3.1 Discrete distributions

A Random variable X is discrete if the range of X is countable.

Uniform $X \sim \text{Uniform}(1, N)$ if
$$P(X=n) = \frac{1}{N}, \quad n=1,\dots,N$$

we write: $P(X=n \mid N) = \frac{1}{N}$

$\uparrow$ parameter

Easy to verify that
$$\mathbb{E}X = \frac{1}{N}(1+\dots+N) = \frac{1}{N}\frac{N(N+1)}{2} = \frac{N+1}{2}$$

$$\mathbb{E}X^2 = \frac{1}{N}(1^2+\dots+N^2) = \frac{1}{N}\frac{N(N+1)(2N+1)}{6} = \frac{(N+1)(2N+1)}{6}$$

$$\operatorname{var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{(N+1)(2N+1)}{6} - \frac{(N+1)^2}{4}$$
$$= \frac{(N+1)(N-1)}{12}$$

Hypergeometric distributions

Experiment: N balls : M red , $N-M$ green

Pick K balls uniformly at Random (without replacement)

Let X = number of reds ,

$$\boxed{P(X=x \mid N, M, K) = \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}} \quad , x = 0, \ldots, K}$$

Notes:

- The counting argument implies $\sum_{x=0}^{K} \binom{M}{x}\binom{N-M}{K-x} = \binom{N}{K}$.

- $$\mathbb{E}X = \sum_{x=0}^{K} \underbrace{x\binom{M}{x}}\binom{N-M}{K-x} \underbrace{\frac{1}{\binom{N}{K}}}$$

$$= \sum_{x=1}^{K} M\binom{M-1}{x-1}\binom{N-M}{K-x}\left(\frac{N}{K}\binom{N-1}{K-1}\right)^{-1}$$

$$= \frac{KM}{N} \sum_{x=1}^{K} \binom{M-1}{x-1}\binom{N-M}{K-x} \Big/ \binom{N-1}{K-1}$$

$y = x-1$

$$= \frac{KM}{N} \sum_{y=0}^{K-1} \binom{M-1}{y}\binom{(N-1)-(M-1)}{K-1-y} \Big/ \binom{N-1}{K-1}$$

$$= \frac{KM}{N} \sum^{K-1} P(Y=y \mid N-1, M-1, K-1)$$

$$= \frac{KM}{N} \sum_{y=0}^{K-1} P(Y=y \mid N-1, M-1, K-1)$$

$$= KM/N.$$

- $\mathrm{Var}\, X = \frac{KM}{N} \frac{(N-M)(N-K)}{N(N-1)}.$

Example. $N = 25$ items
$M = \#$ defective items

We sample K items from the lot, say $K=10$, and find that none is defective. What can we say about M?

From above $X \mid N, M, K \sim \text{Hypergeometric}(N, M, K)$

So

$$P(X=0 \mid N=25, M=6, K=10) = \frac{\binom{M}{0}\binom{N-M}{K-0}}{\binom{N}{K}} = \frac{\binom{6}{0}\binom{19}{10}}{\binom{25}{10}} = 0.028$$

$$\Rightarrow P(X=0 \mid N=25, K=10, M \geq 6) \leq 0.028.$$

"if $M \geq 6$ then the observed event "$X=0$" is highly unlikely"

Bernoulli

$X \sim \text{Bernoulli}(p)$, $0 \le p \le 1$

if $\begin{cases} P(X=1 \mid p) = p \\ P(X=0 \mid p) = 1-p \end{cases}$

$$\mathbb{E}X = 1.p + 0.(1-p) = p$$

$$\mathbb{E}X^2 = \mathbb{E}X = p$$

$$\text{var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = p - p^2 = p(1-p)$$

Binomial

Experiment : Perform n independent and identically distributed (i.i.d.) Bernoulli trials (p).

Let Y be the number of 1's (1 = success)

we call

$$Y \sim \text{Binomial}(n,p)$$

$$P(Y=y \mid n,p) = \binom{n}{y} p^y (1-p)^{n-y}.$$

$\begin{cases} \mathbb{E}Y = np \\ \text{var}Y = np(1-p) \\ M_X(t) = (pe^t + 1-p)^n \end{cases}$ from last chapter.

Poisson $X \sim \text{Poisson}(\lambda)$, $\lambda > 0$

if

$$P(X = x \mid \lambda) = e^{-\lambda} \frac{\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

Note : $e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \ldots + \frac{\lambda^x}{x!} + \ldots$

Poisson distribution arises in modelling the following experiments:

- the number of buses arriving within a window of time
- the number of stars observed in a $\square$ of skymap
- the number of incidents found in a segment of a network ...

Easy Computation

$$\mathbb{E}X = \sum_{x=0}^{\infty} x \, e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}$$

$$= e^{-\lambda} \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda.$$

Similar computation gives $\operatorname{var} X = \lambda$.

Last chapter : $M_X(t) = e^{\lambda(e^t - 1)}$.

Question. "A call operator handles on average 5 calls/3 min
what is the probability that there'll be
NO calls in the next minute?".

Solution.

- Let X be the number of calls in the next minute.
- Assume $X \sim$ Poisson (λ)

Then $\mathbb{E}X = \lambda = 5/3$.

$$P(X=0 \mid \lambda = 5/3) = e^{-5/3} \frac{\left(\frac{5}{3}\right)^0}{0!} = e^{-5/3} = .189.$$

Recall

Poisson approximation of binomial distributions

if $X_n \sim$ Binomial (n, p_n)

and $np_n \to \lambda$ as $n \to \infty$

Then $X_n \xrightarrow{d} Y$ where $Y \sim$ Poisson (λ).

Question: - A typesetter on average makes one error per 500 words typeset.

- A typical page has 300 words

what is the probability of ≤ 2 errors in a five-page essay?

Solution

- Let X be the number of error in five pages
- Assume

$$X \sim \text{Binomial}(n, p)$$

where $n = 300 \times 5 = 1500$

$p = 1/500$.

Then

$$P(X \leq 2 \mid n, p) = \sum_{x=0}^{2} \binom{1500}{x} \left(\frac{1}{500}\right)^{x} \left(\frac{499}{500}\right)^{1500-x}$$

$$= 0.4230$$

- Use Poisson approximation : $X \overset{d}{\approx} Y$

$Y \sim \text{Poisson}(\lambda)$, $\lambda = np = 3$.

then

$$P(X \leq 2 \mid \lambda) \approx P(Y \leq 2 \mid \lambda)$$

$$= e^{-3}\left(1 + 3 + \frac{3^2}{2!}\right) = 0.4232.$$

Negative Binomial

Experiment: Count number of independent Bernoulli (p) trials until obtaining r successes (1's).

$$X \sim \text{Neg Binomial}(r, p), \quad 0 \le p \le 1, \ r \in \mathbb{N}_+$$

if $$\boxed{P(X = x \mid p, r) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = r, r+1, \ldots}$$

Let $Y = X - r$ (# of failures) then

$$P(Y = y \mid p, r) = P(X = y + r \mid p, r) = \binom{y+r-1}{r-1} p^r (1-p)^y$$

$$y = 0, 1, 2, \ldots$$

Since

$$\binom{y+r-1}{r-1} = \binom{y+r-1}{y} = \frac{(y+r-1)\ldots(r+1)r}{y!}$$

$$= (-1)^y \frac{(-r)(-r-1)\ldots(-r-y+1)}{y!}$$

$$=: (-1)^y \binom{-r}{y}$$

$$\boxed{P(Y = y \mid p, r) = (-1)^y \binom{-r}{y} p^r (1-p)^y}$$

this gives the name "negative binomial"

Additional facts

$$EY = \sum_{y=0}^{\infty} y \binom{y+r-1}{r-1} p^r (1-p)^y$$

$$= r(1-p)/p \qquad \text{(check!)}$$

$$\text{var} Y = r(1-p)/p^2 \qquad \text{(check!)}$$

set $\mu = r(1-p)/p \Rightarrow \frac{1-p}{p} = \frac{\mu}{r} \Rightarrow \frac{1}{p} = \frac{\mu+r}{r}$

Hence $\text{var} Y = \frac{EY}{p} = \frac{\mu(\mu+r)}{r}$

$$\boxed{\text{var} Y = \frac{1}{r}\mu^2 + \mu}$$

Quadratic Relation

Poisson Approximation

if $r \to \infty$, $p \to 1$ such that $r(1-p) \to \lambda$

Then $\begin{cases} EY \longrightarrow \lambda \\ \text{var} Y \longrightarrow \lambda \end{cases}$

Moreover $Y \xrightarrow{d} \text{Poisson}(\lambda)$ (exercise!)

Geometric

$X \sim \text{Geometric}(p)$

if $P(X=x \mid p) = p(1-p)^{x-1}$, $x = 1, 2, \ldots$

This is special case of Neg Binomial $(p, r=1)$

So $\begin{cases} \mathbb{E}X = (1-p)/p \\ \text{var}X = (1-p)/p^2. \end{cases}$

Note: $P(X > x) = \sum_{k=x+1}^{\infty} p(1-p)^{k-1} = (1-p)^x.$

So if $s > t$

$$P(X>s \mid X>t) = \frac{P(\{X>t\} \cap \{X>s\})}{P(X>t)}$$

$$= \frac{P(X>s)}{P(X>t)} = (1-p)^{s-t}$$

$$\boxed{P(X>s \mid X>t) = P(X > s-t)}$$

this is called the "Memoryless" property