\ M+\tfrac{N}{2}+1 \leq j \leq M+N\right\},\\
\Gamma^{\mathrm{I}}_{1,0a} &= \left\{(i,j) \mid M+1 \leq i \leq M+\tfrac{N}{2},\ \left\lfloor \tfrac{M}{2} \right\rfloor+1 \leq j \leq M\right\},
\end{aligned}$$

$$\Gamma^{\mathrm{I}}_{1,0b} = \left\{(i,j) \mid M+\tfrac{N}{2}+1 \le i \le M+N,\ \left\lceil \tfrac{M}{2} \right\rceil+1 \le j \le M\right\},$$

$$\Gamma^{\mathrm{II}}_{0,1} = \left\{(i,j) \mid 1 \le i \le M,\ M+\tfrac{N}{2}+1 \le j \le M+N\right\},$$
$$\Gamma^{\mathrm{II}}_{1,0} = \left\{(i,j) \mid M+1 \le i \le M+\tfrac{N}{2},\ 1 \le j \le M\right\},$$

and assign $\Gamma^{\mathrm{I}}_{0,1} = \Gamma^{\mathrm{I}}_{0,1a} \cup \Gamma^{\mathrm{I}}_{0,1b}$, $\Gamma^{\mathrm{I}}_{1,0} = \Gamma^{\mathrm{I}}_{1,0a} \cup \Gamma^{\mathrm{I}}_{1,0b}$ to define

$$\Gamma_{\text{even}} := \Gamma_{0,0} \cup \Gamma_{1,1}, \quad \Gamma_{\text{odd[I]}} := \Gamma^{\mathrm{I}}_{0,1} \cup \Gamma^{\mathrm{I}}_{1,0}, \quad \Gamma_{\text{odd[II]}} := \Gamma^{\mathrm{II}}_{0,1} \cup \Gamma^{\mathrm{II}}_{1,0}.$$

At last, we can finally define the following sets

$$\Lambda^+_{\mathrm{I}} := \Gamma_{\text{even}} \cup \Gamma_{\text{odd[I]}}, \qquad \Lambda^+_{\mathrm{II}} := \Gamma_{\text{even}} \cup \Gamma_{\text{odd[II]}}. \tag{2.17}$$

Visually, if we regard the elements of $(\mathbb{Z}^+_{M+N})^2$ as an index set for an algebra of $(M+N) \times (M+N)$ matrices, then those indices that correspond to the shaded red regions of the following diagrams are exactly those that occur in Λ^+_{I} and Λ^+_{II}, respectively.

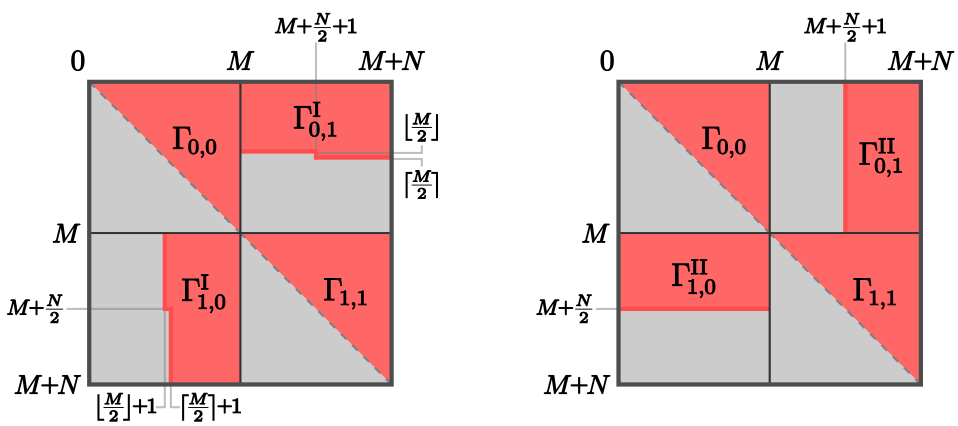

Figure 2.1: Visualizations of Λ^+_{I} and Λ^+_{II}, respectively

Further, let us consider the sets

$$\Lambda^\circ := \left\{(i,i) \mid i \in \mathbb{Z}^+_{M+N}\right\}, \tag{2.18}$$

$$\Lambda^-_{\mathrm{I}} := (\mathbb{Z}^+_{M+N})^2 \setminus \left(\Lambda^+_{\mathrm{I}} \cup \Lambda^\circ\right), \quad \text{and} \quad \Lambda^-_{\mathrm{II}} := (\mathbb{Z}^+_{M+N})^2 \setminus \left(\Lambda^+_{\mathrm{II}} \cup \Lambda^\circ\right). \tag{2.19}$$

If we let Θ denote either I or II, then we will have the triangular decomposition $\mathfrak{osp}_{M|N} = \mathfrak{n}_\Theta^- \oplus \mathfrak{b}_\Theta = \mathfrak{n}_\Theta^- \oplus \mathfrak{h} \oplus \mathfrak{n}_\Theta^+$, where $\mathfrak{h}$ is the Cartan subalgebra, $\mathfrak{b}_\Theta = \mathfrak{h} \oplus \mathfrak{n}_\Theta^+$ is the Borel subalgebra, and

$$\mathfrak{n}_\Theta^- = \operatorname{span}_{\mathbb{C}} \left\{F_{ij}\right\}_{(i,j)\in\Lambda_\Theta^-}, \quad \mathfrak{n}_\Theta^+ = \operatorname{span}_{\mathbb{C}} \left\{F_{ij}\right\}_{(i,j)\in\Lambda_\Theta^+}.$$

Letting $(\mathfrak{osp}_{M|N})_\alpha = \{X \in \mathfrak{osp}_{M|N} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$ denote the root space corresponding to $\alpha \in \mathfrak{h}^*$, then

$$\mathfrak{n}_\Theta^- = \bigoplus_{\alpha\in\Phi_\Theta^-} (\mathfrak{osp}_{M|N})_\alpha \quad \text{and} \quad \mathfrak{n}_\Theta^+ = \bigoplus_{\alpha\in\Phi_\Theta^+} (\mathfrak{osp}_{M|N})_\alpha.$$

Furthermore, the action of the Cartan subalgebra on $\mathbf{X}(\mathfrak{osp}_{M|N})$ described by (**??**) results in the following decomposition for the extended super Yangian $\mathbf{X}(\mathfrak{osp}_{M|N})$ in terms of the root lattice $\mathbb{Z}\Phi$:

$$\mathbf{X}(\mathfrak{osp}_{M|N}) = \bigoplus_{\alpha\in\mathbb{Z}\Phi} \mathbf{X}(\mathfrak{osp}_{M|N})_\alpha = \mathfrak{N}_\Theta^{\mathbf{X},-} \oplus \mathfrak{B}_\Theta^{\mathbf{X}} = \mathfrak{N}_\Theta^{\mathbf{X},-} \oplus \mathfrak{H}_\Theta^{\mathbf{X}} \oplus \mathfrak{N}_\Theta^{\mathbf{X},+},$$

where $\mathbf{X}(\mathfrak{osp}_{M|N})_\alpha = \{X \in \mathbf{X}(\mathfrak{osp}_{M|N}) \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$ is the root space for $\alpha \in \mathbb{Z}\Phi$, the positive/negative root lattices $\bigoplus_{\alpha\in\mathbb{Z}^+\Phi_\Theta^\pm} \mathbf{X}(\mathfrak{osp}_{M|N})_\alpha$ are denoted $\mathfrak{N}_\Theta^{\mathbf{X},\pm}$, the *lattice Cartan subalgebra* is $\mathfrak{H}_\Theta^{\mathbf{X}} = \mathbf{X}(\mathfrak{osp}_{M|N})_0$ (here, 0 refers to the zero linear functional in $\mathfrak{h}^*$, and not $\bar{0} \in \mathbb{Z}_2$), and $\mathfrak{B}_\Theta^{\mathbf{X}} = \mathfrak{H}_\Theta^{\mathbf{X}} \oplus \mathfrak{N}_\Theta^{\mathbf{X},+}$ is the *lattice Borel subalgebra*. Further, we define the *restricted Borel subalgebra* $\mathfrak{B}_\Theta^{\mathbf{X},r}$ as the subalgebra of $\mathfrak{B}_\Theta^{\mathbf{X}}$ generated by the elements $T_{ij}(u)$, $(i, j) \in \Lambda_\Theta^+$, and $\mathcal{Z}(u)$.

Similarly, there is a decomposition for $\mathbf{Y}(\mathfrak{osp}_{M|N})$ in terms of the root lattice $\mathbb{Z}\Phi$:

$$\mathbf{Y}(\mathfrak{osp}_{M|N}) = \bigoplus_{\alpha\in\mathbb{Z}\Phi} \mathbf{Y}(\mathfrak{osp}_{M|N})_\alpha = \mathfrak{N}_\Theta^- \oplus \mathfrak{B}_\Theta = \mathfrak{N}_\Theta^- \oplus \mathfrak{H}_\Theta \oplus \mathfrak{N}_\Theta^+,$$

where $\mathbf{Y}(\mathfrak{osp}_{M|N})_\alpha = \{Y \in \mathbf{Y}(\mathfrak{osp}_{M|N}) \mid [H, Y] = \alpha(H)Y \text{ for all } H \in \mathfrak{h}\}$ is the root space for $\alpha \in \mathbb{Z}\Phi$, the positive/negative root lattices $\bigoplus_{\alpha\in\mathbb{Z}^+\Phi_\Theta^\pm} \mathbf{Y}(\mathfrak{osp}_{M|N})_\alpha$ are denoted $\mathfrak{N}_\Theta^\pm$, the *lattice Cartan subalgebra* is $\mathfrak{H}_\Theta = \mathbf{Y}(\mathfrak{osp}_{M|N})_0$, and $\mathfrak{B}_\Theta = \mathfrak{H}_\Theta \oplus \mathfrak{N}_\Theta^+$ is the *lattice Borel subalgebra*. Further, we define the *restricted Borel subalgebra* $\mathfrak{B}_\Theta^r$ as the subalgebra of $\mathfrak{B}_\Theta$ generated by the elements $\mathcal{T}_{ij}(u)$ for $(i,j) \in \Lambda_\Theta^+$.

2.4.2 Highest Weight Representations

Via the embedding (2.25), any representation of $\mathbf{X}(\mathfrak{osp}_{M|N})$ can be pulled back to be a representation for the Lie superalgebra $\mathfrak{osp}_{M|N}$. Therefore, we have the familiar notions of weights and weight vectors for representations V of $\mathbf{X}(\mathfrak{osp}_{M|N})$: for $\mu \in \mathfrak{h}^*$, if $V_\mu := \{v \in V \mid H \cdot v = \mu(H)v \text{ for all } H \in \mathfrak{h}\} \neq 0$ then μ is called a *weight* of V, V_μ is called a *weight space*, and non-zero vectors in V_μ are called *weight vectors*. Selecting a system of positive roots Φ_{I}^+ or Φ_{II}^+, we can endow a partial ordering '$\preccurlyeq$' on the set of weights of V via the rule $\omega \preccurlyeq \mu \Leftrightarrow \mu - \omega$ is an $\mathbb{N}$-linear combination of positive roots of $\mathfrak{osp}_{M|N}$. Furthermore, since

$$\mathbf{X}(\mathfrak{osp}_{M|N})_\alpha\big(V_\mu\big) \subseteq V_{\mu+\alpha},$$

then

$$\mathbf{X}(\mathfrak{osp}_{M|N})_\alpha\Big(\bigoplus\nolimits_{\mu\in\mathfrak{h}^*} V_\mu\Big) \subseteq \bigoplus\nolimits_{\mu\in\mathfrak{h}^*} V_\mu. \tag{2.20}$$

Definition 2.4.1. Let Θ denote either I or II. A representation V of the extended super Yangian $\mathbf{X}(\mathfrak{osp}_{M|N})$ is defined as an X_Θ*-highest weight representation* if there exists a nonzero vector $\xi \in V$ such that $\mathbf{X}(\mathfrak{osp}_{M|N})\xi = V$, and

$$\begin{aligned} T_{ij}(u)\xi &= 0 \qquad \text{for all} \quad (i,j) \in \Lambda_\Theta^+ \\ \text{and} \quad T_{kk}(u)\xi &= \lambda_k(u)\xi \qquad \text{for all} \quad k \in \mathbb{Z}_{M+N}^+, \end{aligned} \tag{2.21}$$

where $\lambda_k(u)$ is some formal series

$$\lambda_k(u) = 1 + \sum_{n=1}^{\infty} \lambda_k^{(n)} u^{-n} \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]. \tag{2.22}$$

We say that ξ is the X_Θ*-highest weight vector* of V and call the $M+N$-tuple $(\lambda_k(u))_{k=1}^{M+N}$ of formal series as the X_Θ*-highest weight* of V.

To prove the first theorem of this section, we will need the following lemma:

Lemma 2.4.2. *Let* Θ *denote either* I *or* II. *If* $\mathcal{I}_\Theta$ *is the left ideal of* $\mathbf{X}(\mathfrak{osp}_{M|N})$ *generated by the coefficients of* $T_{ij}(u)$*, where* $(i,j) \in \Lambda_\Theta^+$*, then*

(i) *for all* $(i,j) \in \Lambda_\Theta^+$ *and* $k \in \mathbb{Z}_{M+N}^+$*,*

$$T_{ij}(u)T_{kk}(v) \equiv 0 \mod \mathcal{I}_\Theta \tag{2.23}$$

(ii) *for all* $(k,l) \in (\mathbb{Z}_{M+N}^+)^2$*,*

$$\big[T_{kk}(u), T_{ll}(v)\big] \equiv 0 \mod \mathcal{I}_\Theta \tag{2.24}$$

Proof. For brevity, we shall only use '$\equiv$' to denote equivalence of elements in $\mathbf{X}(\mathfrak{osp}_{M|N})$ modulo $\mathcal{I}_\Theta$.

(i) We shall provide a proof for $\Theta = \mathrm{I}$ and leave the case $\Theta = \mathrm{II}$ in the Appendix; accordingly, throughout the proof we shall suppose $(i,j) \in \Lambda_{\mathrm{I}}^+$ and $k \in \mathbb{Z}_{M+N}^+$.

Step 1. To verify equation (2.23), we demarcate the problem into the two situations when $(i,k) \in \Lambda_{\mathrm{I}}^+$ and when $(i,k) \notin \Lambda_{\mathrm{I}}^+$. If $(i,k) \in \Lambda_{\mathrm{I}}^+$ such that $k \neq \bar{\imath}$, $k \neq \bar{\jmath}$, then equation (2.23) is immediate from $T_{ij}(u)T_{kk}(v) \equiv \big[T_{ij}(u), T_{kk}(v)\big]$. Furthermore, if $(i,k) \notin \Lambda_{\mathrm{I}}^+$, then $(k,j) \in \Lambda_{\mathrm{I}}^+$. Indeed,

- For $(i,j) \in \Gamma_{0,0}$, then $(i,k) \notin \Lambda_{\mathrm{I}}^+$ if and only if $k \leq i$; $\lfloor \frac{M}{2} \rfloor + 1 \leq i \leq M$, $M+1 \leq k \leq M+\frac{N}{2}$; or $\lceil \frac{M}{2} \rceil + 1 \leq i \leq M$, $M+\frac{N}{2}+1 \leq k \leq M+N$. For the first inequality, $(k,j) \in \Gamma_{0,0}$; for the second, $(k,j) \in \Gamma_{1,0a}$; and for the third, $(k,j) \in \Gamma_{1,0b}$.

- For $(i,j) \in \Gamma_{0,1}^{\mathrm{I}}$, then $(i,k) \notin \Lambda_{\mathrm{I}}^+$ if and only if $k \leq i$ or when M is odd, $(i,j) \in \Gamma_{0,1b}^{\mathrm{I}}$, $M+1 \leq k \leq M+\frac{N}{2}$. For the first case, $(k,j) \in \Gamma_{0,1}^{\mathrm{I}}$, and for the second, $(k,j) \in \Gamma_{1,1}$.

- For $(i,j) \in \Gamma_{1,0}^{\mathrm{I}}$, then $(i,k) \notin \Lambda_{\mathrm{I}}^+$ if and only if $M+1 \leq k \leq i$; $(i,j) \in \Gamma_{1,0a}^{\mathrm{I}}$, $1 \leq k \leq \lfloor \frac{M}{2} \rfloor$; or $(i,j) \in \Gamma_{1,0b}^{\mathrm{I}}$, $1 \leq k \leq \lceil \frac{M}{2} \rceil$. For the first case, $(k,j) \in \Gamma_{1,1}$, and for the second and third, $(k,j) \in \Gamma_{0,0}$.

• For $(i,j) \in \Gamma_{1,1}$, then $(i,k) \notin \Lambda_{\mathrm{I}}^{+}$ if and only if $M+1 \le k \le i$; $M+1 \le i \le M+\frac{N}{2}$, $1 \le k \le \lfloor \frac{M}{2} \rfloor$; or $M+\frac{N}{2}+1 \le i \le M+N$, $1 \le k \le \lceil \frac{M}{2} \rceil$. For the first inequality, $(k,j) \in \Gamma_{1,1}$, and for the second and third, $(k,j) \in \Gamma_{0,1}^{\mathrm{I}}$.

Hence, if we further assume $k \ne \bar{\imath}$, $k \ne \bar{\jmath}$, then equation (2.23) follows from $T_{ij}(u)T_{kk}(v) \equiv -\big[T_{kk}(v), T_{ij}(u)\big] \equiv 0$. Therefore, the remaining steps of the proof will be to show that equation (2.23) is true in the exceptional cases when $k = \bar{\imath}$ or $k = \bar{\jmath}$.

Step 2. Let us suppose $(i,k) \in \Lambda_{\mathrm{I}}^{+}$ and $k = \bar{\imath}$. These conditions necessarily imply that either $1 \le i \le \lfloor \frac{M}{2} \rfloor$ or $M+1 \le i \le M+\frac{N}{2}$.

When $1 \le i \le \lfloor \frac{M}{2} \rfloor$, we must have $\lceil \frac{M}{2} \rceil + 1 \le k \le M$ and $(i,j) \in \Gamma_{0,0} \cup \Gamma_{0,1}^{\mathrm{I}}$. Because $j \ne i = \bar{k}$ and $(\bar{k}, k) \in \Gamma_{0,0}$, the defining relations (4.10) imply

$$T_{\bar{k}j}(u)T_{kk}(v) \equiv -\frac{1}{u-v-\kappa} \sum_{p=1}^{\bar{k}} (-1)^{[k][j]+[k]+[j][p]} \theta_{\bar{k}} \theta_p T_{pj}(u) T_{\bar{p}k}(v). \tag{2.25}$$

Since $\bar{k} < j$, each index $1 \le p \le \bar{k} \le \lfloor \frac{M}{2} \rfloor$ satisfies $(p,j) \in \Gamma_{0,0} \cup \Gamma_{0,1}^{\mathrm{I}}$; thus, because we have $(p,k) \in \Gamma_{0,0}$ and $k \ne \bar{\jmath}$, for such indices we can compute

$$T_{pj}(u)T_{\bar{p}k}(v) \equiv -\frac{1}{u-v-\kappa} \sum_{q=1}^{\bar{k}} (-1)^{[p][j]+[p]+[j][q]} \theta_p \theta_q T_{qj}(u) T_{\bar{q}k}(v)$$

and therefore, $(-1)^{[p][j]+[p]} \theta_p T_{pj}(u) T_{\bar{p}k}(v) \equiv (-1)^{[k][j]+[k]} \theta_{\bar{k}} T_{\bar{k}j}(u) T_{kk}(v)$. Equation (2.25) hence implies

$$\left(1 + \frac{\bar{k}}{u-v-\kappa}\right) T_{\bar{k}j}(u) T_{kk}(v) \equiv 0,$$

and so $T_{\bar{k}j}(u)T_{kk}(v) \equiv 0$.

When $M+1 \le i \le M+\frac{N}{2}$, we have $M+\frac{N}{2}+1 \le k \le M+N$ and $(i,j) \in \Gamma_{1,0a}^{\mathrm{I}} \cup \Gamma_{1,1}$.

Since $j \neq i = \bar{k}$ and $(\bar{k}, k) \in \Gamma_{1,1}$, the defining relations give

$$T_{\bar{k}j}(u)T_{kk}(v) \equiv -\frac{1}{u-v-\kappa} \sum_{\substack{1 \leq p \leq \lfloor \frac{M}{2} \rfloor \\ M+1 \leq p \leq \bar{k}}} (-1)^{[k][j]+[k]+[j][p]} \theta_{\bar{k}} \theta_p T_{pj}(u) T_{\bar{p}k}(v). \tag{2.26}$$

Note that if $1 \leq p \leq \lfloor \frac{M}{2} \rfloor$, then $(p, j) \in \Gamma_{0,0} \cup \Gamma_{0,1}^{\mathrm{I}}$; whilst for $M+1 \leq p \leq \bar{k}$, we have $(p, j) \in \Gamma_{1,0a}^{\mathrm{I}}$ if $(i, j) \in \Gamma_{1,0a}^{\mathrm{I}}$ and $(p, j) \in \Gamma_{1,1}$ if $(i, j) \in \Gamma_{1,1}$ since $p \leq \bar{k} = i < j$. For these indices, we can then compute

$$T_{pj}(u)T_{\bar{p}k}(v) \equiv -\frac{1}{u-v-\kappa} \sum_{\substack{1 \leq q \leq \lfloor \frac{M}{2} \rfloor \\ M+1 \leq q \leq \bar{k}}} (-1)^{[p][j]+[p]+[j][q]} \theta_p \theta_q T_{qj}(u) T_{\bar{q}k}(v).$$

Therefore, by using a similar argument as before but with equation (2.26), one obtains

$$\left(1 + \frac{M + \lfloor \frac{M}{2} \rfloor - \bar{k}}{u-v-\kappa}\right) T_{\bar{k}j}(u)T_{kk}(v) \equiv 0,$$

so $T_{\bar{k}j}(u)T_{kk}(v) \equiv 0$.

Step 3. Now suppose $(i, k) \in \Lambda_{\mathrm{I}}^{+}$ and $k = \bar{\jmath}$. Necessarily, these conditions impose that either $(i, j) \in \Gamma_{0,1a}^{\mathrm{I}}$ or $(i, j) \in \Gamma_{0,1b}^{\mathrm{I}}$, $i \leq \lfloor \frac{M}{2} \rfloor$.

When $(i, j) \in \Gamma_{0,1a}^{\mathrm{I}}$, we have $M+1 \leq k \leq M+\frac{N}{2}$. Since $(k, \bar{k}) \in \Gamma_{1,1}$ and $k \neq \bar{\imath}$, the defining relations give

$$\begin{aligned} T_{i\bar{k}}(u)T_{kk}(v) &\equiv -\left[T_{kk}(v), T_{i\bar{k}}(u)\right] \\ &\equiv -\frac{1}{v-u-\kappa} \sum_{\substack{1 \leq p \leq \lfloor \frac{M}{2} \rfloor \\ M+1 \leq p \leq k}} (-1)^{[k]+[k][p]+[p]} \theta_k \theta_p T_{i\bar{p}}(u) T_{kp}(v). \end{aligned} \tag{2.27}$$

Now, for indices $1 \leq p \leq \lfloor \frac{M}{2} \rfloor$, we have $(i, \bar{p}) \in \Gamma_{0,0}$ and $(k, \bar{p}) \in \Gamma_{1,0a}^{\mathrm{I}}$; whilst the indices

$M+1 \leq p \leq k$ satisfy $(i, \bar{p}) \in \Gamma^{\mathrm{I}}_{0,1b}$ and $(k, \bar{p}) \in \Gamma_{1,1}$. Hence, we can compute

$$\begin{aligned} T_{i\bar{p}}(u)T_{kp}(v) &\equiv -\big[T_{kp}(v), T_{i\bar{p}}(u)\big] \\ &\equiv -\frac{1}{v-u-\kappa} \sum_{\substack{1\leq q\leq \lfloor \frac{M}{2} \rfloor \\ M+1\leq q\leq k}} (-1)^{[k][i]+[p][i]+[p]+[k][q]+[q]} \theta_p\theta_q T_{i\bar{q}}(u)T_{kq}(v), \end{aligned} \tag{2.28}$$

and therefore, $(-1)^{[k][i]+[p][i]+[p]}\theta_p T_{i\bar{p}}(u)T_{kp}(v) \equiv (-1)^{[k]}\theta_k T_{i\bar{k}}(u)T_{kk}(v)$. Equation (2.27) therefore implies

$$\left(1 + \frac{M+\lfloor \frac{M}{2} \rfloor - k}{v-u-\kappa}\right) T_{i\bar{k}}(u)T_{kk}(v) \equiv 0, \tag{2.29}$$

and so $T_{i\bar{k}}(u)T_{kk}(v) \equiv 0$.

Otherwise, when $(i, j) \in \Gamma^{\mathrm{I}}_{0,1b}$ and $i \leq \lfloor \frac{M}{2} \rfloor$, we have $M+\frac{N}{2}+1 \leq k \leq M+N$. Since $(i, k) \in \Gamma^{\mathrm{I}}_{0,1b}$ and $k \neq \bar{\imath}$, the defining relations imply

$$T_{i\bar{k}}(u)T_{kk}(v) \equiv \frac{1}{u-v-\kappa} \sum_{p=1}^{i} (-1)^{[i][k]+[i][p]+[p]} \theta_{\bar{k}}\theta_p T_{k\bar{p}}(v)T_{ip}(u). \tag{2.30}$$

Now, for each index $1 \leq p \leq i \leq \lfloor \frac{M}{2} \rfloor$ we have $(k, \bar{p}) \in \Gamma^{\mathrm{I}}_{1,0b}$ and $(i, \bar{p}) \in \Gamma_{0,0}$. Since $k \neq \bar{\imath}$, we have

$$\begin{aligned} T_{k\bar{p}}(v)T_{ip}(u) &\equiv -\big[T_{ip}(u), T_{k\bar{p}}(v)\big] \\ &\equiv -\frac{1}{u-v-\kappa} \sum_{q=1}^{i} (-1)^{[i][k]+[p][k]+[p]+[i][q]+[q]} \theta_p\theta_q T_{k\bar{q}}(v)T_{iq}(u). \end{aligned} \tag{2.31}$$

Hence, $-\theta_{\bar{k}} T_{i\bar{k}}(u)T_{kk}(v) \equiv (-1)^{[p][k]+[p]}\theta_p T_{k\bar{p}}(v)T_{ip}(u)$ and so equation (2.30) implies

$$\left(1 + \frac{i}{u-v-\kappa}\right) T_{i\bar{k}}(u)T_{kk}(v) \equiv 0, \tag{2.32}$$

and therefore $T_{i\bar{k}}(u)T_{kk}(v) \equiv 0$.

Step 4. Here, we assume $(i, k) \notin \Lambda^+_{\mathrm{I}}$ and $k = \bar{\imath}$. These conditions require that either $\lceil \frac{M}{2} \rceil \leq i \leq M$ or $M+\frac{N}{2}+1 \leq i \leq M+N$.

When $\lceil \frac{M}{2} \rceil \leq i \leq M$, we must have either $(i,j) \in \Gamma_{0,0}$ or $(i,j) \in \Gamma^{\mathrm{I}}_{0,1b}$ when $i = \lceil \frac{M}{2} \rceil$ and M is odd. In the first case, we therefore have $1 \leq k \leq \lceil \frac{M}{2} \rceil$, $(k,j) \in \Gamma_{0,0}$, and $k \neq \bar{\jmath}$ since $k = \bar{\imath}$. Hence, the defining relations (4.10) imply

$$T_{\bar{k}j}(u)T_{kk}(v) \equiv -\big[T_{kk}(v), T_{\bar{k}j}(u)\big] \equiv \frac{1}{v-u-\kappa}\sum_{p=1}^{\bar{\jmath}}(-1)^{[k][p]}\theta_k\theta_p T_{pk}(v)T_{\bar{p}j}(u). \quad (2.33)$$

For each index $1 \leq p \leq \bar{\jmath}$, since $\lceil \frac{M}{2} \rceil \leq i < j \leq M$ and $k = \bar{\imath}$, then both (p,k) and (p,j) lie in $\Gamma_{0,0}$. Hence,

$$T_{pk}(v)T_{\bar{p}j}(u) \equiv -\frac{1}{v-u-\kappa}\sum_{q=1}^{\bar{\jmath}}(-1)^{[p][k]+[p]+[k][q]}\theta_p\theta_q T_{qk}(v)T_{\bar{q}j}(u), \quad (2.34)$$

and therefore $-\theta_k T_{\bar{k}j}(u)T_{kk}(v) \equiv (-1)^{[p][k]+[p]}\theta_p T_{pk}(v)T_{\bar{p}j}(u)$, implying

$$\left(1 + \frac{\bar{\jmath}}{v-u-\kappa}\right) T_{\bar{k}j}(u)T_{kk}(v) \equiv 0, \quad (2.35)$$

from equation (2.33) and therefore $T_{\bar{k}j}(u)T_{kk}(v) \equiv 0$.

In the second case, $i = \bar{k} = k = \lceil \frac{M}{2} \rceil$ and $(k,j) \in \Gamma^{\mathrm{I}}_{1,0b}$, so we similarly obtain the equivalence

$$T_{\bar{k}j}(u)T_{kk}(v) \equiv -\big[T_{kk}(v), T_{\bar{k}j}(u)\big] \equiv \frac{1}{v-u-\kappa}\sum_{\substack{1 \leq p \leq \lfloor \frac{M}{2} \rfloor \\ M+1 \leq p \leq \bar{\jmath}}}(-1)^{[k][p]}\theta_k\theta_p T_{pk}(v)T_{\bar{p}j}(u). \quad (2.36)$$

Each index $1 \leq p \leq \lfloor \frac{M}{2} \rfloor$, satisfies $(p,k) \in \Gamma_{0,0}$ (since M is odd) and $(p,j) \in \Gamma^{\mathrm{I}}_{0,1b}$; whilst each index $M+1 \leq p \leq \bar{\jmath}$ satisfies $(p,k) \in \Gamma^{\mathrm{I}}_{1,0a}$ and $(p,j) \in \Gamma_{1,1}$. For such indices,

$$T_{pk}(v)T_{\bar{p}j}(u) \equiv -\frac{1}{v-u-\kappa}\sum_{\substack{1 \leq q \leq \lfloor \frac{M}{2} \rfloor \\ M+1 \leq q \leq \bar{\jmath}}}(-1)^{[p][k]+[p]+[k][q]}\theta_p\theta_q T_{qk}(v)T_{\bar{q}j}(u), \quad (2.37)$$

and a similar argument to before shows

$$\left(1+\frac{M+\left\lfloor\frac{M}{2}\right\rfloor-\bar{\jmath}}{v-u-\kappa}\right)T_{\bar{k}j}(u)T_{kk}(v)\equiv 0. \tag{2.38}$$

When $M+\frac{N}{2}+1\le i\le M+N$, we have $(i,j)\in\Gamma^{\mathrm{I}}_{1,0b}\cup\Gamma_{1,1}$ and $M+1\le k\le M+\frac{N}{2}$. If we first assume $(i,j)\in\Gamma^{\mathrm{I}}_{1,0b}$, then $(k,j)\in\Gamma^{\mathrm{I}}_{1,0a}$. Consequently, we yield the equivalence (2.33). Since for each index $1\le p\le\bar{\jmath}$, we find that $(p,k)\in\Gamma^{\mathrm{I}}_{0,1a}$ and $(p,j)\in\Gamma_{0,0}$, we also get equation (2.34) and can therefore deduce (2.35).

If instead $(i,j)\in\Gamma_{1,1}$, then $(k,j)\in\Gamma_{1,1}$, so we can deduce (2.36). Each index $1\le p\le\left\lfloor\frac{M}{2}\right\rfloor$ satisfies $(p,k)\in\Gamma^{\mathrm{I}}_{0,1a}$ and $(p,j)\in\Gamma^{\mathrm{I}}_{0,1b}$; whilst for each index $M+1\le p\le\bar{\jmath}$, both (p,k) and (p,j) lie in $\Gamma_{1,1}$ since $M+\frac{N}{2}+1\le\bar{k}=i<j\le M+N$. Therefore, we get the equation (2.37) and hence (2.38).

Step 5. Finally, we consider the case when $(i,k)\notin\Lambda^{+}_{\mathrm{I}}$ and $k=\bar{\jmath}$. Necessarily, these conditions imply that either $\left\lceil\frac{M}{2}\right\rceil\le i\le M$, $(i,j)\in\Gamma^{\mathrm{I}}_{1,0}$ where $\left\lceil\frac{M}{2}\right\rceil+1\le j$, or $M+\frac{N}{2}+1\le i<j\le M+N$.

When $\left\lceil\frac{M}{2}\right\rceil\le i\le M$, it must be that either $(i,j)\in\Gamma_{0,0}$ or $(i,j)\in\Gamma^{\mathrm{I}}_{0,1b}$ when $i=\left\lceil\frac{M}{2}\right\rceil$ and M is odd. In the first case, we therefore have $1\le k\le\left\lfloor\frac{M}{2}\right\rfloor$ and $(k,\bar{k})\in\Gamma_{0,0}$, so the defining relations (4.10) yield

$$\begin{aligned}T_{i\bar{k}}(u)T_{kk}(v)&\equiv-\left[T_{kk}(v),T_{i\bar{k}}(u)\right]\\&\equiv-\frac{1}{v-u-\kappa}\sum_{p=1}^{k}(-1)^{[k]+[k][p]+[p]}\theta_k\theta_pT_{i\bar{p}}(u)T_{kp}(v).\end{aligned} \tag{2.39}$$

Since for indices $1\le p\le k$, we have both $(i,\bar{p})$ and $(k,\bar{p})$ in $\Gamma_{0,0}$, we can compute

$$\begin{aligned}T_{i\bar{p}}(u)T_{kp}(v)&\equiv-\left[T_{kp}(v),T_{i\bar{p}}(u)\right]\\&\equiv-\frac{1}{v-u-\kappa}\sum_{q=1}^{k}(-1)^{[k][i]+[p][i]+[p]+[k][q]+[q]}\theta_p\theta_qT_{i\bar{q}}(u)T_{kq}(v),\end{aligned} \tag{2.40}$$

and therefore, $(-1)^{[k][i]+[p][i]+[p]}\theta_p T_{i\bar{p}}(u)T_{kp}(v) \equiv (-1)^{[k]}\theta_k T_{i\bar{k}}(u)T_{kk}(v)$. Equation (2.39) therefore implies

$$\left(1+\frac{k}{v-u-\kappa}\right)T_{i\bar{k}}(u)T_{kk}(v) \equiv 0, \tag{2.41}$$

and so $T_{i\bar{k}}(u)T_{kk}(v) \equiv 0$.

In the second case we have $M+1 \leq k \leq M+\frac{N}{2}$, so $(k,\bar{k}) \in \Gamma_{1,1}$ and $k \neq \bar{\imath}$, which yields the equivalence (2.27). For indices $1 \leq p \leq \lfloor \frac{M}{2} \rfloor$, we have $(i,\bar{p}) \in \Gamma_{0,0}$ and $(k,\bar{p}) \in \Gamma^{\mathrm{I}}_{1,0a}$; whilst the indices $M+1 \leq p \leq k$ satisfy $(i,\bar{p}) \in \Gamma^{\mathrm{I}}_{0,1b}$ and $(k,\bar{p}) \in \Gamma_{1,1}$. Thus, we obtain equation (2.28) and ultimately deduce (2.29).

When $(i,j) \in \Gamma^{\mathrm{I}}_{1,0}$ where $\lceil \frac{M}{2} \rceil + 1 \leq j$, we have $1 \leq k \leq \lfloor \frac{M}{2} \rfloor$. Since $(k,\bar{k}) \in \Gamma_{0,0}$, the get the equivalence (2.39). For indices $1 \leq p \leq k$, we have both $(i,\bar{p}) \in \Gamma^{\mathrm{I}}_{1,0}$ and $(k,\bar{p}) \in \Gamma_{0,0}$, so we can deduce (2.40).

Lastly, when $M+\frac{N}{2}+1 \leq i < j \leq M+N$, we have $M+1 \leq k \leq M+\frac{N}{2}$. Since $(k,\bar{k}) \in \Gamma_{1,1}$ and $i \neq j = \bar{k}$, we also get the equivalence (2.27). For indices $1 \leq p \leq \lfloor \frac{M}{2} \rfloor$, we have $(i,\bar{p}) \in \Gamma^{\mathrm{I}}_{1,0b}$ and $(k,\bar{p}) \in \Gamma^{\mathrm{I}}_{1,0a}$; whilst for the indices $M+1 \leq p \leq k$, both $(i,\bar{p})$ and $(k,\bar{p})$ lie in $\Gamma_{1,1}$. Thus, we obtain equation (2.28) and which implies

$$\left(1-\frac{\lfloor \frac{M}{2} \rfloor + k - M}{v-u-\kappa}\right)T_{i\bar{k}}(u)T_{kk}(v) \equiv 0. \tag{2.42}$$

(ii) Again, we shall only provide a proof for $\Theta = \mathrm{I}$ and leave the case $\Theta = \mathrm{II}$ in the Appendix; accordingly, throughout the proof we shall suppose $(i,j) \in \Lambda^+_{\mathrm{I}}$ and $k \in \mathbb{Z}^+_{M+N}$.

Step 1. Recall that we have the decomposition $(\mathbb{Z}^+_{M+N})^2 = \Lambda^+_{\mathrm{I}} \cup \Lambda^\circ \cup \Lambda^-_{\mathrm{I}}$. Assuming $k \neq \lceil \frac{M}{2} \rceil$ (i.e., $k \neq \bar{k}$) when M is odd, we observe

$$\left[T_{kk}(u), T_{kk}(v)\right] = \frac{1}{u-v}(-1)^{[k]}\Big(T_{kk}(u)T_{kk}(v) - T_{kk}(v)T_{kk}(u)\Big)$$

and hence, $\big[T_{kk}(u), T_{kk}(v)\big] = 0$. Furthermore, if $(k,l) \in \Lambda_{\mathrm{I}}^{+}$ such that $k \neq \bar{l}$, we have

$$\big[T_{kk}(u), T_{ll}(v)\big] = \frac{1}{u-v}(-1)^{[k]}\Big(T_{lk}(u)T_{kl}(v) - T_{lk}(v)T_{kl}(u)\Big) \equiv 0. \tag{2.43}$$

Since $(k,l) \in \Lambda_{\mathrm{I}}^{+}$ if and only if $(l,k) \in \Lambda_{\mathrm{I}}^{-}$, and $\big[T_{kk}(u), T_{ll}(v)\big] = -\big[T_{ll}(v), T_{kk}(u)\big]$, all that remains to verify (2.24) is to examine when $k = \bar{l}$. To this end, it suffices to show $\big[T_{ll}(u), T_{\bar{l}\bar{l}}(v)\big] \equiv 0$ for $1 \le l \le \lceil \frac{M}{2} \rceil$ and $M+1 \le l \le M+\frac{N}{2}$. Moreover, for the remaining steps we shall define

$$\mathbf{A}_{kl} := T_{kl}(u)T_{\bar{k}\bar{l}}(v) - (-1)^{[k]+[l]}T_{\bar{l}\bar{k}}(v)T_{lk}(u). \tag{2.44}$$

for any $(k,l) \in (\mathbb{Z}_{M+N}^{+})^2$.

Step 2. First suppose $1 \le l \le \lfloor \frac{M}{2} \rfloor$. Since $\mathbf{A}_{ll} = \big[T_{ll}(u), T_{\bar{l}\bar{l}}(v)\big]$, the defining relations (4.10) gives

$$\mathbf{A}_{ll} \equiv -\frac{1}{u-v-\kappa}\sum_{p=1}^{l}\Big(T_{pl}(u)T_{\bar{p}\bar{l}}(v) - T_{\bar{l}\bar{p}}(v)T_{lp}(u)\Big) \equiv -\frac{1}{u-v-\kappa}\sum_{k=1}^{l}\mathbf{A}_{kl}. \tag{2.45}$$

For $1 \le k < l \le \lfloor \frac{M}{2} \rfloor$, we have

$$\big[T_{kl}(u), T_{\bar{k}\bar{l}}(v)\big] \equiv -\frac{1}{u-v-\kappa}\sum_{p=1}^{l}T_{pl}(u)T_{\bar{p}\bar{l}}(v) + \frac{1}{u-v-\kappa}\sum_{p=1}^{k}T_{\bar{k}\bar{p}}(v)T_{kp}(u)$$

and

$$\big[T_{lk}(u), T_{\bar{l}\bar{k}}(v)\big] \equiv -\frac{1}{u-v-\kappa}\sum_{p=1}^{k}T_{pk}(u)T_{\bar{p}\bar{k}}(v) + \frac{1}{u-v-\kappa}\sum_{p=1}^{l}T_{\bar{l}\bar{p}}(v)T_{lp}(u).$$

Since for such indices, $\mathbf{A}_{kl} \equiv \big[T_{kl}(u), T_{\bar{k}\bar{l}}(v)\big] + \big[T_{lk}(u), T_{\bar{l}\bar{k}}(v)\big]$, we have

$$\mathbf{A}_{kl} \equiv -\frac{1}{u-v-\kappa}\sum_{p=1}^{k}\mathbf{A}_{pk} - \frac{1}{u-v-\kappa}\sum_{p=1}^{l}\mathbf{A}_{pl}. \tag{2.46}$$

The equivalences (2.45) and (2.46) therefore imply the relation $\mathbf{A}_{kl} \equiv \mathbf{A}_{kk} + \mathbf{A}_{ll}$ for indices $1 \le k < l \le \lfloor \frac{M}{2} \rfloor$. Using this resulting relation along with (2.45), we derive the formula

$$\left(1 + \frac{l}{u - v - \kappa}\right) \mathbf{A}_{ll} + \frac{1}{u - v - \kappa} \sum_{k=1}^{l-1} \mathbf{A}_{kk} \equiv 0. \tag{2.47}$$

Hence, an inductive argument will show $\mathbf{A}_{ll} \equiv 0$ for all $1 \le l \le \lfloor \frac{M}{2} \rfloor$.

Step 3. Let us now suppose $M+1 \le l \le M+\frac{N}{2}$. Since $\mathbf{A}_{ll} = \left[T_{ll}(u), T_{\bar{l}\bar{l}}(v)\right]$, the defining relations (4.10) imply

$$\begin{aligned} \mathbf{A}_{ll} &\equiv -\frac{1}{u - v - \kappa} \sum_{\substack{1 \le p \le \lfloor \frac{M}{2} \rfloor \\ M+1 \le p \le l}} (-1)^{[p][l]} \Big(T_{pl}(u) T_{\bar{p}\bar{l}}(v) - (-1)^{[p]+[l]} T_{\bar{l}\bar{p}}(v) T_{lp}(u) \Big) \\ &\equiv -\frac{1}{u - v - \kappa} \sum_{\substack{1 \le k \le \lfloor \frac{M}{2} \rfloor \\ M+1 \le k \le l}} (-1)^{[k][l]} \mathbf{A}_{kl}. \end{aligned} \tag{2.48}$$

Supposing $M+1 \le k < l \le M+\frac{N}{2}$, we have

$$\begin{aligned} \left[T_{kl}(u), T_{\bar{k}\bar{l}}(v)\right] \equiv -\frac{1}{u - v - \kappa} &\sum_{\substack{1 \le p \le \lfloor \frac{M}{2} \rfloor \\ M+1 \le p \le l}} (-1)^{[p][l]} T_{pl}(u) T_{\bar{p}\bar{l}}(v) \\ &+ \frac{1}{u - v - \kappa} \sum_{\substack{1 \le p \le \lfloor \frac{M}{2} \rfloor \\ M+1 \le p \le k}} (-1)^{[p][k]+[p]+[k]} T_{\bar{k}\bar{p}}(v) T_{kp}(u) \end{aligned}$$

and

$$\begin{aligned} \left[T_{lk}(u), T_{\bar{l}\bar{k}}(v)\right] \equiv -\frac{1}{u - v - \kappa} &\sum_{\substack{1 \le p \le \lfloor \frac{M}{2} \rfloor \\ M+1 \le p \le k}} (-1)^{[p][k]} T_{pk}(u) T_{\bar{p}\bar{k}}(v) \\ &+ \frac{1}{u - v - \kappa} \sum_{\substack{1 \le p \le \lfloor \frac{M}{2} \rfloor \\ M+1 \le p \le l}} (-1)^{[p][l]+[p]+[l]} T_{\bar{l}\bar{p}}(v) T_{lp}(u). \end{aligned}$$

For such indices, $\mathbf{A}_{kl} \equiv \left[T_{kl}(u), T_{\bar{k}\bar{l}}(v)\right] + \left[T_{lk}(u), T_{\bar{l}\bar{k}}(v)\right]$, so

$$\mathbf{A}_{kl} \equiv -\frac{1}{u - v - \kappa} \sum_{\substack{1 \le p \le \lfloor \frac{M}{2} \rfloor \\ M+1 \le p \le k}} (-1)^{[p][k]} \mathbf{A}_{pk} - \frac{1}{u - v - \kappa} \sum_{\substack{1 \le p \le \lfloor \frac{M}{2} \rfloor \\ M+1 \le p \le l}} (-1)^{[p][l]} \mathbf{A}_{pl}. \tag{2.49}$$

The equivalences (2.48) and (2.49) imply $\mathbf{A}_{kl} \equiv \mathbf{A}_{kk} + \mathbf{A}_{ll}$ for $M+1 \le k < l \le M+\frac{N}{2}$.

Furthermore, if we suppose $1 \leq k \leq \lfloor \frac{M}{2} \rfloor$ and $M+1 \leq l \leq M+\frac{N}{2}$, then

$$\left[T_{kl}(u), T_{\bar{k}\bar{l}}(v)\right] \equiv -\frac{1}{u-v-\kappa} \sum_{\substack{1 \leq p \leq \lfloor \frac{M}{2} \rfloor \\ M+1 \leq p \leq l}} (-1)^{[p][l]} T_{pl}(u) T_{\bar{p}\bar{l}}(v) - \frac{1}{u-v-\kappa} \sum_{1 \leq p \leq k} T_{\bar{k}\bar{p}}(v) T_{kp}(u)$$

and

$$\left[T_{lk}(u), T_{\bar{l}\bar{k}}(v)\right] \equiv \frac{1}{u-v-\kappa} \sum_{1 \leq p \leq k} T_{pk}(u) T_{\bar{p}\bar{k}}(v) + \frac{1}{u-v-\kappa} \sum_{\substack{1 \leq p \leq \lfloor \frac{M}{2} \rfloor \\ M+1 \leq p \leq l}} (-1)^{[p][l]+[p]+[l]} T_{\bar{l}\bar{p}}(v) T_{lp}(u).$$

For such indices, $\mathbf{A}_{kl} \equiv \left[T_{kl}(u), T_{\bar{k}\bar{l}}(v)\right] + \left[T_{lk}(u), T_{\bar{l}\bar{k}}(v)\right]$, so

$$\mathbf{A}_{kl} \equiv \frac{1}{u-v-\kappa} \sum_{1 \leq p \leq k} \mathbf{A}_{pk} - \frac{1}{u-v-\kappa} \sum_{\substack{1 \leq p \leq \lfloor \frac{M}{2} \rfloor \\ M+1 \leq p \leq l}} (-1)^{[p][l]} \mathbf{A}_{pl}. \tag{2.50}$$

The equivalences (2.45), (2.48), and (2.50) therefore imply $\mathbf{A}_{kl} \equiv \mathbf{A}_{ll} - \mathbf{A}_{kk}$ for indices $1 \leq k \leq \lfloor \frac{M}{2} \rfloor$ and $M+1 \leq l \leq M+\frac{N}{2}$.

By combing these new relations with (2.48), we can deduce the formula

$$\left(1 + \frac{\lfloor \frac{M}{2} \rfloor + M - l}{u-v-\kappa}\right) \mathbf{A}_{ll} - \frac{1}{u-v-\kappa} \sum_{k=M+1}^{l-1} \mathbf{A}_{kk} \equiv 0 \tag{2.51}$$

since $\mathbf{A}_{kk} \equiv 0$ for $1 \leq k \leq \lfloor \frac{M}{2} \rfloor$ by Step 2. Hence, an inductive argument will prove $\mathbf{A}_{ll} \equiv 0$ for $M+1 \leq l \leq M+\frac{N}{2}$.

Step 4. In the special case when $l = \lceil \frac{M}{2} \rceil$ and M is odd, $\mathbf{A}_{ll} = \big[T_{ll}(u), T_{ll}(v)\big]$, so

$$\begin{aligned}\mathbf{A}_{ll} &\equiv \frac{1}{u-v}\Big(T_{ll}(u)T_{ll}(v) - T_{ll}(v)T_{ll}(u)\Big) \\ &\quad - \frac{1}{u-v-\kappa} \sum_{\substack{1\le p\le l \\ M+1\le p\le M+\frac{N}{2}}} (-1)^{[p][l]}\Big(T_{pl}(u)T_{\bar{p}l}(v) - (-1)^{[p]+[l]}T_{l\bar{p}}(v)T_{lp}(u)\Big) \\ &\equiv \frac{1}{u-v}\mathbf{A}_{ll} - \frac{1}{u-v-\kappa} \sum_{\substack{1\le k\le l \\ M+1\le k\le M+\frac{N}{2}}} (-1)^{[k][l]}\mathbf{A}_{kl}\end{aligned} \tag{2.52}$$

For indices $1 \le k < l = \lceil \frac{M}{2} \rceil$, we have

$$\begin{aligned}\big[T_{kl}(u), T_{\bar{k}l}(v)\big] \equiv -\frac{1}{u-v-\kappa} &\sum_{\substack{1\le p\le l \\ M+1\le p\le M+\frac{N}{2}}} (-1)^{[p][l]}T_{pl}(u)T_{\bar{p}l}(v) \\ &\qquad + \frac{1}{u-v-\kappa}\sum_{1\le p\le k} T_{\bar{k}\bar{p}}(v)T_{kp}(u)\end{aligned}$$

and

$$\begin{aligned}\big[T_{lk}(u), T_{l\bar{k}}(v)\big] \equiv -\frac{1}{u-v-\kappa}&\sum_{1\le p\le k} T_{pk}(u)T_{\bar{p}\bar{k}}(v) \\ &+ \frac{1}{u-v-\kappa} \sum_{\substack{1\le p\le l \\ M+1\le p\le M+\frac{N}{2}}} (-1)^{[p][l]+[p]+[l]}T_{l\bar{p}}(v)T_{lp}(u).\end{aligned}$$

Since $\mathbf{A}_{kl} \equiv \big[T_{kl}(u), T_{\bar{k}l}(v)\big] + \big[T_{lk}(u), T_{l\bar{k}}(v)\big]$ for such indices, we have

$$\mathbf{A}_{kl} \equiv -\frac{1}{u-v-\kappa}\sum_{1\le p\le k}\mathbf{A}_{pk} - \frac{1}{u-v-\kappa}\sum_{\substack{1\le p\le l \\ M+1\le p\le M+\frac{N}{2}}} (-1)^{[p][l]}\mathbf{A}_{pl}. \tag{2.53}$$

Hence, the equivalences (2.45), (2.52), and (2.53) imply $\mathbf{A}_{kl} \equiv \mathbf{A}_{kk} + \frac{u-v-1}{u-v}\mathbf{A}_{ll}$ for $1 \le k < l = \lceil \frac{M}{2} \rceil$. Furthermore, for indices $M+1 \le k \le M+\frac{N}{2}$, we have

$$\begin{aligned}\big[T_{kl}(u), T_{\bar{k}l}(v)\big] \equiv \frac{1}{u-v-\kappa} &\sum_{\substack{1\le p\le l \\ M+1\le p\le M+\frac{N}{2}}} (-1)^{[p][l]}T_{pl}(u)T_{\bar{p}l}(v) \\ &+ \frac{1}{u-v-\kappa} \sum_{\substack{1\le p\le \lfloor\frac{M}{2}\rfloor \\ M+1\le p\le k}} (-1)^{[p][k]+[p]+[k]}T_{\bar{k}\bar{p}}(v)T_{kp}(u)\end{aligned}$$

and

$$\left[T_{lk}(u), T_{l\bar{k}}(v)\right] \equiv -\frac{1}{u-v-\kappa} \sum_{\substack{1 \le p \le \lfloor \frac{M}{2} \rfloor \\ M+1 \le p \le k}} (-1)^{[p][k]} T_{pk}(u) T_{\bar{p}\bar{k}}(v) - \frac{1}{u-v-\kappa} \sum_{\substack{1 \le p \le l \\ M+1 \le p \le M+\frac{N}{2}}} (-1)^{[p][l]+[p]+[l]} T_{l\bar{p}}(v) T_{lp}(u).$$

Since $\mathbf{A}_{kl} \equiv \left[T_{kl}(u), T_{\bar{k}l}(v)\right] + \left[T_{lk}(u), T_{l\bar{k}}(v)\right]$ for such indices, we have

$$\mathbf{A}_{kl} \equiv -\frac{1}{u-v-\kappa} \sum_{\substack{1 \le p \le \lfloor \frac{M}{2} \rfloor \\ M+1 \le p \le k}} \mathbf{A}_{pk} + \frac{1}{u-v-\kappa} \sum_{\substack{1 \le p \le l \\ M+1 \le p \le M+\frac{N}{2}}} (-1)^{[p][l]} \mathbf{A}_{pl}. \tag{2.54}$$

Thus, the equivalences (2.48), (2.52), and (2.54) imply $\mathbf{A}_{kl} \equiv \mathbf{A}_{kk} + \frac{1-(u-v)}{u-v} \mathbf{A}_{ll}$ for $M+1 \le k \le M+\frac{N}{2}$. Combining these new relations with (2.52) will yield

$$\left(1 - \frac{1}{u-v} + \frac{1}{u-v-\kappa} + \frac{\left(l-1-\frac{N}{2}\right)(u-v-1)}{(u-v)(u-v-\kappa)}\right) \mathbf{A}_{ll} \equiv 0 \tag{2.55}$$

since $\mathbf{A}_{kk} \equiv 0$ for $1 \le k \le \left\lfloor \frac{M}{2} \right\rfloor$ and $M+1 \le k \le M+\frac{N}{2}$ by Steps 2 and 3. □

Theorem 2.4.3. *Let Θ denote either* I *or* II. *Every finite-dimensional irreducible representation V of the extended super Yangian $\mathbf{X}(\mathfrak{osp}_{M|N})$ is an X_Θ-highest weight representation. Moreover, V contains a unique, up to scalar multiples, X_Θ-highest weight vector.*

Proof. Letting V denote a finite-dimensional irreducible representation of $\mathbf{X}(\mathfrak{osp}_{M|N})$, we define

$$V^0 := \left\{ v \in V \mid T_{ij}(u)v = 0 \text{ for all } (i,j) \in \Lambda_\Theta^+ \right\} \tag{2.56}$$

and we shall show that $V^0 \neq 0$. To this end, we endow the partial ordering '$\preccurlyeq$' on the set of weights of any $\mathfrak{osp}_{M|N}$-module via the rule that for any weights $\alpha, \beta \in \mathfrak{h}^*$, we write $\alpha \preccurlyeq \beta$ if $\beta - \alpha$ is an $\mathbb{N}$-linear combination of positive roots in Φ_Θ^+.

Via the embedding (2.25), we may view V as an $\mathfrak{osp}_{M|N}$-module and therefore consider its set of weights. This set of weights is non-empty since the collection $\{F_{aa} \mid \lceil \frac{M}{2} \rceil + 1 \leq a \leq M,\ M + \frac{N}{2} + 1 \leq a \leq M + N\}$ is a family of pairwise commuting elements, and therefore is a family of pairwise commuting operators on V, and $\dim V < \infty$, so these operators share a simultaneous eigenvector: Furthermore, the finite-dimensionality of V infers that such set of weights is finite; thus, V must have a maximal weight μ with respect to the partial ordering '$\preccurlyeq$'.

Letting v be a weight vector corresponding to μ, the assertion follows if $v \in V^0$, so we may assume $v \notin V^0$ and therefore $T_{ij}^{(n)} v \neq 0$ for some $(i,j) \in \Lambda_\Theta^+$. However, since

$$F_{aa} T_{ij}^{(n)} v = T_{ij}^{(n)} F_{aa} v + \left[F_{aa}, T_{ij}^{(n)}\right] v,$$

we conclude from equation (**??**) that the weight of $T_{ij}^{(n)} v$ is of the form $\mu + \alpha$ for some positive root $\alpha \in \Phi_\Theta^+$, contradicting the maximality of μ.

Now, by Lemma (2.4.2) the set $\{T_{kk}^{(n)} \mid 1 \leq k \leq M+N\ n \in \mathbb{Z}^+\}$ is a family of pairwise commuting operators on V^0. Therefore, since V^0 is a non-trivial subspace of V and $\dim V < \infty$, there exists a simultaneous eigenvector $0 \neq \xi \in V^0$ for such operators: $T_{kk}^{(n)} \xi = \lambda_k^{(n)} \xi$ for complex eigenvalues $\lambda_k^{(n)}$, $1 \leq k \leq M+N$, $n \in \mathbb{Z}^+$. Via the irreducibility of V, we conclude $\mathbf{X}(\mathfrak{osp}_{M|N})\xi = V$, and by collecting these eigenvalues into power series $\lambda_k(u) = 1 + \sum_{n=1}^{\infty} \lambda_k^{(n)} u^{-n}$ we observe the vector ξ satisfies the conditions (2.21), so V is a highest weight representation with highest weight vector ξ and highest weight $(\lambda_k(u))_{k=1}^{M+N}$.

It remains to show that ξ is unique up to scalar multiplication. Recalling the PBW Theorem (2.2.5) for $\mathbf{X}(\mathfrak{osp}_{M|N})$, we fix a total order '$\preceq$' on the set X in such a way that for any $T_{i_1 j_1}^{(n_1)}, T_{i_2 j_2}^{(n_2)}, T_{i_3 j_3}^{(n_3)} \in \mathsf{X}$ satisfying $(i_1, j_1) \in \mathcal{B}_{M|N} \cap \Lambda_\Theta^-$, $(i_2, j_2) \in \mathcal{B}_{M|N} \cap \Lambda^\circ$, and $(i_3, j_3) \in \mathcal{B}_{M|N} \cap \Lambda_\Theta^+$, then $T_{i_1 j_1}^{(n_1)} \preceq T_{i_2 j_2}^{(n_2)} \preceq T_{i_3 j_3}^{(n_3)}$. Now, since V is irreducible and finite-dimensional, Schur's lemma infers that each generator $\mathscr{Z}_r$ of the center $\mathbf{ZX}(\mathfrak{osp}_{M|N})$ acts by a scalar. Therefore, by the total ordering on X, we conclude that V is spanned by ordered elements of the form

$$T_{i_1 j_1}^{(n_1)} \cdots T_{i_k j_k}^{(n_k)} \xi, \tag{2.57}$$

where $k \in \mathbb{N}$, $(i_p, j_p) \in \mathcal{B}_{M|N} \cap \Lambda_\Theta^-$, and $n_p \in \mathbb{Z}^+$ for $1 \leq p \leq k$. Furthermore, since $F_{aa} = (-1)^{[a]} T_{aa}^{(1)} - \frac{1}{2}(-1)^{[a]} \mathcal{Z}_1$, then ξ is also a weight vector of some weight ν. By (**??**), the elements (2.57) will therefore be weight vectors with corresponding weights of the form $\nu + \sum_{p=1}^{k} \alpha_p$, where $\alpha_p \in \Phi_\Theta^-$.

Hence, there is a weight space decomposition $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$ where each weight $\mu \neq \nu$ is of the form $\nu - \sum_{p=1}^{k} \alpha_p$ for $\alpha_p \in \Phi_\Theta^+$. Consequently, V_ν is 1-dimensional; namely, $V_\nu = \operatorname{span}_{\mathbb{C}}\{\xi\}$.

Now, if $\tilde{\xi}$ is another highest weight vector of V of highest weight $(\lambda_k(u))_{k=1}^{M+N}$, then a replicative argument as above will show that $\tilde{\xi} \in V_\nu$. Hence, $\tilde{\xi} = c\xi$ for some $c \in \mathbb{C}^*$. □

As we saw in the proof of Theorem (2.4.3), we used a consequence of Schur's lemma that central elements in $\mathbf{X}(\mathfrak{osp}_{M|N})$ act on finite-dimensional irreducible representations by scalars. Instead, we could have used the more explicit (and now more general) result described in the following proposition.

Proposition 2.4.4. *Let Θ denote either* I *or* II *and let V be an X_Θ-highest weight representation of $\mathbf{X}(\mathfrak{osp}_{M|N})$ with highest weight $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$. For $\Theta = \mathrm{I}$, $\mathcal{Z}(u)$ acts on V by $\lambda_1(u+\kappa)\lambda_M(u)$ and for $\Theta = \mathrm{II}$, $\mathcal{Z}(u)$ acts on V by $\lambda_{M+1}(u+\kappa)\lambda_{M+N}(u)$.*

Proof. Let ξ_Θ be a highest weight vector of V so that $V = \mathbf{X}(\mathfrak{osp}_{M|N})\xi_\Theta$. If $\Theta = \mathrm{I}$, setting $i = j = M$ in equation (2.51) gives $\mathcal{Z}(u) = \sum_{k=1}^{M+N} (-1)^{[k]} T_{\bar{k}1}(u+\kappa) T_{kM}(u)$, so

$$\mathcal{Z}(u)\xi_{\mathrm{I}} = T_{11}(u+\kappa) T_{MM}(u)\xi_{\mathrm{I}} = \lambda_1(u+\kappa)\lambda_M(u)\xi_{\mathrm{I}}.$$

Otherwise when $\Theta = \mathrm{II}$, we may designate $i = j = M+N$ in equation (2.51) to provide

$\mathcal{Z}(u) = -\sum_{k=1}^{M+N} T_{\bar{k},M+1}(u+\kappa)T_{k,M+N}(u)$, so

$$\mathcal{Z}(u)\xi_{\mathrm{II}} = T_{M+1,M+1}(u+\kappa)T_{M+N,M+N}(u)\xi_{\mathrm{II}} = \lambda_{M+1}(u+\kappa)\lambda_{M+N}(u)\xi_{\mathrm{II}}.$$

□

Furthermore, we can deduce some relations of the components of highest weights:

Proposition 2.4.5. *Let* Θ *denote either* I *or* II *and let* V *be an* X_Θ*-highest weight representation of* $\mathbf{X}(\mathfrak{osp}_{M|N})$ *with highest weight* $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ *and highest weight vector* ξ_Θ*. If* $\Theta = \mathrm{I}$ *and* $M \geq 4$*, then*

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{\lambda_{M-1}(u-\kappa+1)}{\lambda_M(u-\kappa+1)}, \tag{2.58}$$

or if $\Theta = \mathrm{II}$ *and* $N \geq 4$*, then*

$$\frac{\lambda_{M+1}(u)}{\lambda_{M+2}(u)} = \frac{\lambda_{M+N-1}(u-\kappa-1)}{\lambda_{M+N}(u-\kappa-1)}. \tag{2.59}$$

Proof. By first assuming $\Theta = \mathrm{I}$ and $M \geq 4$, the defining relations (4.10) infer

$$\begin{aligned} T_{12}(u)T_{M,M-1}(v)\xi_{\mathrm{I}} &= \big[T_{12}(u), T_{M,M-1}(v)\big]\xi_{\mathrm{I}} \\ &= -\frac{1}{u-v-\kappa}\left(\sum_{p=1}^{M+N}\theta_p T_{p2}(u)T_{\bar{p},M-1}(v) - \sum_{p=1}^{M+N}(-1)^{[p]}\theta_p T_{M\bar{p}}(v)T_{1p}(u)\right)\xi_{\mathrm{I}} \\ &= -\frac{1}{u-v-\kappa}\Big(T_{12}(u)T_{M,M-1}(v) + \lambda_2(u)\lambda_{M-1}(v) - \lambda_1(u)\lambda_M(v)\Big)\xi_{\mathrm{I}}, \end{aligned}$$

so

$$(u-v-\kappa+1)T_{12}(u)T_{M,M-1}(v)\xi_{\mathrm{I}} = \big(\lambda_2(u)\lambda_{M-1}(v) - \lambda_1(u)\lambda_M(v)\big)\xi_{\mathrm{I}}.$$

Setting $v = u-\kappa+1$ then yields the desired relation. Similarly, if instead $\Theta = \mathrm{II}$ and $N \geq 4$, the defining relations (4.10) show

$$T_{M+1,M+2}(u)T_{M+N,M+N-1}(v)\xi_{\mathrm{II}} = \big[T_{M+1,M+2}(u), T_{M+N,M+N-1}(v)\big]\xi_{\mathrm{II}}$$

$$= -\frac{1}{u-v-\kappa}\Bigg(\sum_{p=1}^{M+N}(-1)^{[p]}\theta_p T_{p,M+2}(u)T_{\bar{p},M+N-1}(v) + \sum_{p=1}^{M+N}\theta_p T_{M+N,\bar{p}}(v)T_{M+1,p}(u)\Bigg)\xi_{\mathrm{II}}$$
$$= \frac{1}{u-v-\kappa}\Big(T_{M+1,M+2}(u)T_{M+N,M+N-1}(v)+\lambda_{M+2}(u)\lambda_{M+N-1}(v)-\lambda_{M+1}(u)\lambda_{M+N}(v)\Big)\xi_{\mathrm{II}},$$

so setting $v = u-\kappa-1$ in the equation

$$(u-v-\kappa-1)T_{M+1,M+2}(u)T_{M+N,M+N-1}(v)\xi_{\mathrm{II}} = \big(\lambda_{M+2}(u)\lambda_{M+N-1}(v)-\lambda_{M+1}(u)\lambda_{M+N}(v)\big)\xi_{\mathrm{II}}$$

will yield the desired relation. □

Definition 2.4.6. Let Θ denote either I or II. Given an $M+N$-tuple $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ of the form (2.22) we define the X_Θ-*Verma module* $M_\Theta(\lambda(u))$ to be the quotient $\mathbf{X}(\mathfrak{osp}_{M|N})/\mathcal{I}_\Theta(\lambda(u))$, where $\mathcal{I}_\Theta(\lambda(u))$ is the left graded ideal of $\mathbf{X}(\mathfrak{osp}_{M|N})$ generated by the coefficients of $T_{ij}(u)$, $(i,j)\in\Lambda_\Theta^+$ and $T_{kk}(u)-\lambda_k(u)\mathbf{1}$, $k\in\mathbb{Z}_{M+N}^+$.

One aptly observes that when $M_\Theta(\lambda(u))$ is non-trivial, it is an X_Θ-highest weight representation of $\mathbf{X}(\mathfrak{osp}_{M|N})$ with X_Θ-highest weight $\lambda(u)$ and X_Θ-highest weight vector $\mathbf{1}_{\lambda(u)}$, the image of $\mathbf{1}$ in the canonical projection $\mathbf{X}(\mathfrak{osp}_{M|N})\to M_\Theta(\lambda(u))$. Furthermore, if L is an X_Θ-highest weight representation of $\mathbf{X}(\mathfrak{osp}_{M|N})$ with highest weight $\lambda(u)$ and highest weight vector ξ, then there is a surjective $\mathbf{X}(\mathfrak{osp}_{M|N})$-module morphism $\varphi\colon M_\Theta(\lambda(u))\to L$ induced by the assignment $\mathbf{1}_{\lambda(u)}\mapsto\xi$; thus, $L\cong M_\Theta(\lambda(u))/\ker\varphi$.

By (2.20) , $\bigoplus_{\mu\in\mathfrak{h}^*}M_\Theta(\lambda(u))_\mu$ is invariant under the action of $\mathbf{X}(\mathfrak{osp}_{M|N})$. Therefore, since $\mathbf{1}_{\lambda(u)}$ is contained in $M_\Theta(\lambda(u))_{\lambda^{(1)}}\subset\bigoplus_{\mu\in\mathfrak{h}^*}M_\Theta(\lambda(u))_\mu$, where $\lambda^{(1)}\in\mathfrak{h}^*$ is the linear functional given by $\lambda^{(1)}(F_{kk})=\lambda_k^{(1)}$, we have the weight space decomposition

$$M_\Theta(\lambda(u)) = \bigoplus_{\mu\in\mathfrak{h}^*}M_\Theta(\lambda(u))_\mu \tag{2.60}$$

and each weight μ is of the form $\lambda^{(1)}-\omega$, where ω is a $\mathbb{Z}^+$-linear combination of positive roots in Φ_Θ^+. Indeed, recalling the PBW Theorem (2.2.5) for $\mathbf{X}(\mathfrak{osp}_{M|N})$, we fix a total order '$\preceq$' on the set X in such a way that for any $T_{i_1j_1}^{(n_1)}, T_{i_2j_2}^{(n_2)}, T_{i_3j_3}^{(n_3)}\in\mathsf{X}$

satisfying $(i_1, j_1) \in \mathcal{B}_{M|N} \cap \Lambda_\Theta^-$, $(i_2, j_2) \in \mathcal{B}_{M|N} \cap \Lambda^\circ$, and $(i_3, j_3) \in \mathcal{B}_{M|N} \cap \Lambda_\Theta^+$, then $T_{i_1 j_1}^{(n_1)} \preceq T_{i_2 j_2}^{(n_2)} \preceq T_{i_3 j_3}^{(n_3)}$. Therefore, by the total ordering on X and Proposition (2.4.4), we conclude that $M_\Theta(\lambda(u))$ is spanned by ordered elements of the form

$$T_{i_1 j_1}^{(n_1)} \cdots T_{i_k j_k}^{(n_k)} \mathbf{1}_{\lambda(u)}, \tag{2.61}$$

where $k \in \mathbb{N}$, $(i_p, j_p) \in \mathcal{B}_{M|N} \cap \Lambda_\Theta^-$, and $n_p \in \mathbb{Z}^+$ for $1 \leq p \leq k$. In particular, we conclude that $M_\Theta(\lambda(u))_{\lambda^{(1)}}$ is 1-dimensional; i.e., $M_\Theta(\lambda(u))_{\lambda^{(1)}} = \mathrm{span}_{\mathbb{C}}\{\mathbf{1}_{\lambda(u)}\}$.

Any submodule P of $M_\Theta(\lambda(u))$ also has a weight space decomposition $P = \bigoplus_{\mu \in \mathfrak{h}^*} P_\mu$, where $P_\mu = P \cap M_\Theta(\lambda(u))_\mu$. Indeed, if we suppose to the contrary that $P \neq \bigoplus_{\mu \in \mathfrak{h}^*} P_\mu$, then there is some element $x \in P$ such that $x = \sum_{i=1}^n m_i$, where each m_i lies in $M_\Theta(\lambda(u))_{\mu_i} \backslash P_{\mu_i}$, and we consider such an element x wherein the positive integer n is minimal. Since $\mu_{n-1} \neq \mu_n$, there is some $H \in \mathfrak{h}$ such that $\mu_{n-1}(H) \neq \mu_n(H)$. Since $Hx \in P$, then $Hx - \mu_n(H)x = \sum_{i=1}^{n-1} (\mu_i(H) - \mu_n(H)) m_i \in P$, where $(\mu_{n-1}(H) - \mu_n(H)) m_{n-1}$ lies in $M_\Theta(\lambda(u))_{\mu_{n-1}} \setminus P_{\mu_{n-1}}$ and $(\mu_i(H) - \mu_n(H)) m_i \in \big(M_\Theta(\lambda(u))_{\mu_i} \setminus P_{\mu_i}\big) \cup \{0\}$ for $i = 1, 2, \ldots, n-2$, contradicting the minimality of n.

Hence, given any proper submodule P of $M_\Theta(\lambda(u))$, we conclude that if $P_{\lambda^{(1)}} \neq 0$, then $\mathbf{1}_{\lambda(u)} \in P_{\lambda^{(1)}}$ since $\dim M_\Theta(\lambda(u))_{\lambda^{(1)}} = 1$; hence, $P \subseteq \bigoplus_{\lambda^{(1)} \neq \mu \in \mathfrak{h}^*} M_\Theta(\lambda(u))_\mu$ and so the sum of all proper submodules $K = \sum_{P < M_\Theta(\lambda(u))} P$ is the unique maximal submodule of $M_\Theta(\lambda(u))$.

Definition 2.4.7. When the X_Θ-Verma module $M(\lambda(u))$ is non-trivial, we define the *X_Θ-irreducible highest weight representation* $L_\Theta(\lambda(u))$ of $\mathbf{X}(\mathfrak{osp}_{M|N})$ with X_Θ-highest weight $\lambda(u)$ as the quotient of the X_Θ-Verma module $M_\Theta(\lambda(u))$ by its unique maximal proper submodule.

2.4.3 Restriction Functors of $\mathbf{X}(\mathfrak{osp}_{M|N})$

First, let us consider the following subsets of $(\mathbb{Z}^+_{M+N})^2$:

$$\begin{gathered}
\mathsf{M}_{0,0} = \big\{(i,j) \mid 2 \le i,j \le M-1\big\}, \quad \mathsf{M}_{1,1} = \big\{(i,j) \mid M+1 \le i,j \le M+N\big\}, \\
\mathsf{M}_{0,1} = \big\{(i,j) \mid 2 \le i \le M-1,\ M+1 \le j \le M+N\big\} \\
\mathsf{M}_{1,0} = \big\{(i,j) \mid M+1 \le i \le M+N,\ 2 \le j \le M-1\big\}, \\
\mathsf{N}_{0,0} = \big\{(i,j) \mid 1 \le i,j \le M\big\}, \quad \mathsf{N}_{1,1} = \big\{(i,j) \mid M+2 \le i,j \le M+N-1\big\}, \\
\mathsf{N}_{0,1} = \big\{(i,j) \mid 1 \le i \le M,\ M+2 \le j \le M+N-1\big\} \\
\mathsf{N}_{1,0} = \big\{(i,j) \mid M+2 \le i \le M+N-1,\ 1 \le j \le M\big\},
\end{gathered}$$

so we can define

$$\begin{aligned}
\mathsf{M} &:= \mathsf{M}_{0,0} \cup \mathsf{M}_{0,1} \cup \mathsf{M}_{1,0} \cup \mathsf{M}_{1,1} \\
\text{and} \quad \mathsf{N} &:= \mathsf{N}_{0,0} \cup \mathsf{N}_{0,1} \cup \mathsf{N}_{1,0} \cup \mathsf{N}_{1,1}
\end{aligned} \tag{2.62}$$

We may now consider the Lie sub-superalgebras $\mathfrak{m} \cong \mathfrak{osp}_{(M-2)|N}$ and $\mathfrak{n} \cong \mathfrak{osp}_{M|(N-2)}$ of $\mathfrak{osp}_{M|N}$ of ranks $\lfloor \frac{M}{2} \rfloor + \frac{N}{2} - 1$ given by

$$\mathfrak{m} = \operatorname{span}_{\mathbb{C}}\big\{F_{ij}\big\}_{(i,j)\in\mathsf{M}} \quad \text{and} \quad \mathfrak{n} = \operatorname{span}_{\mathbb{C}}\big\{F_{ij}\big\}_{(i,j)\in\mathsf{N}}$$

An observation to note is that the natural embeddings

$$\begin{gathered}
\mathbf{X}(\mathfrak{osp}_{(M-2)|N}) \to \operatorname{span}_{\mathbb{C}}\{T^{(n)}_{ij} \mid (i,j) \in \mathsf{M},\ n \in \mathbb{N}\} \subset \mathbf{X}(\mathfrak{osp}_{M|N}) \\
\text{and} \quad \mathbf{X}(\mathfrak{osp}_{M|(N-2)}) \to \operatorname{span}_{\mathbb{C}}\{T^{(n)}_{ij} \mid (i,j) \in \mathsf{N},\ n \in \mathbb{N}\} \subset \mathbf{X}(\mathfrak{osp}_{M|N})
\end{gathered}$$

are not subalgebras of $\mathbf{X}(\mathfrak{osp}_{M|N})$. Instead, we will see that one can construct superalgebra morphisms from $\mathbf{X}(\mathfrak{osp}_{(M-2)|N})$ or $\mathbf{X}(\mathfrak{osp}_{M|(N-2)})$ to suitable quotients $\mathbf{X}(\mathfrak{osp}_{M|N})/\mathcal{I}^+$ and $\mathbf{X}(\mathfrak{osp}_{M|N})/\mathcal{I}_+$, respectively, for certain left graded ideals $\mathcal{I}^+$ and $\mathcal{I}_+$.

Proposition 2.4.8. (i) *Let $\mathcal{I}^+$ be the left graded ideal of $\mathbf{X}(\mathfrak{osp}_{M|N})$ generated by the coefficients of $T_{1k}(u)$ and $T_{lM}(u)$ for indices $2 \le k \le M$, $1 \le l \le M-1$, and*

$M+1 \le k, l \le M+N$. There is a superalgebra morphism from $\mathbf{X}(\mathfrak{osp}_{(M-2)|N})$ to $\mathbf{X}(\mathfrak{osp}_{M|N})/\mathcal{I}^+$ given by the assignment

$$\mathsf{T}_{ij}(u) \mapsto \begin{cases} T_{i+1,j+1}(u) \mod \mathcal{I}^+ & \text{if} \quad 1 \le i, j \le M-2, \\ T_{i+2,j+2}(u) \mod \mathcal{I}^+ & \text{if} \quad M-1 \le i, j \le M+N-2, \\ T_{i+1,j+2}(u) \mod \mathcal{I}^+ & \text{if} \quad 1 \le i \le M-2,\ M-1 \le j \le M+N-2, \\ T_{i+2,j+1}(u) \mod \mathcal{I}^+ & \text{if} \quad M-1 \le i \le M+N-2,\ 1 \le j \le M-2, \end{cases}$$

where $\mathsf{T}_{ij}(u)$ denotes a generating series for $\mathbf{X}(\mathfrak{osp}_{(M-2)|N})$.

(ii) *Let $\mathcal{I}_+$ be the left graded ideal of $\mathbf{X}(\mathfrak{osp}_{M|N})$ generated by the coefficients of $T_{M+1,k}(u)$ and $T_{l,M+N}(u)$ for indices $1 \le k, l \le M$, $M+2 \le k \le M+N$, and $M+1 \le l \le M+N-1$. There is a superalgebra morphism from $\mathbf{X}(\mathfrak{osp}_{M|(N-2)})$ to $\mathbf{X}(\mathfrak{osp}_{M|N})/\mathcal{I}_+$ given by the assignment*

$$\mathsf{T}_{ij}(u) \mapsto \begin{cases} T_{ij}(u) \mod \mathcal{I}_+ & \text{if} \quad 1 \le i, j \le M, \\ T_{i+1,j+1}(u) \mod \mathcal{I}_+ & \text{if} \quad M+1 \le i, j \le M+N-2, \\ T_{i,j+1}(u) \mod \mathcal{I}_+ & \text{if} \quad 1 \le i \le M,\ M+1 \le j \le M+N-2, \\ T_{i+1,j}(u) \mod \mathcal{I}_+ & \text{if} \quad M+1 \le i \le M+N-2,\ 1 \le j \le M, \end{cases}$$

where $\mathsf{T}_{ij}(u)$ denotes a generating series for $\mathbf{X}(\mathfrak{osp}_{M|(N-2)})$.

Proof. We shall provide a proof for part (i) and leave the proof for part (ii) in the Appendix. Accordingly, we shall suppose $(i,j),(k,l) \in \mathsf{M}$ for the duration of the proof and shall use '$\equiv$' to denote equivalence of elements in $\mathbf{X}(\mathfrak{osp}_{M|N})$ modulo $\mathcal{I}^+$ for brevity. By the defining relations (4.10), we have

$$\begin{aligned}\big[T_{ij}(u), T_{kl}(v)\big] \equiv{}& \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\Big(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)\Big) \\ &- \frac{1}{u-v-\kappa}\Bigg(\delta_{\bar{\imath}k} \sum_{\substack{2 \le p \le M-1, \\ M+1 \le p \le M+N}} (-1)^{[i][j]+[i]+[j][p]}\theta_i\theta_p T_{pj}(u)T_{\bar{p}l}(v)\end{aligned}$$

$$
- \delta_{\bar{j}l} \sum_{\substack{2 \le p \le M-1, \\ M+1 \le p \le M+N}} (-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]} \theta_j \theta_p T_{k\bar{p}}(v) T_{ip}(u) \Big)
$$

$$
- \frac{1}{u-v-\kappa} \Big(\delta_{\bar{i}k} (-1)^{[i][j]+[i]} \theta_i T_{1j}(u) T_{Ml}(v)
- \delta_{\bar{j}l} (-1)^{[i][k]+[j][k]+[j]} \theta_j T_{kM}(v) T_{i1}(u) \Big).
$$

Using the relations (4.10) again, since $T_{1j}(u)T_{Ml}(v) \equiv \big[T_{1j}(u), T_{Ml}(v)\big]$, we have

$$
T_{1j}(u)T_{Ml}(v) \equiv - \frac{1}{u-v-\kappa} \sum_{\substack{2 \le p \le M-1, \\ M+1 \le p \le M+N}} (-1)^{[j][p]} \theta_p T_{pj}(u) T_{\bar{p}l}(v)
- \frac{1}{u-v-\kappa} \Big(T_{1j}(u)T_{Ml}(v) - \delta_{\bar{j}l} (-1)^{[j]} \theta_j T_{MM}(v) T_{11}(u) \Big),
$$

and therefore

$$
T_{1j}(u)T_{Ml}(v) \equiv - \frac{1}{u-v-\kappa+1} \sum_{\substack{2 \le p \le M-1, \\ M+1 \le p \le M+N}} (-1)^{[j][p]} \theta_p T_{pj}(u) T_{\bar{p}l}(v)
+ \frac{1}{u-v-\kappa+1} \delta_{\bar{j}l} (-1)^{[j]} \theta_j T_{MM}(v) T_{11}(u).
$$

Analogously, we compute

$$
\begin{aligned}
T_{kM}(v)T_{i1}(u) &\equiv -(-1)^{([i]+[1])([k]+[M])} \big[T_{i1}(u), T_{kM}(v)\big] \\
&\equiv \frac{1}{u-v-\kappa} \Big(\delta_{\bar{i}k} \theta_i T_{11}(u) T_{MM}(v) - T_{kM}(v) T_{i1}(u) \Big) \\
&\quad - \frac{1}{u-v-\kappa} \sum_{\substack{2 \le p \le M-1, \\ M+1 \le p \le M+N}} (-1)^{[i][p]+[p]} \theta_p T_{k\bar{p}}(v) T_{ip}(u)
\end{aligned}
$$

and hence

$$
T_{kM}(v)T_{i1}(u) \equiv \frac{1}{u-v-\kappa+1} \delta_{\bar{i}k} \theta_i T_{11}(u) T_{MM}(v)
- \frac{1}{u-v-\kappa+1} \sum_{\substack{2 \le p \le M-1, \\ M+1 \le p \le M+N}} (-1)^{[i][p]+[p]} \theta_p T_{k\bar{p}}(v) T_{ip}(u).
$$

Combining everything, we obtain

$$
\begin{aligned}
\big[T_{ij}(u), T_{kl}(v)\big] \equiv \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\Big(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)\Big) \\
- \frac{1}{u-v-\kappa+1}\Bigg(\delta_{\bar{\imath}k} \sum_{\substack{2\le p\le M-1,\\ M+1\le p\le M+N}} (-1)^{[i][j]+[i]+[j][p]}\theta_i\theta_p T_{pj}(u)T_{\bar{p}l}(v) \\
- \delta_{\bar{\jmath}l} \sum_{\substack{2\le p\le M-1,\\ M+1\le p\le M+N}} (-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]}\theta_j\theta_p T_{k\bar{p}}(v)T_{ip}(u)\Bigg) \\
+ \frac{1}{(u-v-\kappa)(u-v-\kappa+1)}\delta_{\bar{\imath}k}\delta_{\bar{\jmath}l}(-1)^{[i][j]+[i]+[j]}\theta_i\theta_j\big[T_{11}(u), T_{MM}(v)\big].
\end{aligned}
$$

Lastly, the relations (4.10) imply

$$
\big[T_{11}(u), T_{MM}(v)\big] \equiv -\frac{1}{u-v-\kappa}\big[T_{11}(u), T_{MM}(v)\big],
$$

meaning $\big[T_{11}(u), T_{MM}(v)\big] \equiv 0$ and therefore the desired relations are satisfied for the operators $T_{ij}(u)$, $(i,j) \in \mathsf{M}$, on V^+, since $\kappa_{M-2,N} = \kappa_{M,N} - 1$ is the parameter associated to the Lie superalgebra $\mathfrak{m} \cong \mathfrak{osp}_{(M-2)|N}$. □

Given a representation V of $\mathbf{X}(\mathfrak{osp}_{M|N})$, we consider the subspaces

$$
\begin{aligned}
V_1 &= \big\{\eta \in V \mid T_{1k}(u)\eta = 0 \ \text{ for } \ 2 \le k \le M \ \text{ and } \ M+1 \le k \le M+N\big\}, \\
V_M &= \big\{\eta \in V \mid T_{kM}(u)\eta = 0 \ \text{ for } \ 1 \le k \le M-1 \ \text{ and } \ M+1 \le k \le M+N\big\}, \\
V_{M+1} &= \big\{\eta \in V \mid T_{M+1,k}(u)\eta = 0 \ \text{ for } \ 1 \le k \le M \ \text{ and } \ M+2 \le k \le M+N\big\}, \\
V_{M+N} &= \big\{\eta \in V \mid T_{k,M+N}(u)\eta = 0 \ \text{ for } \ 1 \le k \le M \ \text{ and } \ M+1 \le k \le M+N-1\big\},
\end{aligned}
$$

to define

$$
V^+ := V_1 \cap V_M \quad \text{and} \quad V_+ := V_{M+1} \cap V_{M+N} \tag{2.63}
$$

Note that these intersections may be trivial, but if V is an X_{I}-highest weight representation, then V^+ contains an X_{I}-highest weight vector and if V is an X_{II}-highest weight representation, then V_+ contains an X_{II}-highest weight vector. In particular, if V is finite-dimensional and irreducible, then Theorem (2.4.3) ensures that V^+ and V_+ will be non-trivial.

Lemma 2.4.9. *Let $\mathcal{I}^+$ and $\mathcal{I}_+$ be the left graded ideals of $\mathbf{X}(\mathfrak{osp}_{M|N})$ as in Proposition (2.4.8).*

(i) $T_{1k}(v)T_{ij}(u) \equiv T_{lM}(v)T_{ij}(u) \equiv 0 \mod \mathcal{I}^+$ *for indices* $(i,j) \in \mathsf{M}$, $2 \leq k \leq M$, $1 \leq l \leq M-1$, *and* $M+1 \leq k, l \leq M+N$.

(ii) $T_{M+1,k}(v)T_{ij}(u) \equiv T_{l,M+N}(v)T_{ij}(u) \equiv 0 \mod \mathcal{I}_+$ *for* $(i,j) \in \mathsf{N}$, $1 \leq k, l \leq M$, $M+2 \leq k \leq M+N$, *and* $M+1 \leq l \leq M+N-1$.

Proof. We shall provide the proof for (i) and leave the proof for (ii) in the Appendix. Accordingly, we shall suppose $(i,j), (k,l) \in \mathsf{M}$ for the duration of the proof and shall use '$\equiv$' to denote equivalence of elements in $\mathbf{X}(\mathfrak{osp}_{M|N})$ modulo $\mathcal{I}^+$ for brevity.

First supposing $2 \leq k \leq M$ and $(i,j) \in \mathsf{M}_{0,1} \cup \mathsf{M}_{1,1}$, or $M+1 \leq k \leq M+N$ and $(i,j) \in \mathsf{M}_{0,0} \cup \mathsf{M}_{1,0}$, relations (4.10) imply $T_{1k}(u)T_{ij}(v) \equiv \big[T_{1k}(u), T_{ij}(v)\big] \equiv 0$. Alternatively, when $2 \leq k \leq M$ and $(i,j) \in \mathsf{M}_{0,0} \cup \mathsf{M}_{1,0}$, or $M+1 \leq k \leq M+N$ and $(i,j) \in \mathsf{M}_{0,1} \cup \mathsf{M}_{1,1}$, the defining relations (4.10) yield

$$T_{1k}(u)T_{ij}(v) \equiv \big[T_{1k}(u), T_{ij}(v)\big] \equiv \frac{\delta_{\bar{k}j}}{u-v-\kappa}(-1)^{[i][k]+[k]}\theta_k T_{iM}(v)T_{11}(u).$$

By the same defining relations,

$$T_{iM}(v)T_{11}(u) \equiv -\big[T_{11}(u), T_{iM}(v)\big] \equiv -\frac{1}{u-v-\kappa}T_{iM}(v)T_{11}(u),$$

which show $T_{iM}(v)T_{11}(u) \equiv 0$ and hence $T_{1k}(u)T_{ij}(v)\eta \equiv 0$

Now, when $1 \leq l \leq M-1$ and $(i,j) \in \mathsf{M}_{1,0} \cup \mathsf{M}_{1,1}$, or $M+1 \leq l \leq M+N$ and $(i,j) \in \mathsf{M}_{0,0} \cup \mathsf{M}_{0,1}$, relations (4.10) provide $T_{lM}(v)T_{ij}(u) \equiv -\big[T_{ij}(u), T_{lM}(v)\big] \equiv 0$. Otherwise, if $1 \leq l \leq M-1$ and $(i,j) \in \mathsf{M}_{0,0} \cup \mathsf{M}_{0,1}$, or $M+1 \leq l \leq M+N$ and $(i,j) \in \mathsf{M}_{1,0} \cup \mathsf{M}_{1,1}$, then by (4.10) we have

$$T_{lM}(v)T_{ij}(u) \equiv -\big[T_{ij}(u), T_{lM}(v)\big] \equiv \frac{\delta_{\bar{\imath}k}}{u-v-\kappa}(-1)^{[i][j]}\theta_i T_{1j}(u)T_{MM}(v)\eta.$$

The defining relations (4.10) again yield

$$T_{1j}(u)T_{MM}(v) \equiv \left[T_{1j}(u), T_{MM}(v)\right] \equiv -\frac{1}{u-v-\kappa}T_{1j}(u)T_{MM}(v),$$

implying $T_{1j}(u)T_{MM}(v) \equiv 0$ and therefore $T_{lM}(v)T_{ij}(u)\eta \equiv 0$, proving the lemma. $\square$

For a superalgebra $\mathcal{A}$, we shall let $\operatorname{Rep}\mathcal{A}$ denote its category of representations.

Proposition 2.4.10. *There are functors $\mathcal{F}^+\colon \operatorname{Rep}\mathbf{X}(\mathfrak{osp}_{M|N}) \to \operatorname{Rep}\mathbf{X}(\mathfrak{osp}_{(M-2)|N})$ and $\mathcal{F}_+\colon \operatorname{Rep}\mathbf{X}(\mathfrak{osp}_{M|N})) \to \operatorname{Rep}\mathbf{X}(\mathfrak{osp}_{M|(N-2)})$ given by $\mathcal{F}^+(V) = V^+$ and $\mathcal{F}_+(V) = V_+$ for $\mathbf{X}(\mathfrak{osp}_{M|N})$-modules V and $\mathcal{F}^+(\phi) = \phi|_{V^+}$ and $\mathcal{F}_+(\phi) = \phi|_{V_+}$ for $\mathbf{X}(\mathfrak{osp}_{M|N})$-module morphisms ϕ.*

Proof. Let V denote a representation of $\mathbf{X}(\mathfrak{osp}_{M|N})$. By the definition of the left graded ideal $\mathcal{I}^+$, we know $\mathcal{I}^+ \cdot V^+ = 0$; hence, there is a well-defined action of the quotient $\mathbf{X}(\mathfrak{osp}_{M|N})/\mathcal{I}^+$ on V^+. It is not clear that V^+ is closed under the action of $\mathbf{X}(\mathfrak{osp}_{M|N})/\mathcal{I}^+$, however Lemma (2.4.9) ensures that V^+ is closed under the action of the image of $\mathbf{X}(\mathfrak{osp}_{(M-2)|N})$ in $\mathbf{X}(\mathfrak{osp}_{M|N})/\mathcal{I}^+$ described by the superalgebra morphism detailed in Proposition (2.4.8).

Given an $\mathbf{X}(\mathfrak{osp}_{M|N})$-module morphism $\phi\colon V \to W$, the $\mathbf{X}(\mathfrak{osp}_{M|N})$-linearity of ϕ implies $\phi|_{V^+}(V^+) \subseteq W^+$. A similar discussion to the above also shows that $\phi|_{V^+}$ is $\mathbf{X}(\mathfrak{osp}_{(M-2)|N})$-linear. $\square$

2.4.4 Conditions When $M_\Theta(\lambda(u))$ is Non-Trivial

Lemma 2.4.11. *Let Θ denote either* I *or* II*. Given any $M+N$-tuple $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ of formal power series of the form (2.22), one can derive series $\widetilde{\lambda}_k(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$, where $\widetilde{\lambda}_k(u) = \lambda_k(u)$ for $(k,k) \in \mathcal{B}_{M|N}$, such that the $M+N$-tuple $\widetilde{\lambda}(u) = (\widetilde{\lambda}_k(u))_{k=1}^{M+N}$*

yields a non-trivial Verma module $M_\Theta(\widetilde{\lambda}(u))$*. Furthermore, if* $\Theta = \mathrm{I}$ *then* $\widetilde{\lambda}_1(u) = \lambda_1(u)$*, and if* $\Theta = \mathrm{II}$ *then* $\widetilde{\lambda}_{M+1}(u) = \lambda_{M+1}(u)$*.*

Proof. Let $\mathscr{J}_\Theta(\lambda(u))$ denote the left ideal of $\mathbf{X}(\mathfrak{osp}_{M|N})$ generated by the coefficients of the series $T_{ij}(u)$, $(i,j) \in \Lambda^+_\Theta \cap \mathcal{B}_{M|N}$, $T_{kk}(u) - \lambda_k(u)\mathbf{1}$, $(k,k) \in \mathcal{B}_{M|N}$, and the central series $\mathcal{Z}(u) - \lambda_1(u+\kappa)\lambda_M(u)\mathbf{1}$ if $\Theta = \mathrm{I}$ or $\mathcal{Z}(u) - \lambda_{M+1}(u+\kappa)\lambda_{M+N}(u)\mathbf{1}$ if $\Theta = \mathrm{II}$. We therefore define the quotient

$$\widetilde{M}_\Theta(\lambda(u)) := \mathbf{X}(\mathfrak{osp}_{M|N})/\mathscr{J}_\Theta(\lambda(u)).$$

(Finish argument: Such quotient $\widetilde{M}_\Theta(\lambda(u))$ is non-trivial: Since $\mathfrak{osp}_{M|N}[z] = \mathfrak{n}^-_\Theta[z] \oplus \mathfrak{b}_\Theta[z]$ is a direct sum of sub-Lie superalgebras, we can write $\mathfrak{U}(\mathfrak{osp}_{M|N}[z]) \cong \mathfrak{U}(\mathfrak{n}^-_\Theta[z]) \otimes \mathfrak{U}(\mathfrak{b}_\Theta[z])$. Regarding $\mathfrak{U}(\mathfrak{b}_\Theta[z]) \subset \mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ as a sub-superalgebra, we may consider its image under the isomorphism (2.12). Such image $\Psi\big(\mathfrak{U}(\mathfrak{b}_\Theta[z])\big)$ will be the sub-superalgebra of $\operatorname{gr}\mathbf{Y}(\mathfrak{osp}_{M|N})$ generated by the elements $\overline{\mathcal{T}}^{(n)}_{ij}$, where $(i,j) \in \Lambda^+_\Theta$ and $n \in \mathbb{Z}^+$. In particular, we realize that the associated graded superalgebra $\operatorname{gr}\mathfrak{B}^r_\Theta$ coincides with $\Psi\big(\mathfrak{U}(\mathfrak{b}_\Theta[z])\big)$.)

We shall now show that $\widetilde{M}_\Theta(\lambda(u))$ can be realized as an X_Θ-Verma module $M_\Theta(\widetilde{\lambda}(u))$ for some highest weight $\widetilde{\lambda}(u)$. To do so, let $\tilde{\mathbf{1}}_{\lambda(u)}$ denote the image of $\mathbf{1}$ in $\widetilde{M}_\Theta(\lambda(u))$. By Corollary (2.2.5), we may choose a total ordering '$\preceq$' on the set X such that $T^{(n)}_{ij} \preceq T^{(b)}_{aa} \preceq T^{(m)}_{kl}$ for indices $(i,j) \in \Lambda^-_\Theta \cap \mathcal{B}_{M|N}$, $(a,a) \in \mathcal{B}_{M|N}$, $(k,l) \in \Lambda^+_\Theta \cap \mathcal{B}_{M|N}$ and $m,b,n \in \mathbb{Z}^+$. Via the embedding (2.25), $\mathbf{X}(\mathfrak{osp}_{M|N})$ is an $\mathfrak{osp}_{M|N}$-module whose action described by (2.3) and the Cartan subalgebra $\mathfrak{h}$ acts via (**??**).

Recalling $\mathbf{X}(\mathfrak{osp}_{M|N}) = \bigoplus_{\alpha\in\mathbb{Z}\Phi}\mathbf{X}(\mathfrak{osp}_{M|N})_\alpha$, each generator $T^{(n)}_{ij}$, $(i,j) \in \Lambda^+_\Theta$, will lie in the root space $\mathbf{X}(\mathfrak{osp}_{M|N})_{\alpha_{ij}}$ for some positive root $\alpha_{ij} \in \Phi^+_\Theta$. By writing each such generator $T^{(n)}_{ij}$ as a unique linear combination of ordered monomials in X as described by Corollary (2.2.5) and the total order '$\preceq$', say $T^{(n)}_{ij} = \sum_k \sum_{m_1,\ldots,m_k} \gamma_{m_1,\ldots,m_k} X_{m_1}\ldots X_{m_k}$, then each monomial $X_{m_1}\ldots X_{m_k}$ must also lie in $\mathbf{X}(\mathfrak{osp}_{M|N})_{\alpha_{ij}}$.

Indeed, if we suppose to the contrary that there exists some monomials in the expression of $T_{ij}^{(n)}$ that do not lie in $\mathbf{X}(\mathfrak{osp}_{M|N})_{\alpha_{ij}}$, then we may write $T_{ij}^{(n)} = X + Y$, where $X \in \mathbf{X}(\mathfrak{osp}_{M|N})_{\alpha_{ij}}$ and $Y = \sum_k \sum_{m_1,\ldots,m_k} \mathsf{y}_{m_1,\ldots,m_k} X_{m_1} \ldots X_{m_k}$ is the sum of ordered monomials that do not lie in $\mathbf{X}(\mathfrak{osp}_{M|N})_{\alpha_{ij}}$. However, since $Y = T_{ij}^{(n)} - X$ will lie in $\mathbf{X}(\mathfrak{osp}_{M|N})_{\alpha_{ij}}$, we have

$$\begin{aligned}\alpha_{ij}(H)Y = [H, Y] &= \sum_k \sum_{m_1,\ldots,m_k} \mathsf{y}_{m_1,\ldots,m_k} [H, X_{m_1} \ldots X_{m_k}] \\ &= \sum_k \sum_{m_1,\ldots,m_k} \alpha_{m_1,\ldots,m_k}(H) \mathsf{y}_{m_1,\ldots,m_k} X_{m_1} \ldots X_{m_k}\end{aligned}$$

for the corresponding root lattice elements $\alpha_{m_1,\ldots,m_k} \in \mathbb{Z}\Phi$. Hence, the linear independence of these ordered monomials infer α_{ij} is equal to each $\alpha_{m_1,\ldots,m_k}$, a contradiction.

Since α_{ij} is positive, it is necessary that the the last term in each ordered monomial $X_{m_1} \ldots X_{m_k}$ is equal to $T_{kl}^{(m)}$ for some $(k, l) \in \Lambda_\Theta^+ \cap \mathcal{B}_{M|N}$ and $m \in \mathbb{Z}^+$ by definition of the total order '$\preceq$'. Hence, since each monomial in the expression of $T_{ij}^{(n)}$, $(i, j) \in \Lambda_\Theta^+$, is annihilated by $\tilde{\mathbf{1}}_{\lambda(u)}$, then so is $T_{ij}^{(n)}$.

Similarly, each generator $T_{aa}^{(b)}$, $(a, a) \in \Lambda^\circ$, will lie in the space $\mathbf{X}(\mathfrak{osp}_{M|N})_0$ (where 0 refers to the zero element in $\mathfrak{h}^*$). By writing each such generator $T_{aa}^{(b)}$ as a unique linear combination of ordered monomials in X as described by Corollary (2.2.5) and the total order '$\preceq$', say $T_{aa}^{(n)} = \sum_k \sum_{m_1,\ldots,m_k} \gamma_{m_1,\ldots,m_k} X_{m_1} \ldots X_{m_k}$, then each monomial $X_{m_1} \ldots X_{m_k}$ must also lie in $\mathbf{X}(\mathfrak{osp}_{M|N})_0$.

Indeed, if we suppose to the contrary that there exists some monomials in the expression of $T_{aa}^{(b)}$ that do not lie in $\mathbf{X}(\mathfrak{osp}_{M|N})_0$, then we may write $T_{aa}^{(b)} = X + Y$, where $X \in \mathbf{X}(\mathfrak{osp}_{M|N})_0$ and $Y = \sum_k \sum_{m_1,\ldots,m_k} \mathsf{y}_{m_1,\ldots,m_k} X_{m_1} \ldots X_{m_k}$ is the sum of ordered monomials that do not lie in $\mathbf{X}(\mathfrak{osp}_{M|N})_0$. However, since $Y = T_{aa}^{(b)} - X$ will lie in $\mathbf{X}(\mathfrak{osp}_{M|N})_0$, we have

$$\begin{aligned}0 = [H, Y] &= \sum_k \sum_{m_1,\ldots,m_k} \mathsf{y}_{m_1,\ldots,m_k} [H, X_{m_1} \ldots X_{m_k}] \\ &= \sum_k \sum_{m_1,\ldots,m_k} \alpha_{m_1,\ldots,m_k}(H) \mathsf{y}_{m_1,\ldots,m_k} X_{m_1} \ldots X_{m_k}\end{aligned}$$

for the corresponding root lattice elements $\alpha_{m_1,\dots,m_k} \in \mathbb{Z}\Phi$. Hence, the linear independence of these ordered monomials infer $\alpha_{m_1,\dots,m_k} = 0$, a contradiction.

Hence, by definition of the total order '$\preceq$' it is necessary that each ordered monomial $X_{m_1} \dots X_{m_k}$ in the expression of $T_{aa}^{(b)}$ is of the form

$$\big(T_{i_1j_1}^{(n_1)} \cdots T_{i_rj_r}^{(n_r)}\big)\big(T_{c_1c_1}^{(d_1)} \cdots T_{c_ec_e}^{(d_e)}\big)\big(T_{k_1l_1}^{(m_1)} \cdots T_{k_sl_s}^{(m_s)}\big),$$

where $r, s \in \mathbb{N}$, $(i_q, j_q) \in \Lambda_\Theta^- \cap \mathcal{B}_{M|N}$, $(c_f, c_f) \in \Lambda^\circ \cap \mathcal{B}_{M|N}$, $(k_p, l_p) \in \Lambda_\Theta^+ \cap \mathcal{B}_{M|N}$, the element $T_{k_1l_1}^{(m_1)} \cdots T_{k_sl_s}^{(m_s)}$ lies in $\mathbf{X}(\mathfrak{osp}_{M|N})_{\sum_{p=1}^s \alpha_{k_pl_p}}$ for positive roots $\alpha_{k_1l_1}, \dots, \alpha_{k_sl_s} \in \Phi_\Theta^+$, and $T_{i_1j_1}^{(n_1)} \cdots T_{i_rj_r}^{(n_r)}$ lies in $\mathbf{X}(\mathfrak{osp}_{M|N})_{-\sum_{p=1}^s \alpha_{k_pl_p}}$. Hence, since $\tilde{\mathbf{1}}_{\lambda(u)}$ is an eigenvector for each ordered monomial in the expression of $T_{aa}^{(b)}$, $(a,a) \in \Lambda^\circ$, then $\tilde{\mathbf{1}}_{\lambda(u)}$ is an eigenvector for $T_{aa}^{(b)}$ as well.

Therefore, for each $(k,k) \in \Lambda^\circ$, we can write $T_{kk}(u)\tilde{\mathbf{1}}_{\lambda(u)} = \widetilde{\lambda}_k(u)\tilde{\mathbf{1}}_{\lambda(u)}$ for some formal power series $\widetilde{\lambda}_k(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$, where $\widetilde{\lambda}_k(u) = \lambda_k(u)$ for $(k,k) \in \mathcal{B}_{M|N}$.

Defining the ideal $\mathcal{I}_\Theta\big(\widetilde{\lambda}(u)\big)$ as in Definition (2.4.6), the above argument shows $\mathcal{I}_\Theta\big(\widetilde{\lambda}(u)\big) \subseteq \mathcal{J}_\Theta(\lambda(u))$. To prove the reverse inclusion, all that is left to show is that $\mathcal{Z}(u) - \lambda_1(u+\kappa)\lambda_M(u)\mathbf{1}$ lies in $\mathcal{I}_{\mathrm{I}}\big(\widetilde{\lambda}(u)\big)$ and $\mathcal{Z}(u) - \lambda_{M+1}(u+\kappa)\lambda_{M+N}(u)\mathbf{1}$ lies in $\mathcal{I}_{\mathrm{II}}\big(\widetilde{\lambda}(u)\big)$. First supposing $\Theta = \mathrm{I}$, setting $i = j = M$ in equation (2.51) yields

$$\begin{aligned}
&\mathcal{Z}(u) - \lambda_1(u+\kappa)\lambda_M(u)\mathbf{1} \\
&\quad= T_{11}(u+\kappa)T_{MM}(u) - \lambda_1(u+\kappa)\lambda_M(u)\mathbf{1} + \sum_{k\neq M}(-1)^{[k]}T_{\bar{k}1}(u+\kappa)T_{kM}(u) \\
&\quad= \big(T_{11}(u+\kappa) - \lambda_1(u+\kappa)\mathbf{1}\big)\big(T_{MM}(u) - \lambda_M(u)\mathbf{1}\big) + \lambda_1(u+\kappa)\big(T_{MM}(u) - \lambda_M(u)\mathbf{1}\big) \\
&\qquad\quad + \lambda_M(u)\big(T_{11}(u+\kappa) - \lambda_1(u+\kappa)\mathbf{1}\big) + \sum_{k\neq M}(-1)^{[k]}T_{\bar{k}1}(u+\kappa)T_{kM}(u)
\end{aligned}$$

which lies in $\mathcal{I}_{\mathrm{I}}\big(\widetilde{\lambda}(u)\big)$. The case for $\Theta = \mathrm{II}$ is similar. Hence, $\widetilde{M}_\Theta(\lambda(u)) = M_\Theta\big(\widetilde{\lambda}(u)\big)$.

Lastly, via the definition of $\mathcal{J}_{\mathrm{I}}(\lambda(u))$, we know $\mathcal{Z}(u)\mathbf{1}_{\widetilde{\lambda}(u)} = \lambda_1(u+\kappa)\lambda_M(u)\mathbf{1}_{\widetilde{\lambda}(u)}$. At the same time, we know by Proposition (2.4.4), that $\mathcal{Z}(u)\mathbf{1}_{\widetilde{\lambda}(u)} = \widetilde{\lambda}_1(u+\kappa)\lambda_M(u)\mathbf{1}_{\widetilde{\lambda}(u)}$;

hence, $\widetilde{\lambda}_1(u) = \lambda_1(u)$. A similar argument can be made when $\Theta = \mathrm{II}$ to conclude $\widetilde{\lambda}_{M+1}(u) = \lambda_{M+1}(u)$. □

Proposition 2.4.12. *The Verma module $M_{\mathrm{I}}(\lambda(u))$ is non-trivial if and only if the components of the highest weight $\lambda(u) = (\lambda_i(u))_{i=1}^{M+N}$ satisfy the consistency conditions*

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \frac{\lambda_{M-i}(u-\kappa+i)}{\lambda_{M+1-i}(u-\kappa+i)} \quad \text{for} \quad i = 1, 2, \ldots, \lfloor \tfrac{M}{2} \rfloor - 1 \ \text{ when } \ M \geq 4, \tag{2.64}$$

$$\frac{\lambda_{M+j}(u)}{\lambda_{M+j+1}(u)} = \frac{\lambda_{M+N-j}(u-\kappa-j+\lfloor \frac{M}{2} \rfloor)}{\lambda_{M+N+1-j}(u-\kappa-j+\lfloor \frac{M}{2} \rfloor)} \quad \text{for} \quad j = 1, 2, \ldots, \tfrac{N}{2} - 1, \tag{2.65}$$

and when M is odd:

$$\frac{\lambda_{\lfloor \frac{M}{2} \rfloor}(u)}{\lambda_{M+1}(u)} = \frac{\lambda_{M+N}(u-\kappa+\lfloor \frac{M}{2} \rfloor)}{\lambda_{\lceil \frac{M}{2} \rceil+1}(u-\kappa+\lfloor \frac{M}{2} \rfloor)} \quad \text{when} \ \ M \geq 3 \tag{2.66}$$

$$\text{and} \quad \frac{\lambda_{\lceil \frac{M}{2} \rceil}(u)}{\lambda_{M+\frac{N}{2}+1}(u)} = \frac{\lambda_{M+\frac{N}{2}}\left(u-\kappa+\lfloor \frac{M}{2} \rfloor - \frac{N}{2}\right)}{\lambda_{\lceil \frac{M}{2} \rceil}\left(u-\kappa+\lfloor \frac{M}{2} \rfloor - \frac{N}{2}\right)}, \tag{2.67}$$

or when M is even:

$$\frac{\lambda_{\frac{M}{2}}(u)}{\lambda_{M+1}(u)} = \frac{\lambda_{M+N}\left(u-\kappa+\frac{M}{2}-\frac{N}{2}\right)}{\lambda_{\frac{M}{2}+1}\left(u-\kappa+\frac{M}{2}-\frac{N}{2}\right)}. \tag{2.68}$$

Proof. "⇒" Let us assume $M_{\mathrm{I}}(\lambda(u))$ is non-trivial. We shall prove the consistency conditions via induction on $M \in 2\mathbb{Z}^+ - 1$ and $M \in 2\mathbb{Z}^+$.

For the base case $M = 1$, consistency conditions were found in [?] for the presentation $\mathbf{X}^{\mathbf{d}}(\mathfrak{osp}_{1|N})$, where $\mathbf{d} = \{1+\frac{N}{2}\} \subset \{1, 2, \ldots, 1+N\}$. For the bijection $\sigma \in \mathfrak{S}_{1+N}$, where $\sigma(k) = 1+k$ for $k = 1, \ldots, \frac{N}{2}$, $\sigma(1+\frac{N}{2}) = 1$ and $\sigma(k) = k$ for $k = 2+\frac{N}{2}, \ldots, 1+N$, the mapping $T^{\mathbf{d}}_{ij}(u) \mapsto T_{\sigma(i)\sigma(j)}(u)$ induces an isomorphism $\mathbf{X}^{\mathbf{d}}(\mathfrak{osp}_{1|N}) \xrightarrow{\sim} \mathbf{X}(\mathfrak{osp}_{1|N})$. Under such isomorphism, highest weight representations for $\mathbf{X}^{\mathbf{d}}(\mathfrak{osp}_{1|N})$ defined in Section 3 of [?] coincide with X_{I}-highest weight representations for $\mathbf{X}(\mathfrak{osp}_{1|N})$, and the consistency conditions stated in Proposition 3.3 in [?] are equivalent to (2.65) and (2.67) when $M = 1$.

For the base case $M = 2$, consistency conditions were found in [?] for the presentation $\mathbf{X}^{\mathbf{d}}(\mathfrak{osp}_{2|N})$, where $\mathbf{d} = \{1, 2+N\} \subset \{1, 2, \ldots, 2+N\}$. For the bijection $\sigma \in \mathfrak{S}_{2+N}$, where $\sigma(1) = 1$, $\sigma(k) = 1+k$ for $k = 2, \ldots, 1+N$, and $\sigma(2+N) = 2$, the mapping $T^{\mathbf{d}}_{ij}(u) \mapsto T_{\sigma(i)\sigma(j)}(u)$ induces an isomorphism $\mathbf{X}^{\mathbf{d}}(\mathfrak{osp}_{2|N}) \xrightarrow{\sim} \mathbf{X}(\mathfrak{osp}_{2|N})$. Under such isomorphism, highest weight representations for $\mathbf{X}^{\mathbf{d}}(\mathfrak{osp}_{2|N})$ defined in Section 2 of [?] coincide with X_{I}-highest weight representations for $\mathbf{X}(\mathfrak{osp}_{2|N})$, and the consistency conditions stated in Proposition 2.2 in [?] are equivalent to (2.65) and (2.68) when $M = 2$.

The base case for condition (2.66) is when $M = 3$, and in this case we have

$$
\begin{aligned}
T_{14}(u)T_{3,3+N}(v)\xi_{\mathrm{I}} &= \big[T_{14}(u), T_{3,3+N}(v)\big]\xi_{\mathrm{I}} \\
&= -\frac{1}{u-v-\kappa}\Bigg(\sum_{p=1}^{M+N}(-1)^{[p]}\theta_p T_{p4}(u)T_{\bar{p},3+N}(v) + \sum_{p=1}^{M+N}(-1)^{[p]}\theta_p T_{3\bar{p}}(v)T_{1p}(u)\Bigg)\xi_{\mathrm{I}} \\
&= -\frac{1}{u-v-\kappa}\Big(T_{14}(u)T_{3,3+N}(v) + \lambda_4(u)\lambda_{3+N}(v) - \lambda_1(u)\lambda_3(v)\Big)\xi_{\mathrm{I}},
\end{aligned}
$$

so

$$
(u-v-\kappa+1)T_{13}(u)T_{3,3+N}(v)\xi_{\mathrm{I}} = \big(\lambda_4(u)\lambda_{3+N}(v) - \lambda_1(u)\lambda_3(v)\big)\xi_{\mathrm{I}}.
$$

Setting $v = u - \kappa + 1$ then yields the desired relation.

Lastly, the base cases for relations (2.64) is when $M = 4$ and $M = 5$, but such relations are guaranteed by Proposition (2.4.5).

Therefore, let us assume the consistency conditions hold up to $M-2$. By Proposition (2.4.8), $M_{\mathrm{I}}(\lambda(u))^+$ is a non-trivial $\mathbf{X}(\mathfrak{osp}_{(M-2)|N})$-module that contains the X_{I}-highest weight vector $\mathbf{1}_{\lambda(u)}$, Moreover, the $\mathbf{X}(\mathfrak{osp}_{(M-2)|N})$-submodule $\mathbf{X}(\mathfrak{osp}_{(M-2)|N})\mathbf{1}_{\lambda(u)}$ contained in $M_{\mathrm{I}}(\lambda(u))^+$ will be an X_{I}-highest weight module for $\mathbf{X}(\mathfrak{osp}_{(M-2)|N})$ with highest weight vector $\mathbf{1}_{\lambda(u)}$ and highest weight

$$
\mu(u) = (\mu_k(u))_{k=1}^{M-2+N} = \big(\lambda_2(u), \ldots, \lambda_{M-1}(u), \lambda_{M+1}(u), \ldots, \lambda_{M+N}(u)\big).
$$

Noting the formula $\kappa_{M-2,N} = \kappa_{M|N} - 1$, when $M-2 \geq 4$ our induction hypothesis for $i = 1, 2, \ldots, \left\lfloor \frac{M-2}{2} \right\rfloor - 1 = \left\lfloor \frac{M}{2} \right\rfloor - 2$ gives

$$\frac{\mu_i(u)}{\mu_{i+1}(u)} = \frac{\mu_{M-2-i}(u - \kappa_{M-2,N} + i)}{\mu_{M-2+1-i}(u - \kappa_{M-2,N} + i)} \Leftrightarrow \frac{\lambda_{i+1}(u)}{\lambda_{i+2}(u)} = \frac{\lambda_{M-(i+1)}(u - \kappa_{M|N} + (i+1))}{\lambda_{M+1-(i+1)}(u - \kappa_{M|N} + (i+1))},$$

proving the relations (2.64) for $i = 2, 3, \ldots, \left\lfloor \frac{M}{2} \right\rfloor - 1$. The case $i = 1$ is guaranteed by Proposition (2.4.5).

Similarly, we know by induction that for $j = 1, 2, \ldots, \frac{N}{2} - 1$,

$$\begin{aligned}\frac{\mu_{M-2+j}(u)}{\mu_{M-2+j+1}(u)} &= \frac{\mu_{M-2+N-j}(u - \kappa_{M-2,N} - j + \left\lfloor \frac{M-2}{2} \right\rfloor)}{\mu_{M-2+N+1-j}(u - \kappa_{M-2,N} - j + \left\lfloor \frac{M-2}{2} \right\rfloor)} \\ &\Leftrightarrow \frac{\lambda_{M+j}(u)}{\lambda_{M+j+1}(u)} = \frac{\lambda_{M+N-j}(u - \kappa_{M|N} - j + \left\lfloor \frac{M}{2} \right\rfloor)}{\lambda_{M+N+1-j}(u - \kappa_{M|N} - j + \left\lfloor \frac{M}{2} \right\rfloor)}.\end{aligned}$$

Now assume M is odd. The induction hypothesis indicates

$$\begin{aligned}\frac{\mu_{\lceil \frac{M-2}{2} \rceil}(u)}{\mu_{M-2+\frac{N}{2}+1}(u)} &= \frac{\mu_{M-2+\frac{N}{2}}\left(u - \kappa_{M-2,N} + \left\lfloor \frac{M-2}{2} \right\rfloor - \frac{N}{2}\right)}{\mu_{\lceil \frac{M-2}{2} \rceil}\left(u - \kappa_{M-2,N} + \left\lfloor \frac{M-2}{2} \right\rfloor - \frac{N}{2}\right)} \\ &\Leftrightarrow \frac{\lambda_{\lceil \frac{M}{2} \rceil}(u)}{\lambda_{M+\frac{N}{2}+1}(u)} = \frac{\lambda_{M+\frac{N}{2}}\left(u - \kappa_{M|N} + \left\lfloor \frac{M}{2} \right\rfloor - \frac{N}{2}\right)}{\lambda_{\lceil \frac{M}{2} \rceil}\left(u - \kappa_{M|N} + \left\lfloor \frac{M}{2} \right\rfloor - \frac{N}{2}\right)},\end{aligned}$$

and if $M-2 \geq 3$,

$$\begin{aligned}\frac{\mu_{\lfloor \frac{M-2}{2} \rfloor}(u)}{\mu_{M-2+1}(u)} &= \frac{\mu_{M-2+N}(u - \kappa_{M-2,N} + \left\lfloor \frac{M-2}{2} \right\rfloor)}{\mu_{\lceil \frac{M-2}{2} \rceil+1}(u - \kappa_{M-2,N} + \left\lfloor \frac{M-2}{2} \right\rfloor)} \\ &\Leftrightarrow \frac{\lambda_{\lfloor \frac{M}{2} \rfloor}(u)}{\lambda_{M+1}(u)} = \frac{\lambda_{M+N}(u - \kappa + \left\lfloor \frac{M}{2} \right\rfloor)}{\lambda_{\lceil \frac{M}{2} \rceil+1}(u - \kappa + \left\lfloor \frac{M}{2} \right\rfloor)}.\end{aligned}$$

Lastly, if M is even, then

$$\begin{aligned}\frac{\mu_{\frac{M-2}{2}}(u)}{\mu_{M-2+1}(u)} &= \frac{\mu_{M-2+N}\left(u - \kappa_{M-2,N} + \frac{M-2}{2} - \frac{N}{2}\right)}{\mu_{\frac{M-2}{2}+1}\left(u - \kappa_{M-2,N} + \frac{M-2}{2} - \frac{N}{2}\right)} \\ &\Leftrightarrow \frac{\lambda_{\frac{M}{2}}(u)}{\lambda_{M+1}(u)} = \frac{\lambda_{M+N}\left(u - \kappa_{M|N} + \frac{M}{2} - \frac{N}{2}\right)}{\lambda_{\frac{M}{2}+1}\left(u - \kappa_{M|N} + \frac{M}{2} - \frac{N}{2}\right)}.\end{aligned}$$

"$\Leftarrow$" Conversely, let us suppose the highest weight $\lambda(u)$ satisfies the conditions (2.64), (2.65), and (2.66), (2.67) if M is odd or (2.68) if M is even. By the Lemma, we obtain a non-trivial X_{I}-Verma module $M_{\mathrm{I}}(\widetilde{\lambda}(u))$ To finish the proof, it therefore suffices to show $\widetilde{\lambda}(u) = \lambda(u)$. As $\widetilde{\lambda}_k(u) = \lambda_k(u)$ for $(k,k) \in \mathcal{B}_{M|N}$ and $k = 1$, we need to show the equality in the remaining cases.

Furthermore, since $M_{\mathrm{I}}(\widetilde{\lambda}(u))$ is non-trivial and the first conditional statement of the Proposition has been proven, the highest weight components of $\widetilde{\lambda}(u) = (\widetilde{\lambda}_k(u))_{k=1}^{M+N}$ satisfy the relations

$$\frac{\lambda_1(u)}{\widetilde{\lambda}_2(u)} = \frac{\lambda_{M-1}(u-\kappa+i)}{\lambda_M(u-\kappa+i)} \quad \text{and} \quad \frac{\widetilde{\lambda}_i(u)}{\widetilde{\lambda}_{i+1}(u)} = \frac{\lambda_{M-i}(u-\kappa+i)}{\lambda_{M+1-i}(u-\kappa+i)}, \tag{2.69}$$

for $i = 2, \ldots, \lfloor \frac{M}{2} \rfloor - 1$ and $M \geq 4$,

$$\frac{\widetilde{\lambda}_{M+j}(u)}{\widetilde{\lambda}_{M+j+1}(u)} = \frac{\lambda_{M+N-j}(u-\kappa-j+\lfloor \frac{M}{2} \rfloor)}{\lambda_{M+N+1-j}(u-\kappa-j+\lfloor \frac{M}{2} \rfloor)} \tag{2.70}$$

for $j = 1, 2, \ldots, \frac{N}{2} - 1$, and if M is odd:

$$\frac{\widetilde{\lambda}_{\lfloor \frac{M}{2} \rfloor}(u)}{\widetilde{\lambda}_{M+1}(u)} = \frac{\lambda_{M+N}(u-\kappa+\lfloor \frac{M}{2} \rfloor)}{\lambda_{\lceil \frac{M}{2} \rceil+1}(u-\kappa+\lfloor \frac{M}{2} \rfloor)} \quad \text{when } M \geq 3, \tag{2.71}$$

$$\text{and} \quad \frac{\widetilde{\lambda}_{\lceil \frac{M}{2} \rceil}(u)}{\widetilde{\lambda}_{M+\frac{N}{2}}(u)} = \frac{\lambda_{M+\frac{N}{2}+1}(u-\kappa+\lfloor \frac{M}{2} \rfloor-\frac{N}{2})}{\widetilde{\lambda}_{\lceil \frac{M}{2} \rceil}(u-\kappa+\lfloor \frac{M}{2} \rfloor-\frac{N}{2})}, \tag{2.72}$$

or if M is even:

$$\frac{\widetilde{\lambda}_{\frac{M}{2}}(u)}{\widetilde{\lambda}_{M+1}(u)} = \frac{\lambda_{M+N}(u-\kappa+\frac{M}{2}-\frac{N}{2})}{\lambda_{\frac{M}{2}+1}(u-\kappa+\frac{M}{2}-\frac{N}{2})}. \tag{2.73}$$

When $M = 1$, equations (2.67) and (2.72) yield $\widetilde{\lambda}_{M+\frac{N}{2}}(u) = \lambda_{M+\frac{N}{2}}(u)$. Therefore, by combining (2.65) and (2.70), we obtain $\widetilde{\lambda}_k(u) = \lambda_k(u)$ for $k = M+1, \ldots, M+\frac{N}{2}-1$.

When $M = 2$, (2.68) and (2.73) infer $\widetilde{\lambda}_{M+1}(u) = \lambda_{M+1}(u)$. Thus, combining (2.65) with (2.70) show $\widetilde{\lambda}_k(u) = \lambda_k(u)$ for $k = M+2, \ldots, M+\frac{N}{2}$.

When $M = 3$, (2.66) and (2.71) similarly infer $\widetilde{\lambda}_{M+1}(u) = \lambda_{M+1}(u)$. Hence, combining (2.65) with (2.70) show $\widetilde{\lambda}_k(u) = \lambda_k(u)$ for $k = M+2, \ldots, M+\frac{N}{2}$.

Now assume $M \geq 4$. In this case, relations (2.64) and (2.69) show $\widetilde{\lambda}_k(u) = \lambda_k(u)$ for $k = 2, \ldots, \lfloor \frac{M}{2} \rfloor$. Furthermore, by combining (2.66) and (2.71) if M is odd, or (2.68) and (2.73) if M is even, one deduces $\widetilde{\lambda}_{M+1}(u) = \lambda_{M+1}(u)$. Thus, combining (2.65) and (2.70) will show $\widetilde{\lambda}_k(u) = \lambda_k(u)$ for $k = M+2, \ldots, M+\frac{N}{2}$. Finally, when M is odd, applying an analogue of Proposition (2.3.2) to (2.67) and (2.72) will yield the last equality $\widetilde{\lambda}_{\lceil \frac{M}{2} \rceil}(u) = \lambda_{\lceil \frac{M}{2} \rceil}(u)$. □

Corollary 2.4.13. *The X_{I}-irreducible highest weight representation $L_{\mathrm{I}}(\lambda(u))$ of $\mathbf{X}(\mathfrak{osp}_{M|N})$ exists if and only if $\lambda(u)$ satisfies the consistency conditions stated in Proposition* (2.4.12).

Theorem 2.4.14 (R. B. Zhang). *An irreducible highest weight representation V of $\mathbf{Y}(\mathfrak{gl}_{m|n})$ with highest weight $\lambda(u) = (\lambda_k(u))_{k=1}^{m+n}$ is finite-dimensional if and only if there exists monic polynomials $\widetilde{Q}(u)$, $Q(u)$, and $P_k(u)$, $k \in \{1, 2, \ldots, m+n-1\} \setminus \{m\}$, such that*

$$\frac{\lambda_k(u)}{\lambda_{k+1}(u)} = \frac{P_k(u + (-1)^{[k]})}{P_k(u)} \quad \text{for} \quad k \in \{1, 2, \ldots, m+n-1\} \setminus \{m\}, \tag{2.74}$$

and

$$\frac{\lambda_m(u)}{\lambda_{m+1}(u)} = \frac{\widetilde{Q}(u)}{Q(u)}, \tag{2.75}$$

where $\widetilde{Q}(u)$ and $Q(u)$ are coprime and are of the same polynomial degree.

Let $m = \lfloor \frac{M}{2} \rfloor$ and $n = \frac{N}{2}$. If $t_{ij}(u)$ denotes a generating series for $\mathbf{Y}(\mathfrak{gl}_{m|n})$, them the mapping

$$t_{ij}(u) \mapsto \begin{cases} T_{ij}(u) & \text{if} \quad 1 \leq i,j \leq \lfloor \frac{M}{2} \rfloor, \\ T_{i,\lceil \frac{M}{2} \rceil + j}(u) & \text{if} \quad 1 \leq i \leq \lfloor \frac{M}{2} \rfloor,\ \lfloor \frac{M}{2} \rfloor + 1 \leq j \leq \lfloor \frac{M}{2} \rfloor + \frac{N}{2}, \\ T_{\lceil \frac{M}{2} \rceil + i, j}(u) & \text{if} \quad \lfloor \frac{M}{2} \rfloor + 1 \leq i \leq \lfloor \frac{M}{2} \rfloor + \frac{N}{2},\ 1 \leq j \leq \lfloor \frac{M}{2} \rfloor, \\ T_{\lceil \frac{M}{2} \rceil + i, \lceil \frac{M}{2} \rceil + j}(u) & \text{if} \quad \lfloor \frac{M}{2} \rfloor + 1 \leq i,j \leq \lfloor \frac{M}{2} \rfloor + \frac{N}{2}, \end{cases}$$

will define an embedding $\mathbf{Y}(\mathfrak{gl}_{m|n}) \hookrightarrow \mathbf{X}(\mathfrak{osp}_{M|N})$. Furthermore, any X_{I}-highest weight representation V of $\mathbf{X}(\mathfrak{osp}_{M|N})$ will restrict to a highest weight representation of $\mathbf{Y}(\mathfrak{gl}_{m|n})$ under such embedding.

Hence, if the X_{I}-irreducible highest weight representation $L_{\mathrm{I}}(\lambda(u))$ of $\mathbf{X}(\mathfrak{osp}_{M|N})$ is finite-dimensional, then

$$\frac{\lambda_k(u)}{\lambda_{k+1}(u)} = \frac{P_k(u+1)}{P_k(u)} \quad \text{for} \quad 1 \leq k \leq \lfloor \tfrac{M}{2} \rfloor - 1, \tag{2.76}$$

$$\frac{\lambda_k(u)}{\lambda_{k+1}(u)} = \frac{P_k(u-1)}{P_k(u)} \quad \text{for} \quad M+1 \leq k \leq M + \tfrac{N}{2} - 1, \tag{2.77}$$

and

$$\frac{\lambda_{\lfloor \frac{M}{2} \rfloor}(u)}{\lambda_{M+1}(u)} = \frac{\widetilde{Q}(u)}{Q(u)}, \tag{2.78}$$

From Molev's recent work on the representation theory of $\mathbf{X}(\mathfrak{osp}_{1|N})$ and $\mathbf{X}(\mathfrak{osp}_{2|N})$, we can use inductive arguments to get additional relations for the Drinfel'd polynomials. In particular, if $L_{\mathrm{I}}(\lambda(u))$ is finite-dimensional, then when M is even,

$$\frac{\lambda_{M+\frac{N}{2}+1}(u)}{\lambda_{M+\frac{N}{2}}(u)} = \frac{P_{M+\frac{N}{2}}(u+2)}{P_{M+\frac{N}{2}}(u)} \tag{2.79}$$

or when M is odd,

$$\frac{\lambda_{\lceil \frac{M}{2} \rceil}(u)}{\lambda_{M+\frac{N}{2}-1}(u)} = \frac{P_{\lceil \frac{M}{2} \rceil}(u+1)}{P_{\lceil \frac{M}{2} \rceil}(u)}. \tag{2.80}$$

Hence, I suspect that there should be about $m+n$ or $m+n+1$ Drinfel'd polynomials depending on how large M is. We also have the following:

Proposition 2.4.15. *Let V and W be two X_{I}-highest weight representations of $\mathbf{X}(\mathfrak{osp}_{M|N})$ with respective highest weights $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$, $\mu(u) = (\mu_k(u))_{k=1}^{M+N}$ and highest weight vectors $\xi_{\lambda(u)}$, $\xi_{\mu(u)}$. The tensor product $V \otimes W$ will be an X_{I}-highest weight representation of $\mathbf{X}(\mathfrak{osp}_{M|N})$ as well with highest weight $\lambda(u)\mu(u) = (\lambda_k(u)\mu_k(u))_{k=1}^{M+N}$ and highest weight vector $\xi_{\lambda(u)} \otimes \xi_{\mu(u)}$.*

Proof. Via the comultiplication map (2.46) on $\mathbf{X}(\mathfrak{osp}_{M|N})$, the generators $T_{ij}(u)$ with indices $(i,j) \in \Lambda_{\mathrm{I}}^{+}$ will annihilate $\xi_{\lambda(u)} \otimes \xi_{\mu(u)}$ since for such indices, $(i,k) \notin \Lambda_{\mathrm{I}}^{+}$ implies $(k,j) \in \Lambda_{\mathrm{I}}^{+}$. Once can also verify $T_{kk}(u) \cdot (\xi_{\lambda(u)} \otimes \xi_{\mu(u)}) = \lambda_k(u)\mu_k(u)(\xi_{\lambda(u)} \otimes \xi_{\mu(u)})$ for all integers $1 \le k \le M+N$. $\square$

For any $a \in \mathbb{C}$, we recall the representation (2.10) of $\mathbf{X}(\mathfrak{osp}_{M|N})$ on $\mathbb{C}^{M|N}$. On the level of power series, such representation is given by

$$\varrho_a \colon T_{ij}(u) \mapsto \delta_{ij}\,\mathrm{id} + \frac{(-1)^{[j]}E_{ij}}{u-a} - \frac{(-1)^{[i][j]}\theta_{\bar{\imath}}\theta_{\bar{\jmath}}E_{\bar{\jmath}\bar{\imath}}}{u+\kappa-a}.$$

For notation, we shall let the juxtaposition $T_{ij}(u)v$ for $v \in \mathbb{C}^{M|N}$ denote the action $\varrho_0(T_{ij}(u))v$. Tensoring the representations $\varrho_{0\to(-d+1)} := \Delta_{d-1} \circ (\bigotimes_{i=0}^{d-1} \varrho_{-i})$ gives rise to a representation of $\mathbf{X}(\mathfrak{osp}_{M|N}$ on $(\mathbb{C}^{M|N})^{\otimes d}$. On even basis vectors $e_{p_1}, \dots, e_{p_d}$ of $\mathbb{C}^{M|N}$, we can describe the action of $T_{ij}(u)$ on $e_{p_1} \otimes \cdots \otimes e_{p_d}$ via the formula

$$\begin{aligned} &T_{ij}(u)(e_{p_1} \otimes \cdots \otimes e_{p_d}) \\ &\quad = \sum_{a_1,\dots,a_{d-1}=1}^{M+N} T_{ia_1}(u)e_{p_1} \otimes T_{a_1a_2}(u+1)e_{p_2} \otimes \cdots \otimes T_{a_{d-1}j}(u+(d-1))e_{p_d} \end{aligned} \tag{2.81}$$

For any integer $1 \le d \le \left\lfloor \frac{M}{2} \right\rfloor$, we define

$$\xi_d = \sum_{\sigma \in \mathfrak{S}_d} (\operatorname{sgn}\sigma) e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(d)} \tag{2.82}$$

One can show that with the above construction, $(\mathbb{C}^{M|N})^{\otimes d}$ will be an X_{I}-highest weight representation with highest weight vector ξ_d and highest weight $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ given by

$$\lambda_k(u) = \begin{cases} \frac{u+d}{u+d-1} & \text{if} \quad 1 \leq k \leq d, \\ 1 & \text{if} \quad d+1 \leq k \leq M-d, \\ \frac{u+\kappa-1}{u+\kappa} & \text{if} \quad M-d+1 \leq k \leq M, \\ 1 & \text{if} \quad M+1 \leq k \leq M+N, \end{cases} \tag{2.83}$$

For $d \in \{1, 2, \ldots, \lfloor \frac{M}{2} \rfloor\}$, we have

$$\frac{\lambda_k(u)}{\lambda_{k+1}(u)} = 1$$

for $1 \leq k \leq d-1$, $d+1 \leq k \leq \lfloor \frac{M}{2} \rfloor - 1 \leq M-d$, and $M+1 \leq k \leq M+\frac{N}{2}-1$, so $P_k(u)$ for such indices are constant polynomials. Furthermore,

$$\frac{\lambda_d(u)}{\lambda_{d+1}(u)} = \frac{u+d}{u+d-1}$$

so $P_d(u) = u + d - 1$. When M is even,

$$\frac{\lambda_{M+\frac{N}{2}+1}(u)}{\lambda_{M+\frac{N}{2}}(u)} = 1$$

so $P_{M+\frac{N}{2}}(u)$ is constant and when M is odd,

$$\frac{\lambda_{\lceil \frac{M}{2} \rceil}(u)}{\lambda_{M+\frac{N}{2}-1}(u)} = 1$$

so $P_{\lceil \frac{M}{2} \rceil}(u)$ is constant. Now, in the case $d \in \{1, 2, \ldots, \lfloor \frac{M}{2} \rfloor - 1\}$,

$$\frac{\lambda_{\lfloor \frac{M}{2} \rfloor}(u)}{\lambda_{M+1}(u)} = 1$$

then $\widetilde{Q}(u) = Q(u)$ must be the same constant polynomial. However, when $d = \lfloor \frac{M}{2} \rfloor$, then

$$\frac{\lambda_{\lfloor \frac{M}{2} \rfloor}(u)}{\lambda_{M+1}(u)} = \frac{u+d}{u+d-1}$$

so $\widetilde{Q}(u) = u+d$ and $Q(u) = u+d-1$ which are coprime and of the same polynomial degree.

2.5 Gauss Decomposition of $\mathbf{Y}(\mathfrak{osp}_{1|2})$

In this section, we shall consider the presentation of the extended Yangian $\mathbf{X}^{\mathbf{d}}(\mathfrak{osp}_{1|2})$ and Yangian $\mathbf{Y}^{\mathbf{d}}(\mathfrak{osp}_{1|2})$, where $\mathbf{d} = \{2\}$ as in Definitions (2.1.1) and (2.1.2). Therefore, we consider the super vector space $\mathbb{C}^{1|2}$ graded as $[e_i] = [i]_{\mathbf{d}}$, where $[2]_{\mathbf{d}} = \bar{0}$ and $[1]_{\mathbf{d}} = [3]_{\mathbf{d}} = \bar{1}$. Further, the conjugate index $\{1,2,3\} \to \{1,2,3\}$ is given by $i \mapsto \bar{\imath}^{\mathbf{d}} = 3-i$ and we have the signs $\theta_1^{\mathbf{d}} = \theta_2^{\mathbf{d}} = 1$ and $\theta_3^{\mathbf{d}} = -1$. We note that the transposition $\sigma = (1\ 2) \in \mathfrak{S}_3$ induces an isomorphism from $\mathbf{X}(\mathfrak{osp}_{1|2})$ to $\mathbf{X}^{\mathbf{d}}(\mathfrak{osp}_{1|2})$ as in Remark (2.1.9).

For the remainder of this section, we shall for convenience drop the sub/super script $\mathbf{d}$. A Drinfeld-type presentation of the Yangian of $\mathfrak{osp}_{1|2}$ was given in previously with the use of the Gauss decomposition of the matrix $T(u)$. We will be using some calculations produced therein and derive consistency relations for the Gaussian generators. Apply the Gauss decomposition to the generator matrix $T(u)$ for $\mathbf{X}(\mathfrak{osp}_{1|2})$,

$$T(u) = F(u)H(u)E(u), \tag{2.1}$$

where $F(u)$, $H(u)$ and $E(u)$ are uniquely determined matrices of the form

$$F(u) = \begin{pmatrix} \mathbf{1} & 0 & 0 \\ f_{21}(u) & \mathbf{1} & 0 \\ f_{31}(u) & f_{32}(u) & \mathbf{1} \end{pmatrix}, \qquad E(u) = \begin{pmatrix} \mathbf{1} & e_{12}(u) & e_{13}(u) \\ 0 & \mathbf{1} & e_{23}(u) \\ 0 & 0 & \mathbf{1} \end{pmatrix}, \tag{2.2}$$

and

$$
H(u) = \begin{pmatrix} h_1(u) & 0 & 0 \\ 0 & h_2(u) & 0 \\ 0 & 0 & h_3(u) \end{pmatrix} \tag{2.3}
$$

Explicit formulas for the entries of the matrices $F(u)$, $H(u)$, and $E(u)$ can be written with the use of the Gelfand–Retakh quasideterminants. In this way,

$$
h_1(u) = T_{11}(u), \quad h_2(u) = \begin{vmatrix} T_{11}(u) & T_{12}(u) \\ T_{21}(u) & \boxed{T_{22}(u)} \end{vmatrix}, \quad h_3(u) = \begin{vmatrix} T_{11}(u) & T_{12}(u) & T_{13}(u) \\ T_{21}(u) & T_{22}(u) & T_{23}(u) \\ T_{31}(u) & T_{32}(u) & \boxed{T_{33}(u)} \end{vmatrix},
$$

where

$$
e_{12}(u) = h_1(u)^{-1} T_{12}(u), \qquad e_{23}(u) = h_2(u)^{-1} \begin{vmatrix} T_{11}(u) & T_{13}(u) \\ T_{21}(u) & \boxed{T_{23}(u)} \end{vmatrix},
$$

and

$$
f_{21}(u) = T_{21}(u) h_1(u)^{-1}, \qquad f_{32}(u) = \begin{vmatrix} T_{11}(u) & T_{12}(u) \\ T_{31}(u) & \boxed{T_{32}(u)} \end{vmatrix} h_2(u)^{-1}.
$$

Via the Gauss decomposition, we can write

$$
T(u) = \big(T_{ij}(u)\big)_{i,j=1}^{3} = \begin{pmatrix} h_1(u) & h_1(u)e_{12}(u) & h_1(u)e_{13}(u) \\ f_{21}(u)h_1(u) & f_{21}(u)h_1(u)e_{12}(u) + h_2(u) & T_{23}(u) \\ f_{31}(u)h_1(u) & T_{32}(u) & T_{33}(u) \end{pmatrix},
$$

where

$$
T_{23}(u) = f_{21}(u)h_1(u)e_{13}(u) + h_2(u)e_{23}(u), \quad T_{32}(u) = f_{31}(u)h_1(u)e_{12}(u) + f_{32}(u)h_2(u)
$$
$$
\text{and} \quad T_{33}(u) = f_{31}(u)h_1(u)e_{13}(u) + f_{32}(u)h_2(u)e_{23}(u) + h_3(u).
$$

Proposition 2.5.1. *In* $\mathbf{X}(\mathfrak{osp}_{1|2})$, *we have*

$$
\big[h_1(u), h_1(v)\big] = 0, \quad \big[h_1(u), h_2(v)\big] = 0, \tag{2.4}
$$

$$\big[h_1(u), e_{12}(v)\big] = \frac{1}{u-v} h_1(u)\big(e_{12}(u) - e_{12}(v)\big), \tag{2.5}$$

$$\text{and} \quad \big[h_1(u), f_{21}(v)\big] = \frac{1}{u-v}\big(f_{21}(v) - f_{21}(u)\big)h_1(u). \tag{2.6}$$

Consequently,

$$h_1(u)e_{12}(u) = e_{12}(u+1)h_1(u) \tag{2.7}$$

$$\text{and} \quad f_{21}(u)h_1(u) = h_1(u)f_{21}(u+1). \tag{2.8}$$

Proof. We will be using the defining relations (4.10) in several places during this proof.

Step 1. The first relation in (2.4) follows directly from the equation

$$\big[T_{11}(u), T_{11}(v)\big] = -\frac{1}{u-v}\Big(T_{11}(u)T_{11}(v) - T_{11}(v)T_{11}(u)\Big) = -\frac{1}{u-v}\big[T_{11}(u), T_{11}(v)\big]$$

since $T_{11}(u) = h_1(u)$. For the second, note that by the Gauss decomposition we have $T_{22}(v) = f_{21}(v)h_1(v)e_{12}(v) + h_2(v)$, so

$$\big[h_1(u), h_2(v)\big] = \big[T_{11}(u), T_{22}(v)\big] - \big[T_{11}(u), T_{21}(v)T_{11}(v)^{-1}T_{12}(v)\big] \tag{2.9}$$

To compute $\big[T_{11}(u), T_{21}(v)T_{11}(v)^{-1}T_{12}(v)\big]$, we first realize

$$\big[T_{11}(u), T_{21}(v)\big] = -\frac{1}{u-v}\Big(T_{21}(u)T_{11}(v) - T_{21}(v)T_{11}(u)\Big),$$

so $T_{11}(u)T_{21}(v) = \frac{u-v+1}{u-v}T_{21}(v)T_{11}(u) - \frac{1}{u-v}T_{21}(u)T_{11}(v)$, which implies

$$T_{11}(u)T_{21}(v)T_{11}(v)^{-1}T_{12}(v) = \frac{u-v+1}{u-v}T_{21}(v)T_{11}(u)T_{11}(v)^{-1}T_{12}(v) - \frac{1}{u-v}T_{21}(u)T_{12}(v) \tag{2.10}$$

Similarly,

$$\big[T_{11}(u), T_{12}(v)\big] = -\frac{1}{u-v}\Big(T_{11}(u)T_{12}(v) - T_{11}(v)T_{12}(u)\Big),$$

so $-T_{12}(v)T_{11}(u) = \frac{1}{u-v}T_{11}(v)T_{12}(u) - \frac{u-v+1}{u-v}T_{11}(u)T_{12}(v)$, which implies

$$-T_{21}(v)T_{11}(v)^{-1}T_{12}(v)T_{11}(u) = \frac{1}{u-v}T_{21}(v)T_{12}(u) - \frac{u-v+1}{u-v}T_{21}(v)T_{11}(u)T_{11}(v)^{-1}T_{12}(v), \tag{2.11}$$

since $T_{11}(v)^{-1}T_{11}(u) = T_{11}(u)T_{11}(v)^{-1}$. Hence,

$$\left[T_{11}(u), T_{21}(v)T_{11}(v)^{-1}T_{12}(v)\right] = -\frac{1}{u-v}\Big(T_{21}(u)T_{12}(v) - T_{21}(v)T_{12}(u)\Big) = \left[T_{11}(u), T_{22}(v)\right]$$

Step 2. To get equation (2.5), we observe

$$\left[T_{11}(u), T_{12}(v)\right] = -\frac{1}{u-v}\Big(T_{11}(u)T_{12}(v) - T_{11}(v)T_{12}(u)\Big),$$

which re-written is

$$\left[h_1(u), h_1(v)e_{12}(v)\right] = \frac{1}{u-v}\Big(h_1(v)h_1(u)e_{12}(u) - h_1(u)h_1(v)e_{12}(v)\Big).$$

The desired relation therefore follows from (2.4) and multiplying the above equation on the left by $h_1(v)^{-1}$. Similarly, to obtain equation (2.6), we have

$$\left[T_{11}(u), T_{21}(v)\right] = -\frac{1}{u-v}\Big(T_{21}(u)T_{11}(v) - T_{21}(v)T_{11}(u)\Big),$$

which re-written is

$$\left[h_1(u), f_{21}(v)h_1(v)\right] = \frac{1}{u-v}\Big(f_{21}(v)h_1(v)h_1(u) - f_{21}(u)h_1(u)h_1(v)\Big).$$

The desired relation therefore follows from (2.4) and multiplying the above equation on the right by $h_1(v)^{-1}$. $\square$

Now, since $T(u)^{-1} = E(u)^{-1}H(u)^{-1}F(u)^{-1}$, we have

$$T(u)^{-1} = \big(T^\bullet_{ij}(u)\big)_{i,j=1}^{3} = \begin{pmatrix} T^\bullet_{11}(u) & T^\bullet_{12}(u) & T^\bullet_{13}(u) \\ T^\bullet_{21}(u) & e_{23}(u)h_3(u)^{-1}f_{32}(u) + h_2(u)^{-1} & -e_{23}(u)h_3(u)^{-1} \\ T^\bullet_{31}(u) & -h_3(u)^{-1}f_{32}(u) & h_3(u)^{-1} \end{pmatrix},$$

where

$$\begin{aligned} T^\bullet_{11}(u) &= h_1(u)^{-1} + e_{12}(u)h_2(u)^{-1}f_{21}(u) \\ &\qquad + \big(e_{12}(u)e_{23}(u)h_3(u)^{-1} - e_{13}(u)h_3(u)^{-1}\big)\big(f_{32}(u)f_{21}(u) - f_{31}(u)\big), \\ T^\bullet_{12}(u) &= -e_{12}(u)h_2(u)^{-1} - \big(e_{12}(u)e_{23}(u)h_3(u)^{-1} - e_{13}(u)h_3(u)^{-1}\big)f_{32}(u), \\ T^\bullet_{13}(u) &= \big(e_{12}(u)e_{23}(u) - e_{13}(u)\big)h_3(u)^{-1}, \\ T^\bullet_{21}(u) &= -h_2(u)^{-1}f_{21}(u) - e_{23}(u)h_3(u)^{-1}\big(f_{32}(u)f_{21}(u) - f_{31}(u)\big), \\ \text{and}\quad T^\bullet_{31}(u) &= h_3(u)^{-1}\big(f_{32}(u)f_{21}(u) - f_{31}(u)\big). \end{aligned}$$

Since we are viewing $T(w)$ as a matrix, the matrix $T^t(w)$ is given by (2.38). That is,

$$\begin{aligned} T^t(w) = \big((-1)^{[i][j]+[j]}\theta_i\theta_j T_{\bar{j}\bar{i}}(w)\big)_{i,j=1}^{3} &= \begin{pmatrix} T_{33}(w) & T_{23}(w) & -T_{13}(w) \\ -T_{32}(w) & T_{22}(w) & T_{12}(w) \\ -T_{31}(w) & -T_{21}(w) & T_{11}(w) \end{pmatrix} \\ &= \begin{pmatrix} T_{33}(w) & T_{23}(w) & -h_1(w)e_{13}(w) \\ -T_{32}(w) & f_{21}(w)h_1(w)e_{12}(w) + h_2(w) & h_1(w)e_{12}(w) \\ -f_{31}(w)h_1(w) & -f_{21}(w)h_1(w) & h_1(w) \end{pmatrix}. \end{aligned}$$

Recalling $Z(u) = T^t(u+\kappa)T(u)$, the matrix equation $T^t(u+\kappa) = Z(u)T(u)^{-1}$ forces relations such as

$$h_1(u+\kappa) = \mathcal{Z}(u)h_3(u)^{-1}, \tag{2.12}$$

$$h_1(u+\kappa)e_{12}(u+\kappa) = -\mathcal{Z}(u)e_{23}(u)h_3(u)^{-1}, \tag{2.13}$$

$$f_{21}(u+\kappa)h_1(u+\kappa) = \mathcal{Z}(u)h_3(u)^{-1}f_{32}(u), \tag{2.14}$$

and

$$f_{21}(u+\kappa)h_1(u+\kappa)e_{12}(u+\kappa)+h_2(u+\kappa) = \mathcal{Z}(u)\big(e_{23}(u)h_3(u)^{-1}f_{32}(u)+h_2(u)^{-1}\big). \quad (2.15)$$

Proposition 2.5.2. *In* $\mathbf{X}(\mathfrak{osp}_{1|2})$*, we have*

$$e_{12}(u) = -e_{23}(u-\kappa-1) \qquad (2.16)$$

$$\text{and} \quad f_{21}(u) = f_{32}(u-\kappa-1). \qquad (2.17)$$

Proof. For the first equation, we have

$$\begin{aligned} e_{12}(u) &= e_{12}(u)h_1(u-1)h_1(u-1)^{-1} \\ &= h_1(u-1)e_{12}(u-1)h_1(u-1)^{-1} \quad \text{by } (2.7) \\ &= -\mathcal{Z}(u-\kappa-1)e_{23}(u-\kappa-1)h_3(u-\kappa-1)^{-1}h_1(u-1)^{-1} \quad \text{by } (2.13) \\ &= -e_{23}(u-\kappa-1) \quad \text{by } (2.12), \end{aligned}$$

where in the last equality we used the fact that the coefficients of $\mathcal{Z}(u-\kappa-1)$ are central in $\mathbf{X}(\mathfrak{osp}_{1|2})$. Similarly, the second equation is given via the calculation

$$\begin{aligned} f_{21}(u) &= h_1(u-1)^{-1}h_1(u-1)f_{21}(u) \\ &= h_1(u-1)^{-1}f_{21}(u-1)h_1(u-1) \quad \text{by } (2.8) \\ &= h_1(u-1)^{-1}\mathcal{Z}(u-\kappa-1)h_3(u-\kappa-1)^{-1}f_{32}(u-\kappa-1) \quad \text{by } (2.14) \\ &= f_{32}(u-\kappa-1) \quad \text{by } (2.12). \end{aligned}$$

□

Proposition 2.5.3. *In* $\mathbf{X}(\mathfrak{osp}_{1|2})$*, we have*

$$e_{13}(u) = -\big[e_{12}^{(1)}, e_{12}(u)\big] - e_{12}(u)^2 \qquad (2.18)$$

Proof. By relation (2.5), $\big[h_1(u), e_{12}^{(1)}\big] = -h_1(u)e_{12}(u)$, or re-written, $e_{12}^{(1)}h_1(u) =$

$h_1(u)e_{12}^{(1)} + h_1(u)e_{12}(u)$. Furthermore, by the defining relations (4.10), we have

$$\big[T_{12}(u), T_{12}(v)\big] = -\frac{1}{u-v}\Big(T_{12}(u)T_{12}(v) - T_{12}(v)T_{12}(u)\Big) \\ - \frac{1}{u-v-\kappa}\Big(T_{13}(v)T_{11}(u) + T_{12}(v)T_{12}(u) - T_{11}(v)T_{13}(u)\Big),$$

so $\big[T_{12}^{(1)}, T_{12}(v)\big] = -T_{13}(v)$, and therefore

$$\begin{aligned} T_{13}(u) = -\big[T_{12}^{(1)}, T_{12}(u)\big] &= -\big[e_{12}^{(1)}, h_1(u)e_{12}(u)\big] \\ &= -e_{12}^{(1)}h_1(u)e_{12}(u) - h_1(u)e_{12}(u)e_{12}^{(1)} \\ &= -h_1(u)e_{12}^{(1)}e_{12}(u) - h_1(u)e_{12}(u)e_{12}^{(1)} - h_1(u)e_{12}(u)^2 \\ &= -h_1(u)\big(\big[e_{12}^{(1)}, e_{12}(u)\big] + e_{12}(u)^2\big). \end{aligned}$$

□

Proposition 2.5.4. *In* $\mathbf{X}(\mathfrak{osp}_{1|2})$, *we have*

$$e_{13}(u) = e_{12}(u+\kappa+1)^2 - e_{12}(u)e_{12}(u+\kappa+1) - \big[e_{12}^{(1)}, e_{12}(u+\kappa+1)\big].$$

Proof. Step 1. We note that

$$\big[T_{12}(u), T_{23}^{\bullet}(v)\big] = \frac{1}{u-v}\Big(T_{11}(u)T_{13}^{\bullet}(v) + T_{12}(u)T_{23}^{\bullet}(v) + T_{13}(u)T_{33}^{\bullet}(v)\Big) \\ + \frac{1}{u-v-\kappa}\Big(T_{12}(u)T_{23}^{\bullet}(v) - T_{23}^{\bullet}(v)T_{12}(u)\Big),$$

so $\big[T_{12}^{(1)}, T_{23}^{\bullet}(v)\big] = T_{13}^{\bullet}(v)$. Written in terms of Gaussian generators, this relation says

$$-\big[e_{12}^{(1)}, e_{23}(v)h_3(v)^{-1}\big] = e_{12}(v)e_{23}(v)h_3(v)^{-1} - e_{13}(v)h_3(v)^{-1}.$$

Furthermore, by multiplying the above relation by the central series $\mathcal{Z}(v)$ and using relations (2.12) and (2.16), we obtain

$$\begin{aligned}&\big[e_{12}^{(1)}, e_{12}(v+\kappa+1)h_1(v+\kappa)\big]\\ &\qquad\qquad = -e_{12}(v)e_{12}(v+\kappa+1)h_1(v+\kappa) - e_{13}(v)h_1(v+\kappa).\end{aligned}$$

Step 2. By using relation (2.5), one has

$$\big[e_{12}(u), h_1(v+\kappa)\big] = -\big[h_1(v+\kappa), e_{12}(u)\big] = \frac{1}{v-u+\kappa} h_1(v+\kappa)\big(e_{12}(u) - e_{12}(v+\kappa)\big)$$

and hence

$$\begin{aligned}\big[e_{12}^{(1)}, h_1(v+\kappa)\big] = h_1(v+\kappa)e_{12}(v+\kappa) &= -\mathcal{Z}(v)e_{23}(v)h_3(v)^{-1} \quad \text{by } (2.13)\\ &= e_{12}(v+\kappa+1)h_1(v+\kappa) \quad \text{by } (2.12), (2.16)\end{aligned}$$

Finally, since we can write

$$\begin{aligned}&\big[e_{12}^{(1)}, e_{12}(v+\kappa+1)h_1(v+\kappa)\big]\\ &\quad = e_{12}^{(1)}e_{12}(v+\kappa+1)h_1(v+\kappa) + e_{12}(v+\kappa+1)h_1(v+\kappa)e_{12}^{(1)}\\ &\quad = e_{12}^{(1)}e_{12}(v+\kappa+1)h_1(v+\kappa) + e_{12}(v+\kappa+1)e_{12}^{(1)}h_1(v+\kappa)\\ &\qquad\qquad + e_{12}(v+\kappa+1)h_1(v+\kappa)e_{12}^{(1)} - e_{12}(v+\kappa+1)e_{12}^{(1)}h_1(v+\kappa)\\ &\quad = \big[e_{12}^{(1)}, e_{12}(v+\kappa+1)\big]h_1(v+\kappa) + e_{12}(v+\kappa+1)\big[h_1(v+\kappa), e_{12}^{(1)}\big]\\ &\quad = \big[e_{12}^{(1)}, e_{12}(v+\kappa+1)\big]h_1(v+\kappa) - e_{12}(v+\kappa+1)^2h_1(v+\kappa),\end{aligned}$$

we get

$$\big[e_{12}^{(1)}, e_{12}(v+\kappa+1)\big] - e_{12}(v+\kappa+1)^2 = -e_{12}(v)e_{12}(v+\kappa+1) - e_{13}(v).$$

□

Proposition 2.5.5. *In* $\mathbf{X}(\mathfrak{osp}_{1|2})$, *we have*

$$\big[e_{12}(u), f_{21}(v)\big] = \frac{1}{u-v}\Big(h_1(u)^{-1}h_2(u) - h_1(v)^{-1}h_2(v)\Big). \tag{2.19}$$

Corollary 2.5.6. *In* $\mathbf{X}(\mathfrak{osp}_{1|2})$, *we have*

$$h_1(u)h_3(u-\kappa-1) = h_2(u)h_2(u-\kappa-1). \tag{2.20}$$

Proposition 2.5.7. *In* $\mathbf{X}(\mathfrak{osp}_{1|2})$, *we have*

$$\big[e_{12}(v), e_{12}(v)\big] = \tag{2.21}$$

Proof. By the defining relations (4.10), we have

$$\begin{aligned}\big[T_{12}(u), T_{12}(v)\big] = &-\frac{\theta_0}{u-v}\Big(T_{12}(u)T_{12}(v) - T_{12}(v)T_{12}(u)\Big) \\ &- \frac{1}{u-v-\kappa}\sum_{p=1}^{3}(-\theta_0)^{[p]}(-1)^{[p]}\theta_p T_{1\bar{p}}(v)T_{1p}(u)\end{aligned}$$

and

$$\begin{aligned}\big[T_{11}(u), T_{13}(v)\big] = &-\frac{\theta_0}{u-v}\Big(T_{11}(u)T_{13}(v) - T_{11}(v)T_{13}(u)\Big) \\ &- \frac{\theta_0}{u-v-\kappa}\sum_{p=1}^{3}(-\theta_0)^{[p]}(-1)^{[p]}\theta_p T_{1\bar{p}}(v)T_{1p}(u).\end{aligned}$$

Since the sums above are equal up to θ_0, we therefore have the equality

$$\begin{aligned}&\big[T_{12}(u), T_{12}(v)\big] + \frac{\theta_0}{u-v}\Big(T_{12}(u)T_{12}(v) - T_{12}(v)T_{12}(u)\Big) \\ &\qquad = \theta_0\big[T_{11}(u), T_{13}(v)\big] + \frac{1}{u-v}\Big(T_{11}(u)T_{13}(v) - T_{11}(v)T_{13}(u)\Big),\end{aligned}$$

which re-written is

$$\begin{aligned}\big[T_{12}(u), T_{12}(v)\big] = &\frac{\theta_0}{u-v}\Big(T_{12}(v)T_{12}(u) - T_{12}(u)T_{12}(v)\Big) \\ &+ \frac{\theta_0(u-v)+1}{u-v}T_{11}(u)T_{13}(v) - \frac{1}{u-v}T_{11}(v)T_{13}(u) - \theta_0 T_{13}(v)T_{11}(u).\end{aligned} \tag{2.22}$$

Multiplying equation (2.5) on the left by $h_1(v)$ yields the relation

$$h_1(v)e_{12}(v)h_1(u) = h_1(u)h_1(v)e_{12}(v) + \frac{\theta_0}{u-v}h_1(u)h_1(v)\big(e_{12}(v) - e_{12}(u)\big).$$

Since $T_{12}(v)T_{12}(u) = h_1(v)e_{12}(v)h_1(u)e_{12}(u)$, the above equation implies

$$T_{12}(v)T_{12}(u) = h_1(u)h_1(v)e_{12}(v)e_{12}(u) + \frac{\theta_0}{u-v}h_1(u)h_1(v)\big(e_{12}(v)e_{12}(u) - e_{12}(u)^2\big).$$

Similarly, by swapping $u \leftrightarrow v$ in relation (2.5) and multiplying this subsequent equation on the left by $h_1(u)$, we get

$$h_1(u)e_{12}(u)h_1(v) = h_1(u)h_1(v)e_{12}(u) + \frac{\theta_0}{u-v}h_1(u)h_1(v)\big(e_{12}(v) - e_{12}(u)\big).$$

Hence,

$$T_{12}(u)T_{12}(v) = h_1(u)h_1(v)e_{12}(u)e_{12}(v) + \frac{\theta_0}{u-v}h_1(u)h_1(v)\big(e_{12}(v)^2 - e_{12}(u)e_{12}(v)\big).$$

Therefore, the left hand side of equation (2.22) is given by

$$\begin{aligned}\big[T_{12}(u), T_{12}(v)\big] &= T_{12}(u)T_{12}(v) + T_{12}(v)T_{12}(u)\\ &= h_1(u)h_1(v)\Big(\big[e_{12}(u), e_{12}(v)\big] + \frac{\theta_0}{u-v}\big(e_{12}(v) - e_{12}(u)\big)\big(e_{12}(v) + e_{12}(u)\big)\Big)\end{aligned}$$

Let us now determine the right hand side of equation (2.22). The element $(A.1)$ $\frac{\theta_0}{u-v}\big(T_{12}(v)T_{12}(u) - T_{12}(u)T_{12}(v)\big)$ is equal to

$$\begin{aligned}&\frac{\theta_0}{u-v}h_1(u)h_1(v)e_{12}(v)e_{12}(u) + \frac{1}{(u-v)^2}h_1(u)h_1(v)\big(e_{12}(v)e_{12}(u) - e_{12}(u)^2\big)\\ &\quad - \frac{\theta_0}{u-v}h_1(u)h_1(v)e_{12}(u)e_{12}(v) - \frac{1}{(u-v)^2}h_1(u)h_1(v)\big(e_{12}(v)^2 - e_{12}(u)e_{12}(v)\big)\\ &= \frac{\theta_0}{u-v}h_1(u)h_1(v)\big(e_{12}(v)e_{12}(u) - e_{12}(u)e_{12}(v)\big) - \frac{1}{(u-v)^2}h_1(u)h_1(v)\big(e_{12}(u) - e_{12}(v)\big)^2\end{aligned}$$

Since $T_{13}(u) = h_1(u)\big(\theta_0 e_{12}(u+\kappa+\theta_0)^2 - \theta_0 e_{12}(u)e_{12}(u+\kappa+\theta_0) - \big[e_{12}^{(1)}, e_{12}(u+\kappa+\theta_0)\big]\big)$,

$(A.2)$ $\frac{\theta_0(u-v)+1}{u-v}T_{11}(u)T_{13}(v)$ is given by

$$\theta_0 h_1(u)h_1(v)\big(\theta_0 e_{12}(v+\kappa+\theta_0)^2 - \theta_0 e_{12}(v)e_{12}(v+\kappa+\theta_0) - \big[e_{12}^{(1)}, e_{12}(v+\kappa+\theta_0)\big]\big)$$
$$+\frac{1}{u-v}h_1(u)h_1(v)\big(\theta_0 e_{12}(v+\kappa+\theta_0)^2 - \theta_0 e_{12}(v)e_{12}(v+\kappa+\theta_0) - \big[e_{12}^{(1)}, e_{12}(v+\kappa+\theta_0)\big]\big)$$

and similarly, $(A.3)$ $\frac{1}{u-v}T_{11}(v)T_{13}(u)$ is

$$\frac{1}{u-v}h_1(u)h_1(v)\big(\theta_0 e_{12}(u+\kappa+\theta_0)^2 - \theta_0 e_{12}(u)e_{12}(u+\kappa+\theta_0) - \big[e_{12}^{(1)}, e_{12}(u+\kappa+\theta_0)\big]\big)$$

To compute $(A.4)$ $\theta_0 T_{13}(v)T_{11}(u)$, we first need some preliminary computations. By (2.5), one gets

$$e_{12}(v)h_1(u) = h_1(u)e_{12}(v) + \frac{\theta_0}{u-v}h_1(u)\big(e_{12}(v) - e_{12}(u)\big)$$

and so

$$e_{12}(v-\tfrac{\theta_0}{2})h_1(u) = h_1(u)e_{12}(v-\tfrac{\theta_0}{2}) + \frac{\theta_0}{u-v+\frac{\theta_0}{2}}h_1(u)\big(e_{12}(v-\tfrac{\theta_0}{2}) - e_{12}(u)\big)$$

Therefore,

$$\begin{aligned}
&e_{12}(v)e_{12}(v-\tfrac{\theta_0}{2})h_1(u)\\
&= e_{12}(v)\Big(h_1(u)e_{12}(v-\tfrac{\theta_0}{2}) + \frac{\theta_0}{u-v+\frac{\theta_0}{2}}h_1(u)\big(e_{12}(v-\tfrac{\theta_0}{2}) - e_{12}(u)\big)\Big)\\
&= \Big(h_1(u)e_{12}(v) + \frac{\theta_0}{u-v}h_1(u)\big(e_{12}(v) - e_{12}(u)\big)\Big)e_{12}(v-\tfrac{\theta_0}{2})\\
&\qquad + \frac{\theta_0}{u-v+\frac{\theta_0}{2}}\Big(h_1(u)e_{12}(v) + \frac{\theta_0}{u-v}h_1(u)\big(e_{12}(v) - e_{12}(u)\big)\Big)\big(e_{12}(v-\tfrac{\theta_0}{2}) - e_{12}(u)\big)\\
&= h_1(u)\Big(e_{12}(v)e_{12}(v-\tfrac{\theta_0}{2}) + \frac{\theta_0}{u-v}\big(e_{12}(v)e_{12}(v-\tfrac{\theta_0}{2}) - e_{12}(u)e_{12}(v-\tfrac{\theta_0}{2})\big)\\
&\qquad\quad + \frac{\theta_0}{u-v+\frac{\theta_0}{2}}e_{12}(v)\big(e_{12}(v-\tfrac{\theta_0}{2}) - e_{12}(u)\big)\\
&\qquad\qquad + \frac{1}{(u-v)(u-v+\frac{\theta_0}{2})}\big(e_{12}(v) - e_{12}(u)\big)\big(e_{12}(v-\tfrac{\theta_0}{2}) - e_{12}(u)\big)\Big)
\end{aligned}$$

$$e_{12}(v)^2h_1(u) = e_{12}(v)\Big(h_1(u)e_{12}(v) + \frac{\theta_0}{u-v}h_1(u)\big(e_{12}(v) - e_{12}(u)\big)\Big)$$

$$
\begin{aligned}
&= \Big(h_1(u)e_{12}(v) + \frac{\theta_0}{u-v}h_1(u)\big(e_{12}(v) - e_{12}(u)\big)\Big)e_{12}(v) \\
&\qquad + \frac{\theta_0}{u-v}\Big(h_1(u)e_{12}(v) + \frac{\theta_0}{u-v}h_1(u)\big(e_{12}(v) - e_{12}(u)\big)\Big)\big(e_{12}(v) - e_{12}(u)\big) \\
&= h_1(u)\Big(e_{12}(v)^2 + \frac{\theta_0}{u-v}\big(2e_{12}(v)^2 - e_{12}(u)e_{12}(v) - e_{12}(v)e_{12}(u)\big) \\
&\qquad\qquad + \frac{1}{(u-v)^2}\big(e_{12}(v) - e_{12}(u)\big)^2\Big)
\end{aligned}
$$

so

$$
\begin{aligned}
&e_{12}(v - \tfrac{\theta_0}{2})^2 h_1(u) \\
&\quad = h_1(u)\Big(e_{12}(v - \tfrac{\theta_0}{2})^2 + \frac{\theta_0}{u - v + \frac{\theta_0}{2}}\big(2e_{12}(v - \tfrac{\theta_0}{2})^2 - e_{12}(u)e_{12}(v - \tfrac{\theta_0}{2}) - e_{12}(v - \tfrac{\theta_0}{2})e_{12}(u)\big) \\
&\qquad\quad + \frac{1}{(u - v + \frac{\theta_0}{2})^2}\big(e_{12}(v - \tfrac{\theta_0}{2}) - e_{12}(u)\big)^2\Big)
\end{aligned}
$$

Furthermore, since $e_{12}^{(1)}h_1(u) = h_1(u)e_{12}^{(1)} + \theta_0 h_1(u)e_{12}(u)$, then

$$
\begin{aligned}
e_{12}^{(1)}e_{12}(v)h_1(u) &= e_{12}^{(1)}\Big(h_1(u)e_{12}(v) + \frac{\theta_0}{u-v}h_1(u)\big(e_{12}(v) - e_{12}(u)\big)\Big) \\
&= \big(h_1(u)e_{12}^{(1)} + \theta_0 h_1(u)e_{12}(u)\big)e_{12}(v) \\
&\qquad + \frac{\theta_0}{u-v}\big(h_1(u)e_{12}^{(1)} + \theta_0 h_1(u)e_{12}(u)\big)\big(e_{12}(v) - e_{12}(u)\big) \\
&= h_1(u)\big(e_{12}^{(1)}e_{12}(v) + \theta_0 e_{12}(u)e_{12}(v)\big) + \frac{\theta_0}{u-v}h_1(u)\big(e_{12}^{(1)} + \theta_0 e_{12}(u)\big)\big(e_{12}(v) - e_{12}(u)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
e_{12}(v)e_{12}^{(1)}h_1(u) &= e_{12}(v)\big(h_1(u)e_{12}^{(1)} + \theta_0 h_1(u)e_{12}(u)\big) \\
&= \Big(h_1(u)e_{12}(v) + \frac{\theta_0}{u-v}h_1(u)\big(e_{12}(v) - e_{12}(u)\big)\Big)e_{12}^{(1)} \\
&\qquad + \theta_0\Big(h_1(u)e_{12}(v) + \frac{\theta_0}{u-v}h_1(u)\big(e_{12}(v) - e_{12}(u)\big)\Big)e_{12}(u) \\
&= h_1(u)\big(e_{12}(v)e_{12}^{(1)} + \theta_0 e_{12}(v)e_{12}(u)\big) + \frac{\theta_0}{u-v}h_1(u)\big(e_{12}(v) - e_{12}(u)\big)\big(e_{12}^{(1)} + \theta_0 e_{12}(u)\big),
\end{aligned}
$$

so

$$\big[e_{12}^{(1)}, e_{12}(v)\big] h_1(u) = h_1(u)\big[e_{12}^{(1)} + \theta_0 e_{12}(u), e_{12}(v)\big] + \frac{\theta_0}{u-v} h_1(u)\big[e_{12}^{(1)} + \theta_0 e_{12}(u), e_{12}(v) - e_{12}(u)\big]$$

$(A.4)$ $-\theta_0 T_{13}(v)T_{11}(u)$ is given by $h_1(v)\big(\theta_0\big[e_{12}^{(1)}, e_{12}(v)\big] + e_{12}(v)^2\big)h_1(u)$ which is equal to

$$\begin{aligned}
&h_1(u)h_1(v)\big[\theta_0 e_{12}^{(1)} + e_{12}(u), e_{12}(v)\big] + \frac{1}{u-v} h_1(u)h_1(v)\big[e_{12}^{(1)} + \theta_0 e_{12}(u), e_{12}(v) - e_{12}(u)\big] \\
&\qquad + h_1(u)h_1(v)\Big(e_{12}(v)^2 + \frac{\theta_0}{u-v}\big(2e_{12}(v)^2 - e_{12}(u)e_{12}(v) - e_{12}(v)e_{12}(u)\big) \\
&\qquad\qquad + \frac{1}{(u-v)^2}\big(e_{12}(v) - e_{12}(u)\big)^2\Big)
\end{aligned}$$

Thus,

$$\begin{aligned}
\big[e_{12}(u), e_{12}(v)\big] = &-\frac{\theta_0}{u-v}\big(e_{12}(v) - e_{12}(u)\big)\big(e_{12}(v) + e_{12}(u)\big)\Big) \\
&+ \frac{\theta_0}{u-v}\big(e_{12}(v)e_{12}(u) - e_{12}(u)e_{12}(v)\big) - \frac{1}{(u-v)^2}\big(e_{12}(u) - e_{12}(v)\big)^2 \\
&- \theta_0\big(\big[e_{12}^{(1)}, e_{12}(v)\big] + \theta_0 e_{12}(v)^2\big) - \frac{1}{u-v}\big(\big[e_{12}^{(1)}, e_{12}(v)\big] + \theta_0 e_{12}(v)^2\big) \\
&+ \frac{1}{u-v}\big(\big[e_{12}^{(1)}, e_{12}(u)\big] + \theta_0 e_{12}(u)^2\big) \\
&\quad + \big[\theta_0 e_{12}^{(1)} + e_{12}(u), e_{12}(v)\big] + \frac{1}{u-v}\big[e_{12}^{(1)} + \theta_0 e_{12}(u), e_{12}(v) - e_{12}(u)\big] \\
&+ e_{12}(v)^2 + \frac{\theta_0}{u-v}\big(2e_{12}(v)^2 - e_{12}(u)e_{12}(v) - e_{12}(v)e_{12}(u)\big) + \frac{1}{(u-v)^2}\big(e_{12}(v) - e_{12}(u)\big)^2
\end{aligned}$$

So:

$$\big[e_{12}(u), e_{12}(v)\big] = E_0(u,v) + \frac{1}{u-v}E_1(u,v) + \frac{1}{(u-v)^2}E_2(u,v),$$

where

$$E_1(u,v) = \big[e_{12}(u), e_{12}(v)\big],$$

$$E_2(u,v) = \theta_0\big(e_{12}(u) - e_{12}(v)\big)\big(e_{12}(v) + e_{12}(u)\big) + \theta_0\big(e_{12}(v)e_{12}(u) - e_{12}(u)e_{12}(v)\big)$$

$\square$

Chapter 3

Periplectic Super Yangians

3.1 Strange Lie Superalgebras

Let us fix our ground field to be $\mathbb{C}$. Unless otherwise stated, all linear algebraic notions are formulated with respect to this fixed base field $\mathbb{C}$ (i.e., vector space = $\mathbb{C}$-vector space, algebra = $\mathbb{C}$-algebra, linear map = $\mathbb{C}$-linear map, $\otimes = \otimes_{\mathbb{C}}$, etc...).

Given a positive integer N we define the set $I_N = \{\, i \in \mathbb{Z} \setminus \{0\} \mid -N \leq i \leq N \,\}$ and introduce the *gradation index* $[\cdot]\colon I_N \to \mathbb{Z}_2$ defined as $[i] = 0$ for $i > 0$ and $[i] = 1$ for $i < 0$. Denote $\mathbb{C}^{N|N}$ to be the vector space $\mathbb{C}^{2N}$ equipped with the $\mathbb{Z}_2$-grading by assigning $[e_i] = [i]$, where $\{e_i\}_{i \in I_N}$ is the standard ordered basis of $\mathbb{C}^{2N}$ enumerated from $-N$ to N omitting 0. Note that the gradation index comes equipped with the property

$$(-1)^{[i]} = -(-1)^{[-i]}$$

and consequently the relations

$$(-1)^{[i][j]} = (-1)^{[-i][j]+[j]} \quad \text{and} \quad (-1)^{[i][j]+[-i][-j]} = (-1)^{[i]+[-j]}$$

for $i, j \in I_N$.

Consider now the super vector space $M_{N|N}(\mathbb{C}) \cong \operatorname{End} \mathbb{C}^{N|N}$ of $2N \times 2N$ matrices and let its canonical homogeneous basis given by the collection of standard matrix units be denoted $\{E_{ij}\}_{i,j \in I_N}$ where $[E_{ij}] = [i] + [j]$. We denote by $\mathfrak{gl}_{N|N}$ to mean the Lie superalgebra over $\mathbb{C}$ whose underlying super vector space is $M_{N|N}(\mathbb{C})$ and which is equipped with the Lie superbracket $[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il}(-1)^{([i]+[j])([k]+[l])} E_{kj}$, where δ_{ij} is the Kronecker delta. Writing an element $X \in \mathfrak{gl}_{N|N}$ as a supermatrix

$$\begin{pmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{pmatrix}$$

where $X_{ij} \in M_{N,N}(\mathbb{C})$, we recall that the *super trace* is the mapping $\operatorname{str}\colon \mathfrak{gl}_{N|N} \to \mathbb{C}$ given by $X \mapsto \operatorname{tr}(X_{00}) - \operatorname{tr}(X_{11})$. In fact, the bilinear form on $\mathfrak{gl}_{N|N}$ sending $(X, Y) \mapsto \operatorname{str}(XY)$ induced by the super trace is non-degenerate, super symmetric, and $\mathfrak{gl}_{N|N}$-invariant. The dual basis $\{E_{ij}^*\}_{i,j \in I_N}$ of the homogeneous basis $\{E_{ij}\}_{i,j \in I_N}$ of $\mathfrak{gl}_{N|N}$ with respect to this bilinear form is given by $E_{ij}^* = (-1)^{[i]} E_{ji}$. Therefore, the *quadratic Casimir element* of $\mathfrak{gl}_{N|N}$ is

$$\Omega = \sum_{i,j \in I_N} (-1)^{[E_{ij}]} E_{ij} \otimes (-1)^{[i]} E_{ji} = \sum_{i,j \in I_N} (-1)^{[j]} E_{ij} \otimes E_{ji}.$$

Also recall that the *super-transpose* is the morphism $(-)^{st}\colon \mathfrak{gl}_{N|N} \to \mathfrak{gl}_{N|N}$ given by $E_{ij} \mapsto (-1)^{[i]([j]+1)} E_{ji}$. Let us now consider the involutory automorphism of $\mathfrak{gl}_{N|N}$ given by

$$\iota^P := -\eta \circ (-)^{st}, \quad \text{where} \quad \eta\colon E_{ij} \mapsto E_{-i,-j}. \tag{3.1}$$

Furthermore, the map $j^P := -\iota^P$, is an anti-automorphism of the superalgebra $\operatorname{End} \mathbb{C}^{N|N}$ and so therefore satisfies $j^P(A_1 A_2) = (-1)^{[A_1][A_2]} j^P(A_2) j^P(A_1)$ for homogeneous matrices $A_1, A_2 \in \operatorname{End} \mathbb{C}^{N|N}$.

If we denote $(\mathfrak{gl}_{N|N})^{\iota^P}$ to be the collection of fixed elements under the automorphism ι^P (that is, $(\mathfrak{gl}_{N|N})^{\iota^P} = \{X \in \mathfrak{gl}_{N|N} \mid \iota^P(X) = X\}$), then $(\mathfrak{gl}_{N|N})^{\iota^P}$ carries a natural Lie sub-superalgebra structure. From this, we have the following definition:

Definition 3.1.1. The *periplectic Lie superalgebra* $\mathfrak{p}_N$ is the Lie sub-superalgebra of fixed elements of $\mathfrak{gl}_{N|N}$ under the involutive automorphism ι^P.

The periplectic Lie superalgebra is spanned by the elements

$$\mathsf{E}_{ij} := E_{ij} + \iota^P(E_{ij}) = E_{ij} - (-1)^{[i]([j]+1)} E_{-j,-i} \tag{3.2}$$

for $i, j \in I_N$ and these satisfy the relations

$$\begin{aligned}[\mathsf{E}_{ij}, \mathsf{E}_{kl}] = \delta_{jk}\mathsf{E}_{il} - \delta_{il}(-1)^{([i]+[j])([k]+[l])}\mathsf{E}_{kj} \\ - \delta_{i,-k}(-1)^{[i]([j]+1)}\mathsf{E}_{-j,l} + \delta_{j,-l}(-1)^{([i]+[j])[k]}\mathsf{E}_{k,-i}\end{aligned} \tag{3.3}$$

and

$$\mathsf{E}_{ij} + (-1)^{[i]([j]+1)}\mathsf{E}_{-j,-i} = 0. \tag{3.4}$$

Note that a basis of $\mathfrak{p}_N$ is provided by those elements E_{ij} such that the indices $i, j \in I_N$ satisfy either one of

$$1 \leq |i| < |j| \leq N, \quad 1 \leq i = j \leq N, \quad \text{or} \quad 1 \leq -i = j \leq N.$$

We note that $\mathsf{E}_{ij} = 0$ for $-N \leq -i = j \leq -1$.

3.2 Yangians of Type P

3.2.1 Definitions and Preliminaries

Let us now denote $\mathfrak{gl}_{N|N}[z]$ to be the *polynomial current Lie superalgebra* associated to $\mathfrak{gl}_{N|N}$. That is, $\mathfrak{gl}_{N|N}[z] = \mathfrak{gl}_{N|N} \otimes \mathbb{C}[z]$ as a super vector space equipped with the

Lie superbracket

$$[X \otimes f(z), Y \otimes g(z)] = [X, Y] \otimes f(z)g(z) \quad \text{for} \;\; X, Y \in \mathfrak{gl}_{N|N} \;\; \text{and} \;\; f(z), g(z) \in \mathbb{C}[z]. \tag{3.1}$$

Equivalently, $\mathfrak{gl}_{N|N}[z]$ is the Lie superalgebra of polynomial maps $f\colon \mathbb{C} \to \mathfrak{gl}_{N|N}$ with Lie superbracket given point-wise.

Now, the involution ι^P may be extended to an involutory automorphism of $\mathfrak{gl}_{N|N}[z]$, also denoted ι^P, by assigning

$$\iota^P(X \otimes f(z)) = \iota^P(X) \otimes f(-z) \quad \text{for} \;\; f(z) \in \mathbb{C}[z]. \tag{3.2}$$

Definition 3.2.1. The *twisted current Lie superalgebra* $\mathfrak{gl}_{N|N}[z]^{\iota^P}$ is defined as the Lie sub-superalgebra of $\mathfrak{gl}_{N|N}[z]$ fixed by the involutive automorphism ι^P.

Using the identification $Xz^n = X \otimes z^n$ for elements in $\mathfrak{gl}_{N|N}[z]$, the elements

$$\mathsf{E}_{ij}^{(n)}(z) := E_{ij}z^n + \iota^P(E_{ij})(-z)^n \tag{3.3}$$

generate $\mathfrak{gl}_{N|N}[z]^{\iota^P}$ and satisfy the relations

$$\begin{aligned}\left[\mathsf{E}_{ij}^{(m)}(z), \mathsf{E}_{kl}^{(n)}(z)\right] &= \delta_{jk}\mathsf{E}_{il}^{(m+n)}(z) - \delta_{il}(-1)^{([i]+[j])([k]+[l])}\mathsf{E}_{kj}^{(m+n)}(z) \\ &\quad - \delta_{i,-k}(-1)^{[i]([j]+1)+m}\mathsf{E}_{-j,l}^{(m+n)}(z) + \delta_{j,-l}(-1)^{([i]+[j])[k]+m}\mathsf{E}_{k,-i}^{(m+n)}(z)\end{aligned} \tag{3.4}$$

and

$$\mathsf{E}_{ij}^{(n)}(z) + (-1)^{[i]([j]+1)+n}\mathsf{E}_{-j,-i}^{(n)}(z) = 0. \tag{3.5}$$

A basis of $\mathfrak{gl}_{N|N}[z]^{\iota^P}$ is given by those elements $\mathsf{E}_{ij}^{(n)}(z)$ such that the indices $i, j \in I_N$ and power $n \in \mathbb{N}$ satisfy either one of

$$1 \le |i| < |j| \le N, \; n \in \mathbb{N}; \quad 1 \le i = j \le N, \; n \in \mathbb{N};$$
$$1 \le -i = j \le N, \; n \in 2\mathbb{N}; \quad \text{or} \quad -N \le -i = j \le -1, \; n \in 2\mathbb{Z}^+ - 1.$$

Let the *super permutation operator* in $\mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2}$ be given by

$$P := \sum_{i,j=-N}^{N} (-1)^{[j]} E_{ij} \otimes E_{ji}. \tag{3.6}$$

By setting $\iota_1^P = \iota^P \otimes \mathrm{id}$ and $\iota_2^P = \mathrm{id} \otimes \iota^P$, we note that these maps satisfy

$$\iota_2^P(P) = -\iota_1^P(P) \quad \text{and} \quad (\iota_2^P \circ \iota_1^P)(P) = (\iota_1^P \circ \iota_2^P)(P) = -P.$$

Let us define

$$Q := (\iota_2^P)(P) = -\sum_{i,j=-N}^{N} (-1)^{[i][j]} E_{ij} \otimes E_{-i,-j}. \tag{3.7}$$

These operators satisfy the relations

$$P^2 = \mathrm{id}^{\otimes 2}, \quad PQ = Q, \quad QP = -Q, \quad \text{and} \quad Q^2 = 0. \tag{3.8}$$

Now, define the *quantum R-matrix* $R(u,v) \in \mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2}(u,v)$ to be the rational function in the formal parameters $u, v \in \mathbb{C}$ with coefficients in $\mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2}$ given by

$$R(u,v) := \mathrm{id}^{\otimes 2} - \frac{P}{u-v} + \frac{Q}{u+v}. \tag{3.9}$$

Regarding ι_1^P and ι_2^P as maps lifted to act on the space $\mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2}(u,v)$, we have

$$\iota_1^P(R(u,v)) = -R(u,-v), \tag{3.10}$$
$$\iota_2^P(R(u,v)) = -R(-u,v). \tag{3.11}$$

Letting $\mathcal{A}$ denote a superalgebra, we note that for the index $1 \le k \le m$ there is an injective? morphism of superalgebras

$$\begin{aligned} \varphi_k \colon \mathrm{End}\,\mathbb{C}^{N|N} \otimes \mathcal{A} &\to \left(\mathrm{End}\,\mathbb{C}^{N|N}\right)^{\otimes m} \otimes \mathcal{A} \\ \psi \otimes w &\mapsto \mathrm{id}^{\otimes(k-1)} \otimes \psi \otimes \mathrm{id}^{\otimes(m-k)} \otimes w, \end{aligned}$$

and set $X_k = \varphi_k(X)$ for an element $X \in \operatorname{End}\mathbb{C}^{N|N} \otimes \mathcal{A}$. If $\mathcal{A}$ is a formal power series super algebra or if $X = X(u)$ is a rational function in the formal parameter u with coefficients in $\operatorname{End}\mathbb{C}^{N|N} \otimes \mathcal{A}$, we shall write $X_k(u)$ instead of $X(u)_k$ for the element $\varphi_k(X(u))$.

Analogously, we will like to express elements of $\left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes 2}$ in $\left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes m}$ for some integer $m \geq 3$. For indices $1 \leq k < l \leq m$, we also have the injective morphism of super algebras

$$\begin{aligned}\varphi_{kl}\colon \left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes 2} &\to \left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes m}\\ a \otimes b &\mapsto \mathbf{1}^{\otimes(k-1)} \otimes a \otimes \mathbf{1}^{\otimes(l-k-1)} \otimes b \otimes \mathbf{1}^{\otimes(m-l)}\end{aligned}$$

and set $X_{kl} = \varphi_{kl}(X)$ for an element $X \in \left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes 2}$. Similarly, if $X = X(u)$ is a rational function in the formal parameter u with coefficients in $\left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes 2}$, then we write $X_{kl}(u)$ instead of $X(u)_{kl}$ for the element $\varphi_{kl}(X(u))$.

The quantum R-matrix (4.3) satisfies the *super quantum Yang-Baxter equation* (SQYBE):

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v). \tag{3.12}$$

Furthermore, the quantum R-matrix satisfies the conditions of *symmetry* and *unitarity*

$$PR(u,v)P = R(-v,-u), \tag{3.13}$$

$$R(u,v)R(-u,-v) = \left(1 - \frac{1}{(u-v)^2}\right)\mathrm{id}^{\otimes 2}. \tag{3.14}$$

Definition 3.2.2. The *extended super Yangian* $\mathbf{X}(\mathfrak{p}_N)$ of $\mathfrak{p}_N$ is the unital associative $\mathbb{C}$-superalgebra on generators $\{T_{ij}^{(n)} \mid i,j \in I_N,\ n \in \mathbb{Z}^+\}$, with $\mathbb{Z}_2$-grade $\left[T_{ij}^{(n)}\right] := [i]+[j]$ for all $n \in \mathbb{Z}^+$, subject to the defining *RTT-relation*

$$\begin{gathered}R(u,v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u,v),\\ \text{in}\quad \left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes 2} \otimes \mathbf{X}(\mathfrak{p}_N)[\![u^{\pm 1}, v^{\pm 1}]\!],\end{gathered} \tag{3.15}$$

where $R(u,v)$ is identified with $R(u,v)\otimes\mathbf{1}$ and

$$T(u) = \sum_{i,j=-N}^{N} E_{ij}\otimes T_{ij}(u) \in \left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes 2}\otimes \mathbf{X}(\mathfrak{p}_N)[\![u^{\pm1},v^{\pm1}]\!] \tag{3.16}$$

is the *generating matrix* consisting of the *generating series*

$$T_{ij}(u) = \sum_{n=0}^{\infty} T_{ij}^{(n)}u^{-n} \in \mathbf{X}(\mathfrak{p}_N)[\![u^{-1}]\!], \quad \text{where} \quad T_{ij}^{(0)} = \delta_{ij}\mathbf{1}. \tag{3.17}$$

Written in terms of power series, the RTT-relation has the form

$$\begin{aligned}\left[T_{ij}(u),T_{kl}(v)\right] &= \frac{1}{u-v}(-1)^{[i][k]+[i][l]+[k][l]}\left(T_{kj}(u)T_{il}(v)-T_{kj}(v)T_{il}(u)\right)\\ &\quad -\frac{1}{u+v}\Bigg(\delta_{i,-k}\sum_{p=-N}^{N}(-1)^{[i][l]+[l][p]+[p]}T_{pj}(u)T_{-p,l}(v)\\ &\qquad -\delta_{j,-l}\sum_{p=-N}^{N}(-1)^{[j]+[i][l]+[i][k]+[k][p]}T_{k,-p}(v)T_{ip}(u)\Bigg)\end{aligned} \tag{3.18}$$

in $\mathbf{X}(\mathfrak{p}_N)[\![u^{\pm1},v^{\pm1}]\!]$, where $[\,\cdot\,,\cdot\,]$ is understood as the Lie superbracket

$$\left[T_{ij}(u),T_{kl}(v)\right] = T_{ij}(u)T_{kl}(v)-(-1)^{([i]+[j])([k]+[l])}T_{kl}(v)T_{ij}(u).$$

Note that we may regard $T(u)$ in definition (3.2.2) as a formal power series in u^{-1} whose coefficients lie in $\operatorname{End}\mathbb{C}^{N|N}\otimes\mathbf{X}(\mathfrak{p}_N)$. Since the constant term of the power series $T(u)$ is invertible, the series itself is so, whose inverse shall be denoted $T(u)^{-1}$. Furthermore, let us define $T^{j^P}(u) = (j^P\otimes\mathbf{1})T(u)$ and $T_{ij}^{j^P}(u) = (-1)^{[i]([j]+1)}T_{-j,-i}(u)$ so that $T^{j^P}(u) = \sum_{i,j=-N}^{N} E_{ij}\otimes T_{ij}^{j^P}(u)$.

Consider now a formal series $f(u) = 1+\sum_{n=0}^{\infty} f_n u^{-n}\in\mathbb{C}[\![u^{-1}]\!]$. The mappings

$$j^p\colon T(u)\mapsto T^{j^P}(u) \tag{3.19}$$

$$\mu_f\colon T(u) \mapsto f(u)T(u) \tag{3.20}$$

define superalgebra automorphisms of $\mathbf{X}(\mathfrak{p}_N)$. Moreover, each mapping

$$\sigma\colon T(u) \mapsto T(-u) \tag{3.21}$$

$$S\colon T(u) \mapsto T(u)^{-1} \tag{3.22}$$

induces a superalgebra anti-automorphism of $\mathbf{X}(\mathfrak{p}_N)$.

Note that there exists a Hopf superalgebra structure on $\mathbf{X}(\mathfrak{p}_N)$ given by the comultiplication

$$\Delta\colon \mathbf{X}(\mathfrak{p}_N) \to \mathbf{X}(\mathfrak{p}_N) \otimes \mathbf{X}(\mathfrak{p}_N), \quad T(u) \mapsto T_{[1]}(u)T_{[2]}(u) \tag{3.23}$$

the counit

$$\varepsilon\colon \mathbf{X}(\mathfrak{p}_N) \to \mathbb{C}, \quad T(u) \mapsto \mathbb{1} \tag{3.24}$$

and the antipode

$$S\colon \mathbf{X}(\mathfrak{p}_N) \to \mathbf{X}(\mathfrak{p}_N), \quad T(u) \mapsto T(u)^{-1}. \tag{3.25}$$

Note that the comultiplication may be written in terms of generating series as the map $\Delta\colon T_{ij}(u) \mapsto \sum_{k=-N}^{N}(-1)^{([i]+[k])([j]+[k])}T_{ik}(u) \otimes T_{kj}(u)$.

Let us define $Z(u) := T^{j^P}(u)T(-u)$ and further consider the series $\mathcal{Z}(u)$ lying in $\mathbf{X}(\mathfrak{p}_N)[\![u^{-1}]\!]$ such that $\mathrm{id} \otimes \mathcal{Z}(u) = Z(u)$. Multiplying both sides of the RTT-relation by $-(u+v)$ and setting $v = -u$ yields the equation

$$QT_1(u)T_2(-u) = T_2(-u)T_1(u)Q. \tag{3.26}$$

Using that $QT_1(u) = QT_2^{j^P}(u)$ and $T_1(u)Q = T_2^{j^P}(u)Q$ and applying the map j^P to the first tensor factor of (3.26), we deduce

$$P \otimes \mathcal{Z}(u) = PT_2^{j^P}(u)T_2(-u) = T_2(-u)T_2^{j^P}(u)P.$$

Multiplying the above on the right by P, we obtain $\mathrm{id}^{\otimes 2} \otimes \mathcal{Z}(u) = T_2(-u)T_2^{j^P}(u)$.

Therefore,

$$Z(u) = T^{j^P}(u)T(-u) = T(-u)T^{j^P}(u), \tag{3.27}$$

or rather put,

$$\delta_{ik}\mathcal{Z}(u) = \sum_{q=-N}^{N} (-1)^{[i]+[i][k]+[k][q]+[q]} T_{-q,-i}(u)T_{qk}(-u)$$

$$= \sum_{q=-N}^{N} (-1)^{[i][k]+[i][q]} T_{iq}(-u)T_{-k,-q}(u), \tag{3.28}$$

where

$$\mathcal{Z}(u) = \mathbf{1} + \sum_{n=1}^{\infty} \mathcal{Z}_n u^{-n} \in \mathbf{X}(\mathfrak{p}_N)[\![u^{-1}]\!] \tag{3.29}$$

We note that the coefficients of $\mathcal{Z}(u)$ are homogeneous of even degree, so all coefficients of $\mathcal{Z}(u)$ lies within the even subalgebra of $\mathbf{X}(\mathfrak{p}_N)$. Let us denote $(\mathcal{Z}(u) - \mathbf{1})$ to mean the two-sided ideal of $\mathbf{X}(\mathfrak{p}_N)$ generated by the coefficients of $\mathcal{Z}(u) - \mathbf{1}$ and let $\mathbf{ZX}(\mathfrak{p}_N)$ denote the subalgebra generated by the coefficients of $\mathcal{Z}(u)$. The ideal $(\mathcal{Z}(u) - \mathbf{1})$ is graded as it is generated by homogeneous elements.

Proposition 3.2.3. *The coefficients of the series $\mathcal{Z}(u) \in \mathbf{1} + u^{-1}\mathbf{X}(\mathfrak{p}_N)[\![u^{-1}]\!]$ given by the equation $T^{j^P}(u)T(-u) = T(-u)T^{j^P}(u) = \mathrm{id} \otimes \mathcal{Z}(u)$ lie in the center of $\mathbf{X}(\mathfrak{p}_N)$. Furthermore,*

$$\Delta \colon \mathcal{Z}(u) \mapsto \mathcal{Z}(u) \otimes \mathcal{Z}(u), \tag{3.30}$$

where Δ is the comultiplication map (3.23), and $(\mathcal{Z}(u) - \mathbf{1})$ is a graded bi-ideal.

Proof. First observe

$$(\mathrm{id} \otimes Z(u))T_2(v) = T_1^{j^P}(u)T_1(-u)T_2(v) = T_1^{j^P}(u)(R(-u,v))^{-1}T_2(v)T_1(-u)R(-u,v),$$

by the RTT-relation. Now, by applying j^P to the first tensor factor of the RTT-relation (4.7) and using properties (3.10) and (4.5), we obtain

$$T_1^{j^P}(u)R(-u,v)^{-1}T_2(v) = T_2(v)R(-u,v)^{-1}T_1^{j^P}(u).$$

Therefore,

$$\begin{aligned}(\mathrm{id} \otimes Z(u))T_2(v) &= T_2(v)R(-u,v)^{-1}T_1^{j^P}(u)T_1(-u)R(-u,v) \\ &= T_2(v)R(-u,v)^{-1}(\mathrm{id} \otimes Z(u))R(-u,v) = T_2(v)(\mathrm{id} \otimes Z(u)),\end{aligned}$$

since $(\mathrm{id} \otimes Z(u))$ commutes with $R(-u,v) \otimes \mathbf{1}$. Furthermore, $\Delta\colon \mathcal{Z}(u) \mapsto \mathcal{Z}(u) \otimes \mathcal{Z}(u)$ is readily verified since $\Delta(\mathcal{Z}(u))$ is provided by the sum

$$\begin{aligned}&\sum_{q=-N}^{N} (-1)^{[k]+[k][q]} \Delta\left(T_{kq}(-u)T_{-k,-q}(u)\right) \\ &\quad = \sum_{a,b,q=-N}^{N} (-1)^{\substack{[k]+[k][q]+([k]+[a])([q]+[a]) \\ +([-k]+[b])([-q]+[b])}} \big(T_{ka}(-u) \otimes T_{aq}(-u)\big)\big(T_{-k,b}(u) \otimes T_{b,-q}(u)\big) \\ &\quad = \sum_{a,b,q=-N}^{N} (-1)^{\substack{[a]([k]+[q]+1)+[b]([k]+[q]+1) \\ +[k][q]+[-q]+([a]+[q])([-k]+[b])}} T_{ka}(-u)T_{-k,b}(u) \otimes T_{aq}(-u)T_{b,-q}(u) \\ &\quad = \sum_{a,b=-N}^{N} (-1)^{[k][-a]} T_{ka}(-u)T_{-k,-a}(u) \otimes \delta_{a,-b}\mathcal{Z}(u) = \mathcal{Z}(u) \otimes \mathcal{Z}(u).\end{aligned}$$

Let us set $\mathcal{I} = (\mathcal{Z}(u) - \mathbf{1})$. One may verify that

$$\varepsilon\colon \mathcal{Z}(u) \mapsto 1,$$

and so $\varepsilon(\mathcal{I}) = 0$. Moreover, since $\Delta(\mathcal{Z}_n) = \sum_{a+b=n} \mathcal{Z}_a \otimes \mathcal{Z}_b$ (where $\mathcal{Z}_0 = \mathbf{1}$), then for $X \in \mathbf{X}(\mathfrak{p}_N)$ we have $\Delta(X\mathcal{Z}_n), \Delta(\mathcal{Z}_n X) \in \mathcal{I} \otimes \mathbf{X}(\mathfrak{p}_N) + \mathbf{X}(\mathfrak{p}_N) \otimes \mathcal{I}$, so $\mathcal{I}$ is a coideal. $\square$

Now, by the axioms of the Hopf superalgebra structure, the image of $\mathcal{Z}(u)$ under the antipode is given by

$$S\colon \mathcal{Z}(u) \mapsto \mathcal{Z}(u)^{-1}. \tag{3.31}$$

We therefore have the immediate corollary:

Corollary 3.2.4. $\mathbf{ZX}(\mathfrak{p}_N)$ *is a sub-Hopf superalgebra and* $(\mathcal{Z}(u) - \mathbf{1})$ *is a graded Hopf ideal of* $\mathbf{X}(\mathfrak{p}_N)$.

By equation (3.27) we know that the inverse of $T(u)$ is given by

$$T(u)^{-1} = \mathcal{Z}(-u)^{-1} T^{j^P}(-u), \tag{3.32}$$

so the antipode on $\mathbf{X}(\mathfrak{p}_N)$ is the mapping

$$S\colon T(u) \mapsto \mathcal{Z}(-u)^{-1} T^{j^P}(-u). \tag{3.33}$$

In particular, the square of the antipode is given by

$$S^2\colon T(u) \mapsto \frac{\mathcal{Z}(-u)}{\mathcal{Z}(u)} T(u). \tag{3.34}$$

Definition 3.2.5. The *super Yangian* $\mathbf{Y}(\mathfrak{p}_N)$ of $\mathfrak{p}_N$ is the quotient of $\mathbf{X}(\mathfrak{p}_N)$ by the two-sided ideal $(\mathcal{Z}(u) - \mathbf{1})$, i.e.,

$$\mathbf{Y}(\mathfrak{p}_N) := \mathbf{X}(\mathfrak{p}_N)/(\mathcal{Z}(u) - \mathbf{1}).$$

Letting $\mathcal{T}_{ij}^{(n)}$ denote the image of the generator $T_{ij}^{(n)}$ under the canonical projection $\mathbf{X}(\mathfrak{p}_N) \twoheadrightarrow \mathbf{Y}(\mathfrak{p}_N)$, the super Yangian $\mathbf{Y}(\mathfrak{p}_N)$ can be equivalently described as the unital associative $\mathbb{C}$-superalgebra on generators $\{\mathcal{T}_{ij}^{(n)} \mid i, j \in I_N,\ n \in \mathbb{Z}^+\}$, with $\mathbb{Z}_2$-grade $\left[\mathcal{T}_{ij}^{(n)}\right] := [i]+[j]$ for all $n \in \mathbb{Z}^+$, subject to the relations

$$\begin{aligned} &R(u,v)\mathcal{T}_1(u)\mathcal{T}_2(v) = \mathcal{T}_2(v)\mathcal{T}_1(u)R(u,v) \\ &\text{in} \quad \left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes 2} \otimes \mathbf{Y}(\mathfrak{p}_N)[\![u^{\pm 1}, v^{\pm 1}]\!] \end{aligned} \tag{3.35}$$

$$\text{and} \quad \mathcal{T}(-u)\mathcal{T}^{j^P}(u) = \mathbb{1} \quad \text{in} \quad \left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes 2} \otimes \mathbf{Y}(\mathfrak{p}_N)[\![u^{-1}]\!], \tag{3.36}$$

where $R(u,v)$ is identified with $R(u,v)\otimes\mathbf{1}$, $\mathcal{T}(u) = \sum_{i,j=-N}^{N} E_{ij}\otimes\mathcal{T}_{ij}(u)$ is the *generating matrix*, and $\mathcal{T}_{ij}(u) = \delta_{ij}\mathbf{1} + \sum_{n=1}^{\infty} \mathcal{T}_{ij}^{(n)} u^{-n}$ is the *generating series.*

Since $(\mathcal{Z}(u) - \mathbf{1})$ is a graded Hopf ideal, the Yangian $\mathbf{Y}(\mathfrak{p}_N)$ has a unique Hopf superalgebra structure such that the canonical projection $\mathbf{X}(\mathfrak{p}_N) \twoheadrightarrow \mathbf{Y}(\mathfrak{p}_N)$ is a morphism of Hopf superalgebras. That is, the Hopf superalgebra structure on $\mathbf{Y}(\mathfrak{p}_N)$ is given by

$$\Delta\colon \mathbf{Y}(\mathfrak{p}_N) \to \mathbf{Y}(\mathfrak{p}_N) \otimes \mathbf{Y}(\mathfrak{p}_N), \quad \mathcal{T}(u) \mapsto \mathcal{T}_{[1]}(u)\mathcal{T}_{[2]}(u) \tag{3.37}$$

the counit

$$\varepsilon\colon \mathbf{Y}(\mathfrak{p}_N) \to \mathbb{C}, \quad \mathcal{T}(u) \mapsto \mathbb{1} \tag{3.38}$$

and the antipode

$$S\colon \mathbf{Y}(\mathfrak{p}_N) \to \mathbf{Y}(\mathfrak{p}_N), \quad \mathcal{T}(u) \mapsto \mathcal{T}(u)^{-1} \tag{3.39}$$

In particular, we see that the antipode of $\mathbf{Y}(\mathfrak{p}_N)$ is an involutory anti-automorphism given by

$$S\colon \mathcal{T}(u) \mapsto \mathcal{T}^{j^P}(-u). \tag{3.40}$$

3.3 Poincaré-Birkhoff-Witt Theorem

Let us now introduce an ascending algebra filtration $\mathbf{F} = \{\mathbf{F}_n\}_{n\in\mathbb{N}}$ on $\mathbf{Y}(\mathfrak{p}_N)$ by declaring $\deg_{\mathbf{F}} \mathcal{T}_{ij}^{(n)} = n-1$ for $i,j \in I_N$, $n \in \mathbb{Z}^+$. That is, the filtration is given by the assignment

$$\mathbf{F}_n = \operatorname{span}_{\mathbb{C}} \left\{ \textstyle\prod_{\alpha=1}^{\gamma_\alpha} \mathcal{T}_{i_\alpha j_\alpha}^{(k_\alpha)} \,\middle|\, \gamma_\alpha \in \mathbb{Z}^+,\ i_\alpha, j_\alpha \in I_N,\ \sum_{\alpha=1}^{\gamma_\alpha} k_\alpha \leq n + \gamma_\alpha \right\}$$

for $n \in \mathbb{Z}^+$ and

$$\mathbf{F}_0 = \operatorname{span}_{\mathbb{C}} \left\{ \textstyle\prod_{\alpha=1}^{\gamma_\alpha} \mathcal{T}_{i_\alpha j_\alpha}^{(k_\alpha)} \,\middle|\, \gamma_\alpha \in \mathbb{Z}^+,\ i_\alpha, j_\alpha \in I_N,\ k_\alpha \in \{0,1\} \right\}.$$

We may therefore consider the associated graded algebra of $mbY(\mathfrak{p}_N)$:

$$\operatorname{gr} \mathbf{Y}(\mathfrak{p}_N) = \bigoplus_{n\in\mathbb{N}} \operatorname{gr}_n \mathbf{Y}(\mathfrak{p}_N) = \bigoplus_{n\in\mathbb{N}} \mathbf{F}_n/\mathbf{F}_{n-1}.$$

We note that $\operatorname{gr} \mathbf{Y}(\mathfrak{p}_N)$ inherits a $\mathbb{Z}_2$-graded structure from $\mathbf{Y}(\mathfrak{p}_N)$ by assigning the $\mathbb{Z}_2$-grade $[i]+[j]$ to the image of $\mathcal{T}_{ij}^{(n)}$ in $\operatorname{gr}_{n-1} \mathbf{Y}(\mathfrak{p}_N)$. Let us denote this image by $\overline{\mathcal{T}}_{ij}^{(n)}$.

Note that the ascending algebra filtration on $\mathbf{Y}(\mathfrak{p}_N)$ endows one on its tensor product $\mathbf{Y}(\mathfrak{p}_N)^{\otimes 2}$ by setting $\mathbf{F}_n^{\otimes 2} = \bigoplus_{i+j=n} \mathbf{F}_i \otimes \mathbf{F}_j$. Moreover, the Hopf superalgebra structure on $\mathbf{Y}(\mathfrak{p}_N)$ induces one on its associated graded $\operatorname{gr}\mathbf{Y}(\mathfrak{p}_N)$ with respect to these filtrations. Its structure maps are given by the comultiplication

$$\operatorname{gr}\Delta\colon \operatorname{gr}\mathbf{Y}(\mathfrak{p}_N) \to \operatorname{gr}\mathbf{Y}(\mathfrak{p}_N) \otimes \operatorname{gr}\mathbf{Y}(\mathfrak{p}_N), \quad \overline{\mathcal{T}}_{ij}^{(n)} \mapsto \overline{\mathcal{T}}_{ij}^{(n)} \otimes \bar{\mathbf{1}} + \bar{\mathbf{1}} \otimes \overline{\mathcal{T}}_{ij}^{(n)}, \quad n \in \mathbb{Z}^+$$

the counit

$$\operatorname{gr}\varepsilon\colon \operatorname{gr}\mathbf{Y}(\mathfrak{p}_N) \to \mathbb{C}, \quad \overline{\mathcal{T}}_{ij}^{(n)} \mapsto \delta_{0n}\delta_{ij}$$

and the antipode

$$\operatorname{gr}S\colon \operatorname{gr}\mathbf{Y}(\mathfrak{p}_N) \to \operatorname{gr}\mathbf{Y}(\mathfrak{p}_N), \quad \overline{\mathcal{T}}_{ij}^{(n)} \mapsto -\overline{\mathcal{T}}_{ij}^{(n)}, \quad n \in \mathbb{Z}^+.$$

Letting $\mathfrak{U}(\mathfrak{g})$ denote the universal enveloping algebra of a Lie superalgebra $\mathfrak{g}$, we recall that it is equipped with a Hopf superalgebra structure given by the comultiplication $\Delta\colon \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g}),\ X \mapsto X \otimes 1 + 1 \otimes X$, counit $\varepsilon\colon \mathfrak{U}(\mathfrak{g}) \to \mathbb{C},\ X \mapsto 0$ and the antipode $S\colon \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}),\ X \mapsto -X$ for $X \in \mathfrak{g}$.

Furthermore, we note that $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^P})$ is $\mathbb{N}$-graded with respect to the polynomial degree in z. We now have the following:

Proposition 3.3.1. *There is a surjective morphism of $\mathbb{N}$-graded Hopf superalgebras $\Psi\colon \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^P}) \to \operatorname{gr} Y(\mathfrak{p}_N)$ defined by the assignment*

$$\Psi\colon \mathsf{E}_{ij}^{(n-1)}(z) \mapsto -(-1)^{[i]}\overline{\mathcal{T}}_{ji}^{(n)}. \tag{3.1}$$

Proof. Using the expansion

$$\frac{1}{u \mp v} = \frac{u^{-1}}{1 \mp u^{-1}v} = u^{-1}\sum_{c=0}^{\infty}(\pm u^{-1}v)^c,$$

then the relations (3.35) are written as

$$\sum_{m,n=0}^{\infty}\big[\mathcal{T}_{ij}^{(m)},\mathcal{T}_{kl}^{(n)}\big]u^{-m}v^{-n}$$
$$=(-1)^{[i][k]+[i][l]+[k][l]}\sum_{a,b,c=0}^{\infty}\left(\mathcal{T}_{kj}^{(a)}\mathcal{T}_{il}^{(b)}-\mathcal{T}_{kj}^{(b)}\mathcal{T}_{il}^{(a)}\right)u^{-a-c-1}v^{c-b}$$
$$-\delta_{i,-k}\sum_{a,b,c=0}^{\infty}\sum_{p=-N}^{N}(-1)^{c+[i][l]+[l][p]+[p]}\mathcal{T}_{pj}^{(a)}\mathcal{T}_{-p,l}^{(b)}u^{-a-c-1}v^{c-b}$$
$$+\delta_{j,-l}\sum_{a,b,c=0}^{\infty}\sum_{p=-N}^{N}(-1)^{c+[j]+[i][l]+[i][k]+[k][p]}\mathcal{T}_{k,-p}^{(b)}\mathcal{T}_{ip}^{(a)}u^{-a-c-1}v^{c-b}.$$

The coefficient of $u^{-m}v^{-n}$ is therefore given by

$$\big[\mathcal{T}_{ij}^{(m)},\mathcal{T}_{kl}^{(n)}\big]=(-1)^{[i][k]+[i][l]+[k][l]}\sum_{a+c=m}\sum_{b-c=n}\left(\mathcal{T}_{kj}^{(a-1)}\mathcal{T}_{il}^{(b)}-\mathcal{T}_{kj}^{(b)}\mathcal{T}_{il}^{(a-1)}\right)$$
$$-\delta_{i,-k}\sum_{p=-N}^{N}\sum_{a+c=m}\sum_{b-c=n}(-1)^{[i][l]+[l][p]+[p]+c}\mathcal{T}_{pj}^{(a-1)}\mathcal{T}_{-p,l}^{(b)}$$
$$+\delta_{j,-l}\sum_{p=-N}^{N}\sum_{a+c=m}\sum_{b-c=n}(-1)^{[j]+[i][l]+[i][k]+[k][p]+c}\mathcal{T}_{k,-p}^{(b)}\mathcal{T}_{ip}^{(a-1)},$$

where $a\geq 1$, $b,c\geq 0$ (and so $a\leq m$). Passing to the associated graded algebra, we obtain

$$\big[\overline{\mathcal{T}}_{ij}^{(m)},\overline{\mathcal{T}}_{kl}^{(n)}\big]=\delta_{jk}(-1)^{[i][k]+[i][l]+[k][l]}\overline{\mathcal{T}}_{il}^{(m+n-1)}-\delta_{il}(-1)^{[i]}\overline{\mathcal{T}}_{kj}^{(m+n-1)}$$
$$-\delta_{i,-k}(-1)^{[i][l]+[l][j]+[j]+m-1}\overline{\mathcal{T}}_{-j,l}^{(m+n-1)}+\delta_{j,-l}(-1)^{[j]+[i][l]+m-1}\overline{\mathcal{T}}_{k,-i}^{(m+n-1)}.$$

In particular, $\big[\Psi\big(\mathsf{E}_{ij}^{(m-1)}(z)\big),\Psi\big(\mathsf{E}_{kl}^{(n-1)}(z)\big)\big]$ is given by

$$(-1)^{[i]+[k]}\big[\overline{\mathcal{T}}_{ji}^{(m)},\overline{\mathcal{T}}_{lk}^{(n)}\big]$$
$$=(-1)^{[i]+[k]}\Big(\delta_{il}(-1)^{[j][l]+[j][k]+[k][l]}\overline{\mathcal{T}}_{jk}^{(m+n-1)}-\delta_{jk}(-1)^{[j]}\overline{\mathcal{T}}_{li}^{(m+n-1)}$$
$$-\delta_{j,-l}(-1)^{[j][k]+[k][i]+[i]+m-1}\overline{\mathcal{T}}_{-i,k}^{(m+n-1)}+\delta_{i,-k}(-1)^{[i]+[j][k]+m-1}\overline{\mathcal{T}}_{l,-j}^{(m+n-1)}\Big)$$
$$=\delta_{jk}\left(-(-1)^{[i]}\overline{\mathcal{T}}_{li}^{(m+n-1)}\right)-\delta_{il}(-1)^{([i]+[j])([k]+[l])}\left(-(-1)^{[k]}\overline{\mathcal{T}}_{jk}^{(m+n-1)}\right)$$

$$
\begin{aligned}
&\quad - \delta_{i,-k}(-1)^{[i]([j]+1)+m-1}\Big(-(-1)^{[-j]}\overline{\mathcal{T}}^{(m+n-1)}_{l,-j}\Big)\\
&\qquad + \delta_{j,-l}(-1)^{([i]+[j])[k]+m-1}\Big(-(-1)^{[k]}\overline{\mathcal{T}}^{(m+n-1)}_{-i,k}\Big)\\
&= \delta_{jk}\Psi\big(\mathsf{E}^{(m-1)}_{il}(z)\big) - \delta_{il}(-1)^{([i]+[j])([k]+[l])}\Psi\big(\mathsf{E}^{(m-1)}_{kj}(z)\big)\\
&\quad - \delta_{i,-k}(-1)^{[i]([j]+1)+m-1}\Psi\big(\mathsf{E}^{(m-1)}_{-j,l}(z)\big) + \delta_{j,-l}(-1)^{([i]+[j])[k]+m-1}\Psi\big(\mathsf{E}^{(m-1)}_{k,-i}(z)\big)
\end{aligned}
$$

Now, by relation (3.36), we have

$$
\sum_{i,j=-N}^{N} E_{ij}\otimes(-1)^{[i]([j]+1)}\mathcal{T}^{(n)}_{-j,-i} = (-1)^n\sum_{s=1}^{n}(-1)^s\sum\nolimits_{\sum_{i=1}^{s}k_i=n}\prod_{p=k_1}^{k_s}\Bigg(\sum_{i,j=-N}^{N}E_{ij}\otimes\mathcal{T}^{(k_p)}_{ij}\Bigg)
$$

and therefore,

$$
(-1)^{[i]([j]+1)}\overline{\mathcal{T}}^{(n)}_{-j,-i} = -(-1)^n\overline{\mathcal{T}}^{(n)}_{ij}. \tag{3.2}
$$

Rewriting the above, we have $(-1)^{[j]([i]+1)+[i]+n-1}\overline{\mathcal{T}}^{(n)}_{-i,-j} = (-1)^{[i]}\overline{\mathcal{T}}^{(n)}_{ji}$. Hence, we have

$$
\begin{aligned}
&\Psi\big(\mathsf{E}^{(n-1)}_{ij}(z)\big) + (-1)^{[i]([j]+1)+n-1}\Psi\big(E^{(n-1)}_{-j,-i}(z)\big)\\
&\qquad\qquad = -(-1)^{[i]}\overline{\mathcal{T}}^{(n)}_{ji} + (-1)^{[i]([j]+1)+[j]+n-1}\overline{\mathcal{T}}^{(n)}_{-i,-j} = 0.
\end{aligned}
$$

The mapping is seen to preserves the $\mathbb{Z}_2$-grading by definition and also preserves the Hopf superalgebra structures. Since the elements $\overline{\mathcal{T}}^{(n)}_{ij}$ generate $\operatorname{gr}\mathbf{Y}(\mathfrak{p}_N)$, the mapping is surjective. □

Let us consider now the *evaluation representation* of $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^P})$ at $a\in\mathbb{C}$ given by

$$
\begin{aligned}
\rho_a\colon \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^P}) &\to \operatorname{End}\mathbb{C}^{N|N}\\
\mathsf{E}^{(n)}_{ij}(z) &\mapsto a^n E_{ij} + (-a)^n\iota^P(E_{ij}).
\end{aligned}
\tag{3.3}
$$

For any complex numbers $a_1,\dots,a_n\in\mathbb{C}$, we may consider the following tensor product of the evaluation representations of $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^P})$:

$$
\rho_{a_1\to a_n} := (\otimes_{i=1}^{n}(\rho_{a_i}\otimes\rho_{-a_i}))\circ\Delta_{2n-1}, \tag{3.4}
$$

where $\Delta_{2n-1}\colon \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^P}) \to \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^P})^{\otimes 2n}$ is the unique $(2n-1)$-fold coproduct sending $X \in \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^P})$ to the element $\sum_X X_{(1)} \otimes X_{(2)} \otimes \cdots \otimes X_{(2n)}$ in Sweedler notation.

Lemma 3.3.2. *We have* $\bigcap_{n\in\mathbb{Z}^+} \bigcap_{(a_1,\dots,a_n)\in\mathbb{C}^n} \ker(\rho_{a_1 \to a_n}) = 0$ *in* $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^P})$.

Proof. Let us consider the vector representation $\widetilde{\rho}\colon \mathfrak{gl}_{N|N} \to \mathfrak{gl}(\mathbb{C}^{N|N})$. For any $a \in \mathbb{C}$, we have the corresponding evaluation morphism $\mathrm{ev}_a\colon \mathfrak{gl}_{N|N}[z] \to \mathfrak{gl}_{N|N}$ induced by the assignment $z \mapsto a$. We therefore may consider the evaluation representation of $\mathfrak{U}(\mathfrak{gl}_{N|N}[z])$ at $a \in \mathbb{C}$ as $\widetilde{\rho}_a := \widetilde{\rho} \circ \mathrm{ev}_a\colon \mathfrak{U}(\mathfrak{gl}_{N|N}[z]) \to \operatorname{End}\mathbb{C}^{N|N}$ as given by the assignment $E_{ij}z^n \mapsto a^n E_{ij}$.

By setting $\widetilde{\rho}_{a_1\to a_n} := (\bigotimes_{i=1}^n(\widetilde{\rho}_{a_i} \otimes \widetilde{\rho}_{-a_i})) \circ \Delta_{2n-1}$, it then suffices to prove that $\bigcap_{n\in\mathbb{Z}^+} \bigcap_{(a_1,\dots,a_n)\in\mathbb{C}^n} \ker(\widetilde{\rho}_{a_1\to a_n}) = 0$ in $\mathfrak{U}(\mathfrak{gl}_{N|N}[z])$. Indeed, since we we have a superalgebra embedding $i\colon \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^P}) \hookrightarrow \mathfrak{U}(\mathfrak{gl}_{N|N}[z])$, we note that $\rho_a = \widetilde{\rho}_a \circ i$. Consequently, if $\rho_{a_1\to a_n}(X) = 0$, then $\widetilde{\rho}_{a_1\to a_n}(X) = 0$ as well.

Let $X_1, \dots, X_{4N^2}$ be an ordered homogeneous basis of $\mathfrak{gl}_{N|N}$ such that $X_1 = \mathrm{id}$, and let us write $\chi_i = \widetilde{\rho}(X_i)$ for $i = 1, \dots, 4N^2$. There is an induced total ordering '$\preceq$' on the collection of basis elements $X_b z^m$ of $\mathfrak{gl}_{N|N}[z]$ such that $X_b z^m \preceq X_b z^{m+1}$ and $X_b z^m \preceq X_{b+1} z^n$ for all $m, n \in \mathbb{N}$. Consider now a nonzero element $A \in \mathfrak{U}(\mathfrak{gl}_{N|N}[z])$. By the PBW Theorem for Lie superalgebras, we may write the element A as a linear combination of ordered monomials with respect to the total ordering '$\preceq$'. Let M_k, $k = 1, \dots, p$, be the collection of maximal length ordered monomials occurring in A. That is, $M_k = X_{b_{k,1}} z^{m_{k,1}} \cdots X_{b_{k,n}} z^{m_{k,n}}$ such that $X_{b_{k,j}} z^{m_{k,j}} \preceq X_{b_{k,j+1}} z^{m_{k,j+1}}$ for indices $j = 1, \dots, n-1$ and $X_{b_{k,j}} z^{m_{k,j}} \neq X_{b_{k,j+1}} z^{m_{k,j+1}}$ if $X_{b_{k,j}}$ is odd.

Let us set $\widehat{z}_{2k-1} = z_k$ and $\widehat{z}_{2k} = -z_k$. For integers $1 \le r_1 < \cdots < r_n \le 2n$, consider the embedding

$$\begin{aligned} \nu_{r_1,\dots,r_n}\colon \mathfrak{U}(\mathfrak{gl}_{N|N}[z])^{\otimes n} &\to \textstyle\bigotimes_{i=1}^{2n} \mathfrak{U}(\mathfrak{gl}_{N|N}[\widehat{z}_i]) \\ Y_1(z) \otimes \cdots \otimes Y_n(z) &\mapsto 1^{\otimes(r_1-1)} \otimes Y_1(\widehat{z}_{r_1}) \otimes 1^{\otimes(r_2-r_1-1)} \otimes \cdots \otimes Y_n(\widehat{z}_{r_n}) \otimes 1^{\otimes(2n-r_n)}, \end{aligned}$$

where $Y_j(z) = y_{j,1}(z) \cdots y_{j,h_j}(z) \in \mathfrak{U}(\mathfrak{gl}_{N|N}[z])$.

By setting $M'_k = X_{b_{k,1}} z^{m_{k,1}} \otimes \cdots \otimes X_{b_{k,n}} z^{m_{k,n}} \in (\mathfrak{gl}_{N|N}[z])^{\otimes n}$, we then consider an associated element

$$M_k^\sigma := \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma, M'_k) \left(\sum_{1 \le r_1 < \cdots < r_n \le 2n} \nu_{r_1,\ldots,r_n} \big(\bigotimes_{j=1}^{n} X_{b_{k,\sigma(j)}} z^{m_{k,\sigma(j)}} \big) \right), \tag{3.5}$$

where $\epsilon \colon \mathfrak{S}_n \times (\mathfrak{gl}_{N|N}[z])^{\otimes n} \to \{\pm 1\}$ is the Koszul sign given by the assignment $\epsilon(\sigma, x) = \prod_{(k,l) \in \mathrm{Inv}(\sigma)} (-1)^{[x_{\sigma(k)}][x_{\sigma(l)}]}$, where $x = (x_1, \ldots, x_n) \in (\mathfrak{gl}_{N|N}[z])^{\otimes n}$ is homogeneous and $\mathrm{Inv}(\sigma) = \{(k,l) \,|\, k < l,\, \sigma(k) > \sigma(l)\}$.

By endowing total orders on the collection of basis elements $X_b \widehat{z}_i^m$ of each Lie superalgebra $\mathfrak{gl}_{N|N}[\widehat{z}_i]$ in a similar way as before, we obtain a basis $\mathfrak{B}$ consisting of elements of the form $\otimes_{i=1}^{2n} X_{c_{i,1}} \widehat{z}_i^{m_{i,1}} \cdots X_{c_{i,h_i}} \widehat{z}_i^{m_{i,h_i}}$ for the space $\bigotimes_{i=1}^{2n} \mathfrak{U}(\mathfrak{gl}_{N|N}[\widehat{z}_i])$, where $X_{c_{i,1}} \widehat{z}_i^{m_{i,1}} \cdots X_{c_{i,h_i}} \widehat{z}_i^{m_{i,h_i}}$ is an element of the PBW basis for $\mathfrak{gl}_{N|N}[\widehat{z}_i]$. Considering now the linear map

$$\begin{aligned} \phi \colon \bigotimes_{i=1}^{2n} \mathfrak{U}(\mathfrak{gl}_{N|N}[\widehat{z}_i]) &\to \mathfrak{U}(\mathfrak{gl}_{N|N})^{\otimes 2n}[\widehat{z}_1, \ldots, \widehat{z}_{2n}] \\ \otimes_{i=1}^{2n} X_{c_{i,1}} \widehat{z}_i^{m_{i,1}} \cdots X_{c_{i,h_i}} \widehat{z}_i^{m_{i,h_i}} &\mapsto \big(\otimes_{i=1}^{2n} X_{c_{i,1}} \cdots X_{c_{i,h_i}} \big) \textstyle\prod_{i=1}^{2n} \widehat{z}_i^{m_{i,1}} \cdots \widehat{z}_i^{m_{i,h_i}}, \end{aligned}$$

we claim that the elements $\phi(M_k^\sigma)$, $k = 1, \ldots, p$, are linearly independent. Noting that each term in the sum $\phi(M_k^\sigma)$ is an element of the basis $\mathfrak{B}$ up to sign, it suffices to show that there exists basis element (up to scaling) in each expression $\phi(M_k^\sigma)$ that does not occur in any other expressions $\phi(M_i^\sigma)$ for $i \neq k$. We observe that such a candidate would be

$$\phi\big(\nu_{1,3,\ldots,2n-1}\big(\bigotimes_{j=1}^{n} X_{b_{k,j}} z^{m_{k,j}} \big)\big) = (X_{k,1} \otimes 1 \otimes \cdots \otimes X_{k,n} \otimes 1) z_1^{m_{k,1}} \cdots z_n^{m_{k,n}}. \tag{3.6}$$

Indeed, first observe that by the total ordering on the basis elements of $\mathfrak{gl}_{N|N}[z]$, there is no such permutation $\sigma \in \mathfrak{S}_N$ such that $\phi\big(\epsilon(\sigma, M'_k)\nu_{1,3,\ldots,2n-1}\big(\otimes_{j=1}^{n} X_{b_{k,\sigma(j)}} z^{m_{k,\sigma(j)}} \big)\big)$ yields $-\phi\big(\nu_{1,3,\ldots,2n-1}\big(\otimes_{j=1}^{n} X_{b_{k,j}} z^{m_{k,j}} \big)\big)$. Therefore, some nonzero scalar multiple of the element (3.6) occurs in $\phi(M_k^\sigma)$. Furthermore, it can be seen that there is no permutation $\sigma \in \mathfrak{S}_N$ such that the element $\phi\big(\epsilon(\sigma, M_i')\nu_{1,3,\ldots,2n-1}\big(\bigotimes_{j=1}^n X_{b_{i,\sigma(j)}} z^{m_{i,\sigma(j)}}\big)\big)$ yields $\pm\phi\big(\nu_{1,3,\ldots,2n-1}\big(\bigotimes_{j=1}^n X_{b_{k,j}} z^{m_{k,j}}\big)\big)$ for $i \neq k$.

Let us set $\widehat{a}_{2k-1} = a_k$ and $\widehat{a}_{2k} = -a_k$. Now, since

$$\widetilde{\rho}_{a_1 \to a_n} \colon X_b z^m \mapsto \sum_{q=1}^{2n} \widehat{a}_q^m \chi_b^{[q]}, \qquad \chi_b^{[q]} := \mathrm{id}^{\otimes(q-1)} \otimes \chi_b \otimes \mathrm{id}^{\otimes(n-q)} \in \mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n},$$

then the image of the monomial M_k under $\widetilde{\rho}_{a_1 \to a_n}$ is given by

$$\sum_{q_1,\ldots,q_{2n}=1}^{2n} \widehat{a}_{q_1}^{m_{k,1}} \cdots \widehat{a}_{q_n}^{m_{k,n}} \chi_{b_{k,1}}^{[q_1]} \cdots \chi_{b_{k,n}}^{[q_n]} \in \mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n}. \tag{3.7}$$

Consider now the subspace W_{2n} of $\mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n}$ given by the span of all monomials of the form $\chi_{i_1} \otimes \cdots \otimes \chi_{i_{2n}}$, $1 \leq i_j \leq 4N^2$ such that the identity $\chi_1 = \mathrm{id}$ occurs in at least $n+1$ tensor factors of each monomial. Therefore, the image of any monomial in $\mathfrak{U}_{n-1}(\mathfrak{gl}_{N|N}[z])$ (that is, any monomial of length $< n$) under $\widetilde{\rho}_{a_1 \to a_n}$ will be contained in W_{2n}. For integers $1 \leq r_1 < \cdots < r_n \leq 2n$ we consider now the embedding

$$\begin{aligned} \widetilde{\nu}_{r_1,\ldots,r_n} \colon\ & \mathrm{End}(\mathbb{C}^{N|N})^{\otimes n} \to \mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n} \\ & \varphi_1 \otimes \cdots \otimes \varphi_n \mapsto \mathrm{id}^{\otimes(r_1-1)} \otimes \varphi_1 \otimes \mathrm{id}^{\otimes(r_2-r_1-1)} \otimes \cdots \otimes \varphi_n \otimes \mathrm{id}^{\otimes(2n-r_n)}, \end{aligned}$$

From (3.7), the image of the monomial M_k under $\widetilde{\rho}_{a_1 \to a_n}$ may therefore be written as

$$\sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma, M_k') \left(\sum_{1 \leq r_1 < \cdots < r_n \leq 2n} \widehat{a}_{r_1}^{m_{k,\sigma(1)}} \cdots \widehat{a}_{r_n}^{m_{k,\sigma(n)}} \widetilde{\nu}_{r_1,\ldots,r_n}\Big(\bigotimes_{j=1}^n \chi_{b_{k,\sigma(j)}} \Big) \right) \quad \mathrm{mod}\ W_{2n}. \tag{3.8}$$

Now, since $\widetilde{\rho}$ is a faithful representation, then so is $\widetilde{\rho}^{\otimes 2n} \colon \mathfrak{U}(\mathfrak{gl}_{N|N})^{\otimes 2n} \to \mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n}$ and its extension $\widetilde{\rho}^{\otimes n}[\widehat{z}_1, \ldots, \widehat{z}_n] \colon \mathfrak{U}(\mathfrak{gl}_{N|N})^{\otimes 2n}[\widehat{z}_1, \ldots, \widehat{z}_n] \to \mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n}[\widehat{z}_1, \ldots, \widehat{z}_n]$. Therefore, since the elements $\phi(M_k^\sigma)$, $k = 1, \ldots, p$, are linearly independent, then their images under $\widetilde{\rho}^{\otimes n}[\widehat{z}_1, \ldots, \widehat{z}_n]$ are so. That is, a nonzero linear combination

$\sum_{k=1}^{p} \lambda_k \phi(M_k^\sigma)$ implies that the sum of polynomials

$$\sum_{k=1}^{p} \lambda_k \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma, M_k') \left(\sum_{1 \le r_1 < \cdots < r_n \le 2n} \widetilde{\nu}_{r_1,\ldots,r_n} \Big(\bigotimes_{j=1}^{n} \chi_{b_{k,\sigma(j)}} \Big) \right) \widehat{z}_{r_1}^{m_{k,\sigma(1)}} \cdots \widehat{z}_{r_n}^{m_{k,\sigma(n)}}$$

is nonzero. Hence, there exists complex numbers $a_1, \ldots, a_n \in \mathbb{C}$ such that

$$\sum_{k=1}^{p} \lambda_k \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma, M_k') \left(\sum_{1 \le r_1 < \cdots < r_n \le 2n} \widehat{a}_{r_1}^{m_{k,\sigma(1)}} \cdots \widehat{a}_{r_n}^{m_{k,\sigma(n)}} \widetilde{\nu}_{r_1,\ldots,r_n} \Big(\bigotimes_{j=1}^{n} \chi_{b_{k,\sigma(j)}} \Big) \right)$$

is nonzero. Comparing this with (3.8), we conclude that that image of $\widetilde{\rho}_{a_1 \to a_n}(A)$ in the quotient $\mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n}/W_{2n}$ is nonzero and therefore $\widetilde{\rho}_{a_1 \to a_n}(A) \neq 0$, proving the lemma. □

For any $a \in \mathbb{C}$, we may consider the R-matrix representation of the extended Yangian $\mathbf{X}(\mathfrak{p}_N)$ given by

$$\begin{aligned} \varphi_a \colon \mathbf{X}(\mathfrak{p}_N) &\to \mathrm{End}\, \mathbb{C}^{N|N} \\ T(u) &\mapsto R(u,a), \end{aligned} \tag{3.9}$$

or more explicitly,

$$\varphi_a \colon T_{ji}(u) \mapsto -(-1)^{[i]} \left(-(-1)^{[i]} \mathbf{1} + \frac{E_{ij}}{u-a} - \frac{(-1)^{[i]([j]+1)} E_{-j,-i}}{u+a} \right),$$

hence $\varphi_a \colon T_{ji}^{(n)} \mapsto -(-1)^{[i]} \rho_a(\mathsf{E}_{ij}^{(n-1)}(z))$ for $n \in \mathbb{Z}^+$.

By the Hopf superalgebra structure on $\mathbf{X}(\mathfrak{p}_N)$, we may consider the tensor product of these representations: $(\varphi_a \otimes \varphi_{-a}) \circ \Delta \colon T(u) \mapsto R_{12}(u,a) R_{13}(u,-a)$. Considering now the rational function $f_a(u) = \frac{(u+a)^2}{(u+a)^2 - 1}$, we may post-compose the this representation with the automorphism μ_{f_a} to yield a representation ϕ_a given by

$$\begin{aligned} \phi_a \colon X(\mathfrak{p}_N) &\to \mathrm{End}\left(\mathbb{C}^{N|N}\right)^{\otimes 2} \\ T(u) &\mapsto f_a(u) R_{12}(u,a) R_{13}(u,-a). \end{aligned}$$

In particular, we see that $\phi_a(Z(u)) = \mathrm{id}^{\otimes 2}$. Indeed, by (3.10) and (4.5) we have

$$\begin{aligned}\phi_a(T^{j^P}(u)T(-u)) &= f_a(u)f_a(-u)R_{1,3}(u,-a)^{j_1^P}R_{12}(u,a)^{j_1^P}R_{12}(-u,a)R_{13}(-u,-a)\\ &= f_a(u)f_a(-u)f_a(u)^{-1}f_a(-u)^{-1}\,\mathrm{id}^{\otimes 2n} = \mathrm{id}^{\otimes 2n}\,.\end{aligned}$$

Therefore, the representation ϕ_a factors through to a representation of the Yangian:

$$\begin{aligned}\Phi_a\colon Y(\mathfrak{p}_N) &\to \mathrm{End}\left(\mathbb{C}^{N|N}\right)^{\otimes 2}\\ \mathcal{T}(u) &\mapsto f_a(u)R_{12}(u,a)R_{13}(u,-a),\end{aligned}$$

or more explicitly,

$$\begin{aligned}\Phi_a\colon \sum_{n=0}^{\infty}\mathcal{T}_{ji}^{(n)}u^{-n} \mapsto f_a(u)&\Bigg(\mathrm{id}^{\otimes 2}+\sum_{n=1}^{\infty}\left(-(-1)^{[i]}\rho_a(\mathsf{E}_{ij}^{(n-1)}(z))\right)\otimes\mathrm{id}\,u^{-n}\Bigg)\\ &\Bigg(\mathrm{id}^{\otimes 2}+\sum_{n=1}^{\infty}\mathrm{id}\otimes\left(-(-1)^{[i]}\rho_{-a}(\mathsf{E}_{ij}^{(n-1)}(z))\right)u^{-n}\Bigg).\end{aligned}$$

Furthermore, for complex numbers $a_1,\dots,a_n\in\mathbb{C}$, let us set $\Phi_{a_1\to a_n|} := (\bigotimes_{i=1}^{n}\Phi_{a_i})\circ\Delta_{n-1}$. By writing $f_a(u)=\sum_{n=0}^{\infty}f_a^{(n)}u^{-n}$, we observe that since

$$\begin{aligned}f_a(u) = \sum_{n=1}^{\infty}(u+a)^{-2n} &= \sum_{n=1}^{\infty}u^{-2n}\left(\sum_{k=0}^{\infty}(-a)^k u^{-k}\right)^{2n}\\ &= \sum_{n=1}^{\infty}\sum_{p=0}^{\infty}\binom{2n+p-1}{2n-1}(-a)^p u^{-2n-p},\end{aligned}$$

then the polynomial degree of $f_a^{(n)}$ in the indeterminate 'a' is $n-2$ for $n\geq 2$. In particular, we observe that the polynomial degree of $\Phi_a(-(-1)^{[i]}\mathcal{T}_{ji}^{(n)})$ is $n-1$ whose highest degree term is $\rho_a(\mathsf{E}_{ij}^{(n-1)}(z))\otimes\mathrm{id}+\mathrm{id}\otimes\rho_{-a}(\mathsf{E}_{ij}^{(n-1)}(z))$. Consequently, the image of any element in $\mathbf{F}_n$ under Φ_a will be a polynomial in 'a' of degree $\leq n-1$.

Theorem 3.3.3. *The mapping*

$$\Psi\colon \mathsf{E}_{ij}^{(n-1)}(z)\mapsto -(-1)^{[i]}\overline{\mathcal{T}}_{ji}^{(n)}. \tag{3.10}$$

is an isomorphism of $\mathbb{N}$*-graded Hopf superalgebras* $\Psi\colon\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^P})\to \mathrm{gr}\,\mathbf{Y}(\mathfrak{p}_N)$.

Proof. By Proposition (3.3.1), we only need to show that this map is injective. To this end, let $A \in \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^P})$ be a nonzero element that is homogeneous of degree d with respect to the $\mathbb{N}$-gradation. That is,

$$A = \sum A^{k_1;\dots;k_m}_{i_1,j_1;\dots;i_m,j_m} \mathsf{E}^{(k_1-1)}_{i_1j_1}(z)\cdots\mathsf{E}^{(k_m-1)}_{i_mj_m}(z), \quad A^{k_1;\dots;k_m}_{i_1,j_1;\dots;i_m,j_m} \in \mathbb{C}, \tag{3.11}$$

summed over the indices i_b, j_b, k_b, $1 \le b \le m$, where m may vary over a finite subset of $\mathbb{Z}^+$ and such that $i_b, j_b \in I_N$, $\sum_{b=1}^m k_b = d+m$. Considering the element

$$\widetilde{A} = \sum (-1)^{[i_1]+\cdots+[i_m]+m} A^{k_1;\dots;k_m}_{i_1,j_1;\dots;i_m,j_m} \mathcal{T}^{(k_1)}_{j_1i_1}\cdots\mathcal{T}^{(k_m)}_{j_mi_m} \in \mathbf{Y}(\mathfrak{p}_N)$$

whose summation is given over the same indices as in (3.11), then $\Psi(A)$ coincides with the image of $\widetilde{A}$ in $\operatorname{gr}\mathbf{Y}(\mathfrak{p}_N)$, so it suffices to prove that the filtration degree of $\widetilde{A}$ is d.

Let us equip $\mathbf{Y}(\mathfrak{p}_N)^{\otimes n}$ with the ascending algebra filtration $\mathbf{F}^{\otimes n} = \{\mathbf{F}^{\otimes n}_h\}_{h\in\mathbb{N}}$ induced by the one on $\mathbf{Y}(\mathfrak{p}_N)$: $\mathbf{F}^{\otimes n}_h = \bigoplus_{\sum_{i=1}^n k_i = h} \mathbf{F}_{k_1}\otimes\cdots\otimes\mathbf{F}_{k_n}$. Supposing $\sum_{i=1}^m k_i = d+m$, then the element $\Delta_{n-1}((-1)^{[i_1]+\cdots+[i_m]+m}\mathcal{T}^{(k_1)}_{j_1i_1}\cdots\mathcal{T}^{(k_m)}_{j_mi_m})$ may be written as

$$\sum_{q_1,\dots,q_m=1}^{n} (-1)^{[i_1]+\cdots+[i_m]+m} (\mathcal{T}^{(k_1)}_{j_1i_1})^{\langle q_1\rangle}\cdots(\mathcal{T}^{(k_m)}_{j_mi_m})^{\langle q_m\rangle} \mod \mathbf{F}^{\otimes n}_{d-1}$$

where $(\mathcal{T}^{(k_b)}_{j_bi_b})^{\langle q_b\rangle} = \mathbf{1}^{\otimes(q_b-1)}\otimes\mathcal{T}^{(k_b)}_{j_bi_b}\otimes\mathbf{1}^{\otimes(n-q_b)}$. Therefore, the image of the monomial $(-1)^{[i_1]+\cdots+[i_m]+m}\mathcal{T}^{(k_1)}_{j_1i_1}\cdots\mathcal{T}^{(k_m)}_{j_mi_m}$ under the representation $\Phi_{a_1\to a_n}$ will a polynomial in the indeterminates $a_1,\dots,a_n$ of degree $\le d$. In particular, this image may be written as

$$\sum_{q_1,\dots,q_m=1}^{n}\prod_{b=1}^{m}\Big(\rho_{a_{q_b}}(\mathsf{E}^{(k_b-1)}_{i_bj_b}(z))\otimes\mathrm{id}+\mathrm{id}\otimes\rho_{-a_{q_b}}(\mathsf{E}^{(k_b-1)}_{i_bj_b}(z))\Big)^{\langle\langle q_b\rangle\rangle}$$

modulo the subspace $\big(\mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n}\big)[a]_{d-1}$, where $X^{\langle\langle q_b\rangle\rangle} = \mathrm{id}^{\otimes(2q_b-2)}\otimes X\otimes\mathrm{id}^{\otimes(2n-2q_b)}$. In particular, the image of the element $\widetilde{A}$ under $\Phi_{a_1\to a_n}$ will be a polynomial in $a_1,\dots,a_n$ of degree $\le d$ given by

$$\sum A^{k_1;\dots;k_m}_{i_1,j_1;\dots;i_m,j_m}\sum_{q_1,\dots,q_m=1}^{n}\prod_{b=1}^{m}\Big(\rho_{a_{q_b}}(\mathsf{E}^{(k_b-1)}_{i_bj_b}(z))\otimes\mathrm{id}+\mathrm{id}\otimes\rho_{-a_{q_b}}(\mathsf{E}^{(k_b-1)}_{i_bj_b}(z))\Big)^{\langle\langle q_b\rangle\rangle}$$

modulo the subspace $\left(\mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n}\right)[a]_{d-1}$. That is,

$$\rho_{a_1 \to a_n}(A) = \Phi_{a_1 \to a_n}(\widetilde{A}) \mod \left(\mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n}\right)[a]_{d-1}$$

By the previous lemma we know that there exists $a_1, \ldots, a_n \in \mathbb{C}$ such that $\rho_{a_1 \to a_n}(A) \neq 0$, thus $\widetilde{A}$ under $\Phi_{a_1 \to a_n}$ will be a polynomial in $a_1, \ldots, a_n$ of degree d and hence the filtration degree of $\widetilde{A}$ is d. $\square$

Corollary 3.3.4. *Given any total ordering on the collection $\mathcal{X}$ of generators $\mathcal{T}_{ji}^{(n)}$ satisfying either one of*

$$1 \leq |i| < |j| \leq N,\ n \in \mathbb{Z}^+; \quad 1 \leq i = j \leq N,\ n \in \mathbb{Z}^+;$$
$$1 \leq -i = j \leq N,\ n \in 2\mathbb{Z}^+ - 1; \quad \textit{or} \quad -N \leq -i = j \leq -1,\ n \in 2\mathbb{Z}^+,$$

then the set of all ordered monomials of the form $\prod_{k=1}^{m} \mathcal{T}_{j_k i_k}^{(n_k)}$ with $\mathcal{T}_{j_k i_k}^{(n_k)} \in \mathcal{X}$ such that $\mathcal{T}_{j_k i_k}^{(n_k)} \preceq \mathcal{T}_{j_{k+1} i_{k+1}}^{(n_{k+1})}$ and $\mathcal{T}_{j_k i_k}^{(n_k)} \neq \mathcal{T}_{j_{k+1} i_{k+1}}^{(n_{k+1})}$ if $\mathcal{T}_{j_k i_k}^{(n_k)}$ is odd, forms a basis for the super Yangian $\mathbf{Y}(\mathfrak{p}_N)$.

Proposition 3.3.5. *There is an injective morphism of superalgebras*

$$\begin{aligned} \varphi \colon \mathfrak{U}(\mathfrak{p}_N) &\hookrightarrow \mathbf{Y}(\mathfrak{p}_N) \\ \mathsf{E}_{ij} &\mapsto -(-1)^{[i]} \mathcal{T}_{ji}^{(1)}. \end{aligned}$$

Proof. Let us show that the images of the generators of $\mathfrak{U}(\mathfrak{p}_N)$ satisfy the desired relations. First, we observe that

$$\varphi(\mathsf{E}_{ij}) + (-1)^{[i]([j]+1)} \varphi(\mathsf{E}_{-j,-i}) = -(-1)^{[i]} \mathcal{T}_{ji}^{(1)} + (-1)^{[i]([j]+1)+[j]} \mathcal{T}_{-i,-j}^{(1)} = 0$$

by the relation (3.36). Now, by taking the coefficient of $u^{-1}v^{-1}$ in (3.18), we observe

$$\begin{aligned} \left[\varphi(E_{ij}), \varphi(E_{kl})\right] &= (-1)^{[i]+[k]} \left[\mathcal{T}_{ji}^{(1)}, \mathcal{T}_{lk}^{(1)}\right] \\ &= (-1)^{[i]+[k]+[j][l]+[j][k]+[k][l]} (\delta_{li} \mathcal{T}_{jk}^{(1)} - \mathcal{T}_{li}^{(1)} \delta_{jk}) \end{aligned}$$

$$
\begin{aligned}
&- (-1)^{[i]+[k]} \Bigg(\delta_{j,-l} \sum_{p=-N}^{N} (-1)^{[j][k]+[k][p]+[p]} \delta_{pi} \mathcal{T}^{(1)}_{-p,k} \\
&\qquad\qquad - \delta_{i,-k} \sum_{p=-N}^{N} (-1)^{[i]+[j][k]+[j][l]+[l][p]} \mathcal{T}^{(1)}_{l,-p} \delta_{jp} \Bigg) \\
&= \delta_{jk} \left(-(-1)^{[i]} \mathcal{T}^{(1)}_{li} \right) - \delta_{il} (-1)^{([i]+[j])([k]+[l])} \left(-(-1)^{[k]} \mathcal{T}^{(1)}_{jk} \right) \\
&\quad - \delta_{i,-k} (-1)^{[i]([j]+1)} \left(-(-1)^{[-j]} \mathcal{T}^{(1)}_{l,-j} \right) + \delta_{j,-l} (-1)^{([i]+[j])[k]} \left(-(-1)^{[k]} \mathcal{T}^{(1)}_{-i,k} \right) \\
&= \delta_{jk} \varphi(\mathsf{E}_{il}) - \delta_{il} (-1)^{([i]+[j])([k]+[l])} \varphi(\mathsf{E}_{kj}) \\
&\quad - \delta_{i,-k} (-1)^{[i]([j]+1)} \varphi(\mathsf{E}_{-j,l}) + \delta_{j,-l} (-1)^{([i]+[j])[k]} \varphi(\mathsf{E}_{k,-i}).
\end{aligned}
$$

Injectivity follows from the previous corollary. □

3.4 The Schur-Weyl Functor

In this section, we wish connect the representation theories of the extended super Yangian $\mathbf{X}(\mathfrak{p}_N)$ and a somewhat less complicated algebra known as the *affine periplectic Brauer algebra* $\widehat{P_d}$ as defined in [?]. Here, we shall instead refer to this new algebra as the *degenerate* affine periplectic Brauer algebra and denote it as $\mathfrak{B}_d^{\mathrm{aff}}$, as we find this to be a more suitable in nature. We recall the definition below.

Definition 3.4.1. For $d \in \mathbb{N}$, the *degenerate affine periplectic Brauer algebra*, denoted $\mathfrak{B}_d^{\mathrm{aff}}$, is the free unital associative algebra over $\mathbb{C}$ on indeterminates s_a, ε_a, y_k, $1 \le a \le d-1$, $1 \le k \le d$, subject to the relations

$(P.1)$ (i) $s_a^2 = \mathbf{1}$, (ii) $\varepsilon_a^2 = 0$, (iii) $s_a\varepsilon_a = \varepsilon_a$, (iv) $\varepsilon_a s_a = -\varepsilon_a$, $1 \leq a \leq d-1$;
$(P.2)$ (i) $s_a s_b = s_b s_a$, (ii) $\varepsilon_a\varepsilon_b = \varepsilon_b\varepsilon_a$, (iii) $s_a\varepsilon_b = \varepsilon_b s_a$, $|a-b| \geq 2$;
$(P.3)$ (i) $s_a s_{a+1} s_a = s_{a+1} s_a s_{a+1}$, (ii) $\varepsilon_a\varepsilon_{a+1}\varepsilon_a = -\varepsilon_a$,
(iii) $\varepsilon_{a+1}\varepsilon_a\varepsilon_{a+1} = -\varepsilon_{a+1}$, $1 \leq a \leq d-1$;
'123' $(P.4)$ $(i) s_a\varepsilon_{a+1}\varepsilon_a = -s_{a+1}\varepsilon_a$, (ii) $\varepsilon_{a+1}\varepsilon_a s_{a+1} = -\varepsilon_{a+1} s_a$, $1 \leq a \leq d-1$;
$(P.5)$ $y_j y_k = y_k y_j$, $1 \leq j, k \leq d$;
$(P.6)$ (i) $s_a y_k = y_k s_a$, (ii) $\varepsilon_a y_k = y_k \varepsilon_a$, $1 \leq a \leq d-1$, $k \notin \{a, a+1\}$;
$(P.7)$ (i) $s_a y_a - y_{a+1} s_a = \varepsilon_a - \mathbf{1}$, (ii) $y_a s_a - s_a y_{a+1} = -\varepsilon_a - \mathbf{1}$, $1 \leq a \leq d-1$;
$(P.8)$ (i) $\varepsilon_a(y_a - y_{a+1}) = \varepsilon_a$, (ii) $(y_a - y_{a+1})\varepsilon_a = -\varepsilon_a$, $1 \leq a \leq d-1$.

We note that the relations $(P.1)-(P.4)$ are precisely those that define the periplectic Brauer algebra A_d in [?]; therefore, $\mathfrak{B}_d^{\text{aff}}$ may be regarded as an affine analogue of A_d. We shall also regard $\mathfrak{B}_d^{\text{aff}}$ as a superalgebra, but with only the trivial $\mathbb{Z}_2$-grading where all elements are even. Also note that the relations $(P.1)\,(i)$, $(P.2)\,(i)$, and $(P.3)\,(i)$ define the subalgebra $\mathbb{C}[\mathfrak{S}_d] \subset \mathfrak{B}_d^{\text{aff}}$, where $\mathfrak{S}_d$ is the symmetric group on d letters.

The following equations are immediate from the defining relations of $\mathfrak{B}_d^{\text{aff}}$, but are nonetheless useful:

$(C.1)$ (i) $s_{a+1}\varepsilon_a s_{a+1} = s_a\varepsilon_{a+1} s_a$, (ii) $\varepsilon_a y_a \varepsilon_a = \varepsilon_a y_{a+1}\varepsilon_a$, $1 \leq a \leq d-1$;
$(C.2)$ (i) $\varepsilon_a s_{a+1}\varepsilon_a = -\varepsilon_a$, (ii) $\varepsilon_{a+1} s_a \varepsilon_{a+1} = \varepsilon_{a+1}$, $1 \leq a \leq d-1$;
$(C.3)$ $(i) s_a y_a s_a = y_{a+1} - (s_a + \varepsilon_a)$, (ii) $s_a y_{a+1} s_a = y_a + (s_a - \varepsilon_a)$, $1 \leq a \leq d-1$.

For integers $1 \leq j < k \leq d$, we define the following elements in $\mathfrak{B}_d^{\text{aff}}$:

$$\sigma_{j,j+1} := s_j, \quad \sigma_{j,k} := s_j s_{j+1} \cdots s_{k-2} s_{k-1} s_{k-2} \cdots s_{j+1} s_j \quad \text{for } k-j > 1 \tag{3.1}$$

$$\beta_{j,j+1} := \varepsilon_j, \quad \beta_{j,k} := s_j s_{j+1} \cdots s_{k-2}\varepsilon_{k-1} s_{k-2} \cdots s_{j+1} s_j \quad \text{for } k-j > 1. \tag{3.2}$$

Here, observe that $\sigma_{j,k}$ is simply the transposition $(j\ k) \in \mathfrak{S}_d$. We now introduce new generators $\widetilde{y}_1, \ldots, \widetilde{y}_d$, defined by

$$\widetilde{y}_1 := y_1, \quad \widetilde{y}_k := y_k - \sum_{j=1}^{k-1} (\sigma_{j,k} + \beta_{j,k}) = y_k - \sum_{j=1}^{k-1} (\mathbf{1} - \beta_{j,k})\sigma_{j,k}$$

Note that these generators satisfy the following relations:

$$\tau \widetilde{y}_k \tau^{-1} = \widetilde{y}_{\tau(k)} \quad \text{for } \tau \in \mathfrak{S}_d \tag{3.3}$$

$$\beta_{p,d} \widetilde{y}_k = \widetilde{y}_k \beta_{p,d} \quad \text{for } 1 \leq k \neq p < d \tag{3.4}$$

$$y_p \widetilde{y}_k = \widetilde{y}_k y_p \quad \text{for } 1 \leq k < p \leq d \tag{3.5}$$

Consider now the ideal $\mathcal{I}_d$ of $\mathfrak{B}_d^{\mathrm{aff}}$ generated by elements of the form

$$\varepsilon_a y_{a+1} + y_{a+1} \varepsilon_a - \sum_{j=1}^{a-1} \big(\varepsilon_a (\sigma_{j,a} + \beta_{j,a}) + (\sigma_{j,a} + \beta_{j,a}) \varepsilon_a \big), \quad a = 1, \ldots, d-1.$$

In the quotient $\mathfrak{B}_d^{\mathrm{aff}} / \mathcal{I}_d$, we then have the relation

$$\beta_{k,d} \widetilde{y}_k = -\widetilde{y}_k \beta_{k,d} \quad \text{for } 1 \leq k < d. \tag{3.6}$$

Letting v be a formal variable, we may consider the slight generalization of the R-matrix representation (3.9) given by

$$\begin{aligned} \varphi_v \colon \mathbf{X}(\mathfrak{p}_N) &\to \mathrm{End}\left(\mathbb{C}^{N|N}\right)[v] \\ T(u) &\mapsto R(u,v) \end{aligned} \tag{3.7}$$

Given pairwise commuting formal variables $v_1, \ldots, v_d$, we will take the tensor product $\varphi_{v_1 \to v_d} = (\bigotimes_{j=1}^d \varphi_{v_j}) \circ \Delta_{n-1}$ of the representations $\varphi_{v_1}, \ldots, \varphi_{v_d}$ as below:

$$\begin{aligned} \varphi_{v_1 \to v_d} \colon \mathbf{X}(\mathfrak{p}_N) &\to \mathrm{End}\left(\mathbb{C}^{N|N}\right)^{\otimes d}[v_1, \ldots, v_d] \\ T(u) &\mapsto \textstyle\prod_{j=1}^d R_{1,j+1}(u, v_j) \end{aligned}$$

Now, since $y_1, \ldots, y_d \in \mathfrak{B}_d^{\text{aff}}/\mathcal{I}_d$ are pairwise commuting elements, the assignment

$$\sum_{j_1,\ldots,j_d} f_{j_1,\ldots,j_d} v_1^{j_1} \cdots v_d^{j_d} \mapsto f_{j_1,\ldots,j_d} \otimes y_1^{j_1} \cdots y_d^{j_d}$$

defines an algebra morphism $\operatorname{End}\left(\mathbb{C}^{N|N}\right)^{\otimes d}[v_1, \ldots, v_d] \to \operatorname{End}\left(\mathbb{C}^{N|N}\right)^{\otimes d} \otimes \mathfrak{B}_d^{\text{aff}}/\mathcal{I}_d$. Post-composing this morphism with the representation $\varphi_{v_1 \to v_d}$, we get an algebra morphism

$$\begin{aligned} \varphi_{y_1 \to y_d} \colon \mathbf{X}(\mathfrak{p}_N) &\to \operatorname{End}\left(\mathbb{C}^{N|N}\right)^{\otimes d} \otimes \mathfrak{B}_d^{\text{aff}}/\mathcal{I}_d \\ T(u) &\mapsto \prod_{p=1}^{d} \left(\mathrm{id}^{\otimes(d+1)} \otimes \mathbf{1} - P_{1,p+1} \otimes \frac{\mathbf{1}}{u - y_p} + Q_{1,p+1} \otimes \frac{\mathbf{1}}{u + y_p} \right). \end{aligned}$$

Proposition 3.4.2. *Let $\mathcal{J}_d$ be the two-sided ideal in* $\operatorname{End}\left(\mathbb{C}^{N|N}\right)^{\otimes(d+1)} \otimes \left(\mathfrak{B}_d^{\text{aff}}/\mathcal{I}_d\right)[\![u^{-1}]\!]$ *generated by all elements* $\mathrm{id}^{\otimes(d+1)} \otimes \sigma_{p,q} - P_{p+1,q+1} \otimes \mathbf{1}$ *and* $\mathrm{id}^{\otimes(d+1)} \otimes \beta_{p,q} - Q_{p+1,q+1} \otimes \mathbf{1}$ *with $1 \leq p < q \leq d$. Then*

$$\varphi_{y_1 \to y_d}(T(u)) \equiv \mathrm{id}^{\otimes(d+1)} \otimes \mathbf{1} - \sum_{p=1}^{d} \left(P_{1,p+1} \otimes \frac{\mathbf{1}}{u - \widetilde{y}_p} - Q_{1,p+1} \otimes \frac{\mathbf{1}}{u + \widetilde{y}_p} \right) \mod \mathcal{J}_d \tag{3.7}$$

Proof. Proceeding via induction on d, the induction base is satisfied by the equality $\widetilde{y}_1 = y_1$.

Let us first note that if $\mathcal{A}$ is an algebra and $\mathcal{I} \subseteq \mathcal{A}$ is some right ideal, and if there exists elements $a_1, \ldots, a_d$, $b_1, \ldots, b_d \in \mathcal{A}$ such that $\prod_{p=1}^{d-1}(1 - a_p) - \left(1 - \sum_{p=1}^{d-1} b_p\right) \in \mathcal{I}$, then $\prod_{p=1}^{d}(1 - a_p) - \left(1 - \sum_{p=1}^{d} b_p\right) \equiv b_d - \left(1 - \sum_{p=1}^{d-1} b_p\right) a_d \mod \mathcal{I}$. Indeed, one can see this from the equality

$$\begin{aligned} \prod_{p=1}^{d}(1 - a_p) - \left(1 - \sum_{p=1}^{d} b_p\right) = \left(\prod_{p=1}^{d-1}(1 - a_p) - \left(1 - \sum_{p=1}^{d-1} b_p\right)\right)(1 - a_d) \\ + b_d - \left(1 - \sum_{p=1}^{d-1} b_p\right) a_d. \end{aligned}$$

Assuming the hypothesis holds for $d-1$, then

$$\varphi_{y_1\to y_{d-1}}(T(u)) - \left(\mathrm{id}^{\otimes d}\otimes \mathbf{1} - \sum_{p=1}^{d-1}\left(P_{1,p+1}\otimes \frac{\mathbf{1}}{u-\widetilde{y}_p} - Q_{1,p+1}\otimes\frac{\mathbf{1}}{u+\widetilde{y}_p}\right)\right) \in \mathcal{J}_{d-1}$$

If we consider now the embedding γ from $\mathrm{End}\left(\mathbb{C}^{N|N}\right)^{\otimes d}\otimes\left(\mathfrak{B}^{\mathrm{aff}}_{d-1}/\mathcal{I}_{d-1}\right)[\![u^{-1}]\!]$ to $\mathrm{End}\left(\mathbb{C}^{N|N}\right)^{\otimes(d+1)}\otimes\left(\mathfrak{B}^{\mathrm{aff}}_{d}\right.$ determined by $f_1\otimes\cdots\otimes f_d\otimes b\mapsto f_1\otimes\cdots f_d\otimes\mathrm{id}\otimes b$, the image of the above element under γ is given by

$$\prod_{p=1}^{d-1}\left(\mathrm{id}^{\otimes(d+1)}\otimes\mathbf{1}-P_{1,p+1}\otimes\frac{\mathbf{1}}{u-y_p}+Q_{1,p+1}\otimes\frac{\mathbf{1}}{u+y_p}\right)$$
$$-\left(\mathrm{id}^{\otimes(d+1)}\otimes\mathbf{1}-\sum_{p=1}^{d-1}\left(P_{1,p+1}\otimes\frac{\mathbf{1}}{u-\widetilde{y}_p}-Q_{1,p+1}\otimes\frac{\mathbf{1}}{u+\widetilde{y}_p}\right)\right)$$

and this element lies in the ideal $\mathcal{J}_d$ since $\gamma(\mathcal{J}_{d-1})\subset\mathcal{J}_d$. Therefore, the difference $\mathrm{Diff} := \varphi_{y_1\to y_d}(T(u)) - \left(\mathrm{id}^{\otimes(d+1)}\otimes\mathbf{1}-\sum_{p=1}^{d}\left(P_{1,p+1}\otimes\frac{\mathbf{1}}{u-\widetilde{y}_p}-Q_{1,p+1}\otimes\frac{\mathbf{1}}{u+\widetilde{y}_p}\right)\right)$ is given by

$$\left(P_{1,d+1}\otimes\frac{\mathbf{1}}{u-\widetilde{y}_d}-Q_{1,d+1}\otimes\frac{\mathbf{1}}{u+\widetilde{y}_d}\right)-\left(P_{1,d+1}\otimes\frac{\mathbf{1}}{u-y_d}-Q_{1,d+1}\otimes\frac{\mathbf{1}}{u+y_d}\right)$$
$$+\sum_{p=1}^{d-1}\left(P_{1,p+1}\otimes\frac{\mathbf{1}}{u-\widetilde{y}_p}-Q_{1,p+1}\otimes\frac{\mathbf{1}}{u+\widetilde{y}_p}\right)\left(P_{1,d+1}\otimes\frac{\mathbf{1}}{u-y_d}-Q_{1,d+1}\otimes\frac{\mathbf{1}}{u+y_d}\right)$$

modulo $\mathcal{J}_d$. Now, by using the equations $P_{1,p+1}P_{1,d+1}=P_{1,d+1}P_{p+1,d+1}$, $Q_{1,p+1}P_{1,d+1}=-P_{1.d+1}Q_{p+1,d+1}$, $P_{1,p+1}Q_{1,d+1}=-Q_{p+1,d+1}Q_{1,d+1}$, and $Q_{1,p+1}Q_{1,d+1}=P_{p+1,d+1}Q_{1,d+1}$, we observe that Diff is equivalent to

$$\left(P_{1,d+1}\otimes\frac{\mathbf{1}}{u-\widetilde{y}_d}-Q_{1,d+1}\otimes\frac{\mathbf{1}}{u+\widetilde{y}_d}\right)-\left(P_{1,d+1}\otimes\frac{\mathbf{1}}{u-y_d}-Q_{1,d+1}\otimes\frac{\mathbf{1}}{u+y_d}\right)$$
$$+\sum_{p=1}^{d-1}\left(P_{1,d+1}P_{p+1,d+1}\otimes\frac{\mathbf{1}}{u-\widetilde{y}_p}\cdot\frac{\mathbf{1}}{u-y_d}+P_{1,d+1}Q_{p+1,d+1}\otimes\frac{\mathbf{1}}{u+\widetilde{y}_p}\cdot\frac{\mathbf{1}}{u-y_d}\right)$$
$$+\sum_{p=1}^{d-1}\left(Q_{p+1,d+1}Q_{1,d+1}\otimes\frac{\mathbf{1}}{u-\widetilde{y}_p}\cdot\frac{\mathbf{1}}{u+y_d}+P_{p+1,d+1}Q_{1,d+1}\otimes\frac{\mathbf{1}}{u+\widetilde{y}_p}\cdot\frac{\mathbf{1}}{u+y_d}\right),$$

which re-written is

$$
\begin{aligned}
P_{1,d+1} \otimes \left(\frac{\mathbf{1}}{u-\widetilde{y}_d} - \frac{\mathbf{1}}{u-y_d} \right) + Q_{1,d+1} \otimes \left(\frac{\mathbf{1}}{u+y_d} - \frac{\mathbf{1}}{u+\widetilde{y}_d} \right) \\
+ \sum_{p=1}^{d-1} \left(P_{1,d+1} \otimes \frac{\mathbf{1}}{u-\widetilde{y}_p} \cdot \frac{\mathbf{1}}{u-y_d} \right) \left(P_{p+1,d+1} \otimes \mathbf{1} \right) \\
+ \sum_{p=1}^{d-1} \left(P_{1,d+1} \otimes \frac{\mathbf{1}}{u+\widetilde{y}_p} \cdot \frac{\mathbf{1}}{u-y_d} \right) \left(Q_{p+1,d+1} \otimes \mathbf{1} \right) \\
+ \sum_{p=1}^{d-1} \left(Q_{p+1,d+1} \otimes \mathbf{1} \right) \left(Q_{1,d+1} \otimes \frac{\mathbf{1}}{u-\widetilde{y}_p} \cdot \frac{\mathbf{1}}{u+y_d} \right) \\
+ \sum_{p=1}^{d-1} \left(P_{p+1,d+1} \otimes \mathbf{1} \right) \left(Q_{1,d+1} \otimes \frac{\mathbf{1}}{u+\widetilde{y}_p} \cdot \frac{\mathbf{1}}{u+y_d} \right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{Diff} \equiv P_{1,d+1} \otimes \left(\frac{\mathbf{1}}{u-\widetilde{y}_d} - \frac{\mathbf{1}}{u-y_d} + \frac{\mathbf{1}}{u-y_d} \sum_{p=1}^{d-1} \left(\frac{\mathbf{1}}{u-\widetilde{y}_p} - \frac{\mathbf{1}}{u+\widetilde{y}_p} \beta_{p,d} \right) \sigma_{p,d} \right) \\
+ Q_{1,d+1} \otimes \left(\frac{\mathbf{1}}{u+y_d} - \frac{\mathbf{1}}{u+\widetilde{y}_d} + \sum_{p=1}^{d-1} \sigma_{p,d} \left(\frac{\mathbf{1}}{u+\widetilde{y}_p} + \beta_{p,d} \frac{\mathbf{1}}{u-\widetilde{y}_p} \right) \frac{\mathbf{1}}{u+y_d} \right)
\end{aligned}
$$

mod $\mathcal{J}_d$, where we used the fact that y_d commutes with $\widetilde{y}_p$, $p = 1, \ldots, d-1$. It now suffices to show that the elements

$$
\begin{aligned}
B_1 &= \frac{\mathbf{1}}{u-\widetilde{y}_d} - \frac{\mathbf{1}}{u-y_d} + \frac{\mathbf{1}}{u-y_d} \sum_{p=1}^{d-1} \left(\frac{\mathbf{1}}{u-\widetilde{y}_p} - \frac{\mathbf{1}}{u+\widetilde{y}_p} \beta_{p,d} \right) \sigma_{p,d} \\
B_2 &= \frac{\mathbf{1}}{u+y_d} - \frac{\mathbf{1}}{u+\widetilde{y}_d} + \sum_{p=1}^{d-1} \sigma_{p,d} \left(\frac{\mathbf{1}}{u+\widetilde{y}_p} + \beta_{p,d} \frac{\mathbf{1}}{u-\widetilde{y}_p} \right) \frac{\mathbf{1}}{u+y_d}
\end{aligned}
$$

lying in $\left(\mathfrak{B}_d^{\mathrm{aff}} / \mathcal{I}_d\right)[\![u^{-1}]\!]$ are both zero. To this end, we observe that

$$
(u-y_d) B_1 (u-\widetilde{y}_d) = (u-y_d) - (u-\widetilde{y}_d) + \sum_{p=1}^{d-1} (1-\beta_{p,d}) \sigma_{p,d} = 0
$$

by the definition of $\widetilde{y}_d$. Similarly, we have

$$(u+\widetilde{y}_d)B_2(u+y_d) = (u+\widetilde{y}_d) - (u+y_d) + \sum_{p=1}^{d-1} \sigma_{p,d}(1+\beta_{p,d}) = 0,$$

so $\mathrm{Diff} \equiv 0$. $\square$

Consider now any representation $\zeta\colon \mathfrak{B}_d^{\mathrm{aff}}/\mathcal{I}_d \to \operatorname{End} M$.

Chapter 4

Twisted Super Yangians

4.1 The Super Yangian $\mathbf{Y}(\mathfrak{gl}_{M|N})$

By convention, $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, \dots\}$, $\mathbb{Z}$ is the set of all integers, $\mathbb{Z}^+$ denotes the set of positive integers, $\mathbb{C}$ is the field of complex numbers, $\mathbb{Q}$ is the field of rational numbers, and $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ denotes the field of two elements. Let us fix our ground field to be $\mathbb{C}$. Unless otherwise stated, all linear algebraic notions are formulated with respect to this fixed base field $\mathbb{C}$ (i.e., vector space = $\mathbb{C}$-vector space, algebra = $\mathbb{C}$-algebra, linear map = $\mathbb{C}$-linear map, $\otimes = \otimes_{\mathbb{C}}$, etc...).

Given a integers $M, N \in \mathbb{N}$ such that $M+N > 0$, we introduce the *gradation index*

$$[\cdot]\colon \{1, 2, \dots, M+N\} \to \mathbb{Z}_2 \tag{4.1}$$

given by $[i] = \bar{0}$ if $1 \leq i \leq M$ and $[i] = \bar{1}$ if $M+1 \leq i \leq M+N$.

We will often be working with some super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ that is graded with respect to the gradation index (4.1), but we shall also denote the gradation of homogeneous elements in V with the similar notation: $[\,\cdot\,]\colon V_{\bar{0}} \sqcup V_{\bar{1}} \to \mathbb{Z}_2, v \mapsto [v]$, where $[v] = \gamma \in \mathbb{Z}_2$ if $v \in V_\gamma$. For instance, denote $\mathbb{C}^{M|N}$ to be the vector space $\mathbb{C}^{M+N}$ equipped with the $\mathbb{Z}_2$-grading by assigning $[e_i] = [i]$, where $\{e_i\}_{i=1}^{M+N}$ is the standard ordered basis of $\mathbb{C}^{M+N}$. Note that in a super vector space V, elements in $V_{\bar{0}}$ are said to be *even* and elements in $V_{\bar{1}}$ are said to be *odd*.

The space of $\mathbb{C}$-linear maps $\mathbb{C}^{M|N} \to \mathbb{C}^{M|N}$, denoted $\operatorname{End}_{\mathbb{C}} \mathbb{C}^{M|N}$ (or $\operatorname{End} \mathbb{C}^{M|N}$ for short), carries the unique and natural $\mathbb{Z}_2$-grading via the assignment $[E_{ij}] := [i]+[j]$, where $\{E_{ij}\}_{i,j=1}^{M+N}$ is the collection of the matrix units of $\operatorname{End} \mathbb{C}^{M|N}$ with respect to the standard basis $\{e_i\}_{i=1}^{M+N}$.

Furthermore, $\operatorname{End} \mathbb{C}^{M|N}$ carries the structure of a Lie superalgebra with the Lie superbracket given by the super-commutator

$$[E_{ij}, E_{kl}] := \delta_{jk} E_{il} - (-1)^{([i]+[j])([k]+[l])} \delta_{li} E_{kj}.$$

When endowed with this extra structure, we shall denote the space $\operatorname{End} \mathbb{C}^{M|N}$ as $\mathfrak{gl}_{M|N} = \mathfrak{gl}(\mathbb{C}^{M|N})$ and call it the *general Lie superalgebra.*

Every space in this work will be regarded as an object in the symmetric monoidal category of super vector spaces over $\mathbb{C}$, denoted $\mathrm{sVect}_{\mathbb{C}}$, which is equipped with the *super-braiding* σ. As such, given any two objects X and Y in $\mathrm{sVect}_{\mathbb{C}}$ we have the isomorphism $\sigma_{X,Y}\colon X \otimes Y \to Y \otimes X$, $x \otimes y \mapsto (-1)^{[x][y]} y \otimes x$ on homogeneous elements $x \in X$ and $y \in Y$.

If we are given homogeneous linear maps $\varphi \in \operatorname{End} X$, $\psi \in \operatorname{End} Y$, then their (super) tensor product is the homogeneous linear map in $\operatorname{End} X \otimes \operatorname{End} Y \cong \operatorname{End}(X \otimes Y)$, denoted $\varphi \otimes \psi$, given by

$$\begin{aligned} \varphi \otimes \psi \colon X \otimes Y &\to X \otimes Y \\ x \otimes y &\mapsto (-1)^{[\psi][x]} \varphi(x) \otimes \psi(y) \end{aligned}$$

Note that when φ and ψ are even (or just ψ), then their (super) tensor product is just the traditional tensor product of linear maps.

If we further suppose that X and Y are superalgebras in $\mathrm{sVect}_{\mathbb{C}}$ with multiplication maps $\mu_X \colon X \otimes X \to X$ and $\mu_Y \colon Y \otimes Y \to Y$, then multiplication in $X \otimes Y$ is defined by the composition $(\mu_X \otimes \mu_Y) \circ (\mathrm{id}_X \otimes \sigma \otimes \mathrm{id}_Y)$. Explicitly, this multiplication is given by $(x_1 \otimes y_1)(x_2 \otimes y_2) = (-1)^{[y_1][x_2]} x_1 x_2 \otimes y_1 y_2$ on homogeneous elements. If the superalgebra X is also associative, then it may be given a Lie superalglebra structure via the Lie superbracket $[\cdot\,,\cdot] = \mu_X + \mu_X \circ \sigma$.

The *super-transpose* is the linear map defined by

$$\begin{aligned}(-)^{st} \colon \operatorname{End}\mathbb{C}^{M|N} &\to \operatorname{End}\mathbb{C}^{M|N}\\ E_{ij} &\mapsto E_{ij}^{st} := (-1)^{[i][j]+[i]} E_{ji}\end{aligned}$$

and satisfies the property $(A_1 A_2)^{st} = (-1)^{[A_1][A_2]} A_2^{st} A_1^{st}$ for homogeneous elements $A_1, A_2 \in \operatorname{End}\mathbb{C}^{M|N}$.

We note that operators $E_{ij} \otimes E_{kl}$ in $\operatorname{End}\mathbb{C}^{M|N} \otimes \operatorname{End}\mathbb{C}^{M|N}$ act on basis elements $e_a \otimes e_b$ via the formula

$$(E_{ij} \otimes E_{kl})\, e_a \otimes e_b = \delta_{ja}\delta_{lb}(-1)^{([k]+[l])[a]} e_i \otimes e_k$$

Let the *super permutation operator* in $\operatorname{End}\mathbb{C}^{M|N} \otimes \operatorname{End}\mathbb{C}^{M|N}$ be given by

$$P := \sum_{i,j=1}^{M+N} (-1)^{[j]} E_{ij} \otimes E_{ji}, \tag{4.2}$$

and note that this operator satisfies the relations

$$P^2 = \mathrm{id}^{\otimes 2} \quad \text{and} \quad P^{st_1 st_2} = P,$$

and acts on the standard basis vectors and matrix units via

$$P(e_a \otimes e_b) = (-1)^{[a][b]} e_b \otimes e_a \quad \text{and} \quad P(E_{ij} \otimes E_{kl})P = (-1)^{([i]+[j])([k]+[l])} E_{kl} \otimes E_{ij}.$$

Now, define the *quantum R-matrix* $R(u) \in \left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2}(u)$ to be the rational function in the formal parameter $u \in \mathbb{C}$ with coefficients in $\left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2}$ given by

$$R(u) := \mathrm{id}^{\otimes 2} - \frac{P}{u}. \tag{4.3}$$

The quantum R-matrix (4.3) satisfies the relations

$$PR(u)P = R(u) = R^{st_2 \circ st_1}(u) = R^{st_1 \circ st_2}(u) \quad \text{(symmetry)}, \tag{4.4}$$

$$R(u)R(-u) = \left(1 - \frac{1}{u^2}\right)\mathrm{id}^{\otimes 2} \quad \text{(unitarity)}. \tag{4.5}$$

Letting $\mathcal{A}$ denote a superalgebra, we note that for the index $1 \leq k \leq m$ there is an injective? morphism of superalgebras

$$\begin{aligned}\varphi_k \colon \operatorname{End}\mathbb{C}^{M|N} \otimes \mathcal{A} &\to \left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes m} \otimes \mathcal{A} \\ \psi \otimes w &\mapsto \mathrm{id}^{\otimes(k-1)} \otimes \psi \otimes \mathrm{id}^{\otimes(m-k)} \otimes w,\end{aligned}$$

and set $X_k = \varphi_k(X)$ for an element $X \in \operatorname{End}\mathbb{C}^{M|N} \otimes \mathcal{A}$. If $\mathcal{A}$ is a formal power series super algebra or if $X = X(u)$ depends on a formal parameter $u \in \mathbb{C}$, we shall write $X_k(u)$ instead of $X(u)_k$ for the element $\varphi_k(X(u))$.

Analogously, we will like to express elements of $\left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2}$ in $\left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes m}$ for some integer $m \geq 3$. For indices $1 \leq k < l \leq m$, we also have the injective? morphism of superalgebras

$$\begin{aligned}\varphi_{kl} \colon \left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2} &\to \left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes m} \\ a \otimes b &\mapsto \mathbf{1}^{\otimes(k-1)} \otimes a \otimes \mathbf{1}^{\otimes(l-k-1)} \otimes b \otimes \mathbf{1}^{\otimes(m-l)}\end{aligned}$$

and set $X_{kl} = \varphi_{kl}(X)$ for an element $X \in \left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2}$. Similarly, if $X = X(u)$ depends on a formal parameter $u \in \mathbb{C}$, then we write $X_{kl}(u)$ instead of $X(u)_{kl}$ for the element $\varphi_{kl}(X(u))$.

The quantum R-matrix (4.3) satisfies the *super quantum Yang-Baxter equation*

(SQYBE)

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u), \tag{4.6}$$

where the above relation lies in the space $\left(\mathrm{End}\,\mathbb{C}^{M|N}\right)^{\otimes 3}[\![u^{\pm 1}, v^{\pm 1}]\!]$.

Definition 4.1.1. The *super Yangian* $\mathbf{Y}(\mathfrak{gl}_{M|N})$ of $\mathfrak{gl}_{M|N}$ is the unital associative $\mathbb{C}$-superalgebra on generators $\{T_{ij}^{(n)} \mid 1 \leq i,j \leq M+N,\ n \in \mathbb{Z}^+\}$, with $\mathbb{Z}_2$-grade $[T_{ij}^{(n)}] := [i]+[j]$ for all $n \in \mathbb{Z}^+$, subject to the defining RTT-relation

$$\begin{aligned} R(u-v)T_1(u)T_2(v) &= T_2(v)T_1(u)R(u-v) \\ \text{in} \quad \left(\mathrm{End}\,\mathbb{C}^{M|N}\right)^{\otimes 2} &\otimes \mathbf{Y}(\mathfrak{gl}_{M|N})[\![u^{\pm 1}, v^{\pm 1}]\!], \end{aligned} \tag{4.7}$$

where $R(u-v)$ is the quantum R-matrix (4.3) identified with $R(u-v) \otimes \mathbf{1}$ and

$$T(u) := \sum_{i,j=1}^{M+N} E_{ij} \otimes T_{ij}(u) \in \mathrm{End}\,\mathbb{C}^{M|N} \otimes \mathbf{Y}(\mathfrak{gl}_{M|N})[\![u^{-1}]\!] \tag{4.8}$$

is the *generating matrix* consisting of the *generating series*

$$T_{ij}(u) := \sum_{n=0}^{\infty} T_{ij}^{(n)} u^{-n} \in \mathbf{Y}(\mathfrak{gl}_{M|N})[\![u^{-1}]\!], \quad \text{where} \quad T_{ij}^{(0)} = \delta_{ij}\mathbf{1}, \tag{4.9}$$

and $T_1(u)$, $T_2(u)$ are elements in $\left(\mathrm{End}\,\mathbb{C}^{M|N}\right)^{\otimes 2} \otimes \mathbf{Y}(\mathfrak{gl}_{M|N})[\![u^{-1}]\!]$ given by

$$T_1(u) = \sum_{i,j=1}^{M+N} E_{ij} \otimes \mathrm{id} \otimes T_{ij}(u), \quad T_2(u) = \sum_{i,j=1}^{M+N} \mathrm{id} \otimes E_{ij} \otimes T_{ij}(u).$$

Note that the $\mathbb{Z}_2$-grading on $\mathbf{Y}(\mathfrak{gl}_{M|N})$ induces one on $\mathbf{Y}(\mathfrak{gl}_{M|N})[\![u^{\pm 1}, v^{\pm 1}]\!]$ such that $[T_{ij}(u)] = [i]+[j]$. On the level of power series, the defining relations of the super Yangian are equivalent to the following relations in $\mathbf{Y}(\mathfrak{gl}_{M|N})[\![u^{\pm 1}, v^{\pm 1}]\!]$:

$$\left[T_{ij}(u), T_{kl}(v)\right] = \frac{1}{u-v}(-1)^{[i][k]+[i][l]+[k][l]}\left(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)\right), \tag{4.10}$$

where $[\cdot,\cdot]$ is understood as the Lie superbracket

$$\big[T_{ij}(u), T_{kl}(v)\big] = T_{ij}(u)T_{kl}(v) - (-1)^{([i]+[j])([k]+[l])}T_{kl}(v)T_{ij}(u).$$

In terms of relations on the explicit generators, the above relation is equivalent to

$$\big[T_{ij}^{(a+1)}, T_{kl}^{(b)}\big] - \big[T_{ij}^{(a)}, T_{kl}^{(b+1)}\big] = (-1)^{[i][k]+[i][l]+[k][l]}\big(T_{kj}^{(a)}T_{il}^{(b)} - T_{kj}^{(b)}T_{il}^{(a)}\big), \tag{4.11}$$

for $a, b \in \mathbb{N}$.

Letting $\mathfrak{U}(\mathfrak{gl}_{M|N})$ denote the universal enveloping superalgebra of $\mathfrak{gl}_{M|N}$, then there is a superalgebra epimorphism

$$\begin{aligned} \mathrm{ev}\colon \mathbf{Y}(\mathfrak{gl}_{M|N}) &\to \mathfrak{U}(\mathfrak{gl}_{M|N}) \\ T_{ij}(u) &\mapsto \delta_{ij} + (-1)^{[i][j]}E_{ij}u^{-1} \end{aligned} \tag{4.12}$$

called the *evaluation morphism.*

We shall consider two types of filtrations $\mathbf{F}^{\mathrm{Ab}} = \{\mathbf{F}_n^{\mathrm{Ab}}\}_{n\in\mathbb{N}}$ and $\mathbf{F} = \{\mathbf{F}_n\}_{n\in\mathbb{N}}$ on $\mathbf{Y}(\mathfrak{gl}_{M|N})$ via the respective filtration degree assignments

$$\deg_{\mathrm{Ab}} T_{ij}^{(n)} = n \quad \text{and} \quad \deg_{\mathbf{F}} T_{ij}^{(n)} = n-1. \tag{4.13}$$

By (4.10), the associated graded superalgebra $\mathrm{gr}_{\mathrm{Ab}}\,\mathbf{Y}(\mathfrak{gl}_{M|N}) = \bigoplus_{n\in\mathbb{N}} \mathbf{F}_n^{\mathrm{Ab}}/\mathbf{F}_{n-1}^{\mathrm{Ab}}$ is supercommutative.

Also note that the super Yangian $\mathbf{Y}(\mathfrak{gl}_{M|N})$ comes equipped with a Hopf superalgebra structure given by the comultiplication, counit, and antipode:

$$\Delta\colon T(u) \mapsto T_{[1]}(u)T_{[2]}(u), \quad \varepsilon\colon T(u) \mapsto \mathbb{1}, \quad S\colon T(u) \mapsto T(u)^{-1}, \tag{4.14}$$

where $T_{[1]}(u) = \sum_{i,j=1}^{M+N} E_{ij} \otimes T_{ij}(u) \otimes \mathbf{1}$ and $T_{[2]}(u) = \sum_{i,j=1}^{M+N} E_{ij} \otimes \mathbf{1} \otimes T_{ij}(u)$.

4.2 Reflection Superalgebras and Twisted Super Yangians

Consider now integers $M, N \in \mathbb{N}$ such that $M+N \geq 1$ and let us decompose these into sums of two non-negative integers: $M = p+k_M$ and $N = q+k_N$. Define now a matrix $\mathcal{G}_{M|N} = \mathcal{G} = (\mathcal{G}_{ij})_{i,j=1}^{M+N} \in \mathrm{Mat}_{M+N}(\mathbb{C})$ via the formula

$$\mathcal{G}_{ij} = \begin{cases} \delta_{ij} & \text{if } 1 \leq i \leq p \text{ or } M+1 \leq i \leq M+q \\ -\delta_{ij} & \text{if } p+1 \leq i \leq M \text{ or } M+q+1 \leq i \leq M+N \end{cases} \tag{4.1}$$

Definition 4.2.1. The *extended reflection superalgebra* $\mathbf{XB}(\mathfrak{gl}_{M|N}, p, q)$ of $\mathfrak{gl}_{M|N}$ is the unital associative $\mathbb{C}$-superalgebra on generators $\{\mathsf{B}_{ij}^{(n)} \mid 1 \leq i, j \leq M+N,\ n \in \mathbb{Z}^+\}$, with $\mathbb{Z}_2$-grade $[\mathsf{B}_{ij}^{(n)}] := [i]+[j]$ for all $n \in \mathbb{Z}^+$, subject to the defining *super reflection equation*

$$\begin{aligned} R(u-v)\mathsf{B}_1(u)R(u+v)\mathsf{B}_2(v) &= \mathsf{B}_2(v)R(u+v)\mathsf{B}_1(u)R(u-v) \\ \text{in} \quad &\left(\mathrm{End}\,\mathbb{C}^{M|N}\right)^{\otimes 2} \otimes \mathbf{XB}(\mathfrak{gl}_{M|N}, p, q)[\![u^{\pm 1}, v^{\pm 1}]\!], \end{aligned} \tag{4.2}$$

where $R(u-v)$ is the quantum R-matrix (4.3) identified with $R(u-v) \otimes \mathbf{1}$ and

$$\mathsf{B}(u) := \sum_{i,j=1}^{M+N} E_{ij} \otimes \mathsf{B}_{ij}(u) \in \mathrm{End}\,\mathbb{C}^{M|N} \otimes \mathbf{XB}(\mathfrak{gl}_{M|N}, p, q)[\![u^{-1}]\!] \tag{4.3}$$

is the *generating matrix* consisting of the *generating series*

$$\mathsf{B}_{ij}(u) := \sum_{n=0}^{\infty} \mathsf{B}_{ij}^{(n)} u^{-n} \in \mathbf{XB}(\mathfrak{gl}_{M|N}, p, q)[\![u^{-1}]\!], \quad \text{where} \quad \mathsf{B}_{ij}^{(0)} = \mathcal{G}_{ij}\mathbf{1}, \tag{4.4}$$

and $\mathsf{B}_1(u)$, $\mathsf{B}_2(u)$ are elements in $\left(\mathrm{End}\,\mathbb{C}^{M|N}\right)^{\otimes 2} \otimes \mathbf{XB}(\mathfrak{gl}_{M|N}, p, q)[\![u^{-1}]\!]$ given by

$$\mathsf{B}_1(u) = \sum_{i,j=1}^{M+N} E_{ij} \otimes \mathrm{id} \otimes \mathsf{B}_{ij}(u), \quad \mathsf{B}_2(u) = \sum_{i,j=1}^{M+N} \mathrm{id} \otimes E_{ij} \otimes \mathsf{B}_{ij}(u).$$

The $\mathbb{Z}_2$-grading on $\mathbf{XB}(\mathfrak{gl}_{M|N},p,q)$ induces one on $\mathbf{XB}(\mathfrak{gl}_{M|N},p,q)[\![u^{\pm 1},v^{\pm 1}]\!]$ such that $\big[\mathsf{B}_{ij}(u)\big]=[i]+[j]$. Furthermore, on the level of power series, the super reflection equation (4.2) is given by

$$
\begin{aligned}
\big[\mathsf{B}_{ij}(u),\mathsf{B}_{kl}(v)\big] = {} & \frac{1}{u-v}(-1)^{[i][k]+[i][l]+[k][l]}\Big(\mathsf{B}_{kj}(u)\mathsf{B}_{il}(v)-\mathsf{B}_{kj}(v)\mathsf{B}_{il}(u)\Big) \\
& + \frac{1}{u+v}\Bigg((-1)^{[j]([i]+[l])}\delta_{jk}\sum_{a=1}^{M+N}(-1)^{([i]+[l])[a]+[a]}\mathsf{B}_{ia}(u)\mathsf{B}_{al}(v) \\
& \qquad - (-1)^{[i]+[j][k]}\delta_{il}\sum_{a=1}^{M+N}(-1)^{([k]+[j])[a]+[a]}\mathsf{B}_{ka}(v)\mathsf{B}_{aj}(u)\Bigg) \\
& - \frac{1}{u^2-v^2}\delta_{ij}\Bigg(\sum_{a=1}^{M+N}(-1)^{([k]+[a])([a]+[l])}\mathsf{B}_{ka}(u)\mathsf{B}_{al}(v) \\
& \qquad - \sum_{a=1}^{M+N}(-1)^{([k]+[a])([a]+[l])}\mathsf{B}_{ka}(v)\mathsf{B}_{al}(u)\Bigg)
\end{aligned} \tag{4.5}
$$

in the space $\mathbf{XB}(\mathfrak{gl}_{M|N},p,q)[\![u^{\pm 1},v^{\pm 1}]\!]$, where $[\,\cdot\,,\cdot\,]$ is understood as the Lie superbracket

$$\big[\mathsf{B}_{ij}(u),\mathsf{B}_{kl}(v)\big]=\mathsf{B}_{ij}(u)\mathsf{B}_{kl}(v)-(-1)^{([i]+[j])([k]+[l])}\mathsf{B}_{kl}(v)\mathsf{B}_{ij}(u).$$

Proposition 4.2.2. *There is a series* $\mathsf{f}(u)=\mathbf{1}+\sum_{n=1}^{\infty}\mathsf{f}^{(2n)}u^{-2n}\in\mathbf{XB}(\mathfrak{gl}_{M|N},p,q)[\![u^{-2}]\!]$ *of homogeneous $\mathbb{Z}_2$-degree $\bar{0}$ such that*

$$\mathsf{B}(u)\mathsf{B}(-u)=\mathsf{B}(-u)\mathsf{B}(u)=\mathrm{id}\otimes\mathsf{f}(u) \tag{4.6}$$

and the coefficients of $\mathsf{f}(u)$ *lie in the center of* $\mathbf{XB}(\mathfrak{gl}_{M|N},p,q)$.

Proof. By multiplying the defining relations (4.5) with the polynomial u^2-v^2 and substituting $v=-u$, we yield the relation

$$2u\Bigg((-1)^{[j]([i]+[l])}\delta_{jk}\sum_{a=1}^{M+N}(-1)^{([i]+[l])[a]+[a]}\mathsf{B}_{ia}(u)\mathsf{B}_{al}(-u)$$

$$- (-1)^{[i]+[j][k]}\delta_{il}\sum_{a=1}^{M+N}(-1)^{([k]+[j])[a]+[a]}\mathsf{B}_{ka}(-u)\mathsf{B}_{aj}(u)\Bigg)$$

$$= \delta_{ij}\left(\sum_{a=1}^{M+N}(-1)^{([k]+[a])([a]+[l])}\mathsf{B}_{ka}(u)\mathsf{B}_{al}(-u) - \sum_{a=1}^{M+N}(-1)^{([k]+[a])([a]+[l])}\mathsf{B}_{ka}(-u)\mathsf{B}_{al}(u)\right) \tag{4.7}$$

For $i = j$, $k \neq j = i$, and $l \neq i$ (thus, $k \neq l$), the above equation becomes

$$\sum_{a=1}^{M+N}(-1)^{([k]+[a])([a]+[l])}\mathsf{B}_{ka}(u)\mathsf{B}_{al}(-u) = \sum_{a=1}^{M+N}(-1)^{([k]+[a])([a]+[l])}\mathsf{B}_{ka}(-u)\mathsf{B}_{al}(u).$$

Alternatively, when $i = j = k = l$, one can factor out the term $(2(-1)^{[k]}u - 1)$ to obtain

$$\sum_{a=1}^{M+N}(-1)^{[k]+[a]}\mathsf{B}_{ka}(u)\mathsf{B}_{ak}(-u) = \sum_{a=1}^{M+N}(-1)^{[k]+[a]}\mathsf{B}_{ka}(-u)\mathsf{B}_{ak}(u).$$

Thus, from these last two relations we get

$$\sum_{k,l,a=1}^{M+N}(-1)^{([k]+[a])([a]+[l])}E_{kl}\otimes\mathsf{B}_{ka}(u)\mathsf{B}_{al}(-u)$$
$$= \sum_{k,l,a=1}^{M+N}(-1)^{([k]+[a])([a]+[l])}E_{kl}\otimes\mathsf{B}_{ka}(-u)\mathsf{B}_{al}(u),$$

the expanded form of $\mathsf{B}(u)\mathsf{B}(-u) = \mathsf{B}(-u)\mathsf{B}(u)$. Now, when $i \neq j$, $k = j$, and $l = i$ (thus, $k \neq l$), then

$$2u\sum_{a=1}^{M+N}(-1)^{[a]}\mathsf{B}_{la}(u)\mathsf{B}_{al}(-u) = 2u(-1)^{[k]+[l]}\sum_{a=1}^{M+N}(-1)^{[a]}\mathsf{B}_{ka}(-u)\mathsf{B}_{ak}(u)$$

and hence,

$$\sum_{a=1}^{M+N}(-1)^{[l]+[a]}\mathsf{B}_{la}(u)\mathsf{B}_{al}(-u) = \sum_{a=1}^{M+N}(-1)^{[k]+[a]}\mathsf{B}_{ka}(-u)\mathsf{B}_{ak}(u).$$

Therefore, we may introduce the following well-defined series for any $1 \leq k \leq M+N$

by the previous equality:

$$\mathsf{f}(u) = \sum_{a=1}^{M+N} (-1)^{[k]+[a]} \mathsf{B}_{ka}(u)\mathsf{B}_{ak}(-u) = \sum_{a=1}^{M+N} (-1)^{[k]+[a]} \mathsf{B}_{ka}(-u)\mathsf{B}_{ak}(u).$$

Note that the coefficients of $\mathsf{f}(u)$ are of $\mathbb{Z}_2$-grade $\bar{0}$ and its constant term is provided by $\sum_{a=1}^{M+N}(-1)^{[k]+[a]}\mathcal{G}_{ka}\mathcal{G}_{ak}\mathbf{1} = \mathcal{G}_{kk}^2\mathbf{1} = \mathbf{1}$. Now, if $k \neq j$, $i = l$, and $i \neq j$, then

$$-2u(-1)^{[i]+[j][k]} \sum_{a=1}^{M+N} (-1)^{([k]+[j])[a]+[a]} \mathsf{B}_{ka}(-u)\mathsf{B}_{aj}(u) = 0,$$

so

$$\sum_{a=1}^{M+N} (-1)^{([k]+[a])([a]+[j])} \mathsf{B}_{ka}(-u)\mathsf{B}_{aj}(u) = 0. \tag{4.8}$$

Hence, $\mathsf{B}(-u)\mathsf{B}(u)$ is given by

$$\sum_{k=1}^{M+N} E_{kk} \otimes \sum_{a=1}^{M+N} (-1)^{[k]+[a]} \mathsf{B}_{ka}(-u)\mathsf{B}_{ak}(u) \\ + \sum_{\substack{k,l=1,\\ k\neq l}}^{M+N} E_{kl} \otimes \sum_{a=1}^{M+N} (-1)^{([k]+[a])([a]+[j])} \mathsf{B}_{ka}(-u)\mathsf{B}_{aj}(u),$$

which is consequently equal to $\mathrm{id} \otimes \mathsf{f}(u)$. Multiplying the super reflection equation (4.2) on the right by $\mathsf{B}_2(-v)$ yields the equation

$$R(u-v)\mathsf{B}_1(u)R(u+v)\left(\mathrm{id}^{\otimes 2} \otimes \mathsf{f}(v)\right) = \mathsf{B}_2(v)R(u+v)\mathsf{B}_1 R(u-v)\mathsf{B}_2(-v).$$

However, by translating $v \mapsto -v$ in the super reflection equation, we therefore have

$$\begin{aligned} \mathsf{B}_2(v)R(u+v)_1 R(u-v)\mathsf{B}_2(-v) &= \mathsf{B}_2(v)\mathsf{B}_2(-v)R(u-v)\mathsf{B}_1(u)R(u+v) \\ &= \left(\mathrm{id}^{\otimes 2} \otimes \mathsf{f}(v)\right) R(u-v)\mathsf{B}_1(u)R(u+v), \end{aligned}$$

resulting in the equation

$$R(u-v)\mathsf{B}_1(u)R(u+v)\left(\mathrm{id}^{\otimes 2} \otimes \mathsf{f}(v)\right) = \left(\mathrm{id}^{\otimes 2} \otimes \mathsf{f}(v)\right) R(u-v)\mathsf{B}_1(u)R(u+v)$$

Now, multiply the above equation on the left by $R(v-u)\left(\frac{(u-v)^2}{(u-v)^2-1}\right)$ and on the right by $R(-u-v)\left(\frac{(u+v)^2}{(u+v)^2-1}\right)$. By equation (4.5) and the the fact that all the maps in tensor factors of $\mathrm{id}^{\otimes 2}\otimes \mathsf{f}(v)$ are of $\mathbb{Z}_2$-grade $\bar{0}$, we yield the relation

$$\mathsf{B}_1(u)\left(\mathrm{id}^{\otimes 2}\otimes \mathsf{f}(v)\right) = \left(\mathrm{id}^{\otimes 2}\otimes \mathsf{f}(v)\right)\mathsf{B}_1(u),$$

and therefore $\mathsf{B}_{ij}(u)\mathsf{f}(v) = \mathsf{f}(v)\mathsf{B}_{ij}(u)$ for all $1 \le i,j \le M+N$. □

Let $(\mathsf{f}(u)-\mathbf{1})$ denote the two-sided ideal of $\mathbf{XB}(\mathfrak{gl}_{M|N},p,q)$ generated by the coefficients of $\mathsf{f}(u)-\mathbf{1}$.

Definition 4.2.3. The *reflection superalgebra* $\mathbf{B}(\mathfrak{gl}_{M|N},p,q)$ of $\mathfrak{gl}_{M|N}$ is the quotient of $\mathbf{XB}(\mathfrak{gl}_{M|N},p,q)$ by the two-sided ideal $(\mathsf{f}(u)-\mathbf{1})$:

$$\mathbf{B}(\mathfrak{gl}_{M|N},p,q) = \mathbf{XB}(\mathfrak{gl}_{M|N},p,q)/\left(\mathsf{f}(u)-\mathbf{1}\right).$$

Equivalently, the reflection superalgebra $\mathbf{B}(\mathfrak{gl}_{M|N},p,q)$ of $\mathfrak{gl}_{M|N}$ is the unital associative $\mathbb{C}$-superalgebra on generators $\left\{B_{ij}^{(n)} \mid 1 \le i,j \le M+N,\ n \in \mathbb{Z}^+\right\}$, with $\mathbb{Z}_2$-grade $\left[B_{ij}^{(n)}\right] := [i]+[j]$ for all $n \in \mathbb{Z}^+$, subject to the defining *super reflection equation*

$$\begin{aligned} R(u-v)B_1(u)R(u+v)B_2(v) &= B_2(v)R(u+v)B_1(u)R(u-v) \\ \text{in} \quad &\left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2} \otimes \mathbf{B}(\mathfrak{gl}_{M|N},p,q)[\![u^{\pm 1},v^{\pm 1}]\!], \end{aligned} \tag{4.9}$$

and *unitary condition*

$$B(u)B(-u) = \mathbb{1} \in \operatorname{End}\mathbb{C}^{M|N} \otimes \mathbf{B}(\mathfrak{gl}_{M|N},p,q)[\![u^{\pm 1}]\!] \tag{4.10}$$

where $R(u-v)$ is the quantum R-matrix (4.3) identified with $R(u-v)\otimes \mathbf{1}$ and

$$B(u) := \sum_{i,j=1}^{M+N} E_{ij}\otimes B_{ij}(u) \in \operatorname{End}\mathbb{C}^{M|N} \otimes \mathbf{B}(\mathfrak{gl}_{M|N},p,q)[\![u^{-1}]\!] \tag{4.11}$$

is the *generating matrix* consisting of the *generating series*

$$B_{ij}(u) := \sum_{n=0}^{\infty} B_{ij}^{(n)} u^{-n} \in \mathbf{B}(\mathfrak{gl}_{M|N}, p, q)[\![u^{-1}]\!], \quad \text{where} \quad B_{ij}^{(0)} = \mathcal{G}_{ij}\mathbf{1}, \tag{4.12}$$

and $B_1(u)$, $B_2(u)$ are elements in $\big(\operatorname{End}\mathbb{C}^{M|N}\big)^{\otimes 2} \otimes \mathbf{B}(\mathfrak{gl}_{M|N}, p, q)[\![u^{-1}]\!]$ given by

$$B_1(u) = \sum_{i,j=1}^{M+N} E_{ij} \otimes \mathrm{id} \otimes B_{ij}(u), \quad B_2(u) = \sum_{i,j=1}^{M+N} \mathrm{id} \otimes E_{ij} \otimes B_{ij}(u).$$

We shall consider two types of filtrations $\mathbf{E}^{\mathrm{Ab}} = \{\mathbf{E}_n^{\mathrm{Ab}}\}_{n\in\mathbb{N}}$ and $\mathbf{E} = \{\mathbf{E}_n\}_{n\in\mathbb{N}}$ on $\mathbf{B}(\mathfrak{gl}_{M|N}, p, q)$ via the respective filtration degree assignments

$$\deg_{\mathrm{Ab}} B_{ij}^{(n)} = n \quad \text{and} \quad \deg_{\mathbf{E}} B_{ij}^{(n)} = n-1. \tag{4.13}$$

Due to an analogue of relations (4.5) for generators $B_{ij}^{(n)}$, the associated graded algebra $\mathrm{gr}_{\mathrm{Ab}}\, \mathbf{B}(\mathfrak{gl}_{M|N}, p, q) = \bigoplus_{n\in\mathbb{N}} \mathbf{E}_n^{\mathrm{Ab}}/\mathbf{E}_{n-1}^{\mathrm{Ab}}$ is supercommutative.

We now wish to establish some notation: Set $\mathbb{Z}^+_{M+N} = \mathbb{Z} \cap [1, M+N]$ and consider the subset $C \subset (\mathbb{Z}^+_{M+N})^2$ consisting of all pairs (i,j) that satisfy any of the following inequalities:

$$\begin{gathered} 1 \le i,j \le p; \quad p+1 \le i,j \le M; \quad M+1 \le i,j \le M+q; \quad M+q+1 \le i,j \le M+N; \\ 1 \le i \le p,\ M+1 \le j \le M+q; \quad M+1 \le i \le M+q,\ 1 \le j \le p; \\ p+1 \le i \le M,\ M+q+1 \le j \le M+N; \quad M+q+1 \le i \le M+N,\ p+1 \le j \le M. \end{gathered} \tag{4.14}$$

Finally, we define the subset $\mathcal{K} \subset (\mathbb{Z}^+_{M+N})^2 \times \mathbb{Z}^+$ to be the collection of all 3-tuples (i,j,n) that satisfy either

$$(i,j) \in C,\ n \in 2\mathbb{Z}^+ - 1 \quad \text{or} \quad (i,j) \in (\mathbb{Z}^+_{M+N})^2 \setminus C,\ n \in 2\mathbb{Z}^+. \tag{4.15}$$

Proposition 4.2.4. *The set* $\big\{B_{ij}^{(n)}\big\}_{(i,j,n)\in\mathcal{K}}$ *generates* $\mathbf{B}(\mathfrak{gl}_{M|N}, p, q)$.

Proof. Let $\mathcal{A}$ denote the subalgebra generated by the elements $B_{ij}^{(n)}$ where $(i,j,n) \in \mathcal{K}$. We shall prove the following statement for all $1 \leq r, s \leq M+N$ via induction on $m \in \mathbb{N}$:

$$B_{rs}^{(m)} \in \mathcal{A}$$

First, by the unitary condition (4.10) we have

$$\sum_{k+l=m} (-1)^l \sum_{a=1}^{M+N} (-1)^{([i]+[a])([a]+[j])} B_{ia}^{(k)} B_{aj}^{(l)} = 0 \quad \text{for} \quad m \in \mathbb{Z}^+.$$

For instance, when $m = 1$ this equation becomes $(\mathcal{G}_{jj} - \mathcal{G}_{ii}) B_{ij}^{(1)} = 0$. Therefore,

$$B_{ij}^{(1)} = 0 \quad \text{for all} \quad (i,j) \in (\mathbb{Z}_{M+N}^+)^2 \setminus C. \tag{4.16}$$

For $m = 2$, we get

$$(\mathcal{G}_{ii} + \mathcal{G}_{jj}) B_{ij}^{(2)} = \sum_{a=1}^{M+N} (-1)^{([i]+[a])([a]+[j])} B_{ia}^{(1)} B_{aj}^{(1)}.$$

Therefore, by (4.16) we have $B_{ij}^{(2)} \in \mathcal{A}$ for $(i,j) \in C$. Let us assume now the induction hypothesis holds for $m-1$. In this case,

$$(\mathcal{G}_{jj} + (-1)^m \mathcal{G}_{ii}) B_{ij}^{(m)} = -\sum_{l=1}^{m-1} (-1)^l \sum_{a=1}^{M+N} (-1)^{([i]+[a])([a]+[j])} B_{ia}^{(m-l)} B_{aj}^{(l)},$$

so by induction hypothesis, $B_{ij}^{(m)} \in \mathcal{A}$ for $(i,j) \in (\mathbb{Z}_{M+N}^+)^2 \setminus C$ if m is odd and $(i,j) \in C$ if m is even. $\square$

Definition 4.2.5. The *twisted super Yangian* $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}$ of $\mathfrak{gl}_{M|N}$ is the sub-superalgebra of $\mathbf{Y}(\mathfrak{gl}_{M|N})$ generated by the coefficients $\{S_{ij}^{(n)} \mid 1 \leq i, j \leq M+N,\ n \in$

$\mathbb{Z}^+\}$ of

$$S(u) = T(u)\mathcal{G}T(-u)^{-1} \in \left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2} \otimes \mathbf{Y}(\mathfrak{gl}_{M|N})[\![u^{-1}]\!],$$
$$\text{where} \quad S(u) := \sum_{i,j=1}^{M+N} E_{ij} \otimes S_{ij}(u) \quad \text{and} \quad S_{ij}(u) := \sum_{n=0}^{\infty} S_{ij}^{(n)} u^{-n}. \tag{4.17}$$

If we write $T(u)^{-1} = \sum_{i,j=1}^{M+N} E_{ij} \otimes T_{ij}^{\bullet}(u)$ such that $T_{ij}^{\bullet}(u) = \delta_{ij}\mathbf{1} + \sum_{n=1}^{\infty} T_{ij}^{\bullet(n)} u^{-n}$, then more explicitly,

$$S_{ij}(u) = \sum_{a=1}^{M+N} (-1)^{([i]+[a])([a]+[j])} \mathcal{G}_{aa} T_{ia}(u) T_{aj}^{\bullet}(-u). \tag{4.18}$$

To see this, expand $T(u)\mathcal{G}T(-u)^{-1}$ by

$$\begin{aligned}
&\left(\sum_{i,k=1}^{M+N} E_{ik} \otimes T_{ik}(u)\right)\left(\sum_{a=1}^{M+N} \mathcal{G}_{aa} E_{aa} \otimes \mathbf{1}\right)\left(\sum_{l,j=1}^{M+N} E_{lj} \otimes T_{lj}^{\bullet}(-u)\right) \\
&\qquad = \left(\sum_{i,a=1}^{M+N} \mathcal{G}_{aa} E_{ia} \otimes T_{ia}(u)\right)\left(\sum_{l,j=1}^{M+N} E_{lj} \otimes T_{lj}^{\bullet}(-u)\right) \\
&\qquad = \sum_{i,j=1}^{M+N} E_{ij} \otimes \sum_{a=1}^{M+N} (-1)^{([i]+[a])([a]+[j])} \mathcal{G}_{aa} T_{ia}(u) T_{aj}^{\bullet}(-u).
\end{aligned}$$

Furthermore, note that the coefficients of $T_{ij}^{\bullet}(u)$ are provided by

$$T_{ij}^{\bullet(n)} = -T_{ij}^{(n)} + \sum_{s=2}^{n} (-1)^s \sum_{\sum_{j=1}^{s} k_j = n} \left(\sum_{a_1,\ldots,a_{s-1}=1}^{M+N} (-1)^{\phi(i,j;a_1,\ldots,a_{s-1})} T_{ia_1}^{(k_1)} T_{a_1a_2}^{(k_2)} \cdots T_{a_{s-1}j}^{(k_s)}\right),$$

where $\phi(i,j;a_1,\ldots,a_{s-1}) = [i][a_1]+[i][j]+[a_{s-1}][j]+\sum_{b=1}^{s-1}[a_b]+\sum_{1\le b<c\le s-1}[a_b][a_c]$ and $k_j \in \mathbb{Z}^+$ for each k_j in the sum $\sum_{j=1}^{s} k_j = n$. Hence, $S_{ij}(u)$ is homogeneous of degree $[S_{ij}(u)] = [i]+[j]$ since $[T_{aj}^{\bullet}(-u)] = [a]+[j]$.

Note that the filtrations $\mathbf{F}^{\mathrm{Ab}}$ and $\mathbf{F}$ on $\mathbf{Y}(\mathfrak{gl}_{M|N})$ will endow filtrations on the twisted super Yangian $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}$, which we will also denote $\mathbf{F}^{\mathrm{Ab}}$ and $\mathbf{F}$ respectively, via the filtration degree assignments

$$\deg_{\mathrm{Ab}} S_{ij}^{(n)} = n \quad \text{and} \quad \deg_{\mathbf{F}} S_{ij}^{(n)} = n-1. \tag{4.19}$$

Lemma 4.2.6. *Let $\bar{S}_{ij}^{(n)}$ denote the image of $S_{ij}^{(n)}$ in n-th graded component $\mathbf{F}_n^{\mathrm{Ab}} / \mathbf{F}_{n-1}^{\mathrm{Ab}}$ of the graded superalgebra $\mathrm{gr}_{\mathrm{Ab}} \mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw} = \bigoplus_{n\in\mathbb{N}} \mathbf{F}_n^{\mathrm{Ab}} / \mathbf{F}_{n-1}^{\mathrm{Ab}}$. Fixing a total order $\preceq$ on the index set $\mathcal{K}$ defined by (4.15), then the collection of all ordered monomials of the form*

$$\bar{S}_{i_1 j_1}^{(n_1)} \bar{S}_{i_2 j_2}^{(n_2)} \cdots \bar{S}_{i_k j_k}^{(n_k)} \tag{4.20}$$

with $(i_a, j_a, n_a) \in \mathcal{K}$, $(i_a, j_a, n_a) \preceq (i_{a+1}, j_{a+1}, n_{a+1})$, and $(i_a, j_a, n_a) \neq (i_{a+1}, j_{a+1}, n_{a+1})$ if $[i_a]+[j_a] = 1$, are linearly independent.

Proof. We recall the associated graded superalgebra $\mathrm{gr}_{\mathrm{Ab}} \mathbf{Y}(\mathfrak{gl}_{M|N})$ and let us have $\overline{T}_{ij}^{(n)}$ denote the image of the generator $T_{ij}^{(n)}$ in the n-th graded component. As was shown in the proof of Theorem 1 in Section 2 in [?], if we endow a total order $\preceq$ on the set $\mathcal{I} = (\mathbb{Z}_{M+N}^{+})^2 \times \mathbb{Z}^{+}$, then the collection of all ordered monomials of the form

$$\overline{T}_{i_1 j_1}^{(n_1)} \overline{T}_{i_2 j_2}^{(n_2)} \cdots \overline{T}_{i_k j_k}^{(n_k)} \tag{4.21}$$

with $(i_a, j_a, n_a) \in \mathcal{I}$, $(i_a, j_a, n_a) \preceq (i_{a+1}, j_{a+1}, n_{a+1})$, and $(i_a, j_a, n_a) \neq (i_{a+1}, j_{a+1}, n_{a+1})$ if $[i_a]+[j_a] = 1$, forms a basis for $\mathrm{gr}_{\mathrm{Ab}} \mathbf{Y}(\mathfrak{gl}_{M|N})$.

Let us now introduce another filtration $\overline{\mathbf{F}} = \{\overline{\mathbf{F}}_n\}_{n\in\mathbb{N}}$ but instead on $\mathrm{gr}_{\mathrm{Ab}} \mathbf{Y}(\mathfrak{gl}_{M|N})$ via $\deg_{\overline{\mathbf{F}}} \overline{T}_{ij}^{(n)} = n-1$. Therefore, from the description (4.18), we have

$$\bar{S}_{ij}^{(n)} \equiv (\mathcal{G}_{jj} + (-1)^{n+1} \mathcal{G}_{ii}) \overline{T}_{ij}^{(n)} \mod \overline{\mathbf{F}}_{n-2}, \tag{4.22}$$

so $\bar{S}_{ij}^{(n)} \equiv \pm 2 \overline{T}_{ij}^{(n)} \bmod \overline{\mathbf{F}}_{n-2}$ if and only if $(i, j, k) \in \mathcal{K}$.

If we do assume to the contrary that there exists a non-trivial linear combination A of ordered monomials of the form (4.20) such that $A = 0$, then let R denote the linear combination of those monomials occurring in A of maximal $\overline{\mathbf{F}}$-filtration degree, say α

(so each monomial occuring in R lies in $\overline{\mathbf{F}}_\alpha \setminus \overline{\mathbf{F}}_{\alpha-1}$). However, the equivalence (4.22) would imply $0 = A \equiv R \equiv R' \operatorname{mod} \overline{\mathbf{F}}_{\alpha-1}$, where R' is a non-trivial linear combination of ordered monomials of the form (4.21) of $\overline{\mathbf{F}}$-filtration degree α, so $R' = R''$ for some $R'' \in \overline{\mathbf{F}}_{\alpha-1}$. Hence, since no monomial in R' occurs in $\overline{\mathbf{F}}_{\alpha-1}$, then $R' - R''$ is a non-trivial linear combination of ordered monomials of the form (4.21) that equals 0, contradicting linear independence. □

Theorem 4.2.7. *There is a superalgebra isomorphism*

$$\begin{aligned} \varphi\colon \mathbf{B}(\mathfrak{gl}_{M|N}, p, q) &\to \mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw} \\ B(u) &\mapsto S(u) \end{aligned} \tag{4.23}$$

Proof. It is immediate that the map φ is homogeneous and surjective. To show φ is a superalgebra morphism, it is fast to check $S(u)S(-u) = \mathbb{1}$ is satisfied since $\mathcal{G}^2 = \mathrm{id}$. To show to $S(u)$ satisfies the super reflection equation, we require the use of the following equations obtained from the RTT-relation:

$$T_1(-u)^{-1}R(u+v)T_2(v) = T_2(v)R(u+v)T_1(-u)^{-1} \tag{4.24}$$

$$R(u-v)T_1(-u)^{-1}T_2(-v)^{-1} = T_2(-v)^{-1}T_1(-u)^{-1}R(u-v) \tag{4.25}$$

$$T_1(u)R(u+v)T_2(-v)^{-1} = T_2(-v)^{-1}R(u+v)T_1(u) \tag{4.26}$$

To obtain the first equation (4.24) above, we swap $u \mapsto -u$ in the RTT-relation to get

$$\begin{aligned} T_2(v)T_1(-u)R(-u-v) = R(-u-v)T_1(-u)T_2(v)& \\ \Rightarrow R(u+v)T_2(v)T_1(-u) &= T_1(-u)T_2(v)R(u+v) \\ \Rightarrow T_1(-u)^{-1}R(u+v)T_2(v) &= T_2(v)R(u+v)T_1(-u)^{-1}, \end{aligned}$$

where the first implication follows from multiplying on the left by $\frac{(u+v)^2}{(u+v)^2-1}R(u+v)$ and the right by $R(u+v)$. Equation (4.25) follows from (4.24) from the swap $v \mapsto -v$. Equation (4.26) follows from the RTT-relation by swapping $v \mapsto -v$ to obtain

$$T_2(-v)T_1(u)R(u+v) = R(u+v)T_1(u)T_2(-v)$$

$$\Rightarrow T_1(u)R(u+v)T_2(-v)^{-1} = T_2(-v)^{-1}R(u+v)T_1(u).$$

Now, by using (4.24) and the fact that $\mathcal{G}_i$ commutes with $T_j(u)$ and $T_j(-u)^{-1}$ for integers $1 \leq i \neq j \leq 2$, the expression $R(u-v)S_1(u)R(u+v)S_2(v)$ is given by

$$\begin{aligned}
&R(u-v)T_1(u)\mathcal{G}_1T_1(-u)^{-1}R(u+v)T_2(v)\mathcal{G}_2T_2(-v)^{-1} \\
&\qquad = R(u-v)T_1(u)\mathcal{G}_1T_2(v)R(u+v)T_1(-u)^{-1}\mathcal{G}_2T_2(-v)^{-1} \\
&\qquad = R(u-v)T_1(u)T_2(v)\mathcal{G}_1R(u+v)\mathcal{G}_2T_1(-u)^{-1}T_2(-v)^{-1} \\
&\qquad = T_2(v)T_1(u)R(u-v)\mathcal{G}_1R(u+v)\mathcal{G}_2T_1(-u)^{-1}T_2(-v)^{-1},
\end{aligned}$$

where we used the RTT-relation in the last equality. Furthermore, since $\mathcal{G}_1\mathcal{G}_2 = \mathcal{G}_2\mathcal{G}_1$ and $\mathcal{G}_iP = P\mathcal{G}_i$ for $i = 1, 2$, then $R(u-v)\mathcal{G}_1R(u+v)\mathcal{G}_2 = \mathcal{G}_2R(u+v)\mathcal{G}_1R(u-v)$. Therefore, by using this equality, equations (4.25), (4.26), and techniques from before, the above expression becomes

$$\begin{aligned}
&T_2(v)T_1(u)\mathcal{G}_2R(u+v)\mathcal{G}_1R(u-v)T_1(-u)^{-1}T_2(-v)^{-1} \\
&\qquad = T_2(v)T_1(u)\mathcal{G}_2R(u+v)\mathcal{G}_1T_2(-v)^{-1}T_1(-u)^{-1}R(u-v) \\
&\qquad = T_2(v)\mathcal{G}_2T_1(u)R(u+v)T_2(-v)^{-1}\mathcal{G}_1T_1(-u)^{-1}R(u-v) \\
&\qquad = T_2(v)\mathcal{G}_2T_2(-v)^{-1}R(u+v)T_1(u)\mathcal{G}_1T_1(-u)^{-1}R(u-v),
\end{aligned}$$

which is the expression $S_2(v)R(u+v)S_1(u)R(u-v)$.

To show the injectivity of the map φ, we shall prove that its kernel must be zero. To this end, recall the filtrations $\mathbf{E}^{\mathrm{Ab}} = \{\mathbf{E}^{\mathrm{Ab}}_n\}_{n\in\mathbb{N}}$ and $\mathbf{F}^{\mathrm{Ab}} = \{\mathbf{F}^{\mathrm{Ab}}_n\}_{n\in\mathbb{N}}$ on $\mathbf{B}(\mathfrak{gl}_{M|N}, p, q)$ and $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}$, respectively, via the filtration degree assignments $\deg_{\mathrm{Ab}} B_{ij}^{(n)} = n$ and $\deg_{\mathrm{Ab}} S_{ij}^{(n)} = n$. Since φ is filtration preserving, it therefore defines an superalgebra morphism on the associated graded superalgebras

$$\operatorname{gr}\varphi\colon\ \operatorname{gr}_{\mathrm{Ab}}\mathbf{B}(\mathfrak{gl}_{M|N}, p, q) \to \operatorname{gr}_{\mathrm{Ab}}\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw} \tag{4.27}$$

By Proposition (4.2.4), $\{\bar{B}_{ij}^{(n)}\}_{(i,j,n)\in\mathcal{K}}$ generates $\operatorname{gr}_{\mathrm{Ab}}\mathbf{B}(\mathfrak{gl}_{M|N}, p, q)$, so the map $\operatorname{gr}\varphi$ is injective by Lemma (4.2.6). ☐

Corollary 4.2.8 (PBW Theorem)**.** *Fix a total order $\preceq$ on the index set $\mathcal{K}$ defined by* (4.15)*.*

(i) *The collection of all ordered monomials of the form*

$$B_{i_1 j_1}^{(n_1)} B_{i_2 j_2}^{(n_2)} \cdots B_{i_k j_k}^{(n_k)} \tag{4.28}$$

w/ $(i_a, j_a, n_a) \in \mathcal{K}$, $(i_a, j_a, n_a) \preceq (i_{a+1}, j_{a+1}, n_{a+1})$, and $(i_a, j_a, n_a) \neq (i_{a+1}, j_{a+1}, n_{a+1})$ if $[i_a]+[j_a] = 1$, forms a basis for $\mathbf{B}(\mathfrak{gl}_{M|N}, p, q)$.

(ii) *The collection of all ordered monomials of the form*

$$S_{i_1 j_1}^{(n_1)} S_{i_2 j_2}^{(n_2)} \cdots S_{i_k j_k}^{(n_k)} \tag{4.29}$$

w/ $(i_a, j_a, n_a) \in \mathcal{K}$, $(i_a, j_a, n_a) \preceq (i_{a+1}, j_{a+1}, n_{a+1})$, and $(i_a, j_a, n_a) \neq (i_{a+1}, j_{a+1}, n_{a+1})$ if $[i_a]+[j_a] = 1$, forms a basis for $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}$.

Corollary 4.2.9. *The twisted super Yangian $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}$ is the unital associative $\mathbb{C}$-superalgebra on generators $\{S_{ij}^{(n)} \mid 1 \leq i, j \leq M+N,\ n \in \mathbb{Z}^+\}$, with $\mathbb{Z}_2$-grade $[S_{ij}^{(n)}] := [i]+[j]$ for all $n \in \mathbb{Z}^+$, subject to the defining relations*

$$\begin{aligned}
\big[S_{ij}(u), S_{kl}(v)\big] = {} & \frac{1}{u-v}(-1)^{[i][k]+[i][l]+[k][l]}\Big(S_{kj}(u)S_{il}(v) - S_{kj}(v)S_{il}(u)\Big) \\
& + \frac{1}{u+v}\Bigg((-1)^{[j]([i]+[l])}\delta_{jk}\sum_{a=1}^{M+N}(-1)^{([i]+[l])[a]+[a]}S_{ia}(u)S_{al}(v) \\
& \qquad - (-1)^{[i]+[j][k]}\delta_{il}\sum_{a=1}^{M+N}(-1)^{([k]+[j])[a]+[a]}S_{ka}(v)S_{aj}(u)\Bigg) \\
& - \frac{1}{u^2-v^2}\delta_{ij}\Bigg(\sum_{a=1}^{M+N}(-1)^{([k]+[a])([a]+[l])}S_{ka}(u)S_{al}(v) \\
& \qquad - \sum_{a=1}^{M+N}(-1)^{([k]+[a])([a]+[l])}S_{ka}(v)S_{al}(u)\Bigg)
\end{aligned} \tag{4.30}$$

and

$$\sum_{a=1}^{M+N}(-1)^{([i]+[a])([a]+[j])}S_{ia}(u)S_{aj}(-u)=\delta_{ij}\mathbf{1}. \tag{4.31}$$

Proposition 4.2.10. *The sub-superalgebra $\mathbf{Y}(\mathfrak{gl}_{M|N},p,q)^{tw}$ is a left coideal of $\mathbf{Y}(\mathfrak{gl}_{M|N})$:*

$$\Delta(\mathbf{Y}(\mathfrak{gl}_{M|N},p,q)^{tw})\subseteq\mathbf{Y}(\mathfrak{gl}_{M|N})\otimes\mathbf{Y}(\mathfrak{gl}_{M|N},p,q)^{tw}$$

Proof. By virtue that Δ is a morphism of superalgebras, having it act on the equality $T(u)T(u)^{-1}=\mathrm{id}\otimes\mathbf{1}$ yields

$$\Delta(T(u)^{-1})=(T_{[2]}(u))^{-1}(T_{[1]}(u))^{-1}.$$

Since $(T_{[2]}(u))^{-1}=\sum_{i,a=1}^{M+N}E_{ia}\otimes\mathbf{1}\otimes T^{\bullet}_{ia}(u)$ and $(T_{[1]}(u))^{-1}=\sum_{a,j=1}^{M+N}E_{aj}\otimes T^{\bullet}_{aj}(u)\otimes\mathbf{1}$, we get

$$\Delta(T^{\bullet}_{ij}(u))=\sum_{b=1}^{M+N}T^{\bullet}_{bj}(u)\otimes T^{\bullet}_{ib}(u) \tag{4.32}$$

Hence,

$$\begin{aligned}
&\Delta(S_{ij}(u))\\
&\quad=\sum_{c=1}^{M+N}(-1)^{([i]+[c])([c]+[j])}\mathcal{G}_{cc}\Delta(T_{ic}(u))\Delta(T^{\bullet}_{cj}(-u))\\
&\quad=\sum_{a,b,c=1}^{M+N}(-1)^{([i]+[c])([c]+[j])+([i]+[a])([a]+[c])}\mathcal{G}_{cc}(T_{ia}(u)\otimes T_{ac}(u))(T^{\bullet}_{bj}(-u)\otimes T^{\bullet}_{cb}(-u))\\
&\quad=\sum_{a,b,c=1}^{M+N}(-1)^{([i]+[c])([c]+[j])+([i]+[a]+[b]+[j])([a]+[c])}\mathcal{G}_{cc}T_{ia}(u)T^{\bullet}_{bj}(-u)\otimes T_{ac}(u)T^{\bullet}_{cb}(-u)\\
&\quad=\sum_{a,b=1}^{M+N}(-1)^{([i]+[c])([c]+[j])+([i]+[a]+[c]+[j])([a]+[c])}T_{ia}(u)T^{\bullet}_{bj}(-u)\otimes S_{ab}(u)\\
&\quad=\sum_{a,b=1}^{M+N}(-1)^{([i]+[a])([a]+[j])}T_{ia}(u)T^{\bullet}_{bj}(-u)\otimes S_{ab}(u),
\end{aligned} \tag{4.33}$$

completing the proof. $\square$

4.3 Highest Weight Representations

Note that from (4.18) we have $\mathcal{G}_{kk}S^{(1)}_{kk} = 2T^{(1)}_{kk}$ and so the span of $S^{(1)}_{kk}$, $1 \le k \le M+N$ is the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}_{M|N}$ identified under the superalgebra embedding $\mathfrak{U}(\mathfrak{gl}_{M|N}) \hookrightarrow \mathbf{Y}(\mathfrak{gl}_{M|N})$, $E_{ij} \mapsto (-1)^{[i][j]}T^{(1)}_{ij}$.

We consider the basis $\{\mathcal{H}_k\}_{k=1}^{M+N}$ of $\mathfrak{h}$, given by $\mathcal{H}_k = (-1)^{[k]}\mathcal{G}_{kk}S^{(1)}_{kk}$, with its dual basis $\{\varepsilon_k\}_{k=1}^{M+N} \subset \mathfrak{h}^*$ to yield the root system $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \le i, j \le M+N,\, i \ne j\}$ of $\mathfrak{gl}_{M|N}$. We also use the standard root system decomposition $\Phi = \Phi^+ \sqcup \Phi^-$, where $\Phi^+ = \{\varepsilon_i - \varepsilon_j \in \Phi \mid i < j\}$ and $\Phi^- = \{\varepsilon_i - \varepsilon_j \in \Phi \mid i > j\}$ and call the linear functionals in $\Phi^\pm$ as *positive/negative roots*. Note that via the relations (4.30), we have

$$\big[S^{(1)}_{kl}, S_{ij}(u)\big] = (-1)^{[i][k]+[i][l]+[k][l]}(\mathcal{G}_{kk} + \mathcal{G}_{ll})\big(\delta_{il}S_{kj}(u) - \delta_{jk}S_{il}(u)\big), \tag{4.1}$$

and hence

$$\big[\mathcal{H}_k, S_{ij}(u)\big] = 2(\delta_{ik} - \delta_{jk})S_{ij}(u) = 2(\varepsilon_i - \varepsilon_j)(\mathcal{H}_k)S_{ij}(u), \tag{4.2}$$

or more generally, since $\big[\mathcal{H}_k, S^{(n_1)}_{i_1j_1}\cdots S^{(n_k)}_{i_kj_k}\big] = \mathcal{H}_kS^{(n_1)}_{i_1j_1}\cdots S^{(n_b)}_{i_bj_b} - S^{(n_1)}_{i_1j_1}\cdots S^{(n_b)}_{i_bj_b}\mathcal{H}_k$, then

$$\begin{aligned}
\big[\mathcal{H}_k, S^{(n_1)}_{i_1j_1}\cdots S^{(n_b)}_{i_bj_b}\big] &= \big[\mathcal{H}_k, S^{(n_1)}_{i_1j_1}\big]S^{(n_2)}_{i_2j_2}\cdots S^{(n_b)}_{i_bj_b} + S^{(n_1)}_{i_1j_1}\mathcal{H}_kS^{(n_2)}_{i_2j_2}\cdots S^{(n_b)}_{i_bj_b} - S^{(n_1)}_{i_1j_1}\cdots S^{(n_b)}_{i_bj_b}\mathcal{H}_k \\
&= 2(\varepsilon_{i_1} - \varepsilon_{j_1})(\mathcal{H}_k)S^{(n_1)}_{i_1j_1}\cdots S^{(n_b)}_{i_bj_b} + S^{(n_1)}_{i_1j_1}\big[\mathcal{H}_k, S^{(n_2)}_{i_2j_2}\big]S^{(n_3)}_{i_3j_3}\cdots S^{(n_b)}_{i_bj_b} \\
&\quad + S^{(n_1)}_{i_1j_1}S^{(n_2)}_{i_2j_2}\mathcal{H}_kS^{(n_3)}_{i_3j_3}\cdots S^{(n_b)}_{i_bj_b} - S^{(n_1)}_{i_1j_1}\cdots S^{(n_b)}_{i_bj_b}\mathcal{H}_k \\
&\ \ \vdots \\
&= 2\Big(\sum_{a=1}^{b}(\varepsilon_{i_a} - \varepsilon_{j_a})\Big)(\mathcal{H}_k)S^{(n_1)}_{i_1j_1}\cdots S^{(n_b)}_{i_bj_b},
\end{aligned}$$

so we have the following analogue of a root space decomposition:

$$\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw} = \left(\sum_{n\in\mathbb{N}}\sum_{k=1}^{M+N}\mathbb{C}S^{(n)}_{kk}\right) \oplus \bigoplus_{\alpha\in\Phi}\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}_{2\alpha},$$

where $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}_{2\alpha} = \{X \in \mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw} \mid [H, X] = 2\alpha(H)X \text{ for all } H \in \mathfrak{h}\}$.

We also have familiar notions of weights and weight vectors for representations V of $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}$: for $\lambda \in \mathfrak{h}^*$, if $V_\lambda := \{v \in V \mid H \cdot v = \lambda(H)v \text{ for all } H \in \mathfrak{h}\} \neq 0$ then λ is called a *weight* of V, V_λ is called a *weight space*, and non-zero vectors in V_λ are called *weight vectors*. We endow a partial ordering $\preceq$ on the set of weights of V via the rule $\omega \preceq \lambda \Leftrightarrow \lambda - \omega$ is an $\mathbb{N}$-linear combination of positive roots of $\mathfrak{gl}_{M|N}$. Furthermore, since

$$\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}_{2\alpha}(V_\lambda) \subseteq V_{\lambda+2\alpha},$$

then

$$\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}_{2\alpha}\Big(\bigoplus\nolimits_{\lambda\in\mathfrak{h}^*} V_\lambda\Big) \subseteq \bigoplus\nolimits_{\lambda\in\mathfrak{h}^*} V_\lambda. \tag{4.3}$$

Definition 4.3.1. A representation V of the twisted super Yangian $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}$ is defined as a *highest weight representation* if there exists a non-zero vector $\xi \in V$ such that $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}\xi = V$, and

$$\begin{aligned} S_{ij}(u)\xi &= 0 \qquad \text{for all} \quad 1 \leq i < j \leq M+N \\ \text{and} \quad S_{kk}(u)\xi &= \lambda_k(u)\xi \qquad \text{for all} \quad 1 \leq k \leq M+N, \end{aligned} \tag{4.4}$$

where $\lambda_k(u)$ is some formal series

$$\lambda_k(u) = \mathcal{G}_{kk} + \sum_{n=1}^{\infty} \lambda_k^{(n)} u^{-n} \in 1 + \mathbb{C}[\![u^{-1}]\!]u^{-1}. \tag{4.5}$$

We say that ξ is the *highest weight vector* of V and call the $M+N$-tuple $(\lambda_k(u))_{k=1}^{M+N}$ of formal series the *highest weight* of V.

To prove the first theorem of this section, we need the following lemma:

Lemma 4.3.2. *Let $\mathcal{J}$ be the left ideal of $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}$ generated by the coefficients of $S_{ij}(u)$, where $1 \leq i < j \leq M+N$. Then*

(i) $S_{ij}(u)S_{kk}(v) \equiv 0 \mod \mathcal{J}$ *for all* $1 \leq i < j \leq M+N$ *and* $1 \leq k \leq M+N$,

(ii) $[S_{kk}(u), S_{ll}(v)] \equiv 0 \mod \mathcal{J}$ *for all* $1 \leq k, l \leq M+N$.

Proof. For brevity, we shall use '$\equiv$' to denote equivalence of elements in $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}$ modulo $\mathcal{J}$.

(i) We shall prove the statement via reverse strong induction on k. For the initial case $k = M+N$, the relations (4.30) imply $S_{ij}(u)S_{M+N,M+N}(v) \equiv 0$ for $i < j < M+N$. When $i < j = M+N$, the same relations imply

$$S_{i,M+N}(u)S_{M+N,M+N}(v) \equiv \frac{1}{u+v}(-1)^{[M+N]}S_{i,M+N}(u)S_{M+N,M+N}(v),$$

so

$$\frac{u+v-(-1)^{[k]}}{u+v}S_{i,M+N}(u)S_{M+N,M+N}(v) \equiv 0$$

and hence $S_{i,M+N}(u)S_{M+N,M+N}(v) \equiv 0$.

Suppose now the hypothesis holds down to $k+1$. We first note that if $i < l$, then when $i < j$ the relations (4.30) imply

$$(-1)^{[j]([i]+[l])}S_{ij}(u)S_{jl}(v) \equiv \frac{1}{u+v}\sum_{a=l}^{M+N}(-1)^{([i]+[l])[a]+[a]}S_{ia}(u)S_{al}(v). \tag{4.6}$$

Since the right side of the equivalence (4.6) is independent of the index j, we have the following equivalence for all indices $i < j_1, j_2, l$:

$$S_{ij_1}(u)S_{j_1l}(v) \equiv (-1)^{([j_1]+[j_2])([i]+[l])}S_{ij_2}(u)S_{j_2l}(v). \tag{4.7}$$

If $i < j < k$, then the defining relations (4.30) immediately show $S_{ij}(u)S_{kk}(v) \equiv 0$, so let us now suppose $i < j = k$. By the equivalences (4.6) and (4.7), we obtain

$$\begin{aligned} S_{ik}(u)S_{kk}(v) &\equiv \frac{1}{u+v}(-1)^{[k]([i]+[k])}\sum_{a=k}^{M+N}(-1)^{([i]+[k])[a]+[a]}S_{ia}(u)S_{ak}(v) \\ &\equiv \frac{(-1)^{[k]}(M+1-k)-N}{u+v}S_{ik}(u)S_{kk}(v), \end{aligned}$$

where $\sum_{a=k}^{M+N}(-1)^{[a]} = (-1)^{[k]}(M+1-k)-N$, so $S_{ik}(u)S_{kk}(v) \equiv 0$.

Remark 4.3.3. We have $\sum_{a=k}^{M+N}(-1)^{[a]} = (M+1-k)-N$ if $k \leq M$ and $\sum_{a=k}^{M+N}(-1)^{[a]} = -(M+N+1)+k$ if $k > M$, so $\sum_{a=k}^{M+N}(-1)^{[a]} = (-1)^{[k]}(M+1-k)-N$.

When $k = i < j$, relations (4.30) gives

$$\begin{aligned} S_{kj}(u)S_{kk}(v) \equiv & \frac{1}{u-v}(-1)^{[k]}\big(S_{kj}(u)S_{kk}(v) - S_{kj}(v)S_{kk}(u)\big) \\ & - \frac{1}{u+v}(-1)^{[k]+[j][k]}\sum_{a=j}^{M+N}(-1)^{([k]+[j])[a]+[a]}S_{ka}(v)S_{aj}(u). \end{aligned}$$

However, by the equivalence (4.7) the sum $\sum_{a=j}^{M+N}(-1)^{([k]+[j])[a]+[a]}S_{ka}(v)S_{aj}(u)$ becomes $(-1)^{[j][k]}(M+1-j-(-1)^{[j]}N)S_{kj}(u)S_{jj}(v)$, which is equivalent to zero by our induction hypothesis. We therefore have the relation

$$\frac{u-v-(-1)^{[k]}}{u-v}S_{kj}(u)S_{kk}(v) + \frac{(-1)^{[k]}}{u-v}S_{kj}(v)S_{kk}(u) \equiv 0. \tag{4.8}$$

Furthermore, by exchanging $u \leftrightarrow v$ in (4.8), we get

$$-\frac{(-1)^{[k]}}{u-v}S_{kj}(u)S_{kk}(v) + \frac{u-v+(-1)^{[k]}}{u-v}S_{kj}(v)S_{kk}(u) \equiv 0, \tag{4.9}$$

so by taking the difference of (4.8) and (4.9), one yields $S_{kj}(u)S_{kk}(v) \equiv S_{kj}(v)S_{kk}(u)$. Hence, either (4.8) or (4.9) shows $S_{kj}(u)S_{kk}(v) \equiv 0$.

Lastly, when $k < i < j$, the defining relations (4.30) imply

$$S_{ij}(u)S_{kk}(v) \equiv \frac{1}{u-v}(-1)^{[k]}\big(S_{kj}(u)S_{ik}(v) - S_{kj}(v)S_{ik}(v)\big)$$

and

$$\begin{aligned} S_{kj}(u)S_{ik}(v) \equiv & \frac{1}{u-v}(-1)^{[k]}\big(S_{ij}(u)S_{kk}(v) - S_{ij}(v)S_{kk}(u)\big) \\ & - \frac{1}{u+v}(-1)^{[i][j]+[k]}\sum_{a=j}^{M+N}(-1)^{([i]+[j])[a]+[a]}S_{ia}(v)S_{aj}(u). \end{aligned}$$

By using the equivalence (4.7), then the sum $\sum_{a=j}^{M+N}(-1)^{([i]+[j])[a]+[a]}S_{ia}(v)S_{aj}(u)$ becomes $(-1)^{[i][j]}(M+1-j-(-1)^{[j]}N)S_{ij}(v)S_{jj}(u)$, which is equivalent to zero by the induction hypothesis. Therefore, $S_{kj}(u)S_{ik}(v) \equiv S_{kj}(v)S_{ik}(u)$ and so $S_{ij}(u)S_{kk}(v) \equiv 0$.

(ii) We shall first prove that $[S_{kk}(u), S_{kk}(v)] \equiv 0$ for all $1 \leq k \leq M+N$ via reverse strong induction on k.

We first note that by setting

$$\rho_k(u,v) = \sum_{a=k+1}^{M+N} (-1)^{[a]}\big(S_{ka}(u)S_{ak}(v) - S_{ka}(v)S_{ak}(u)\big), \tag{4.10}$$

then relations (4.30) imply

$$\begin{aligned}\big[S_{kk}(u), S_{kk}(v)\big] \equiv \frac{(-1)^{[k]}}{u-v}\big[S_{kk}(u), S_{kk}(v)\big] + \frac{(-1)^{[k]}}{u+v}\big[S_{kk}(u), S_{kk}(v)\big] + \frac{1}{u+v}\rho_k(u,v)\\ - \frac{1}{u^2-v^2}\big[S_{kk}(u), S_{kk}(v)\big] - \frac{(-1)^{[k]}}{u^2-v^2}\rho_k(u,v)\end{aligned}$$

which becomes

$$\left(1 + \frac{1-2(-1)^{[k]}u}{u^2-v^2}\right)\big[S_{kk}(u), S_{kk}(v)\big] \equiv \frac{u-v-(-1)^{[k]}}{u^2-v^2}\rho_k(u,v). \tag{4.11}$$

Hence, $[S_{kk}(u), S_{kk}(v)] \equiv 0$ if and only if $\rho_k(u,v) \equiv 0$. The base case $k = M+N$ is therefore satisfied since $\rho_{M+N}(u,v)$ is the empty sum, so now suppose the induction statement is true down to $k+1$.

Supposing $k < a$, relations (4.30) assert

$$\begin{aligned}\big[S_{aa}(u), S_{kk}(v)\big] \equiv \frac{1}{u-v}(-1)^{[k]}\big(S_{ka}(u)S_{ak}(v) - S_{ka}(v)S_{ak}(u)\big)\\ - \frac{1}{u^2-v^2}\Big(\big[S_{kk}(u), S_{kk}(v)\big] + (-1)^{[k]}\rho_k(u,v)\Big).\end{aligned}$$

and

$$\big[S_{kk}(v), S_{aa}(u)\big] \equiv \frac{1}{u^2-v^2}\Big(\big[S_{aa}(u), S_{aa}(v)\big] + (-1)^{[a]}\rho_a(u,v)\Big).$$

Therefore, since $[S_{aa}(u), S_{kk}(v)] = -[S_{kk}(v), S_{aa}(u)]$ we get

$$\begin{aligned} &S_{ka}(u)S_{ak}(v) - S_{ka}(v)S_{ak}(u) \\ &\quad \equiv \frac{(-1)^{[k]}}{u+v}\Big(\big[S_{kk}(u), S_{kk}(v)\big] - \big[S_{aa}(u), S_{aa}(v)\big] + (-1)^{[k]}\rho_k(u,v) - (-1)^{[a]}\rho_a(u,v)\Big). \end{aligned} \tag{4.12}$$

Hence, by (4.10),

$$\begin{aligned} \rho_k(u,v) \equiv \frac{M-k-(-1)^{[k]}N}{u+v}&\Big(\big[S_{kk}(u), S_{kk}(v)\big] + (-1)^{[k]}\rho_k(u,v)\Big) \\ &- \frac{(-1)^{[k]}}{u+v}\sum_{a=k+1}^{M+N}\Big((-1)^{[a]}\big[S_{aa}(u), S_{aa}(v)\big] + \rho_a(u,v)\Big), \end{aligned}$$

so we have

$$(u+v+N-(-1)^{[k]}(M-k))\rho_k(u,v) \equiv (M-k-(-1)^{[k]}N)[S_{kk}(u), S_{kk}(v)] \tag{4.13}$$

since $[S_{aa}(u), S_{aa}(v)] \equiv 0 \equiv \rho_a(u,v)$ for $a > k$ by induction. By combining the equivalences (4.11) and (4.13), one concludes $[S_{kk}(u), S_{kk}(v)] \equiv 0 \equiv \rho_k(u,v)$.

To finish the proof, it suffices to show $[S_{kk}(u), S_{ll}(v)] \equiv 0$ for $1 \leq k < l \leq M+N$. To this end, we realize by the relations (4.30) that

$$\big[S_{kk}(u), S_{ll}(v)\big] \equiv -\frac{1}{u^2-v^2}\Big(\big[S_{ll}(u), S_{ll}(v)\big] + (-1)^{[l]}\rho_l(u,v)\Big),$$

which is equivalent to zero by before. □

Theorem 4.3.4. *Every finite-dimensional irreducible representation V of the twisted super Yangian $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}$ is a highest weight representation. Moreover, V a contains unique, up to scalar multiples, highest weight vector.*

Proof. Let V be a finite-dimensional irreducible representation of $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}$ and let us define

$$V^0 := \{v \in V \mid S_{ij}(u)v = 0 \text{ for all } 1 \leq i < j \leq M+N\} \tag{4.14}$$

We first establish that V^0 is non-trivial. From (4.1), we observe that $\{\mathcal{H}_k\}_{k=1}^{M+N}$ is a family of pairwise commuting elements. As operators on V, the finite-dimensionality of V implies that they have a common eigenvector $0 \neq \zeta \in V$: so $\mathcal{H}_k\zeta = \alpha(\mathcal{H}_k)\zeta$ for some non-zero linear functional $\alpha \in \mathfrak{h}^*$. If we assume to the contrary that V^0 is trivial, then there exists an infinite sequence of non-zero vectors

$$\big(S_{i_a j_a}^{(n_a)} \cdots S_{i_1 j_1}^{(n_1)}\zeta\big)_{a=1}^{\infty} \tag{4.15}$$

in V such that $i_a < j_a$ for each $a \in \mathbb{Z}^+$. However, by (4.2), we realize

$$\mathcal{H}_k S_{ij}^{(n)}\zeta = (\alpha + 2(\varepsilon_i - \varepsilon_j))(\mathcal{H}_k) S_{ij}^{(n)}\zeta$$

and more generally, we have

$$\mathcal{H}_k\big(S_{i_a j_a}^{(n_a)} \cdots S_{i_1 j_1}^{(n_1)}\zeta\big) = \Big(\alpha + 2\sum_{b=1}^{a}(\varepsilon_{i_b} - \varepsilon_{j_b})\Big)(\mathcal{H}_k) S_{i_a j_a}^{(n_a)} \cdots S_{i_1 j_1}^{(n_1)}\zeta,$$

so all elements $S_{i_a j_a}^{(n_a)} \cdots S_{i_1 j_1}^{(n_1)}\zeta$ in the sequence (4.15) are weight vectors corresponding to weights $\alpha + 2\sum_{b=1}^{a}(\varepsilon_{i_b} - \varepsilon_{j_b})$. However, all these weights are different by virtue of the partial ordering $\preceq$ on the weights of V (if two such weights were equal, then this implies $\sum_{b=a_1}^{a_2}(\varepsilon_{i_b} - \varepsilon_{j_b}) = 0$ for some positive integers $a_1 < a_2$. However, by writing such expression in terms of the dual basis $\{\varepsilon_k\}_{k=1}^{M+N}$, then the coefficient of ε_{i_m}, where i_m is the minimal index occurring in the expression, is a positive integer $\circledast$) and hence there exists an infinite dimensional subspace $\bigoplus_{a=1}^{\infty} V_{\alpha+2\sum_{b=1}^{a}(\varepsilon_{i_b}-\varepsilon_{j_b})}$ of V, a contradiction.

Now, by Lemma (5.1.10) the set $\{S_{kk}^{(n)} \mid 1 \leq k \leq M+N,\ n \in \mathbb{Z}^+\}$ is a family of pairwise commuting operators on V^0. Therefore, since V^0 is finite-dimensional and non-trivial, there exists a common eigenvector $0 \neq \xi \in V^0$ for such operators: $S_{kk}^{(n)}\xi = \lambda_k^{(n)}\xi$ for complex eigenvalues $\lambda_k^{(n)}$, $1 \leq k \leq M+N$, $n \in \mathbb{Z}^+$. By the irreducibility of V, we observe $\mathbf{Y}(\mathfrak{gl}_{M|N},p,q)^{tw}\xi = V$ and by collecting these eigenvalues into power series $\lambda_k(u) = \mathcal{G}_{kk} + \sum_{n=1}^{\infty}\lambda_k^{(n)}u^{-n}$, we see the vector ξ satisfies the conditions (4.4), so V is a highest weight representation with highest weight vector ξ and highest weight $(\lambda_k(u))_{k=1}^{M+N}$.

It remains to show that ξ is unique up to scalar multiples. Recalling the PBW Theorem (4.2.8), we fix a total order $\preceq$ on the index set $\mathcal{K}$ (4.15) such that for any tuples $(i_1,j_1,n_1),(i_2,j_2,n_2),(i_3,j_3,n_3)\in\mathcal{K}$ satisfying $i_1>j_1$, $i_2=j_2$, $i_3<j_3$, then $(i_1,j_1,n_1)\preceq(i_2,j_2,n_2)\preceq(i_3,j_3,n_3)$. Via this total ordering ordering, we conclude that V is spanned by ordered elements of the form

$$S_{i_1j_1}^{(n_1)}\cdots S_{i_kj_k}^{(n_k)}\xi, \tag{4.16}$$

where $k\in\mathbb{N}$, $i_a>j_a$, and $(i_a,j_a,n_a)\in\mathcal{K}$ for $1\leq a\leq k$. By (4.2), the elements (4.16) are weight vectors of weights

$$\nu+2\sum_{a=1}^{k}(\varepsilon_{i_a}-\varepsilon_{j_a})$$

where ν is the linear functional on $\mathfrak{h}$ given by $\mathcal{H}_k\mapsto(-1)^{[k]}\mathcal{G}_{kk}\lambda_k^{(1)}$. Hence, there is a weight space decomposition $V=\bigoplus_{\mu\in\mathfrak{h}^*}V_\mu$ where each weight $\mu\neq\nu$ is of the form $\nu-2\sum_{a=1}^{k}(\varepsilon_{i_a}-\varepsilon_{j_a})$ for $i_a<j_a$, $a=1,\ldots,k$. Consequently, V_ν is 1-dimensional; namely, $V_\nu=\operatorname{span}_{\mathbb{C}}\{\xi\}$.

Now, if $\tilde{\xi}$ is another highest weight vector of V of highest weight $(\lambda_k(u))_{k=1}^{M+N}$, then a replicative argument as above will show that $\tilde{\xi}\in V_\nu$. Hence, $\tilde{\xi}=c\xi$ for some $c\in\mathbb{C}^*$. □

Definition 4.3.5. Given an $M+N$-tuple $\lambda(u)=(\lambda_k(u))_{k=1}^{M+N}$ of the form (4.5), the *Verma module* $M(\lambda(u))$ is the quotient $\mathbf{Y}(\mathfrak{gl}_{M|N},p,q)^{tw}/\mathcal{I}_{\lambda(u)}$ of $\mathbf{Y}(\mathfrak{gl}_{M|N},p,q)^{tw}$ by the left ideal $\mathcal{I}_{\lambda(u)}$ generated by the coefficients of $S_{ij}(u)$, $1\leq i<j\leq M+N$, and $S_{kk}(u)-\lambda_k(u)\mathbf{1}$, $1\leq k\leq M+N$.

One aptly observes that when $M(\lambda(u))$ is non-trivial, it is a highest weight representation of $\mathbf{Y}(\mathfrak{gl}_{M|N},p,q)^{tw}$ with highest weight $\lambda(u)$ and highest weight vector $\mathbf{1}_{\lambda(u)}$, the image of $\mathbf{1}$ in the canonical projection $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw} \to M(\lambda(u))$. Furthermore, if L is a highest weight representation of $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}$ with highest weight $\lambda(u)$ and highest weight vector ξ, then there is a surjective $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}$-module morphism $\varphi\colon M(\lambda(u)) \to L$ induced by the assignment $\mathbf{1}_{\lambda(u)} \mapsto \xi$; thus, $L \cong M(\lambda(u))/\ker\varphi$.

By (4.3), $\bigoplus_{\mu\in\mathfrak{h}^*} M(\lambda(u))_\mu$ is invariant under the action of $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}$. Therefore, since $\mathbf{1}_{\lambda(u)}$ is contained in $M(\lambda(u))_{\lambda^{(1)}} \subset \bigoplus_{\mu\in\mathfrak{h}^*} M(\lambda(u))_\mu$, where $\lambda^{(1)} \in \mathfrak{h}^*$ is the linear functional given by $\lambda^{(1)}(\mathcal{H}_k) = \lambda_k^{(1)}$, we have the weight decomposition

$$M(\lambda(u)) = \bigoplus_{\mu\in\mathfrak{h}^*} M(\lambda(u))_\mu$$

and each weight μ is of the form $\lambda^{(1)} - \omega$, where ω is a $\mathbb{Z}^+$-linear combination of positive roots of $\mathfrak{gl}_{M|N}$.

Recalling the PBW Theorem (4.2.8), we fix a total order $\preceq$ on the index set $\mathcal{K}$ (4.15) such that for any tuples $(i_1, j_1, n_1), (i_2, j_2, n_2), (i_3, j_3, n_3) \in \mathcal{K}$ satisfying $i_1 > j_1$, $i_2 = j_2$, $i_3 < j_3$, then $(i_1, j_1, n_1) \preceq (i_2, j_2, n_2) \preceq (i_3, j_3, n_3)$. Via this total ordering ordering, we conclude that $M(\lambda(u))$ is spanned by ordered elements of the form

$$S_{i_1 j_1}^{(n_1)} \cdots S_{i_k j_k}^{(n_k)} \mathbf{1}_{\lambda(u)}, \tag{4.17}$$

where $k \in \mathbb{N}$, $i_a > j_a$, and $(i_a, j_a, n_a) \in \mathcal{K}$ for $1 \leq a \leq k$. In particular, we conclude that $M(\lambda(u))_{\lambda^{(1)}}$ is 1-dimensional; $M(\lambda(u))_{\lambda^{(1)}} = \operatorname{span}_{\mathbb{C}}\{\mathbf{1}_{\lambda(u)}\}$.

Any submodule P of $M(\lambda(u))$ also has a weight space decomposition $P = \bigoplus_{\mu\in\mathfrak{h}^*} P_\mu$, where $P_\mu = P \cap M(\lambda(u))_\mu$. Indeed, if we suppose to the contrary that $P \neq \bigoplus_{\mu\in\mathfrak{h}^*} P_\mu$, then there exists some element $x \in P$ such that $x = \sum_{i=1}^n m_i$, where $m_i \in M(\lambda(u))_{\mu_i} \setminus P_{\mu_i}$, and consider such an element x wherein the positive integer n is minimal. Since $\mu_{n-1} \neq \mu_n$, there is some $H \in \mathfrak{h}$ such that $\mu_{n-1}(H) \neq \mu_n(H)$. Since $Hx \in P$, then $Hx - \mu_n(H)x = \sum_{i=1}^{n-1}(\mu_i(H) - \mu_n(H))m_i \in P$, where $(\mu_{n-1}(H) - \mu_n(H))m_{n-1} \in M(\lambda(u))_{\mu_{n-1}} \setminus P_{\mu_{n-1}}$ and $(\mu_i(H) - \mu_n(H))m_i \in \left(M(\lambda(u))_{\mu_i} \setminus P_{\mu_i}\right) \cup \{0\}$ for $i = 1, 2, \ldots, n-2$, contradicting the minimality of n.

Given any proper submodule P of $M(\lambda(u))$, since $P = \bigoplus_{\mu \in \mathfrak{h}^*} P_\mu$ we conclude that if $P_{\lambda^{(1)}} \neq 0$, then $\mathbf{1}_{\lambda(u)} \in P_{\lambda^{(1)}}$ since $\dim M(\lambda(u))_{\lambda^{(1)}} = 1$; hence, $P \subseteq \bigoplus_{\lambda^{(1)} \neq \mu \in \mathfrak{h}^*} M(\lambda(u))_\mu$ and so the sum of all proper submodules $K = \sum_{P < M(\lambda(u))} P$ is the unique maximal submodule of $M(\lambda(u))$.

Definition 4.3.6. When the Verma module $M(\lambda(u))$ is non-trivial, we define the *irreducible highest weight representation* $L(\lambda(u))$ of $\mathbf{Y}(\mathfrak{gl}_{M|N}, p, q)^{tw}$ with highest weight $\lambda(u)$ as the quotient of $M(\lambda(u))$ by its unique maximal proper submodule.

Chapter 5

Appendix

5.1 Orthosymplectic Super Yangians

5.1.1 Ring Theory & Hopf Algebra Theory

Proposition 5.1.1. *If A is a unital ring, then the inverse $a(u)^{-1}$ of the power series $a(u) = \mathbf{1} + \sum_{n=1}^{\infty} a_n u^{-n} \in \mathbf{1} + A[\![u^{-1}]\!]u^{-1}$ is given by*

$$\frac{\mathbf{1}}{\mathbf{1} + \sum_{n=1}^{\infty} a_n u^{-n}} = \mathbf{1} + \sum_{n=1}^{\infty} \left(\sum_{s=1}^{n} (-1)^s \sum_{\sum_{j=1}^{s} k_j = n} a_{k_1} \cdots a_{k_s} \right) u^{-n},$$

where $k_i \in \mathbb{Z}^+$ for $i = 1, \ldots, s$ in the sum $\sum_{j=1}^{s} k_j = n$.

Proof. The inverse of $a(u)$ is the series $b(u) = \mathbf{1} + \sum_{n=1}^{\infty} b_n u^{-n}$, where the coefficients of $b(u)$ are given recursively by the formula

$$b_n = -\sum_{i=1}^{n} a_i b_{n-i} \quad \text{for} \quad n \in \mathbb{Z}^+,$$

where $b_0 = 1$. Let us define $c_n = \sum_{s=1}^{n}(-1)^s \sum_{\sum_{j=1}^{s} k_j = n} a_{k_1} \cdots a_{k_s}$, where $c_0 = 1$, and so we claim that $c_n = b_n$ for all $n \in \mathbb{Z}^+$. For $n = 1$, it is immediate to see $c_1 = -a_1 = b_1$. Assuming the claim holds for integers up to n, we observe that c_{n+1} is the sum

$$\begin{aligned}
\sum_{s=1}^{n+1}(-1)^s \sum_{\sum_{j=1}^{s} k_j = n+1} a_{k_1} \cdots a_{k_s} &= -a_{n+1} - \sum_{i=1}^{n} a_i \sum_{s=2}^{n+2-i} (-1)^{s-1} \sum_{\sum_{j=2}^{s} k_j = n+1-i} a_{k_2} \cdots a_{k_s} \\
&= -a_{n+1} - \sum_{i=1}^{n} a_i \sum_{s=1}^{n+1-i} (-1)^{s} \sum_{\sum_{j=1}^{s} k_j = n+1-i} a_{k_1} \cdots a_{k_s} \\
&= -\sum_{i=1}^{n+1} a_i c_{n+1-i} = -\sum_{i=1}^{n+1} a_i b_{n+1-i} = b_{n+1}.
\end{aligned}$$

$\square$

Proposition 5.1.2. *Let (A, μ_A, η_A) be a superalgebra and let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf superalgebra. If $\varphi \colon A \to H$ is an isomorphism of superalgebras, then A will also be equipped with a Hopf superalgebra with comultiplication $\Delta_A = (\varphi^{-1} \otimes \varphi^{-1}) \circ \Delta \circ \varphi$, counit $\varepsilon_A = \varepsilon \circ \varphi$, and antipode $S_A = \varphi^{-1} \circ S \circ \varphi$.*

Proof. For coassociativity, we have

$$\begin{aligned}
(\Delta_A \otimes \mathrm{id}) \circ \Delta_A &= (((\varphi^{-1} \otimes \varphi^{-1}) \circ \Delta \circ \varphi) \otimes \mathrm{id}) \circ ((\varphi^{-1} \otimes \varphi^{-1}) \circ \Delta \circ \varphi) \\
&= ((\varphi^{-1} \otimes \varphi^{-1}) \otimes \varphi^{-1}) \circ (\Delta \otimes \mathrm{id}) \circ \Delta \circ \varphi \\
&= (\varphi^{-1} \otimes (\varphi^{-1} \otimes \varphi^{-1})) \circ (\mathrm{id} \otimes \Delta) \circ \Delta \circ \varphi \\
&= (\mathrm{id} \otimes ((\varphi^{-1} \otimes \varphi^{-1}) \circ \Delta \circ \varphi) \circ ((\varphi^{-1} \otimes \varphi^{-1}) \circ \Delta \circ \varphi) \\
&= (\mathrm{id} \otimes \Delta_A) \circ \Delta_A.
\end{aligned}$$

For counitality, we compute

$$\begin{aligned}
(\varepsilon_A \otimes \mathrm{id}) \circ \Delta_A &= ((\varepsilon \circ \varphi) \otimes \mathrm{id}) \circ (\varphi^{-1} \otimes \varphi^{-1}) \circ \Delta \circ \varphi \\
&= (\mathrm{id} \otimes \varphi^{-1}) \circ (\varepsilon \otimes \mathrm{id}) \circ \Delta \circ \varphi
\end{aligned}$$

and therefore

$$\begin{aligned}((\varepsilon_A \otimes \mathrm{id}) \circ \Delta_A)(a) &= ((\mathrm{id} \otimes \varphi^{-1}) \circ (\varepsilon \otimes \mathrm{id}) \circ \Delta)(\varphi(a)) \\ &= (\mathrm{id} \otimes \varphi^{-1})(1 \otimes \varphi(a)) = 1 \otimes a.\end{aligned}$$

Similarly, one computes $((\mathrm{id} \otimes \varepsilon_A) \circ \Delta_A)(a) = a \otimes 1$. For the antipodal equations, we have

$$\begin{aligned}\mu_A \circ (S_A \otimes \mathrm{id}) \circ \Delta_A &= \mu_A \circ ((\varphi^{-1} \circ S \circ \varphi) \otimes \mathrm{id}) \circ (\varphi^{-1} \otimes \varphi^{-1}) \circ \Delta \circ \varphi \\ &= \varphi^{-1} \circ \mu \circ (S \otimes \mathrm{id}) \circ \Delta \circ \varphi \\ &= \varphi^{-1} \circ \eta \circ \varepsilon \circ \varphi = \eta_A \circ \varepsilon_A,\end{aligned}$$

and one may similarly compute $\mu_A \circ (\mathrm{id} \otimes S) \circ \Delta_A = \eta_A \circ \varepsilon_A$. Finally, the Hopf structure maps of A are graded since they are compositions of graded maps. $\square$

5.1.2 Super Linear Algebra and Lie Superalgebras

Proposition 5.1.3. *Defining the elements* $F_{ij} := E_{ij} - E_{ij}^t = E_{ij} - (-1)^{[i][j]+[i]}\theta_i\theta_j E_{\bar{j}\bar{\imath}}$, *then* $\mathfrak{osp}_{M|N}$ *is the Lie superalgebra generated by* $\{F_{ij} \mid 1 \leq i, j \leq M+N\}$ *satisfying the relations*

$$\begin{aligned}\left[F_{ij}, F_{kl}\right] = \delta_{jk}F_{il} - \delta_{il}(-1)^{([i]+[j])([k]+[l])}F_{kj} \\ - \delta_{\bar{\imath}k}(-1)^{[i][j]+[i]}\theta_i\theta_j F_{\bar{j}l} + \delta_{\bar{j}l}(-1)^{([i]+[l])[k]}\theta_{\bar{\imath}}\theta_l F_{k\bar{\imath}}\end{aligned}$$

and

$$F_{ij} + (-1)^{[i][j]+[i]}\theta_i\theta_j F_{\bar{j}\bar{\imath}} = 0$$

Proof. Let us first prove the relations. For the first, we calculate

$$\begin{aligned}\left[F_{ij}, F_{kl}\right] = \left[E_{ij}, E_{kl}\right] - (-1)^{[k][l]+[k]}\theta_k\theta_l\left[E_{ij}, E_{\bar{l}\bar{k}}\right] \\ - (-1)^{[i][j]+[i]}\theta_i\theta_j\left[E_{\bar{j}\bar{\imath}}, E_{kl}\right] + (-1)^{[i][j]+[i]+[k][l]+[k]}\theta_i\theta_j\theta_k\theta_j\left[E_{\bar{j}\bar{\imath}}, E_{\bar{l}\bar{k}}\right]\end{aligned}$$

$$
\begin{aligned}
&= \delta_{jk} E_{il} - \delta_{il}(-1)^{([i]+[j])([k]+[l])} E_{kj} \\
&\quad - (-1)^{[k][l]+[k]} \theta_k \theta_l \big(\delta_{\bar{j}l} E_{i\bar{k}} - \delta_{\bar{\imath}k}(-1)^{([i]+[j])([k]+[l])} E_{\bar{l}j} \big) \\
&\qquad - (-1)^{[i][j]+[i]} \theta_i \theta_j \big(\delta_{\bar{\imath}k} E_{\bar{j}l} - \delta_{\bar{j}l}(-1)^{([i]+[j])([k]+[l])} E_{k\bar{\imath}} \big) \\
&\qquad\quad + (-1)^{[i][j]+[i]+[k][l]+[k]} \theta_i \theta_j \theta_k \theta_j \big(\delta_{il} E_{\bar{j}\bar{k}} - \delta_{jk}(-1)^{([i]+[j])([k]+[l])} E_{\bar{l}\bar{\imath}} \big)
\end{aligned}
$$

$$
\begin{aligned}
&= \delta_{jk} \big(E_{il} - (-1)^{[i][j]+[i]+[j][l]+[j]+([i]+[j])([j]+[l])} \theta_i \theta_l E_{\bar{l}\bar{\imath}} \big) \\
&\quad - \delta_{il}(-1)^{([i]+[j])([k]+[l])} \big(E_{kj} - (-1)^{([i]+[j])([k]+[i])+[i][j]+[i]+[k][i]+[k]} \theta_k \theta_j E_{\bar{j}\bar{k}} \big) \\
&\qquad - \delta_{\bar{\imath}k}(-1)^{[i][j]+[i]} \theta_i \theta_j \big(E_{\bar{j}l} - (-1)^{[i][j]+[i][l]+([i]+[j])([i]+[l])} \theta_i \theta_{\bar{\imath}} \theta_j \theta_l E_{\bar{l}j} \big) \\
&\qquad\quad + \delta_{\bar{j}l}(-1)^{([i]+[l])[k]} \theta_{\bar{\imath}} \theta_l \big((-1)^{([i]+[l])[k]+[i][l]+[i]+([i]+[l])([k]+[l])+[i]+[l]} E_{k\bar{\imath}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - (-1)^{[k][i]+[k]} \theta_k \theta_{\bar{\imath}} E_{i\bar{k}} \big) \\
&= \delta_{jk} F_{il} - \delta_{il}(-1)^{([i]+[j])([k]+[l])} F_{kj} \\
&\qquad - \delta_{\bar{\imath}k}(-1)^{[i][j]+[i]} \theta_i \theta_j F_{\bar{j}l} + \delta_{\bar{j}l}(-1)^{([i]+[l])[k]} \theta_{\bar{\imath}} \theta_l F_{k\bar{\imath}},
\end{aligned}
$$

and for the second we have

$$
(-1)^{[i][j]+[i]} \theta_i \theta_j F_{\bar{j}\bar{\imath}} = (-1)^{[i][j]+[i]} \theta_i \theta_j \big(E_{\bar{j}\bar{\imath}} - (-1)^{[i][j]+[j]} \theta_{\bar{\imath}} \theta_{\bar{j}} E_{ij} \big) = -F_{ij}.
$$

By regarding the matrix associated to bilinear form of $\mathfrak{osp}_{M|N} = \mathfrak{osp}(\mathbb{C}^{M|N}, b)$ as $B = \sum_{a=1}^{M+N} \theta_{\bar{a}} E_{a\bar{a}}$, then $G = \sum_{a=1}^{M} E_{a\bar{a}} \in \mathrm{Mat}_M(\mathbb{C})$ and $J = \sum_{a=M+1}^{M+N} \theta_{\bar{a}} E_{a\bar{a}} \in \mathrm{Mat}_N(\mathbb{C})$. Recall that a supermatrix $A = (A_{ij})_{i,j=0}^{1} \in \mathrm{Mat}_{M|N}(\mathbb{C})$ lies in $\mathfrak{osp}_{M|N}$ if and only if $A^{st}B + BA = 0$. Expanding such matrix equation, we have

$$
\begin{pmatrix} A'_{00}G & A'_{10}J \\ -A'_{01}G & A'_{11}J \end{pmatrix} + \begin{pmatrix} GA_{00} & GA_{01} \\ JA_{10} & JA_{11} \end{pmatrix} = \begin{pmatrix} A'_{00}G + GA_{00} & A'_{10}J + GA_{01} \\ -A'_{01}G + JA_{10} & A'_{11}J + JA_{11} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
$$

Hence, $A_{00} \in \mathfrak{so}_M$, $A_{11} \in \mathfrak{sp}_N$, and $A'_{10}J + GA_{01} = 0$. Let us now show that $\mathfrak{osp}_{M|N} = \mathrm{span}_{\mathbb{C}}\{F_{ij} \mid 1 \le i, j \le M+N\}$.

Restricting to $1 \le i, j \le M$, let us regard the generators F_{ij} as matrices in $\mathrm{Mat}_M(\mathbb{C})$ so that we want to show $F_{ij} \in \mathfrak{so}_M$ and that these span the Lie algebra $\mathfrak{so}_M$. Here,

$F_{ij} = E_{ij} - E_{\bar{j}\bar{\imath}}$ and to show that these lie in $\mathfrak{so}_M$ is to show $(F_{ij})'G + GF_{ij} = 0$. This is satisfied since

$$(F_{ij})'G = \sum_{a=1}^{M} \left(\delta_{ia} E_{j\bar{a}} - \delta_{\bar{j}a} E_{\bar{\imath}\bar{a}}\right) = E_{j\bar{\imath}} - E_{\bar{\imath}j}$$

and

$$GF_{ij} = \sum_{a=1}^{M} \left(\delta_{\bar{a}i} E_{aj} - \delta_{\bar{a}\bar{j}} E_{a\bar{\imath}}\right) = E_{\bar{\imath}j} - E_{j\bar{\imath}} = -(F_{ij})'G$$

That $\{F_{ij} \mid 1 \leq i, j \leq M\}$ forms a spanning set follows from the fact that the dimensions of $\mathrm{span}_{\mathbb{C}}\{F_{ij} \mid 1 \leq i, j \leq M\}$ and $\mathfrak{so}_M$ are both $M(M-1)/2$. To see why this is true in the former case, we observe that we have the relation $F_{ij} = -F_{\bar{j}\bar{\imath}}$.

Now restricting to $M+1 \leq i, j \leq M+N$, regard the generators F_{ij} as matrices in $\mathrm{Mat}_N(\mathbb{C})$. We wish to show $F_{ij} \in \mathfrak{sp}_N$ and that these span the Lie algebra $\mathfrak{sp}_N$. Here, we have $F_{ij} = E_{ij} - \theta_i \theta_j E_{\bar{j}\bar{\imath}}$ and we must then show $(F_{ij})'J + JF_{ij} = 0$. Similar calculations yield

$$(F_{ij})'J = \sum_{a=M+1}^{M+N} \theta_{\bar{a}} \left(\delta_{ia} E_{j\bar{a}} - \delta_{\bar{j}a} \theta_i \theta_j E_{\bar{\imath}\bar{a}}\right) = \theta_{\bar{\imath}} E_{j\bar{\imath}} - \theta_i E_{\bar{\imath}j}$$

and

$$JF_{ij} = \sum_{a=M+1}^{M+N} \theta_{\bar{a}} \left(\delta_{\bar{a}i} E_{aj} - \delta_{\bar{a}\bar{j}} \theta_i \theta_j E_{a\bar{\imath}}\right) = \theta_i E_{\bar{\imath}j} + \theta_i E_{j\bar{\imath}} = -(F_{ij})'J,$$

since $\theta_{\bar{\imath}} = -\theta_i$. That these form a spanning set follows from the fact that the dimensions of $\mathrm{span}_{\mathbb{C}}\{F_{ij} \mid M+1 \leq i, j \leq M+N\}$ and $\mathfrak{so}(N)$ are both $N(N+1)/2$. To see why this is true in the former case, we observe that we have the relation $F_{ij} = -\theta_i \theta_j F_{\bar{j}\bar{\imath}}$.

By restricting to $1 \leq i \leq M$, $M+1 \leq j \leq M+N$, we have $F_{ij} = E_{ij} - \theta_j E_{\bar{j}\bar{\imath}}$. By considering $E_{ij} \in \mathrm{Mat}_{M,N}(\mathbb{C})$ and $-\theta_j E_{\bar{j}\bar{\imath}} \in \mathrm{Mat}_{N,M}(\mathbb{C})$, we wish to show the equation $(-\theta_j E_{\bar{j}\bar{\imath}})'J + GE_{ij} = 0$. To this end,

$$GE_{ij} + (-\theta_j E_{\bar{j}\bar{\imath}})'J = \sum_{a=1}^{M} \delta_{i\bar{a}} E_{aj} - \theta_j \sum_{a=M+1}^{M+N} \theta_{\bar{a}} \delta_{\bar{j}a} E_{\bar{\imath}\bar{a}} = E_{\bar{\imath}j} - E_{\bar{\imath}j} = 0,$$

To show that these F_{ij} span the space of all solutions to $(A_{10})'J + GA_{01} = 0$, let us write $A_{01} = \sum_{i,j} x_{ij}E_{ij}$ and $A_{10} = \sum_{k,l} y_{kl}E_{kl}$ where the indices are appropriate. The equation $(A_{10})'J + GA_{01} = 0$ then forces the sum

$$\sum_{i,j}\sum_{a=1}^{M} \delta_{i\bar{a}} x_{ij} E_{aj} + \sum_{k,l}\sum_{a=M+1}^{M+N} \theta_{\bar{a}} \delta_{ka} y_{kl} E_{l\bar{a}} = \sum_{i,j} x_{ij} E_{\bar{\imath}j} + \sum_{k,l} \theta_{\bar{k}} y_{kl} E_{l\bar{k}}$$

to be 0. For this to occur, one needs the coefficients to satisfy $x_{ij} + \theta_j y_{\bar{\jmath}\bar{\imath}} = 0$.

The situation of $M+1 \leq i \leq M+N$, $1 \leq j \leq M$ is similar. □

5.1.3 Generating Matrix Relations

The RTT-Relation

In this appendix section, we compute explicitly relations equivalent to, and related to, the RTT-relation as given at (4.7).

Proposition 5.1.4. *The RTT-relation $R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v)$ as in (4.7) is equivalent to the relations*

$$\begin{aligned}\big[T_{ij}(u), T_{kl}(v)\big] = {} & \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\Big(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)\Big) \\ & - \frac{1}{u-v-\kappa}\Bigg(\delta_{\bar{\imath}k}\sum_{p=1}^{M+N}(-1)^{[i][j]+[i]+[j][p]}\theta_i\theta_p T_{pj}(u)T_{\bar{p}l}(v) \\ & - \delta_{\bar{\jmath}l}\sum_{p=1}^{M+N}(-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]}\theta_j\theta_p T_{k\bar{p}}(v)T_{ip}(u)\Bigg).\end{aligned}$$

as in (4.10).

Proof. For ease of computations, let us normalize $\widehat{T}_{ij}(u) = (-1)^{[i][j]+[j]}T_{ij}(u)$ and collect these into a matrix $\widehat{T}(u) = \sum_{i,j=1}^{M+N} E_{ij} \otimes \widehat{T}_{ij}(u)$. It suffices to find the power series relations corresponding to the system $R(u-v)\widehat{T}_1(u)\widehat{T}_2(v) = \widehat{T}_2(v)\widehat{T}_1(u)R(u-v)$.

Step 1. To start, we have the equations

$$(A.1)\quad \widehat{T}_1(u)\widehat{T}_2(v) = \left(\sum_{i,j=1}^{M+N} E_{ij}\otimes \mathrm{id}\otimes \widehat{T}_{ij}(u)\right)\left(\sum_{k,l=1}^{M+N} \mathrm{id}\otimes E_{kl}\otimes \widehat{T}_{kl}(v)\right)$$
$$= \sum_{i,j,k,l=1}^{M+N} (-1)^{([i]+[j])([k]+[l])} E_{ij}\otimes E_{kl}\otimes \widehat{T}_{ij}(u)\widehat{T}_{kl}(v)$$

and

$$(A.2)\quad \widehat{T}_2(v)\widehat{T}_1(u) = \left(\sum_{k,l=1}^{M+N} \mathrm{id}\otimes E_{kl}\otimes \widehat{T}_{kl}(v)\right)\left(\sum_{i,j=1}^{M+N} E_{ij}\otimes \mathrm{id}\otimes \widehat{T}_{ij}(u)\right)$$
$$= \sum_{i,j,k,l=1}^{M+N} E_{ij}\otimes E_{kl}\otimes \widehat{T}_{kl}(v)\widehat{T}_{ij}(u).$$

Step 2. Now, by writing $P\otimes \mathbf{1} = \sum_{a,b=1}^{M+N}(-1)^{[b]}E_{ab}\otimes E_{ba}\otimes \mathbf{1}$, we obtain the following relations: the first, $(B.1)$ $(P\otimes \mathbf{1})\widehat{T}_1(u)\widehat{T}_2(v)$, is given by

$$\sum_{a,b,i,j,k,l=1}^{M+N} (-1)^{([i]+[j])(([a]+[b])+([k]+[l]))+[b]}\delta_{bi}\delta_{ak}E_{aj}\otimes E_{bl}\otimes \widehat{T}_{ij}(u)\widehat{T}_{kl}(v)$$
$$= \sum_{i,j,k,l=1}^{M+N} (-1)^{[i][j]+[i][l]+[j][l]} E_{kj}\otimes E_{il}\otimes \widehat{T}_{ij}(u)\widehat{T}_{kl}(v)$$
$$= \sum_{i,j,k,l=1}^{M+N} (-1)^{[j][k]+[j][l]+[k][l]} E_{ij}\otimes E_{kl}\otimes \widehat{T}_{kj}(u)\widehat{T}_{il}(v)$$

and $(B.2)$ $\widehat{T}_2(v)\widehat{T}_1(u)(P\otimes \mathbf{1})$ is given by

$$\sum_{a,b,i,j,k,l=1}^{M+N} (-1)^{([k]+[l])([a]+[b])+[b]}\delta_{ja}\delta_{lb}E_{ib}\otimes E_{ka}\otimes \widehat{T}_{kl}(v)\widehat{T}_{ij}(u)$$
$$= \sum_{i,j,k,l=1}^{M+N} (-1)^{[j][k]+[j][l]+[k][l]} E_{il}\otimes E_{kj}\otimes \widehat{T}_{kl}(v)\widehat{T}_{ij}(u)$$

$$= \sum_{i,j,k,l=1}^{M+N} (-1)^{[j][k]+[j][l]+[k][l]} E_{ij} \otimes E_{kl} \otimes \widehat{T}_{kj}(v)\widehat{T}_{il}(u).$$

Step 3. Lastly, since $Q \otimes \mathbf{1} = \sum_{p,q=1}^{M+N} (-1)^{[p][q]} \theta_p \theta_q E_{qp} \otimes E_{\bar{q}\bar{p}} \otimes \mathbf{1}$, we compute the equation $(C.1)$ $(Q \otimes \mathbf{1})\widehat{T}_1(u)\widehat{T}_2(v)$ to be

$$\sum_{i,j,k,l,p,q=1}^{M+N} (-1)^{([i]+[j])(([k]+[l])+([p]+[q]))+[p][q]} \theta_p \theta_q \delta_{pi} \delta_{\bar{p}k} E_{qj} \otimes E_{\bar{q}l} \otimes \widehat{T}_{ij}(u)\widehat{T}_{kl}(v)$$

$$= \delta_{\bar{\imath}k} \sum_{i,j,k,l,p=1}^{M+N} (-1)^{[i][l]+[j][l]+[j][p]} \theta_i \theta_p E_{pj} \otimes E_{\bar{p}l} \otimes \widehat{T}_{ij}(u)\widehat{T}_{\bar{\imath}l}(v)$$

$$= \delta_{\bar{\imath}k} \sum_{i,j,k,l,p=1}^{M+N} (-1)^{[i][j]+[j][l]+[l][p]} \theta_i \theta_p E_{ij} \otimes E_{kl} \otimes \widehat{T}_{pj}(u)\widehat{T}_{\bar{p}l}(v)$$

and $(C.2)$ $\widehat{T}_2(v)\widehat{T}_1(u)(Q \otimes \mathbf{1})$ to be

$$\sum_{i,j,k,l,p,q=1}^{M+N} (-1)^{([k]+[l])([p]+[q])+[p][q]} \theta_p \theta_q \delta_{jq} \delta_{l\bar{q}} E_{ip} \otimes E_{k\bar{p}} \otimes \widehat{T}_{kl}(v)\widehat{T}_{ij}(u)$$

$$= \delta_{\bar{\jmath}l} \sum_{i,j,k,l,p=1}^{M+N} (-1)^{[j][k]+[k][p]+[j]} \theta_j \theta_p E_{ip} \otimes E_{k\bar{p}} \otimes \widehat{T}_{k\bar{\jmath}}(v)\widehat{T}_{ij}(u)$$

$$= \delta_{\bar{\jmath}l} \sum_{i,j,k,l,p=1}^{M+N} (-1)^{[j][k]+[k][p]+[p]} \theta_j \theta_p E_{ij} \otimes E_{kl} \otimes \widehat{T}_{k\bar{p}}(v)\widehat{T}_{ip}(u).$$

Step 4. Now, by the RTT-relation, we have that

$$R(u-v)\widehat{T}_1(u)\widehat{T}_2(v) = \widehat{T}_1(u)\widehat{T}_2(v) - \frac{(P \otimes \mathbf{1})\widehat{T}_1(u)\widehat{T}_2(v)}{u-v} + \frac{(Q \otimes \mathbf{1})\widehat{T}_1(u)\widehat{T}_2(v)}{u-v-\kappa}$$

and

$$\widehat{T}_2(v)\widehat{T}_1(u)R(u-v) = \widehat{T}_2(v)\widehat{T}_1(u) - \frac{\widehat{T}_2(v)\widehat{T}_1(u)(P \otimes \mathbf{1})}{u-v} + \frac{\widehat{T}_2(v)\widehat{T}_1(u)(Q \otimes \mathbf{1})}{u-v-\kappa}$$

are equal, yielding the relation

$$\widehat{T}_1(u)\widehat{T}_2(v) - \widehat{T}_2(v)\widehat{T}_1(u) = \frac{1}{u-v}\Big((P\otimes \mathbf{1})\widehat{T}_1(u)\widehat{T}_2(v) - \widehat{T}_2(v)\widehat{T}_1(u)(P\otimes \mathbf{1})\Big)$$
$$- \frac{1}{u-v-\kappa}\Big((Q\otimes \mathbf{1})\widehat{T}_1(u)\widehat{T}_2(v) - \widehat{T}_2(v)\widehat{T}_1(u)(Q\otimes \mathbf{1})\Big).$$

Therefore, by using equations $(A.1)$, $(A.2)$, $(B.1)$, $(B.2)$, $(C.1)$, and $(C.2)$, the above relation becomes

$$\begin{aligned}
&(-1)^{([i]+[j])([k]+[l])}\widehat{T}_{ij}(u)\widehat{T}_{kl}(v) - \widehat{T}_{kl}(v)\widehat{T}_{ij}(u)\\
&\quad = \frac{1}{u-v}(-1)^{[j][k]+[j][l]+[k][l]}\Big(\widehat{T}_{kj}(u)\widehat{T}_{il}(v) - \widehat{T}_{kj}(v)\widehat{T}_{il}(u)\Big)\\
&\qquad - \frac{1}{u-v-\kappa}\Bigg(\delta_{\bar{\imath}k}\sum_{p=1}^{M+N}(-1)^{[i][j]+[j][l]+[l][p]}\theta_i\theta_p\widehat{T}_{pj}(u)\widehat{T}_{\bar{p}l}(v)\\
&\qquad\qquad - \delta_{\bar{\jmath}l}\sum_{p=1}^{M+N}(-1)^{[j][k]+[k][p]+[p]}\theta_j\theta_p\widehat{T}_{k\bar{p}}(v)\widehat{T}_{ip}(u)\Bigg),
\end{aligned}$$

Finally, multiplying the above equation by the scalar $(-1)^{([i]+[j])([k]+[l])+[i][j]+[j]+[k][l]+[l]}$ yields the desired relations

$$\begin{aligned}
\big[T_{ij}(u), T_{kl}(v)\big] &= \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\Big(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)\Big)\\
&\quad - \frac{1}{u-v-\kappa}\Bigg(\delta_{\bar{\imath}k}\sum_{p=1}^{M+N}(-1)^{[i][j]+[i]+[j][p]}\theta_i\theta_p T_{pj}(u)T_{\bar{p}l}(v)\\
&\qquad - \delta_{\bar{\jmath}l}\sum_{p=1}^{M+N}(-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]}\theta_j\theta_p T_{k\bar{p}}(v)T_{ip}(u)\Bigg).
\end{aligned}$$

□

To find the expression for the commutator $\big[T_{ij}^{(m)}, T_{kl}^{(n)}\big]$, we use the expansions

$$\frac{1}{u-v} = \sum_{r=0}^{\infty} u^{-r-1}v^{r} \quad \text{and} \quad \frac{1}{u-v-\kappa} = \sum_{s=0}^{\infty} u^{-s-1}(v+\kappa)^{s} = \sum_{s=0}^{\infty}\sum_{b+c=s}\binom{s}{b}\kappa^{b}u^{-s-1}v^{c},$$

in the relations (4.10) to yield

$$\sum_{m,n=0}^{\infty}\big[T_{ij}^{(m)},T_{kl}^{(n)}\big]u^{-m}v^{-n}=(-1)^{[i][j]+[i][k]+[j][k]}\sum_{a,d,r=0}^{\infty}\big(T_{kj}^{(a)}T_{il}^{(d)}-T_{kj}^{(d)}T_{il}^{(a)}\big)u^{-r-a-1}v^{r-d}$$
$$-\delta_{\bar{\imath}k}\sum_{a,d,s=0}^{\infty}\sum_{p=1}^{M+N}\sum_{b+c=s}\binom{s}{b}\kappa^{b}(-1)^{[i][j]+[i]+[p]}\theta_i\theta_pT_{pj}^{(a)}T_{\bar{p}l}^{(d)}u^{-s-a-1}v^{c-d}$$
$$+\delta_{\bar{\jmath}l}\sum_{a,d,s=0}^{\infty}\sum_{p=1}^{M+N}\sum_{b+c=s}\binom{s}{b}\kappa^{b}(-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]}\theta_j\theta_pT_{k\bar{p}}^{(d)}T_{ip}^{(a)}u^{-s-a-1}v^{c-d}.$$

The coefficient of $u^{-m}v^{-n}$ is therefore given by

$$\big[T_{ij}^{(m)},T_{kl}^{(n)}\big]=(-1)^{[i][j]+[i][k]+[j][k]}\sum_{a=1}^{m}\big(T_{kj}^{(a-1)}T_{il}^{(m+n-a)}-T_{kj}^{(m+n-a)}T_{il}^{(a-1)}\big)$$
$$-\delta_{\bar{\imath}k}\sum_{p=1}^{M+N}\sum_{a=1}^{m}\sum_{b=0}^{m-a}\binom{m-a}{b}\kappa^{b}(-1)^{[i][j]+[i]+[p]}\theta_i\theta_pT_{pj}^{(a-1)}T_{\bar{p}l}^{(m+n-a-b)}$$
$$+\delta_{\bar{\jmath}l}\sum_{p=1}^{M+N}\sum_{a=1}^{m}\sum_{b=0}^{m-a}\binom{m-a}{b}\kappa^{b}(-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]}\theta_j\theta_pT_{k\bar{p}}^{(m+n-a-b)}T_{ip}^{(a-1)}.$$

However, when $m>n$, we have

$$\sum_{a=n+1}^{m}\big(T_{kj}^{(a-1)}T_{il}^{(m+n-a)}-T_{kj}^{(m+n-a)}T_{il}^{(a-1)}\big)=0.$$

Indeed, by writing $m=n+q$ for some $q\in\mathbb{Z}^{+}$ and setting $g(a)=T_{kj}^{(a-1)}T_{il}^{(m+n-a)}-T_{kj}^{(m+n-a)}T_{il}^{(a-1)}$, then $g(n+e)=T_{kj}^{(n+e-1)}T_{il}^{(m-e)}-T_{kj}^{(m-e)}T_{il}^{(n+e-1)}=-g(m+1-e)$ for any $e=1\dots,q$. Hence,

$$\big[T_{ij}^{(m)},T_{kl}^{(n)}\big]=(-1)^{[i][j]+[i][k]+[j][k]}\sum_{a=1}^{\min(m,n)}\big(T_{kj}^{(a-1)}T_{il}^{(m+n-a)}-T_{kj}^{(m+n-a)}T_{il}^{(a-1)}\big)$$
$$+\delta_{\bar{\jmath}l}\sum_{p=1}^{M+N}\sum_{a=1}^{m}\sum_{b=0}^{m-a}\binom{m-a}{b}\kappa^{b}(-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]}\theta_j\theta_pT_{k\bar{p}}^{(m+n-a-b)}T_{ip}^{(a-1)}$$
$$-\delta_{\bar{\imath}k}\sum_{p=1}^{M+N}\sum_{a=1}^{m}\sum_{b=0}^{m-a}\binom{m-a}{b}\kappa^{b}(-1)^{[i][j]+[i]+[p]}\theta_i\theta_pT_{pj}^{(a-1)}T_{\bar{p}l}^{(m+n-a-b)}.$$

Proposition 5.1.5. *The inverse $T(u)^{-1}$ of the generating matrix $T(u)$ in (4.8) is given by $T(u)^{-1} = \sum_{i,j=1}^{M+N} (-1)^{[i][j]+[j]} E_{ij} \otimes T^{\bullet}_{ij}(u)$, where $T^{\bullet}_{ij}(u) = \mathbf{1} + \sum_{n=1}^{\infty} T^{\bullet(n)}_{ij} u^{-n}$ and*

$$T^{\bullet(n)}_{ij} = -T^{(n)}_{ij} + \sum_{s=2}^{n} (-1)^s \sum_{\sum_{j=1}^{s} k_j = n} \left(\sum_{a_1,\ldots,a_{s-1}=1}^{M+N} T^{(k_1)}_{ia_1} T^{(k_2)}_{a_1a_2} \cdots T^{(k_s)}_{a_{s-1}j} \right)$$

with $k_j \in \mathbb{Z}^+$ for each k_j in the sum $\sum_{j=1}^{s} k_j = n$. Consequently, one may write

$$T^{\bullet}_{ij}(u) = -T^{\dagger}_{ij}(u) + \sum_{s=2}^{n} (-1)^s \left(\sum_{a_1,\ldots,a_{s-1}=1}^{M+N} T^{\dagger}_{ia_1}(u) T^{\dagger}_{a_1a_2}(u) \cdots T^{\dagger}_{a_{s-1}j}(u) \right),$$

where $T^{\dagger}_{ij}(u) := T_{ij}(u) - \delta_{ij}\mathbf{1}$. Consequently, we can conclude $\left[T^{\bullet}_{ij}(u)\right] = \left[T^{\bullet(n)}_{ij}\right] = [i]+[j]$.

Proof. Writing $T(u) = \mathbb{1} + \sum_{n=1}^{\infty} T^{(n)} u^{-n}$, where $T^{(n)} = \sum_{i,j=1}^{M+N} (-1)^{[i][j]+[j]} E_{ij} \otimes T^{(n)}_{ij}$, then we know

$$T(u)^{-1} = \mathbb{1} + \sum_{n=1}^{\infty} \left(\sum_{s=1}^{n} (-1)^s \sum_{\sum_{j=1}^{s} k_j = n} T^{(k_1)} T^{(k_2)} \cdots T^{(k_s)} \right) u^{-n}.$$

<u>*Claim*</u>: For integers $s \geq 2$, the element $T^{(k_1)} T^{(k_2)} \cdots T^{(k_s)}$ is given by

$$\sum_{i,j,a_1,\ldots,a_{s-1}=1}^{M+N} (-1)^{[i][j]+[j]} E_{ij} \otimes T^{(k_1)}_{ia_1} T^{(k_2)}_{a_1a_2} \cdots T^{(k_s)}_{a_{s-1}j}.$$

Let us prove this claim by induction on s: The base case $s = 2$ can be verified to be true, so let us assume the statement holds for s. Therefore, $T^{(k_1)} T^{(k_2)} \cdots T^{(k_{s+1})}$ is given by

$$\left(\sum_{i,a_s,a_1,\ldots,a_{s-1}=1}^{M+N}(-1)^{[i][a_s]+[a_s]}E_{ia_s}\otimes T_{ia_1}^{(k_1)}T_{a_1a_2}^{(k_2)}\cdots T_{a_{s-1}a_s}^{(k_s)}\right)\left(\sum_{a_s,j=1}^{M+N}(-1)^{[a_s][j]+[j]}E_{a_sj}\otimes T_{a_sj}^{(k_s+1)}\right)$$
$$=\sum_{i,j,a_1,\ldots,a_s=1}^{M+N}(-1)^{[i][j]+[j]}E_{ia_s}\otimes T_{ia_1}^{(k_1)}T_{a_1a_2}^{(k_2)}\cdots T_{a_sj}^{(k_s+1)},$$

since $[i][a_s]+[a_s]+([i]+[a_s])([a_s]+[j])+[a_s][j]+[j]=[i][j]+[j]$. $\square$

Note that the RTT-relation (4.7) is equivalent to

$$T_2(v)^{-1}R(u-v)T_1(u)=T_1(u)R(u-v)T_2(v)^{-1}$$

Proposition 5.1.6. *The relation $T_2(v)^{-1}R(u-v)T_1(u)=T_1(u)R(u-v)T_2(v)^{-1}$ is equivalent to the relations*

$$\begin{aligned}\left[T_{ij}(u),T^{\bullet}_{kl}(v)\right]=&\frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\left(\delta_{jk}\sum_{a=1}^{M+N}T_{ia}(u)T^{\bullet}_{al}(v)-\delta_{il}\sum_{a=1}^{M+N}T^{\bullet}_{ka}(v)T_{aj}(u)\right)\\&-\frac{\theta_0}{u-v-\kappa}\Big((-1)^{[j][k]+[l]}\theta_j\theta_kT_{i\bar{k}}(u)T^{\bullet}_{\bar{j}l}(v)\\&\qquad-(-1)^{[i][k]+[j][k]+[j][l]+[j]}\theta_i\theta_lT^{\bullet}_{k\bar{i}}(v)T_{\bar{l}j}(u)\Big)\end{aligned}$$

Proof. For ease of computations, let us normalize $\widehat{T}_{ij}(u)=(-1)^{[i][j]+[j]}T_{ij}(u)$ and $\widehat{T}^{\bullet}_{ij}(u)=(-1)^{[i][j]+[j]}T^{\bullet}_{ij}(u)$ and collect these into matrices $\widehat{T}(u)=\sum_{i,j=1}^{M+N}E_{ij}\otimes\widehat{T}_{ij}(u)$ and $\widehat{T}(u)^{-1}=\sum_{i,j=1}^{M+N}E_{ij}\otimes\widehat{T}^{\bullet}_{ij}(u)$. It suffices to find the power series relations corresponding to the system $\widehat{T}_2(v)^{-1}R(u-v)\widehat{T}_1(u)=\widehat{T}_1(u)R(u-v)\widehat{T}_2(v)^{-1}$.

Step 1. To start, we have the equations

$$(A.1)\quad \widehat{T}_1(u)\widehat{T}_2(v)^{-1}=\left(\sum_{i,j=1}^{M+N}E_{ij}\otimes\mathrm{id}\otimes\widehat{T}_{ij}(u)\right)\left(\sum_{k,l=1}^{M+N}\mathrm{id}\otimes E_{kl}\otimes\widehat{T}^{\bullet}_{kl}(v)\right)$$

$$= \sum_{i,j,k,l=1}^{M+N} (-1)^{([i]+[j])([k]+[l])} E_{ij} \otimes E_{kl} \otimes \widehat{T}_{ij}(u) \widehat{T}^{\bullet}_{kl}(v)$$

and

$$(A.2) \quad \widehat{T}_2(v)^{-1} \widehat{T}_1(u) = \left(\sum_{k,l=1}^{M+N} \mathrm{id} \otimes E_{kl} \otimes \widehat{T}^{\bullet}_{kl}(v) \right) \left(\sum_{i,j=1}^{M+N} E_{ij} \otimes \mathrm{id} \otimes \widehat{T}_{ij}(u) \right)$$

$$= \sum_{i,j,k,l=1}^{M+N} E_{ij} \otimes E_{kl} \otimes \widehat{T}^{\bullet}_{kl}(v) \widehat{T}_{ij}(u).$$

Step 2. Now, by writing $P \otimes \mathbf{1} = \sum_{a,b=1}^{M+N} (-1)^{[b]} E_{ab} \otimes E_{ba} \otimes \mathbf{1}$, we obtain the following relations: the first, $(B.1)$ $\widehat{T}_1(u)(P \otimes \mathbf{1}) \widehat{T}_2(v)^{-1}$ is given by

$$\left(\sum_{a,b,i,j=1}^{M+N} (-1)^{[b]} \delta_{ja} E_{ib} \otimes E_{ba} \otimes \widehat{T}_{ij}(u) \right) \left(\sum_{k,l=1}^{M+N} \mathrm{id} \otimes E_{kl} \otimes \widehat{T}^{\bullet}_{kl}(v) \right)$$

$$= \sum_{a,b,i,k,l=1}^{M+N} (-1)^{([i]+[a])([k]+[l])+[b]} \delta_{ak} E_{ib} \otimes E_{bl} \otimes \widehat{T}_{ia}(u) \widehat{T}^{\bullet}_{kl}(v)$$

$$= \sum_{a,b,i,l=1}^{M+N} (-1)^{([i]+[a])([a]+[l])+[b]} E_{ib} \otimes E_{bl} \otimes \widehat{T}_{ia}(u) \widehat{T}^{\bullet}_{al}(v)$$

$$= \sum_{a,i,j,k,l=1}^{M+N} (-1)^{([i]+[a])([a]+[l])+[j]} \delta_{jk} E_{ij} \otimes E_{kl} \otimes \widehat{T}_{ia}(u) \widehat{T}^{\bullet}_{al}(v).$$

and $(B.2)$ $\widehat{T}_2(v)^{-1}(P \otimes \mathbf{1}) \widehat{T}_1(u)$ is given by

$$\left(\sum_{k,l=1}^{M+N} \mathrm{id} \otimes E_{kl} \otimes \widehat{T}^{\bullet}_{kl}(v) \right) \left(\sum_{a,b,i,j=1}^{M+N} (-1)^{([a]+[b])([i]+[j])+[b]} \delta_{bi} E_{aj} \otimes E_{ba} \otimes \widehat{T}_{ij}(u) \right)$$

$$= \sum_{a,b,j,k,l=1}^{M+N} (-1)^{([a]+[b])([b]+[j]+[k]+[l])+[b]} \delta_{bl} E_{aj} \otimes E_{ka} \otimes \widehat{T}^{\bullet}_{kl}(v) \widehat{T}_{bj}(u)$$

$$= \sum_{a,b,j,k=1}^{M+N} (-1)^{([a]+[b])([j]+[k])+[b]} E_{aj} \otimes E_{ka} \otimes \widehat{T}^{\bullet}_{kb}(v) \widehat{T}_{bj}(u)$$

$$= \sum_{a,i,j,k,l=1}^{M+N} (-1)^{([a]+[i])([j]+[k])+[a]} \delta_{il} E_{ij} \otimes E_{kl} \otimes \widehat{T}^{\bullet}_{ka}(v) \widehat{T}_{aj}(u)$$

Step 3. Lastly, since $Q \otimes \mathbf{1} = \sum_{p,q=1}^{M+N} (-1)^{[p][q]} \theta_p \theta_q E_{qp} \otimes E_{\bar{q}\bar{p}} \otimes \mathbf{1}$, we compute the equation $(C.1)$ $\widehat{T}_1(u)(Q \otimes \mathbf{1})\widehat{T}_2(v)^{-1}$ to be

$$\begin{aligned}
&\left(\sum_{i,j,p,q=1}^{M+N} (-1)^{[p][q]} \theta_p \theta_q \delta_{jq} E_{ip} \otimes E_{\bar{q}\bar{p}} \otimes \widehat{T}_{ij}(u)\right) \left(\sum_{k,l=1}^{M+N} \mathrm{id} \otimes E_{kl} \otimes \widehat{T}^{\bullet}_{kl}(v)\right) \\
&\qquad = \sum_{i,k,l,p,q=1}^{M+N} (-1)^{([i]+[q])([k]+[l])+[p][q]} \theta_p \theta_q \delta_{\bar{p}k} E_{ip} \otimes E_{\bar{q}l} \otimes \widehat{T}_{iq}(u) \widehat{T}^{\bullet}_{kl}(v) \\
&\qquad = \sum_{i,l,p,q=1}^{M+N} (-1)^{([i]+[q])([p]+[l])+[p][q]} \theta_p \theta_q E_{ip} \otimes E_{\bar{q}l} \otimes \widehat{T}_{iq}(u) \widehat{T}^{\bullet}_{\bar{p}l}(v) \\
&\qquad = \sum_{i,j,k,l=1}^{M+N} (-1)^{[i][j]+[i][l]+[k][l]} \theta_j \theta_{\bar{k}} E_{ij} \otimes E_{kl} \otimes \widehat{T}_{i\bar{k}}(u) \widehat{T}^{\bullet}_{\bar{j}l}(v)
\end{aligned}$$

and $(C.2)$ $\widehat{T}_2(v)^{-1}(Q \otimes \mathbf{1})\widehat{T}_1(u)$ to be

$$\begin{aligned}
&\left(\sum_{k,l=1}^{M+N} \mathrm{id} \otimes E_{kl} \otimes \widehat{T}^{\bullet}_{kl}(v)\right) \left(\sum_{i,j,p,q=1}^{M+N} (-1)^{([p]+[q])([i]+[j])+[p][q]} \theta_p \theta_q \delta_{pi} E_{qj} \otimes E_{\bar{q}\bar{p}} \otimes \widehat{T}_{ij}(u)\right) \\
&\qquad = \sum_{j,k,l,p,q=1}^{M+N} (-1)^{([p]+[q])([p]+[j]+[k]+[l])+[p][q]} \theta_p \theta_q \delta_{l\bar{q}} E_{qj} \otimes E_{k\bar{p}} \otimes \widehat{T}^{\bullet}_{kl}(v) \widehat{T}_{pj}(u) \\
&\qquad = \sum_{j,k,p,q=1}^{M+N} (-1)^{([p]+[q])([j]+[k])+[p][q]+[p]+[q]} \theta_p \theta_q E_{qj} \otimes E_{k\bar{p}} \otimes \widehat{T}^{\bullet}_{k\bar{q}}(v) \widehat{T}_{pj}(u) \\
&\qquad = \sum_{i,j,k,l=1}^{M+N} (-1)^{([i]+[l])([j]+[k])+[i][l]} \theta_{\bar{\imath}} \theta_l E_{ij} \otimes E_{kl} \otimes \widehat{T}^{\bullet}_{k\bar{\imath}}(v) \widehat{T}_{\bar{l}j}(u)
\end{aligned}$$

Step 4. We have that

$$\widehat{T}_1(u)R(u-v)\widehat{T}_2(v)^{-1} = \widehat{T}_1(u)\widehat{T}_2(v)^{-1} - \frac{T_1(u)(P \otimes \mathbf{1})\widehat{T}_2(v)^{-1}}{u-v} + \frac{\widehat{T}_1(u)(Q \otimes \mathbf{1})\widehat{T}_2(v)^{-1}}{u-v-\kappa}$$

and

$$\widehat{T}_2(v)^{-1}R(u-v)\widehat{T}_1(u) = \widehat{T}_2(v)^{-1}\widehat{T}_1(u) - \frac{\widehat{T}_2(v)^{-1}(P \otimes \mathbf{1})\widehat{T}_1(u)}{u-v} + \frac{\widehat{T}_2(v)^{-1}(Q \otimes \mathbf{1})\widehat{T}_1(u)}{u-v-\kappa}$$

are equal, yielding the relation

$$\begin{aligned}\widehat{T}_1(u)\widehat{T}_2(v)^{-1} - \widehat{T}_2(v)^{-1}\widehat{T}_1(u) \\ = \frac{1}{u-v-\kappa}\Big(\widehat{T}_2(v)^{-1}(Q\otimes\mathbf{1})\widehat{T}_1(u) - \widehat{T}_1(u)(Q\otimes\mathbf{1})\widehat{T}_2(v)^{-1}\Big) \\ + \frac{1}{u-v}\Big(\widehat{T}_1(u)(P\otimes\mathbf{1})\widehat{T}_2(v)^{-1} - \widehat{T}_2(v)^{-1}(P\otimes\mathbf{1})\widehat{T}_1(u)\Big).\end{aligned}$$

Therefore, by using equations $(A.1)$, $(A.2)$, $(B.1)$, $(B.2)$, $(C.1)$, and $(C.2)$, the above relation becomes

$$\begin{aligned}&(-1)^{([i]+[j])([k]+[l])}\widehat{T}_{ij}(u)\widehat{T}^{\bullet}_{kl}(v) - \widehat{T}^{\bullet}_{kl}(v)\widehat{T}_{ij}(u) \\ &= \frac{1}{u-v-\kappa}\Big((-1)^{([i]+[l])([[j]+[k])+[i][l]}\theta_{\bar{i}}\theta_l\widehat{T}^{\bullet}_{k\bar{i}}(v)\widehat{T}_{\bar{l}j}(u) \\ &\qquad - (-1)^{[i][j]+[i][l]+[k][l]}\theta_j\theta_{\bar{k}}\widehat{T}_{i\bar{k}}(u)\widehat{T}^{\bullet}_{\bar{j}l}(v)\Big) \\ &\quad + \frac{1}{u-v}\Big(\delta_{jk}\sum_{a=1}^{M+N}(-1)^{([i]+[a])([a]+[l])+[j]}\widehat{T}_{ia}(u)\widehat{T}^{\bullet}_{al}(v) \\ &\qquad - \delta_{il}\sum_{a=1}^{M+N}(-1)^{([a]+[i])([j]+[k])+[a]}\widehat{T}^{\bullet}_{ka}(v)\widehat{T}_{aj}(u)\Big).\end{aligned}$$

Finally, multiplying the above equation by the scalar $(-1)^{([i]+[j])([k]+[l])+[i][j]+[j]+[k][l]+[l]}$ yields the desired relations

$$\begin{aligned}\big[T_{ij}(u), T^{\bullet}_{kl}(v)\big] &= \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\Big(\delta_{jk}\sum_{a=1}^{M+N}T_{ia}(u)T^{\bullet}_{al}(v) - \delta_{il}\sum_{a=1}^{M+N}T^{\bullet}_{ka}(v)T_{aj}(u)\Big) \\ &\quad - \frac{\theta_0}{u-v-\kappa}\Big((-1)^{[j][k]+[l]}\theta_j\theta_k T_{i\bar{k}}(u)T^{\bullet}_{\bar{j}l}(v) \\ &\qquad - (-1)^{[i][k]+[j][k]+[j][l]+[j]}\theta_i\theta_l T^{\bullet}_{k\bar{i}}(v)T_{\bar{l}j}(u)\Big)\end{aligned}$$

□

The Opposite RTT-Relation

Proposition 5.1.7. *A map* $(-)^\circ \in \operatorname{End}\mathbf{X}(\mathfrak{osp}_{M|N})$ *is a superalgebra anti-morphism in the category* $\mathrm{sVect}_{\mathbb{C}}$ *if and only if it satisfies the relation*

$$R(u-v)T_2^\circ(v)T_1^\circ(u) = T_1^\circ(u)T_2^\circ(v)R(u-v)$$

Proof. For ease of computations, let us normalize $\widehat{T}_{ij}(u) = (-1)^{[i][j]+[j]}T_{ij}(u)$ and collect these into a matrix $\widehat{T}(u) = \sum_{i,j=1}^{M+N} E_{ij}\otimes \widehat{T}_{ij}(u)$. It suffices to find the power series relations corresponding to the system $R(u-v)\widehat{T}_2^\circ(v)\widehat{T}_1^\circ(u) = \widehat{T}_1^\circ(u)\widehat{T}_2^\circ(v)R(u-v)$.

Step 1. To start, we have the equations

$$\begin{aligned}(A.1)\quad \widehat{T}_1^\circ(u)\widehat{T}_2^\circ(v) &= \left(\sum_{i,j=1}^{M+N} E_{ij}\otimes \mathrm{id}\otimes \widehat{T}_{ij}^\circ(u)\right)\left(\sum_{k,l=1}^{M+N} \mathrm{id}\otimes E_{kl}\otimes \widehat{T}_{kl}^\circ(v)\right)\\ &= \sum_{i,j,k,l=1}^{M+N} (-1)^{([i]+[j])([k]+[l])} E_{ij}\otimes E_{kl}\otimes \widehat{T}_{ij}^\circ(u)\widehat{T}_{kl}^\circ(v)\end{aligned}$$

and

$$\begin{aligned}(A.2)\quad \widehat{T}_2^\circ(v)\widehat{T}_1^\circ(u) &= \left(\sum_{k,l=1}^{M+N} \mathrm{id}\otimes E_{kl}\otimes \widehat{T}_{kl}^\circ(v)\right)\left(\sum_{i,j=1}^{M+N} E_{ij}\otimes \mathrm{id}\otimes \widehat{T}_{ij}^\circ(u)\right)\\ &= \sum_{i,j,k,l=1}^{M+N} E_{ij}\otimes E_{kl}\otimes \widehat{T}_{kl}^\circ(v)\widehat{T}_{ij}^\circ(u).\end{aligned}$$

Step 2. Now, by writing $P\otimes \mathbf{1} = \sum_{a,b=1}^{M+N}(-1)^{[b]}E_{ab}\otimes E_{ba}\otimes \mathbf{1}$, we obtain the following relations: the first, $(B.1)$ $\widehat{T}_1^\circ(u)\widehat{T}_2^\circ(v)(P\otimes\mathbf{1})$, is given by

$$\begin{aligned}&\sum_{i,j,k,l,a,b=1}^{M+N} (-1)^{(([a]+[b])+([i]+[j]))([k]+[l])+[b]}\delta_{ja}\delta_{lb}E_{ib}\otimes E_{ka}\otimes \widehat{T}_{ij}^\circ(u)\widehat{T}_{kl}^\circ(v)\\ &\qquad\qquad = \sum_{i,j,k,l=1}^{M+N} (-1)^{[i][k]+[i][l]+[k][l]} E_{il}\otimes E_{kj}\otimes \widehat{T}_{ij}^\circ(u)\widehat{T}_{kl}^\circ(v)\end{aligned}$$

$$= \sum_{i,j,k,l=1}^{M+N} (-1)^{[i][j]+[i][k]+[j][k]} E_{ij} \otimes E_{kl} \otimes \widehat{T}^{\circ}_{il}(u)\widehat{T}^{\circ}_{kj}(v)$$

and $(B.2)$ $(P \otimes \mathbf{1})\widehat{T}^{\circ}_2(v)\widehat{T}^{\circ}_1(u)$ is given by

$$\sum_{i,j,k,l,a,b=1}^{M+N} (-1)^{([a]+[b])([i]+[j])+[b]} \delta_{bi}\delta_{ak} E_{aj} \otimes E_{bl} \otimes \widehat{T}^{\circ}_{kl}(v)\widehat{T}^{\circ}_{ij}(u)$$

$$= \sum_{i,j,k,l=1}^{M+N} (-1)^{[i][j]+[i][k]+[j][k]} E_{kj} \otimes E_{il} \otimes \widehat{T}^{\circ}_{kl}(v)\widehat{T}^{\circ}_{ij}(u)$$

$$= \sum_{i,j,k,l=1}^{M+N} (-1)^{[i][j]+[i][k]+[j][k]} E_{ij} \otimes E_{kl} \otimes \widehat{T}^{\circ}_{il}(v)\widehat{T}^{\circ}_{kj}(u).$$

Step 3. Lastly, since $Q \otimes \mathbf{1} = \sum_{p,q=1}^{M+N} (-1)^{[p][q]} \theta_p\theta_q E_{qp} \otimes E_{\bar{q}\bar{p}} \otimes \mathbf{1}$, we compute the equation $(C.1)$ $\widehat{T}^{\circ}_1(u)\widehat{T}^{\circ}_2(v)(Q \otimes \mathbf{1})$ to be

$$\sum_{i,j,k,l,p,q=1}^{M+N} (-1)^{(([i]+[j])+([p]+[q]))([k]+[l])+[p][q]} \theta_p\theta_q \delta_{jq}\delta_{l\bar{q}} E_{ip} \otimes E_{k\bar{p}} \otimes \widehat{T}^{\circ}_{ij}(u)\widehat{T}^{\circ}_{kl}(v)$$

$$= \delta_{\bar{j}l} \sum_{i,j,k,l,p=1}^{M+N} (-1)^{[i][j]+[i][k]+[k][p]} \theta_j\theta_p E_{ip} \otimes E_{k\bar{p}} \otimes \widehat{T}^{\circ}_{ij}(u)\widehat{T}^{\circ}_{k\bar{j}}(v)$$

$$= \delta_{\bar{j}l} \sum_{i,j,k,l,p=1}^{M+N} (-1)^{[i][k]+[j][k]+[i][p]} \theta_j\theta_p E_{ij} \otimes E_{kl} \otimes \widehat{T}^{\circ}_{ip}(u)\widehat{T}^{\circ}_{k\bar{p}}(v)$$

and $(C.2)$ $(Q \otimes \mathbf{1})\widehat{T}^{\circ}_2(v)\widehat{T}^{\circ}_1(u)$ to be

$$\sum_{i,j,k,l,p,q=1}^{M+N} (-1)^{([p]+[q])([i]+[j])+[p][q]} \theta_p\theta_q \delta_{pi}\delta_{\bar{p}k} E_{qj} \otimes E_{\bar{q}l} \otimes \widehat{T}^{\circ}_{kl}(v)\widehat{T}^{\circ}_{ij}(u)$$

$$= \delta_{\bar{i}k} \sum_{i,j,k,l,p=1}^{M+N} (-1)^{[i][j]+[i]+[j][p]} \theta_i\theta_p E_{pj} \otimes E_{\bar{p}l} \otimes \widehat{T}^{\circ}_{\bar{i}l}(v)\widehat{T}^{\circ}_{ij}(u)$$

$$= \delta_{\bar{i}k} \sum_{i,j,k,l,p=1}^{M+N} (-1)^{[i][j]+[j][p]+[p]} \theta_i\theta_p E_{ij} \otimes E_{kl} \otimes \widehat{T}^{\circ}_{\bar{p}l}(v)\widehat{T}^{\circ}_{pj}(u).$$

Step 4. We have that

$$\widehat{T}_1^\circ(u)\widehat{T}_2^\circ(v)R(u-v) = \widehat{T}_1^\circ(u)\widehat{T}_2^\circ(v) - \frac{\widehat{T}_1^\circ(u)\widehat{T}_2^\circ(v)(P\otimes\mathbf{1})}{u-v} + \frac{\widehat{T}_1^\circ(u)\widehat{T}_2^\circ(v)(Q\otimes\mathbf{1})}{u-v-\kappa}$$

and

$$R(u-v)\widehat{T}_2^\circ(v)\widehat{T}_1^\circ(u) = \widehat{T}_2^\circ(v)\widehat{T}_1^\circ(u) - \frac{(P\otimes\mathbf{1})\widehat{T}_2^\circ(v)\widehat{T}_1^\circ(u)}{u-v} + \frac{(Q\otimes\mathbf{1})\widehat{T}_2^\circ(v)\widehat{T}_1^\circ(u)}{u-v-\kappa}$$

are equal, yielding the relation

$$\begin{aligned}\widehat{T}_1^\circ(u)\widehat{T}_2^\circ(v) - \widehat{T}_2^\circ(v)\widehat{T}_1^\circ(u) &= \frac{1}{u-v}\Big(\widehat{T}_1^\circ(u)\widehat{T}_2^\circ(v)(P\otimes\mathbf{1}) - (P\otimes\mathbf{1})\widehat{T}_2^\circ(v)\widehat{T}_1^\circ(u)\Big)\\ &\quad + \frac{1}{u-v-\kappa}\Big((Q\otimes\mathbf{1})\widehat{T}_2^\circ(v)\widehat{T}_1^\circ(u) - \widehat{T}_1^\circ(u)\widehat{T}_2^\circ(v)(Q\otimes\mathbf{1})\Big)\end{aligned}$$

Therefore, by using equations $(A.1)$, $(A.2)$, $(B.1)$, $(B.2)$, $(C.1)$, and $(C.2)$, the above relation becomes

$$\begin{aligned}&(-1)^{([i]+[j])([k]+[l])}\widehat{T}_{ij}^\circ(u)\widehat{T}_{kl}^\circ(v) - \widehat{T}_{kl}^\circ(v)\widehat{T}_{ij}^\circ(u)\\ &\qquad = \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\Big(\widehat{T}_{il}^\circ(u)\widehat{T}_{kj}^\circ(v) - \widehat{T}_{il}^\circ(v)\widehat{T}_{kj}^\circ(u)\Big)\\ &\qquad\quad - \frac{1}{u-v-\kappa}\Bigg(\delta_{\bar{\imath}k}\sum_{p=1}^{M+N}(-1)^{[i][j]+[j][p]+[p]}\theta_i\theta_p\widehat{T}_{\bar{p}l}^\circ(v)\widehat{T}_{pj}^\circ(u)\\ &\qquad\qquad\qquad - \delta_{\bar{\jmath}l}\sum_{p=1}^{M+N}(-1)^{[i][k]+[j][k]+[i][p]}\theta_j\theta_p\widehat{T}_{ip}^\circ(u)\widehat{T}_{k\bar{p}}^\circ(v)\Bigg),\end{aligned}$$

Finally, multiplying the above equation by the scalar $(-1)^{([i]+[j])([k]+[l])+[i][j]+[j]+[k][l]+[l]}$ yields the desired relations

$$\begin{aligned}\big[T_{ij}^\circ(u), T_{kl}^\circ(v)\big] &= \frac{1}{u-v}(-1)^{[j][k]+[j][l]+[k][l]}\Big(T_{il}^\circ(u)T_{kj}^\circ(v) - T_{il}^\circ(v)T_{kj}^\circ(u)\Big)\\ &\quad - \frac{1}{u-v-\kappa}\Bigg(\delta_{\bar{\imath}k}\sum_{p=1}^{M+N}(-1)^{[i][j]+[i]+[j][l]+[l][p]+[p]}\theta_i\theta_pT_{\bar{p}l}^\circ(v)T_{pj}^\circ(u)\\ &\qquad\qquad - \delta_{\bar{\jmath}l}\sum_{p=1}^{M+N}(-1)^{[j][k]+[j]+[k][p]}\theta_j\theta_pT_{ip}^\circ(u)T_{k\bar{p}}^\circ(v)\Bigg).\end{aligned}$$

□

5.1.4 Details of Proofs on Extended Yangians

Proposition 5.1.8. *The extended super Yangian* $\mathbf{X}(\mathfrak{osp}_{M|N})$ *has a Hopf superalgebra structure given by the maps* (**??**), (**??**), *and* (2.43).

Proof. Let us first verify the formula for $\Delta(T_{ij}^{(n)})$. To this end,

$$\begin{aligned}\Delta(T(u)) &= \left(\sum_{i,j=1}^{M+N}(-1)^{[i][j]+[j]}E_{ij}\otimes T_{ij}(u)\otimes\mathbf{1}\right)\left(\sum_{k,l=1}^{M+N}(-1)^{[k][l]+[l]}E_{kl}\otimes\mathbf{1}\otimes T_{kl}(u)\right)\\ &= \sum_{i,j,l=1}^{M+N}(-1)^{[i][j]+[j][l]+[j]+[l]+([i]+[j])([j]+[l])}E_{il}\otimes T_{ij}(u)\otimes T_{jl}(u)\\ &= \sum_{i,j,l=1}^{M+N}(-1)^{[i][l]+[l]}E_{il}\otimes T_{ij}(u)\otimes T_{jl}(u)\end{aligned}$$

The maps Δ and ε are grade-preserving but we now show further that they are in fact superalgebra morphisms. That (**??**) is an algebra morphism is equivalent to the condition

$$R(u-v)T_{1[1]}(u)T_{1[2]}(u)T_{2[1]}(v)T_{2[2]}(v)=T_{2[1]}(v)T_{2[2]}(v)T_{1[1]}(u)T_{1[2]}(u)R(u-v),$$

where we identify $R(u-v)$ with $R(u-v)\otimes\mathbf{1}\otimes\mathbf{1}$ in the above equation. Using that $T_{1[2]}(u)$ and $T_{2[1]}(v)$ commute and that $T_{1[1]}(u)$ and $T_{2[2]}(v)$ commute, we may use the RTT-relation to conclude

$$\begin{aligned}R(u-v)T_{1[1]}(u)T_{1[2]}(u)T_{2[1]}(v)T_{2[2]}(v) &= T_{2[1]}(v)T_{1[1]}(u)R(u-v)T_{1[2]}(u)T_{2[2]}(v)\\ &= T_{2[1]}(v)T_{1[1]}(u)T_{2[2]}(v)T_{1[2]}(u)R(u-v).\end{aligned}$$

That the counit is a superalgebra morphism follows trivially by replacing $T(u)$ and $T(v)$

in the RTT-relation by $\mathbb{1}$.

For coassociativity, we observe that both $(\Delta \otimes \mathrm{id}) \circ \Delta$ and $(\mathrm{id} \otimes \Delta) \circ \Delta$ applied to $T_{ij}(u)$ gives $\sum_{k,l=1}^{M+N} T_{ik}(u) \otimes T_{kl}(u) \otimes T_{lj}(u)$. For counitality, $(\varepsilon \otimes \mathrm{id}) \circ \Delta$ and $(\mathrm{id} \otimes \varepsilon) \circ \Delta$ applied to $T_{ij}(u)$ yields $1 \otimes T_{ij}(u)$ and $T_{ij}(u) \otimes 1$, respectively.

We already know that the map S is a superalgebra anti-automorphism, so all that is left to show is the antipode axioms. To this extent, we observe

$$(S \otimes \mathrm{id}) \circ \Delta \colon T(u) \mapsto T_{[1]}(u)^{-1} T_{[2]}(u)$$

and

$$(\mathrm{id} \otimes S) \circ \Delta \colon T(u) \mapsto T_{[1]}(u) T_{[2]}(u)^{-1}$$

and applying the multiplication map $\mu = \mathrm{id} \otimes \mu$ to these yields images $\mathbb{1} = (\varepsilon \circ \eta)(T(u))$, where $\eta = \mathrm{id} \otimes \eta$ is the unit map. $\square$

Theorem 5.1.9. *The mapping*

$$\Psi \colon F_{ij} z^{n-1} \mapsto \overline{\mathcal{T}}_{ij}^{(n)} \tag{5.1}$$

is an isomorphism of $\mathbb{N}$*-graded Hopf superalgebras* $\Psi \colon U(\mathfrak{osp}_{M|N}[z]) \to \operatorname{gr} \mathbf{Y}(\mathfrak{osp}_{M|N})$.

Proof Addendum. Suppose that we have homogeneous elements $A = A_1 \otimes A_2$ and $B = B_1 \otimes B_2$ such that $[A_1] = [A_2]$ and $[B_1] = [B_2]$, which we will denote $[A]$ and $[B]$, respectively. Setting $(-)^{st_3} = \mathrm{id} \otimes \mathrm{id} \otimes (-)^{st}$, then

$$(A_{13} B_{23})^{st_3} = (-1)^{[A][B]} A_1 \otimes B_1 \otimes (A_2 B_2)^{st} = A_1 \otimes B_1 \otimes B_2^{st} A_2^{st} = B_{23}^{st_3} A_{13}^{st_3}$$

and

$$(B_{23} A_{13})^{st_3} = A_1 \otimes B_1 \otimes (B_2 A_2)^{st} = (-1)^{[A][B]} A_1 \otimes B_1 \otimes A_2^{st} B_2^{st} = A_{13}^{st_3} B_{23}^{st_3}$$

We now verify the identities $(P^{st_2})^2 = (M-N)\theta_0 P^{st_2}$, $P^{st_2}Q^{st_2} = Q^{st_2}P^{st_2} = \theta_0 P^{st_2}$, and $(Q^{st_2})^2 = \mathrm{id}^{\otimes 2}$. The first one is simply

$$\begin{aligned}
(P^{st_2})^2 &= \left(\sum_{i,j=1}^{M+N}(-1)^{[i][j]}E_{ij}\otimes E_{ij}\right)\left(\sum_{k,l=1}^{M+N}(-1)^{[k][l]}E_{kl}\otimes E_{kl}\right)\\
&= \sum_{i,j,k,l=1}^{M+N}(-1)^{[i][j]+[k][l]+([i]+[j])([k]+[l])}\delta_{jk}E_{il}\otimes E_{il}\\
&= \sum_{i,j,l=1}^{M+N}(-1)^{[i][l]+[j]}E_{il}\otimes E_{il} = (M-N)\theta_0 P^{st_2}
\end{aligned}$$

For second, we have

$$\begin{aligned}
P^{st_2}Q^{st_2} &= \left(\sum_{i,j=1}^{M+N}(-1)^{[i][j]}E_{ij}\otimes E_{ij}\right)\left(\sum_{k,l=1}^{M+N}(-1)^{[l]}\theta_k\theta_l E_{\bar{l}\,\bar{k}}\otimes E_{kl}\right)\\
&= \sum_{i,j,k,l=1}^{M+N}(-1)^{[i][j]+[l]+([i]+[j])([k]+[l])}\theta_k\theta_l\delta_{j\bar{l}}\delta_{jk}E_{i\bar{k}}\otimes E_{il}\\
&= \sum_{i,l=1}^{M+N}(-1)^{[i][l]+[l]}\theta_{\bar{l}}\theta_l E_{il}\otimes E_{il} = \theta_0 P^{st_2},
\end{aligned}$$

and similarly,

$$\begin{aligned}
Q^{st_2}P^{st_2} &= \left(\sum_{k,l=1}^{M+N}(-1)^{[l]}\theta_k\theta_l E_{\bar{l}\,\bar{k}}\otimes E_{kl}\right)\left(\sum_{i,j=1}^{M+N}(-1)^{[i][j]}E_{ij}\otimes E_{ij}\right)\\
&= \sum_{i,j,k,l=1}^{M+N}(-1)^{[i][j]+[l]+([i]+[j])([k]+[l])}\theta_k\theta_l\delta_{\bar{k}i}\delta_{li}E_{\bar{l}j}\otimes E_{kj}\\
&= \sum_{j,k=1}^{M+N}(-1)^{[k][j]+[k]}\theta_k\theta_{\bar{k}}E_{kj}\otimes E_{kj} = \theta_0 P^{st_2}.
\end{aligned}$$

Lastly,

$$(Q^{st_2})^2 = \left(\sum_{i,j=1}^{M+N}(-1)^{[j]}\theta_i\theta_j E_{\bar{j}\,\bar{i}}\otimes E_{ij}\right)\left(\sum_{k,l=1}^{M+N}(-1)^{[l]}\theta_k\theta_l E_{\bar{l}\,\bar{k}}\otimes E_{kl}\right)$$

$$
\begin{aligned}
&= \sum_{i,j,k,l=1}^{M+N} (-1)^{[j]+[l]+([i]+[j])([k]+[l])} \theta_i\theta_j\theta_k\theta_l \delta_{il}\delta_{jk} E_{\bar{j}\bar{k}} \otimes E_{il} \\
&= \sum_{i,j=1}^{M+N} E_{\bar{j}\bar{j}} \otimes E_{ii} = \mathrm{id}^{\otimes 2} .
\end{aligned}
$$

Now, $R^{st_2}(a-u)(R^t(a-u-\kappa))^{st_2}$ is explicitly given by

$$
\begin{aligned}
&\left(\mathrm{id}^{\otimes 2} + \frac{P^{st_2}}{u-a} - \frac{Q^{st_2}}{u+\kappa-a}\right)\left(\mathrm{id}^{\otimes 2} + \frac{Q^{st_2}}{u+\kappa-a} - \frac{P^{st_2}}{u+2\kappa-a}\right) \\
&\quad = \mathrm{id}^{\otimes 2} + \frac{P^{st_2}}{u-a} - \frac{Q^{st_2}}{u+\kappa-a} + \frac{Q^{st_2}}{u+\kappa-a} + \frac{P^{st_2}Q^{st_2}}{(u-a)(u+\kappa-a)} - \frac{(Q^{st_2})^2}{(u+\kappa-a)^2} \\
&\qquad - \frac{P^{st_2}}{u+2\kappa-a} - \frac{(P^{st_2})^2}{(u-a)(u+2\kappa-a)} + \frac{Q^{st_2}P^{st_2}}{(u+\kappa-a)(u+2\kappa-a)}
\end{aligned}
$$

The coefficient of $P^{st_2}/(u-a)(u+\kappa-a)(u+2\kappa-a)$ is a quadratic polynomial in u

$$
\begin{aligned}
(u+\kappa-a)(u+2\kappa-a) + \theta_0(u+2\kappa-a) &- (u-a)(u+\kappa-a) \\
&- (M-N)\theta_0(u+\kappa-a) + \theta_0(u-a)
\end{aligned}
$$

and we now verify that it is zero. The coefficient of u^2 is $1-1=0$. The coefficient of u is $2\kappa - a + \kappa - a + \theta_0 - \kappa + a + a - (M-N)\theta_0 + \theta_0 = 2\kappa + (2+N-M)\theta_0 = 0$ since $\kappa = (M-N-2)\theta_0/2$. The coefficient of 1 is $(\kappa-a)(2\kappa-a) + (2\kappa-a)\theta_0 + a(\kappa-a) - (M-N)(\kappa-a)\theta_0 - a\theta_0 = 2\kappa^2 - (M-N-2)\theta_0\kappa - 2a\kappa + (M-N-2)\theta_0 a = 0$.

5.1.5 Root Systems of $\mathfrak{osp}_{M|N}$

We recall that the Cartan subalgebra of $\mathfrak{osp}_{M|N}$ is given by

$$
\mathfrak{h} = \mathrm{span}\left\{F_{qq} \;\middle|\; \lceil \tfrac{M}{2} \rceil + 1 \le q \le M,\ M + \tfrac{N}{2} + 1 \le q \le M+N\right\}
$$

and its action on $\mathfrak{osp}_{M|N}$ is given by the formula

$$(\dagger) := \big[F_{qq}, F_{ij}\big] = \delta_{iq}F_{qj} - \delta_{jq}F_{iq} - \delta_{\bar{\imath}q}F_{\bar{q}j} + \delta_{\bar{\jmath}q}F_{i\bar{q}}. \tag{5.2}$$

We now calculate the root systems using the above formula. Note that we require $F_{ij} \neq 0$, and that $F_{i\bar{\imath}} = 0$ when $[i] = 0$. For the even part, we have:

- If $1 \leq i \neq j \leq \lceil \frac{M}{2} \rceil$ and M is even, then $(\dagger) = (\varepsilon_{\bar{\jmath}} - \varepsilon_{\bar{\imath}})(F_{qq})F_{ij}$. In the case when M is odd and $j = \lceil \frac{M}{2} \rceil$, then $(\dagger) = (-\varepsilon_{\bar{\imath}})(F_{qq})F_{ij}$;
- if $1 \leq i \leq \lceil \frac{M}{2} \rceil$ and $\lceil \frac{M}{2} \rceil + 1 \leq j \leq M$ $(i \neq \bar{\jmath})$ and M is even, then $(\dagger) = (-\varepsilon_j - \varepsilon_{\bar{\imath}})(F_{qq})F_{ij}$. In the case when M is odd and $i = \lceil \frac{M}{2} \rceil$, then $(\dagger) = (-\varepsilon_j)(F_{qq})F_{ij}$;
- if $\lceil \frac{M}{2} \rceil + 1 \leq i \leq M$ and $1 \leq j \leq \lceil \frac{M}{2} \rceil$ $(i \neq \bar{\jmath})$ and M is even, then $(\dagger) = (\varepsilon_i + \varepsilon_{\bar{\jmath}})(F_{qq})F_{ij}$. In the case when M is odd and $j = \lceil \frac{M}{2} \rceil$, then $(\dagger) = \varepsilon_i(F_{qq})F_{ij}$;
- if $\lceil \frac{M}{2} \rceil + 1 \leq i \neq j \leq M$, then $(\dagger) = (\varepsilon_i - \varepsilon_j)(F_{qq})F_{ij}$;
- if $M + 1 \leq i \neq j \leq M + \frac{N}{2}$, then $(\dagger) = (\delta_{\bar{\jmath}} - \delta_{\bar{\imath}})(F_{qq})F_{ij}$;
- if $M+1 \leq i \leq M+\frac{N}{2}$ and $M+\frac{N}{2}+1 \leq j \leq M+N$, then $(\dagger) = (-\delta_{\bar{\imath}} - \delta_j)(F_{qq})F_{ij}$ and $(\dagger) = (-2\delta_j)(F_{qq})F_{ij}$ when $i = \bar{\jmath}$;
- if $M + \frac{N}{2} + 1 \leq i \leq M + N$ and $M + 1 \leq j \leq M + \frac{N}{2}$, then $(\dagger) = (\delta_i + \delta_{\bar{\jmath}})(F_{qq})F_{ij}$ and $(\dagger) = (2\delta_i)(F_{qq})F_{ij}$ when $j = \bar{\imath}$;
- if $M + \frac{N}{2} + 1 \leq i \neq j \leq M + N$, then $(\dagger) = (\delta_i - \delta_j)(F_{qq})F_{ij}$.

For the odd part, we have:

- If $1 \leq i \leq \lceil \frac{M}{2} \rceil$ and $M + 1 \leq j \leq M + \frac{N}{2}$ and M is even, then $(\dagger) = (-\varepsilon_{\bar{\imath}} + \delta_{\bar{\jmath}})(F_{qq})F_{ij}$. In the case when M is odd and $i = \lceil \frac{M}{2} \rceil$, then $(\dagger) = \delta_{\bar{\jmath}}(F_{qq})F_{ij}$;
- if $1 \leq i \leq \lceil \frac{M}{2} \rceil$ and $M + \frac{N}{2} + 1 \leq j \leq M + N$ and M is even, then $(\dagger) = (-\varepsilon_{\bar{\imath}} - \delta_j)(F_{qq})F_{ij}$. In the case when M is odd and $i = \lceil \frac{M}{2} \rceil$, then $(\dagger) = (-\delta_j)(F_{qq})F_{ij}$;

- if $\lceil \frac{M}{2} \rceil + 1 \le i \le M$ and $M+1 \le j \le M+\frac{N}{2}$, then $(\dagger) = (\varepsilon_i + \delta_{\bar{j}})(F_{qq})F_{ij}$;
- if $\lceil \frac{M}{2} \rceil + 1 \le i \le M$ and $M+\frac{N}{2}+1 \le j \le M+N$, then $(\dagger) = (\varepsilon_i - \delta_j)(F_{qq})F_{ij}$;
- if $M+1 \le i \le M+\frac{N}{2}$ and $1 \le j \le \lceil \frac{M}{2} \rceil$ and M is even, then $(\dagger) = (-\delta_{\bar{\imath}} + \varepsilon_{\bar{j}})(F_{qq})F_{ij}$. In the case when M is odd and $j = \lceil \frac{M}{2} \rceil$, then $(\dagger) = (-\delta_{\bar{\imath}})(F_{qq})F_{ij}$;
- if $M+1 \le i \le M+\frac{N}{2}$ and $\lceil \frac{M}{2} \rceil + 1 \le j \le M$, then $(\dagger) = (-\delta_{\bar{\imath}} - \varepsilon_j)(F_{qq})F_{ij}$.
- if $M+\frac{N}{2}+1 \le i \le M+N$ and $1 \le j \le \lceil \frac{M}{2} \rceil$ and M is even, then $(\dagger) = (\delta_i + \varepsilon_{\bar{j}})(F_{qq})F_{ij}$. In the case when M is odd and $j = \lceil \frac{M}{2} \rceil$, then $(\dagger) = \delta_i(F_{qq})F_{ij}$;
- if $M+\frac{N}{2}+1 \le i \le M+N$ and $\lceil \frac{M}{2} \rceil + 1 \le j \le M$, then $(\dagger) = (\delta_i - \varepsilon_j)(F_{qq})F_{ij}$.

5.1.6 Details on Weight Representations

for the highest weight lemma:

(ii). The argument is analogous to the one before, just by replacing sums of the form $\sum_{k \le p \le M,\, M+\frac{N}{2}+1 \le p \le M+N}$ with $\sum_{k \le p \le M}$.

old lemma that is really for Λ^+_{II} (: Re-write):

Lemma 5.1.10. *Let V be a finite-dimensional irreducible representation of $\mathbf{X}(\mathfrak{osp}_{M|N})$. Defining $V_I^0 = \{v \in V \,|\, T_{ij}(u)v = 0$ for all $(i,j) \in \Lambda_I^+\}$, then*

(i) $V_I^0 \neq 0$,

(ii) *the operators $T_{kk}^{(n)}$, $n \in \mathbb{N}$, preserve the subspace V_I^0 for all $k \in \mathbb{N}_{N+M}$,*

(iii) *for all $v \in V_I^0$, $[T_{kk}^{(m)}, T_{ll}^{(n)}]v = 0$ for any $k, l \in \mathbb{N}_{M+N}$, $m, n \in \mathbb{N}$.*

Proof. (i) We endow the partial ordering '$\prec$' on the set of weights of any $\mathfrak{osp}_{M|N}$-module via the rule that for any weights $\alpha, \beta \in \mathfrak{h}^*$, we write $\alpha \prec \beta$ if $\beta - \alpha$ is a $\mathbb{Z}^+$-linear combination of positive roots in Φ_I^+.

Let us now consider the set of weights of V, where V is viewed as an $\mathfrak{osp}_{M|N}$-module via the embedding (**??**). Since V is finite-dimensional, there are only a finite amount of these weights and therefore V has a maximal weight μ with respect to the partial ordering '$\prec$'. Letting v be a weight vector corresponding to μ, the assertion follows if $v \in V_I^0$, so we may assume $v \notin V_I^0$ and therefore $T_{ij}^{(n)} v \neq 0$ for some $(i,j) \in \Lambda_I^+$. Since

$$F_{kk} T_{ij}^{(n)} v = T_{ij}^{(n)} F_{kk} v + \left[F_{kk}, T_{ij}^{(n)} \right] v,$$

we conclude from equation (**??**) that the weight of $T_{ij}^{(n)} v$ is of the form $\mu + \alpha$ for some positive root $\alpha \in \Phi_I^+$, contradicting the maximality of μ and proving the assertion.

(ii) This argument follows the one provided in [**?**, Section 5]. Writing '$\equiv$' for an equality of operators on V_I^0, we then wish to show

$$T_{ij}(u) T_{kk}(v) \equiv 0 \tag{5.3}$$

for all $(i,j) \in \Lambda_I^+$.

Step 1. We shall for now assume $i < k$ for steps 1-4. If $(i,k) \in \Lambda_I^+$, then equation (5.3) is immediate from $T_{ij}(u) T_{kk}(v) \equiv \left[T_{ij}(u), T_{kk}(v) \right]$ with exception when $i = \bar{k}$ or $j = \bar{k}$. However, $(i,k) \notin \Lambda_I^+$ when $(i,j) \in \Gamma_{0,0} \cup \Gamma_{0,1}^I$ and $M+1 \leq k \leq M + \frac{N}{2}$. Nonetheless, in this case we have $T_{ij}(u) T_{kk}(v) \equiv -\left[T_{kk}(v), T_{ij}(u) \right] \equiv 0$ (when $j \neq \bar{k}$) since $(k,j) \in \Gamma_{1,0}^I \cup \Gamma_{1,1}$.

Step 2. Now considering the first exception when $i = \bar{k}$, then either $\left\lceil \frac{M}{2} \right\rceil + 1 \leq k \leq M$ or $M + \frac{N}{2} + 1 \leq k \leq M + N$. If $\left\lceil \frac{M}{2} \right\rceil + 1 \leq k \leq M$, then we are restricted to $(\bar{k}, j) \in \Gamma_{0,0} \cup \Gamma_{0,1}^I$. The defining relations yields

$$T_{\bar{k}j}(u) T_{kk}(v) \equiv -\frac{1}{u - v - \kappa} \sum_{\substack{M + \frac{N}{2} + 1 \leq p \leq M+N \\ k \leq p \leq M}} (-1)^{[k][j]+[j][p]+[p]} \theta_{\bar{k}} \theta_{\bar{p}} T_{\bar{p}j}(u) T_{pk}(v), \tag{5.4}$$

while for each $k \leq p \leq M$ and $M + \frac{N}{2} + 1 \leq p \leq M + N$ we have

$$T_{\bar{p}j}(u)T_{pk}(v) \equiv -\frac{1}{u-v-\kappa} \sum_{\substack{M+\frac{N}{2}+1 \leq q \leq M+N \\ k \leq q \leq M}} (-1)^{[p][j]+[j][q]+[q]} \theta_{\bar{p}}\theta_{\bar{q}} T_{\bar{q}j}(u)T_{qk}(v). \tag{5.5}$$

Therefore, $(-1)^{[p][j]}\theta_{\bar{p}}T_{\bar{p}j}(u)T_{pk}(v) \equiv (-1)^{[k][j]}\theta_{\bar{k}}T_{\bar{k}j}(u)T_{kk}(v)$. Equation (5.4) then implies

$$\left(1 + \frac{(M - \frac{N}{2} - k + 1)\theta_0}{u - v - \kappa}\right) T_{\bar{k}j}(u)T_{kk}(v) \equiv 0, \tag{5.6}$$

and thus $T_{\bar{k}j}(u)T_{kk}(v) \equiv 0$. Alternatively, if $M + \frac{N}{2} + 1 \leq k \leq M + N$, then $(\bar{k}, j) \in \Gamma^{I}_{1,0} \cup \Gamma_{1,1}$ and so

$$T_{\bar{k}j}(u)T_{kk}(v) \equiv -\frac{1}{u-v-\kappa} \sum_{p=k}^{M+N} (-1)^{[k][j]+[j][p]+[p]} \theta_{\bar{k}}\theta_{\bar{p}} T_{\bar{p}j}(u)T_{pk}(v), \tag{5.7}$$

and by using the same argument as before, one shows that

$$\left(1 - \frac{(M + N - k + 1)\theta_0}{u - v - \kappa}\right) T_{\bar{k}j}(u)T_{kk}(v) \equiv 0. \tag{5.8}$$

Step 3. Now consider the second case when $j = \bar{k}$ and further assume $j < k$. Then either $\lceil \frac{M}{2} \rceil + 1 \leq k \leq M$ or $M + \frac{N}{2} + 1 \leq k \leq M + N$. If $\lceil \frac{M}{2} \rceil + 1 \leq k \leq M$, then we have $(i, \bar{k}) \in \Gamma_{0,0}$, so

$$T_{i\bar{k}}(u)T_{kk}(v) \equiv \frac{1}{u-v-\kappa} \sum_{\substack{M+\frac{N}{2}+1 \leq p \leq M+N \\ \bar{\imath} \leq p \leq M}} (-1)^{[i][k]+[i][p]+[k]} \theta_{\bar{k}}\theta_{\bar{p}} T_{kp}(v)T_{i\bar{p}}(u). \tag{5.9}$$

Now, for each $\bar{\imath} \leq p \leq M$ and $M + \frac{N}{2} + 1 \leq p \leq M + N$ we have

$$T_{kp}(v)T_{i\bar{p}}(u) \equiv -\big[T_{i\bar{p}}(u), T_{kp}(v)\big]$$

$$\equiv -\frac{1}{u-v-\kappa} \sum_{\substack{M+\frac{N}{2}+1 \le q \le M+N \\ \bar{\imath} \le q \le M}} (-1)^{[i][k]+[i][q]+[p][k]} \theta_{\bar{p}} \theta_{\bar{q}} T_{kq}(v) T_{i\bar{q}}(u), \quad (5.10)$$

and so $-(-1)^{[k]}\theta_{\bar{k}} T_{i\bar{k}}(u) T_{kk}(v) \equiv (-1)^{[p][k]} \theta_{\bar{p}} T_{kp}(v) T_{i\bar{p}}(u)$. Then equation (5.9) implies

$$\left(1 + \frac{(M + \frac{N}{2} - \bar{\imath} + 1)\theta_0}{u - v - \kappa}\right) T_{i\bar{k}}(u) T_{kk}(v) \equiv 0, \tag{5.11}$$

and therefore $T_{i\bar{k}}(u) T_{kk}(v) \equiv 0$. Otherwise when $M + \frac{N}{2} + 1 \le k \le M + N$, consequently $(i, \bar{k}) \in \Gamma_{1,1}$ and so

$$T_{i\bar{k}}(u) T_{kk}(v) \equiv \frac{1}{u - v - \kappa} \sum_{p=\bar{\imath}}^{M+N} (-1)^{[i][k]+[i][p]+[k]} \theta_{\bar{k}} \theta_{\bar{p}} T_{kp}(v) T_{i\bar{p}}(u), \tag{5.12}$$

and a similar argument shows

$$\left(1 - \frac{(M + N - \bar{\imath} + 1)\theta_0}{u - v - \kappa}\right) T_{i\bar{k}}(u) T_{kk}(v) \equiv 0. \tag{5.13}$$

Step 4. Now supposing $j = \bar{k} \ge k$, we have either $1 \le k \le \lceil \frac{M}{2} \rceil$ or $M + 1 \le k \le M + \frac{N}{2}$. Assuming $1 \le k \le \lceil \frac{M}{2} \rceil$, then $(i, \bar{k}) \in \Gamma_{0,0}$. Now, if $k = \lceil \frac{M}{2} \rceil$ and M is odd, then one may use equations (5.9) and (5.10) to deduce (5.11). Otherwise, we have

$$\begin{aligned} T_{i\bar{k}}(u) T_{kk}(v) &\equiv -\left[T_{kk}(v), T_{i\bar{k}}(u)\right] \\ &\equiv -\frac{1}{v - u - \kappa} \sum_{\substack{M+\frac{N}{2}+1 \le p \le M+N \\ \bar{k} \le p \le M}} (-1)^{[k][p]} \theta_k \theta_{\bar{p}} T_{ip}(u) T_{k\bar{p}}(v). \end{aligned} \tag{5.14}$$

For every $\bar{k} \le p \le M$ and $M + \frac{N}{2} + 1 \le p \le M + N$, we have

$$\begin{aligned} T_{ip}(u) T_{k\bar{p}}(v) &\equiv -\left[T_{k\bar{p}}(v), T_{ip}(u)\right] \\ &\equiv -\frac{1}{v - u - \kappa} \sum_{\substack{M+\frac{N}{2}+1 \le q \le M+N \\ \bar{k} \le q \le M}} (-1)^{[k][i]+[k][q]+[p][i]} \theta_{\bar{p}} \theta_{\bar{q}} T_{iq}(u) T_{k\bar{q}}(v), \end{aligned} \tag{5.15}$$

and so $\theta_k T_{i\bar{k}}(u)T_{kk}(v) \equiv (-1)^{[k][i]+[p][i]}\theta_{\bar{p}}T_{ip}(u)T_{k\bar{p}}(v)$. Hence, equation (5.14) implies

$$\left(1+\frac{(M+\frac{N}{2}-\bar{k}+1)\theta_0}{v-u-\kappa}\right)T_{i\bar{k}}(u)T_{kk}(v) \equiv 0, \tag{5.16}$$

and accordingly, $T_{i\bar{k}}(u)T_{kk}(v) \equiv 0$. On the other hand, if $M+1 \leq k \leq M+\frac{N}{2}$, then $(i,\bar{k}) \in \Gamma^I_{0,1} \cup \Gamma_{1,1}$. In this case,

$$T_{i\bar{k}}(u)T_{kk}(v) \equiv -\big[T_{kk}(v), T_{i\bar{k}}(u)\big] \equiv -\frac{1}{v-u-\kappa}\sum_{p=\bar{k}}^{M+N}(-1)^{[k][p]}\theta_k\theta_{\bar{p}}T_{ip}(u)T_{k\bar{p}}(v), \tag{5.17}$$

and identical reasoning shows

$$\left(1-\frac{(M+N-\bar{k}+1)\theta_0}{v-u-\kappa}\right)T_{i\bar{k}}(u)T_{kk}(v) \equiv 0. \tag{5.18}$$

Step 5. We now assume $i \geq k$ for steps 5-8. If $(k,j) \in \Lambda^+_I$, then the assertion is satisfied for those $(i,j) \in \Lambda^+_I$ from the equivalence $T_{ij}(u)T_{kk}(v) \equiv -\big[T_{kk}(v), T_{ij}(u)\big]$ except for the cases when $i = \bar{k}$ and $j = \bar{k}$. However, $(k,j) \notin \Lambda^+_I$ when $(i,j) \in \Gamma^I_{1,0}$ such that $j \leq k \leq M$ and when $(i,j) \in \Gamma_{1,1}$ such that $M+1 \leq i < j \leq M+\frac{N}{2}$, $1 \leq k \leq M$. Nonetheless, in these cases we have $T_{ij}(u)T_{kk}(v) \equiv \big[T_{ij}(u), T_{kk}(v)\big] \equiv 0$ (when $j \neq \bar{k}$) since $(i,k) \in \Gamma^I_{1,0}$.

Step 6. In the first exception when $i = \bar{k}$, then either $1 \leq k \leq \lceil\frac{M}{2}\rceil$ or $M+1 \leq k \leq M+\frac{N}{2}$. Given $1 \leq k \leq \lceil\frac{M}{2}\rceil$, we have $(\bar{k},j) \in \Gamma_{0,0}$ if $k \neq 1$ or $(\bar{k},j) \in \Gamma^I_{0,1}$. In the case $(\bar{k},j) \in \Gamma_{0,0}$, we have

$$\begin{aligned} T_{\bar{k}j}(u)T_{kk}(v) &\equiv -\big[T_{kk}(v), T_{\bar{k}j}(u)\big] \\ &\equiv \frac{1}{v-u-\kappa}\sum_{\substack{M+\frac{N}{2}+1\leq p\leq M+N \\ j\leq p\leq M}}(-1)^{[k]+[k][p]+[p]}\theta_k\theta_{\bar{p}}T_{\bar{p}k}(v)T_{pj}(u). \end{aligned} \tag{5.19}$$

For each $j \leq p \leq M$ and $M + \frac{N}{2} + 1 \leq p \leq M + N$, we have

$$T_{\bar{p}k}(v)T_{pj}(u) \equiv -\frac{1}{v-u-\kappa} \sum_{\substack{M+\frac{N}{2}+1 \leq q \leq M+N \\ j \leq q \leq M}} (-1)^{[p][k]+[k][q]+[q]} \theta_{\bar{p}} \theta_{\bar{q}} T_{\bar{q}k}(v) T_{qj}(v), \quad (5.20)$$

and therefore $-(-1)^{[k]}\theta_k T_{\bar{k}j}(u)T_{kk}(v) \equiv (-1)^{[p][k]}\theta_{\bar{p}} T_{\bar{p}k}(v)T_{pj}(u)$, implying

$$\left(1 + \frac{(M - \frac{N}{2} - j + 1)\theta_0}{v-u-\kappa}\right) T_{\bar{k}j}(u)T_{kk}(v) \equiv 0, \quad (5.21)$$

from equation (5.19) and therefore $T_{\bar{k}j}(u)T_{kk}(v) \equiv 0$. Otherwise, when $(\bar{k}, j) \in \Gamma_{1,0}$, we have

$$T_{\bar{k}j}(u)T_{kk}(v) \equiv -\big[T_{kk}(v), T_{\bar{k}j}(u)\big] \equiv \frac{1}{v-u-\kappa} \sum_{p=j}^{M+N} (-1)^{[k]+[k][p]+[p]} \theta_k \theta_{\bar{p}} T_{\bar{p}k}(v) T_{pj}(u) \quad (5.22)$$

and with similar reasoning conclude

$$\left(1 - \frac{(M + N - j + 1)\theta_0}{v-u-\kappa}\right) T_{\bar{k}j}(u)T_{kk}(v) \equiv 0. \quad (5.23)$$

Alternatively, if $M + 1 \leq k \leq M + \frac{N}{2}$, then $(\bar{k}, j) \in \Gamma_{1,1}$. We may in the same way use equation (5.22) and a similar version of (5.20) to conclude (5.23).

Step 7. Now consider when $j = \bar{k}$ and we shall also assume $j < k$. Therefore, either we have $\lceil \frac{M}{2} \rceil + 1 \leq k \leq M$ or $M + \frac{N}{2} + 1 \leq k \leq M + N$. In the case $\lceil \frac{M}{2} \rceil + 1 \leq k \leq M$, then $(i, \bar{k}) \in \Gamma^I_{1,0}$ and so we may use equation (5.12) and a similar version of (5.10) to deduce (5.13). In the case $M + \frac{N}{2} + 1 \leq k \leq M + N$, actually $(i, \bar{k}) \notin \Lambda^+_I$ for any $i \geq k$, so we may disregard this scenario.

Step 8. Finally, supposing $j \geq k$, then either $1 \leq k \leq \lceil \frac{M}{2} \rceil$ or $M + 1 \leq k \leq M + \frac{N}{2}$. If $1 \leq k \leq \lceil \frac{M}{2} \rceil$, then $(i, \bar{k}) \in \Gamma_{0,0} \cup \Gamma^I_{1,0}$. In this case where M is odd, $k = \lceil \frac{M}{2} \rceil$, and $(i, \bar{k}) \in \Gamma_{0,0}$, then one may use equations (5.9) and (5.10) to conclude (5.11). If M is odd, $k = \lceil \frac{M}{2} \rceil$, and $(i, \bar{k}) \in \Gamma^I_{1,0}$, then one may use equation (5.12) and a similar version of (5.10) to deduce (5.13). Otherwise, equations (5.14) and (5.15) imply

$$\left(1 + \frac{\sum_{\bar{k} \le p \le M,\ M+\frac{N}{2}+1 \le p \le M+N} (-1)^{[k][p]+[k][i]+[p][i]}}{v - u - \kappa}\right) T_{i\bar{k}}(u) T_{kk}(v) \equiv 0, \tag{5.24}$$

and accordingly, $T_{i\bar{k}}(u)T_{kk}(v) \equiv 0$. On the other hand, if $M + 1 \le k \le M + \frac{N}{2}$, then $(i, \bar{k}) \in \Gamma_{1,1}$. In this case, one can use equation (5.17) and a similar version of (5.15) to imply (5.18).

(iii). We now confirm that the operators $T_{kk}(u)$, $k \in \mathbb{N}_{M+N}$, form a commutative family on the space V^0_I. Foremost, we observe that $\left[T_{kk}(u), T_{kk}(v)\right] = 0$ for any $k \in \mathbb{N}_{M+N}$ unless $k = \lceil \frac{M}{2} \rceil$ in the case when M is odd. Moreover, for any $(k, l) \in \Lambda^+_I$ such that $k \ne \bar{l}$, we have

$$\left[T_{kk}(u), T_{ll}(v)\right] = \frac{1}{u - v}(-1)^{[k]} \left(T_{lk}(u)T_{kl}(v) - T_{lk}(v)T_{kl}(u)\right) \equiv 0 \tag{5.25}$$

as operators on V^0_I. Since $\left[T_{kk}(u), T_{ll}(v)\right] = -\left[T_{ll}(v), T_{kk}(u)\right]$, all that is left to check are the cases when $k = \bar{l}$. To this end, for any $\lfloor \frac{M}{2} \rfloor + 1 \le k \le l \le M$ and $M + \frac{N}{2} + 1 \le k \le l \le M + N$, we first define

$$A_{kl} := (-1)^{[k]+[l]} T_{\bar{l}\bar{k}}(u) T_{lk}(v) - T_{kl}(v) T_{\bar{k}\bar{l}}(u). \tag{5.26}$$

Step 1. Let us first suppose $M + \frac{N}{2} + 1 \le k \le M + N$. By relations (3.18), we obtain

$$A_{kk} = \left[T_{\bar{k}\bar{k}}(u), T_{kk}(v)\right] \equiv \frac{\theta_0}{u - v - \kappa} \sum_{l=k}^{M+N} A_{kl}, \tag{5.27}$$

and for $M + \frac{N}{2} + 1 \le k < l \le M + N$, we have

$$A_{kl} \equiv \left[T_{\bar{l}\bar{k}}(u), T_{lk}(v)\right] + \left[T_{\bar{k}\bar{l}}(u), T_{kl}(v)\right] \equiv \frac{\theta_0}{u - v - \kappa} \sum_{p=k}^{M+N} A_{kp} + \frac{\theta_0}{u - v - \kappa} \sum_{q=l}^{M+N} A_{lq}, \tag{5.28}$$

establishing $A_{kl} \equiv A_{kk} + A_{ll}$. Thus, by equation (5.27) we get

$$\left(1 - \frac{(M+N-k+2)\theta_0}{u-v-\kappa}\right) A_{kk} - \frac{\theta_0}{u-v-\kappa} \sum_{l=k+1}^{M+N} A_{ll} \equiv 0. \tag{5.29}$$

Inductively, we derive $A_{kk} = \left[T_{\bar{k}\bar{k}}(u), T_{kk}(v)\right] \equiv 0$ for all $M + \frac{N}{2} + 1 \leq k \leq M + N$.

Step 2. Now, let us fix $\left\lfloor \frac{M}{2} \right\rfloor + 1 \leq k \leq M$. Assuming $k \neq \left\lceil \frac{M}{2} \right\rceil$ when M is odd, then by relations (3.18) we obtain

$$A_{kk} = \left[T_{\bar{k}\bar{k}}(u), T_{kk}(v)\right] \equiv -\frac{1}{u-v-\kappa} \sum_{\substack{M+\frac{N}{2}+1 \leq l \leq M+N \\ k \leq l \leq M}} (-1)^{[k][l]} A_{kl}, \tag{5.30}$$

while for $\left\lfloor \frac{M}{2} \right\rfloor + 1 \leq k < l \leq M$ we have

$$\begin{aligned} A_{kl} &\equiv \left[T_{\bar{l}\bar{k}}(u), T_{lk}(v)\right] + \left[T_{\bar{k}\bar{l}}(u), T_{kl}(v)\right] \\ &\equiv -\frac{1}{u-v-\kappa} \sum_{\substack{M+\frac{N}{2}+1 \leq p \leq M+N \\ k \leq p \leq M}} (-1)^{[k][p]} A_{kp} - \frac{1}{u-v-\kappa} \sum_{\substack{M+\frac{N}{2}+1 \leq q \leq M+N \\ l \leq q \leq M}} (-1)^{[l][q]} A_{lq}, \end{aligned} \tag{5.31}$$

and therefore $A_{kl} \equiv A_{kk} + A_{ll}$. We then derive the formula

$$\left(1 + \frac{(M-k+2)\theta_0 + \frac{N}{2}}{u-v-\kappa}\right) A_{kk} + \frac{\theta_0}{u-v-\kappa} \sum_{l=k+1}^{M} A_{ll} \equiv 0. \tag{5.32}$$

since $A_{ll} \equiv 0$ for all $M+\frac{N}{2}+1 \leq l \leq M+N$. So by induction, $A_{kk} = \left[T_{\bar{k}\bar{k}}(u), T_{kk}(v)\right] \equiv 0$.

Step 3. In the special case when $k = \left\lceil \frac{M}{2} \right\rceil$ and M is odd, we have

$$A_{kk} = \left[T_{kk}(u), T_{kk}(v)\right] \equiv \frac{\theta_0}{u-v} A_{kk} - \frac{1}{u-v-\kappa} \sum_{\substack{M+\frac{N}{2}+1 \leq l \leq M+N \\ k \leq l \leq M}} (-1)^{[k][l]} A_{kl}.$$

Using equation (5.31), then for $k < l \leq M$ we have

$$A_{kl} \equiv \left[T_{\bar{l}k}(u), T_{lk}(v)\right] + \left[T_{k\bar{l}}(u), T_{kl}(v)\right] \equiv \frac{u - v - \theta_0}{u - v - \kappa} A_{kk} + A_{ll}. \tag{5.33}$$

Therefore,

$$\left(1 - \frac{\theta_0}{u-v} + \frac{\theta_0}{u-v-\kappa} + \frac{(u-v-\theta_0)\sum_{k\leq l\leq M,\ M+\frac{N}{2}+1\leq l\leq M+N}(-1)^{[k][l]}}{(u-v-\kappa)^2}\right) A_{kk} \equiv 0 \tag{5.34}$$

since $A_{ll} \equiv 0$ for all $k < l \leq M$ and $M + \frac{N}{2} + 1 \leq l \leq M + N$, completing the proof. □

Re-write the following proof:

Arguing in a similar way, we have the lemma

Lemma 5.1.11. *Letting V be a representation of $\mathbf{X}(\mathfrak{osp}_{M|N})$, then*

(i) *$V(\mathfrak{n})^+$ is stable under the operators $T_{ij}(u)$ for all $(i,j) \in \Lambda(\mathfrak{n})_{a,b}$, $a, b \in \mathbb{Z}_2$,*

(ii) *the operators $T_{ij}(u)$ for $(i,j) \in \Lambda(\mathfrak{n})_{a,b}$, $a, b \in \mathbb{Z}_2$ form a representation of the superalgebra $X(\mathfrak{n})$ on $V(\mathfrak{n})^+$ under the identification*

$$X(\mathfrak{n}) \hookrightarrow \mathbf{X}(\mathfrak{osp}_{M|N})$$

$$T^{\mathfrak{n}}_{ij}(u) \mapsto \begin{cases} T_{i,j}(u) & \text{if} \quad 1 \leq i, j \leq M, \\ T_{i+1,j+1}(u) & \text{if} \quad M+1 \leq i, j \leq M+N-2, \\ T_{i,j+1}(u) & \text{if} \quad 1 \leq i \leq M,\ M+1 \leq j \leq M+N-2, \\ T_{i+1,j}(u) & \text{if} \quad M+1 \leq i \leq M+N-2,\ 1 \leq j \leq M, \end{cases}$$

where $T^{\mathfrak{n}}_{ij}(u)$ is a generating series for $X(\mathfrak{n})$.

Proof. Similar to before, we assume $\eta \in V(\mathfrak{n})^+$ and write '$\equiv$' for elements modulo $V(\mathfrak{n})^+$ for the following arguments.

(i). If $1 \leq k \leq M$ and $(i,j) \in \Lambda(\mathfrak{n})_{1,0} \cup \Lambda(\mathfrak{n})_{1,1}$ or $M+1 \leq k < M+N$ and $(i,j) \in \Lambda(\mathfrak{n})_{0,0} \cup \Lambda(\mathfrak{n})_{0,1}$, then

$$T_{k,M+M}(v)T_{ij}(u)\eta \equiv -\big[T_{ij}(u), T_{k,M+N}(v)\big]\eta \equiv 0.$$

On the other hand, if $1 \leq k \leq M$ and $(i,j) \in \Lambda(\mathfrak{n})_{0,0} \cup \Lambda(\mathfrak{n})_{0,1}$ or $M+1 \leq k < M+N$ and $(i,j) \in \Lambda(\mathfrak{n})_{1,0} \cup \Lambda(\mathfrak{n})_{1,1}$, then by relations (3.18) we obtain

$$\begin{aligned}
T_{k,M+N}(v)T_{ij}(u)\eta &\equiv -\big[T_{ij}(u), T_{k,M+N}(v)\big]\eta \\
&\equiv -\frac{\delta_{i\bar{k}}}{u-v-\kappa}(-1)^{[i][j]+[M+1][j]}\theta_0\theta_i T_{M+1,j}(u)T_{M+N,M+N}(v)\eta.
\end{aligned}$$

Using the defining relations (4.10) again yields

$$\begin{aligned}
T_{M+1,j}(u)T_{M+N,M+N}(v)\eta &\equiv \big[T_{M+1,j}(u), T_{M+N,M+N}(v)\big]\eta \\
&\equiv \frac{1}{u-v-\kappa}\theta_0 T_{M+1,j}(u)T_{M+N,M+N}(v)\eta,
\end{aligned}$$

resulting in $T_{M+1,j}(u)T_{M+N,M+N}(v)\eta \equiv 0$ and therefore $T_{k,M+N}(v)T_{ij}(u)\eta \equiv 0$.

Now, when $1 \leq k \leq M$ and $(i,j) \in \Lambda(\mathfrak{n})_{0,1} \cup \Lambda(\mathfrak{n})_{1,1}$ or $M+1 < k \leq M+N$ and $(i,j) \in \Lambda(\mathfrak{n})_{0,0} \cup \Lambda(\mathfrak{n})_{1,0}$, we obtain

$$T_{M+1,k}(u)T_{ij}(v)\eta \equiv \big[T_{M+1,k}(u), T_{ij}(v)\big]\eta \equiv 0.$$

Alternatively, when $1 \leq k \leq M$ and $(i,j) \in \Lambda(\mathfrak{n})_{0,0} \cup \Lambda(\mathfrak{n})_{1,0}$ or if $M+1 < k \leq M+N$ and $(i,j) \in \Lambda(\mathfrak{n})_{0,1} \cup \Lambda(\mathfrak{n})_{1,1}$, we similarly compute

$$\begin{aligned}
T_{M+1,k}(u)T_{ij}(v)\eta &\equiv \big[T_{M+1,k}(u), T_{ij}(v)\big]\eta \\
&\equiv -\frac{\delta_{k\bar{j}}}{u-v-\kappa}(-1)^{[M+1][i]+[i][k]}\theta_0\theta_k T_{i,M+N}(v)T_{M+1,M+1}(u)\eta.
\end{aligned}$$

By the defining relations, we also conclude

$$T_{i,M+N}(v)T_{M+1,M+1}(u)\eta \equiv -\big[T_{M+1,M+1}(u), T_{i,M+N}(v)\big]\eta$$

$$\equiv \frac{1}{u-v-\kappa}\theta_0 T_{i,M+N}(v)T_{M+1,M+1}(u)\eta,$$

which entails $T_{i,M+N}(v)T_{M+1,M+1}(u)\eta \equiv 0$ and therefore $T_{M+1,k}(u)T_{ij}(v)\eta \equiv 0$, proving the first part of our lemma.

(ii). Let us suppose $(i,j),(k,l) \in \bigcup_{a,b\in\mathbb{Z}_2} \Lambda(\mathfrak{n})_{a,b}$. The defining relations (3.18) yields the equation

$$\begin{aligned}
\big[T_{ij}(u),T_{kl}(v)\big]\eta \equiv\ & \frac{1}{u-v}(-1)^{[i][k]+[i][j]+[j][k]}\Big(T_{kj}(u)T_{il}(v)-T_{kj}(v)T_{il}(u)\Big)\eta \\
& -\frac{1}{u-v-\kappa}\Bigg(\delta_{i\bar{k}}\sum_{\substack{M+1<p<M+N\\1\le p\le M}}(-1)^{[p]+[i][j]+[p][j]}\theta_i\theta_p T_{pj}(u)T_{\bar{p}l}(v) \\
& \qquad -\delta_{j\bar{l}}\sum_{\substack{M+1<p<M+N\\1\le p\le M}}(-1)^{[i][k]+[j][k]+[i][p]}\theta_j\theta_p T_{k\bar{p}}(v)T_{ip}(u)\Bigg)\eta \\
& +\frac{1}{u-v-\kappa}\Big(\delta_{i\bar{k}}(-1)^{[i][j]+[M+1][j]}\theta_0\theta_i T_{M+1,j}(u)T_{M+N,l}(v) \\
& \qquad +\delta_{j\bar{l}}(-1)^{[i][k]+[j][k]+[M+1][i]}\theta_j T_{k,M+N}(v)T_{i,M+1}(u)\Big)\eta.
\end{aligned}$$

By utilizing (4.10) again, we get

$$\begin{aligned}
T_{M+1,j}(u)T_{M+N,l}(v)\eta \equiv\ & \big[T_{M+1,j}(u),T_{M+N,l}(v)\big]\eta \\
\equiv\ & -\frac{1}{u-v-\kappa}\sum_{\substack{M+1<p<M+N\\1\le p\le M}}(-1)^{[p]+[M+1][j]+[p][j]}\theta_p T_{pj}(u)T_{\bar{p}l}(v)\eta \\
& +\frac{1}{u-v-\kappa}\Big(\theta_0 T_{M+1,j}(u)T_{M+N,l}(v) \\
& \qquad +\delta_{j\bar{l}}(-1)^{[M+1][j]}\theta_j T_{M+N,M+N}(v)T_{M+1,M+1}(u)\Big)\eta,
\end{aligned}$$

and therefore

$$\begin{aligned}
T_{M+1,j}(u)T_{M+N,l}(v)\eta \equiv\ & -\frac{1}{u-v-\kappa-\theta_0}\sum_{\substack{M+1<p<M+N\\1\le p\le M}}(-1)^{[p]+[M+1][j]+[p][j]}\theta_p T_{pj}(u)T_{\bar{p}l}(v)\eta \\
& +\frac{1}{u-v-\kappa-\theta_0}\delta_{j\bar{l}}(-1)^{[M+1][j]}\theta_j T_{M+N,M+N}(v)T_{M+1,M+1}(u)\eta.
\end{aligned}$$

In a similar way, we find

$$
\begin{aligned}
T_{k,M+N}(v)T_{i,M+1}(u)\eta &\equiv -(-1)^{([i]+[M+1])([k]+[M+1])}\big[T_{i,M+1}(u),T_{k,M+N}(v)\big]\eta \\
&\equiv -\frac{1}{u-v-\kappa}\Big(\delta_{i\bar{k}}(-1)^{[i]+[M+1][i]}\theta_0\theta_i T_{M+1,M+1}(u)T_{M+N,M+N}(v) \\
&\qquad\qquad - \theta_0 T_{k,M+N}(v)T_{i,M+1}(u)\Big)\eta \\
&\qquad + \frac{1}{u-v-\kappa}\sum_{\substack{M+1<p<M+N\\ 1\le p\le M}}(-1)^{[M+1][i]+[i][p]}\theta_0\theta_p T_{k\bar{p}}(v)T_{ip}(u)\eta
\end{aligned}
$$

and hence

$$
\begin{aligned}
T_{kM}(v)T_{i1}(u)\eta \equiv &-\frac{1}{u-v-\kappa-\theta_0}\delta_{i\bar{k}}(-1)^{[i]+[M+1][i]}\theta_0\theta_i T_{M+1,M+1}(u)T_{M+N,M+N}(v)\eta \\
&+\frac{1}{u-v-\kappa-\theta_0}\sum_{\substack{M+1<p<M+N\\ 1\le p\le M}}(-1)^{[M+1][i]+[i][p]}\theta_0\theta_p T_{k\bar{p}}(v)T_{ip}(u)\eta.
\end{aligned}
$$

All together, we get

$$
\begin{aligned}
\big[T_{ij}(u),T_{kl}(v)\big]\eta \equiv &\frac{1}{u-v}(-1)^{[i][k]+[i][j]+[j][k]}\Big(T_{kj}(u)T_{il}(v)-T_{kj}(v)T_{il}(u)\Big)\eta \\
&-\frac{1}{u-v-\kappa-\theta_0}\Bigg(\delta_{i\bar{k}}\sum_{\substack{M+1<p<M+N\\ 1\le p\le M}}(-1)^{[p]+[i][j]+[p][j]}\theta_i\theta_p T_{pj}(u)T_{\bar{p}l}(v) \\
&\qquad -\delta_{j\bar{l}}\sum_{\substack{M+1<p<M+N\\ 1\le p\le M}}(-1)^{[i][k]+[j][k]+[i][p]}\theta_j\theta_p T_{k\bar{p}}(v)T_{ip}(u)\Bigg)\eta \\
&-\frac{\delta_{i\bar{k}}\delta_{j\bar{l}}(-1)^{[i][j]}}{(u-v-\kappa)(u-v-\kappa-\theta_0)}\theta_0\theta_i\theta_j[T_{M+1,M+1}(u),T_{M+N,M+N}(v)\}\eta.
\end{aligned}
$$

Finally, relations (3.18) implies the equation

$$
\big[T_{M+1,M+1}(u),T_{M+N,M+N}(v)\big]\eta \equiv \frac{\theta_0}{u-v-\kappa}\big[T_{M+1,M+1}(u),T_{M+N,M+N}(v)\big]\eta,
$$

meaning $\big[T_{M+1,M+1}(u),T_{M+N,M+N}(v)\big]\eta \equiv 0$ and therefore the desired relations are satisfied for the operators $T_{ij}(u)$, $(i,j)\in\Lambda(\mathfrak{n})_{a,b}$, $a,b\in\mathbb{Z}_2$, on $V(\mathfrak{n})^+$, since we observe

$\kappa + \theta_0$ is the parameter associated to the Lie superalgebra $\mathfrak{n} \cong \mathfrak{osp}(M|N-2)$. $\square$

5.2 Periplectic Super Yangians

5.2.1 The Periplectic Lie superalgebra

Proposition 5.2.1. *The elements* $\mathsf{E}_{ij} := E_{ij} + \iota^P(E_{ij}) = E_{ij} - (-1)^{[i]([j]+1)} E_{-j,-i}$ *satisfy the relations*

$$\begin{aligned} [\mathsf{E}_{ij}, \mathsf{E}_{kl}] &= \delta_{jk} \mathsf{E}_{il} - \delta_{il} (-1)^{([i]+[j])([k]+[l])} \mathsf{E}_{kj} \\ &\qquad - \delta_{i,-k} (-1)^{[i]([j]+1)} \mathsf{E}_{-j,l} + \delta_{j,-l} (-1)^{([i]+[j])[k]} \mathsf{E}_{k,-i} \end{aligned}$$

and

$$\mathsf{E}_{ij} + (-1)^{[i]([j]+1)} \mathsf{E}_{-j,-i} = 0.$$

Proof. For the first relation, $[\mathsf{E}_{ij}, \mathsf{E}_{kl}]$ is given by the sum

$$\begin{aligned} &[E_{ij}, E_{kl}] - (-1)^{[k][l]+[k]} [E_{ij}, E_{-l,-k}] - (-1)^{[i][j]+[i]} [E_{-j,-i}, E_{kl}] \\ &\qquad + (-1)^{[i][j]+[i]+[k][l]+[k]} [E_{-j,-i}, E_{-l,-k}] \\ &\quad = \delta_{jk} E_{il} - \delta_{il} (-1)^{([i]+[j])([k]+[l])} E_{kj} \\ &\qquad - (-1)^{[k][l]+[k]} \big(\delta_{j,-l} E_{i,-k} - \delta_{i,-k} (-1)^{([i]+[j])([-k]+[-l])} E_{-l,j} \big) \\ &\qquad\quad - (-1)^{[i][j]+[i]} \big(\delta_{-i,k} E_{-j,l} - \delta_{-j,l} (-1)^{([-i]+[-j])([k]+[l])} E_{k,-i} \big) \\ &\qquad\qquad + (-1)^{[i][j]+[i]+[k][l]+[k]} \big(\delta_{-i,-l} E_{-j,-k} - \delta_{-j,-k} (-1)^{([-i]+[-j])([-k]+[-l])} E_{-l,-i} \big) \\ &\quad = \delta_{jk} (E_{il} - (-1)^{[i][j]+[i]+[j][l]+[j]+([-i]+[-j])([-j]+[-l])} E_{-l,-i}) \\ &\qquad - \delta_{il} (-1)^{([i]+[j])([k]+[l])} (E_{kj} - (-1)^{([i]+[j])([k]+[i])+[i][j]+[i]+[k][i]+[k]} E_{-j,-k}) \\ &\qquad\quad - \delta_{i,-k} (-1)^{[i][j]+[i]} (E_{-j,l} - (-1)^{[i][j]+[i]+[-i][l]+[-i]+([i]+[j])([i]+[-l])} E_{-l,j}) \end{aligned}$$

$$
\begin{aligned}
&\quad + \delta_{j,-l}(-1)^{[k]([-i]+[l])}\left((-1)^{\substack{[k]([-i]+[-j])+[i][j]+[i]\\+([-i]+[-j])([k]+[-j])}} E_{k,-i} - (-1)^{[k][-i]+[k]} E_{i,-k}\right)\\
&= \delta_{jk}\mathsf{E}_{il} - \delta_{il}(-1)^{([i]+[j])([k]+[l])}\mathsf{E}_{kj}\\
&\quad - \delta_{i,-k}(-1)^{[i][j]+[i]}\mathsf{E}_{-j,l} + \delta_{j,-l}(-1)^{([-i]+[l])[k]}\mathsf{E}_{k,-i}.
\end{aligned}
$$

For second, we have

$$
\begin{aligned}
&\mathsf{E}_{ij}+(-1)^{[i]([j]+1)}\mathsf{E}_{-j,-i}\\
&\quad = \left(E_{ij} - (-1)^{[i]([j]+1)}E_{-j,-i}\right) + (-1)^{[i]([j]+1)}\left(E_{-j,-i} - (-1)^{[-j]([-i]+1)}E_{ij}\right)\\
&\quad = (1-(-1)^{[i][j]+[-i][-j]+[i]+[-j]})E_{ij} = 0.
\end{aligned}
$$

□

Proposition 5.2.2. *The elements* $\mathsf{E}_{ij}^{(n)}(z) := E_{ij}z^n + \iota^P(E_{ij})(-z)^n$ *satisfy the relations*

$$
\begin{aligned}
\left[\mathsf{E}_{ij}^{(m)}(z), \mathsf{E}_{kl}^{(n)}(z)\right] &= \delta_{jk}\mathsf{E}_{il}^{(m+n)}(z) - \delta_{il}(-1)^{([i]+[j])([k]+[l])}\mathsf{E}_{kj}^{(m+n)}(z)\\
&\quad - \delta_{i,-k}(-1)^{[i]([j]+1)+m}\mathsf{E}_{-j,l}^{(m+n)}(z) + \delta_{j,-l}(-1)^{([i]+[j])[k]+m}\mathsf{E}_{k,-i}^{(m+n)}(z)
\end{aligned}
$$

and

$$
\mathsf{E}_{ij}^{(n)}(z) + (-1)^{[i]([j]+1)+n}E_{-j,-i}^{(n)}(z) = 0.
$$

Proof. The product $\mathsf{E}_{ij}^{(m)}(z)\mathsf{E}_{kl}^{(n)}(z)$ is given by

$$
\begin{aligned}
&\left(E_{ij}z^m - (-1)^{[i]([j]+1)}E_{-j,-i}(-z)^m\right)\left(E_{kl}z^n - (-1)^{[k]([l]+1)}E_{-l,-k}(-z)^n\right)\\
&\quad = \delta_{jk}E_{il}z^{m+n} - \delta_{j,-l}(-1)^{[k]([l]+1)+n}E_{i,-k}z^{m+n}\\
&\qquad - \delta_{i,-k}(-1)^{[i]([j]+1)+m}E_{-j,l}z^{m+n} + \delta_{il}(-1)^{[i]([j]+1)+[k]([l]+1)}E_{-j,-k}(-z)^{m+n}\\
&\quad = \delta_{jk}E_{il}z^{m+n} - \delta_{il}(-1)^{([i]+[j])([k]+[l])}\left(-(-1)^{[k]([j]+1)}E_{-j,-k}(-z)^{m+n}\right)\\
&\qquad - \delta_{i,-k}(-1)^{[i]([j]+1)+m}E_{-j,l}z^{m+n}\\
&\qquad\quad + \delta_{j,-l}(-1)^{([i]+[j])[k]+m}\left(-(-1)^{[k]([-i]+1)+n}E_{i,-k}(-z)^{m+n}\right).
\end{aligned}
$$

Similarly, the product $\mathsf{E}_{kl}^{(n)}(z)\mathsf{E}_{ij}^{(m)}(z)$ is computed as

$$\begin{aligned}
&\left(E_{kl}z^n - (-1)^{[k]([l]+1)}E_{-l,-k}(-z)^n\right)\left(E_{ij}z^m - (-1)^{[i]([j]+1)}E_{-j,-i}(-z)^m\right) \\
&\qquad = \delta_{il}E_{kj}z^{m+n} - \delta_{j,-l}(-1)^{[i]([j]+1)+m}E_{k,-i}z^{m+n} \\
&\qquad\qquad - \delta_{i,-k}(-1)^{[k]([l]+1)+n}E_{-l,j}z^{m+n} + \delta_{jk}(-1)^{[i]([j]+1)+[k]([l]+1)}E_{-l,-i}(-z)^{m+n},
\end{aligned}$$

and therefore $-(-1)^{([i]+[j])([k]+[l])}\mathsf{E}_{kl}^{(n)}(z)\mathsf{E}_{ij}^{(m)}(z)$ is given by

$$\begin{aligned}
&-\delta_{jk}(-1)^{[i]([l]+1)}E_{-l,-i}(-z)^{m+n} - \delta_{il}(-1)^{([i]+[j])([k]+[l])}E_{kj}z^{m+n} \\
&\qquad - \delta_{i,-k}(-1)^{[i]([j]+1)+m}\left(-(-1)^{[-j]([l]+1)}E_{-l,j}(-z)^{m+n}\right) \\
&\qquad\qquad + \delta_{j,-l}(-1)^{([i]+[j])[k]+m}E_{k,-i}z^{m+n}.
\end{aligned}$$

We therefore have

$$\begin{aligned}
\left[\mathsf{E}_{ij}^{(m)}(z), \mathsf{E}_{kl}^{(n)}(z)\right] &= \mathsf{E}_{ij}^{(m)}(z)\mathsf{E}_{kl}^{(n)}(z) - (-1)^{([i]+[j])([k]+[l])}\mathsf{E}_{kl}^{(n)}(z)\mathsf{E}_{ij}^{(m)}(z) \\
&= \delta_{jk}\mathsf{E}_{il}^{(m+n)}(z) - \delta_{il}(-1)^{([i]+[j])([k]+[l])}\mathsf{E}_{kj}^{(m+n)}(z) \\
&\qquad - \delta_{i,-k}(-1)^{[i]([j]+1)+m}\mathsf{E}_{-j,l}^{(m+n)}(z) + \delta_{j,-l}(-1)^{([i]+[j])[k]+m}\mathsf{E}_{k,-i}^{(m+n)}(z).
\end{aligned}$$

For the second relation,

$$\begin{aligned}
&\mathsf{E}_{ij}^{(n)}(z) + (-1)^{[i]([j]+1)+n}E_{-j,-i}^{(n)}(z) \\
&\qquad = \left(E_{ij}z^n - (-1)^{[i]([j]+1)}E_{-j,-i}(-z)^n\right) \\
&\qquad\quad + (-1)^{[i]([j]+1)+n}\left(E_{-j,-i}z^n - (-1)^{[-j]([-i]+1)}E_{ij}(-z)^n\right) \\
&\qquad = 0
\end{aligned}$$

□

5.2.2 RTT-Relation

First, we observe that

$$\begin{aligned} T_1(u)T_2(v) &= \left(\sum_{i,j=-N}^{N} E_{ij} \otimes \mathrm{id} \otimes T_{ij}(u)\right)\left(\sum_{k,l=-N}^{N} \mathrm{id} \otimes E_{kl} \otimes T_{kl}(v)\right) \\ &= \sum_{i,j,k,l=-N}^{N} (-1)^{([i]+[j])([k]+[l])} E_{ij} \otimes E_{kl} \otimes T_{ij}(u)T_{kl}(v) \end{aligned}$$

and

$$\begin{aligned} T_2(v)T_1(u) &= \left(\sum_{k,l=-N}^{N} \mathrm{id} \otimes E_{kl} \otimes T_{kl}(v)\right)\left(\sum_{i,j=-N}^{N} E_{ij} \otimes \mathrm{id} \otimes T_{ij}(u)\right) \\ &= \sum_{i,j,k,l=-N}^{N} E_{ij} \otimes E_{kl} \otimes T_{kl}(v)T_{ij}(u). \end{aligned}$$

Now, since $P = \sum_{a,b=-N}^{N} (-1)^{[b]} E_{ab} \otimes E_{ba}$ then we have the equations

$$\begin{aligned} &(P \otimes \mathbf{1})T_1(u)T_2(v) \\ &\quad= \sum_{i,j,k,l,a,b=-N}^{N} (-1)^{([i]+[j])(([a]+[b])+([k]+[l]))+[b]} \delta_{bi}\delta_{ak} E_{aj} \otimes E_{bl} \otimes T_{ij}(u)T_{kl}(v) \\ &\quad= \sum_{i,j,k,l=-N}^{N} (-1)^{[i][j]+[i][l]+[j][l]} E_{kj} \otimes E_{il} \otimes T_{ij}(u)T_{kl}(v) \\ &\quad= \sum_{i,j,k,l=-N}^{N} (-1)^{[j][k]+[j][l]+[k][l]} E_{ij} \otimes E_{kl} \otimes T_{kj}(u)T_{il}(v) \end{aligned}$$

and

$$T_2(v)T_1(u)\,(P \otimes \mathbf{1}) = \sum_{i,j,k,l,a,b=-N}^{N} (-1)^{([k]+[l])([a]+[b])+[b]} \delta_{ja}\delta_{lb} E_{ib} \otimes E_{ka} \otimes T_{kl}(v)T_{ij}(u)$$

$$= \sum_{i,j,k,l=-N}^{N} (-1)^{[j][k]+[j][l]+[k][l]} E_{il} \otimes E_{kj} \otimes T_{kl}(v) T_{ij}(u)$$
$$= \sum_{i,j,k,l=-N}^{N} (-1)^{[j][k]+[j][l]+[k][l]} E_{ij} \otimes E_{kl} \otimes T_{kj}(v) T_{il}(u).$$

Lastly, since $Q = -\sum_{p,q=-N}^{N} (-1)^{[p][q]} E_{pq} \otimes E_{-p,-q}$, we compute

$$\begin{aligned}
&\big(Q \otimes \mathbf{1}\big) T_1(u) T_2(v) \\
&\quad = -\sum_{i,j,k,l,p,q=-N}^{N} (-1)^{([i]+[j])(([k]+[l])+([-p]+[-q]))+[p][q]} \delta_{qi} \delta_{-q,k} E_{pj} \otimes E_{-p,l} \otimes T_{ij}(u) T_{kl}(v) \\
&\quad = -\delta_{i,-k} \sum_{i,j,k,l,p=-N}^{N} (-1)^{[i]+[j]+[i][l]+[j][l]+[j][p]} E_{pj} \otimes E_{-p,l} \otimes T_{ij}(u) T_{-i,l}(v) \\
&\quad = -\delta_{i,-k} \sum_{i,j,k,l,p=-N}^{N} (-1)^{[j]+[p]+[i][j]+[j][l]+[l][p]} E_{ij} \otimes E_{kl} \otimes T_{pj}(u) T_{-p,l}(v)
\end{aligned}$$

and

$$\begin{aligned}
&T_2(v) T_1(u) \big(Q \otimes \mathbf{1}\big) \\
&\qquad = -\sum_{i,j,k,l,p,q=-N}^{N} (-1)^{[p][q]+([p]+[q])([k]+[l])} \delta_{jp} \delta_{l,-p} E_{iq} \otimes E_{k,-q} \otimes T_{kl}(v) T_{ij}(u) \\
&\qquad = -\delta_{j,-l} \sum_{i,j,k,l,p=-N}^{N} (-1)^{[p]+[j][k]+[k][p]} E_{ip} \otimes E_{k,-p} \otimes T_{k,-j}(v) T_{ij}(u) \\
&\qquad = -\delta_{j,-l} \sum_{i,j,k,l,p=-N}^{N} (-1)^{[j]+[j][k]+[k][p]} E_{ij} \otimes E_{kl} \otimes T_{k,-p}(v) T_{ip}(u).
\end{aligned}$$

Therefore, we have

$$R(u,v) T_1(u) T_2(v)$$

$$
\begin{aligned}
&= \sum_{i,j,k,l=-N}^{N} (-1)^{([i]+[j])([k]+[l])} E_{ij} \otimes E_{kl} \otimes T_{ij}(u) T_{kl}(v) \\
&\quad - \frac{1}{u-v} \sum_{i,j,k,l=-N}^{N} (-1)^{[j][k]+[j][l]+[k][l]} E_{ij} \otimes E_{kl} \otimes T_{kj}(u) T_{il}(v) \\
&\quad + \frac{1}{u+v} \delta_{i,-k} \sum_{i,j,k,l,p=-N}^{N} (-1)^{[j]+[p]+[i][j]+[j][l]+[l][p]} E_{ij} \otimes E_{kl} \otimes T_{pj}(u) T_{-p,l}(v)
\end{aligned}
$$

equals

$$
\begin{aligned}
T_2(v) T_1(u) R(u,v) &= \sum_{i,j,k,l=-N}^{N} E_{ij} \otimes E_{kl} \otimes T_{kl}(v) T_{ij}(u) \\
&\quad - \frac{1}{u-v} \sum_{i,j,k,l=-N}^{N} (-1)^{[j][k]+[j][l]+[k][l]} E_{ij} \otimes E_{kl} \otimes T_{kj}(v) T_{il}(u) \\
&\quad + \frac{1}{u+v} \delta_{j,-l} \sum_{i,j,k,l,p=-N}^{N} (-1)^{[j]+[j][k]+[k][p]} E_{ij} \otimes E_{kl} \otimes T_{k,-p}(v) T_{ip}(u).
\end{aligned}
$$

Therefore, the equation $R(u,v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u,v)$ is given by

$$
\begin{aligned}
&\sum_{i,j,k,l=-N}^{N} E_{ij} \otimes E_{kl} \otimes \big((-1)^{([i]+[j])([k]+[l])} T_{ij}(u) T_{kl}(v) - T_{kl}(v) T_{ij}(u) \big) \\
&\quad = \frac{1}{u-v} \sum_{i,j,k,l=-N}^{N} E_{ij} \otimes E_{kl} \otimes (-1)^{[j][k]+[j][l]+[k][l]} \Big(T_{kj}(u) T_{il}(v) - T_{kj}(v) T_{il}(u) \Big) \\
&\qquad - \frac{1}{u+v} \sum_{i,j,k,l=-N}^{N} E_{ij} \otimes E_{kl} \otimes \Bigg(\delta_{i,-k} \sum_{p=-N}^{N} (-1)^{\substack{[j]+[p]+[i][j] \\ +[j][l]+[l][p]}} T_{pj}(u) T_{-p,l}(v) \\
&\qquad\qquad - \delta_{j,-l} \sum_{p=-N}^{N} (-1)^{[j]+[j][k]+[k][p]} T_{k,-p}(v) T_{ip}(u) \Bigg),
\end{aligned}
$$

hence giving the relation

$$\begin{aligned}\Big((-1)^{([i]+[j])([k]+[l])}T_{ij}(u)T_{kl}(v) - T_{kl}(v)T_{ij}(u)\Big)&\\ = \frac{1}{u-v}(-1)^{[j][k]+[j][l]+[k][l]}\big(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)\big)&\\ - \frac{1}{u+v}\Big(\delta_{i,-k}\sum_{p=-N}^{N}(-1)^{[j]+[p]+[i][j]+[j][l]+[l][p]}T_{pj}(u)T_{-p,l}(v)&\\ - \delta_{j,-l}\sum_{p=-N}^{N}(-1)^{[j]+[j][k]+[k][p]}T_{k,-p}(v)T_{ip}(u)\Big)&\end{aligned}$$

Multiplying by the scalar $(-1)^{([i]+[j])([k]+[l])}$ therefore gives the desired relation

$$\begin{aligned}[T_{ij}(u), T_{kl}(v)] = \frac{1}{u-v}(-1)^{[i][k]+[i][l]+[k][l]}\big(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)\big)&\\ - \frac{1}{u+v}\Big(\delta_{i,-k}\sum_{p=-N}^{N}(-1)^{[i][l]+[l][p]+[p]}T_{pj}(u)T_{-p,l}(v)&\\ - \delta_{j,-l}\sum_{p=-N}^{N}(-1)^{[j]+[i][l]+[i][k]+[k][p]}T_{k,-p}(v)T_{ip}(u)\Big)&\end{aligned}$$

5.2.3 Properties of the Extended Yangian

Proposition 5.2.3. *The mapping*

$$j^P : T(u) \mapsto T^{j^P}(u)$$

induces a superalgebra automorphism of $X(\mathfrak{p}_N)$.

Proof. Let us set $(-)^{j_1^P} = (-)^{j^P} \otimes \mathrm{id}_{\mathrm{End}\,\mathbb{C}^{N|N}} \otimes \mathrm{id}_{X(\mathfrak{p}_N)}$. Since $(-)^{j_1^P}$ only acts on the first tensor factor non-trivially, we have the equations $(R(u,v)T_1(u)T_2(v))^{j_1^P} = (R(u,v)T_1(u))^{j_1^P}\, T_2(v)$ and $(T_2(v)T_1(u)R(u,v))^{j_1^P} = T_2(v)\,(T_1(u)R(u,v))^{j_1^P}$.

Suppose now that we have elements $A = A_1 \otimes A_2$, $B = B_1 \otimes B_2$ such that $[A_1] = [A_2]$

and $[B_1] = [B_2]$, which we will denote $[A]$ and $[B]$, respectively. If φ is an anti-morphism in the category $\mathrm{sVect}_{\mathbb{C}}$ and by setting $\varphi_1 = \varphi \otimes \mathrm{id} \otimes \mathrm{id}$, we observe

$$\begin{aligned}\varphi_1((A \otimes 1)(B_1 \otimes 1 \otimes B_2)) = (-1)^{[A][B]}\varphi(A_1 B_1) \otimes A_2 \otimes B_2 = \varphi(B_1)\varphi(A_1) \otimes A_2 \otimes B_2 \\ = \varphi_1(B_1 \otimes 1 \otimes B_2)\varphi_1(A \otimes 1)\end{aligned}$$

and

$$\begin{aligned}\varphi_1((B_1 \otimes 1 \otimes B_2)(A \otimes 1)) = \varphi(B_1 A_1) \otimes A_2 \otimes B_2 = (-1)^{[A][B]}\varphi(A_1)\varphi(B_1) \otimes A_2 \otimes B_2 \\ = \varphi_1(A \otimes 1)\varphi_1(B_1 \otimes 1 \otimes B_2),\end{aligned}$$

which proves equations $(R(u,v)T_1(u))^{j_1^P} = T_1^{j^P}(u)R^{j_1^P}(u,v)$ and $(T_1(u)R(u,v))^{j_1^P} = R^{j_1^P}(u,v)T_1^{j^P}(u)$, respectively.

Setting $(-)^{j_2^P} = \mathrm{id}_{\mathrm{End}\,\mathbb{C}^{N|N}} \otimes (-)^{j^P} \otimes \mathrm{id}_{X(\mathfrak{p}_N)}$, since $(-)^{j_2^P}$ only acts on the second tensor factor, we analogously have the equations $\left(T_1^{j^P}(u)R^{j_1^P}(u,v)T_2(v)\right)^{j_2^P} = T_1^{j^P}(u)\left(R^{j_1^P}(u,v)T_2(v)\right)^{j_2^P}$ and $\left(T_2(v)R^{j_1^P}(u,v)T_1^{j^P}(u)\right)^{j_2^P} = \left(T_2(v)R^{j_1^P}(u,v)\right)^{j_2^P} T_1^{j^P}(u)$. By setting $\varphi_2 = \mathrm{id} \otimes \varphi \otimes \mathrm{id}$, the formulas

$$\begin{aligned}\varphi_2((A \otimes 1)(1 \otimes B)) = A_1 \otimes \varphi(A_2 B_1) \otimes B_2 = (-1)^{[A][B]} A_1 \otimes \varphi(B_1)\varphi(A_2) \otimes B_2 \\ = \varphi_2(1 \otimes B)\varphi_2(A \otimes 1)\end{aligned}$$

and

$$\begin{aligned}\varphi_2((1 \otimes B)(A \otimes 1)) = (-1)^{[A][B]} A_1 \otimes \varphi(B_1 A_2) \otimes B_2 = A_1 \otimes \varphi(A_2)\varphi(B_1) \otimes B_2 \\ = \varphi_2(A \otimes 1)\varphi_2(1 \otimes B)\end{aligned}$$

similarly imply $\left(R^{j_1^P}(u,v)T_2(v)\right)^{j_2^P} = T_2^{j^P}(v)R(-u,-v)$ and $\left(T_2(v)R^{j_1^P}(u,v)\right)^{j_2^P}$ equalling the operator $R(-u,-v)T_2^{j^P}(v)$ since $(R(u,v))^{j_2^P \circ j_1^P} = R(-u,-v)$. Therefore, we have the equation

$$T_1^{j^P}(u)T_2^{j^P}(v)R(-u,-v) = R(-u,-v)T_2^{j^P}(v)T_1^{j^P}(u).$$

Multiplying both sides of the above equation by the inverse of $R(-u,-v)$, we ultimately yield

$$R(u,v)T_1^{j^P}(u)T_2^{j^P}(v) = T_2^{j^P}(v)T_1^{j^P}(u)R(u,v).$$

□

Let $(-)^\circ \in \operatorname{End} X(\mathfrak{p}_N)$ be a superalgebra anti-morphism in the category $\mathrm{sVect}_{\mathbb{C}}$. For generating series $T_{ij}(u), T_{kl}(u) \in X(\mathfrak{p}_N)[[u^{-1}]]$, we then have the set of equalities $(T_{ij}(u)T_{kl}(v))^\circ = T^\circ_{ij}(u) \cdot_{\mathrm{op}} T^\circ_{kl}(v) = (-1)^{([i]+[j])([k]+[l])}T^\circ_{kl}(v)T^\circ_{ij}(u)$, where $\cdot_{\mathrm{op}}$ stands for multiplication for algebras in the opposite category $\mathrm{sVect}^{\mathrm{op}}_{\mathbb{C}}$. The map $(-)^\circ$ must therefore satisfy the relations

$$\begin{aligned}
&T^\circ_{ij}(u)\cdot_{\mathrm{op}}T^\circ_{kl}(v) - (-1)^{([i]+[j])([k]+[l])}T^\circ_{kl}(v) \cdot_{\mathrm{op}} T^\circ_{ij}(u)\\
&\qquad = \frac{1}{u-v}(-1)^{[i][k]+[i][l]+[k][l]}\Big(T^\circ_{kj}(u) \cdot_{\mathrm{op}} T^\circ_{il}(v) - T^\circ_{kj}(v) \cdot_{\mathrm{op}} T^\circ_{il}(u)\Big)\\
&\qquad\quad - \frac{1}{u+v}\Bigg(\delta_{i,-k}\sum_{p=-N}^{N}(-1)^{[i][l]+[l][p]+[p]}T^\circ_{pj}(u) \cdot_{\mathrm{op}} T^\circ_{-p,l}(v)\\
&\qquad\qquad\qquad - \delta_{j,-l}\sum_{p=-N}^{N}(-1)^{[j]+[i][l]+[i][k]+[k][p]}T^\circ_{k,-p}(v) \cdot_{\mathrm{op}} T^\circ_{ip}(u)\Bigg).
\end{aligned}$$

Rewritten, these relations have the form

$$\begin{aligned}
[T^\circ_{ij}(u), T^\circ_{kl}(v)] &= \frac{1}{u-v}(-1)^{[i][j]+[i][l]+[j][l]}\Big(T^\circ_{il}(u)T^\circ_{kj}(v) - T^\circ_{il}(v)T^\circ_{kj}(u)\Big)\\
&\quad + \frac{1}{u+v}\Bigg(\delta_{i,-k}\sum_{p=-N}^{N}(-1)^{[j]+[i][l]+[j][l]+[j][p]+[p]}T^\circ_{-p,l}(v)T^\circ_{pj}(u)\\
&\qquad\qquad - \delta_{j,-l}\sum_{p=-N}^{N}(-1)^{[j]+[i][j]+[i][p]}T^\circ_{ip}(u)T^\circ_{k,-p}(v)\Bigg).
\end{aligned}$$

Defining $T^\circ(u) = \sum_{i,j=-N}^{N} E_{ij} \otimes T^\circ_{ij}(u)$ and $T^\circ_i(u)$, $i = 1, 2$ in the suitable ways, these relations may be equivalently written as a variant of the RTT-relation:

$$R(u,v)T^\circ_2(v)T^\circ_1(u) = T^\circ_1(u)T^\circ_2(v)R(u,v). \tag{5.1}$$

Proposition 5.2.4. *A grading preserving map* $(-)^\circ \in \operatorname{End} X(\mathfrak{p}_N)$ *is a superalgebra anti-morphism in the category* $\mathrm{sVect}_{\mathbb{C}}$ *if and only if it satisfies the relation*

$$R(u,v)T^\circ_2(v)T^\circ_1(u) = T^\circ_1(u)T^\circ_2(v)R(u,v).$$

Proof. Indeed, since we have

$$T^\circ_1(u)T^\circ_2(v) = \sum_{i,j,k,l=-N}^{N} (-1)^{([i]+[j])([k]+[l])} E_{ij} \otimes E_{kl} \otimes T^\circ_{ij}(u)T^\circ_{kl}(v)$$

and

$$T^\circ_2(v)T^\circ_1(u) = \sum_{i,j,k,l=-N}^{N} E_{ij} \otimes E_{kl} \otimes T^\circ_{kl}(v)T^\circ_{ij}(u),$$

and as $P = \sum_{a,b=-N}^{N} (-1)^{[b]} E_{ab} \otimes E_{ba}$, we compute

$$\begin{aligned}
&T^\circ_1(u)T^\circ_2(v)\,(P \otimes \mathbf{1})\\
&\quad= \sum_{i,j,k,l,a,b=-N}^{N} (-1)^{(([i]+[j])+([a]+[b]))([k]+[l])+[b]} \delta_{ja}\delta_{lb} E_{ib} \otimes E_{ka} \otimes T^\circ_{ij}(u)T^\circ_{kl}(v)\\
&\quad= \sum_{i,j,k,l=-N}^{N} (-1)^{[i][k]+[i][l]+[k][l]} E_{il} \otimes E_{kj} \otimes T^\circ_{ij}(u)T^\circ_{kl}(v)\\
&\quad= \sum_{i,j,k,l=-N}^{N} (-1)^{[i][j]+[i][k]+[j][k]} E_{ij} \otimes E_{kl} \otimes T^\circ_{il}(u)T^\circ_{kj}(v)
\end{aligned}$$

and

$$(P \otimes \mathbf{1})\,T^\circ_2(v)T^\circ_1(u) = \sum_{i,j,k,l,s,t=-N}^{N} (-1)^{[b]+([a]+[b])([i]+[j])} \delta_{bi}\delta_{ak} E_{aj} \otimes E_{bl} \otimes T^\circ_{kl}(v)T^\circ_{ij}(u)$$

$$= \sum_{i,j,k,l=-N}^{N} (-1)^{[i][j]+[i][k]+[j][k]} E_{kj} \otimes E_{il} \otimes T^{\circ}_{kl}(v) T^{\circ}_{ij}(u)$$
$$= \sum_{i,j,k,l=-N}^{N} (-1)^{[i][j]+[i][k]+[j][k]} E_{ij} \otimes E_{kl} \otimes T^{\circ}_{il}(v) T^{\circ}_{kj}(u).$$

Moreover, since $Q = -\sum_{p,q=1}^{M+N} (-1)^{[p][q]} E_{pq} \otimes E_{-p,-q}$, we get

$$\begin{aligned}
&T^{\circ}_1(u) T^{\circ}_2(v) \left(Q \otimes \mathbf{1}\right) \\
&\quad = -\sum_{i,j,k,l,p,q=-N}^{N} (-1)^{(([i]+[j])+([p]+[q]))([k]+[l])+[p][q]} \delta_{jp} \delta_{l,-p} E_{iq} \otimes E_{k,-q} \otimes T^{\circ}_{ij}(u) T^{\circ}_{kl}(v) \\
&\quad = -\delta_{j,-l} \sum_{i,j,k,l,p=-N}^{N} (-1)^{[i][k]+[i][l]+[k][p]+[p]} E_{ip} \otimes E_{k,-p} \otimes T^{\circ}_{ij}(u) T^{\circ}_{k,-j}(v) \\
&\quad = -\delta_{j,-l} \sum_{i,j,k,l,p=-N}^{N} (-1)^{[i]+[j]+[i][k]+[j][k]+[i][p]} E_{ij} \otimes E_{kl} \otimes T^{\circ}_{ip}(u) T^{\circ}_{k,-p}(v)
\end{aligned}$$

and

$$\begin{aligned}
&(Q \otimes \mathbf{1}) T^{\circ}_2(v) T^{\circ}_1(u) \\
&\quad = -\sum_{i,j,k,l,p,q=-N}^{N} (-1)^{([p]+[q])([i]+[j])+[p][q]} \delta_{qi} \delta_{-q,k} E_{pj} \otimes E_{-p,l} \otimes T^{\circ}_{kl}(v) T^{\circ}_{ij}(u) \\
&\quad = -\delta_{i,-k} \sum_{i,j,k,l,p=-N}^{N} (-1)^{[i]+[i][j]+[j][p]} E_{pj} \otimes E_{-p,l} \otimes T^{\circ}_{-i,l}(v) T^{\circ}_{ij}(u) \\
&\quad = -\delta_{i,-k} \sum_{i,j,k,l,p=-N}^{N} (-1)^{[i][j]+[j][p]+[p]} E_{ij} \otimes E_{kl} \otimes T^{\circ}_{-p,l}(v) T^{\circ}_{pj}(u).
\end{aligned}$$

Henceforth, we obtain that

$$T^{\circ}_1(u) T^{\circ}_2(v) R(u,v)$$

$$
\begin{aligned}
&= \sum_{i,j,k,l=-N}^{N} (-1)^{([i]+[j])([k]+[l])} E_{ij} \otimes E_{kl} \otimes T^{\circ}_{ij}(u) T^{\circ}_{kl}(v) \\
&\quad - \frac{1}{u-v} \sum_{i,j,k,l=-N}^{N} (-1)^{[i][j]+[i][k]+[j][k]} E_{ij} \otimes E_{kl} \otimes T^{\circ}_{il}(u) T^{\circ}_{kj}(v) \\
&\quad + \frac{1}{u+v} \delta_{j,-l} \sum_{i,j,k,l,p=-N}^{N} (-1)^{[i]+[j]+[i][k]+[j][k]+[i][p]} E_{ij} \otimes E_{kl} \otimes T^{\circ}_{ip}(u) T^{\circ}_{k,-p}(v)
\end{aligned}
$$

equals

$$
\begin{aligned}
&R(u,v) T^{\circ}_2(v) T^{\circ}_1(u) \\
&\quad = \sum_{i,j,k,l=-N}^{N} E_{ij} \otimes E_{kl} \otimes T^{\circ}_{kl}(v) T^{\circ}_{ij}(u) \\
&\quad - \frac{1}{u-v} \sum_{i,j,k,l=-N}^{N} (-1)^{[i][j]+[i][k]+[j][k]} E_{ij} \otimes E_{kl} \otimes T^{\circ}_{il}(v) T^{\circ}_{kj}(u) \\
&\quad + \frac{1}{u+v} \delta_{i,-k} \sum_{i,j,k,l,p=-N}^{N} (-1)^{[i][j]+[j][p]+[p]} E_{ij} \otimes E_{kl} \otimes T^{\circ}_{-p,l}(v) T^{\circ}_{pj}(u).
\end{aligned}
$$

The equation $R(u,v)T^{\circ}_2(v)T^{\circ}_1(u) = T^{\circ}_1(u)T^{\circ}_2(v)R(u,v)$ is therefore given by

$$
\begin{aligned}
&\sum_{i,j,k,l=-N}^{N} E_{ij} \otimes E_{kl} \otimes \left((-1)^{([i]+[j])([k]+[l])} T^{\circ}_{ij}(u) T^{\circ}_{kl}(v) - T^{\circ}_{kl}(v) T^{\circ}_{ij}(u) \right) \\
&= \frac{1}{u-v} \sum_{i,j,k,l=1}^{M+N} E_{ij} \otimes E_{kl} \otimes (-1)^{[i][j]+[i][k]+[j][k]} \Big(T^{\circ}_{il}(u) T^{\circ}_{kj}(v) - T^{\circ}_{il}(v) T^{\circ}_{kj}(u) \Big) \\
&\quad + \frac{1}{u+v} \sum_{i,j,k,l=-N}^{N} E_{ij} \otimes E_{kl} \otimes \Bigg(\delta_{i,-k} \sum_{p=-N}^{N} (-1)^{[i][j]+[j][p]+[p]} T^{\circ}_{-p,l}(v) T^{\circ}_{pj}(u) \\
&\qquad\qquad - \delta_{j,-l} \sum_{p=1}^{M+N} (-1)^{\substack{[i]+[j]+[i][k] \\ +[j][k]+[i][p]}} T^{\circ}_{ip}(u) T^{\circ}_{k,-p}(v) \Bigg),
\end{aligned}
$$

hence giving the relation

$$\begin{aligned}
&(-1)^{([i]+[j])([k]+[l])}T^{\circ}_{ij}(u)T^{\circ}_{kl}(v) - T^{\circ}_{kl}(v)T^{\circ}_{ij}(u) \\
&\quad = \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\Big(T^{\circ}_{il}(u)T^{\circ}_{kj}(v) - T^{\circ}_{il}(v)T^{\circ}_{kj}(u)\Big) \\
&\qquad + \frac{1}{u+v}\Bigg(\delta_{i,-k}\sum_{p=-N}^{N}(-1)^{[i][j]+[j][p]+[p]}T^{\circ}_{-p,l}(v)T^{\circ}_{pj}(u) \\
&\qquad\qquad - \delta_{j,-l}\sum_{p=-N}^{N}(-1)^{[i]+[j]+[i][k]+[j][k]+[i][p]}T^{\circ}_{ip}(u)T^{\circ}_{k,-p}(v)\Bigg).
\end{aligned}$$

Multiplying the above by the scalar $(-1)^{([i]+[j])([k]+[l])}$ then gives the desired relation:

$$\begin{aligned}
[T^{\circ}_{ij}(u), T^{\circ}_{kl}(v)] &= \frac{1}{u-v}(-1)^{[i][j]+[i][l]+[j][l]}\Big(T^{\circ}_{il}(u)T^{\circ}_{kj}(v) - T^{\circ}_{il}(v)T^{\circ}_{kj}(u)\Big) \\
&\quad + \frac{1}{u+v}\Bigg(\delta_{i,-k}\sum_{p=-N}^{N}(-1)^{[j]+[i][l]+[j][l]+[j][p]+[p]}T^{\circ}_{-p,l}(v)T^{\circ}_{pj}(u) \\
&\qquad - \delta_{j,-l}\sum_{p=-N}^{N}(-1)^{[j]+[i][j]+[i][p]}T^{\circ}_{ip}(u)T^{\circ}_{k,-p}(v)\Bigg).
\end{aligned}$$

□

Proposition 5.2.5. *Each mapping*

$$\sigma\colon T(u) \mapsto T(-u)$$
$$S\colon T(u) \mapsto T^{-1}(u)$$

induces a superalgebra anti-automorphism of $X(\mathfrak{p}_N)$.

Proof. We first observe that the map defined by (3.21) is both grade-preserving and involutive. To show that the induced map is an anti-morphism, it suffices that show that the image of $T(u)$ under (3.21) satisfies the variant RTT-relation (5.1). In order to obtain the relation

$$R(u,v)T_2(-v)T_1(-u) = T_1(-u)T_2(-v)R(u,v),$$

first perform the exchange $(u,v) \mapsto (-u,-v)$ in the RTT-relation (4.7). One then multiplies both sides by $R(u,v)$ and utilizes equation (4.5) to imply the result.

By writing $T(u) = \sum_{n=0}^{\infty} T^{(n)}u^{-n}$, where $T^{(n)} = \sum_{i,j=1}^{M+N}(-1)^{[i][j]}E_{ij} \otimes T_{ij}^{(n)}$, then $T^{-1}(u)$ is the series $\sum_{n=0}^{\infty} \widetilde{T}^{(n)}u^{-n}$ whose coefficients are given recursively by the formula $\widetilde{T}^{(n)} = -\sum_{p=1}^{n} T^{(p)}\widetilde{T}^{(n-p)}$, where $\widetilde{T}^{(0)} = (T^{(0)})^{-1} = T^{(0)}$. One then shows that the map (3.22) is grade-preserving via induction on $n \in \mathbb{N}$.

The base case of $n = 0$ is immediate from the fact that $\widetilde{T}^{(0)} = T^{(0)}$. For the inductive step, we may first write $\widetilde{T}^{(n-p)} = \sum_{k,l=1}^{M+N} E_{kl} \otimes \widetilde{T}_{kl}^{(n-p)} \in \operatorname{End}\mathbb{C}^{M|N} \otimes X(\mathfrak{osp}_{M|N})$, where each $\widetilde{T}_{kl}^{(n-p)}$ is homogeneous of degree $[k]+[l]$ by hypothesis, so that

$$\widetilde{T}^{(n)} = \sum_{i,l=1}^{M+N} E_{il} \otimes \sum_{p=1}^{n}\sum_{j=1}^{M+N}(-1)^{[i][j]+([i]+[j])([j]+[l])}T_{ij}^{(p)}\widetilde{T}_{jl}^{(n-p)},$$

where we note that every term in the second tensor factor is of degree $[i]+[l]$, concluding the induction.

To prove that the map (3.22) is an anti-morphism amounts to observing that the RTT-relation (4.7) is equivalent to the equation

$$R(u,v)T_2^{-1}(v)T_1^{-1}(u) = T_1^{-1}(u)T_2^{-1}(v)R(u,v).$$

To see that the map is bijective, first consider the composition

$$\omega := S \circ \sigma \colon T(u) \mapsto T^{-1}(-u). \tag{5.2}$$

Consequently, ω is a morphism of superalgebras and is in fact involutive. Indeed, ω

acting on the identity

$$\omega(T(u))T(-u) = \mathrm{id} \otimes \mathbf{1}$$

yields

$$\omega^2(T(u))T^{-1}(u) = \mathrm{id} \otimes \mathbf{1},$$

proving that ω^2 is the identity map. Hence, S is bijective as well. □

Proposition 5.2.6. *Let $\varphi\colon \mathcal{A} \to \mathcal{A}$ be an algebra anti-morphism in the category* $\mathrm{sVect}_{\mathbb{C}}$ *and for an integer $n \geq 2$ consider homogeneous elements $X^{(i)} \in \mathcal{A}^{\otimes 2}$, $i = 1, \ldots, n$, given by $X^{(i)} = X_1^{(i)} \otimes X_2^{(i)}$ such that $[X_1^{(i)}] = [X_2^{(i)}]$. By setting $\varphi_1 = \varphi \otimes \mathrm{id}^{\otimes n}$, we then have*

$$\varphi_1(X_{12}^{(1)} X_{13}^{(2)} \cdots X_{1,n+1}^{(n)}) = \varphi_1(X_{1,n+1}^{(n)}) \cdots \varphi_1(X_{13}^{(2)})\varphi_1(X_{12}^{(1)}).$$

Proof. Let us set $[X^{(i)}] = [X_j^{(i)}]$ for $j = 1, 2$. We note that

$$X_{12}^{(1)} X_{13}^{(2)} \cdots X_{1,n+1}^{(n)} = (-1)^{\sum_{j<k}[X^{(j)}][X^{(k)}]} \left(X_1^{(1)} X_1^{(2)} \cdots X_1^{(n)}\right) \otimes X_2^{(1)} \otimes X_2^{(2)} \otimes \cdots \otimes X_2^{(n)},$$

and therefore

$$\begin{aligned} \varphi_1(X_{12}^{(1)} X_{13}^{(2)} \cdots X_{1,n+1}^{(n)}) &= \left(\varphi(X_1^{(n)}) \cdots \varphi(X_1^{(2)})\varphi(X_1^{(1)})\right) \otimes X_2^{(1)} \otimes X_2^{(2)} \otimes \cdots \otimes X_2^{(n)} \\ &= \varphi_1(X_{1,n+1}^{(n)}) \cdots \varphi_1(X_{13}^{(2)})\varphi_1(X_{12}^{(1)}). \end{aligned}$$

□

5.2.4 The Degenerate Periplectic Brauer Algebra

Recall the algebra $\mathfrak{B}_d^{\mathrm{aff}}$ generated by elements s_a, ε_a, y_k satisfying relations $(P.1) - (P.8)$ and the elements $\sigma_{j,k}$, $\beta_{j,k}$ and $\widetilde{y}_k$.

Lemma 5.2.7. (i) *For $k < d$, $s_k\sigma_{j,k}s_k = \sigma_{j,k+1}$, $s_k\beta_{j,k}s_k = \beta_{j,k+1}$ and for $k - j > 1$, $s_{k-1}\sigma_{j,k}s_{k-1} = \sigma_{j,k-1}$, $s_{k-1}\beta_{j,k}s_{k-1} = \beta_{j,k-1}$;*

(ii) *For $k - j > 1$, $s_j\sigma_{j,k}s_j = \sigma_{j+1,k}$, $s_j\beta_{j,k}s_j = \beta_{j+1,k}$ and for $j > 1$, $s_{j-1}\sigma_{j,k}s_{j-1} = \sigma_{j-1,k}$, $s_{j-1}\beta_{j,k}s_{j-1} = \beta_{j-1,k}$;*

(iii) *Letting $k \geq 4$ and $1 \leq j < p < k-1$, then $s_p\sigma_{j,k} = \sigma_{j,k}s_p$, $s_p\beta_{j,k} = \beta_{j,k}s_p$ and $\varepsilon_p\sigma_{j,k} = \sigma_{j,k}\varepsilon_p$, $\varepsilon_p\beta_{j,k} = \beta_{j,k}\varepsilon_p$;*

(iv) *For $p > k$, $y_p\sigma_{j,k} = \sigma_{j,k}y_p$ and $y_p\beta_{j,k} = \beta_{j,k}y_p$.*

Proof. (i) Using relation $(P.3)\,(i)$, one computes

$$\begin{aligned}
s_k\sigma_{j,k}s_k &= s_k(s_j\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_j)s_k \\
&= (s_j\cdots s_{k-2})(s_ks_{k-1}s_k)(s_{k-2}\cdots s_j) \\
&= (s_j\cdots s_{k-2})(s_{k-1}s_ks_{k-1})(s_{k-2}\cdots s_j) \\
&= s_j\cdots s_{k-1}s_ks_{k-1}\cdots s_j = \sigma_{j,k+1},
\end{aligned}$$

and a similar computation using $(C.1)\,(i)$ shows $s_k\beta_{j,k}s_k = \beta_{j,k+1}$. Furthermore,

$$\begin{aligned}
s_{k-1}\sigma_{j,k}s_{k-1} &= s_{k-1}(s_j\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_j)s_{k-1} \\
&= (s_j\cdots s_{k-3})s_{k-1}(s_{k-2}s_{k-1}s_{k-2})s_{k-1}(s_{k-3}\cdots s_j) \\
&= (s_j\cdots s_{k-3})s_{k-1}(s_{k-1}s_{k-2}s_{k-1})s_{k-1}(s_{k-3}\cdots s_j) \\
&= s_j\cdots s_{k-3}s_{k-2}s_{k-3}\cdots s_j = \sigma_{j,k-1},
\end{aligned}$$

and one shows $s_{k-1}\beta_{j,k}s_{k-1} = \beta_{j,k-1}$ in an analogous way.

(ii) Directly,

$$s_j\sigma_{j,k}s_j = s_j(s_j\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_j)s_j = s_{j+1}\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_{j+1} = \sigma_{j+1,k},$$

and $s_j\beta_{j,k}s_j = \beta_{j+1,k}$ is similar. Also,

$$\begin{aligned} s_{j-1}\sigma_{j,k}s_{j-1} &= s_{j-1}(s_j \cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_j)s_{j-1} \\ &= s_{j-1}\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_{j-1} = \sigma_{j-1,k}. \end{aligned}$$

The formula $s_{j-1}\beta_{j,k}s_{j-1} = \beta_{j-1,k}$ is shown similarly.

(iii) For $p > j+1$, observe

$$\begin{aligned} s_p\sigma_{j,k} &= s_p(s_j\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_j) \\ &= (s_j\cdots s_{p-2})(s_ps_{p-1}s_p)(s_{p+1}\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_j) \\ &= (s_j\cdots s_{p-2})(s_{p-1}s_ps_{p-1})(s_{p+1}\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_j) \\ &= (s_j\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_{p+1})(s_{p-1}s_ps_{p-1})(s_{p-2}\cdots s_j) \\ &= (s_j\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_{p+1})(s_ps_{p-1}s_p)(s_{p-2}\cdots s_j) \\ &= (s_j\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_j)s_p = s_p\sigma_{j,k} \end{aligned}$$

and the case of $p = j+1$ is the same. The proof that $s_p\beta_{j,k} = \beta_{j,k}s_p$ is similar. Now, for $p > j+1$,

$$\begin{aligned} \varepsilon_p\sigma_{j,k} &= \varepsilon_p(s_j\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_j) \\ &= (s_j\cdots s_{p-2})\varepsilon_p(s_{p-1}\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_j) \\ &= (s_j\cdots s_{p-2}s_{p-1})(s_{p-1}\varepsilon_ps_{p-1})(s_ps_{p+1}\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_j) \\ &= (s_j\cdots s_{p-2}s_{p-1})(s_p\varepsilon_{p-1}s_p)(s_ps_{p+1}\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_j) \\ &= (s_j\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_{p+1})\varepsilon_{p-1}(s_p\cdots s_j) \\ &= (s_j\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_{p+1}s_p)(s_p\varepsilon_{p-1}s_p)(s_{p-1}s_{p-2}\cdots s_j) \\ &= (s_j\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_{p+1}s_p)(s_{p-1}\varepsilon_ps_{p-1})(s_{p-1}s_{p-2}\cdots s_j) \\ &= (s_j\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_j)\varepsilon_p = \sigma_{j,k}\varepsilon_p \end{aligned}$$

and the case $p = j+1$ is identical. A similar argument shows $\varepsilon_p\beta_{j,k} = \beta_{j,k}\varepsilon_p$.

(iv) follows from relations $(P.6)$ (i) & (ii). □

Corollary 5.2.8. *Let $k \geq 4$ and $1 \leq j < p < q < k$. Then $\sigma_{p,q}\sigma_{j,k} = \sigma_{j,k}\sigma_{p,q}$, $\sigma_{p,q}\beta_{j,k} = \beta_{j,k}\sigma_{p,q}$ and $\beta_{p,q}\sigma_{j,k} = \sigma_{j,k}\beta_{p,q}$, $\beta_{p,q}\beta_{j,k} = \beta_{j,k}\beta_{p,q}$.*

Proposition 5.2.9. (i) *For any permutation $\tau \in \mathfrak{S}_d$ and $1 \leq k \leq d$, one has*

$$\tau\widetilde{y}_k\tau^{-1} = \widetilde{y}_{\tau(k)}$$

(ii) *For $1 \leq k < p \leq d$, we have*

$$y_p\widetilde{y}_k = \widetilde{y}_k y_p$$

Proof. (i) It suffices to prove the statement for all elementary transpositions. That is, it suffices to show $s_q\widetilde{y}_k s_q = \widetilde{y}_{s_q(k)}$ for all $1 \leq q \leq d$. Indeed, since the group $\mathfrak{S}_d$ is generated by such transpositions, any permutation $\tau \in \mathfrak{S}_d$, may be written as some product $\tau = s_{q_1} \cdots s_{q_n}$. Therefore, if the statement is true for elementary transpositions, then

$$\begin{aligned}
\tau\widetilde{y}_k\tau^{-1} &= s_{q_1} \cdots s_{q_n}\widetilde{y}_k s_{q_n} \cdots s_{q_1} \\
&= s_{q_1} \cdots s_{q_{n-1}}\widetilde{y}_{s_{q_n}(k)} s_{q_{n-1}} \cdots s_{q_1} \\
&\ \vdots \\
&= s_{q_1}\widetilde{y}_{s_{q_2}\cdots s_{q_n}(k)} s_{q_1} \\
&= \widetilde{y}_{s_{q_1}\cdots s_{q_n}(k)} = \widetilde{y}_{\tau(k)}.
\end{aligned}$$

- For $q > k$, then $s_q\widetilde{y}_k s_q = \widetilde{y}_k = \widetilde{y}_{s_q(k)}$ is immediate.

- For $q = k$, we have

$$s_k\widetilde{y}_k s_k = s_k y_k s_k - \sum_{j=1}^{k-1}(s_k\sigma_{j,k}s_k + s_k\beta_{j,k}s_k)$$

$$= y_{k+1} - (s_k + \varepsilon_k) - \sum_{j=1}^{k-1}(\sigma_{j,k+1} + \beta_{j,k+1})$$

$$= y_{k+1} - \sum_{j=1}^{k}(\sigma_{j,k+1} + \beta_{j,k+1}) = \widetilde{y}_{k+1} = \widetilde{y}_{s_k(k)}$$

where we used the formulas $s_k\sigma_{j,k}s_k = \sigma_{j,k+1}$ and $s_k\beta_{j,k}s_k = \beta_{j,k+1}$.

- For $q = k-1$, one obtains

$$\begin{aligned} s_{k-1}\widetilde{y}_k s_{k-1} &= s_{k-1}y_k s_{k-1} - \sum_{j=1}^{k-1}(s_{k-1}\sigma_{j,k}s_{k-1} + s_{k-1}\beta_{j,k}s_{k-1}) \\ &= y_{k-1} + (s_{k-1} - \varepsilon_{k-1}) - \sum_{j=1}^{k-2}(\sigma_{j,k-1} + \beta_{j,k-1}) - (s_{k-1} - \varepsilon_{k-1}) \\ &= y_{k-1} - \sum_{j=1}^{k}(\sigma_{j,k-1} + \beta_{j,k-1}) = \widetilde{y}_{k-1} = \widetilde{y}_{s_{k-1}(k)} \end{aligned}$$

- For $q < k-1$, we compute

$$\begin{aligned} s_q\widetilde{y}_k s_q = s_q y_k s_q &- \sum_{j=1}^{q-1}(s_q\sigma_{j,k}s_q + s_q\beta_{j,k}s_q) - (s_q\sigma_{q,k}s_q + s_q\beta_{q,k}s_q) \\ &- (s_q\sigma_{q+1,k}s_q + s_q\beta_{q+1,k}s_q) - \sum_{j=q+2}^{k-1}(s_q\sigma_{j,k}s_q + s_q\beta_{j,k}s_q) \\ = y_k - \sum_{j=1}^{q-1}(\sigma_{j,k} + \beta_{j,k}) &- (\sigma_{q+1,k} + \beta_{q+1,k}) \\ &- (\sigma_{q,k} + \beta_{q,k}) - \sum_{j=q+2}^{k-1}(\sigma_{j,k} + \beta_{j,k}) \\ = y_k - \sum_{j=1}^{k-1}(\sigma_{j,k} + \beta_{j,k}) &= \widetilde{y}_k = \widetilde{y}_{s_q(k)} \end{aligned}$$

(ii) follows from relations $(P.5)$ and $(P.6)\,(i)\,\&\,(ii)$.

$\square$

Lemma 5.2.10. *For $1 \le p \ne q < k$, $\beta_{q,k}\widetilde{y}_p = \widetilde{y}_p\beta_{q,k}$*

Proof. We first prove $\beta_{1,k}\widetilde{y}_{k-1} = \widetilde{y}_{k-1}\beta_{1,k}$. To this end, we note the formula

$$y_{a+1}(s_a\varepsilon_{a+1}s_a) = (s_a\varepsilon_{a+1}s_a)y_{a+1} - s_a\varepsilon_{a+1}\varepsilon_a - s_a\varepsilon_{a+1} - \varepsilon_a\varepsilon_{a+1}s_a + \varepsilon_{a+1}s_a. \tag{5.3}$$

This may be found by the computation:

$$\begin{aligned}
y_{a+1}(s_a\varepsilon_{a+1}s_a) &= (s_ay_a - \varepsilon_a + \mathbf{1})\varepsilon_{a+1}s_a \\
&= (s_a\varepsilon_{a+1}y_a - \varepsilon_a\varepsilon_{a+1} + \varepsilon_{a+1})s_a \\
&= s_a\varepsilon_{a+1}(s_ay_{a+1} - \varepsilon_a - \mathbf{1}) - \varepsilon_a\varepsilon_{a+1}s_a + \varepsilon_{a+1}s_a \\
&= (s_a\varepsilon_{a+1}s_a)y_{a+1} - s_a\varepsilon_{a+1}\varepsilon_a - s_a\varepsilon_{a+1} - \varepsilon_a\varepsilon_{a+1}s_a + \varepsilon_{a+1}s_a.
\end{aligned}$$

Therefore,

$$\begin{aligned}
y_{k-1}\beta_{1,k} &= (s_1\cdots s_{k-3})y_{k-1}(s_{k-2}\varepsilon_{k-1}s_{k-2})(s_{k-2}\cdots s_1) \\
&= s_1\cdots s_{k-3}\Big((s_{k-2}\varepsilon_{k-1}s_{k-2})y_{k-1} - s_{k-2}\varepsilon_{k-1}s_{k-2} - s_{k-2}\varepsilon_{k-1} \\
&\qquad\qquad - \varepsilon_{k-2}\varepsilon_{k-1}s_{k-2} + \varepsilon_{k-1}s_{k-2}\Big)s_{k-3}\cdots s_1 \\
&= (s_1\cdots s_{k-2}\varepsilon_{k-1}s_{k-2}\cdots s_1)y_{k-1} - s_1\cdots s_{k-3}s_{k-2}\varepsilon_{k-1}\varepsilon_{k-2}s_{k-3}\cdots s_1 \\
&\qquad - s_1\cdots s_{k-3}s_{k-2}\varepsilon_{k-1}s_{k-3}\cdots s_1 - s_1\cdots s_{k-3}\varepsilon_{k-2}\varepsilon_{k-1}s_{k-2}s_{k-3}\cdots s_1 \\
&\qquad\quad + s_1\cdots s_{k-3}\varepsilon_{k-1}s_{k-2}s_{k-3}\cdots s_1 \\
&= \beta_{1,k}y_{k-1} - \beta_{1,k}\beta_{1,k-1} - \beta_{1,k}\sigma_{1,k-1} + \beta_{1,k-1}\beta_{1,k} + \sigma_{1,k-1}\beta_{1,k}
\end{aligned}$$

and so

$$\widetilde{y}_{k-1}\beta_{1,k} = y_{k-1}\beta_{1,k} - (\sigma_{1,k-1}\beta_{1,k} + \beta_{1,k-1}\beta_{1,k}) - \sum_{j=2}^{k-2}(\sigma_{j,k-1}\beta_{1,k} + \beta_{j,k-1}\beta_{1,k})$$

$$= \beta_{1,k} y_{k-1} - (\beta_{1,k}\sigma_{1,k-1} + \beta_{1,k}\beta_{1,k-1}) - \sum_{j=2}^{k-2} (\beta_{1,k}\sigma_{j,k-1} + \beta_{1,k}\beta_{j,k-1})$$
$$= \beta_{1,k}\widetilde{y}_{k-1}$$

by a previous corollary and the prior result. Now, the statement $\beta_{q,k}\widetilde{y}_p = \widetilde{y}_p\beta_{q,k}$ is true for $p < q$ by relations $(P.2)$ and $(P.6)$, so suppose that $q < p < k-1$. Conjugating the equation $\beta_{1,k}\widetilde{y}_{k-1} = \widetilde{y}_{k-1}\beta_{1,k}$ by $\sigma_{p,k-1}$ yields the formula

$$\beta_{1,k}\widetilde{y}_p = \widetilde{y}_p\beta_{1,k}$$

by a previous lemma. Then conjugating the above again by the permutation $s_1 \cdots s_{q-1}$ will yield $\beta_{q,k}\widetilde{y}_p = \widetilde{y}_p\beta_{q,k}$. □

Lemma 5.2.11. *In the quotient $\mathfrak{B}_d^{\mathit{aff}}/\mathcal{I}_d$, we have the relation*

$$\beta_{k,d}\widetilde{y}_k = -\widetilde{y}_k\beta_{k,d} \quad \textit{for } 1 \le k < d. \tag{5.4}$$

Proof. Using the relation $s_k\widetilde{y}_k = \widetilde{y}_{k+1}s_k$, we compute

$$\begin{aligned}
\beta_{k,d}\widetilde{y}_k &= (s_k \cdots s_{d-2})\varepsilon_{d-1}\widetilde{y}_{d-1}(s_{d-2}\cdots s_k) \\
&= (s_k \cdots s_{d-2})\varepsilon_{d-1}y_{d-1}(s_{d-2}\cdots s_k) \\
&\qquad - (s_k \cdots s_{d-2}\varepsilon_{d-1})\left(\sum_{j=1}^{d-2}(\sigma_{j,d-1} + \beta_{j,d-1})\right)(s_{d-2}\cdots s_k) \\
&= -(s_k \cdots s_{d-2})y_{d-1}\varepsilon_{d-1}(s_{d-2}\cdots s_k) \\
&\quad + (s_k \cdots s_{d-2})\left(\sum_{j=1}^{d-2}\left(\varepsilon_{d-1}(\sigma_{j,d-1} + \beta_{j,d-1}) + (\sigma_{j,d-1} + \beta_{j,d-1})\varepsilon_{d-1}\right)\right)(s_{d-2}\cdots s_k) \\
&\qquad - (s_k \cdots s_{d-2}\varepsilon_{d-1})\left(\sum_{j=1}^{d-2}(\sigma_{j,d-1} + \beta_{j,d-1})\right)(s_{d-2}\cdots s_k) \\
&= -(s_k \cdots s_{d-2})y_{d-1}\varepsilon_{d-1}(s_{d-2}\cdots s_k) \\
&\qquad + (s_k \cdots s_{d-2})\left(\sum_{j=1}^{d-2}(\sigma_{j,d-1} + \beta_{j,d-1})\right)(\varepsilon_{d-1}s_{d-2}\cdots s_k)
\end{aligned}$$

$$= -(s_k \cdots s_{d-2})\widetilde{y}_{d-1}\varepsilon_{d-1}(s_{d-2}\cdots s_k) = -\widetilde{y}_k\beta_{k,d}$$

□

Lemma 5.2.12. *For integers* $1 < p < n$*, we have the relations*

$$P_{1p}P_{1n} = P_{1n}P_{pn}, \quad Q_{1p}P_{1n} = -P_{1n}Q_{pn},$$
$$Q_{1p}Q_{1n} = P_{pn}Q_{1n}, \quad P_{1p}Q_{1n} = -Q_{pn}Q_{1n}$$

in the space $\mathrm{End}\left(\mathbb{C}^{N|N}\right)^{\otimes n}$.

Proof. The product $P_{1p}P_{1n}$ is given by

$$\begin{aligned}
&\left(\sum_{i,j\in I_N}(-1)^{[j]}E_{ij}\otimes \mathrm{id}^{\otimes(p-2)}\otimes E_{ji}\otimes \mathrm{id}^{\otimes(n-p)}\right)\left(\sum_{k,l\in I_N}(-1)^{[l]}E_{kl}\otimes \mathrm{id}^{\otimes(n-2)}\otimes E_{lk}\right)\\
&\quad= \sum_{i,j,k,l\in I_N}(-1)^{\substack{([i]+[j])([k]+[l])\\+[j]+[l]}}\delta_{jk}E_{il}\otimes \mathrm{id}^{\otimes(p-2)}\otimes E_{ji}\otimes \mathrm{id}^{\otimes(n-p-1)}\otimes E_{lk}\\
&\quad= \sum_{i,j,l\in I_N}(-1)^{[i][j]+[i][l]+[j][l]+[l]}E_{il}\otimes \mathrm{id}^{\otimes(p-2)}\otimes E_{ji}\otimes \mathrm{id}^{\otimes(n-p-1)}\otimes E_{lj}\\
&\quad= \sum_{i,j,k,l\in I_N}(-1)^{\substack{([i]+[l])([j]+[k])\\+[l]+[k]}}E_{il}\otimes \mathrm{id}^{\otimes(p-2)}\otimes E_{jk}\otimes \mathrm{id}^{\otimes(n-p-1)}\otimes \delta_{ik}E_{lj}\\
&\quad= \left(\sum_{i,l\in I_N}(-1)^{[l]}E_{il}\otimes \mathrm{id}^{\otimes(n-2)}\otimes E_{li}\right)\left(\sum_{j,k\in I_N}(-1)^{[k]}\,\mathrm{id}^{\otimes(p-1)}\otimes E_{jk}\otimes \mathrm{id}^{\otimes(n-p-1)}\otimes E_{kj}\right)
\end{aligned}$$

which is simply $P_{1n}P_{pn}$. Similarly, $Q_{1p}P_{1n}$ is provided by

$$\begin{aligned}
&-\left(\sum_{i,j\in I_N}(-1)^{[i][j]}E_{ij}\otimes \mathrm{id}^{\otimes(p-2)}\otimes E_{-i,-j}\otimes \mathrm{id}^{\otimes(n-p)}\right)\left(\sum_{k,l\in I_N}(-1)^{[l]}E_{kl}\otimes \mathrm{id}^{\otimes(n-2)}\otimes E_{lk}\right)\\
&\quad= -\sum_{i,j,k,l\in I_N}(-1)^{\substack{([i]+[j])([k]+[l])\\+[i][j]+[l]}}\delta_{jk}E_{il}\otimes \mathrm{id}^{\otimes(p-2)}\otimes E_{-i,-j}\otimes \mathrm{id}^{\otimes(n-p-1)}\otimes E_{lk}\\
&\quad= -\sum_{i,j,l\in I_N}(-1)^{[i][l]+[j][l]+[j]+[l]}E_{il}\otimes \mathrm{id}^{\otimes(p-2)}\otimes E_{-i,-j}\otimes \mathrm{id}^{\otimes(n-p-1)}\otimes E_{lj}
\end{aligned}$$

$$= - \sum_{i,j,k,l \in I_N} (-1)^{\substack{([i]+[j])([k]+[l]) \\ +[-i][-j]+[l]}} E_{kl} \otimes \mathrm{id}^{\otimes(p-2)} \otimes E_{-i,-j} \otimes \mathrm{id}^{\otimes(n-p-1)} \otimes \delta_{ki} E_{lj}$$

$$= \left(\sum_{k,l \in I_N} (-1)^{[l]} E_{kl} \otimes \mathrm{id}^{\otimes(n-2)} \otimes E_{lk} \right) \times \left(- \sum_{i,j \in I_N} (-1)^{[-i][-j]} \mathrm{id}^{\otimes(p-1)} \otimes E_{-i,-j} \otimes \mathrm{id}^{\otimes(n-p-1)} \otimes E_{ij} \right)$$

which is equal to $P_{1n}Q_{pn}$.

The product $P_{1p}Q_{1n}$ is given by

$$\left(\sum_{i,j \in I_N} (-1)^{[j]} E_{ij} \otimes \mathrm{id}^{\otimes(p-2)} \otimes E_{ji} \otimes \mathrm{id}^{\otimes(n-p)} \right) \times \left(- \sum_{k,l \in I_N} (-1)^{[k][l]} E_{kl} \otimes \mathrm{id}^{\otimes(n-2)} \otimes E_{-k,-l} \right)$$

$$= - \sum_{i,j,k,l \in I_N} (-1)^{\substack{([i]+[j])([k]+[l]) \\ +[j]+[k][l]}} \delta_{jk} E_{il} \otimes \mathrm{id}^{\otimes(p-2)} \otimes E_{ji} \otimes \mathrm{id}^{\otimes(n-p-1)} \otimes E_{-k,-l}$$

$$= - \sum_{i,j,l \in I_N} (-1)^{[i][j]+[i][l]} E_{il} \otimes \mathrm{id}^{\otimes(p-2)} \otimes E_{ji} \otimes \mathrm{id}^{\otimes(n-p-1)} \otimes E_{-j,-l}$$

$$= - \left(- \sum_{j,i \in I_N} (-1)^{[i][j]} \mathrm{id}^{\otimes(p-1)} \otimes E_{ji} \otimes \mathrm{id}^{\otimes(n-p-1)} \otimes E_{-j,-i} \right) \times \left(- \sum_{i,l \in I_N} (-1)^{[i][l]} E_{il} \otimes \mathrm{id}^{\otimes(n-2)} \otimes E_{-i,-l} \right)$$

which is $-Q_{pn}Q_{1n}$ The product $Q_{1p}Q_{1n}$ is given by

$$\left(- \sum_{i,j \in I_N} (-1)^{[i][j]} E_{ij} \otimes \mathrm{id}^{\otimes(p-2)} \otimes E_{-i.-j} \otimes \mathrm{id}^{\otimes(n-p)} \right) \times \left(- \sum_{k,l \in I_N} (-1)^{[k][l]} E_{kl} \otimes \mathrm{id}^{\otimes(n-2)} \otimes E_{-k,-l} \right)$$

$$= \sum_{i,j,k,l \in I_N} (-1)^{\substack{([i]+[j])([k]+[l]) \\ +[i][j]+[k][l]}} \delta_{jk} E_{il} \otimes \mathrm{id}^{\otimes(p-2)} \otimes E_{-i,-j} \otimes \mathrm{id}^{\otimes(n-p-1)} \otimes E_{-k,-l}$$

$$
\begin{aligned}
&= - \sum_{i,j,l \in I_N} (-1)^{[i][l]+[-j]} E_{il} \otimes \mathrm{id}^{\otimes (p-2)} \otimes E_{-i,-j} \otimes \mathrm{id}^{\otimes (n-p-1)} \otimes E_{-j,-l} \\
&= \left(\sum_{j,i \in I_N} (-1)^{[-j]} \mathrm{id}^{\otimes (p-1)} \otimes E_{-i,-j} \otimes \mathrm{id}^{\otimes (n-p-1)} \otimes E_{-j,-i} \right) \\
&\qquad\qquad\qquad \times \left(- \sum_{i,l \in I_N} (-1)^{[i][l]} E_{il} \otimes \mathrm{id}^{\otimes (n-2)} \otimes E_{-i,-l} \right)
\end{aligned}
$$

which is $P_{pn}Q_{1n}$. □

Lemma 5.2.13. *For integers $1 < p < q \leq n$, we have the equations*

$$
\begin{aligned}
P_{1p}P_{pq} = P_{pq}P_{1q}, \quad P_{1q}P_{pq} = P_{pq}P_{1p}, \\
Q_{1p}P_{pq} = P_{pq}Q_{1q}, \quad Q_{1q}P_{pq} = P_{pq}Q_{1p},
\end{aligned}
$$

in the space $\mathrm{End}\left(\mathbb{C}^{N|N}\right)^{\otimes n}$.

Proof. It suffices to prove only the equations $P_{1p}P_{pq} = P_{pq}P_{1q}$ and $Q_{1p}P_{pq} = P_{pq}Q_{1q}$. To this end, $P_{1p}P_{pq}$ is given by

$$
\begin{aligned}
&\left(\sum_{i,j \in I_N} (-1)^{[j]} E_{ij} \otimes \mathrm{id}^{\otimes (p-2)} \otimes E_{ji} \otimes \mathrm{id}^{\otimes (n-p)} \right) \\
&\qquad\qquad \times \left(\sum_{k,l \in I_N} (-1)^{[l]} \mathrm{id}^{\otimes (p-1)} \otimes E_{kl} \otimes \mathrm{id}^{\otimes (q-p-1)} \otimes E_{lk} \otimes \mathrm{id}^{\otimes (n-q)} \right) \\
&= \sum_{i,j,k,l \in I_N} (-1)^{[j]+[l]} E_{ij} \otimes \mathrm{id}^{\otimes (p-2)} \, \delta_{ik} E_{jl} \otimes \mathrm{id}^{\otimes (q-p-1)} \otimes E_{lk} \otimes \mathrm{id}^{\otimes (n-q)} \\
&= \sum_{i,j,l \in I_N} (-1)^{[j]+[l]} E_{ij} \otimes \mathrm{id}^{\otimes (p-2)} \, E_{jl} \otimes \mathrm{id}^{\otimes (q-p-1)} \otimes E_{li} \otimes \mathrm{id}^{\otimes (n-q)} \\
&= \left(\sum_{j,l \in I_N} (-1)^{[l]} \mathrm{id}^{\otimes (p-1)} \otimes E_{jl} \otimes \mathrm{id}^{\otimes (q-p-1)} \otimes E_{lj} \otimes \mathrm{id}^{\otimes (n-q)} \right) \\
&\qquad\qquad \times \left(\sum_{i,j \in I_N} (-1)^{[j]} E_{ij} \otimes \mathrm{id}^{\otimes (q-2)} \otimes E_{ji} \otimes \mathrm{id}^{\otimes (n-q)} \right),
\end{aligned}
$$

which is $P_{pq}P_{1q}$. Now, $Q_{1p}P_{pq}$ is given by

$$\begin{aligned}
&\left(-\sum_{i,j\in I_N} (-1)^{[i][j]} E_{ij} \otimes \mathrm{id}^{\otimes(p-2)} \otimes E_{-i,-j} \otimes \mathrm{id}^{\otimes(n-p)} \right) \\
&\qquad \times \left(\sum_{k,l\in I_N} (-1)^{[l]} \,\mathrm{id}^{\otimes(p-1)} \otimes E_{kl} \otimes \mathrm{id}^{\otimes(q-p-1)} \otimes E_{lk} \otimes \mathrm{id}^{\otimes(n-q)} \right) \\
&= -\sum_{i,j,k,l\in I_N} (-1)^{[i][j]+[l]} E_{ij} \otimes \mathrm{id}^{\otimes(p-2)}\, \delta_{-j,k} E_{-i,l} \otimes \mathrm{id}^{\otimes(q-p-1)} \otimes E_{lk} \otimes \mathrm{id}^{\otimes(n-q)} \\
&= -\sum_{i,j,l\in I_N} (-1)^{[i][j]+[l]} E_{ij} \otimes \mathrm{id}^{\otimes(p-2)}\, E_{-i,l} \otimes \mathrm{id}^{\otimes(q-p-1)} \otimes E_{l,-j} \otimes \mathrm{id}^{\otimes(n-q)} \\
&= \left(\sum_{i,l\in I_N} (-1)^{[l]} \,\mathrm{id}^{\otimes(p-1)} \otimes E_{-i,l} \otimes \mathrm{id}^{\otimes(q-p-1)} \otimes E_{l,-i} \otimes \mathrm{id}^{\otimes(n-q)} \right) \\
&\qquad \times \left(-\sum_{i,j\in I_N} (-1)^{[i][j]} E_{ij} \otimes \mathrm{id}^{\otimes(q-2)} \otimes E_{-i,-j} \otimes \mathrm{id}^{\otimes(n-q)} \right),
\end{aligned}$$

which is $P_{pq}Q_{1q}$. □

Proof. $P_{1p}Q_{pq}$ is given by

$$\begin{aligned}
&\left(\sum_{i,j\in I_N} (-1)^{[j]} E_{ij} \otimes \mathrm{id}^{\otimes(p-2)} \otimes E_{ji} \otimes \mathrm{id}^{\otimes(n-p)} \right) \\
&\qquad \times \left(-\sum_{k,l\in I_N} (-1)^{[k][l]} \,\mathrm{id}^{\otimes(p-1)} \otimes E_{kl} \otimes \mathrm{id}^{\otimes(q-p-1)} \otimes E_{-k,-l} \otimes \mathrm{id}^{\otimes(n-q)} \right) \\
&= -\sum_{i,j,k,l\in I_N} (-1)^{[j]+[k][l]} E_{ij} \otimes \mathrm{id}^{\otimes(p-2)}\, \delta_{ik} E_{jl} \otimes \mathrm{id}^{\otimes(q-p-1)} \otimes E_{-k,-l} \otimes \mathrm{id}^{\otimes(n-q)} \\
&= -\sum_{i,j,l\in I_N} (-1)^{[j]+[i][l]} E_{ij} \otimes \mathrm{id}^{\otimes(p-2)}\, E_{jl} \otimes \mathrm{id}^{\otimes(q-p-1)} \otimes E_{-i,-l} \otimes \mathrm{id}^{\otimes(n-q)} \\
&= \left(-\sum_{i,j\in I_N} (-1)^{[i][j]} E_{ij} \otimes \mathrm{id}^{\otimes(q-2)} \otimes E_{-i,-j} \otimes \mathrm{id}^{\otimes(n-q)} \right) \\
&\qquad \times \left(-\sum_{j,l\in I_N} (-1)^{[j][l]} \,\mathrm{id}^{\otimes(p-1)} \otimes E_{jl} \otimes \mathrm{id}^{\otimes(q-p-1)} \otimes E_{-j,-l} \otimes \mathrm{id}^{\otimes(n-q)} \right),
\end{aligned}$$

which is $Q_{1q}Q_{pq}$. But also,

$$
\begin{aligned}
&-\sum_{i,j,l\in I_N}(-1)^{[j]+[i][l]}E_{ij}\otimes \mathrm{id}^{\otimes(p-2)}\,E_{jl}\otimes \mathrm{id}^{\otimes(q-p-1)}\otimes E_{-i,-l}\otimes \mathrm{id}^{\otimes(n-q)}\\
&\qquad=\left(-\sum_{i,l\in I_N}(-1)^{[i][l]}E_{il}\otimes \mathrm{id}^{\otimes(q-2)}\otimes E_{-i,-l}\otimes \mathrm{id}^{\otimes(n-q)}\right)\\
&\qquad\qquad\times\left(\sum_{l,j\in I_N}(-1)^{[j]}E_{lj}\otimes \mathrm{id}^{\otimes(p-2)}\otimes E_{jl}\otimes \mathrm{id}^{\otimes(n-p)}\right),
\end{aligned}
$$

which is $Q_{1q}P_{1p}$, so $P_{1p}Q_{pq}=Q_{1q}Q_{pq}=Q_{1q}P_{1p}$.

$Q_{1p}Q_{pq}$ is given by

$$
\begin{aligned}
&\left(-\sum_{i,j\in I_N}(-1)^{[i][j]}E_{ij}\otimes \mathrm{id}^{\otimes(p-2)}\otimes E_{-i,-j}\otimes \mathrm{id}^{\otimes(n-p)}\right)\\
&\qquad\times\left(-\sum_{k,l\in I_N}(-1)^{[k][l]}\,\mathrm{id}^{\otimes(p-1)}\otimes E_{kl}\otimes \mathrm{id}^{\otimes(q-p-1)}\otimes E_{-k,-l}\otimes \mathrm{id}^{\otimes(n-q)}\right)\\
&=\sum_{i,j,k,l\in I_N}(-1)^{[i][j]+[k][l]}E_{ij}\otimes \mathrm{id}^{\otimes(p-2)}\,\delta_{-j,k}E_{-i,l}\otimes \mathrm{id}^{\otimes(q-p-1)}\otimes E_{-k,-l}\otimes \mathrm{id}^{\otimes(n-q)}\\
&=\sum_{i,j,l\in I_N}(-1)^{[i][j]+[-j][l]}E_{ij}\otimes \mathrm{id}^{\otimes(p-2)}\,E_{-i,l}\otimes \mathrm{id}^{\otimes(q-p-1)}\otimes E_{j,-l}\otimes \mathrm{id}^{\otimes(n-q)}\\
&=\left(-\sum_{i,l\in I_N}(-1)^{[i][-l]}E_{i,-l}\otimes \mathrm{id}^{\otimes(p-2)}\otimes E_{-i,l}\otimes \mathrm{id}^{\otimes(n-p)}\right)\\
&\qquad\times\left(\sum_{l,j\in I_N}(-1)^{[j]}E_{-l,j}\otimes \mathrm{id}^{\otimes(q-2)}\otimes E_{j,-l}\otimes \mathrm{id}^{\otimes(n-q)}\right),
\end{aligned}
$$

which is $Q_{1p}P_{1q}$. Also, this equals $P_{1q}Q_{pq}$, so $Q_{1p}Q_{pq}=P_{1q}Q_{pq}=Q_{1p}P_{1q}$. $\square$

5.3 Twisted Super Yangians

5.3.1 Reflection Equation

Note that $(-)\otimes\mathbf{1}\colon \big(\operatorname{End}\mathbb{C}^{M|N}\big)^{\otimes 2} \to \big(\operatorname{End}\mathbb{C}^{M|N}\big)^{\otimes 2}\otimes \mathbf{XB}(\mathfrak{gl}_{M|N},p,q)$ is a superalgebra embedding, and therefore the identity $P(E_{ij}\otimes E_{kl})P = (-1)^{([i]+[j])([k]+[l])}E_{kl}\otimes E_{ij}$, implies that the equation $(P\otimes\mathbf{1})\,\mathsf{B}_1(u)\,(P\otimes\mathbf{1}) = \mathsf{B}_2(u)$ is also true. Alternatively, one may compute this directly. For instance, by writing $P=\sum_{a,b=1}^{M+N}(-1)^{[b]}E_{ab}\otimes E_{ba}$, then

$$\begin{aligned}
(P\otimes\mathbf{1})\,\mathsf{B}_1(u) &= \sum_{i,j,a,b=1}^{M+N}(-1)^{([i]+[j])([a]+[b])+[b]}\delta_{bi}E_{aj}\otimes E_{ba}\otimes \mathsf{B}_{ij}(u)\\
&= \sum_{i,j,a=1}^{M+N}(-1)^{[i][j]+([i]+[j])[a]}E_{aj}\otimes E_{ia}\otimes \mathsf{B}_{ij}(u),
\end{aligned}$$

and so

$$\begin{aligned}
(P\otimes\mathbf{1})\,\mathsf{B}_1(u)\,(P\otimes\mathbf{1}) &= \sum_{i,j,a,c,d=1}^{M+N}(-1)^{[i][j]+[E_{ij}][a]+[E_{ia}][E_{cd}]+[d]}\delta_{jc}\delta_{ad}E_{ad}\otimes E_{ic}\otimes \mathsf{B}_{ij}(u)\\
&= \sum_{i,a,c,d=1}^{M+N}(-1)^{[i]([a]+[d])+[a][d]+[d]}\delta_{ad}E_{ad}\otimes E_{ic}\otimes \mathsf{B}_{ic}(u)\\
&= \sum_{i,a,c,d=1}^{M+N}\delta_{ad}E_{ad}\otimes E_{ic}\otimes \mathsf{B}_{ic}(u) = \mathsf{B}_2(u).
\end{aligned}$$

Step 1. First, we write equations $(A.1)-(A.4)$:

$$\begin{aligned}
(A.1)\quad \mathsf{B}_1(u)\mathsf{B}_2(v) &= \left(\sum_{i,j=1}^{M+N}E_{ij}\otimes \mathrm{id}\otimes \mathsf{B}_{ij}(u)\right)\left(\sum_{k,l=1}^{M+N}\mathrm{id}\otimes E_{kl}\otimes \mathsf{B}_{kl}(v)\right)\\
&= \sum_{i,j,k,l=1}^{M+N}(-1)^{([i]+[j])([k]+[l])}E_{ij}\otimes E_{kl}\otimes \mathsf{B}_{ij}(u)\mathsf{B}_{kl}(v)
\end{aligned}$$

and

$$(A.2)\quad \mathsf{B}_2(v)\mathsf{B}_1(u) = \left(\sum_{k,l=1}^{M+N} \mathrm{id} \otimes E_{kl} \otimes \mathsf{B}_{kl}(v)\right)\left(\sum_{i,j=1}^{M+N} E_{ij} \otimes \mathrm{id} \otimes \mathsf{B}_{ij}(u)\right)$$
$$= \sum_{i,j,k,l=1}^{M+N} E_{ij} \otimes E_{kl} \otimes \mathsf{B}_{kl}(v)\mathsf{B}_{ij}(u).$$

Further,

$$(A.3)\quad \mathsf{B}_2(u)\mathsf{B}_2(v) = \left(\sum_{i,j=1}^{M+N} \mathrm{id} \otimes E_{ij} \otimes \mathsf{B}_{ij}(u)\right)\left(\sum_{k,l=1}^{M+N} \mathrm{id} \otimes E_{kl} \otimes \mathsf{B}_{kl}(v)\right)$$
$$= \sum_{i,j,k,l=1}^{M+N} (-1)^{([i]+[j])([k]+[l])}\delta_{jk}\,\mathrm{id} \otimes E_{il} \otimes \mathsf{B}_{ij}(u)\mathsf{B}_{kl}(v)$$
$$= \sum_{k,l,a=1}^{M+N} (-1)^{([k]+[a])([a]+[l])}\,\mathrm{id} \otimes E_{kl} \otimes \mathsf{B}_{ka}(u)\mathsf{B}_{al}(v)$$
$$= \sum_{i,j,k,l,a=1}^{M+N} (-1)^{([k]+[a])([a]+[l])}\delta_{ij}E_{ij} \otimes E_{kl} \otimes \mathsf{B}_{ka}(u)\mathsf{B}_{al}(v)$$

and so

$$(A.4)\quad \mathsf{B}_2(v)\mathsf{B}_2(u) = \sum_{k,l,a=1}^{M+N} (-1)^{([k]+[a])([a]+[l])}\,\mathrm{id} \otimes E_{kl} \otimes \mathsf{B}_{ka}(v)\mathsf{B}_{al}(u)$$
$$= \sum_{i,j,k,l,a=1}^{M+N} (-1)^{([k]+[a])([a]+[l])}\delta_{ij}E_{ij} \otimes E_{kl} \otimes \mathsf{B}_{ka}(v)\mathsf{B}_{al}(u)$$

Step 2. Now, we write equations $(B.1) - (B.4)$:

$$(B.1)\quad (P \otimes \mathbf{1})\mathsf{B}_1(u)\mathsf{B}_2(v)$$
$$= \sum_{i,j,k,l,a,b=1}^{M+N} (-1)^{([i]+[j])(([a]+[b])+([k]+[l]))+[b]}\delta_{bi}\delta_{ak}E_{aj} \otimes E_{bl} \otimes \mathsf{B}_{ij}(u)\mathsf{B}_{kl}(v)$$
$$= \sum_{i,j,k,l=1}^{M+N} (-1)^{[i][j]+[i][l]+[j][l]}E_{kj} \otimes E_{il} \otimes \mathsf{B}_{ij}(u)\mathsf{B}_{kl}(v)$$

$$= \sum_{i,j,k,l=1}^{M+N} (-1)^{[j][k]+[j][l]+[k][l]} E_{ij} \otimes E_{kl} \otimes \mathsf{B}_{kj}(u)\mathsf{B}_{il}(v)$$

and $\mathsf{B}_2(v)\mathsf{B}_1(u)\,(P\otimes \mathbf{1}) = \mathsf{B}_2(v)\,(P\otimes \mathbf{1})\,\mathsf{B}_2(u) = (P\otimes \mathbf{1})\,\mathsf{B}_1(v)\mathsf{B}_2(u)$, so

$$(B.2)\quad \mathsf{B}_2(v)\mathsf{B}_1(u)(P\otimes \mathbf{1}) = \sum_{i,j,k,l=1}^{M+N} (-1)^{[j][k]+[j][l]+[k][l]} E_{ij} \otimes E_{kl} \otimes \mathsf{B}_{kj}(v)\mathsf{B}_{il}(u).$$

Now,

$$\begin{aligned}(B.3)\quad (P\otimes \mathbf{1})\,\mathsf{B}_2(u)\mathsf{B}_2(v) &= \sum_{k,l,a,c,d=1}^{M+N} (-1)^{([k]+[a])([a]+[l])+[d]} \delta_{ck} E_{cd} \otimes E_{dl} \otimes \mathsf{B}_{ka}(u)\mathsf{B}_{al}(v)\\ &= \sum_{l,a,c,d=1}^{M+N} (-1)^{([c]+[a])([a]+[l])+[d]} E_{cd} \otimes E_{dl} \otimes \mathsf{B}_{ca}(u)\mathsf{B}_{al}(v)\\ &= \sum_{i,j,k,l,a=1}^{M+N} (-1)^{([i]+[a])([a]+[l])+[j]} \delta_{jk} E_{ij} \otimes E_{kl} \otimes \mathsf{B}_{ia}(u)\mathsf{B}_{al}(v)\end{aligned}$$

and

$$\begin{aligned}&(B.4)\quad \mathsf{B}_2(v)\mathsf{B}_2(u)\,(P\otimes \mathbf{1})\\ &= \sum_{k,l,a,c,d=1}^{M+N} (-1)^{([k]+[a])([a]+[l])+([k]+[l])([c]+[d])+[d]} \delta_{ld} E_{cd} \otimes E_{kc} \otimes \mathsf{B}_{ka}(v)\mathsf{B}_{al}(u)\\ &= \sum_{k,a,c,d=1}^{M+N} (-1)^{([k]+[d])([a]+[c])+[a]} E_{cd} \otimes E_{kc} \otimes \mathsf{B}_{ka}(v)\mathsf{B}_{ad}(u)\\ &= \sum_{i,j,k,l,a=1}^{M+N} (-1)^{([k]+[j])([a]+[i])+[a]} \delta_{il} E_{ij} \otimes E_{kl} \otimes \mathsf{B}_{ka}(v)\mathsf{B}_{aj}(u)\end{aligned}$$

Step 3. Furthermore, we have the equations $(C.1)$ and $(C.2)$:

$$(C.1)\quad (P\otimes \mathbf{1})\,\mathsf{B}_1(u)\,(P\otimes \mathbf{1})\,\mathsf{B}_2(v) = \mathsf{B}_2(u)\mathsf{B}_2(v)$$

and

$$(C.2)\quad \mathsf{B}_2(v)\,(P\otimes\mathbf{1})\,\mathsf{B}_1(u)\,(P\otimes\mathbf{1}) = \mathsf{B}_2(v)\mathsf{B}_2(u)$$

Step 4. Therefore, we have

$$\begin{aligned}
&R(u-v)\mathsf{B}_1(u)R(u+v)\mathsf{B}_2(v)\\
&\qquad = \mathsf{B}_1(u)\mathsf{B}_2(v) - \frac{(P\otimes\mathbf{1})\,\mathsf{B}_1(u)\mathsf{B}_2(v)}{u-v} - \frac{(P\otimes\mathbf{1})\,\mathsf{B}_2(u)\mathsf{B}_2(v)}{u+v} + \frac{\mathsf{B}_2(u)\mathsf{B}_2(v)}{u^2-v^2}
\end{aligned}$$

and

$$\begin{aligned}
&\mathsf{B}_2(v)R(u+v)\mathsf{B}_1(u)R(u-v)\\
&\qquad = \mathsf{B}_2(v)\mathsf{B}_1(u) - \frac{\mathsf{B}_2(v)\mathsf{B}_1(u)\,(P\otimes\mathbf{1})}{u-v} - \frac{\mathsf{B}_2(v)\mathsf{B}_2(u)\,(P\otimes\mathbf{1})}{u+v} + \frac{\mathsf{B}_2(v)\mathsf{B}_2(u)}{u^2-v^2}
\end{aligned}$$

being equal, yielding the relation

$$\begin{aligned}
\mathsf{B}_1(u)\mathsf{B}_2(v) - \mathsf{B}_2(v)\mathsf{B}_1(u) = \frac{1}{u-v}&\Big((P\otimes\mathbf{1})\mathsf{B}_1(u)\mathsf{B}_2(v) - \mathsf{B}_2(v)\mathsf{B}_1(u)(P\otimes\mathbf{1})\Big)\\
&+\frac{1}{u+v}\Big((P\otimes\mathbf{1})\mathsf{B}_2(u)\mathsf{B}_2(v) - \mathsf{B}_2(v)\mathsf{B}_2(u)(P\otimes\mathbf{1})\Big)\\
&-\frac{1}{u^2-v^2}\Big(\mathsf{B}_2(u)\mathsf{B}_2(v) - \mathsf{B}_2(v)\mathsf{B}_2(u)\Big),
\end{aligned}$$

or rather:

$$\begin{aligned}
&\sum_{i,j,k,l=1}^{M+N} E_{ij}\otimes E_{kl}\otimes\big((-1)^{([i]+[j])([k]+[l])}\mathsf{B}_{ij}(u)\mathsf{B}_{kl}(v) - \mathsf{B}_{kl}(v)\mathsf{B}_{ij}(u)\big)\\
&\quad = \frac{1}{u-v}\sum_{i,j,k,l=1}^{M+N} E_{ij}\otimes E_{kl}\otimes(-1)^{[j][k]+[j][l]+[k][l]}\Big(\mathsf{B}_{kj}(u)\mathsf{B}_{il}(v) - \mathsf{B}_{kj}(v)\mathsf{B}_{il}(u)\Big)\\
&\quad + \frac{1}{u+v}\sum_{i,j,k,l=1}^{M+N} E_{ij}\otimes E_{kl}\otimes\Bigg((-1)^{[j]}\delta_{jk}\sum_{a=1}^{M+N}(-1)^{([i]+[a])([a]+[l])}\mathsf{B}_{ia}(u)\mathsf{B}_{al}(v)\\
&\qquad\qquad - (-1)^{[i]([j]+[k])}\delta_{il}\sum_{a=1}^{M+N}(-1)^{([k]+[j])[a]+[a]}\mathsf{B}_{ka}(v)\mathsf{B}_{aj}(u)\Bigg)\\
&\quad - \frac{1}{u^2-v^2}\sum_{i,j,k,l=1}^{M+N} E_{ij}\otimes E_{kl}\otimes\delta_{ij}\Bigg(\sum_{a=1}^{M+N}(-1)^{([k]+[a])([a]+[l])}\mathsf{B}_{ka}(u)\mathsf{B}_{al}(v)
\end{aligned}$$

$$- \sum_{a=1}^{M+N} (-1)^{([k]+[a])([a]+[l])} \mathsf{B}_{ka}(v)\mathsf{B}_{al}(u)\Bigg),$$

hence giving the relation

$$\begin{aligned}
&\big((-1)^{([i]+[j])([k]+[l])} \mathsf{B}_{ij}(u)\mathsf{B}_{kl}(v) - \mathsf{B}_{kl}(v)\mathsf{B}_{ij}(u)\big) \\
&\quad = \frac{1}{u-v}(-1)^{[j][k]+[j][l]+[k][l]}\Big(\mathsf{B}_{kj}(u)\mathsf{B}_{il}(v) - \mathsf{B}_{kj}(v)\mathsf{B}_{il}(u)\Big) \\
&\qquad + \frac{1}{u+v}\Bigg((-1)^{[j]}\delta_{jk} \sum_{a=1}^{M+N} (-1)^{([i]+[a])([a]+[l])} \mathsf{B}_{ia}(u)\mathsf{B}_{al}(v) \\
&\qquad\qquad - (-1)^{[i]([j]+[k])}\delta_{il} \sum_{a=1}^{M+N} (-1)^{([k]+[j])[a]+[a]} \mathsf{B}_{ka}(v)\mathsf{B}_{aj}(u)\Bigg) \\
&\qquad - \frac{1}{u^2-v^2}\delta_{ij}\Bigg(\sum_{a=1}^{M+N} (-1)^{([k]+[a])([a]+[l])} \mathsf{B}_{ka}(u)\mathsf{B}_{al}(v) \\
&\qquad\qquad - \sum_{a=1}^{M+N} (-1)^{([k]+[a])([a]+[l])} \mathsf{B}_{ka}(v)\mathsf{B}_{al}(u)\Bigg),
\end{aligned}$$

Multiplying by the scalar $(-1)^{([i]+[j])([k]+[l])}$ therefore gives the desired relation

$$\begin{aligned}
\big[\mathsf{B}_{ij}(u), \mathsf{B}_{kl}(v)\big] &= \frac{1}{u-v}(-1)^{[i][k]+[i][l]+[k][l]}\Big(\mathsf{B}_{kj}(u)\mathsf{B}_{il}(v) - \mathsf{B}_{kj}(v)\mathsf{B}_{il}(u)\Big) \\
&\quad + \frac{1}{u+v}\Bigg((-1)^{[j]([i]+[l])}\delta_{jk} \sum_{a=1}^{M+N} (-1)^{([i]+[l])[a]+[a]} \mathsf{B}_{ia}(u)\mathsf{B}_{al}(v) \\
&\qquad - (-1)^{[i]+[j][k]}\delta_{il} \sum_{a=1}^{M+N} (-1)^{([k]+[j])[a]+[a]} \mathsf{B}_{ka}(v)\mathsf{B}_{aj}(u)\Bigg) \\
&\quad - \frac{1}{u^2-v^2}\delta_{ij}\Bigg(\sum_{a=1}^{M+N} (-1)^{([k]+[a])([a]+[l])} \mathsf{B}_{ka}(u)\mathsf{B}_{al}(v) \\
&\qquad - \sum_{a=1}^{M+N} (-1)^{([k]+[a])([a]+[l])} \mathsf{B}_{ka}(v)\mathsf{B}_{al}(u)\Bigg).
\end{aligned}$$

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