3.2c Continuous distributions

Beta distribution $X \sim \text{Beta}(\alpha, \beta)$, $\alpha, \beta > 0$
if its pdf is

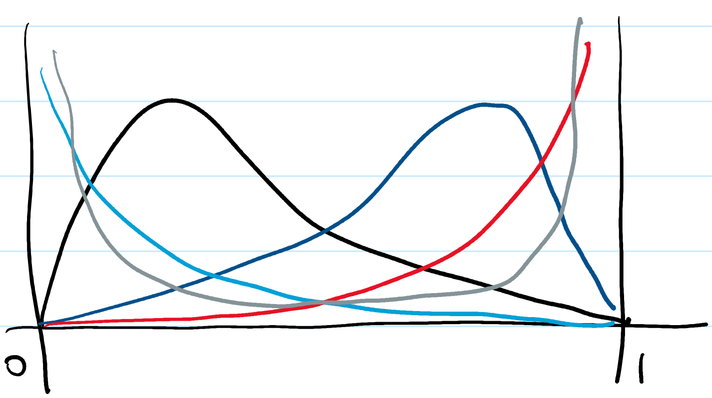
$$f_X(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{for } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

where

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

denotes Beta function.

Fact. $B(\alpha,\beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.



Beta pdf's for different (α,β).

n-th moments

$$\mathbb{E}X^n = \frac{1}{B(\alpha,\beta)} \int_0^1 x^n (1-x)^{\alpha-1} (1-x)^{\beta-1} \, dx$$

$$= \frac{1}{B(\alpha,\beta)} B(\alpha+n, \beta)$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+n)\Gamma(\beta)}{\Gamma(\alpha+\beta+n)}$$

$$= \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(\alpha+\beta+n)} .$$

$n=1$

$$\mathbb{E}X = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(\alpha+\beta+1)}$$

$$= \frac{\Gamma(\alpha+\beta)\, \alpha\Gamma(\alpha)}{\Gamma(\alpha)\,(\alpha+\beta)\Gamma(\alpha+\beta)} = \frac{\alpha}{\alpha+\beta} .$$

$$\mathbb{E}X^2 = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+2)}{\Gamma(\alpha)\Gamma(\alpha+\beta+2)} = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$\operatorname{var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

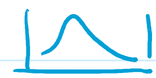
$$= \frac{\alpha}{\ldots} \; \frac{(\alpha+1)(\alpha+\beta) - \alpha(\alpha+\beta+1)}{\ldots}$$

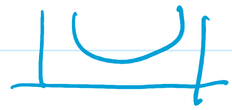
$$= \frac{\alpha}{(\alpha+\beta)} \frac{(\alpha+1)(\alpha+\beta) - \alpha(\alpha+\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

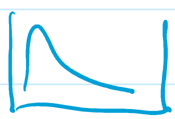
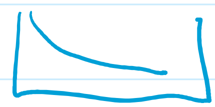
$$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} .$$

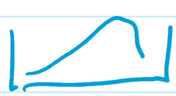
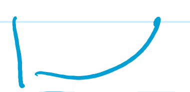
Easy facts:

if $\alpha = \beta = 1$ Beta $(1,1) \equiv$ Uniform $(0,1)$

if $\alpha > 1, \beta > 1$ Beta is unimodal 

if $\alpha < 1, \beta < 1$ Beta is 

if $\alpha < \beta$  or 

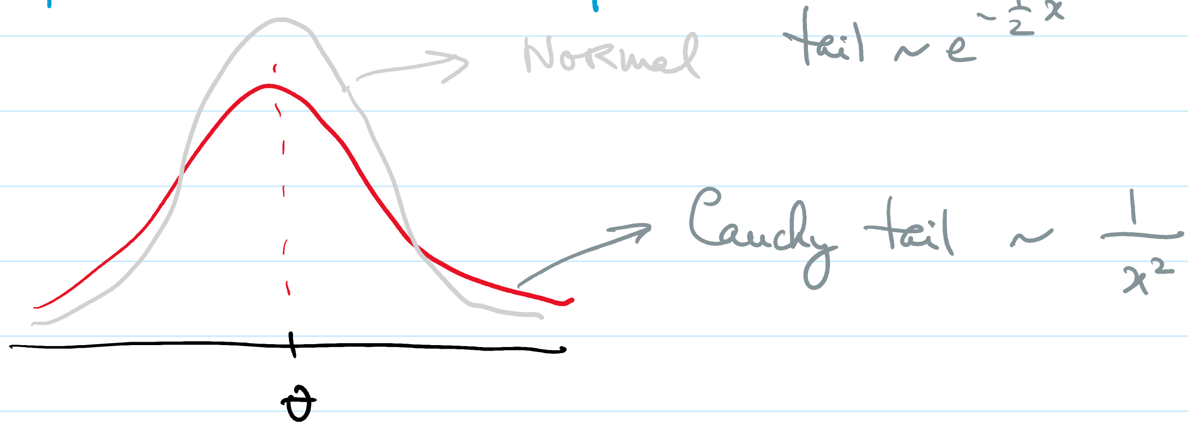
if $\alpha > \beta$  or 

Cauchy distribution

$X \sim$ Cauchy (θ) , $\theta \in \mathbb{R}$ if

$$f_X(x) = \frac{1}{\pi} \frac{1}{1+(x-\theta)^2} , \quad x \in \mathbb{R}$$

Cauchy pdf is bell-shaped too



θ is the median of X
(location)

but $\begin{cases} \mathbb{E}|X| = \infty \\ \mathbb{E}X \text{ does not exist.} \end{cases}$

Fact: if $X, Y \overset{iid}{\sim} N(0,1)$

then $\dfrac{X}{Y} \sim \text{Cauchy}(0)$.

we'll prove this in the next chapter.

Log-normal distribution

Recall

if $Y \sim N(\mu, \sigma^2)$, then

$X = e^Y$ has lognormal distribution

in other words, $Y = \log X \sim N(\mu, \sigma^2)$

$$f_X(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{x} e^{-\frac{1}{2\sigma^2}(\log x - \mu)^2}$$