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Reminder from calculus

- How can you integrate something like this?

$$\int_0^1 2x\sqrt{x^2+1}dx.$$

- Usually, u-substitution is used:

$$u = x^2 + 1$$
$$du = 2xdx = (\frac{du}{dx})dx$$
$$u = 1 \text{ when } x = 0$$
$$u = 2 \text{ when } x = 1$$

so the integral above becomes

$$\int_1^2 \sqrt{u}du = \frac{2}{3}u^{3/2}|_{u=1}^{u=2} = \frac{2}{3}(2^{3/2} - 1)$$

Breakdown of u-substitution

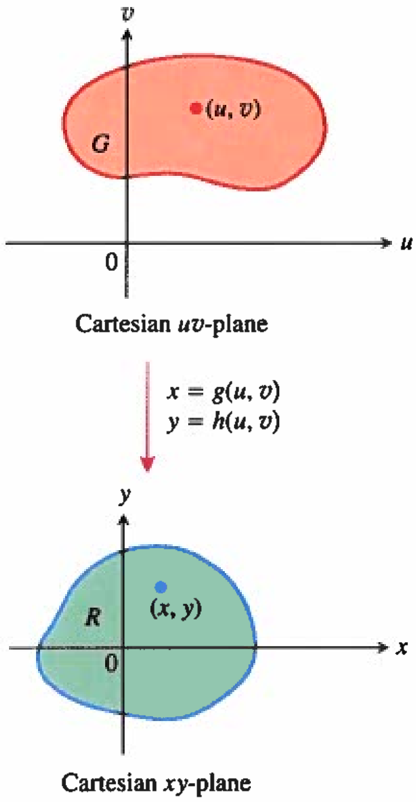
$$\int_0^1 2x\sqrt{x^2+1}dx = \int_1^2 \sqrt{u}du$$

Components of u-substitution

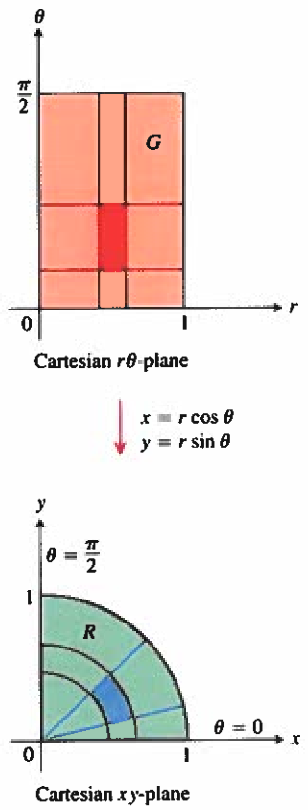
- new variable u is created, one-to-one with $x = g(u)$
- dx and du are related by $dx = g'(u)du = \frac{dx}{du}du$
- There is an interval (c, d) for u so that $x = g(u)$ ranges from (a, b) for u in this range

$$\int_{x=a}^{x=b} f(x)dx = \int_{u=c}^{u=d} f(g(u))\frac{dx}{du}du.$$

Higher dimensional analog



- We extend the idea of u-substitution to multiple integrals
- In general, the situation is very similar to u-substitution, but is slightly more involved because of having several variables.
- We have actually already seen 3 examples of this: Polar, Cylindrical, and Spherical coordinates



- Suppose we have the domain $0 \leq r \leq 1$, $0 \leq \theta \leq \frac{\pi}{2}$.
- We can plot this as if these were Cartesian variables and we get the box on the top left.
- Then, we use the transformation $x = g(r,\theta) = r\cos(\theta)$, $y = h(r,\theta) = r\sin(\theta)$ and we plot the corresponding (x, y) values.
- **Everyone:** Compute the following quantity:

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}$$

$x = r\cos\theta \nearrow \frac{\partial x}{\partial r} = \cos\theta$

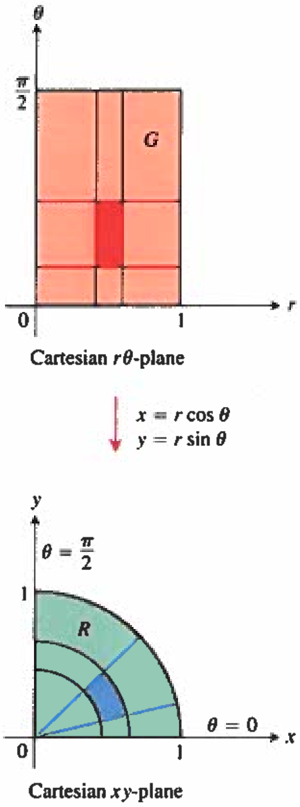
$\searrow \frac{\partial x}{\partial \theta} = -r\sin\theta$

$y = r\sin\theta \nearrow \partial y/\partial r = \sin\theta$

$\searrow \frac{\partial y}{\partial \theta} = r\cos\theta$

$$\rightsquigarrow J(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Polar Coordinates in this Framework



- Suppose we have the domain $0 \leq r \leq 1$, $0 \leq \theta \leq \frac{\pi}{2}$.
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- Then, we use the transformation $x = g(r, \theta) = r\cos(\theta)$, $y = h(r, \theta) = r\sin(\theta)$ and we plot the corresponding (x, y) values.
- **Everyone:** Compute the following quantity:

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}$$

- **Solution:** $J(r, \theta) = r$. Where have we seen this before dealing with polar coordinates?

Substitution in Double Integrals

- The Jacobian determinant or Jacobian of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{\partial(x,y)}{\partial(u,v)}$$

- Suppose $f(x, y)$ is continuous over R. Let G be the set of (u, v) points which are mapped onto R by the transformation $x = g(u, v)$, $y = h(u, v)$. If g and h have continuous first partial derivatives in G, then

$$\iint_R f(x,y)dxdy = \iint_G f(g(u,v), h(u,v))|J(u,v)|dudv$$

- In the new coordinate systems $dA = |J(u, v)|dudv$.

absolute value of the Jacobian

Example 1

Evaluate

$$I = \int_0^4 \int_{y/2}^{(y/2)+1} \overbrace{\frac{2x-y}{2}}^{u}\, dx\,dy$$

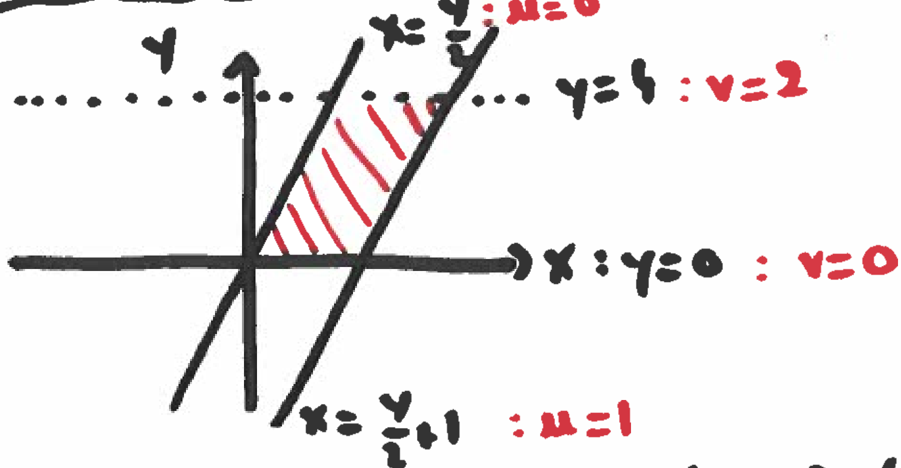
using the transformation

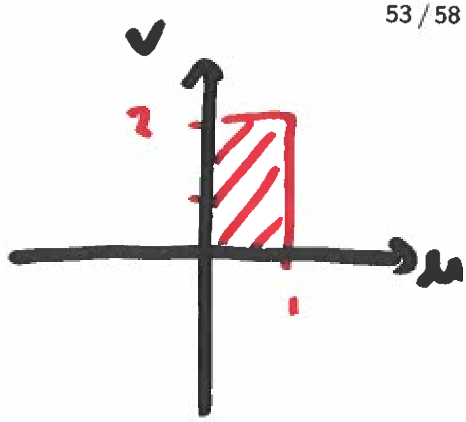
$$u = \frac{2x-y}{2}, \quad v = \frac{y}{2}.$$

Let us reverse the system of equation:

$$\begin{cases} v = \frac{y}{2} \Rightarrow y = 2v \\ u = x - \frac{y}{2} \Rightarrow x = u + \frac{y}{2} = u + v \end{cases}$$

Domain of integration:





Jacobian: $J(u,v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$

Conclusion: $I = \int_0^2 \int_0^1 u \cdot \underbrace{2}_{|J(u,v)|}\, du\,dv = 4\left[\frac{u^2}{2}\right]_0^1 = 2.$

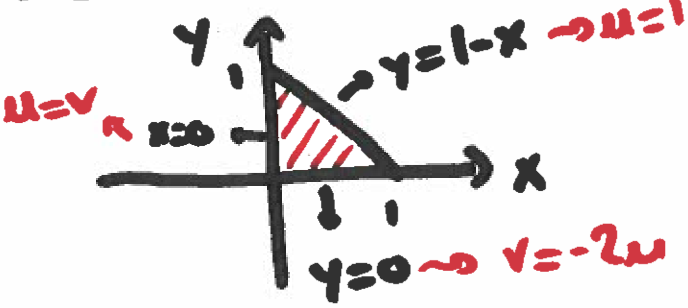
Evaluate

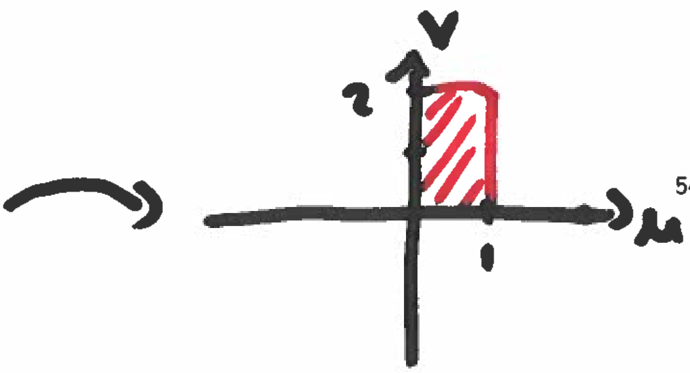
$$I = \int_0^1 \int_0^{1-x} \overbrace{\sqrt{x+y}(y-2x)^2}^{\sqrt{u}\, v^2} dy dx$$

Substitution:

$$\begin{cases} u = x + y \\ v = y - 2x \end{cases} \rightarrow \begin{cases} x = \frac{1}{3}(u - v) \\ y = \frac{1}{3}(2u + v) \end{cases}$$

· Domain:





· Jacobian:

$$J(u,v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{vmatrix} = \frac{1}{9} + \frac{2}{9} =$$

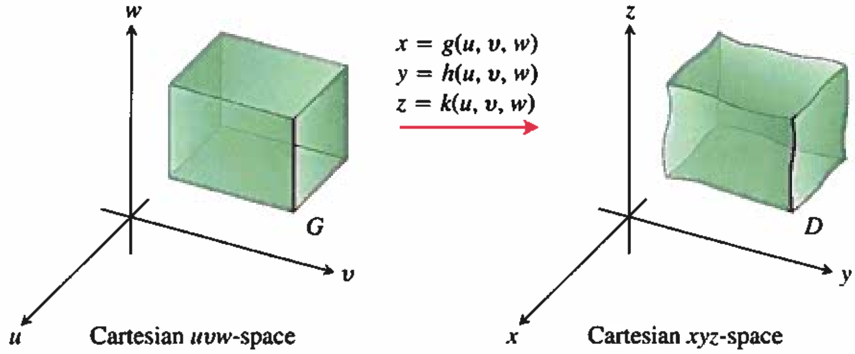
· Conclusion:

$$I = \int_0^1 \int_{-2u}^{u} \sqrt{u}\, v^2 \underbrace{\frac{1}{3}}_{|J(u,v)|} dv du = \int_0^1 \frac{\sqrt{u}}{3} \left[\frac{v^3}{3}\right]_{-2u}^{u} du$$

$$= \frac{1}{9} \int_0^1 u^{1/2}(u^3 + 8u^3) du = \int_0^1 u^{7/2} du = \frac{2}{9} [u^{9/2}]_0^1 =$$

Even Higher Dimensions

- In higher dimensions, the exact same principles apply.



- The only difference is that the Jacobian matrices get larger and larger

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- In the new coordinate system $dV = |J(u,v,w)|dudvdw$.

Spherical Coordinates

This gives a way of showing that $dV = \rho^2 \sin(\phi) d\rho d\phi d\theta$ in spherical coordinates, since we didn't fully derive it.

$$x = \rho \sin(\phi)\cos(\theta), \quad y = \rho \sin(\phi)\sin(\theta), \quad z = \rho\cos(\phi)$$

$$\begin{aligned}
J(\rho,\phi,\theta) &= \begin{vmatrix} \sin(\phi)\cos(\theta) & \rho\cos(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & \rho\cos(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) \\ \cos(\phi) & -\rho\sin(\phi) & 0 \end{vmatrix} \\
&= \cos(\phi)\left(\rho^2\cos(\phi)\sin(\phi)\cos^2(\theta) + \rho^2\cos(\phi)\sin(\phi)\sin^2(\theta)\right) \\
&+ \rho\sin(\phi)\left(\rho\sin^2(\phi)\cos^2(\theta) + \rho\sin^2(\phi)\sin^2(\theta)\right) \\
&= \rho^2\cos^2(\phi)\sin(\phi) + \rho^2\sin^2(\phi)\sin(\phi) \\
&= \rho^2\sin(\phi)
\end{aligned}$$

We could do the exact same calculation for Cylindrical coordinates to find $J(r,\theta,z) = r$.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$\begin{aligned}
J(r,\theta,z) &= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\
&= r(\cos^2\theta + \sin^2\theta) \\
&= r
\end{aligned}$$

Example 3-Neither Spherical nor Cylindrical

Evaluate

$$I = \int_0^3 \int_0^4 \int_{y/2}^{y/2+1} \overbrace{\frac{2x-y}{2}}^{u} + \overbrace{\frac{z}{3}}^{w} dxdydz$$

using the transformation

$$u = \frac{2x-y}{2}, \quad v = \frac{y}{2}, \quad w = \frac{z}{3}.$$

- Substitution : $\begin{cases} x = u+v \\ y = 2v \\ z = 3w \end{cases}$

- Boundaries :

$$\frac{y}{2} \leq x \leq \frac{y}{2}+1 \rightsquigarrow 0 \leq u \leq 1$$

$$0 \leq y \leq 4 \rightsquigarrow 0 \leq v \leq 2$$

$$0 \leq z \leq 3 \rightsquigarrow 0 \leq w \leq 1$$

- Jacobian :

$$J(u,v,w) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6$$

- Conclusion :

$$I = \int_0^1 \int_0^2 \int_0^1 (u+w) \underset{\downarrow}{\overset{|J(u,v,w)|}{6}} \, du dv dw = 6 \int_0^1 \int_0^2 \left[\frac{u^2}{2} + uw\right]_0^1 dv dw$$

$$= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w\right) dv dw = 6 \int_0^1 (1+2w) dw = 6 \left[w + w^2\right]_0^1 = 1$$

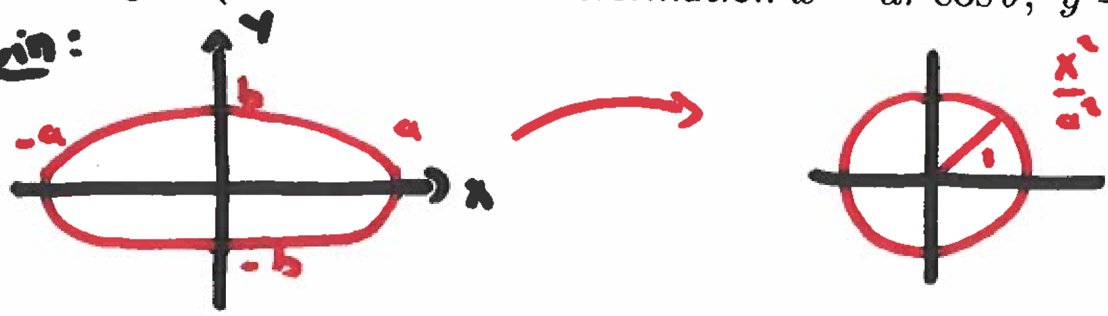
Example 4

A thin plate of constant density covers the region bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > 0, b > 0$$

in the xy-plane. Find the ~~first~~ moment of the plate about the origin. (Hint: use the transformation $x = ar\cos\theta,\ y = br\sin\theta$.)

Domain:



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2 = 1$$

Jacobian:

$$J(r,\theta) = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix}$$

$$= abr(\cos^2\theta + \sin^2\theta) = abr$$

Moment:

$$I_0 = \iint_D \overbrace{(x^2+y^2)}^{f(x,y)}\, dV = \int_0^{2\pi}\int_0^1 \overbrace{r^2(a^2\cos^2\theta + b^2\sin^2\theta)}^{f(r,\theta)}\, \overbrace{abr}^{|J(r,\theta)|}\, dr\, d\theta$$

$$= ab\int_0^{2\pi} (a^2\cos^2\theta + b^2\sin^2\theta)\left[\frac{r^4}{4}\right]_0^1 d\theta$$

$$= \frac{ab}{4}\int_0^{2\pi} (a^2\cos^2\theta + b^2\sin^2\theta)\, d\theta$$

<u>but</u>:

$$\int_0^{2\pi} \cos^2\theta \, d\theta = \int_0^{2\pi} \frac{1+\cos(2\theta)}{2} \, d\theta$$

$$= \left[\frac{1}{2}\theta + \frac{1}{4}\cancel{\sin(2\theta)} \right]_0^{2\pi} = \pi$$

and $\displaystyle\int_0^{2\pi} \sin^2\theta \, d\theta = \int_0^{2\pi} (1-\cos^2\theta) \, d\theta = \pi$

<u>Thus</u>:

$$I_O = \frac{ab}{4} \times (a^2\pi + b^2\pi)$$

$$= \frac{\pi ab}{4}(a^2+b^2)$$