Week 1

1.1: Sets and Probabilities

- Probability theory can be viewed as calculus of random variables
- Random variables are our devices for capturing random phenomena or outcomes of experiments
  - + natural or engineered

Def (Sample space): S is the space of all possible outcomes of a particular experiment

Remark: can be concrete, complicated, or downright abstract

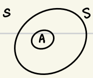
e.g. . coin tossing $\rightarrow S = \{H, T\}$; binary outcome

. SAT scores $\rightarrow S = \{200, \dots, 800\}$; finite outcome

. Human heights $\rightarrow S = (0, 7)$ ft ; continuous (infinite) outcome

. Text messages from chatGPT $\rightarrow S = ?$ } countably infinite

Def (event): A collection of possible outcomes, i.e. any subset of a sample space S



note: $A \subset S$

- We'll be speaking of "probability of an event" $\equiv$ Formally, probability of sets (subsets of S)
- Most events of interest can be described by the usual operations on sets, i.e.

Union	$A \cup B$	commutative, associative
Intersection	$A \cap B$	Distributive, deMorgan's
complementation	A^c	

Set Theory Basics

- Let set S be a collection of elements
- A subset A of S is a collection of elements in S
  - + A is also a set
- Given any two sets A and B,

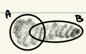
$A \subset B$ i.f.f. $x \in A \Rightarrow x \in B$

$A = B$ i.f.f. $A \subset B$ and $B \subset A$.

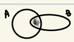
- $\emptyset$: empty set

operations of sets

$\boxed{\text{Union: } A \cup B := \{x \mid x \in A \text{ or } x \in B\}}$



$\boxed{\text{Intersection: } A \cap B := \{x \mid x \in A \text{ and } x \in B\}}$



$\boxed{\text{Complementation: } A^c := \{x \mid x \notin A\}}$

THM (Laws on operations of sets):

Commutativeness: $A \cup B = B \cup A$ $\quad \forall A, B \subset S$
(1) $A \cap B = B \cap A$
(order of operands)

Associativity: $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$ (a)
(2) $A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$
(grouping of operands)

Distributive law: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (a)
(3) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

De Morgan's law: $(A \cup B)^c = A^c \cap B^c$
$(A \cap B)^c = A^c \cup B^c$ (b)

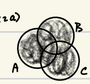
Proof. By def.

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

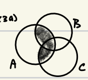
$= \{x \mid x \in B \text{ and } x \in A\}$

$\overset{\text{def}}{=} B \cap A.$ ∎

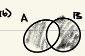
(2a)



(3a)



(4b)



Proof (3a). To show $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$

indeed, if $x \in A \cap (B \cup C)$

then $x \in A$ and $x \in B \cup C$

Now, $x \in B \cup C \Rightarrow x \in B$ or $x \in C$

$\Rightarrow x \in A \cap B$ or $x \in A \cap C$

$\overset{\text{union}}{\Rightarrow} x \in (A \cap B) \cup (A \cap C)$.

Similarly, we can show

$(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$, i.e.

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

and $A \cap C = \{x \mid x \in A \text{ and } x \in C\}$

so, $(A \cap B) \cup (A \cup C)$ implies

$(x \in A \text{ and } x \in B)$ or $(x \in A \text{ and } x \in C)$

$\Rightarrow x \in A$ and $(x \in B$ or $x \in C)$.

$\Rightarrow x \in A \cap (B \cup C)$.

Thus, we have shown equality. ∎

EX: Select a card at random s.t. $S = \{C, D, H, S\}$ (suits)

Suppose $A = \{C, D\}$, $B = \{H\}$, $C = \emptyset$, etc.

EX: Let S = all possible heights of a human.

An event is a subset of real numbers in $(0, 7)$ ft or $(0, 213.36)$ cm

* want to put prob. numbers on (large) number of events

$\rightarrow$ easy if S finite

$\rightarrow$ challenging if S infinite

Countable unions & intersections

Q: Given $A_1, A_2, \ldots \subset S$, (Defining countable union).

What is meant by $A_1 \cup A_2 \cup \cdots = \bigcup_{i=1}^{\infty} A_i$

$A_1 \cup A_2 = \{x \mid x \in A_1 \text{ or } x \in A_2\}$

$A_1 \cup A_2 \cup A_3 = (A_1 \cup A_2) \cup A_3$

$A_1 \cup A_2 \cup \cdots \cup A_k$ is well-defined for any $k < \infty$

$$\boxed{\begin{aligned} \bigcup_{i=1}^{\infty} A_i &= \{x \mid x \in A_i \text{ for some } i < \infty\} \\ &= \{x \in S : x \in A_i \text{ for some } i\} \\ &= \lim_{n \to \infty} \bigcup_{i=1}^{n} A_i \end{aligned}}$$

Now, define countable intersection, i.e.

$$\boxed{\begin{aligned} \bigcap_{i=1}^{\infty} A_i &= \{x \mid x \in A_i \text{ for all } i = 1, \ldots \infty\} \\ &= \{x \in S : x \in A_i \text{ for all } i\} \\ &= \lim_{n \to \infty} \bigcap_{i=1}^{n} A_i \end{aligned}}$$

It can be verified that de Morgan's law holds for infinite collections of subsets too, i.e.

$$\boxed{\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c}$$

Proof. $\omega \notin \bigcup_i A_i \Leftrightarrow \forall i, \omega \notin A \Leftrightarrow \forall i, \omega \in A_i^c \Leftrightarrow \omega \in \bigcap_i A_i^c$. ∎

Sigma Algebra

Def: A collection of subsets of S is called a sigma-algebra (or Borel field) denoted by $\mathcal{B}$, if:

i) $\emptyset \in \mathcal{B}$

ii) $A^c \in \mathcal{B}$ whenever $A \in \mathcal{B}$; $\mathcal{B}$ closed under complementation

iii) If $A_1, A_2, \ldots \in \mathcal{B}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$; $\mathcal{B}$ closed under countable unions

Remarks: • $S \in \mathcal{B}$

• $\mathcal{B}$ is also closed under countable intersections (by deMorgan's)

i.e. If $A_1, A_2, \ldots \in \mathcal{B}$ then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{B}$

• The countable operations allow us to capture sufficiently large collection of events of interest

Proof. $A, B \in \mathcal{B} \Rightarrow A^c \in \mathcal{B}, B^c \in \mathcal{B}$

$\overset{(iii)}{\Rightarrow} A^c \cup B^c \in \mathcal{B}$

$\overset{dM}{\Rightarrow} (A \cap B)^c \in \mathcal{B}$

$\overset{(iii)}{\Rightarrow} A \cap B \in \mathcal{B}$. ∎

Examples:

(1) Let $S = \{1, 2, \ldots, n\}$, a finite set.

Take $\mathcal{B}$ to be a sigma-algebra containing every element of S, i.e.

$\{1\}, \{2\}, \ldots, \{n\} \subset \mathcal{B}$

Then, $\mathcal{B}$ = a collection of all subsets of S

s.t. $|\mathcal{B}| = 2^n$

↳ each of the n elements doubles the number of subsets, hence the cardinality of $\mathcal{B}$ is 2^n

↳ # elements in set

note: singleton is a set containing one element

(2) Let $S = \mathbb{Z}$ (all integers)

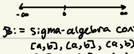
$= \{0, \pm 1, \pm 2, \ldots\}$

where $\mathcal{B}_1 := \{\emptyset, S$, subset of all odd numbers, subset of all even numbers$\}$

$\mathcal{B}$:= collection of all subsets of $\mathbb{Z}$.

Note: $\mathcal{B}_1$ has 4 numbers but $\mathcal{B}$ is infinite and uncountable

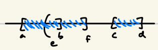
(3) Let $S = (-\infty, \infty) = \mathbb{R}$, the real line.



Let $\mathcal{B}$:= sigma-algebra containing all sets of forms

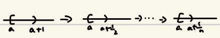
$[a, b], (a, b], (a, b), [a, b)$, $a, b \in \mathbb{R}$.

Q: How to imagine $\mathcal{B}$, e.g.



Let $\mathcal{B}'$:= σ-algebra containing all sets of form $[a, b)$ for $a, b \in \mathbb{Q}$ (rationals)

Proof (via picture).



$\to \{a\}$ for $n \to \infty$.

Proof. I will show that $\{a\} \in \mathcal{B}'$.

observe that $\{a\} = \lim_{n \to \infty} \bigcap_{i=1}^{n} [a, a + \frac{1}{n})$, $n \in \mathbb{N}$

$\Rightarrow \{a\}$ closed under countable intersection.

$\Rightarrow \{a\} \in \mathcal{B}'$. ∎

Recall:

Def (Real Number): any number that can be placed on the number line

$\boxed{\text{Numbers: Real} = \text{Rational} \cup \text{Irrational}}$

Def (Rational Number): Any number that can be written as a fraction

$\frac{p}{q}$ where $\begin{cases} \cdot\ p \text{ and } q \text{ are integers} & (1) \\ \cdot\ q \neq 0 & (2) \end{cases}$,

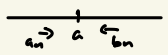
e.g. integers, fractions, terminating & repeating decimals

Def (Irrational Number): real number that violates (1) and/or (2)

s.t. its decimal expansion goes on forever,

e.g. π, e, square roots of non-perfect squares ($\sqrt{2}$)

- Now suppose $\{a\} \in \mathbb{R}$.
  - Q: is $\{a\} \in \mathcal{B}$ also?
  - P. Let a_n, b_n be sequence of rational numbers s.t. $a_n \uparrow a$ and $b_n \downarrow a$



Then, $\{a\} = \lim_{n\to\infty} \bigcap_{i=1}^{n} [a_n, b_n)$. ∎

<u>Remark</u>: It's incredible that

$\mathcal{B} = \mathcal{B}' :=$ Borel sigma-algebra of $\mathbb{R}$.

Probability Functions

<u>Def</u>: Given a sample space S,

a sigma algebra $\mathcal{B}$ associated w/ S,

a <u>probability function</u> (distribution, measure) is a function P on $\mathcal{B}$ that satisfies

1. $P(A) \geq 0 \quad \forall A \in \mathcal{B}$
2. $P(S) = 1$
3. If $A_1, A_2, \dots, \in \mathcal{B}$ are pairwise disjoint then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

<u>Remark</u>:
- A and B are disjoint i.f.f. $A \cap B = \emptyset$.
- the three properties termed <u>Kolmogorov's Axioms of Prob.</u>
- (3) is the axiom of <u>countable additivity</u>

<u>Recall</u>: Finite additivity := $A_1, \dots, A_n$ pairwise disjoint.

then $P\left(\bigcup_{i=1}^{n} A_i\right) = P(A_1) + \cdots + P(A_n)$

<u>Ex</u>: Tossing a fair coin: $S = \{H, T\}$.

$P(\{H\}) := P(\{T\}) := \frac{1}{2}$

Tossing an unfair coin: $\begin{cases} P(\{H\}) = q \\ P(\{T\}) = 1-q \end{cases}$ for some $q \in [0,1]$

Tossing two coins in a row:

$S = \{HH, HT, TH, TT\}$

Take $P(HH) = q_1, \dots, P(TT) = q_4$

where $\begin{cases} q_1 + \cdots + q_4 = 1 \\ q_1, \dots, q_4 \geq 0 \end{cases}$

<u>Recall</u>: a set can be <u>finite</u> or <u>infinite</u>

→ an infinite set is either <u>countable</u> or <u>uncountable</u>

→ a countable set is either <u>finite</u> or <u>countably infinite</u>

Probabilities on countable sets

- relatively easy to define

<u>Thm.</u> Let $S = \{s_1, \dots, s_n\}$ (<u>finite</u>),

$\mathcal{B}$ any sigma-algebra of subsets of S.

Let $p_1, \dots, p_n \in [0,1]$ s.t. $\sum_{i=1}^{n} p_i = 1$.

Define, for any $A \in \mathcal{B}$ $\boxed{P(A) := \sum_{i: s_i \in A} p_i}$

then, P is a valid prob function on $\mathcal{B}$

<u>Prof.</u> check finite additivity, i.e. $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$.

$P(A \cup B) = \sum_{i: s_i \in A \cup B} \underset{A \cap B = \emptyset}{=} \sum_{s_i \in A} p_i + \sum_{s_i \in B} p_i$

$= P(A) + P(B)$. ∎

<u>Thm.</u> Let $S = \{s_1, s_2, \dots, \}$ (countably infinite)

$\mathcal{B}$ any sigma-algebra of subsets of S.

Let $p_1, p_2, \dots \in [0,1]$ s.t. $\sum_{i=1}^{\infty} p_i = 1$.

Define, for any $A \in \mathcal{B}$ $\boxed{P(A) := \sum_{i: s_i \in A} p_i}$

then, P is a valid prob function on $\mathcal{B}$

<u>Prof.</u> check infinite additivity, i.e. $P(A \cup B \cup \cdots) = P(A) + P(B) + \cdots$ if $A \cap B \cap \cdots = \emptyset$.

$P(A \cup B \cup \cdots) = \sum_{i: s_i \in A \cup B \cup \cdots} \underset{\text{pairwise disjoint}}{=} \sum_{s_i \in A} p_i + \sum_{s_i \in B} p_i + \cdots$

$= P(A) + P(B) + \cdots$. ∎