SUBJECT NOTES

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CONTENTS

CHAPTER 1

FINITE GAMES/NASH EQUILIBRIA

Strategic form/normal form/matrix games: Games in which all participants act simultaneously and without knowledge of other players' actions.

- Set of players (agents)
- Set of actions
- Set of payoff/utility functions
- Information structure players can access

1 Finite Games

Definition 1.1: Strategic form of game/Finite game

A strategic forms game is a triplet $\langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ such that

- $\mathcal{I}$: finite number of players, where $N = \mathcal{I} = \{1, 2, \ldots, n\}$
- S_i: set of actions (decisions, strategies) for player i
- $s_i \in S_i$: actions (decisions, strategies) for player i
- $u_i : S \longrightarrow \mathbb{R}$: the payoff (utility) function of player i, where $S = \prod_{i=1}^{n} S_i$ is the set of actions of all players

Notation

- $s = (s_1, \ldots, s_n) \in S = \prod_{i=1}^{n} S_i = S_1 \times S_2 \times \cdots \times S_n$
- s: decision/action/strategy profile
- $s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, s_{i+2}, \ldots, s_n)$

Strategy: Complete description of how to play the game. Requires full contingent planning (full description how to play in every contingency).

General Setup of n-Player Finite Game

- Players: n-players with $i \in N = \{1, 2, \ldots, n\}$
- decision/action/strategy for Player i: $s_i \in S_i$
  - S_i is a finite set
- $s = (s_1, \ldots, s_n) \in S = S_1 \times S_2 \times \cdots \times S_n$
  - s: decision/action/strategy profile
- $s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, s_{i+2}, \ldots, s_n)$
- Payoff function: $u_i(s_i, s_{-i})$ with $u_i : S \to \mathbb{R}$
  - Each player has to maximize u_i over $s_i \in S_i$
- Player 1 and Player 2
- $S_1 = \{1, \ldots, p\}$ finite set
- $S_2 = \{1, \ldots, m\}$ finite set
- $u_1 : p \times m$ matrix
- $u_2 : m \times p$ matrix
- Zero-sums game
  - When $u := u_1 = -u_2$
  - Player 2 plays minimizing u

		Player 2 D	Player 2 E	Player 2 F
Player 1	A	(a, b)	(c, d)	(e, f)
Player 1	B	(g, h)	(i, j)	(k, l)
Player 1	C	(m, n)	(o, p)	(q, r)

- Player 1 chooses row with respect to the first component $X_1 = \{A, B, C\}$
- Player 2 chooses column with respect to the second component $X_2 = \{D, E, F\}$

2 Dominant Equilibrium: Optimality of Game

For i player, a dominant strategy is one that yields the highest payoff, *regardless of other players' actions*. A **Dominant Strategy Equilibrium** occurs when *every* player has a clear best choice irrespective of others' and there is this no incentive for any player to deviate. (e.g., Prisoner's Dilemma).

Nash Equilibrium: Set of strategies, for each player, such that *no player can improve payoff by unilaterally changing only their own strategy* (assuming all players stick to chosen strategies). Note: a game can have multiple Nash equilibria and they don't always mean the best possible collective outcome for all players.

Every dominant strategy equilibrium is also Nash equilibrium. NOT the other way around though.

Definition 2.1: Dominant strategy

A strategy $s_i \in S_i$ is dominant for Player $i \in N$ if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \quad \forall (s_i', s_{-i}) \in S_i \times S_{-i}$$

where, s_{-i} is collection of strategies chosen by all players except player i.

Definition 2.2: Dominant equilibrium

A strategy profile $s^* \in S$ is the dominant strategy equilibrium if for each Player $i \in N$, $s_i^* \in S_i$ is the dominant strategy.

- We observe that "Confess" is the dominant equilibrium in Prisoner's dilemma game
- Rational players will choose the dominant strategy

Definition 2.3: Strictly dominated strategy

A strategy $s_i \in S_i$ is strictly dominated for player $i \in N$ if there exists some $s_i' \in S_i$ such that

$$u(s_i', s_{-i}) > u(s_i, s_{-i}), \quad \forall s_{-i} \in S_{-i}$$

- Can obtain the dominant equilibrium by eliminating strictly dominated strategies (**iterated elimination of strictly dominated strategies (IESDS)**).
- Rational players do not choose the strictly dominated strategy

Therefore, if there exists another strategy s_i' such that choosing s_i' *always* yields a strictly higher payoff for player i. regardless of what strategies other players s_{-i} choose.

IESDS

Method to simplify a game and find a solution/equilibrium.

Let S_j^k be set of strategies for player j that have survived elimination up to iteration k.

Let $S_{-i}^k = X_{j \neq i} S_j^k$ be set of strategy profiles for players other than i using strategies available at iteration k.

Pseudocode:

```
    Initialize S_i_current = S_i for all players i in N
Set strategies_eliminated_this_round = true

WHILE strategies_eliminated_this_round == true:
    Set strategies_eliminated_this_round = false
```

```
    FOR EACH player i in N:
        Let S_i_next_round = S_i_current
        FOR EACH strategy s_prime_i in S_i_current:
            Set is_dominated = false
            FOR EACH strategy s_double_prime_i in S_i_current (where s_double_prime_i != s_prime
                Set s_double_prime_dominates_s_prime = true
                // Check if s_double_prime_i strictly dominates s_prime_i
                // against all combinations of opponents' current strategies S_minus_i_current
                FOR EACH strategy_profile_s_minus_i in S_minus_i_current:
                    IF u_i(s_double_prime_i, s_minus_i) <= u_i(s_prime_i, s_minus_i):
                        s_double_prime_dominates_s_prime = false
                        BREAK // s_double_prime_i does not dominate s_prime_i w.r.t. this s_minu
                IF s_double_prime_dominates_s_prime == true:
                    is_dominated = true
                    BREAK // s_prime_i is dominated by s_double_prime_i
            IF is_dominated == true:
                Remove s_prime_i from S_i_next_round
                strategies_eliminated_this_round = true
        Set S_i_current = S_i_next_round // Update player i's strategy set for this iteration

Output: The final sets S_i_current for all players.
```

3 Nash Equilibrium: Optimality of Game

- N-player noncooperative game
- Rationality and optimality are key underlying assumptions
- No incentive to deviate once every player is in Nash

Definition 3.1: Nash Equilibrium (state)

The strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in S$ is called a Nash equilibrium of the game if for all i, $i = 1, 2, \ldots, n$,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \quad \forall s_i \in S_i$$

Thus, no single player has an incentive to change only their own strategy. If player i unilaterally deviates from s^* to s_i, while $-i$ stick to s^*, player i *will NOT* achieve a strictly better payoff (either same or worse).

Definition 3.2: Best response function (tool)

The best response function **(correspondence)** $B_i(s_{-i})$ is defined by $B_i : S_{-i} \to S_i$

$$B_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$$

$$= \{s_i \in S_i \mid u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall s_i' \in S_i\}$$

- It is sometimes correspondence, since given $s_{-i} \in S_{-i}$, there can be multiple $s_i \in S_i$
- It is a multi-valued (set-valued) function

B_i defines (set) strategy(s) such that player i's payoff is maximized, *given* $-i$ are playing s_{-1}. Output of B_i can be a set of strategies too if multiple yield same maximum payoff.

A strategy $s^* = (s_1^*, s_2^*, ..., s_n^*)$ is **Nash Equilibrium** if every player's strategy in that profile is a best response to the strategies of all other players in that profile. Thus, $\forall i \in N$: $s_i^* \in B_i(s_{-1}^*)$:

Proposition 3.3

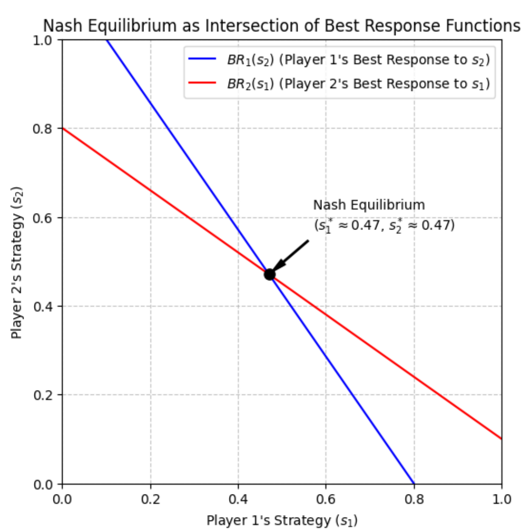
The strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in S$ is a Nash equilibrium of the game if and only if

$$s_i^* \in B_i(s_{-i}^*), \quad \forall i \in N = \{1, 2, \ldots, n\}$$

Proof for Proposition.

- If part: since $s_i^* \in B_i(s_{-i}^*)$ for all $i \in N$, the result is true by definition
- Only if part: since $s^* \in S$ is a NE, the result follows from definition of the best response function

Current strategy maxed out payoff = *No* incentive to unilaterally change strategy.



Definition 3.4: Nash equilibrium for two player game

Simplified to two player game ($n = 2$)
The strategy profile $s^* = (s_1^*, s_2^*) \in S_1 \times S_2$ is called a Nash equilibrium of the game if

$$u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^*), \quad \forall s_1 \in S_1$$
$$u_2(s_1^*, s_2^*) \geq u_2(s_1^*, s_2), \quad \forall s_2 \in S_2$$

Each player wants to choose their strategy to maximize their own payoff, keeping in mind that the other player is also trying to do the same.

Core Idea: No Unilateral Incentive to Deviate (No Regrets)

e.g., Prisoner's Dilemma (PAyoff does not lead to best result at NE):

'The dilemma is that individual rationality (each prisoner choosing their dominant strategy to minimize their own sentence) leads to a collectively suboptimal outcome where both are worse off than if they had managed to cooperate. Even if they had agreed beforehand to Stay Silent, the incentive to betray the other for a chance at freedom (or a reduced sentence if the other also betrays) is very strong. This highlights the conflict between individual incentives and mutual benefit, and the difficulty of achieving cooperation in the absence of trust and binding agreements.'

4 Saddle-Point Equilibrium

Definition 4.1: Saddle-Point Equilibrium

NE for a **2-player zero-sum** game ($u = u_1 = -u_2$)
Strategy profile $s^* = (s_1^*, s_2^*) \in S_1 * S_2$ is a saddle-point equilibrium of 2-player game if

$$u(s_1, s_2^*) \leq u(s_1^*, s_2^*) \leq u(s_1^*, s_2), \forall (u(s_1, s_2) \in S_1 * S_2$$

- $u(s_1^*, s_2^*)$: **value of the game**

Minimax strategy (player 2 - minimizer of player's 1 payoff): For each column, player 2 identifies max. possible payoff Player 1 could achieve if player 2 chooses that column (assuming player 1 will try to maximize their payoff for that column). **Column Maximum*****

Player 2 then chooses strategy (column) that corresponds to the **minimum of these column maximums** = **Minimax value** of the game (From player 1's perspective, representing the maximum payoff player 2 is willing to concede.

$$\textit{Maximin value (Player 1)} = \textit{Minimax value (Player 2)}$$
$$=$$
$$\textbf{\textit{Value of the Game}}\ (\mathbf{V})$$

> ***Example : Saddle Point in Pure Strategies***
>
> Consider the following *PAYOFF MATRIX* for Player 1 in a 2-player 0-sum game. Entries in matrix = Payoff to Player 1. ($u = u1 = u2$)

	Player 2: Strategy Y1	Player 2: Strategy Y2	Row Minimums
Player 1: Strategy X1	4	**2**	2
Player 1: Strategy X2	3	1	1
Column Maximums	4	**2**	

Goal: To find maximin and minimax values to identify saddle point:

1. Player 1's Maximin Strategy (Maximizing minimum guaranteed payoff)

Player 1 looks at minimum payoff they could receive for each of their strategies:

- If Player 1 plays Strategy X1, the minimum payoff is $\min(4, 2) = 2$.
- If Player 1 plays Strategy X2, the minimum payoff is $\min(3, 1) = 1$.

Player 1 wants to choose the strategy that maximizes this minimum payoff. The maximum of $\{2, 1\}$ is 2. Thus, Player 1's maximin strategy is X1, and the **Maximin value** = **2**.

2. Player 2's Minimax Strategy (Minimizing Player 1's maximum possible gain)

Player 2 looks at the maximum payoff Player 1 could achieve for each of Player 2's strategies:

- If Player 2 plays Strategy Y1, the maximum payoff Player 1 can get is $\max(4, 3) = 4$.
- If Player 2 plays Strategy Y2, the maximum payoff Player 1 can get is $\max(2, 1) = 2$.

Player 2 wants to choose the strategy that minimizes maximum payoff for Player 1. The minimum of $\{4, 2\}$ is 2. Thus, Player 2's minimax strategy is Y2, and the **Minimax value** = **2** (from Player 1's perspective).

Saddle Point and Value of the Game

Since Maximin value (2) = Minimax value (2) → Saddle point exists.

$$V = 2$$

Saddle point occurs at strategy profile where Player 1 plays **Strategy X1** and Player 2 plays **Strategy Y2**. The payoff at this point is **2** (to Player 1).

Looking at the entry '2' in the matrix (at the intersection of X1 and Y2):

- It is the minimum value in its row (Row X1: values are $\{4, \mathbf{2}\}$).
- It is the maximum value in its column (Column Y2: values are $\{\mathbf{2}, 1\}$).

Dual property (minimum of its row and maximum of its column) is characteristic of a saddle point in a payoff matrix.

Stability of the Saddle Point (Connection to Nash Equilibrium)

At saddle point (X1, Y2):

- If Player 1 (currently playing X1) unilaterally considers switching to Strategy X2 (while Player 2 continues to play Y2), Player 1's payoff would decrease from 2 to 1. Therefore, Player 1 has no incentive to switch.
- If Player 2 (currently playing Y2) unilaterally considers switching to Strategy Y1 (while Player 1 continues to play X1), Player 1's payoff would increase from 2 to 4 = Player 2's payoff would change from -2 to -4 (zero-sum game), which is worse for Player 2. Therefore, Player 2 has no incentive to switch.

Since neither player has an incentive to unilaterally deviate from strategy profile (X1, Y2), saddle point is also a NE for the above 0-sum game.

Properties of Zero-Game:

- **Value is unique:**
  - There is 1 value which is = upper and lower values of the game.
- **Order Interchangeability:**
  - if (x_1, x_2) and (y_1, y_2) are saddle-point solutions, then (x_1, y_2) and (y_1, x_2) are also saddle-point solution
  - (x_1, x_2) and (y_1, y_2) lead to *same value* of the 0-sum game

5 Mixed Strategies and Mixed Nash Equilibrium

- Probability vector in strategy space
- Randomization of the strategy (action) space
- Payoff becomes expected value

CHAPTER 2

MATHEMATICAL PROOFS

1 Set Theory

Set: *A collection of objects considered as a single object.*

- **Open Interval** ( ): (a, b) represents all $\Re$ x such that $a < x < b$.
- **Closed Interval** [ ]: $[a, b]$ represents all $\Re$ x such that $a \leq x \leq b$.
- **Half-Open/Half-Closed Intervals**: $[a, b)$ means $a \leq x < b$, and $(a, b]$ means $a < x \leq b$.

Disjoint: $A \cap B = \emptyset$

Difference: $A - B$ or $A/B = \{x : x \in A \text{ and } x \notin B\}$

Example : Set operations
Let $A = \{x \in \mathbb{R} : |x| \leq 3\}$, $B = \{x \in \mathbb{R} : |x| > 2\}$ and $C = \{x \in \mathbb{R} : |x - 1| \leq 4\}$.

1. Express A, B and C using interval notation.
2. Determine $A \cap B$, $A - B$, $B \cap C$, $B \cup C$, $B - C$ and $C - B$.

Solution

1. $A = [-3, 3]$, $B = (-\infty, -2) \cup (2, \infty)$ and $C = [-3, 5]$ *(For C, $-4 \leq x - 1 \leq 4$)*.
2. $A \cap B = [-3, -2) \cup (2, 3]$, $A - B = [-2, 2]$, $B \cap C = [-3, -2) \cup (2, 5]$, $B \cup C = (-\infty, \infty)$, $B - C = (-\infty, -3) \cup (5, \infty)$ and $C - B = [-2, 2]$.

Complement:

All elements that are *not in* the given set but are *within* a defined **universal set**.

Consider universal set U. For a set A, its **complement** is: $\overline{A} = U - A = \{x : x \in U \text{ and } x \notin A\}$.
If $U = \mathbb{Z}$, then $\overline{\mathbb{N}} = \{0, -1, -2, \dots\}$; while if $U = \mathbb{R}$, then $\overline{\mathbb{Q}} = \mathbb{I}$.

Key Properties of Complements

Let U be the universal set and A and B be subsets of U.

- **Union with Original Set:** A set and its complement, when united, form the universal set:

$$A \cup \overline{A} = U$$

- **Intersection with Original Set:** A set and its complement are always disjoint (they have no elements in common):

$$A \cap \overline{A} = \emptyset$$

- **Double Complement:** The complement of the complement of a set is the original set itself:

$$\overline{(\overline{A})} = A$$

- **Complement of Universal Set:** The complement of the universal set is the empty set:

$$\overline{U} = \emptyset$$

- **Complement of Empty Set:** The complement of the empty set is the universal set:

$$\overline{\emptyset} = U$$

- **De Morgan's Laws:** These important laws relate complements to unions and intersections:
  - The complement of a union is the intersection of the complements:

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$

  - The complement of an intersection is the union of the complements:

$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

1.1 Indexed Collections of Sets