University of Michigan, Dept of Statistics

Stats 510, Instructor: Long Nguyen

Solution to Midterm 1

Please do not distribute.

1. [5 points] Prove the de Morgan's law $(A \cup B)^c = (A^c \cap B^c)$, by using the definition of the basic set operations (complement, union and intersection).

   Solution:

   $(A \cup B)^c = \{x \in \Omega : x \notin (A \cup B)\} = \{x \in \Omega : x \notin A \text{ and } x \notin B\} = \{x \in \Omega : x \in A^c \text{ and } x \in B^c\} = A^c \cap B^c$

   Common Mistakes:

   - $\{x \in \Omega : x \notin (A \cup B)\} = \{x \in \Omega : x \notin A \text{ or } x \notin B\}$ (2 points deduction)
   - Showing only one direction like $(A \cup B)^c \subset (A^c \cap B^c)$. (2 points deduction)

2. [10 points] Let $\mathcal{B}$ denote the sigma of subsets of the real line that are generated by sets of the form (a, b), where a and b are rationals.

   (i) Show that every singleton, i.e., set of a single element $\{a\}$ where a is real, is a member of $\mathcal{B}$.

   (ii) Is the set of all real numbers a member of $\mathcal{B}$? Prove or disprove.

   Solution:

   (i)
   There exists two sequence of rationals $\{p_n\}$ and $\{q_n\}$ such that p_n is an increasing sequence converges to a from below and q_n is a decreasing sequence converges to a from above. Since $(p_n, q_n) \in \mathcal{B}$,

   $$\{a\} = \cap_{n=1}^{\infty}(p_n, q_n) \in \mathcal{B}.$$

   (ii)
   Since $(-n, n) \in \mathcal{B}$ for all $n \in \mathbb{N}$,

   $$\mathbb{R} = \cup_{n=1}^{\infty}(-n, n) \in \mathcal{B}.$$

   Common Mistakes:

   - Let $(S, \mathcal{B})$ be the sigma algebra defined in the problem. We only have $S \subset \mathbb{R}$, and $(a, b) \in \mathcal{B}$ for all $a, b \in \mathbb{Q}$.
     Since it is not guaranteed that $S = \mathbb{R}$, proofs using complement are mostly wrong in both (i) and (ii), because $(a, b)^c \neq (-\infty, a] \cup [b, \infty)$ and $\{a\}^c \neq \mathbb{R} \setminus \{a\}$. (no points given)
   - $(a, b) \in \mathcal{B}$ does not imply that $(-\infty, b) \in \mathcal{B}$, since ∞ is neither rational nor real number. (no points given)
   - For (i), using $(a - \frac{1}{n}, a + \frac{1}{n})$ or something similar is wrong because $a \pm \frac{1}{n}$ is not guaranteed to be a rational (3 points deduction)

3. [10 points] Consider the experiment of tossing a coin two times, and let X be the number of heads and Y the number of tails after the two tosses. Construct a suitable sample space, and define a valid probability function on the two tosses. Show that X and Y have identical distributions, but the two coin tosses are neither independent nor identically distributed.

Solution: The sample space for two coin tosses is

$$\Omega = \{HH, HT, TH, TT\}. \quad \text{(2 points)}$$

Here, HT means the result of coin 1 is head, the result of coin 2 is tail, and so on. Note that it is not correct to write down sample space as

$$\Omega = \{(X,Y) = (0,2), (X,Y) = (1,1), (X,Y) = (2,0)\}.$$

Because if we write down the sample space like this, we can not analyze the distribution of the two coins.

In order X and Y have the same distribution, we need

$$P(HH) = P(TT) \quad \text{(3 points)}$$

In order the two coins have dependent distribution, we must have

$$P(\text{coin } 2 = T | \text{coin } 1 = H) \neq P(\text{coin } 2 = T | \text{coin } 1 = T),$$

so

$$\frac{P(HT)}{P(HT) + P(HH)} \neq \frac{P(TT)}{P(TH) + P(TT)},$$

so the construction must satisfies

$$P(TT)P(HH) \neq P(HT)P(TH) \quad \text{(3 points)}$$

In order the two coins have not identical distribution, we must have

$$P(\text{coin } 2 = T) \neq P(\text{coin } 1 = T)$$

so the construction must satisfies

$$P(TT) + P(TH) \neq P(HT) + P(TT) \iff P(TH) \neq P(HT) \quad \text{(3 points)}$$

In summary, your answer is correct as long as your construct satisfies

- $P(TT) = P(HH)$
- $P(HT) \neq P(TH)$
- $P(HH)P(TT) \neq P(HT)P(TH)$

For example, $P(TT) = P(HH) = 1/4, P(HT) = 1/3, P(TH) = 1/6$ is a valid answer.

4. [15 points] Suppose that we throw 5 identical balls into 3 identical boxes independently (and in the same random manner), such that every ball will end up in one of the three boxes. In the following, it is sufficient to write your answers in math expressions without explicit calculations.

   (i) Define a sample space of outcomes. Based on your definition, what are the probabilities allocated to elements of your sample space?

   (ii) What is the probability of the event that there is no empty box?

(iii) What is the probability of the event that no box contains more than 3 balls?

Solution:

(i) Sample space and probabilities. Since both balls and boxes are identical, an outcome is determined by the (unordered) occupancy triple of box counts (sorted nonincreasing):

$$\mathcal{S} = \{(5,0,0), (4,1,0), (3,2,0), (3,1,1), (2,2,1)\}. \quad \text{[3 points]}$$

To assign probabilities, view the underlying experiment as each of the 5 balls independently choosing one of 3 (temporarily labeled) boxes uniformly, so each microstate has probability 3^{-5}. The probability of a macro-outcome is the number of labeled allocations collapsing to it divided by 3^5.

Then, (You can get at most 8 points in this calculation)

$$\begin{aligned}
\mathbb{P}(5,0,0) &= \frac{3\cdot\binom{5}{5}}{3^5}\text{[2 points]} = \frac{3}{243} = \frac{1}{81},\\
\mathbb{P}(4,1,0) &= \frac{6\cdot\binom{5}{4}}{3^5}\text{[2 points]} = \frac{30}{243} = \frac{10}{81},\\
\mathbb{P}(3,2,0) &= \frac{6\cdot\binom{5}{3}}{3^5}\text{[2 points]} = \frac{60}{243} = \frac{20}{81},\\
\mathbb{P}(3,1,1) &= \frac{3\cdot\binom{5}{3}\cdot\binom{2}{1}}{3^5}\text{[2 points]} = \frac{60}{243} = \frac{20}{81},\\
\mathbb{P}(2,2,1) &= \frac{3\cdot\binom{5}{2}\cdot\binom{3}{2}}{3^5}\text{[2 points]} = \frac{90}{243} = \frac{10}{27}.
\end{aligned}$$

(ii) No empty box. This corresponds to the outcomes with all counts ≥ 1, i.e. $(3,1,1)$ and $(2,2,1)$:

$$\mathbb{P}(\text{no empty box}) = \mathbb{P}(3,1,1) + \mathbb{P}(2,2,1)\text{[2 points]} = \frac{60+90}{243} = \frac{50}{81}.$$

You can also use the inclusion-exclusion theorem: the number of outcomes is $3^5 - 3\cdot 2^5 + 3\cdot 1^5 = 150$.

(iii) No box contains more than 3 balls. Allowable occupancy types are $(3,2,0), (3,1,1), (2,2,1)$, hence

$$\mathbb{P}(\max \leq 3) = \mathbb{P}(3,1,1) + \mathbb{P}(2,2,1) + \mathbb{P}(3,2,0)\text{[2 points]} = \frac{60+60+90}{243} = \frac{70}{81}.$$

Remark: If you didn't write down the sample space correctly and did the question (ii) and (iii) in another way, I will not grade according to the points I distribute above. For example, if you did (ii) and (iii) all correct, I will deduct only 3 points (from the sample space).

Remark: Some attempt scores (at most 5) are given if none of the analysis is correct but you've showed enough work.

5. [10 points] Recall the "three prisoners" story: Out of three prisoners A, B and C, one was chosen uniformly at random to be pardoned. A asked the warden: "Which among B and C will be executed?", to which the warden responded: "B is to be executed". Here is a twist: given a correct answer to say either "B" or "C", we happen to know that the warden has the tendency to say "B" with probability p (and "C" with probability $1-p$). Let W denote the event that "the warden tells A that B dies".

   (i) Provide a formula for $P(W)$.

   (ii) What is the conditional probability $P(A \text{ is pardoned} \mid W)$? Comment on whether or not the warden's answer affects our updated knowledge about A's fate, depending on p.

Solution: Let E_A, E_B, E_C be the events that A, B, C is pardoned, respectively. Then $\mathbb{P}(E_A) = \mathbb{P}(E_B) = \mathbb{P}(E_C) = 1/3$.

(i) Compute $P(W)$.

$$\mathbb{P}(W \mid E_A) = p, \qquad \mathbb{P}(W \mid E_B) = 0, \qquad \mathbb{P}(W \mid E_C) = 1.$$

Hence

$$\mathbb{P}(W) = \sum_{g \in \{A,B,C\}} \mathbb{P}(W \mid E_g)\mathbb{P}(E_g) = \tfrac{1}{3}p + \tfrac{1}{3} \cdot 0 + \tfrac{1}{3} \cdot 1 = \frac{p+1}{3}. \quad \text{[4 points]}$$

(ii) Posterior $P(A$ pardoned $\mid W)$. By Bayes formula,

$$P(E_A \mid W) = \frac{P(W \mid E_A)P(E_A)}{P(W)} = \frac{p \cdot (1/3)}{(p+1)/3} = \frac{p}{p+1}. \quad \text{[4 points]}$$

Comment. The warden's answer is informative unless $p = \frac{1}{2}$ [2 points]:

$$p = \tfrac{1}{2} \Rightarrow P(E_A \mid W) = \tfrac{1}{3} \text{ (no update);}$$

$$p = 1 \Rightarrow P(E_A \mid W) = \tfrac{1}{2} \text{ (A more likely pardoned);}$$

$$p = 0 \Rightarrow P(E_A \mid W) = 0 \text{ (A certainly not pardoned).}$$