SUBJECT NOTES

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CONTENTS

CHAPTER 1

GAME THEORY

Strategic form/normal form/matrix games: Games in which all participants act simultaneously and without knowledge of other players' actions.

- Set of players (agents)
- Set of actions
- Set of payoff/utility functions
- Information structure players can access

1 Finite Games/Nash Equilibria

1.1 Finite Games

Definition 1.1: Strategic form of game/Finite game

A strategic forms game is a triplet $\langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ such that

- $\mathcal{I}$: finite number of players, where $N = \mathcal{I} = \{1, 2, \ldots, n\}$
- S_i: set of actions (decisions, strategies) for player i
- $s_i \in S_i$: actions (decisions, strategies) for player i
- $u_i : S \longrightarrow \mathbb{R}$: the payoff (utility) function of player i, where $S = \prod_{i=1}^{n} S_i$ is the set of actions of all players

Notation

- $s = (s_1, \ldots, s_n) \in S = \prod_{i=1}^{n} S_i = S_1 \times S_2 \times \cdots \times S_n$
- s: decision/action/strategy profile
- $s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, s_{i+2}, \ldots, s_n)$

Strategy: Complete description of how to play the game. Requires full contingent planning (full description how to play in every contingency).

General Setup of n-Player Finite Game

- Players: n-players with $i \in N = \{1, 2, \ldots, n\}$
- decision/action/strategy for Player i: $s_i \in S_i$
  - S_i is a finite set
- $s = (s_1, \ldots, s_n) \in S = S_1 \times S_2 \times \cdots \times S_n$
  - s: decision/action/strategy profile
- $s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, s_{i+2}, \ldots, s_n)$
- Payoff function: $u_i(s_i, s_{-i})$ with $u_i : S \to \mathbb{R}$
  - Each player has to maximize u_i over $s_i \in S_i$
- Player 1 and Player 2
- $S_1 = \{1, \ldots, p\}$ finite set
- $S_2 = \{1, \ldots, m\}$ finite set
- $u_1 : p \times m$ matrix
- $u_2 : m \times p$ matrix
- Zero-sums game
  - When $u := u_1 = -u_2$
  - Player 2 plays minimizing u

		Player 2 D	E	F
	A	(a, b)	(c, d)	(e, f)
Player 1	B	(g, h)	(i, j)	(k, l)
	C	(m, n)	(o, p)	(q, r)

- Player 1 chooses row with respect to the first component $X_1 = \{A, B, C\}$
- Player 2 chooses column with respect to the second component $X_2 = \{D, E, F\}$

1.2 Dominant Equilibrium: Optimality of Game

For i player, a dominant strategy is one that yields the highest payoff, *regardless of other players' actions*. A **Dominant Strategy Equilibrium** occurs when *every* player has a clear best choice irrespective of others' and there is this no incentive for any player to deviate. (e.g., Prisoner's Dilemma).

Nash Equilibrium: Set of strategies, for each player, such that *no player can improve payoff by unilaterally changing only their own strategy* (assuming all players stick to chosen strategies). Note: a game can have multiple Nash equilibria and they don't always mean the best possible collective outcome for all players.

Every dominant strategy equilibrium is also Nash equilibrium. NOT the other way around though.

Definition 1.2: Dominant strategy

A strategy $s_i \in S_i$ is dominant for Player $i \in N$ if

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}), \quad \forall (s'_i, s_{-i}) \in S_i \times S_{-i}$$

where, s_{-i} is collection of strategies chosen by all players except player i.

Definition 1.3: Dominant equilibrium

A strategy profile $s^* \in S$ is the dominant strategy equilibrium if for each Player $i \in N$, $s^*_i \in S_i$ is the dominant strategy.

- We observe that "Confess" is the dominant equilibrium in Prisoner's dilemma game
- Rational players will choose the dominant strategy

Definition 1.4: Strictly dominated strategy

A strategy $s_i \in S_i$ is strictly dominated for player $i \in N$ if there exists some $s'_i \in S_i$ such that

$$u(s'_i, s_{-i}) > u(s_i, s_{-i}), \quad \forall s_{-i} \in S_{-i}$$

- Can obtain the dominant equilibrium by eliminating strictly dominated strategies (**iterated elimination of strictly dominated strategies (IESDS)**).
- Rational players do not choose the strictly dominated strategy

Therefore, if there exists another strategy s'_i such that choosing s'_i *always* yields a strictly higher payoff for player i. regardless of what strategies other players s_{-i} choose.

IESDS

Method to simplify a game and find a solution/equilibrium.

Let S^k_j be set of strategies for player j that have survived elimination up to iteration k.

Let $S^k_{-i} = X_{j \neq i} S^k_j$ be set of strategy profiles for players other than i using strategies available at iteration k.

Pseudocode:

```
    Initialize S_i_current = S_i for all players i in N
Set strategies_eliminated_this_round = true
```

```
WHILE strategies_eliminated_this_round == true:
    Set strategies_eliminated_this_round = false
    FOR EACH player i in N:
        Let S_i_next_round = S_i_current
        FOR EACH strategy s_prime_i in S_i_current:
            Set is_dominated = false
            FOR EACH strategy s_double_prime_i in S_i_current (where s_double_prime_i != s_prime
                Set s_double_prime_dominates_s_prime = true
                // Check if s_double_prime_i strictly dominates s_prime_i
                // against all combinations of opponents' current strategies S_minus_i_current
                FOR EACH strategy_profile_s_minus_i in S_minus_i_current:
                    IF u_i(s_double_prime_i, s_minus_i) <= u_i(s_prime_i, s_minus_i):
                        s_double_prime_dominates_s_prime = false
                        BREAK // s_double_prime_i does not dominate s_prime_i w.r.t. this s_minu
                IF s_double_prime_dominates_s_prime == true:
                    is_dominated = true
                    BREAK // s_prime_i is dominated by s_double_prime_i
            IF is_dominated == true:
                Remove s_prime_i from S_i_next_round
                strategies_eliminated_this_round = true
        Set S_i_current = S_i_next_round // Update player i's strategy set for this iteration

Output: The final sets S_i_current for all players.
```

1.3 Nash Equilibrium: Optimality of Game

- N-player noncooperative game

- Rationality and optimality are key underlying assumptions

- No incentive to deviate once every player is in Nash

Definition 1.5: Nash Equilibrium (state)

The strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in S$ is called a Nash equilibrium of the game if for all i, $i = 1, 2, \ldots, n$,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \quad \forall s_i \in S_i$$

Thus, no single player has an incentive to change only their own strategy. If player i unilaterally deviates from s^* to s_i, while $-i$ stick to s^*, player i *will NOT* achieve a strictly better payoff (either same or worse).

Definition 1.6: Best response function (tool)

The best response function **(correspondence)** $B_i(s_{-i})$ is defined by $B_i : S_{-i} \to S_i$

$$B_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$$

$$= \{s_i \in S_i \mid u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall s_i' \in S_i\}$$

- It is sometimes correspondence, since given $s_{-i} \in S_{-i}$, there can be multiple $s_i \in S_i$
- It is a multi-valued (set-valued) function

B_i defines (set) strategy(s) such that player i's payoff is maximized, *given* $-i$ are playing s_{-1}.

Output of B_i can be a set of strategies too if multiple yield same maximum payoff.

A strategy $s^* = (s_1^*, s_2^*, ..., s_n^*)$ is **Nash Equilibrium** if every player's strategy in that profile is a best response to the strategies of all other players in that profile. Thus, $\forall i \in N$: $s_i^* \in B_i(s_{-1}^*)$:

Proposition 1.7

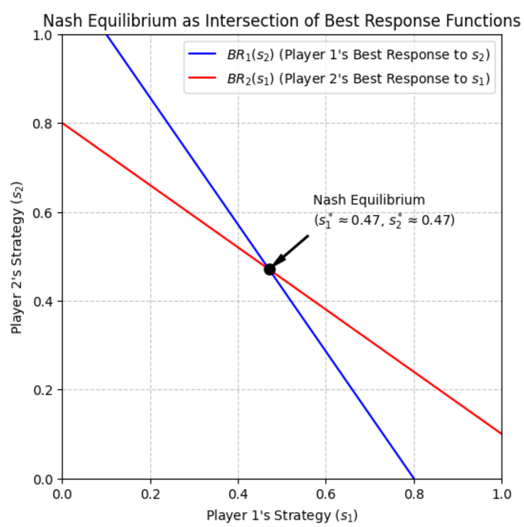
The strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in S$ is a Nash equilibrium of the game if and only if

$$s_i^* \in B_i(s_{-i}^*), \quad \forall i \in N = \{1, 2, \ldots, n\}$$

Proof for Proposition.

- If part: since $s_i^* \in B_i(s_{-i}^*)$ for all $i \in N$, the result is true by definition
- Only if part: since $s^* \in S$ is a NE, the result follows from definition of the best response function

Current strategy maxed out payoff = *No* incentive to unilaterally change strategy.



Definition 1.8: Nash equilibrium for two player game

Simplified to two player game ($n = 2$)
The strategy profile $s^* = (s_1^*, s_2^*) \in S_1 \times S_2$ is called a Nash equilibrium of the game if

$$u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^*), \quad \forall s_1 \in S_1$$
$$u_2(s_1^*, s_2^*) \geq u_2(s_1^*, s_2), \quad \forall s_2 \in S_2$$

Each player wants to choose their strategy to maximize their own payoff, keeping in mind that the other player is also trying to do the same.

Core Idea: No Unilateral Incentive to Deviate (No Regrets)

e.g., Prisoner's Dilemma (PAyoff does not lead to best result at NE):

'The dilemma is that individual rationality (each prisoner choosing their dominant strategy to minimize their own sentence) leads to a collectively suboptimal outcome where both are worse off than if they had managed to cooperate. Even if they had agreed beforehand to Stay Silent, the incentive to betray the other for a chance at freedom (or a reduced sentence if the other also betrays) is very strong. This highlights the conflict between individual incentives and mutual benefit, and the difficulty of achieving cooperation in the absence of trust and binding agreements.'

1.4 Saddle-Point Equilibrium

Definition 1.9: Saddle-Point Equilibrium

NE for a **2-player zero-sum** game ($u = u_1 = -u_2$)
Strategy profile $s^* = (s_1^*, s_2^*) \in S_1 * S_2$ is a saddle-point equilibrium of 2-player game if

$$u(s_1, s_2^*) \leq u(s_1^*, s_2^*) \leq u(s_1^*, s_2), \forall (u(s_1, s_2) \in S_1 * S_2$$

- $u(s_1^*, s_2^*)$: **value of the game**

Minimax strategy (player 2 - minimizer of player's 1 payoff): For each column, player 2 identifies max. possible payoff Player 1 could achieve if player 2 chooses that column (assuming player 1 will try to maximize their payoff for that column). **Column Maximum*****

Player 2 then chooses strategy (column) that corresponds to the **minimum of these column maximums** = **Minimax value** of the game (From player 1's perspective, representing the maximum payoff player 2 is willing to concede.

Maximin value (Player 1) = *Minimax value (Player 2)*

=

Value of the Game (**V**)

Example : Saddle Point in Pure Strategies

Consider the following *PAYOFF MATRIX* for Player 1 in a 2-player 0-sum game. Entries in matrix = Payoff to Player 1. ($u = u1 = u2$)

	Player 2: Strategy Y1	**Player 2: Strategy Y2**	**Row Minimums**
Player 1: Strategy X1	4	**2**	2
Player 1: Strategy X2	3	1	1
Column Maximums	4	**2**	

Goal: To find maximin and minimax values to identify saddle point:

1. Player 1's Maximin Strategy (Maximizing minimum guaranteed payoff)

Player 1 looks at minimum payoff they could receive for each of their strategies:

- If Player 1 plays Strategy X1, the minimum payoff is $\min(4, 2) = 2$.
- If Player 1 plays Strategy X2, the minimum payoff is $\min(3, 1) = 1$.

Player 1 wants to choose the strategy that maximizes this minimum payoff. The maximum of $\{2, 1\}$ is 2. Thus, Player 1's maximin strategy is X1, and the **Maximin value** = **2**.

2. Player 2's Minimax Strategy (Minimizing Player 1's maximum possible gain)

Player 2 looks at the maximum payoff Player 1 could achieve for each of Player 2's strategies:

- If Player 2 plays Strategy Y1, the maximum payoff Player 1 can get is $\max(4, 3) = 4$.
- If Player 2 plays Strategy Y2, the maximum payoff Player 1 can get is $\max(2, 1) = 2$.

Player 2 wants to choose the strategy that minimizes maximum payoff for Player 1. The minimum of $\{4, 2\}$ is 2. Thus, Player 2's minimax strategy is Y2, and the **Minimax value** = **2** (from Player 1's perspective).

Saddle Point and Value of the Game

Since Maximin value (2) = Minimax value (2) → Saddle point exists.

$$V = 2$$

Saddle point occurs at strategy profile where Player 1 plays **Strategy X1** and Player 2 plays **Strategy Y2**. The payoff at this point is **2** (to Player 1).

Looking at the entry '2' in the matrix (at the intersection of X1 and Y2):

- It is the minimum value in its row (Row X1: values are $\{4, \mathbf{2}\}$).

- It is the maximum value in its column (Column Y2: values are $\{\mathbf{2}, 1\}$).

Dual property (minimum of its row and maximum of its column) is characteristic of a saddle point in a payoff matrix.

Stability of the Saddle Point (Connection to Nash Equilibrium)

At saddle point (X1, Y2):

- If Player 1 (currently playing X1) unilaterally considers switching to Strategy X2 (while Player 2 continues to play Y2), Player 1's payoff would decrease from 2 to 1. Therefore, Player 1 has no incentive to switch.

- If Player 2 (currently playing Y2) unilaterally considers switching to Strategy Y1 (while Player 1 continues to play X1), Player 1's payoff would increase from 2 to 4 = Player 2's payoff would change from -2 to -4 (zero-sum game), which is worse for Player 2. Therefore, Player 2 has no incentive to switch.

Since neither player has an incentive to unilaterally deviate from strategy profile (X1, Y2), saddle point is also a NE for the above 0-sum game.

Properties of Zero-Game:

- **Value is unique:**
  - There is 1 value which is = upper and lower values of the game.
- **Order Interchangeability:**
  - if (x_1, x_2) and (y_1, y_2) are saddle-point solutions, then (x_1, y_2) and (y_1, x_2) are also saddle-point solution
  - (x_1, x_2) and (y_1, y_2) lead to *same value* of the 0-sum game

1.5 Mixed Strategies and Mixed Nash Equilibrium

- Probability vector in strategy space
- Randomization of the strategy (action) space
- Payoff becomes expected value

Mixed Strategies and Expected Payoff

- Let Σ_i be set of probabilities on S_i.
- Let $\sigma_i \in \Sigma_i$ is probability on S_i. σ_i (also called the simplex on Σ_i)
- Note $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$
- Similarly σ_{-i} and Σ_{-i}

- With $\sigma_i \in \Sigma_i$, strategy set S_i can be randomized. **Randomization is independent**.

- Expected payoff is given by $u_i : \Sigma \to \mathbb{R}$ with

$$u_i(\sigma) = u_i(\sigma_i, \sigma_{-i}) = E[u_i(s_i, s_{-i})] = \sum_{s \in S} \left(\prod_{j=1}^{n} \sigma_j(s_j) \right) u_i(s)$$

(Note: $s = (s_1, \ldots, s_n)$ is a pure strategy profile and $\sigma_j(s_j)$ is probability player j plays pure strategy s_j.)

Let $|S_1| = p$ and $|S_2| = q$

$\sigma_1 = (\sigma_1^{(1)}, ..., \sigma_1^{(p)})$ with $\sum_{i=1}^{p} \sigma_1^i = 1$ and $\sigma_1^i \in [0, 1]$

$\sigma_2 = (\sigma_2^{(1)}, ..., \sigma_2^{(p)})$ with $\sum_{i=1}^{p} \sigma_2^i = 1$ and $\sigma_2^i \in [0, 1]$

σ_1 and σ_2 are probability measures on S_1 and S_2. They are probability mass functions

Always choosing pure strategy (same option) is predictable and can be *exploited.* A **mixed strategy** is a way to be unpredictable. Player chooses a *probability distribution* over their available actions.

Example : Mixed

- If $S_1 = \{A, B\}$, then player 1 selects A with probability $\sigma_1^{(1)}$ and B with probability $1 - \sigma_1^{(1)}$
- $\sigma_1 = (\sigma_1^{(1)}, 1 - \sigma_1^{(1)})$
- If $S_2 = \{C, D\}$ then $\sigma_2 = (\sigma_2^{(1)}, 1 - \sigma_2^{(1)})$
- **Expected/probabilistic payoff** for Player 1 is:

$$\begin{aligned} u_1(\sigma_1, \sigma_2) = &\sigma_1^{(1)} \sigma_2^{(1)} u_1(A, C) \\ &+ (1 - \sigma_1^{(1)}) \sigma_2^{(1)} u_1(B, C) \\ &+ \sigma_1^{(1)} (1 - \sigma_2^{(1)}) u_1(A, D) \\ &+ (1 - \sigma_1)(1 - \sigma_2^{(1)}) u_1(B, D) \end{aligned}$$

$\sigma_1^1 \sigma_1^2 u_1(A, C)$: Chance Player 1 plays A and Player 2 plays C times Player 1's payoff $u(A, C)$ if that happens and so on x4 for all the possible combinations

Definition 1.10: Mixed Nash equilibrium

A mixed strategy profile $\sigma^* \in \Sigma$ is a mixed (strategy) Nash equilibrium if for any player $i \in N$,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*), \quad \forall \sigma_i \in \Sigma_i$$

Remark.

- Previous definition NE is deterministic, **pure NE**.
- Mixed NE = Randomization
- Every finite (matrix) game admits mixed strategy NE

Theorem 1.11: Mixed NE with best response

A mixed strategy profile $\sigma^* \in \Sigma$ is a mixed NE *if and only if* for any player $i \in N$,

$$\sigma_i^* \in B_i(\sigma_{-i}^*) = \arg\max_{\sigma_i \in \Sigma_i} u_i(\sigma_i, \sigma_{-i}^*) = \{\sigma_i \in \Sigma_i \mid u_i(\sigma_i, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*), \forall \sigma_i' \in \Sigma_i\}$$

Remark.

- B_i: Best response correspondence of Player i
- Proof is analogous to that of pure NE case

Check player i's strategy σ_i^* against *every other possible mixed strategy* σ_i' *but there are infinitely ways to mix probabilities therefore not feasible.* The solution is to check that player i's mixed strategy σ_i^* gives a payoff that is **at least as good as any of the player's individual pure strategies** (s_i'). Thus, a situation is NE when every player's chosen s_i is a B_i to S_{-i} = *equilibrium* = No reason to change strategy:

Proposition 1.12

A mixed strategy profile $\sigma^* \in \Sigma$ is a mixed NE *if and only if* for any player $i \in N$,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*), \quad \forall s_i' \in S_i$$

Proof for Proposition.

$$u_i(\sigma_i', \sigma_{-i}^*) = \sum_{j=1}^{|S_i|} \sigma_i'^{(j)} u_i(s_j, \sigma_{-i}^*)$$

$$\sum_{j=1}^{|S_i|} \sigma_i'^{(j)} u_i(s_j, \sigma_{-i}^*) \leq \sum_{j=1}^{|S_i|} \sigma_i'^{(j)} u_i(\sigma_i^*, \sigma_{-i}^*)$$

$$= u_i(\sigma_i^*, \sigma_{-i}^*) \sum_{j=1}^{|S_i|} \sigma_i'^{(j)} = u_i(\sigma_i^*, \sigma_{-i}^*)$$

∎

Support of a mixed strategy: Any pure strategy s_i that is played with a probability > 0.

Proposition 1.13

A mixed strategy profile $\sigma^* \in \Sigma$ is a mixed strategy NE if and only if for any $i \in N$, every pure strategy in the support $\sigma^* \in \Sigma_i$ is the best response to $\sigma^*_{-i} \in \Sigma^*_{-i}$.

For any $s_i \in \Sigma_i$ with:

$$\mathbb{P}(s_i) = \sigma_i^*(s_i) > 0$$
$$s_i \in \arg\max_{s_j \in S_i} u_i(s_j, \sigma^*_{-i}) \iff s_i \in B_i(\sigma^*_{-i})$$

Remark.

- Each pure strategy s_i is the best response to the mixed strategies of other players σ^*_i
- Important for characterization of mixed NE

The support is the set of all pure strategies that are actually played with a **non-zero probability** while the the proposition says that a mixed strategy is a NE if and only if **every pure strategy the player is actively using (i.e., in the support) is itself a best response.**

For every **pure** strategy in the support to be a best response, they must all yield the exact **same expected payoff**. If $1/pure_strategies$ in mix gave a higher expected payoff than another, player would have incentive to shift all probability to that better strategy.

Indifference Principle

The equation says that in a mixed NE, the expected payoff for playing **any pure strategy in the support** is the same.

Proposition 1.14

Under strategy of $(\sigma_i^*, \sigma^*_{-i}) \in \Sigma_i * \Sigma_{-i}$, if $s_i, s_i' \in S_i$ are supports of $\sigma_i^* \in \Sigma_i$, then:

$$u_i(s_i, \sigma^*_{-i}) = u_i(s_i', \sigma^*_{-i}) = u_i(\sigma_i^*, \sigma^*_{-i}), \quad \forall i = 1, \ldots, n \text{ (since mixed's payoff is weighted average of identical payoffs)}$$

Proof for Proposition.

if $u_i(s_i, \sigma^*_{-i}) > u_i(s_i', \sigma^*_{-i})$ then reducing probability of playing s_i' leads to increasing probability of playing s_i which implies that σ_i^* is not the best response to σ^*_{-1}. ■

Thus, in a Mixed NE player must be indifferent to all the pure strategies player is actively mixing between.

Let A be a payoff matrix for Player 1 and B for Player 2, where Player 1 chooses row and Player 2 chooses column for optimal decisions:

Definition 1.15: Matrix representation of NE (2 players)

A pair $(\sigma_1^*, \sigma_2^*) \in \Sigma_1$ x Σ_2 is said to constitute a NE in mixed strategies if:

- Player 1 $\sigma_1^{*\top} A \sigma_2^* \geq \sigma_1^\top A \sigma_2^*, \quad \forall \sigma_1 \in \Sigma_1$
- Player 2 $\sigma_1^{*\top} B \sigma_2^* \geq \sigma_1^\top B \sigma_2, \quad \forall \sigma_2 \in \Sigma_2$

Let $A = -B$. Then the definition of mixed NE is equivalent to the mixed saddle-point equilibrium:

Definition 1.16: Mixed saddle-point equilibrium

A pair $(\sigma_1^*, \sigma_2^*) \in \Sigma_1 \times \Sigma_2$ is said to constitute a saddle-point solution in mixed strategies if

$$\sigma_1^\top A \sigma_2^* \leq (\sigma_1^*)^\top A \sigma_2^* \leq (\sigma_1^*)^\top A \sigma_2, \quad \forall (\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2$$

- Note
$$(\sigma_1^*)^\top A \sigma_2^* = \max_{\sigma_1 \in \Sigma_1} \min_{\sigma_2 \in \Sigma_2} \sigma_1^\top A \sigma_2 = \min_{\sigma_2 \in \Sigma_2} \max_{\sigma_1 \in \Sigma_1} \sigma_1^\top A \sigma_2$$
- This is the value of the zero-sum game
- For zero-sum games, a NE = Saddle-Point

Example : Matching Penny Game

		Player 2	
		Head	Tail
Player 1	Head	$(1, -1)$	$(-1, 1)$
	Tail	$(-1, 1)$	$(1, -1)$

- There does not exist pure NE (or saddle-point equilibrium).
  - If they play (Head, Head), Player 2 gets -1 and would prefer to switch to Tail to get $+1$
  - If they play (Head, Tail), Player 1 gets -1 and would prefer to switch to Tail to get $+1$
  - so on...

  In each case player has incentive to switch = no NE, thus a mixed strategy must be found:
- Compute mixed NE (or saddle-point equilibrium)
- Apply previous proposition 1.14

Recall Proposition 1.14

Theorem 1.17: Mixed Nash equilibrium with best response | Practical Application

When a player is indifferent, they are willing to mix their strategies. A mixed strategy profile $\sigma^* \in \Sigma$ is a mixed (strategy) NE if and only if for any player $i \in N$,

$$\sigma_i^* \in B_i(\sigma_{-i}^*) = \arg\max_{\sigma_i \in \Sigma_i} u_i(\sigma_i, \sigma_{-i}^*)$$

$$= \{\sigma_i \in \Sigma_i \mid u_i(\sigma_i, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*), \forall \sigma_i' \in \Sigma_i\}$$

Mixed Strategies and Expected Payoffs

Let Player 1's mixed strategy be $\sigma_1 = (p, 1-p)$ and Player 2's be $\sigma_2 = (q, 1-q)$, with $p, q \in [0, 1]$.

For a given σ_2, the expected payoff of Player 1 can be written as:

$$\text{(1) Player 1 playing Head: } E_1(\text{Head}) = q \cdot (1) + (1-q) \cdot (-1) = 2q - 1$$
$$\text{(2) Player 1 playing Tail: } E_1(\text{Tail}) = q \cdot (-1) + (1-q) \cdot (1) = 1 - 2q$$

Principle of indifference: a player will only be willing to play a mixed strategy i.e., choose probability, if they are perfectly indifferent between their pure strategies.

Player 1 is indifferent when $2q - 1 = 1 - 2q \implies q = \frac{1}{2}$.

- When $q < \frac{1}{2} \implies E_1(\text{Head}) < E_1(\text{Tail})$
- When $q = \frac{1}{2} \implies E_1(\text{Head}) = E_1(\text{Tail})$
- When $q > \frac{1}{2} \implies E_1(\text{Head}) > E_1(\text{Tail})$

For a given σ_1, the expected payoff of Player 2 is:

$$\text{(3) Player 2 playing Head: } E_2(\text{Head}) = p \cdot (-1) + (1-p) \cdot (1) = 1 - 2p$$
$$\text{(4) Player 2 playing Tail: } E_2(\text{Tail}) = p \cdot (1) + (1-p) \cdot (-1) = 2p - 1$$

Player 2 is indifferent when $1 - 2p = 2p - 1 \implies p = \frac{1}{2}$.

- When $p < \frac{1}{2} \implies E_2(\text{Head}) > E_2(\text{Tail})$
- When $p = \frac{1}{2} \implies E_2(\text{Head}) = E_2(\text{Tail})$
- When $p > \frac{1}{2} \implies E_2(\text{Head}) < E_2(\text{Tail})$

Thus, zero-sum game with no *Pure Strategy NE* since there is no stable outcome if P_1 knows P_2 and P_2 knows P_1 leading to cycle of responses. Players must therefore be **unpredictable** by adopting a *mixed strategy* choosing action based on **probability distribution**.

Player i's strategy σ_i: Plays head with probability p and tail with $1-p$ $\sigma_i = (p, 1-p)$.

Best B_i and NE

Player's' optimal strategy for every possible strategy of the opponent.

$$\begin{cases} p = 0 \text{ (playing Tail)} & \text{if } q < \frac{1}{2} \\ p \in [0,1] & \text{if } q = \frac{1}{2} \\ p = 1 \text{ (playing Head)} & \text{if } q > \frac{1}{2} \end{cases}$$

The best response of Player 2 for σ_1 is:

$$B_2(\sigma_1) = \begin{cases} q = 1 \text{ (playing Head)} & \text{if } p < \frac{1}{2} \\ q \in [0,1] & \text{if } p = \frac{1}{2} \\ q = 0 \text{ (playing Tail)} & \text{if } p > \frac{1}{2} \end{cases}$$

Best response functions leads to Mixed Strategy NE $(\sigma_1^*, \sigma_2^*) = ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$. (Neither player has an incentive to deviate)

1.6 Existence of NE

Nash's Existence

Recall ▶ **Minimax Theorem:** There exists a mixed strategy saddle-point solution for the finite zero-sum game.

Consider the zero-sum finite game A then from *minimax theorem*, there exists $(x^*, y^*) \in \Sigma_1 \times \Sigma_2$ such that, for any two-player, zero-sum game represented by player's 1's payoff matrix A:

$$\max_x \min_y x^T Ay = \min_y \max_x (x^T)Ay = x^{*T} Ay^*$$

where $x^T Ay$ is expected payoff for Player 1 when the players use strategies x *and* y and $x^{*T} Ay^* = v^*$ is value of the game.

Indeed, $(x^*, y^*) \in \Sigma_1 \times \Sigma_2$ is the saddle-point equilibrium

Recall saddle-point equilibrium for zero-sum = NE for zero-sum

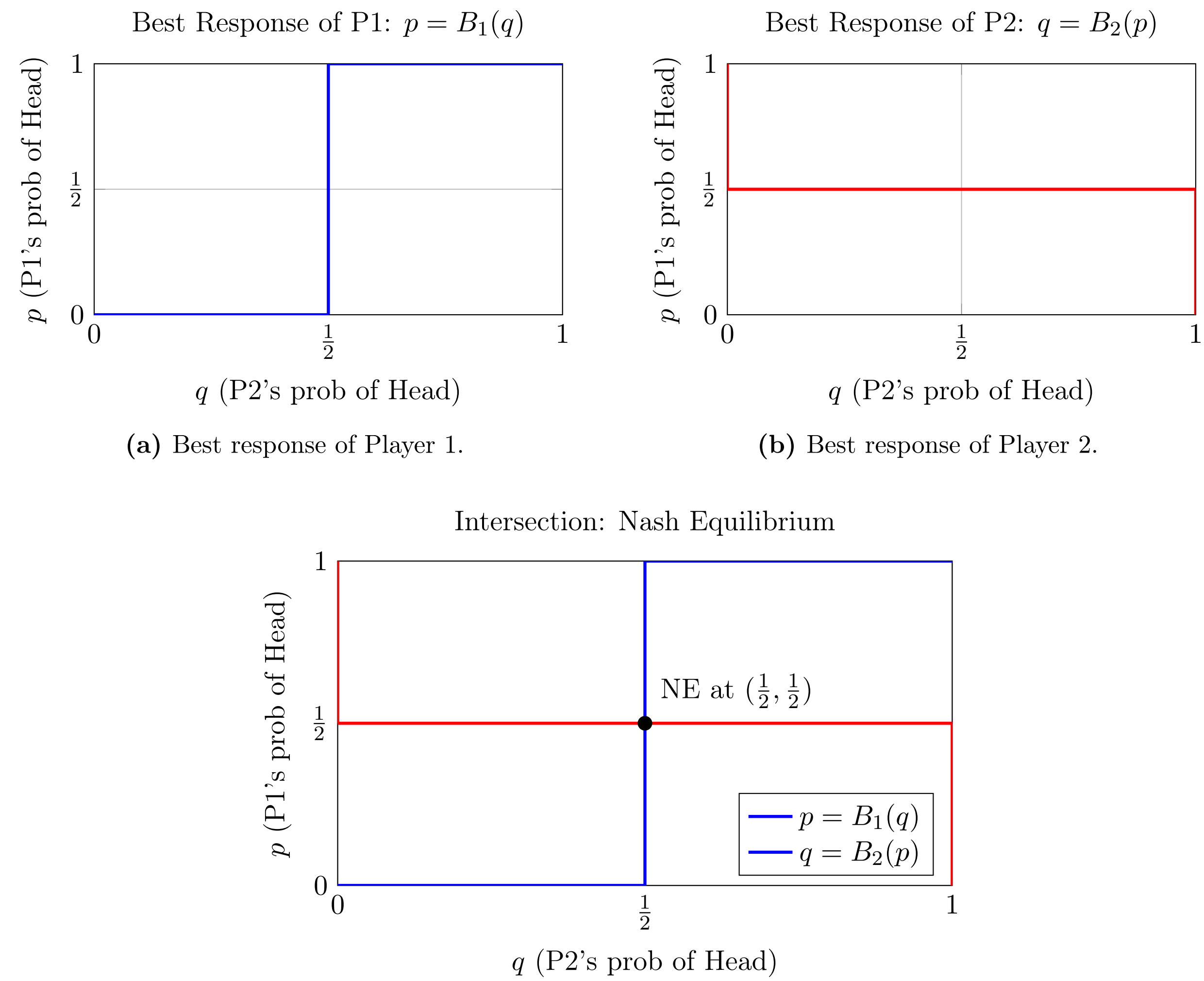
▶ A characterization of saddle-point equilibrium, the finite zero-sum game can be formulated via ***linear program***.

1. Formulate player 1's *maximin* as linear program

2. Find x to solve $max_x(min_y x^T Ay)$

3. Player 2 will respond by *minimize* $x^T Ay$. Therefore, only consider Player 2's *pure strategies.* Player 2 picks *column* j that yields *lowest value*. Thus, $min_y x^T Ay = min_j \sum_{i=1}^n a_{ij} x_{ij}$

4. Let z represent **guaranteed minimum payoff**. Player 1 chooses mixed strategy x that makes guaranteed payoff as high as possible. Thus z must be constraint to $z \le min\{set\}$



(a) Best response of Player 1.

(b) Best response of Player 2.

(c) Intersection of best response functions.

Figure 1.1: Graphical representation of Best Response functions and the resulting Nash Equilibrium.

5. Apply same logic to player 2 (Dual LP).

The **Strong Duality Theorem** of linear programming states: if primal LP has optimal solution, then its dual has optimal solution and thus both objective values are equal.

$$max\ z = min\ w = Minimax$$

Proof. Note $\sum y_i = \sum x_i = 1$ Hence,

$$\min_y x^T Ay = \min_j \sum_{i=1}^n a_{ij}x_i, \quad z \leq \sum_{i=1}^n a_{ij}x_i, \forall j$$
$$\max_x x^T Ay = \max_i \sum_{j=1}^n a_{ij}y_j, \quad \sum_{j=1}^n a_{ij}y_j \leq w \quad \forall i$$

Primal Linear Program (Player 1)

Hence,

$$\max_x \min_y x^T Ay = \max_x \min_j \sum_{i=1}^n a_{ij}x_i$$

Equivalent to

$$\begin{aligned} \max \quad & z \\ \text{subject to} \quad & \\ & z - \sum_{i=1}^n a_{ij}x_i \leq 0, \quad j = 1, 2, \ldots, n \\ & \sum_{i=1}^n x_i = 1, \quad x_i \geq 0, \quad i = 1, 2, \ldots, n \end{aligned}$$

Dual Linear Program (Player 2) Primal linear program for x^* is:

$$\begin{aligned} \max \quad & z \\ \text{subject to} \quad & \\ & z - \sum_{i=1}^n a_{ij}x_i \leq 0, \quad j = 1, 2, \ldots, n \\ & \sum_{i=1}^n x_i = 1, \quad x_i \geq 0, \quad i = 1, 2, \ldots, n \end{aligned}$$

and for y^* (Dual program) is:

$$\begin{aligned} \min \quad & w \\ \text{subject to} \quad & \\ & w - \sum_{j=1}^n a_{ij}y_j \geq 0, \quad i = 1, 2, \ldots, n \\ & \sum_{j=1}^n y_j = 1, \quad y_j \geq 0, \quad j = 1, 2, \ldots, n \end{aligned}$$

▶ Above two LPs are dual to each other

Converting into Standard LP Form: LP formulations into standard matrix

$$\max_x a^T x, \quad \text{subject to} \quad Ax \le c, x \ge 0$$

Let $z = z_1 - z_2$, where $z_1 = z$ and $z_2 = w$ Then

$$\max_{x \in \mathbb{R}^n, z_1, z_2 \in \mathbb{R}} \begin{pmatrix} 1 & -1 & 0_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ x \end{pmatrix}$$

$$\text{subject to} \quad \begin{pmatrix} -1_n & 1_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \ge Ax,$$

$$1_n^T x \le 1, \quad -1_n^T x \le -1$$

Then transform it to standard LP form.

Equivalently 0-$\sum$ game = linear program:

$$\max_{x \in \mathbb{R}^n, z_1, z_2 \in \mathbb{R}} \begin{pmatrix} 0_n & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ z_1 \\ z_2 \end{pmatrix}$$

$$\text{subject to} \quad \begin{pmatrix} 1_n^T & 0 & 0 \\ -1_n^T & 0 & 0 \\ A & 1_n & -1_n \end{pmatrix} \begin{pmatrix} x \\ z_1 \\ z_2 \end{pmatrix} \le \begin{pmatrix} 1 \\ -1 \\ 0_n \end{pmatrix}$$

□

▶ Finding the solution to the saddle-point equilibrium to a 2−player, 0−sum game is = solving a pair of dual linear programs.

Consider the Mathematical Framework of a Game:

1. Strategy Space:
   - $N = \{1, 2, ..., n\}$
   - $S_i = \{s_{i1}, s_{i2}, ..., s_{ik}\}$ (pure strategy) for $i \in N$
   - $u_i : S \to \Re, S = S_1 * S_2 * ... * S_n$ for $i \in N$

   For i, σ_i **mixed strategy** is probability distribution over pure strategies. $\sigma_i = (p_{i1}, p_{i2}, ..., pik)$. p_{ij} is probability of playing pure strategy s_{ij}.

   (a) $p_{ij} \ge 0 \forall j \in \{1, ..., k_i\}$

   (b) $\sum_{j=1}^{k_i} P_{ij} = 1$

Set of all possible mixed strategies for i forms a **standard simplex** Δ_i. *Simplex*: Generalization of triangle to higher dimensions. e.g., $3 * S_i$ is triangle in $3D$ space.

Set of all possible mixed strategy *profiles* is the **Cartesian Product of the individual player's strategy simplices:**

$$\Sigma = \Delta_1 * \Delta_2 * ... * \Delta_n$$

A point $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n) \in \Sigma$ is full profile of mixed strategies, 1 for each $i \in N$

2. Properties of Strategy Space Σ

Consider **Kakutani's Fixed-Point Theorem**:

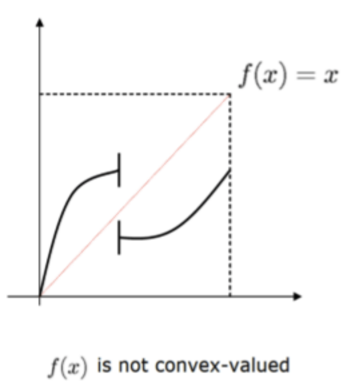
Let $S \subset \Re^n$. Let $f : S \to S$ be the set-valued mapping (correspondence function) with for $x \in S \to f(x) \subseteq S$. Assume the following holds:

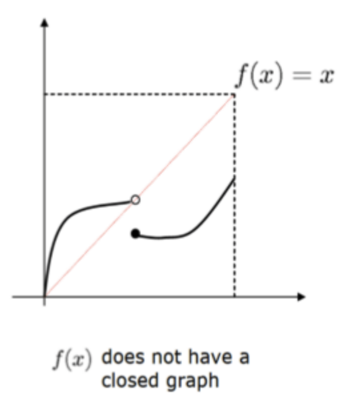
- ▶ S is convex and compact
- ▶ f is non-empty for $x \in S$
- ▶ For any $x \in S, f$ is convex set, i.e., f is a convex-valued mapping (correspondence).
- ▶ f is a closed graph, i.e., if $(x_k, y_k) \to (x, y)$ as $k \to \infty$ with $y_k \in f(x_k)$ then $y \in f(x)$

Then f has a fixed point, i.e., there is $x \in S$ such that $x \in f(x)$

Examples Graphical Illustration Kakutani's:

Conditions of theorem are violated and therefore no fixed point exists. Red line is *identity line*. A fixed point exists wherever the graph of the mapping $f(x)$ intersects identity line





Right: Mapping has a jump. As one approaches jump from left, output values on the curve approach the position of the open circle. AT that very input value the function's output is the filled circle. Therefore, the limit of the outputs(hollow) **is not** an element of the actual output set.

Goal: To find a fixed point. A **fixed point** of a mapping is an input that is also part of its own output. Thus a fixed point exists if $x \in f(x)$, this corresponds to NE. A set-valued mapping (correspondence) maps x to a whole *set of points*, $f(x)$.

3. Best-Response (set-valued mapping) Correspondence

$$u_i(\sigma) = \sum_{s \in S} \left(\prod_{j=1}^{n} \sigma_j(s_j) \right) u_i(s) \tag{1.1}$$

where $\sigma_j(s_j)$ is probability player j plays pure strategy s_j. Function is linear in each player's own probabilities and therefore continuous over entire space Σ.

Define**best-response correspondence** for player i, B_i. It takes the strategies of all other players, $\sigma_{-i} \in \Sigma_{-i}$, as input and returns set of all of player i's mixed strategies that yield the maximum possible payoff.

$$B_i(\sigma_{-i}) = \{\sigma_i^* \in \Delta_i \mid u_i(\sigma_i^*, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i}) \text{ for all } \sigma_i \in \Delta_i\} \tag{1.2}$$

Term **correspondence** or **set-valued function** instead of "function" because there might be multiple best responses e.g., when player is indifferent between several strategies.

2 Correlated Equilibria

2.1 Correlated Equilibrium

► Introduction of **Correlating Device**: It sends private signals to each player and signals from different players can be correlated. e.g., Traffic light is correlating device: if it signals Green to north-south traffic, it simultaneously signals Red to east-west traffic.

► *Solution concept* that generalizes NE. Description of stable outcome in a game where players coordinate their actions based on shared, external, and random correlating signal.

Signal recommends a pure strategy to each player and equilibrium holds if no player has incentive to unilaterally deviate from recommended strategy.

Incentive Compatibility (Mechanism): if $\forall i \in N$ can achieve own best outcome by reporting their *true* preferences.

Definition 2.1: Correlated Equilibrium

Consider a finite n-player game $\langle \mathcal{N}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$

- $N = \{1, ..., n\}$
- Set S_i of pure strategies for player i. $s = (s_1, ..., s_n)$
- Set $S = S_1 * ... * S_n$ of pure strategy profiles
- $u_i : S \to \mathbb{R}$

A **Correlated Equilibrium** is probability distribution p over set of all pure strategy profiles S.

$$p : S \to [0, 1] \text{ such that } \textstyle\sum_{s\in S} P(s) = 1$$

where $p(s)$ is probability recommended by correlating device. p must satisfy *incentive compatibility* constraint $\forall i \in N$ and $\forall s_i \in S_i$.

$$\sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} p(s_i', s_{-i}) u_i(s_i', s_{-i}) \quad \forall i \in N, \forall s_i, s_i' \in S_i \tag{1.3}$$

CHAPTER 2

MATHEMATICAL PROOFS

1 Set Theory

Set: *A collection of objects considered as a single object.*

- **Open Interval** ( ): (a, b) represents all $\Re$ x such that $a < x < b$.
- **Closed Interval** [ ]: $[a, b]$ represents all $\Re$ x such that $a \leq x \leq b$.
- **Half-Open/Half-Closed Intervals**: $[a, b)$ means $a \leq x < b$, and $(a, b]$ means $a < x \leq b$.

Disjoint: $A \cap B = \emptyset$

Difference: $A - B$ or $A/B = \{x : x \in A \text{ and } x \notin B\}$

Example : Set operations
Let $A = \{x \in \mathbb{R} : |x| \leq 3\}$, $B = \{x \in \mathbb{R} : |x| > 2\}$ and $C = \{x \in \mathbb{R} : |x - 1| \leq 4\}$.

1. Express A, B and C using interval notation.
2. Determine $A \cap B$, $A - B$, $B \cap C$, $B \cup C$, $B - C$ and $C - B$.

Solution

1. $A = [-3, 3]$, $B = (-\infty, -2) \cup (2, \infty)$ and $C = [-3, 5]$ *(For C, $-4 \leq x - 1 \leq 4$).*
2. $A \cap B = [-3, -2) \cup (2, 3]$, $A - B = [-2, 2]$, $B \cap C = [-3, -2) \cup (2, 5]$, $B \cup C = (-\infty, \infty)$, $B - C = (-\infty, -3) \cup (5, \infty)$ and $C - B = [-2, 2]$.

Complement:

All elements that are *not in* the given set but are *within* a defined **universal set**.

Consider universal set U. For a set A, its **complement** is: $\overline{A} = U - A = \{x : x \in U \text{ and } x \notin A\}$.

If $U = \mathbb{Z}$, then $\overline{\mathbb{N}} = \{0, -1, -2, \dots\}$; while if $U = \mathbb{R}$, then $\overline{\mathbb{Q}} = \mathbb{I}$.

Key Properties of Complements

Let U be the universal set and A and B be subsets of U.

- **Union with Original Set:** A set and its complement, when united, form the universal set:

$$A \cup \overline{A} = U$$

- **Intersection with Original Set:** A set and its complement are always disjoint (they have no elements in common):

$$A \cap \overline{A} = \emptyset$$

- **Double Complement:** The complement of the complement of a set is the original set itself:

$$\overline{(\overline{A})} = A$$

- **Complement of Universal Set:** The complement of the universal set is the empty set:

$$\overline{U} = \emptyset$$

- **Complement of Empty Set:** The complement of the empty set is the universal set:

$$\overline{\emptyset} = U$$

- **De Morgan's Laws:** These important laws relate complements to unions and intersections:
  - The complement of a union is the intersection of the complements:

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$

  - The complement of an intersection is the union of the complements:

$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

1.1 Indexed Collections of Sets

Definition 1.1: Union $A \cup B \cup C$

$$A \cup B \cup C = \{x : x \in A_i \text{ or } x \in B, \text{ or } x \in C\}$$

Definition 1.2: Union of sets (set of sets)

To consider the union of several sets: The union of $n \geq 2$ sets $A_1, A_2, \ldots, A_n$ is denoted by $A_1 \cup A_2 \cup \cdots \cup A_n$ or $\bigcup_{i=1}^{n} A_i$,

$$\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for some } i, 1 \leq i \leq n\}.$$

Thus, for element a to belong to $\bigcup_{i=1}^{n} A_i$, a must belong to at least one of the sets $A_1, A_2, \ldots, A_n$.

Example : Union of sets

Let $B_1 = \{1,2\}$, $B_2 = \{2,3\}$, $\ldots$, $B_{10} = \{10,11\}$; that is, $B_i = \{i, i+1\}$ for $i = 1, 2, \ldots, 10$. Determine each of the following:

(a) $\bigcup_{i=1}^{5} B_i$.

(b) $\bigcup_{i=1}^{10} B_i$.

(c) $\bigcup_{i=3}^{7} B_i$.

(d) $\bigcup_{i=j}^{k} B_i$, where $1 \leq j \leq k \leq 10$.

Solution

(a) $\bigcup_{i=1}^{5} B_i = \{1, 2, \ldots, 6\}$.

(b) $\bigcup_{i=1}^{10} B_i = \{1, 2, \ldots, 11\}$.

(c) $\bigcup_{i=3}^{7} B_i = \{3, 4, \ldots, 8\}$.

(d) $\bigcup_{i=j}^{k} B_i = \{j, j+1, \ldots, k+1\}$.

1.2 Partitions of Sets

Recall 2 sets are disjoint if their intersection is the empty set. A collection S of subsets of a set A is **pairwise disjoint** if every 2 distinct subsets that belong to S are disjoint. ($element_1 \cap element_2 \cap element_n = \emptyset$).

Partition of A: Collection S of nonempty subsets of A such that $\forall x_i \in A$ belongs exactly 1 subset in S.

1. $X \neq \emptyset \ \forall set X \in S$

2. for every 2 sets $X, Y \in S$, either $X = Y$ or $X \cap Y = \emptyset$

3. $\bigcup_{X \in S} X = A$

Example : Partition

Consider collection of subsets of set $A = \{1, 2, 3, 4, 5, 6\}$:

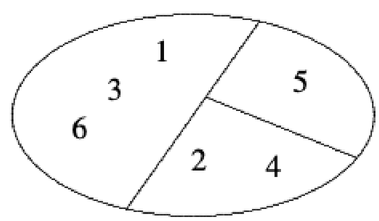
$$S_1 = \{\{1, 3, 6\}, \{2, 4\}, \{5\}\};$$
$$S_2 = \{\{1, 2, 3\}, \{4\}, \emptyset, \{5, 6\}\};$$
$$S_3 = \{\{1, 2\}, \{3, 4, 5\}, \{5, 6\}\};$$
$$S_4 = \{\{1, 4\}, \{3, 5\}, \{2\}\}.$$

Determine which of these sets are partitions of A.

The set S_1 is a partition of A. The set S_2 is not a partition of A since $\emptyset$ is one of the elements of S_2. Set S_3 is not a partition of A since the element 5 belongs to two distinct subsets in S_3 ($\{3, 4, 5\}$, $\{5, 6\}$). S_4 is not a partition of A because element 6 belongs to no subset in S_4. ■

A partition of a nonempty set A is a division of A into nonempty subsets.

Partition S_1 of set A:



The set $\mathbb{Z}$ of integers can be partitioned into the set of even integers and the set of odd integers. The set $\mathbb{R}$ of real numbers can be partitioned into the set $\mathbb{R}^+$ of positive real numbers, the set of negative real numbers and the set $\{0\}$ consisting of the number 0. $\mathbb{R}$ can also be partitioned into the set $\mathbb{Q}$ of rational numbers and the set $\mathbb{I}$ of irrational numbers. ■

1.3 Cartesian Products of Sets

The **Cartesian product** $A * B$ of 2 *sets* A and B is the set consisting of all **ordered pairs** whose first coordinate belongs to A and whose second belongs to B:

$A * B = \{(a, b) : a \in A \text{ and } b \in B\}$

Example : Cartesian
If $A = \{x, y\}$ and $B = \{1, 2, 3\}$, then

$$A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\},$$

while

$$B \times A = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}.$$

Since, for example, $(x, 1) \in A \times B$ and $(x, 1) \notin B \times A$, these two sets do not contain the same elements; so $A \times B \neq B \times A$. Also,

$$A \times A = \{(x, x), (x, y), (y, x), (y, y)\}$$

and

$$B \times B = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}. \quad \blacksquare$$

Note if $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$. The Cartesian product $\mathbb{R} \times \mathbb{R}$ is the set of all points in the Euclidean plane.

Consider: The graph of the straight line $y = 2x + 3$ is the set

$$\{(x,y) \in \mathbb{R} \times \mathbb{R} : y = 2x + 3\}.$$

For the sets $A = \{x, y\}$ and $B = \{1,2,3\}$, $|A| = 2$ and $|B| = 3$; while $|A \times B| = 6$. Indeed, for all finite sets A and B,

$$|A \times B| = |A| \cdot |B|.$$

2 Logic

'Are there connections between 2 given mathematical concepts? If so, what are they?', 'Under what conditions does an object possess a particularly property?'...

2.1 Stements

P, Q, R used to denote statements:

► P_1 : The integer 3 is odd and P_2: The integer 57 is prime are statements where P_1 has truth value T and P_2 has truth value F

Imperative (commands) sentences, interrogative or exclamatory are NOT statements since they're not declarative.

Open Sentence: Declarative sentence that contains one or more variables, each representing a value in some prescribed set, the **domain** of the variable, and which becomes a statement when values from their respective domains are substituted for these variables.

Example : Open Sentence

$$P(x,y) : |x+1| + |y| = 1$$

Suppose domain of x is $S = \{-2,-1,0\}$ and domain of y is $T = \{-1,0,1\}$

Then,

$$P(-1,1) : |-1+1| + |1| = 1$$

is $TRUE$, while

$$P(1,-1) : |1+1| + |-1| = 1$$

is $FALSE$

$P(x, y)$ is a *true* statement when

$$(x, y) \in \{(-2, 9), (-1, -1), (-1, -1), (0, 0)\}$$

while it is a $false$ statement for all other elements $(x, y) \in S * T$ ■

The possible values of a statement are usually listed in a **truth table**. There are 2 possible truth values for P and Q, thus there are 4 possible combinations of truth values for P and Q

P
T
F

Q
T
F

P	*Q*
T	T
T	F
F	T
F	F

P	*Q*	*R*
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

► A truth table involving n statements $P_1, P_2, ..., P_n$ contains 2^n possible combinations of truth values for statements and a truth table would have n columns and 2^n rows.

2.2 Negations

Example : negation
For the statement

$$P_1 : \text{The integer 3 is odd.}$$

described above, we have

$$\sim P_1 : \text{The integer 3 is not odd.}$$

or better yet to write

$$\sim P_1 : \text{The integer 3 is even.}$$

Similarly, the negation of the statement

$$P_2 : \text{The integer 57 is prime.}$$

considered above is

$$\sim P_2 : \text{The integer 57 is not prime.}$$

Note that $\sim P_1$ is false, while $\sim P_2$ is true. ♦

Indeed, the negation of a true statement is always false and the negation of a false statement is always true; that is, the truth value of $\sim P$ is opposite to that of P. Truth table for $\sim P$ (in terms of the possible truth values of P):

P	$\sim P$
T	F
F	T

2.3 Disjunctions and Conjunctions

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

2.4 Implications/Conditional

A statement formed from 2 given statements. For statements P and Q, the **implication** is the statement:

If P, **then** Q or *implies* $\Longrightarrow$

P	**Q**	$\mathbf{P \Rightarrow Q}$
T	T	T
T	F	F
F	T	T
F	F	T

►A true premise cannot lead to a false statement in a valid deductive argument

The validity of a material implication is determined by its **truth table**. Let P be the premise and Q be the conclusion. The statement $P \Rightarrow Q$ is considered false only *when a true premise leads to a false conclusion.* In all other cases, the implication is true.

Example : Material Implication

Define the statements:

- **P**: You earn an A on the final exam
- **Q**: You receive an A for your final grade

The instructor's promise is the implication: "If you earn an A on the final exam, then you will receive an A for your final grade," or $P \Rightarrow Q$.

1. **Case 1: P is True, Q is Truth**
   You get an A on the exam, and you get an A in the course. The instructor kept their promise. The implication $P \Rightarrow Q$ is **true**.

2. **Case 2: P is True, Q is False.**

You get an A on the exam, but you do not get an A in the course. The instructor broke their promise. This is the only scenario where the promise was not kept. The implication $P \Rightarrow Q$ is **false**.

3. **Case 3: P is False, Q is True.**
   You do not get an A on the exam, but you still get an A in the course. The instructor did not break their promise. The promise was only about what would happen *if* you got an A on the final. It didn't say what would happen if you didn't. Therefore, the implication $P \Rightarrow Q$ is **true**. This is often called the **Law of Implication** or the **Principle of Vacuous Truth**. The promise was not tested, so it cannot have been broken.

4. **Case 4: P is False, Q is False.**
   You do not get an A on the exam, and you do not get an A in the course. Instructor's promise was not broken because the condition (getting an A on the exam) was not met. The implication $P \Rightarrow Q$ is **true**.

Phrasing and Terminology in Proofs

The material implication $P \Rightarrow Q$ is the backbone of most mathematical theorems

- **If P, then Q.** This is the most direct phrasing.

- **P implies Q.** This is synonymous with "If P, then Q."

- **P only if Q.** This means that P can only be true when Q is also true. If Q were false, P could not be true. This captures the essence of the second row of the truth table (T, F, F).

- **P is sufficient for Q.** The truth of P is enough (sufficient) to guarantee the truth of Q. Knowing P is true means you know Q must also be true. This is the direct meaning of a proof: showing the premises are sufficient for the conclusion.

- **Q is necessary for P.** The truth of Q is required (is a necessity) for P to be true. If Q is false, then P must also be false. This is also known as the **contrapositive**. The statement $P \Rightarrow Q$ is logically equivalent to its contrapositive, $\neg Q \Rightarrow \neg P$ (If not Q, then not P).

CHAPTER 3

COMBINATORICS

1 Basic Methods

Theorem 1.1: Pigeon-hole Principle

Let n and k be positive integers, and let $n > k$. Suppose we have to place n identical balls into k identical boxes. Then there will be at least one box in which we place at least two balls.

Proof for Theorem.

Assume that statement $= FALSE =$ No box ≥ 2 balls. Therefore, k boxes contain either 0 or 1 ball.

Let m be the number of boxes that have zero balls. Then $m \geq 0$.

Number boxes with 1 ball $= k - m$. Total number balls placed in k boxes is $1 \cdot (k - m) + 0 \cdot m = k - m$.

We are given that we placed n balls into the boxes. So, total number of balls is n. Therefore, we must have $n = k - m$.

Since $m \geq 0$, it follows that $k - m \leq k$. Thus, $n \leq k$.

This contradicts our initial assumption that $n > k$. Therefore, our assumption that there is no box with at least two balls must have been false, and consequently, there is at least one box with at least two balls. ■

'If you have more items ("pigeons") than you have containers ("pigeonholes"), then at least one container must hold more than one item.'

Example : Pigeonhole Principle and modular arithmetic
For infinite sequence of numbers $a1 = 7, a_2 = 77, a_3 = 777, ...$, proof there is at least 1 number that is perfectly divisible by 2023. Proof even stronger statement: One of the first 2023 elements of the sequence must be divisible by 2023.

1. **Proof by Contradiction:** Assume **no element** in the sequence is divisible by 2023. Show

assumption leads to logical impossibility.

2. **Pigeons and pigeonholes:** First 2023 = pigeons. Now, consider the **remainder** of numbers divided by 2023. Based on initial assumption, remainder **cannot be 0** and therefore only possible remainders are integers from 1 t0 2022. Thus exactly **2022 possible values** for remainder = Pigeonholes.

   e.g.,

   10 / 5 = 2 remainder 0
   11 / 5 = 2 remainder 1
   12 / 5 = 2 remainder 2
   13 / 5 = 2 remainder 3
   14 / 5 = 2 remainder 4
   15 / 5 = 3 remainder

   Remainder can never be ≥ 5. Therefore, for any divisor D remainder set = $\{0, 1, 2, ..., 2022\}$ (2023 elements).

3. **Apply principle:** Pigeons > pigeonhole. The principle guarantees $\geq$ 2 pigeons in 1 pigeonhole. Meaning, 2 elements $\geq$ must be placed in the same pigeon's hole a_j and a_i, where $j > i$ must have **exact same remainder** when divided by 2023.

   $a_j = 2023 * k_j + r$

   $a_i = 2023 * k_i + r$ (where k_j and k_i are integers)

   $a_j - a_i = (2023 * k_j + r) - (2023 * k_i + r) = 2023(k_j - k_i)$, proving difference $a_i - a_j$ **must be perfectly divisible by 2023.**

   e.g., given $j = 5$ and $i = 2$, $a_5 = 77777$
   $a_2 = 77$
   $a_5 - a_2 = 77700 = a_3$
   $777 * 100 = a_3 * 10^2$

   $a_j - a_i = a_{j-i} * 10^i$

4. **Final contradiction:** a_{j-1} must be divisible by 2023. *'If integer N divides product $A * B$ and N shares no common factors **(relatively prime)** with B, then N must divide A.* Thus 2023 and 10^i are relatively prime and 2023 divides product $a_{j-i} * 10^i$ and prime to 10^i, it **must divide** a_{j-i}.

   a_{j-i} consists of $(j - i)$ sevens which means it is element of original sequence $(a1 = 7, a_2 = 77, a_3 = 777, ...,)$. THUS assumption MUST be $FALSE$ and original statement MUST be $TRUE$.

■

Theorem 1.2: General version, Pigeon-Hole

Let n, m, r be positive integers so that $n > rm$, and let us distribute n identical balls into m identical boxes. There will be at least 1 box into which we place at least $r + 1$ balls.

Proof for Theorem.

Assume contrary statement. Then each of the m boxes can hold at most r balls, so all boxes can hold at most $rm < n$ balls, which contradicts the requirement that we distribute n balls. ■

Example : Geometric application

$\geq 2/10$ within 0.48

Given:

- Square of unit size (1*1)
- 10 points placed anywhere within square

Prove:

- There must be $\geq 2/10$ points that are closer to each other than a distance of 0.48
- There must be $\geq 3/10$ points that can be covered by a single disk of radius 0.5

Apply pigeon hole principle:

- **Pigeon holes:** First divide unit square into 9, thus each box 1/3 side length $\rightarrow$ pigeon holes. 10 total points **(pigeons)** into 9 holes. *At least* 1 *hole must contain* $>$ *1 pigeon.* Therefore, **at least** $1/9$ **squares must contain** $2/10$ **points.**

Calculate max. distance within square(hole):

- Longest distance between any 2 points inside a square is length of diagonal.
- Since $a^2 + b^2 = c^2$ then diagonal$^2 = 2/9$. Diagonal is then $\sqrt{2}/3 \approx 0.4714$. Diagonal < 0.48.
- Maximum possible distance between 2 points in same square is < 0.48 thus 2 points *must* exist in the square and these are closer to each other than 0.48

$\geq 3/10$ covered by disk of radius 0.5

Apply pigeonhole principle:

- Divide square into 4 equal triangles using 2 main diagonals (**pigeonholes**).
- If N items are put into k containers then at least 1 container must hold at least $\lceil N/k \rceil$ items, where $\lceil \cdot \rceil$ denotes ceiling function (rounding up)

  $N = 10$ (points) and $k = 4$ (triangles). The calculation is:

$$\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{10}{4} \right\rceil = \lceil 2.5 \rceil = 3$$

Thus at least 1/4 triangles must contain *at least* 3 *points*.

Geometric Argument - The Circumcircle The **circumcircle** of a triangle is the unique circle that passes through all three of its vertices. Key property is entire area of triangle is contained within its circumcircle. Therefore, proof that the circumcircle of each of 4 triangles has radius ≤ 0.5.

- Imagine coordinates with set $V = \{(0,0),(1,0),(1,1),(0,1)\}$. 2 diagonals intersect at center of square $(0.5, 0.5)$.
- Consider e.g., triangle with vertexes at $(0,0),(0,1)$ and center $(0.5,0.5)$. Circumcircle is thus circle that passes through said points.
- Center of circle can be found at $(0.5, 0)$. R is thus

$$R = \text{distance}((0.5,0),(0,0)) = \sqrt{(0.5-0)^2 + (0-0)^2} = \sqrt{0.5^2} = 0.5$$

- By symmetry, all triangles formed by diagonals have circumcircle of $R = 0.5$. It then follows that there must be 3 points that can be covered by a disk of radius 0.5.

Example : Integers
Prove that among eight integers, there are always two whose difference is divisible by seven.

1. Identify remainders (pigeonholes) = 7
2. Assign integers to remainders (pigeons to pigeonholes) = 8
3. Apply principle $\rightarrow$ at least 2 integers must have the **same remainder**
4. $a = 7k_1 + r$ and $b = 7k_2 + r = a - b = 7(k_1 - k_2)$ Difference is also integer thus $a - b$ is a multiple of 7

CHAPTER 4

LINEAR ALGEBRA

1 Linear Algebra in Probability & Statistics

1.1 Recap

$$V = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} = \text{diagonal covariance matrix}$$

where, σ_1^2 and σ_2^2 are variances of variables $\rightarrow$n spread from their mean

$$\sigma_{12} = \sum_i \sum_j (p_i)(p_j)(x_i - m_1)(y_j - m_2) = \left[\sum_i (p_i)(x_i - m_1)\right] \left[\sum_j (p_j)(y_j - m_2)\right] = [0][0]$$

and σ_{12} ($= \sigma_{21}$) is **covariance** between 2 variables - how both change together. Thus, $\sigma_{21} = 0 =$ uncorrelated

$$V = \sum \sum V_{ij} \quad V = \sum_{\text{all } i,j} p_{ij} \begin{bmatrix} (x_i - m_1)^2 & (x_i - m_1(y_j - m_2) \\ (x_i - m_1)(y_j - m_2) & (y_j - m_2)^2 \end{bmatrix}$$

► A real symmetric matrix A is positive **semidefinite** if for any non-zero column vector z:

$z^T A z \geq 0$ thus A *all eigenvalues are non-negative.*

This ensures that transformation doesn't reflect or invert space in a way that produces negative scaling

If inequality strict ($z^T A z > 0$) $\forall z \neq 0$ matrix is **positive definite**.

$$V = \iiint p(x, y, z) U U^T dx\, dy\, dz \quad \text{with} \quad U = \begin{bmatrix} x - \bar{x} \\ y - \bar{y} \\ z - \bar{z} \end{bmatrix}$$

$p(x, y, z) = p_1(x)p_2(y)p_3(z)$

Perfect linear dependency: $p(x, y, z) = 0 \quad \text{except when} \quad cx+dy+ez = 0$ Cvarian matrix is singular (det $= 0$) and $\neg$ diagonal

$$UU^T = \begin{bmatrix} (x-\bar{x})^2 & (x-\bar{x})(y-\bar{y}) & (x-\bar{x})(z-\bar{z}) \\ (y-\bar{y})(x-\bar{x}) & (y-\bar{y})^2 & (y-\bar{y})(z-\bar{z}) \\ (z-\bar{z})(x-\bar{x}) & (z-\bar{z})(y-\bar{y}) & (z-\bar{z})^2 \end{bmatrix}$$

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y} = \text{covariance of } \frac{x}{\sigma_x} \text{ and } \frac{y}{\sigma_y}$$

$$-1 \leq \rho_{xy} \leq 1$$

$$R = \begin{bmatrix} 1 & \rho_{xy} \\ \rho_{xy} & 1 \end{bmatrix}$$

where ρ_{xy} **Pearson correlation coefficient**: measure of strength and direction of linear association between 2 random variables. Value is always bounded $|1|$.

Standardization reframes it as measure of how 2 variables move together independent of their scales.

$$R = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{when } y = -x$$

$\rightarrow$ perfect but inverse linear dependency

Thus, *Covariance* (unbounded covariances and variances $[0, \infty)$) is raw directional relationship between variables whereas *Correlation* (bounded) is standardized scale-independent measure of such linear relationship.

1.2 Multivariate Gaussian and Weighted Least Squares

Generalization of normal distribution to multiple dimensions:

$$p(\mathbf{x}) = \frac{1}{(\sqrt{2\pi})^M \sqrt{\det V}} e^{-(\mathbf{x}-\mathbf{m})^T V^{-1} (\mathbf{x}-\mathbf{m})/2}$$

For vector X containing M variables, $X = [x_1, x_2, ..., x_M]^T$

►Shape/orientation of ellipsoidal distribution is determined by covariance matrix V