FUZZY TOPOLOGICAL SPACE

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FUZZY TOPOLOGICAL SPACE

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DECLARATION

I hereby declare that this thesis is based on my original work except for citations and quotations which have been duly acknowledged. I also declare that it has not been previously and concurrently submitted for any other degree or award at Xiamen University Malaysia or other institutions.

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APPROVAL FOR SUBMISSION

I certify that this project/thesis, entitled “FUZZY TOPOLOGICAL SPACE", and prepared by BRYAN NG ZHENG QUAN, has met the required standard for submission in partial fulfilment of the requirements for the award of Bachelor of Science in Mathematics and Applied Mathematics (Honours) at Xiamen University Malaysia.

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ABSTRACT

This thesis studies the theory of fuzzy topological spaces and its relation to general topological spaces. We begin with an introduction to fuzzy sets and then work on two well-known definition for fuzzy topology: Chang's original definition and Lowen's improved version, which solves key problems in terms of continuity and compactness. Furthermore, we study the mutually generative relations between fuzzy and general topologies by the function ω and ι, emphasizing their connections to each other. Finally, we systematically analyze the concepts of continuity and compactness under these two definitions and we study in depth the relations and implications between them.

Keywords: Fuzzy Set, Fuzzy Topological Space, Fuzzy Continuous, Fuzzy Compactness, Topological Space.

CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS / ABBREVIATIONS

μ	:	Membership Function	[2.1.1]
χ	:	Characteristic Function	[2.1.5]
$\emptyset$	:	Empty Set	
$\big(X, \mu(x)\big)$	:	Fuzzy Set	[2.1.2]
$\big(X, \chi(x)\big)$	:	Crisp Set	[2.1.6]
$\emptyset_X$	:	Empty Fuzzy Set	[2.1.9]
$\sqcup_X$	:	Fully Included Fuzzy Set	[2.1.9]
$(X, \mathscr{T})$	:	Topological Space	[3.1.1]
(X, δ) or fts	:	Fuzzy Topological Space	[3.2.1]
$nbhd$	:	Neighboudhood of a Fuzzy Set	[3.2.6]
$\mathscr{J}(X)$	:	Set of All Topologies on X	[4.2.1]
$\mathscr{W}(X)$	:	Set of All Fuzzy Topologies on X	[4.2.1]
f	:	Function Between Two Sets	[4.3.1]
$\tilde{f}$	:	Function Between fts	[4.4.1]
F-continuous	:	Fuzzy Continuous	[4.4.1]
F-compact	:	Fuzzy Compact	[5.3.2]

CHAPTER 1

INTRODUCTION

The theory of fuzzy topological spaces is an extension of general topology. Traditionally, topological concepts use exact binary logic to express whether they belong to a set or not. However, many real-life concepts are not 'either/or'; for example, the boundaries of the ocean change with the ebb and flow of the tides, and therefore cannot simply be represented by a line. Pests and diseases tend to start from a single original spot and gradually spread to the surrounding areas, and the degree of damage is also gradually decreasing, so a disaster cannot simply be evaluated by whether it is affected or not.

In order to express information more objectively, there is an acute need to solve the problem of applications involving imprecise or uncertain data. This need for a more detailed and nuanced understanding of the membership of sets led to the development of fuzzy set theory, introduced by Lotfi Zadeh in the 1960s [Zad65], which in turn led to the study of fuzzy topology.

The theory of fuzzy topological spaces was originated from the fundamental work of Zadeh, and mathematicians such as C. L. Chang and Robert Lowen played a key role in establishing and refining the field.In 1968, Chang introduced the concept of a fuzzy topological space [Cha68], which adapts the general topology to the degree of membership rather than the degree of binary inclusion. His framework allowed elements to belong to open sets with different degrees of membership, providing a more flexible model for analyzing situations characterized by uncertainty.Chang's foundational work has opened up new methods for the application of topology in fuzzy environments and laid the groundwork for further research. However, over time, the limitations of Chang's framework became apparent, especially with respect to the continuity of constant functions. Aware of these problems, Robert Lowen introduced a revised definition [Low76] in 1976 to ensure the continuity of constant functions in fuzzy topological spaces. Lowen's improvements brought fuzzy topological spaces closer to general topology, strengthened the theoretical foundations of the field, and led to its recognition as a powerful area of mathematical research.

Fuzzy topological spaces provide a framework that combines fuzzy set theory with general topology, allowing the possibility of exploring spaces where the degree of membership of each element is between 0 and 1. In this way, we can apply the extension of classical concepts such as continuity and compactness to environments characterized by degrees of membership that are either graded or partial, opening up new possibilities for analysis and applications in uncertain or imprecise environments.

The aim of this thesis is to systematically study the fundamentals and properties of fuzzy topological spaces, focusing on fuzzy continuous functions and compactness in this broad framework. The main objective is to contribute to the theoretical foundations of fuzzy topology and to explore its practical implications. Chapter 2 introduces the basic background of fuzzy sets, compares them with classical ''crisp'' sets, and discusses the basic operations of fuzzy set theory. Chapter 3 delves into the concepts of fuzzy topology (Chang's), defines fuzzy topological spaces, and examines their properties and structure.

The next chapters build on these foundations to explore advanced topics such as fuzzy continuity and compactness. Chapter 4 discusses Lowen's definition and the fuzzy continuous functions, which are an extension of classical continuity and are fundamental to understanding mappings between fuzzy topological spaces. Chapter 5 explores the concept of compactness in fuzzy topology, its meaning and potential applications. Finally, Chapter 6 discusses the applications and potential for the future of fuzzy mathematics.

Through a comprehensive study of these topics, we seeks to deepen the understanding of fuzzy topology as a theoretical framework and tool for applied mathematics. The findings of this thesis are intended to help the wider mathematical community understand how fuzzy topological concepts can be applied to practical real-world scenarios where uncertainty plays a central role.

CHAPTER 2

FUZZY SET

The concept of fuzzy set is the fundamental of fuzzy topological space. It was first introduced by Lotfi Zadeh [Zad65] in 1964 as an extension of classical set. Let us begin with the concept of fuzzy set.

2.1 Fuzzy Set and Crisp Set

Let X be a non-empty set of objects or points, with an arbitrary element denoted by x. Thus, we express X as $X = \{x\}$. For any set X, we use the notation $x \in X$ to indicate that x is an element of X, and $x \notin X$ to indicate that x is not an element of X.

We begin by introducing the concept of a membership function, which is fundamental of the theory of fuzzy sets and allows us to define degrees of membership for elements in X.

Definition 2.1.1. Membership Function
Let X be a subset of the real numbers $\mathbb{R}$, or $X \subseteq \mathbb{R}$. A **membership function** $\mu(x)$ on X is any function from X to a real closed unit interval $I = [0, 1]$. Namely, for all $x \in X$

$$\mu : X \to [0, 1].$$

The value of μ at x representing the "grade of membership", quantifies the degree to all $x \in X$, with $\mu(x) = 0$ indicating no membership and $\mu(x) = 1$ representing the full membership.

Definition 2.1.2. Fuzzy Set [Zad65, ZN21]
Let $X \subseteq \mathbb{R}$, and let $\mu_A : X \to [0, 1]$ be the membership function for a set $A \subseteq X$. A **fuzzy set** A is defined as the set of ordered pairs

$$A = (X, \mu_A) = \{(x, \mu_A(x)) | x \in X\}$$

where $\mu_A(x)$ represents the degree of membership of the element $x \in A$. The set A is characterized by function μ_A, which assigns a degree of membership to each $x \in X$.

Example 2.1.3. Consider the closed interval $X = [0, 90]$. Define the membership function $\mu : X \to [0, 1]$ as the follows:

$$\mu(x) = \frac{x}{120}, \qquad \forall x \in X.$$

Then (X, μ) is a fuzzy set.

Example 2.1.4. Consider $X \subseteq \mathbb{R}$. The membership function $\mu : X \to [0, 1]$ is defined as:

$$\mu(x) = \frac{1}{1 + x^2}, \qquad \forall x \in X.$$

Since $\mu(x) \in [0, 1]$ for all $x \in X$, (X, μ) is a fuzzy set.

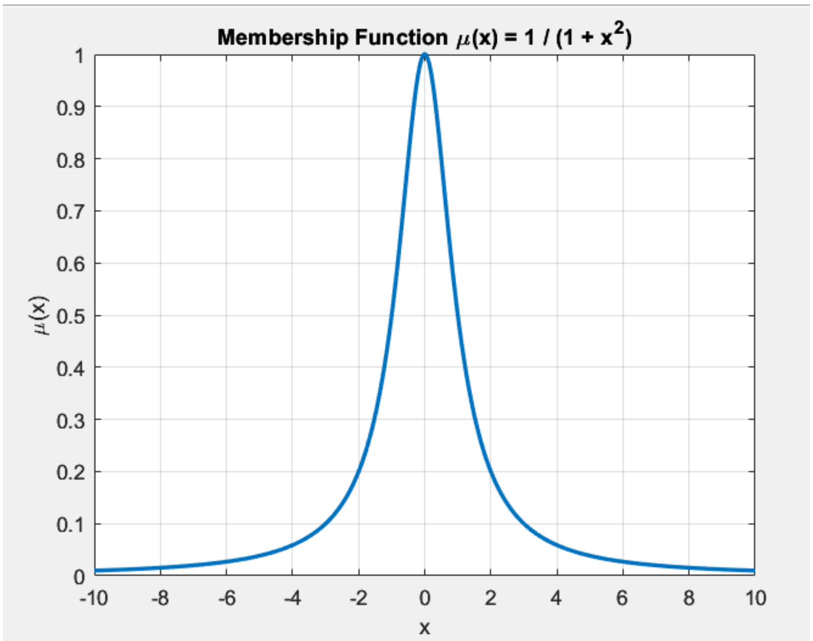

Figure 2.1: Example 2.1.4

Definition 2.1.5. Characteristic Function

Let $A \subseteq \mathbb{R}$. The **characteristic function** of the set A, denoted as χ_A, is a function $\chi_A : A \to \{0, 1\}$ defined as

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Definition 2.1.6. Crisp Set [ZN21]

Let $X \subseteq \mathbb{R}$. A **crisp set** in X is a set of ordered pairs $A = (X, \chi_A)$, where $\chi_A(x)$ is the characteristic function.

Example 2.1.7. Let $X = [0, 20]$, and define $A = \{x \in X | x \in \mathbb{Q}\}$. Then for each $x \in X$, the characteristic function is given by:

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Then, (X, χ_A) is a crisp set.

Remark 2.1.8. Crisp sets are indeed a special subclass of fuzzy set, where the membership function $\mu(x)$ is restricted to take only the values 0 or 1, corresponding to full exclusion or inclusion in the set.

Let us look at the following definitions:

Definition 2.1.9.
Given a fuzzy set $A = (X, \mu_A)$ and for all $x \in X$,

(a) Two fuzzy sets A and B are said to be equal or $A = B$ if $\mu_A(x) = \mu_B(x)$.

(b) A is a subset of B or $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$.

(c) A is an empty fuzzy set or $A = \emptyset_X$ if $\mu_A(x) = 0$.

(d) A is an fully included fuzzy set or $A = \sqcup_X$ if $\mu_A(x) = 1$.

2.2 Basic Operations of Fuzzy Set

In this section, we will study the compliment of fuzzy set and the union and intersection of fuzzy sets.

Definition 2.2.1. Compliment of a Fuzzy Set
Given a fuzzy set $A = (X, \mu_A)$, its **compliment** A^c is defined by the following membership function:

$$\forall x \in X, \mu_{A^c}(x) = 1 - \mu_A(x).$$

Example 2.2.2. Let $X = [0, 90]$ and define $\mu_A : X \to [0, 1]$ by

$$\mu_A(x) = \frac{x}{90}, \quad \forall x \in X.$$

Then $A = (X, \mu_A)$ is a fuzzy set. The compliment of A is defined as $A^c = (X, \mu_{A^c})$ is defined as

$$\mu_{A^c}(x) = 1 - \frac{x}{90}, \quad \forall x \in X.$$

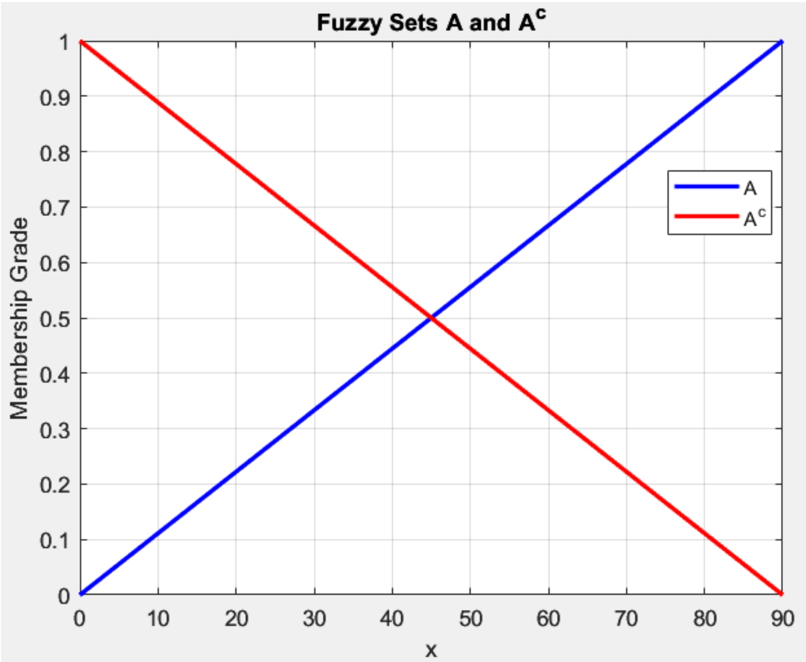

Figure 2.2: Fuzzy set A and its complement A^c.

2.2.1 Union and Intersection of Fuzzy Set

Definition 2.2.3. Union and Intersection

Given an index set $\mathcal{J}$ and a family of fuzzy sets $A_j = (X, \mu_{A_j})_{j \in \mathcal{J}}$, we define

- the **union** of fuzzy sets as

  $$\bigcup_{j \in \mathcal{J}} A_j = (X, \mu_{\bigcup_{j \in \mathcal{J}} A_j})$$

  where

  $$\mu_{\bigcup_{j \in \mathcal{J}} A_j}(x) := \sup_{j \in \mathcal{J}} \{\mu_{A_j}(x)\} \text{ for all x} \in X.$$

  In particular, when $\mathcal{J}$ is finite, then

  $$\mu_{\bigcup_{j \in \mathcal{J}} A_j}(x) = \max_{j \in \mathcal{J}} \{\mu_{A_j}(x)\} \text{ for all } x \in X.$$

- the **intersection** of fuzzy sets as

$$\bigcap_{j \in \mathcal{J}} A_j = (X, \mu_{\underset{j \in \mathcal{J}}{\cap} A_j})$$

where

$$\mu_{\underset{j \in \mathcal{J}}{\cap} A_j}(x) := \inf_{j \in \mathcal{J}} \{\mu_{A_j}(x)\} \text{ for all x } \in X.$$

In particular, when $\mathcal{J}$ is finite, then

$$\mu_{\underset{j \in \mathcal{J}}{\cap} A_j}(x) = \min_{j \in \mathcal{J}} \{\mu_{A_j}(x)\} \text{ for all } x \in X.$$

Theorem 2.2.4. If A and B are two fuzzy sets, then

(1) $(A \cup B)^c = A^c \cap B^c$ (by De Morgan's Laws),

(2) $(A \cap B)^c = A^c \cup B^c$,

(3) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (by Distributive Laws),

(4) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. Consider $1 - \max\{\mu_A, \mu_B\} = \min\{1 - \mu_A, 1 - \mu_B\}$ for the case $\mu_A < \mu_B$ and $\mu_A > \mu_B$. For case $\mu_A < \mu_B$, then $A \subseteq B$ and hence $1 - \mu_A > 1 - \mu_B$. This shows $B^c \subseteq A^c$. Similarly, for case $\mu_A > \mu_B$, then $B \subseteq A$ and hence $1 - \mu_B > 1 - \mu_A$. This shows $A^c \subseteq B^c$ and hence we have (1). Similarly for (2).

For (3), we have $\max\{\mu_A, \min\{\mu_B, \mu_C\}\} = \min\{\max\{\mu_A, \mu_B\}, \max\{\mu_A, \mu_C\}\}$. It is easy to check this equality by considering six cases: $\mu_A > \mu_B > \mu_C, \mu_A > \mu_C > \mu_B, \mu_B > \mu_A > \mu_C, \mu_B > \mu_C > \mu_A, \mu_C > \mu_A > \mu_B$ and $\mu_C > \mu_B > \mu_A$. Similarly for (4). $\square$

CHAPTER 3

FUZZY TOPOLOGICAL SPACE

In this chapter, we will study the well-established concepts of general topology onto fuzzy sets. Before we study fuzzy topological spaces, it is important to first recall the fundamental of general topology, as they serve as the basis for this generalization.

3.1 Topological Space

Definition 3.1.1. General Topological Space

Given a set X, a **topology** on X is a collection of $\mathscr{T}$ of subsets of X that satisfies the following properties:

(1) $\emptyset, X \in \mathscr{T}$.

(2) Any arbitrary union of members of τ is belongs to $\mathscr{T}$.

(3) The intersection of finite number of members of τ is belongs to $\mathscr{T}$.

If τ is a topology for X, then the pair $(X, \mathscr{T})$ is a **topological space**.

Given a topological space $(X, \mathscr{T})$, we say that a subset U of X is an **open set** of X if $U \in \mathscr{T}$.

Definition 3.1.2. Discrete Topology and Indiscrete Topology
Let X be a nonempty set. Then

1. The family of all subsets of X, known as the power set $P(X)$, is a topology on X and it is defined as **discrete topology**.
2. The collection of set consisting $\emptyset$ and X is a topology on X and it is defined as **indiscrete topology**.

Definition 3.1.3. Neighbourhood
Let X be a topological space and let x be a point in X. A set U is a **neighbourhood** of x if U is an open set in X that contains x.

Definition 3.1.4. Interior

Given that X is a topological space and A is a subset of X, then the **interior** of A is the union of all the open sets contained in A. It is denoted as int(A) or A^0.

Definition 3.1.5. Finer and Coarser

Suppose $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on a given set X. If $\mathscr{T}_1 \subseteq \mathscr{T}_2$, we say $\mathscr{T}_2$ is **finer** than $\mathscr{T}_1$ or $\mathscr{T}_1$ is **coarser** than $\mathscr{T}_2$. If $\mathscr{T}_1 \subset \mathscr{T}_2$, we say $\mathscr{T}_2$ is **strictly finer** than $\mathscr{T}_1$ or $\mathscr{T}_1$ is **strictly coarser** than $\mathscr{T}_2$.

Definition 3.1.6. Basis of Topology

Given set X, let $\mathcal{B}$ be the collection of subsets of X. Then $\mathcal{B}$ is called a **basis** for a topology on X if $\mathcal{B}$ satisfies the following properties:

(1) $\forall x \in X, \exists B \in \mathcal{B}$ such that $x \in B$.

(2) If B_1 and B_2 are elements of $\mathcal{B}$ and x is an element of X such that $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology $\mathscr{T}$ generated by $\mathcal{B}$ as follows: Given $U \subset X$, then $U \in \mathscr{T}$ if $\forall x \in U$, there $\exists B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

Definition 3.1.7. Subbasis of Topology

A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union is equals to X. The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathscr{T}$ of all union of finite intersections of elements of $\mathcal{S}$.

3.2 Fuzzy Topological Space

Now, it is enough to study the concepts of fuzzy topological spaces.

Definition 3.2.1. Fuzzy Topological Space [Cha68, PL80a]

Given a set X, a **fuzzy topology** on X is a family $\delta = \{A_j | j \in \mathcal{J}\}$ of fuzzy sets that satisfies the following properties:

(1) $\emptyset_X, \sqcup_X \in \delta$.

(2) Closed under finite intersection: If $A_1, A_2 \in \delta$, then $A_1 \cap A_2 \in \delta$.

(3) Closed under arbitrary union: If $A_j \in \delta$ for all $j \in \delta$, then $\bigcup\limits_{j \in \mathcal{J}} A_i \in \delta$.

If δ is a fuzzy topology for X, then the pair (X, δ) is a **fuzzy topological space**, or fts in short. Moreover, every member of δ is called an δ-**open** fuzzy set. A fuzzy set is **closed** if and only if its complement is δ-open. In the sequel, if there is no confusion is likely to raise, we will simply call δ-open (closed) fuzzy set as an open (closed) set. Same as general topology, the **indiscrete fuzzy topology** contains $\emptyset_X$ and $\sqcup_X$ only, while the **discrete fuzzy topology** contains all fuzzy sets.

Let δ, γ be two fuzzy topologies for X with $\delta \subset (\subseteq)\gamma$, then we say γ is **finer** (strictly) than δ or δ is **coarser** (strictly) than γ.

Remark 3.2.2. For the definition of fuzzy topology, we know

Definition(1): It is clear that the empty fuzzy set and fully included fuzzy set must in the family δ.

Definition(2): Since the finite intersection between fuzzy sets is taking the minimum of different fuzzy sets' membership function on same point x, it is also clear that the finite intersection must in the family δ.

Definition(3): The infinite union of A_i will raise a problem. If there is a function such that $\mu(x) = \frac{x-1}{x}$ and it is the largest membership grade for all membership function, then the infinite union of all fuzzy subset will not bounded above. But since by definition (1), $\sqcup_X \in \delta$, and $\sqcup_X$ is the infinite union of any fuzzy subsets from δ. Hence the problem solved.

Definition 3.2.3. Basis of Fuzzy Topological Space [PL80a]
Let (X, δ) be a fts. A subfamily β of δ is called **basis** for δ if for each $A \in \delta$, there exists $\beta_A \subseteq \beta$ such that $A = \bigcup \beta_A$. A subfamily σ of δ is called a **subbasis** for δ if the family $\beta = \{\bigcap \mathcal{K} | \mathcal{K}$ is a finite subset of $\sigma\}$ is a basis for δ.

Example 3.2.4. Given two fuzzy sets on X such that μ_A, μ_B are both membership function defined on X and $X \subseteq \mathbb{R}$ such that

$$\mu_A = \begin{cases} 0, & \text{when } x \leq 0 \text{ and } x \geq 50, \\ \dfrac{x}{20}, & \text{when } 0 < x < 20, \\ 1, & \text{when } 20 \leq x \leq 30, \\ \dfrac{50-x}{20}, & \text{when } 30 < x < 50. \end{cases}$$

And,

$$\mu_B = \begin{cases} 0, & \text{when } x \leq -20 \text{ and } x \geq 30, \\ \dfrac{20+x}{20}, & \text{when } -20 < x < 0, \\ 1, & \text{when } 0 \leq x \leq 10, \\ \dfrac{30-x}{20}, & \text{when } 10 < x < 30. \end{cases}$$

Then the union and intersection of two fuzzy sets are

$$\mu_{A\cup B} = \max\{\mu_A(x), \mu_B(x)\} \text{ and } \mu_{A\cap B} = \min\{\mu_A(x), \mu_B(x)\}.$$

The family of fuzzy sets $\delta = \{\mu_A, \mu_B, \mu_{A\cup B}, \mu_{A\cap B}\}$ defined on X forms a fts.

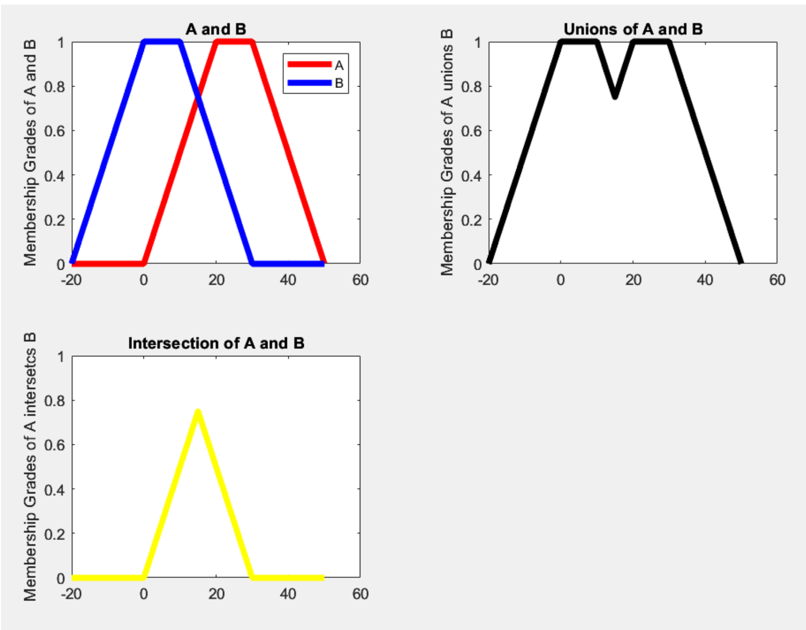

Figure 3.1: Fuzzy Topology defined on X

Example 3.2.5. Let $X = (x, y, z)$ and define $\mu_i : X \to I^3$ such that $\mu_A = (0.4, 0.6.0.9)$, $\mu_B = (0.2, 0.7, 0.8)$, $\mu_C = (0.4, 0.7, 0.9)$, $\mu_D = (0.2, 0.6, 0.8)$. Then $\delta = \{\mu_A, \mu_B, \mu_C, \mu_D\}$ forms a fuzzy topology.

Now,

$$\begin{aligned}
\mu_A \cup \mu_B &= (0.4, 0.7, 0.9) = \mu_C \in \delta, \\
\mu_A \cup \mu_C &= (0.4, 0.7, 0.9) = \mu_C \in \delta, \\
\mu_A \cup \mu_D &= (0.4, 0.6.0.9) = \mu_A \in \delta, \\
\mu_B \cup \mu_C &= (0.4, 0.7.0.9) = \mu_C \in \delta, \\
\mu_B \cup \mu_D &= (0.2, 0.7.0.8) = \mu_B \in \delta, \\
\mu_C \cup \mu_D &= (0.4, 0.7, 0.9) = \mu_C \in \delta.
\end{aligned}$$

And,

$$\begin{aligned}
\mu_A \cap \mu_B &= (0.2, 0.6, 0.8) = \mu_D \in \delta, \\
\mu_A \cap \mu_C &= (0.4, 0.7, 0.9) = \mu_A \in \delta, \\
\mu_A \cap \mu_D &= (0.2, 0.6, 0.9) = \mu_D \in \delta, \\
\mu_B \cap \mu_C &= (0.2, 0.7, 0.8) = \mu_B \in \delta, \\
\mu_B \cap \mu_D &= (0.2, 0.6, 0.8) = \mu_D \in \delta, \\
\mu_C \cap \mu_D &= (0.2, 0.6, 0.8) = \mu_D \in \delta.
\end{aligned}$$

And also

$$\begin{aligned}
\mu_A \cup \mu_B \cup \mu_C &= (0.2, 0.7, 0.9) = \mu_C \in \delta, \\
\mu_A \cup \mu_B \cup \mu_D &= (0.2, 0.7, 0.9) = \mu_C \in \delta, \\
\mu_B \cup \mu_C \cup \mu_D &= (0.2, 0.7, 0.9) = \mu_C \in \delta, \\
\mu_A \cup \mu_B \cup \mu_C \cup \mu_D &= (0.2, 0.7, 0.9) = \mu_C \in \delta, \\
\mu_A \cap \mu_B \cap \mu_C &= (0.2, 0.6, 0.8) = \mu_D \in \delta, \\
\mu_A \cap \mu_B \cap \mu_D &= (0.2, 0.6, 0.8) = \mu_D \in \delta, \\
\mu_B \cap \mu_C \cap \mu_D &= (0.2, 0.6, 0.8) = \mu_D \in \delta, \\
\mu_A \cap \mu_D \cap \mu_C \cap \mu_D &= (0.2, 0.6, 0.8) = \mu_D \in \delta.
\end{aligned}$$

Hence, (X, δ) forms a fts.

Definition 3.2.6. Open Neighbourhood [Cha68]

A fuzzy set U in a fts (X, δ) is a neighbourhood, or $nbhd$ for short, of a fuzzy set A if there exists an open fuzzy set O such that $A \subseteq O \subseteq U$.

Theorem 3.2.7. Open Fuzzy Set [Cha68]

A fuzzy set A is open if and only if for every fuzzy set B contained in A, A is a $nbhd$ of B.

Proof. "$\Rightarrow$": Assume A is open. By definition 3.2.6, A is an open fuzzy set, then every fuzzy set B contained in A is contained in an open set. Hence, A is the $nbhd$ of B.
"$\Leftarrow$": Assume that for every fuzzy set B that contained in A, A is a $nbhd$ of B, since $A \subseteq A$, A is a neighbourhood of itself. Then there is a open set O such that $A \subseteq O \subseteq A$. This shows $A = O$. Hence A is an open fuzzy set. $\square$

Theorem 3.2.8. Intersection of Finite Neighbourhood [Cha68]

If U is a $nbhd$ system of a fuzzy set A, then the finite intersection of members of U belongs to U, and each fuzzy set which contains a member of U belongs to U.

Proof. Suppose $U_1, U_2, \ldots, U_n$ are the members of $nbhd$ system U of a fuzzy set A, then there exists $O_1, O_2, \ldots, O_n$ be respectively open fuzzy set such that $A \subset O_i \subset U_i$ where $i = 1, 2, \ldots, n$. Since $U_1 \cap U_2 \cap \ldots \cap U_n$ contains $O_1 \cap O_2 \cap \ldots \cap O_n$ and it is a $nbhd$ of A, this shows $U_1 \cap U_2 \cap \ldots \cap U_n$ is the $nbhd$ system of A. Hence, the finite intersection of members of U is belongs to U. If a fuzzy set B contains a member of U, then it contains an open $nbhd$ of A, and B is an open $nbhd$ of A. Hence, B is belongs to U. $\square$

Remark 3.2.9. This theorem ensures that the finite intersection of open $nbhd$ of a fuzzy set A is still an open $nbhd$ of A.

Definition 3.2.10. Interior Fuzzy Set [Cha68]

Let A, B be fuzzy sets in a fts (X, δ) and let $B \subseteq A$. Then B is called an **interior fuzzy set** of A if A is a $nbhd$ of B. The union of all interior fuzzy sets of A is called the interior of A and is denoted by $\text{int}(A)$.

Theorem 3.2.11. [Cha68]
Let A be a fuzzy set in the fts (X, δ). Then $\text{int}(A)$ is open and is the largest open fuzzy set contained in A.

Proof. For a given fuzzy set A, let B_α be the interior fuzzy sets of A where $\alpha \in \mathcal{A}$. Let O_λ be the open fuzzy sets where $\lambda \in \Lambda$ such that $O_\lambda \subseteq A$. Let λ be the map from $\mathcal{A}$ to Λ. By definition 3.2.6 and definition 3.2.10, for every B_α there is a $O_{\lambda(\alpha)}$ such that $B_\alpha \subseteq O_{\lambda(\alpha)} \subseteq A$. Notice that the choice of $O_{\lambda(\alpha)}$ might not be unique since it is possible that $O_{\lambda(\alpha_1)} = O_{\lambda(\alpha_2)}$ for $\alpha_1 \neq \alpha_2$ where $\alpha_1, \alpha_2 \in \mathcal{A}$. In particular, every O_λ is also an interior fuzzy set of A. Since $O_\lambda \subseteq O_\lambda \subseteq A$. This shows $\{O_\lambda\}_{\lambda\in\Lambda}$ is a subcollection of $\{B_\alpha\}_{\alpha\in\mathcal{A}}$. Hence,

$$\text{int}(A) = \bigcup_{\alpha\in\mathcal{A}} B_\alpha \subseteq \bigcup_{\alpha\in\mathcal{A}} O_{\lambda(\alpha)} \subseteq \bigcup_{\lambda\in\Lambda} O_\lambda,$$

and since $\{O_\lambda\}_{\lambda\in\Lambda}$ is a subcollection of $\{B_\alpha\}_{\alpha\in\mathcal{A}}$, it shows

$$\bigcup_{\lambda\in\Lambda} O_\lambda \subseteq \bigcup_{\alpha\in\mathcal{A}} B_\alpha = \text{int}(A).$$

Therefore,

$$\text{int}(A) = \bigcup_{\alpha\in\mathcal{A}} B_\alpha = \bigcup_{\lambda\in\Lambda} O_\lambda = \text{int}(A).$$

Since $O = \bigcup_{\lambda\in\Lambda} O_\lambda$ is a union of open fuzzy sets, it is also an open fuzzy sets and contains all open fuzzy sets O_λ in A. Hence, int(A) is open and it is the largest open fuzzy set contained in A. □

Since int(A) is the largest open fuzzy set contained in A, it leads us to the following corollary.

Corollary 3.2.12.

The fuzzy set A is open if and only if $A = \text{int}(A)$.

Proof. If A is open, then $A \subseteq \text{int}(A)$. Since int$(A)$ is the largest open fuzzy set contained in A, then $A = $ int(A). Conversely, if $A = $ int(A), A is open by theorem 3.2.11. □

3.3 Sequences of Fuzzy Set

Definition 3.3.1. Convergence of Fuzzy Set [Cha68]

A sequence of fuzzy sets $\{A_n, n = 1, 2, \ldots\}$ is said to be **eventually contained** in a fuzzy set A if there is an integer m such that, if $n \geq m$, then $A_n \subset A$. If the sequence $\{A_n\}$

is in a fts (X, δ), then we say the sequence converges to a fuzzy set A if and only if it is eventually contained in each nbhd of A.

Definition 3.3.2. Cluster Fuzzy Set [Cha68]

A sequence of fuzzy sets $\{A_n, n = 1, 2, \ldots\}$ is said to be **frequently contained** in a fuzzy set A if for each integer m there is an integer n such that $n \geq m$, and $A_n \subset A$. If the sequence $\{A_n\}$ is in fts (X, δ), then a fuzzy set A is a **cluster fuzzy set** of $\{A_n\}$ if and only if $\{A_n\}$ is frequently contained in every nbhd of A.

Definition 3.3.3. [Cha68]

Let N be the map from the set of non-negative integers to the set of non-negative integers. Then the sequence $\{B_i, i = 1, 2, \ldots\}$ is a subsequence of a sequence $\{A_n, n = 1, 2, \ldots\}$ if there is a map N such that $B_i = A_{N(i)}$ and for each integer m there is an integer n such that $N(i) \geq m$ whenever $i \geq n$.

Theorem 3.3.4. [Cha68]

If the $nbhd$ system of each fuzzy set in a fts (X, δ) is countable, then

(a) A fuzzy set A is open if and only if each sequence of fuzzy set $\{A_n, n = 1, 2, \ldots\}$ which converges to a fuzzy set B contained in A is eventually contained in A.

(b) If A is a cluster fuzzy set of a sequence $\{A_n, n = 1, 2, \ldots\}$ of fuzzy sets, then there is a subsequence of the sequence converging to A.

Proof. (a) "$\Rightarrow$": Assume A is open, then for every fuzzy set B that contained in A, A is a $nbhd$ of B. By definition 3.3.2, if $\{A_n\}$ is converges to B, then there is an integer $m \in \mathbb{Z}$ such that for all $n \geq m$, $A_n \subseteq B$. Since A is a nbhd of B, $\{A_n\}$ is eventually contained in A.

"$\Leftarrow$": Assume that each sequence of fuzzy set $\{A_n, n = 1, 2, \ldots\}$ which converges to a fuzzy set B contained in A is eventually contained in A. Then for each $B \subseteq A$, let $\{U_n | n = 1, 2, \ldots\}$ be the $nbhd$ system of B and let $V_n = \bigcap_{i=1}^{n} U_i$. Then $\{V_n | n = 1, 2, \ldots\}$ is a sequence of $nbhd$ that converges to B. By definition 3.3.2, it is eventually contained in each $nbhd$ of B. This shows $\exists m \in \mathbb{N}$ such that $\forall n \geq m$, $\{V_n\} \subseteq B \subseteq A$. Since V_n are nbhd of B, $\{V_n\} \subseteq A$, by theorem 3.2.7, A is open.

(b) Let $\{U_n | n = 1, 2, \ldots\}$ be the $nbhd$ system of A. Then, let $S_n = \bigcup_{i=1}^{n} U_i$ such that $S_{n+1} \subseteq S_n$ for each $n \in \mathbb{N}$. This shows the sequence $\{S_n\}$ is decreasing. Then for every non-negative i, by definition $3.3.3$, we may choose $N(i)$ such that $N(i) \geq i$ and $A_{N(i)} \subset S_i$. This shows $N(i)$ maps i to an index set in $\{A_n\}$ such that $A_{N(i)} \subset S_i$. Clearly, $\{A_{N(i)}\}$ is a subsequence of $\{A_n\}$. Since $\{S_n\}$ is converges to A, $\{A_{N(i)}\}$ must converges to A.

$\square$

CHAPTER 4

FUZZY CONTINUOUS FUNCTION

The concept of continuous is very important in Mathematics. In this chapter, we will study the continuous function between a fuzzy topological space to another fuzzy topological space.

4.1 Lowen's Fuzzy Topological Space

Before we study fuzzy continuous, let us look at another definition of fuzzy topological space. Let us recall the definition of Chang's[Cha68].

Definition 4.1.1.
Given a set X, a fuzzy topology on X is a family $\delta = \{A_j | j \in \mathcal{J}\}$ of fuzzy sets that satisfies the following properties:

(1) $\emptyset_X, \sqcup_X \in \delta$.

(2) Closed under finite intersection: If $A_1, A_2 \in \delta$, then $A_1 \cap A_2 \in \delta$.

(3) Closed under arbitrary union: If $A_j \in \mathcal{F}$ for all $j \in \delta$, then $\bigcup_{j \in \mathcal{J}} A_i \in \delta$.

If δ is a fuzzy topology for X, then the pair (X, δ) is a fuzzy topological space.

However, Lowen [Low76] pointed out that under Chang's definition, constant functions between fuzzy topological spaces are not necessarily continuous. This is a significant deviation from general topology, where constant functions are trivially continuous. Lowen supports his argument by providing concrete examples, which highlight the shortcomings of Chang's framework. To address this issue, Lowen proposed an alternative definition that guarantees the continuity of constant functions, ensuring greater consistency with general topological concepts.

Definition 4.1.2. Lowen's Fuzzy Topological Space [Low76]
Given a set X, a fuzzy topology on X is a family $\delta = \{A_j | j \in \mathcal{J}\}$ of fuzzy sets that satisfies the following properties:

(1) $\forall\ \alpha$, the constant function $\alpha \in \delta$.

(2) Closed under finite intersection: If $A_1, A_2 \in \delta$, then $A_1 \cap A_2 \in \delta$.

(3) Closed under arbitrary union: If $A_j \in \delta$ for all $j \in \delta$, then $\bigcup\limits_{j \in \mathcal{J}} A_i \in \delta$.

If δ is a fuzzy topology for X, then the pair (X, δ) is a fuzzy topological space.

We will use this concept of fuzzy topology throughout the sequel. For Chang's definition, we will refer as **quasi fuzzy topology**.

4.2 The function ω and ι

Let $\mathscr{J}(X)$ be the family of all topologies on X and $\mathscr{W}(X)$ be the set of all fuzzy topologies on X. On $\mathbb{R}$, we consider the topology $\mathscr{J}_r = \{(\alpha, \infty) \cup \{\emptyset\} | \alpha \in \mathbb{R}\}$. The topological space one obtains unit interval I the induced topology on $\mathscr{J}_r$ is denoted as I_r. Then we define the following maps.

Definition 4.2.1. Topologically Generated [Low76]
For the sets $\mathscr{J}(X)$ and $\mathscr{W}(X)$, we define the mapping

$$\begin{aligned} \iota : \mathscr{W}(X) &\to \mathscr{J}(X) \\ \delta &\mapsto \iota(\delta) \quad \forall \delta \in \mathscr{W}(X), \end{aligned}$$

where $\iota(\delta)$ is a initial topology on X for the family of "function" δ and the topological space I_r. Then we define the mapping

$$\begin{aligned} \omega : \mathscr{J}(X) &\to \mathscr{W}(X) \\ \mathscr{T} &\mapsto \omega(\mathscr{T}) \quad \forall \mathscr{T} \in \mathscr{J}(X), \end{aligned}$$

where $\omega(\mathscr{J}) = \mathscr{G}(\mathscr{J}, I_r)$ is a continuous function from $(X, \mathscr{J})$ to I_r. For every $\delta \in \mathscr{W}(X)$, δ is said to be **topologically generated** if $\delta = \omega(\mathscr{T})$ for some $\mathscr{T} \in \mathscr{J}(X)$.

Remark 4.2.2.

(a) $\mathscr{J}_r$ is indeed a topology.

- $\emptyset$ is open. $\mathbb{R} = \bigcup\limits_{k \in \mathbb{R}} (-k, \infty)$ is open.

- Any union of open set is an open set. Finite intersection of open set is also an open set. Namely, $\bigcap_{m=1}^{n}(\alpha_m, \infty) = (\max\{\alpha_1, \dots, \alpha_n\}, \infty)$ is an open set in $\mathbb{R}$.

(b) $I_r = \{(\alpha, 1] | \forall \alpha \in \mathbb{R}\}$ forms a topology.

- For $\alpha \leq 0$, $[0, 1]$ is an open in I_r.
- For $0 < \alpha < 1$, $(\alpha, 1)$ is an open set in I_r.
- For $\alpha \geq 1$, $\emptyset$ is an open set in I_r.

(c) $\iota(\delta) = \mathscr{T}$ is a topology.

- Fixed r where $(r, 1] \in I_r$. Then $\iota(\delta = \alpha) = \begin{cases} \emptyset, & \text{if } \alpha \geq r, \\ X, & \text{if } \alpha < r. \end{cases}$
- $\iota(\bigcup_i \delta_i) = \bigcup_i \iota(\delta_i)$ and $\iota(\bigcap_i^n \delta_i) = \bigcap_i^n \iota(\delta_i)$ are both open set in $\mathscr{T}$.

Hence, we can conclude for every $\delta \in \mathscr{W}(X)$,

$$\iota(\delta) = \{\iota_r(\mu) \,|\, \forall r \in [0, 1],\ \forall \mu \in \delta\} \text{ where } \iota_r(\mu) = \{x \in X, \mu(x) > r\}.$$

Hence $\iota_r : \delta \to \iota_r(\delta)$ is indeed a continuous function since it maps from an open set in δ to another open set in $\mathscr{T}$.

(d) Notice that $\omega(\mathscr{T})$ is a continuous function from $(X, \mathscr{T})$ to I_r if and only if for every $r \in [0, 1]$,

$$\mu^{-1}((r, 1]) = \{x \in X \,|\, \mu(x) > r\} \text{ is open in } X.$$

Moreover, $\omega(\mathscr{T})$ is a lower semicontinuous function from $(X, \mathscr{T})$ to I which equipped by Euclidean topology for every $\mathscr{T} \in \mathscr{J}(X)$.

Proposition 4.2.3.

(1) $\iota \circ \omega = id_{\mathscr{T}(X)}$.

(2) ι and ω are respectively an isotone surjection and isotone injection.

(3) $\omega \circ \iota(\delta)$ is the smallest topologically generated fuzzy topology which contains δ and it is denoted as $\bar{\delta}$.

(4) δ is topologically generated if and only if $\delta = \bar{\delta}$.

Proof. (1) Since $\omega(\mathscr{J}) = \mathscr{G}(X, I_r)$, by remark (c) and (d) we have $\iota(\omega(\mathscr{J})) = id_{\mathscr{J}(X)}$.

(2) Notice that if $\mathscr{T}_1 \subseteq \mathscr{T}_2$, then $\mathscr{T}_2$ is a refinement of $\mathscr{T}_1$.

For ι, if $\delta_1 \subseteq \delta_2$, then δ_2 has more open fuzzy set than δ_1. Then $\iota(\delta_2)$ has more open set than $\iota(\delta_1)$. This shows $\iota(\delta_1) \subseteq \iota(\delta_2)$. This shows ι is an isotone map. Now, for every $\mathscr{T} \in \mathscr{J}$, $\omega(\mathscr{T}) \in \mathscr{W}(X)$ is the element such that $\iota(\omega(\mathscr{T})) = \mathscr{T}$. Hence, ι is an isotone surjection.

For ω, for every $\mathscr{T}_1, \mathscr{T}_2 \in \mathscr{J}$, if $\mathscr{T}1 \subseteq \mathscr{T}_2$, then $\omega(\mathscr{T}_1) \subseteq \omega(\mathscr{T}_2)$. Hence, ω is an isotone map. Assume $\omega(\mathscr{T}_1) = \omega(\mathscr{T}_2)$, then we have

$$\iota(\omega(\mathscr{T}_1)) = \iota(\omega(\mathscr{T}_2) \Leftrightarrow \mathscr{T}_1 = \mathscr{T}_2$$

Hence, ω is an isotone injection.

(3) For every $\delta \in \mathscr{W}(X)$, $\iota(\delta) \in \mathscr{J}(X)$ is a topology on X. Hence, $\omega(\iota(\delta))$ is a topologically generated fuzzy topology. Now, we want to show $\bar{\delta} = \omega(\iota(\delta))$ is the smallest topologically generated fuzzy topology that contains δ.

Assume $\Delta = \omega(\mathscr{T})$ is a topologically generated fuzzy topology for some $\mathscr{T} \in \mathscr{J}(X)$. If $\delta \subseteq \Delta$, we want to show $\bar{\delta} \subseteq \Delta$. Then, we have

$$\delta \subseteq \omega(\mathscr{T}) = \Delta$$
$$\iota(\delta) \subseteq \iota(\omega(\mathscr{T})) = \mathscr{T}$$
$$\bar{\delta} = \omega(\iota(\delta)) \subseteq \omega(\mathscr{T}) = \Delta$$

This shows $\bar{\delta}$ is the smallest topologically generated fuzzy topology that contains δ.

Moreover, $\bar{\delta} = \bigcap_{\delta \subseteq \Delta} \Delta$ where Δ is topologically generated.

(4) If δ is topologically generated, then $\bar{\delta}$ is the smallest topologically generated fuzzy topology that contains δ, then $\bar{\delta} = \delta$. Conversely, if $\delta = \bar{\delta}$, then δ is topologically generated.

□

Lemma 4.2.4. The continuous function $f \in \mathscr{G}(I_r, I_r)$ is a strictly monotonic.

Proof. Let $f : (\alpha, 1] \to (a, 1]$ and $f : (\beta, 1] \to (b, 1]$. Then assume $a > b$. If $\alpha \leq \beta$, then $(\beta, 1] \subseteq (\alpha, 1]$ and $f(\beta, 1] = (b, 1] \subseteq (a, 1] = f(\alpha, 1]$ which contradicts to $a > b$. Hence, f is strictly monotonic. $\square$

Theorem 4.2.5.
(X, δ) is topologically generated if and only if for each continuous function $f \in \mathscr{G}(I_r, I_r)$ and for each $\nu \in \delta$, $f \circ \nu \in \delta$.

Proof. Assume (X, δ) is topologically generated. Since $\nu \in \mathscr{G}(\mathscr{T}, I_r) = \delta$ and $f \in \mathscr{G}(I_r, I_r)$, then $f \circ \nu \in \mathscr{G}(\mathscr{T}, I_r) = \delta$.

Conversely, assume $\mu \in \bar{\delta}$. Recall that $\bar{\delta} = \omega \circ \iota(\delta)$. This shows $\mu \in \mathscr{G}\big(\iota(\delta), I_r\big)$. Since a basis for $\iota(\delta)$ is provided by the finite intersections

$$\bigcap_{i=1}^{n} \nu_i^{-1}\big((r_i, 1]\big) \quad \text{for some } \nu_i \in \delta,\ r_i \in I;$$

this is equivalent to saying for any $r \in I$, any $x \in \mu^{-1}\big((r, 1]\big)$, $(r, 1]$ is open in I_r and $\mu^{-1}(r, 1]$ is open in $\iota(\delta)$ since $\mu \in \mathscr{G}\big(\iota(\delta), I_r\big)$. Hence, for every $x \in \mu^{-1}(r, 1]$, there exists finite open set $(r_i, 1]$ such that

$$x \in \bigcap_{i \in I_{r,x}} \nu_i^{-1}\big((r_i, 1]\big) \subseteq \mu_i^{-1}\big((r_i, 1]\big).$$

Now, we want to show μ is closed under some finite intersection and arbitrary union of basis of δ. Fix x and let $\mu(x) = k_x \in (r, 1]$, then $\forall x < k_x$, $\exists$ a finite index set I_r such that

$$x \in \bigcap_{i \in I_r} \nu_i^{-1}\big((r_i, 1]\big) \subseteq \mu_i^{-1}\big((r_i, 1]\big).$$

Then, $\forall r < k_x$ and $\forall i \in I_r$, let

$$\mu_{i,r}(y) = \big((r\chi_{(i_i,1]}) \circ \nu_i\big)(y) = \begin{cases} r, \text{ if } \nu_i(y) > r_i, \\ 0, \text{ if } \nu_i(y) \leq r_i. \end{cases}$$

where $\mu_{i,r} \in \delta$ and $r\chi_{(i_i,1]} \in \mathscr{G}(I_r, I_r)$. This indeed follows from our assumption $f \circ \nu$ for all $f \in \mathscr{G}(I_r, I_r)$ and $\nu \in \delta$. Then, let $\nu_r^x = \inf_{i \in I_r} \mu_{i,r} \in \delta$. Since I_r is finite and hence $\nu_r^x = \min_{i \in I_r} \mu_{i,r}$, then we have

$$\nu_r^x(y) = \begin{cases} r, \text{ if } \ \forall i \in I_r \text{ we have } \nu_i(y) > r_i, \\ 0, \text{ if } \ \exists j \in I_r \text{ such that } \nu_i(y) \leq r_j. \end{cases}$$

Hence if $\nu_r^x(y) = r$, then $\nu_i(y) > r_i$ for all $i \in I_r$. Since

$$y \in \bigcap_{i=1}^{n} \nu_i^{-1}(r_i, 1] \subseteq \mu^{-1}(r, 1],$$

therefore for every $y \in [0, 1]$ we have $\mu(y) > r$. This shows $\mu \geq \nu_r^x$ for every $x \in \mu^{-1}(r, 1]$ and $r < k_x$. Now, it is easily to seen that

$$\mu = \sup_{x \in X} \sup_{r < k_x} \nu_r^x(y) \in \delta.$$

Hence, if every $\mu \in \bar{\delta}$, then $\mu \in \delta$. This shows $\bar{\delta} = \delta$ which implies (X, δ) is topologically generated. $\square$

4.3 Function Between Two Spaces

Definition 4.3.1. Function Between Two Sets [Cha68, PL80b]
Let f be a function from X to Y. Let B be a fuzzy set in Y with membership function $\mu_B(x)$ for all x in X. Then the inverse of B, denoted as $f^{-1}[B]$, is a fuzzy set in X whose membership function is given by

$$\mu_{f^{-1}[B]}(x) = \mu_B(f(x)) \text{ for all } x \in X.$$

Conversely, let A be a fuzzy set in X with membership function $\mu_A(x)$ for all x in X. Then, the image of A, denoted as $f[A]$, is a fuzzy set in Y whose membership function is given by

$$\mu_{f[A]}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\mu_A(x)\} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

for all x in X and y in Y where $f^{-1}(y) = \{x | f(x) = y\}$.

Clearly, since $f : U_1 \to U_2$ is well-defined, this two equalities are true for all x in X. Now, let us prove some theorem.

Theorem 4.3.2. [Cha68, PL80b]
Let f be a function from X to Y. Then,

(a) $f^{-1}[B^c] = \{f^{-1}[B]\}^c$ for all $B \in Y$.

(b) $\{f[A]\}^c \subseteq f[A^c]$ for all $A \in X$.

(c) If $B_1 \subseteq B_2$, then $f^{-1}[B_1] \subseteq f^{-1}[B_2]$ for all $B_1, B_2 \in Y$.

(d) If $A_1 \subseteq A_2$, then $f[A_1] \subseteq f[A_2]$ for all $A_1, A_2 \in X$.

(e) $f[f^{-1}[B]] \subseteq B$ for all $B \in Y$.

(f) $A \subseteq f^{-1}[f[A]]$ for all $A \in X$.

(g) Let $f : X \to Y$ and $g : Y \to Z$. Then, $(g \circ f)^{-1}[C] = f^{-1}[g^{-1}[C]]$ for all $C \in Z$, where $g \circ f$ is the composition of g and f.

Proof. (a) For all fuzzy sets B in Y, by definition 4.1.2, we have

$$\begin{aligned}\mu_{f^{-1}[B^c]}(x) &= \mu_{B^c}(f(x)) \\ &= 1 - \mu_B(f(x)) \\ &= 1 - \mu_{f^{-1}[B]}(x) \\ &= \mu_{f^{-1}[B]^c}(x) \quad \forall x \in X.\end{aligned}$$

Then, we have proved the equality.

(b) For all fuzzy sets A in X, let us consider two cases:

(i) If $f^{-1}(y) \neq \emptyset$, then we have

$$\begin{aligned}\mu_{f[A]^c}(y) &= 1 - \mu_{f[A]}(y)) \\ &= 1 - \sup_{x \in f^{-1}(y)} \{\mu_A(x)\} \quad \forall x \in X.\end{aligned}$$

And,

$$\begin{aligned}\mu_{f[A^c]}(y) &= \sup_{x \in f^{-1}(y)} \{\mu_{A^c}(x)\} \\ &= \sup_{x \in f^{-1}(y)} \{1 - \mu_A(x)\} \\ &= 1 - \inf_{x \in f^{-1}(y)} \{\mu_A(x)\} \quad \forall x \in X.\end{aligned}$$

This shows $\mu_{f[A]^c}(y) \leq \mu_{f[A^c]}(y)$ for all $y \in Y$.

(ii) If $f^{-1}(y) = \emptyset$, by definition we have $\mu_{f[A]}(y)) = 0$. Then $\mu_{f[A]^c}(y) = \mu_{f[A^c]}(y) = 1$.

We has concluded that $\mu_{f[A]^c}(y) \leq \mu_{f[A^c]}(y)$ for all $y \in Y$. and hence we have the inequality.

(c) For all fuzzy sets B_1, B_2 in Y where $B_1 \subseteq B_2$, then we have $\mu_{B_1}(f(x)) \leq \mu_{B_2}(f(x))$ for all $x \in X$. Since $\mu_{f^{-1}[B]}(x) = \mu_B(f(x))$, then $\mu_{f^{-1}[B_1]}(x) \leq \mu_{f^{-1}[B_2]}(x)$. Then we have proved the inequality.

(d) For all fuzzy sets A_1, A_2 in X where $A_1 \subseteq A_2$, then we have $\mu_{A_1}(x) \leq \mu_{A_2}(x)$ for all $x \in X$. Let us consider two cases:

(i) If $f^{-1}(y) \neq \emptyset$, then $\mu_{f[A]}(y) = \sup\limits_{x \in f^{-1}(y)} \{\mu_A(x)\}$. Then, we have $\mu_{f[A_1]}(y) \leq \mu_{f[A_2]}(y)$ for all $y \in Y$.

(ii) If $f^{-1}(y) = \emptyset$,$\mu_{f[A_1]}(y) = \mu_{f[A_2]}(y) = 0$ for all $y \in Y$.

Hence, we have proved the inequality.

(e) For all fuzzy sets B in Y, let us consider two cases:

(i) If $f^{-1}(y) \neq \emptyset$, then we have

$$\begin{aligned}\mu_{f[f^{-1}[B]]}(y) &= \sup_{x \in f^{-1}(y)} \{\mu_{f^{-1}[B]}(x)\} \\ &= \sup_{x \in f^{-1}(y)} \{\mu_B(f(x))\} \\ &= \mu_B(y) \qquad \text{for all } y \in Y.\end{aligned}$$

(ii) If $f^{-1}(y) = \emptyset$, then $\mu_{f^{-1}[f[B]]}(y) = 0$ for all $y \in Y$. This shows $\mu_{f^{-1}[f[B]]}(y) \leq \mu_B(y)$ for all $y \in Y$.

Then, we have proved the inequality.

(f) For all fuzzy sets A in X,

$$\begin{aligned}\mu_{f^{-1}[f[A]]}(x) &= \mu_{f[A]}(f(x)) \\ &= \sup_{x \in f^{-1}(y)} \{\mu_A(x)\} \quad \text{for all } x \in X.\end{aligned}$$

This shows $\mu_{f^{-1}[f[A]]}(x) \geq \mu_A(x)$ for all $x \in X$. Hence, we have proved the inequality.

(g) Let f be a function from X to Y and g be a function from Y to Z. Let C be a fuzzy set in U_3, then

$$\begin{aligned}\mu_{(g\circ f)^{-1}[C]}(x) &= \mu_C(g(f(x))) \\ &= \mu_{g^{-1}[C]}(f(x)) \\ &= \mu_{f^{-1}[g^{-1}[C]]}(x) \qquad \text{for all } x \in X.\end{aligned}$$

Then, we have proved $(g \circ f)^{-1}[C] = f^{-1}[g^{-1}[C]]$ for all $C \in Z$.

□

Let us study an example.

Example 4.3.3.

Consider two fts $U = (X, \delta)$ and $V = (Y, \gamma)$. Define a function $f : U \to V$, where f is a constant function between two fts, such that $f(x) = y_0$ for all $x \in X$.

Now, consider two fuzzy sets: $A \subseteq V$ and $B \subseteq U$.

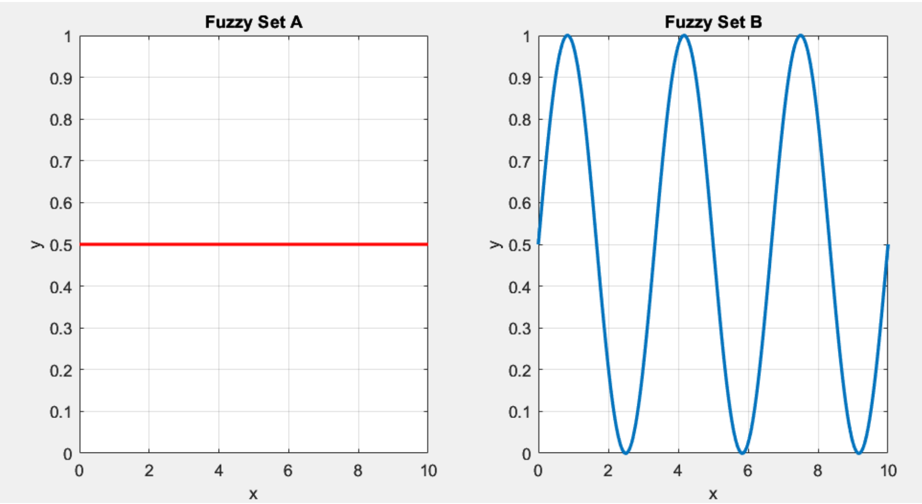

Figure 4.1: Fuzzy Topological Space U and V

By definition 4.3.1, consider any point on Y, e.g. $x = 2$. Then we have $\mu_B(2) = \mu_{f^{-1}[B]}(x)$ and $\mu_{f^{-1}[B]}(x) = 0.5$ is a constant function for all $x \in X$.Under Chang's definition, this constant function might not necessarily be continuous due to the lack of conditions ensuring the continuity of constant functions. In Chang's framework, only $\emptyset_X$ and $\sqcup_X$ are guaranteed to be part of the fuzzy topology.

By contrast, Lowen's definition rectifies this by explicitly requiring that constant functions be included in the fuzzy topology. This strengthens the overall structure, ensuring that the function f, defined as a constant between two fuzzy sets, is continuous by default. In doing so, Lowen addresses the deficiencies in Chang's original framework and provides a more robust foundation for fuzzy topological spaces.

4.4 Fuzzy Continuous

Definition 4.4.1. Fuzzy Continuous [Cha68, PL80b, Low76]
A function $\tilde{f}$ from a $fts\, U_1 = (X, \delta)$ to another $fts\, U_2 = (Y, \gamma)$ is said to be **fuzzy continuous**, or F-continuous in short, if every inverse of each γ-open fuzzy set is an δ-open fuzzy set. Namely, $\tilde{f}$ is F-continuous if and only if

$$\mu_{f^{-1}[\nu]} \in \delta \quad \forall \nu \in \gamma.$$

Remark 4.4.2. $f : X \to Y$ has dictates the relationship between (X, μ_A) $(Y, \mu_{f[A]})$ for all fuzzy set A in X which is the map $\tilde{f}$.

Corollary 4.4.3. [Cha68, PL80b]
If $\tilde{f}$ is a function from a $fts\ U_1 = (X, \delta)$ to another $fts\ U_2 = (Y, \gamma)$ and $\tilde{g}$ is a function from $U_2 = (Y, \gamma)$ to $U_3 = (Z, \lambda)$ where are both F-continuous, then the composition of the functions $\tilde{g} \circ \tilde{f}$ is also F-continuous.

Proof. If $\tilde{f} : U_1 \to U_2$ and $\tilde{g} : U_2 \to U_3$ are both F-continuous, then

$$[\tilde{g} \circ \tilde{f}]^{-1}[U_3] = \tilde{f}^{-1}[\tilde{g}^{-1}[U_3]].$$

For every $\mu \in \delta$, $\nu \in \gamma$ and $\eta \in \lambda$, since $\tilde{g}^{-1}[\eta]$ is γ-open and $\tilde{f}^{-1}[\nu]$ is δ-open, hence $[\tilde{g} \circ \tilde{f}]^{-1}[\eta] = \tilde{f}^{-1}[\tilde{g}^{-1}[\eta]]$ is δ-open. Hence, we have proved the composition of F-continuous function is still a F-continuous function. $\square$

Definition 4.4.4. [Low76]
A function $\tilde{f} : (X, \delta) \to (Y, \gamma)$ is said to be continuous if $\tilde{f} : \big(X, \iota(\delta)\big) \to \big(Y, \iota(\gamma)\big)$ is continuous. Namely,

$$f \in \mathscr{G}\big((X, \mathscr{T}), (Y, \mathscr{P})\big)$$

where $\mathscr{T} = \iota(\delta)$ and $\mathscr{P} = \iota(\gamma)$.

Remark 4.4.5. If δ and γ is topologically generated, then $\delta = \omega(\mathscr{T})$ and $\gamma = \omega(\mathscr{P})$ for some $\mathscr{T} \in \mathscr{J}(X)$ and $\mathscr{P} \in \mathscr{J}(Y)$. Then we have $\iota(\omega(\mathscr{T})) = \mathscr{T}$ and $\iota(\omega(\mathscr{P})) = \mathscr{P}$. This lead us to study the continuous property between general topologies and fuzzy topologies.

Proposition 4.4.6. [Low76]
Consider the following properties for $\tilde{f} : (X, \delta) \to (Y, \gamma)$:

(1) $\tilde{f}$ is F-continuous.

(2) $\tilde{f}$ is continuous.

(3) $\tilde{f} : (X, \bar{\delta}) \to (Y, \bar{\gamma})$ is F-continuous.

(4) $\tilde{f} : (X, \bar{\delta}) \to (Y, \gamma)$ is F-continuous.

then we have $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$.

Proof. For $(1) \Rightarrow (2)$, assume a basis for $\iota(\delta)$ and $\iota(\gamma)$ respectively are given by $\bigcap_{i=1}^{n} \mu_i^{-1}(r_i, 1]$ and $\bigcap_{j=1}^{n} \nu_j^{-1}(r_j, 1]$ where $\mu_i \in \delta$ and $\nu_j \in \gamma$. Recall that for $f : X \to Y$,

$$\mu_i(x) = f^{-1}\big(\nu_j(x)\big) = \nu_j\big(f(x)\big) \ \ \forall x \in X.$$

Then if $\tilde{f}$ is F-continuous, we have

$$f^{-1}(\nu_j) = \nu_j\big(f(x)\big) = \mu_i(x) \in \delta \ \ \forall \nu_j \in \gamma.$$

Recall that for each $y \in \nu_j^{-1}(r_j, 1]$ for some j, we have $\nu_j(y) \in (r_j, 1]$. Since $\nu_j^{-1}(r_j, 1]$ is an open set in $\big(Y, \iota(\gamma)\big)$, we have

$$\mu_i(x) = f^{-1}(\nu_j) = \nu_j\big(f(x)\big) \in (r_j, 1] \Rightarrow x \in \mu_i^{-1}(r_j, 1] \ \ \forall x \in X$$

where $\mu_i^{-1}(r_j, 1]$ is an open set in $\big(X, \iota(\delta)\big)$. Hence $\tilde{f}$ is continuous.

For $(2) \Leftrightarrow (3)$, assume $\tilde{f} : (X, \delta) \to (Y, \gamma)$ is continuous. Since $\omega = \mathscr{G}(\mathscr{J}, I_r)$, we have $\tilde{\mu} \in \mathscr{G}\big((X, \mathscr{T}), I_r\big)$ and $\tilde{\nu} \in \mathscr{G}\big((Y, \mathscr{P}), I_r\big)$ are both continuous. Since $\tilde{f}$ is continuous, then we have $\tilde{\mu}_i^{-1}(r_j, 1] = f^{-1}\big(\tilde{\nu}_j^{-1}(r_j, 1]\big) \in (X, \mathcal{J})$ for every $\tilde{\nu}_j(r_j, 1] \in \bar{\gamma}$. Since $\tilde{\mu}_i \in \mathscr{G}\big((X, \mathscr{T}), I_r\big)$, then we have $\tilde{f} : (X, \bar{\delta}) \to (Y, \bar{\gamma})$ is F-continuous. Conversely, assume $\tilde{f} : (X, \bar{\delta}) \to (Y, \bar{\gamma})$ is F-continuous. Notice that $\omega\big(\iota(\gamma)\big) = \bar{\gamma}$. Since $\tilde{f} : (X, \bar{\delta}) \to (Y, \bar{\gamma})$

is F-continuous and $\bar{\delta}$ is the smallest topologically generated fuzzy topology that contains δ, hence $\tilde{f}$ is continuous.

For (3) $\Leftrightarrow$ (4), assume $\tilde{f} : (X, \bar{\delta}) \to (Y, \bar{\gamma})$ is F-continuous. Then define a function g as $g : Y \to Y$ and define $\tilde{g} : (Y, \bar{\gamma}) \to (Y, \gamma)$. Notice that $\bar{\gamma}$ is the smallest topologically generated fuzzy topology that contains γ, namely

$$\nu \in \bar{\gamma} \ \ , \forall \nu \in \gamma.$$

Then $\tilde{g}$ is F-continuous. Since $\tilde{f}$ and $\tilde{g}$ are both F-continuous, then $\tilde{g} \circ \tilde{f} : (X, \bar{\delta}) \to (Y, \gamma)$ is F-continuous. Conversely, assume $\tilde{f} : (X, \bar{\delta}) \to (Y, \gamma)$ is F-continuous. Since (1) $\Rightarrow$ (2), then we have $\tilde{f} : \big(X, \iota(\bar{\delta})\big) \to \big(Y, \iota(\gamma)\big)$ is continuous. By (2) $\Rightarrow$ (3), we have $\tilde{f} : (X, \bar{\delta}) \to (Y, \bar{\gamma})$ is F-continuous.

□

Corollary 4.4.7. Let $\mathscr{G}(X, Y)$ be the set of all continuous function from (X, δ) to (Y, γ) and $\mathscr{G}_\omega(X, Y)$ be the set of all F-continuous function from (X, δ) to (Y, γ).If δ is topologically generated, then

$$\mathscr{G}(X, Y) = \mathscr{G}_\omega(X, Y).$$

Proof. By proposition 4.4.6, we know $\tilde{f} : (X, \bar{\delta}) \to (Y, \gamma)$ is F-continuous if and only if $\tilde{f} : (X, \delta) \to (Y, \gamma)$ is continuous. If $\delta = \bar{\delta}$, then we have $\mathscr{G}(X, Y) = \mathscr{G}_\omega(X, Y)$. □

Example 4.4.8. The inverse of corollary 4.4.7 is not true.
Consider two fts (X, δ) and (Y, γ) where $X = I = [0, 1]$ and Y is arbitrary. Let δ be the fuzzy topology on X with the subbasis

$$\{\text{constant } \alpha \,|\, \forall \alpha\} \cup \{y = x \,|\, \forall x \in [0, 1]\}$$

and let γ be the discrete fuzzy topology on Y, i.e., $\gamma = I^Y$. Since γ is a discrete fuzzy topology, then $\iota(\gamma)$ is discrete. Since $X = [0, 1]$ and it is generated by the subbasis, we have $\iota(\delta)$ is connected. Since $\iota(\delta)$ is connected and $\iota(\gamma)$ is discrete, any function maps $\iota(\delta)$ to $\iota(\gamma)$ must be a constant function. Hence we have

$$\mathscr{G}(X, Y) = \{\text{constant function from } X \text{ to } Y\}.$$

By proposition 4.4.6, we have $\mathscr{G}_\omega(X,Y) \subset \mathscr{G}(X,Y)$. Since constant functions are continuous, hence we have $\mathscr{G}_\omega(X,Y) = \mathscr{G}(X,Y)$. However, since $\iota(\delta) = I_r$ and $\mathscr{G}(I_r, I_r)$ is finer than δ, hence δ is not topologically generated.

Now, let us look at some theorem.

Theorem 4.4.9. [Cha68, PL80b]
Given $\tilde{f}$ is a function from $fts\ U_1 = (X, \delta)$ to $fts\ U_2 = (Y, \gamma)$, then we have the following statements and their relations: $(a) \Leftrightarrow (b), (c) \Leftrightarrow (d), (a) \Rightarrow (c)$ and $(d) \Rightarrow (e)$ where

(a) The function $\tilde{f}$ is F-continuous.

(b) The inverse of every fuzzy closed set is closed.

(c) For each fuzzy set A in U_1, the inverse of every nbhd of $\tilde{f}[A]$ is a $nbhd$ of A.

(d) For each fuzzy set A in U_1 and each $nbhd$ V of $\tilde{f}[A]$, there is a $nbhd$ W of A such that $\tilde{f}[W] \subseteq V$.

(e) For each sequence of fuzzy sets $\{A_n | n = 1, 2, \ldots\}$ in U_1 which converges to a fuzzy set A in U_1, the sequence $\{\tilde{f}[A_n] | n = 1, 2, \ldots\}$ converges to $\tilde{f}[A]$.

Proof. (i) $(a) \Leftrightarrow (b)$:
If $\tilde{f} : U_1 \to U_2$ is F-continuous and B is a fuzzy set in U_2, then $\tilde{f}^{-1}[B]$ is open. Since $\tilde{f}^{-1}[B^c] = \{\tilde{f}^{-1}[B]\}^c$, then $\tilde{f}^{-1}[B^c]$ is a closed fuzzy set. Conversely, if $\tilde{f}^{-1}[B^c]$ is closed, then $\tilde{f}^{-1}[B]$ is open and hence $\tilde{f}$ is F-continuous.

(ii) $(a) \Rightarrow (c)$
If $\tilde{f}$ is F-continuous and A is a fuzzy set in U_1, let V be the $nbhd$ of $\tilde{f}[A]$. Then, V contains an open $nbhd$ W of $\tilde{f}[A]$. Namely, $\tilde{f}[A] \subseteq W \subseteq V$ and hence $\tilde{f}[\tilde{f}^{-1}[A]] \subseteq \tilde{f}[W] \subseteq \tilde{f}[V]$. Since $\tilde{f}$ is F-continuous, then $\tilde{f}^{-1}[W]$ is open. By definition 3.2.1, $\tilde{f}^{-1}V$ is a $nbhd$ of A.

(iii) $(c) \Leftrightarrow (d)$
$\forall A \in U_1$, if V is a $nbhd$ of $\tilde{f}[A]$ and $\tilde{f}^{-1}[V]$ is $nbhd$ of A, then there is an open fuzzy set W such that $A \subseteq W \subseteq \tilde{f}^{-1}[V]$. By theorem 3.2.7 W is open if and only if it is a $nbhd$ of A, hence W is a $nbhd$ of A where $\tilde{f}[W] \subseteq V$. Conversely, for every fuzzy set $A \in U_1$ and every $nbhd$ V of $\tilde{f}[A]$, there is a $nbhd$ W of A such that $\tilde{f}[W] \subseteq V$.

Since $A \subseteq \tilde{f}^{-1}[\tilde{f}[A]]$, then $A \subseteq W \subseteq \tilde{f}^{-1}[f[W]] \subseteq \tilde{f}^{-1}[V]$. This shows $\tilde{f}^{-1}[V]$ is a $nbhd$ of A.

(iv) $(d) \Rightarrow (e)$

For each fuzzy set $A \in U_1$ and every $nbhd$ V of $\tilde{f}[A]$,there is a $nbhd$ W of A such that $\tilde{f}[W] \subseteq V$. For each sequence of fuzzy sets $\{A_n | n = 1, 2, \ldots\} \in U_1$ which converges to A, by definition $3.2.6$, $\{A_n\}$ is eventually contained in each $nbhd$ of A. Then $\exists m \in \mathbb{N}$ such that $\forall n \geq m, A_n \subseteq W$. Since $\tilde{f}[A_n] \subseteq \tilde{f}[W] \subseteq V$ for all $n \geq m$, then $\tilde{f}[A_N]$ is eventually contained in every $nbhd$ of $f[A]$. This shows $\{\tilde{f}[A_n] | n = 1, 2, \ldots\}$ is converges to $\tilde{f}[A]$.

□

Now, we are enough to define the fuzzy homeomorphism.

Definition 4.4.10. Fuzzy Homeomorphism [Cha68]

A **fuzzy homeomorphism** is an F-continuous one-to-one map of a fts $U_1 = (X, \delta)$ onto another fts $U_2 = (Y, \gamma)$ such that the inverse of the map is also F-continuous. Then, we said U_1 is F**-homeomorphic** to U_2 if there is a fuzzy homeomorphism $\tilde{f} : U_1 \to U_2$. If U_1 is F-homeomorphic to U_2, then we said U_1 is F**-topologically equivalent** to U_2.

Remark 4.4.11. We can consider the category of fuzzy topological spaces and fuzzy continuous mapping in the same way as the category of general topological spaces and continuous mapping. Moreover, the functions ω and ι induce two covariant functors between these two categories. Let $\mathscr{G}$ be the category of general topological spaces and $\mathscr{F}$ be the category of fuzzy topological spaces, then we define

$$\tilde{\omega} : \mathscr{G} \to \mathscr{F} \text{ where } \tilde{\omega}(X, \mathcal{J}) = \big(X, \omega(\mathcal{J})\big), \tilde{\omega}(f) = f \text{ and}$$
$$\tilde{\gamma} : \mathscr{F} \to \mathscr{G} \text{ where } \tilde{\gamma}(X, \delta) = \big(X, \omega(\delta)\big), \tilde{\gamma}(f) = f.$$

By corollary $4.4.7$, $\tilde{\omega}(\mathscr{G})$ is a full subcategory of $\mathscr{F}$.

CHAPTER 5

COMPACT FUZZY TOPOLOGICAL SPACE

In this chapter, we will study the compactness on quasi fuzzy topological space.

5.1 Compact Space in General Topology

Let us recall the concept of compactness from general topological spaces.

Definition 5.1.1. Cover and Subcover in General Topological Space [Mun00, Teo23]
In general topology, a collection $\mathcal{A}$ of subsets of a topological space $(X, \mathscr{T})$ is said to cover X, or to be a **covering** of X, if the union of the elements of $\mathcal{A}$ is equals to X. Namely,

$$X \subseteq \bigcup_{A \in \mathcal{A}} A$$

In case of a subcollection of $\mathcal{A}$ also cover X, we call it a **subcover**.

Definition 5.1.2. Open Covering
For a topological space $(X, \mathscr{T})$, a covering $\mathcal{A}$ is called an **open covering** of X if every members of $\mathcal{A}$ is an open subsets of X.

Definition 5.1.3. Compact Space
A topological space $(X, \mathscr{T})$ is said to be **compact** if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Example 5.1.4. Let $a, b \in \mathbb{R}$ with $a < b$, then the closed interval $[a, b]$ is compact.

Proof. Fixed $\epsilon > 0$. An open covering of (a, b) is given by the following:

$$(a, b) \subseteq O = \bigcup_{n=2}^{\infty} (a + \frac{b-a}{n}, b).$$

Then we define $S = O \cup (a - \epsilon, a + \epsilon) \cup (b - \epsilon, b + \epsilon)$ and S is an open covering of $[a, b]$. Now, fixed $n \in \mathbb{N}$ large enough such that $\frac{1}{n} < \epsilon$. Then

$$S' = \bigcup_{m=2}^{n} (a + \frac{b-a}{m}, b) \cup (a - \epsilon, a + \epsilon) \cup (b - \epsilon, b + \epsilon)$$

is a finite subcover of S that covers $[a, b]$. $\square$

Definition 5.1.5. Finite Intersection Property
Let X be a topological space, and let $\mathcal{C} = \{C_\alpha | \alpha \in J\}$ be a collection of subsets of X. Then, $\mathcal{C}$ has the **finite intersection property** if $\alpha_1, \ldots, \alpha_n$ are finitely many elements of J, the intersection $\bigcap_{k=1}^{n} C_{\alpha_k} \neq \emptyset$.

Theorem 5.1.6.
Let X be a topological space. Then the following statements are equivalent:

(a) X is compact.

(b) If $\mathcal{C} = \{C_\alpha | \alpha \in J\}$ is a collection of closed subset of X that has finite intersection property, then $\bigcap_{\alpha \in J} C_\alpha \neq \emptyset$

.

Proof. This theorem is general. Proof of this theorem will be referred to Munkres's book $Topology$ [Mun00], Chapter 3, Section 26, Theorem 26.9, Page 169-170. □

Theorem 5.1.7. Alexander Subbasis Theorem
Let $(X, \mathscr{T})$ be a topological space. If X has subbasis $\mathscr{B}$ such that every cover $\mathcal{A} = \{B_\alpha \,|\, \alpha \in \mathscr{B}\}$of X by elements of $\mathscr{B}$ has a finite subcover, then X is compact.

Proof. This theorem is general. Proof of this theorem will be referred to Mueger's book $Topology\, for\, the\, Working\, Mathematician$ [Müg20], Chapter 7, Page 128. □

5.2 Compact Space in Quasi Fuzzy Topological Space

In this section, let us discuss the compact structure on fuzzy topology. Similar with the general topology, let us first define the open covering of fuzzy space.

Definition 5.2.1. Cover and Subcover in Fuzzy Space [Cha68]
Let $\mathcal{A}$ be a collection of fuzzy sets on X. For a fuzzy set B on X, $\mathcal{A}$ is said to be a **covering** of B if

$$B \subseteq \bigcup_{A \in \mathcal{A}} A.$$

Moreover, this is equivalent to

$$\sup_{A \in \mathcal{A}} \{\mu_A(x)\} \geq \mu_B(x) \quad \forall x \in X.$$

A **subcover** $\mathcal{A}'$ of $\mathcal{A}$ is a subfamily of $\mathcal{A}$ which ia also a covering.

Definition 5.2.2. Open Covering in Fuzzy Topological Space [Cha68]
For a fuzzy set B on X and a fts $U = (X, \delta)$, a covering $\mathcal{A}$ is called an **open covering** of B if every member of $\mathcal{A}$ is an open fuzzy set, or $\mathcal{A} \subset \delta$.

Definition 5.2.3. Quasi Fuzzy Compactness [Cha68]
A fts $U = (X, \delta)$ is said to be **quasi fuzzy compact**, or quasi F-compact in short, if each open covering $\mathcal{A}$ of $\sqcup_X$ has a finite subcover $\mathcal{A}'$.

Remark 5.2.4. The following are equivalent:

- A fts $U = (X, \delta)$ is quasi F-compact.
- For every open covering $\mathcal{A}$ of $\sqcup_X$ such that

  $$\bigcup_{A \in \mathcal{A}} A = \sqcup_X,$$

  there is an finite open subcover $\mathcal{A}'$ of $\sqcup_X$ such that

  $$\bigcup_{A \in \mathcal{A}'} A = \sqcup_X.$$

- For every open covering $\mathcal{A}$ of $\sqcup_X$ such that

  $$\sup_{A \in \mathcal{A}} \{\mu_A(x)\} = 1 \quad \forall x \in X,$$

  there is an finite open subcover $\mathcal{A}'$ of $\sqcup_X$ such that

  $$\max_{A \in \mathcal{A}'} \{\mu_A(x)\} = \sup_{A \in \mathcal{A}'} \{\mu_A(x)\} = 1 \quad \forall x \in X.$$

Definition 5.2.5. Finite Intersection Property [Cha68]
Let U be a fts and let $\mathcal{A} = \{A_\alpha | \alpha \in J\}$ be a family of fuzzy sets. Then, $\mathcal{A}$ has **finite intersection property** if and only if there is a finite subfamily $\mathcal{A}' \subseteq \mathcal{A}$ such that the intersection $\bigcap_{A \in \mathcal{A}'} A \neq \emptyset_X$.

Theorem 5.2.6. [Cha68]
A fts U is quasi F-compact if and only if each family of closed fuzzy sets which has finite intersection property has a nonempty intersection.

Proof. Assume U is quasi F-compact and let $\mathcal{A}$ be a family of closed fuzzy sets in U. Suppose the intersection of all closed sets in $\mathcal{A}$ is empty. Namely,

$$\bigcap_{A \in \mathcal{A}} A = \emptyset_X.$$

By De Morgan's Law, we have

$$\bigcup_{A \in \mathcal{A}} A^c = \sqcup_X.$$

Since U is quasi F-compact and the open sets A^c forms an open cover of U, there exists a finite family $\mathcal{A}' \subseteq \mathcal{A}$ such that

$$\bigcup_{A \in \mathcal{A}'} A = \sqcup_X.$$

By applying De Morgan's Law again, we have

$$\bigcap_{A \in \mathcal{A}'} A = \emptyset_X.$$

This contradicts the finite intersection property. Hence, the intersection of all closed sets in $\mathcal{A}$ must be nonempty.

Conversely, assume that every family of closed fuzzy sets that satisfies the finite intersection property also has a nonempty intersection. Let $\mathcal{A}$ be an open cover of U. Then by taking the compliment, let $\mathcal{B} = \mathcal{A}^c$ be the family of closed fuzzy sets corresponding to $\mathcal{A}$. Since $\mathcal{A}$ covers U, we must have

$$\bigcap_{A^c \in \mathcal{B}} A^c = \emptyset_X.$$

By definition 5.2.5, there exists a finite family $\mathcal{B}' \subseteq \mathcal{B}$ such that

$$\bigcap_{A^c \in \mathcal{B}'} A^c = \emptyset_X.$$

By De Morgan's Law, there exists a finite family $\mathcal{A}' \subseteq \mathcal{A}$ corresponding to $\mathcal{B}'$ such that

$$\bigcup_{A \in \mathcal{A}'} A = \sqcup_X.$$

Hence, U is quasi F-compact. $\square$

Theorem 5.2.7. [Cha68]
Let f be a F-continuous function mapping the quasi F-compact fts $U_1 = (X, \delta)$ onto fts $U_2 = (Y, \gamma)$. Then, U_2 is quasi F-compact.

Proof. Let $\mathcal{B}$ be an open cover of U_2. Then, we have

$$\bigcup_{B \in \mathcal{B}} \mu_{f^{-1}[B]}(x) = \sup_{B \in \mathcal{B}} \{\mu_{f^{-1}[B]}(x)\} = \sup_{B \in \mathcal{B}} \{\mu_B f(x)\} = 1 \quad \forall x \in X.$$

Then, for all $B \in \mathcal{B}$, the family of all fuzzy sets of the form $f^{-1}[B]$, is an open cover of X which has finite subcover. Since f is onto, for all $B \in \mathcal{B}$, there exists an open fuzzy set $A \in U_1$ such that $f[A] = B$ and we have $A = f^{-1}[B]$. Hence, the family of images of members of the subcover is a finite subfamily of $\mathcal{B}$ is covers U_2 and hence U_2 is quasi F-compact. $\square$

5.3 Compact Space in Fuzzy Topological Space

In [Cha68], Chang give a definition of compactness for quasi fts which formally follow the definition of compactness in general topology. Chang's definition also be used in [Won73] and [Gog73]. However, under this definition $(X, \mathscr{T})$ is compact does not implies $\big(X, \omega(\mathscr{T})\big)$ is compact.

Let us look at a counter-example.

Example 5.3.1. Consider $(X, \mathscr{T}) = I_r$ be the unit interval $X = I$ with the usual topology. A function y is linear if it has the form $y = mx + c$. Then for every $x \in I$ where $x \neq 0$ and $x \neq 1$, we define $\mu_x(y)$ by the following:

$$\mu_x(y) = \begin{cases} 1 & \text{, if } y = x, \\ 0 & \text{, if } y \in [0, \frac{x}{2}] \cup [\frac{x+1}{2}, 1], \\ \frac{2}{x}y - 1 & \text{, if } y \in [\frac{x}{2}, x], \\ -\frac{2}{1-x}y + 1 + \frac{2x}{1-x} & \text{, if } y \in [x, \frac{x+1}{2}]. \end{cases}$$

We also define for

$$x = 0,\ \mu_0(y) = -y + 1 \ \forall y \in I,$$
$$x = 1,\ \mu_1(y) = y \qquad \forall y \in I.$$

Then for all $x \in I$, we have $\mu_x \in \omega(\mathscr{T})$ and the following property:

$$\sup_{x \in I} \mu_x(y) = 1 \ \ \forall y \in I.$$

However, there is not subfamily of $\omega(\mathscr{T})$ has this property.

Hence, Lowen has induced another form of compactness in [Low76].

Definition 5.3.2. Fuzzy Compact as A Set[Low76]
Let (X, δ) be a fts (or quasi fts). A fuzzy set $B = (X, \mu_B)$ is said to be **fuzzy compact**, or F-compact in short, if for all family $\mathcal{A} \subset \delta$ such that

$$\sup_{A \in \mathcal{A}} \mu_A(x) \geq \mu_B(x) \quad \text{for all } x \in X,$$

and for all $\epsilon > 0$, there exists a finite subfamily $\mathcal{A}' \subset \mathcal{A}$ such that

$$\max_{A \in \mathcal{A}'} \{\mu_A(x)\} = \sup_{A \in \mathcal{A}'} \mu_A(x) \geq \mu_B(x) - \epsilon \quad \text{for all } x \in X.$$

Definition 5.3.3. Fuzzy Compact as A Fuzzy Topological Space[Low76]
A fts (or quasi fts) (X, δ) ia said to be **fuzzy compact** if each constant fuzzy set in (X, δ) is F-compact.

Example 5.3.4. No fts can be quasi F-compact.
For a fts, $\sqcup_X$ has a covering $\mathcal{A} = \{$ all constant $\alpha \,|\, \forall \alpha \in [0, 1)\}$. But there is no finite subfamily $\mathcal{A}'$ of $\mathcal{A}$ such that $\sqcup_X = \bigcup\limits_{A \in \mathcal{A}'} A$.

Definition 5.3.5. Weakly Fuzzy Compact[Low76]
A fts (or quasi fts) (X, δ) is said to be **weakly *F*-compact** if $\sqcup_X$ is F-compact.

Remark 5.3.6.

- If (X, δ) is a quasi F-compact fts, then the constant fuzzy set $\sqcup_X$ is F-compact.
- Every quasi F-compact fts is weakly F-compact.

Theorem 5.3.7. [Low76]
A fuzzy topological space $\big(X, \omega(\mathscr{T})\big)$ if F-compact if and only if $(X, \mathscr{T})$ is compact.

Proof. Recall that a fts is F-compact if every constant fuzzy set α in fts is F-compact. Assume $(X, \mathscr{T})$ is compact. Let $\beta \subset \omega(\mathscr{T})$ such that $\sup\limits_{\mu \in \beta} \geq \alpha > 0$ and let $\epsilon > 0$ such that

$\alpha > \epsilon > 0$. For all $\mu \in \beta$, let $\mu^\epsilon = \mu + \epsilon$ and $[0, \alpha] = I_\alpha$. Then, $\forall \mu \in \beta$,

$$\mathscr{Q}(\mu^\epsilon) = \{(x, \alpha) \,|\, \mu^\epsilon(x) > \alpha\}$$

is an open set in $X \times I_\alpha$. Since $\bigcup_{\mu \in \beta} \mu^\epsilon = \sup_{\mu \in \beta} \mu^\epsilon = \sup_{\mu \in \beta}\{\mu\} + \epsilon \geq \alpha$, hence we have

$$X \times I_\alpha \subset \bigcup_{\mu \in \beta} \mathscr{Q}(\mu^\epsilon).$$

This shows the family $\{\mathscr{Q}(\mu^\epsilon) \,|\, \mu \in \beta\}$ forms an open cover of $X \times I_\alpha$. Moreover, $X \times I_\alpha$ is compact, then there exists a finite subfamily $\beta' \subset \beta$ such that

$$X \times I_\alpha \subset \bigcup_{\mu \in \mathcal{B}'} \mathscr{Q}(\mu^\epsilon).$$

Hence we have $\sup_{\mu \in \beta} \mu \geq \alpha - \epsilon$ which shows $\big(X, \omega(\mathscr{T})\big)$ is F-compact.

Conversely, assume $\big(X, \omega(\mathscr{T})\big)$ is F-compact. Suppose $\mathcal{A} \subset \mathscr{T}$ is an open cover of X. Then we have

$$X \subset \bigcup_{A \in \mathcal{A}} A \Leftrightarrow \sup_{A \in \mathcal{A}} \chi_A(x) = 1, \quad \forall x \in X$$

where χ_A is the characteristic function of A. Then, choose $\epsilon \in (0, 1)$. Notice that χ_A is a special case of fuzzy set. Since $\big(X, \omega(\mathscr{T})\big)$ is F-compact, then there exists $\mathcal{A}' \subset \mathcal{A}$ such that

$$\sup_{A \in \mathcal{A}'} \chi_A \geq 1 - \epsilon.$$

By definition of F-compact, $\mathcal{A}'$ is a finite subfamily of $\mathcal{A}$. Hence, $(X, \mathscr{T})$ is compact. $\square$

Proposition 5.3.8. [Low76]
Given $f : (X, \delta) \to (Y, \gamma)$ is F-continuous, if $\mu \in \delta$ is a F-compact fuzzy set, then $f(\mu)$ is F-compact in (Y, γ).

Proof. Let $\beta \subset \gamma$ such that $\sup_{\nu \in \beta} \nu \geq f(\mu)$, then we have $\sup_{\nu \in \beta} f^{-1}(\nu) \geq \mu$. Since f is F-continuous, then $f^{-1}(\nu) \in \delta$. Notice that μ is a F-compact set, then for all $\epsilon > 0$ there exists a finite subfamily $\beta' \subset \beta$ such that

$$\sup_{\nu \in \beta'} f^{-1}(\nu) \geq \mu - \epsilon.$$

Then we have

$$\sup_{\nu \in \beta'} \nu \geq f(\mu) - \epsilon$$

which shows $f(\mu)$ is F-compact in (Y, γ). $\square$

Moreover, since the inverse of constant fuzzy set is still a constant function, then we have the following proposition.

Proposition 5.3.9. [Low76]
If (X, δ) is F-compact and $f : (X, \delta) \to (Y, \gamma)$ is F-continuous, then (Y, γ) is F-compact.

Proof. Recall that a fts is F-compact if every constant fuzzy set is F-compact set. If (X, δ) is F-compact, then every $\mu \in \delta$ is a F-compact set. By our assumption, f is F-continuous and it shows f is an onto function. Then by proposition 5.3.8, we have every $\nu \in \gamma$, there is come F-compact set $\mu \in \delta$ such that $\nu = f(\mu)$ is a F-compact set. Hence we have (Y, γ) is F-compact. $\square$

Proposition 5.3.10. [Low76]
If (X, δ) is topologically generated F-compact, i.e. there exists a compact topology $\mathscr{T}$ such that $\delta = \omega(\mathscr{T})$, then every closed fuzzy set is a F-compact set.

Proof. Consider constant α such that $\alpha^c \in \omega(\mathscr{T})$ and $\beta \subset \omega(\mathscr{T})$ such that $\sup_{\mu \in \beta} \mu \geq \alpha$. Since $\alpha^c \in \omega(\mathscr{T})$, then we have

$$\mathscr{U}(\alpha) = \{(x, r) \,|\, \alpha(x) < r\} \text{ is an open set in } X \times I.$$

Notice that $X \times I = \mathscr{U}(\alpha) \cup \mathscr{U}(\alpha)^c$. Since $X \times I$ is compact and $\mathscr{U}(\alpha)^c$ is a closed subset of $X \times I$, hence $\mathscr{U}(\alpha)^c$ is compact. Choose $\epsilon > 0$ and let

$$\mu^\epsilon = \mu + \epsilon.$$

Then $\forall \mu \in \beta$, we define

$$\mathscr{Q}(\mu^\epsilon) = \{(x, r) \,|\, \mu^\epsilon(x) > r\} \subset X \times I$$

is an open set in $X \times I$. Then we have $\mathscr{U}(\alpha)^c \subset \bigcup_{\mu \in \beta} \mathscr{Q}(\mu^\epsilon)$. This shows the family $\{\mathscr{Q}(\mu^\epsilon) \,|\, \mu \in \beta\}$ is an open covering of $\mathscr{U}(\alpha)^c$. Since $\mathscr{U}(\alpha)^c$ is compact, then there exists finite subfamily $\beta' \subset \beta$ such that

$$\mathscr{U}(\alpha)^c \subset \bigcup_{\mu \in \beta'} \mathscr{Q}(\mu^\epsilon)$$

Since $\sup_{\mu \in \beta} \mu \geq \alpha$, then $\sup_{\mu \in \beta} \mu^\epsilon \geq \alpha$. This shows $\sup_{\mu \in \beta} \mu \geq \alpha - \epsilon$. Hence, every closed fuzzy set is F-compact. □

Remark 5.3.11. A F-compact (X, δ) is not necessary that every closed fuzzy set is a F-compact set. Counter-example will be given later.

Let us study the fuzzy form of Alexander Subbasis Theorem on Section 5.1, theorem 5.1.7. It will be used to study the products of F-compact space.

Theorem 5.3.12. [Low76]
(X, δ) is F-compact if and only if for any subbasis σ for δ, for any $\beta \subset \sigma$ and for any $\alpha > \epsilon > 0$ such that $\sup_{\mu \in \beta} \mu \geq \alpha$, there exists a finite subfamily $\beta' \subset \beta$ such that

$$\sup_{\mu \in \beta'} \mu \geq \alpha - \epsilon.$$

Proof. By definition 5.3.3, a fts is F-compact if every constant fuzzy set in (X, δ) is F-compact fuzzy set. Clearly, every constant function in δ must contained in σ, otherwise it cannot be contained in δ by definition of subbasis fuzzy topology. Hence, the only if part is trivial.

Conversely, assume (X, δ) is F-compact. Definition of F-compactness implies that for all family $\beta \subset \delta$ and for all constant $\alpha \in \beta$ such that

$$\sup_{\mu \in \beta} \mu(x) \ngeq \alpha \quad \text{for all } x \in X,$$

there exists a finite subfamily $\beta' \subset \beta$ and $\alpha > \epsilon > 0$ such that

$$\max_{\mu \in \beta'} \{\mu(x)\} = \sup_{\mu \in \beta'} \mu(x) \ngeq \alpha - \epsilon \quad \text{for all } x \in X.$$

Now, consider $\mathscr{G} = \{\beta \subset \delta \mid \sup_{\mu \in \beta'} \mu(x) \ngeq \alpha - \epsilon \text{ for all } \beta' \subset \beta\}$. Clearly, $|\mathscr{G}| < \infty$. This shows $\mathscr{G}$ is a finite set. Hence, for each $\beta \in \mathscr{G}$ there exists a maximal family $\tilde{\beta} \in \mathscr{G}$ that containing β. Now, if there exists $\mu \in \tilde{\beta}$ such that for all $x \in X$,

$$\mu(x) \geq \min_{1 \leq i \leq n} \{\gamma_i(x)\} \text{ where } \gamma_1, \gamma_2, \ldots, \gamma_n \in \delta.$$

then there exists $k \in \mathbb{N}$ such that $\gamma_k \in \tilde{\beta}$. Then we consider $\tilde{\beta} \cap \sigma$. Notice that, any basis of δ must be an open covering of δ. By our assumption from theorem, since $\beta \subset \sigma$, then we have $\sup \tilde{\beta} \cap \sigma \not\geq \alpha$. Now, we want to show $\sup \tilde{\beta} \leq \sup \tilde{\beta} \cap \sigma$.

For all $\mu \in \tilde{\beta}$, all $x \in X$ such that $\mu(x) > 0$, and all $\alpha > 0$ such that $\mu(x) > \alpha$, there exists $\nu_1^\alpha, \nu_2^\alpha, \ldots, \nu_n^\alpha \in \sigma$ such that

$$\mu(x) - \alpha < \min_{1 \leq i \leq n} \{\nu_i^\alpha(x)\} \leq \mu(x) \;\; \forall x \in X.$$

Since $\mu \in \tilde{\beta}$ and $\tilde{\beta}$ is largest set for some $\nu_k^\alpha \in \tilde{\beta}$, hence for all $a > 0$, there exists ν_k^α such that $\nu_k^\alpha > \mu(x) - \alpha$ and $\nu_k^\alpha \in \tilde{\beta} \cap \sigma$. Now, fix $x \in X$. Then for all $\mu \in \tilde{\beta}$ such that $\mu(x) > 0$ and for all $\alpha > 0$ such that $\mu(x) > \alpha$, there exists $\nu^\alpha \in \tilde{\beta} \cap \sigma$ such that $\nu^\alpha(x) > \mu(x) - \alpha$. This implies for all $\mu \in \tilde{\beta}$,

$$\sup(\tilde{\beta} \cap \sigma)(x) \geq \mu(x) \Rightarrow \sup \tilde{\beta} \cap \sigma \geq \sup \tilde{\beta}.$$

This shows $\sup \tilde{\beta} \not\geq \alpha$, which implies $\sup \beta \not\geq \alpha$. Hence, (X, δ) is F-compact. $\square$

Theorem 5.3.13. [Low76]
(X, δ) is weakly F-compact if and only if for any subbasis σ for δ, for any $\beta \subset \sigma$ such that $\sup_{\mu \in \beta} \mu = 1$ and for all $\epsilon > 0$, there exists a finite subset $\beta' \subset \beta$ such that

$$\sup_{\mu \in \beta'} \mu \geq 1 - \epsilon.$$

Proof. Proof of this theorem is similar to the proof of theorem 5.3.12. Fix $\alpha = 1$, then we will prove this theorem. $\square$

Example 5.3.14. In remark 5.3.11, we have noticed that if μ is closed in a F-compact (X, δ), then μ need not to be F-compact set. Let us look at an example:

Let $X = I$ and δ be the fuzzy topology with subbasis

$$\{\text{constant } \alpha\} \cup \{\mu_n \,|\, n \in \mathbb{N}\} \cup \{\mu \cup \mu^c\}$$

where for all $n \in \mathbb{N}$,

$$\mu_n(x) = \begin{cases} \frac{1}{3}, & \text{for all } x \in [0, \frac{1}{2} - \frac{1}{n+1}] \cup [\frac{1}{2} + \frac{1}{n+1}, 1]; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu(x) = \begin{cases} 0, & \text{if } x = \frac{1}{2}; \\ \frac{1}{3}, & \text{otherwise.} \end{cases}$$

By theorem 5.3.12, for any constant a, there exists some a finite family β' of σ such that

$$\sup_{\alpha \in \beta'} \alpha \geq a - \epsilon.$$

Hence, (X, δ) is F-compact. Notice that $\sup_{n \in \mathbb{N}} \mu_n = \mu$ and μ is a closed fuzzy set since $\mu^c \in \delta$. However, there is no finite subfamily of $\{\mu_n \,|\, n \in \mathbb{N}\}$ that covers μ.

Let us look at another example. By proposition 5.3.8, $\big(X, \omega(\mathscr{T})\big)$ is F-compact if and only if $(X, \mathscr{T})$ is compact. Obviously, if $\big(X, \iota(\delta)\big)$ is compact, then (X, δ) is F-compact. However, the converse is not true.

Example 5.3.15. Let $X = I$ and δ be the fuzzy topology with subbasis

$$\{\text{constant } \alpha\} \cup \{\nu \,|\nu(x) = x \text{ or } 0,\, \forall x \in X\} \cup \{\chi_0\}$$

where χ_0 is the Dirac function at 0:

$$\chi_0(x) = \begin{cases} 1, & \text{if } x = 0; \\ 0, & \text{otherwise.} \end{cases}$$

By theorem 5.3.12, it is obvious to see that (X, δ) is F-compact by taking finite subfamily $\{\alpha' \,|\, \alpha' = \alpha - \epsilon\}$ where $\{\alpha'\}$ only contains one elements. However, since $\{\nu\} \in \sigma$, $\iota(\delta)$ is discrete. Hence, $\iota(\delta)$ is not compact.

CHAPTER 6

APPLICATIONS AND POTENTIAL STUDY

6.1 Application of Fuzzy Mathematics

One significant application of fuzzy topological spaces is fuzzy decision making system [BY20]. This decision-making-system is the collection of single or multicriteria techniques aiming at selecting the best alternative in case of imprecise, incomplete, and vague data.

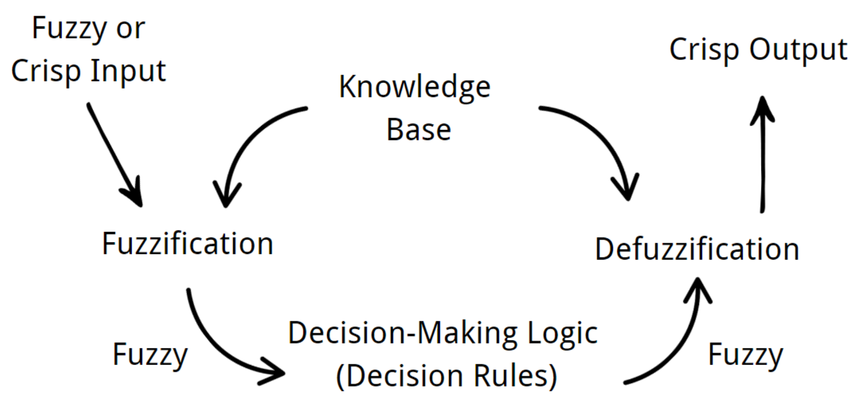

Figure 6.1: Basic Configuration of Fuzzy Decision-Making-System

Moreover, in control systems, fuzzy topological concepts are used to design fuzzy controllers that manage complex systems, like autonomous vehicles or industrial machinery, where exact parameters may be unavailable.

In image processing [Per15], fuzzy topology improves edge detection and noise reduction by considering gradual transitions between image regions instead of sharp boundaries. In artificial intelligence and machine learning, it supports inference, clustering and classification tasks under uncertainty, especially for datasets with fuzzy boundaries.

Additionally, fuzzy topological spaces find applications in economics, biological sciences, and social network analysis, where relationships and behaviors often exhibit uncertainty.

Their ability to generalize traditional topology while incorporating fuzziness makes them a powerful tool for modeling real-world systems with inherent imprecision.

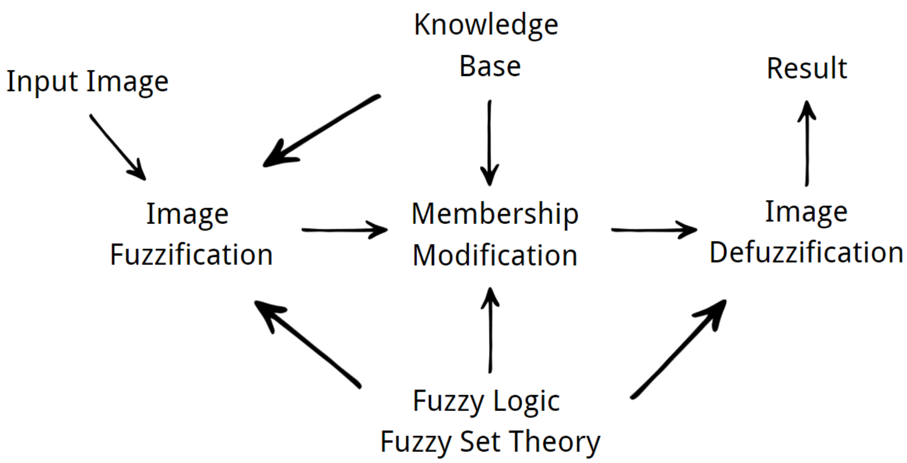

Figure 6.2: Basic Configuration of Fuzzy Image Processing

6.2 Potential Study

The study of fuzzy topological spaces offers great potential for the development of theoretical and applied mathematics. It also provides valuable extensions to general topology. By incorporating fuzziness into topological structures, fuzzy topology provides a powerful framework for modeling uncertainty and imprecision in complex systems. This work highlights the fundamental properties of fuzzy topological spaces and their applications in various fields.

In modern world, AI plays an increasingly important role such as OpenAI's Chat GPT and Google's Gemini, so that the importance of fuzzy mathematics cannot be overemphasized. By generalizing general topological concepts to fuzzy environments, we can deepen our understanding of the continuity, compactness and convergence of spaces reflecting the real-world dimensions of ambiguity. Moreover, since computer computing systems, especially in the field of fuzzy control theory and decision making, need to operate under fuzzy conditions, fuzzy topology will definitely play an important role in the future.

In the fields of data science, machine learning and artificial intelligence, fuzzy topologies provide a natural way to deal with noisy, incomplete or imprecise data, leading to more robust analyses and decisions. Their application also extends to the fields of biological, social and economic modeling, where plays a crucial role in fuzzy system behavior.

Finally, the continued exploration of fuzzy topological spaces will lead to significant innovations in mathematical theory and practical applications, especially in areas where a more flexible and detailed description of reality is required. As this area of research continues to grow, it is expected to further our understanding of mathematical structures and the complex systems we have tried to model and control.

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