4.2b Independence

independent Random variables

Def. Let (X,Y) be a bivariate Random vector, with joint pdf / pmf $f_{XY}(x,y)$ and marginal pdf / pmf $f_X(x)$ and $f_Y(y)$.

Then X and Y are called **independent variables**, if for every $x, y \in \mathbb{R}$

$$\boxed{f_{XY}(x,y) = f_X(x)\, f_Y(y)}.$$

Remark.

1. if $X \perp\!\!\!\perp Y$ then

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{\cancel{f_X(x)}\, f_Y(y)}{\cancel{f_X(x)}}$$

$$= f_Y(y)$$

does NOT depend on X

does not depend on X

Moreover, $\forall A, B \subset \mathbb{R}$

$$P(Y \in B \mid X \in A) = \frac{\int_B \int_A f_{XY}(x,y)\, dx\, dy}{\int_A f_X(x)\, dx}$$

$$= \frac{\int_B \int_A f_X(x) f_Y(y)\, dx\, dy}{\int_A f_X(x)\, dx}$$

$$= \frac{\left(\int_B f_Y(y)\, dy\right) \cancel{\left(\int_A f_X(x)\, dx\right)}}{\cancel{\left(\int_A f_X(x)\, dx\right)}}$$

$$= \int_B f_Y(y)\, dy$$

$$= P(Y \in B)$$

Hence the event $\{Y \in B\}$ is independent of $\{X \in A\}$ for any A, B.

Recall

$$X \perp\!\!\!\perp Y \quad \text{if} \quad f_{XY}(x,y) = f_X(x)\, f_Y(y) \quad \underline{\underline{\forall}}\, x,y \in \mathbb{R}$$

$$\Leftrightarrow \quad P(X \in A,\, Y \in B) = P(X \in A)\, P(Y \in B)$$

$$\underline{\underline{\forall}}\, A, B \subset \mathbb{R}$$

To verify independence, need to check the above identity for all x, y (or all A, B)

To show non-independence, need to identify a pair (x, y) or (A, B) where the identity is not satisfied.

Example.

if $f_{XY}(x,y)$ is given by the table

$x \backslash y$	1	2	3
10	1/10	1/5	1/5
20	1/10	1/10	3/10

all numbers add up to 1.

$$\Rightarrow \quad f_X(10) = f_{XY}(10,1) + f_{XY}(10,2) + f_{XY}(10,3)$$
$$= 1/10 + 1/5 + 1/5$$
$$= 1/2$$

$$f_Y(3) = 1/5 + 3/10$$
$$= 1/2$$

but $f_{XY}(10,3) = 1/5 \neq f_X(10) \times f_Y(3)$

So X, Y are not independent.

$\boxed{\text{Lemma}}$ Let $(X,Y) \sim f_{XY}$

Then $X \perp\!\!\!\perp Y$ if and only if there exist functions $g(x)$ and $h(y)$ such that

$$f_{XY}(x,y) = g(x)\,h(y) \qquad \forall x,y \in \mathbb{R}$$

Proof.

"only if": when $X \perp\!\!\!\perp Y$, $f_{XY}(x,y) = \underbrace{f_X(x)}_{g(x)}\,\underbrace{f_Y(y)}_{h(y)}$

"if": given that $f_{XY}(x,y) = g(x)\,h(y)$

$$1 = \iint_{\mathbb{R}\,\mathbb{R}} f_{XY}(x,y)\,dx\,dy$$

$$= \iint g(x)\,h(y)\,dx\,dy$$

$$= \Big(\underbrace{\int g(x)\,dx}_{c}\Big)\Big(\underbrace{\int h(y)\,dy}_{d}\Big)$$

$=:$

Now

$$f_X(x) = \int_{\mathbb{R}} f_{XY}(x,y)\,dy$$

$$= g(x)\,d$$

$$f_Y(y) = \int_{\mathbb{R}} f_{XY}(x,y)\, dy$$
$$= h(y)\, c.$$

Hence

$$f_X(x)\, f_Y(y) = g(x)\, h(y)\, cd$$
$$= g(x)\, h(y)$$
$$= f_{XY}(x,y) \qquad \square.$$

Example:

- if

$$f_{XY}(xy) = \frac{1}{384} x^2 y^4 e^{-y-(x/2)}, \quad x,y > 0$$

then $X \perp\!\!\!\perp Y$.

Theorem. if $X \perp\!\!\!\perp Y$ then

① $\forall A \subset \mathbb{R}, B \subset \mathbb{R}$

$$P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B) \checkmark$$

② For any function $g(x)$ (only of x)
and $h(y)$ (only of y)

$$\mathbb{E}\, g(X) h(Y) = \mathbb{E}\, g(X)\ \mathbb{E}\, h(Y).$$

Prog.
① was already proved in the last section
② is similarly proved.

Remark.
This simple theorem is remarkably powerful, as we'll now see.

Theorem if $X \perp\!\!\!\perp Y$, with mgf M_X and M_Y.

Then the Random variable $Z := X+Y$ has the mgf

$$\boxed{M_Z(t) = M_X(t) + M_Y(t) \qquad \forall t}.$$

Proof.

$$M_Z(t) = \mathbb{E}\, e^{tZ}$$

$$= \mathbb{E}\, e^{t(X+Y)}$$

$$= \mathbb{E}\, e^{tX} e^{tY}$$

$X \perp\!\!\!\perp Y$ previous theorem

$$= \left(\mathbb{E}\, e^{tX}\right)\left(\mathbb{E}\, e^{tY}\right)$$

$$= M_X(t)\, M_Y(t) \qquad \square$$

As an application, we'll now prove that

$\boxed{\text{Theorem}}$ The sum of two independent normal random variables is again normal.

Proof.

① Let $X \sim \text{Normal}(0,1)$

Some basic calculations:

$$M_X(t) = \mathbb{E}\, e^{tX}$$

$$= \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int e^{tx - \frac{1}{2}x^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(x-t)^2} e^{\frac{1}{2}t^2} dx$$

$$= e^{\frac{1}{2}t^2}.$$

if $X \sim \text{Normal}(\mu, \sigma^2)$ then

$$Z := \frac{1}{\sigma}(X - \mu) \sim N(0,1), \quad \text{so}$$

$$M_X(t) = \mathbb{E}\, e^{tX}$$
$$= \mathbb{E}\, e^{t(\mu + \sigma Z)}$$

$$= e^{t\mu}\, \mathbb{E}\, e^{t\sigma Z}$$

$$= e^{t\mu}\, M_Z(t\sigma)$$

$$= e^{t\mu + \frac{1}{2}t^2\sigma^2}.$$

② Now suppose $X \sim \text{Normal}(\mu_1, \sigma_1^2)$

$Y \sim \text{Normal}(\mu_2, \sigma_2^2)$

$X \perp\!\!\!\perp Y$

Let $Z = X + Y$. Then,

$$M_Z(t) = M_X(t)\, M_Y(t) \quad (\text{since } X \perp\!\!\!\perp Y)$$

above calculation

$$= e^{t\mu_1 + \frac{1}{2}t^2\sigma_1^2}\; e^{t\mu_2 + \frac{1}{2}t^2\sigma_2^2}$$

$$= e^{t(\mu_1+\mu_2) + \frac{1}{2}t^2(\sigma_1^2+\sigma_2^2)}$$

This is the mgf of $N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

Since a random variable's distribution is determined by its MGF (see chapter 2),

$$Z \sim \text{Normal}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2) \quad \square$$