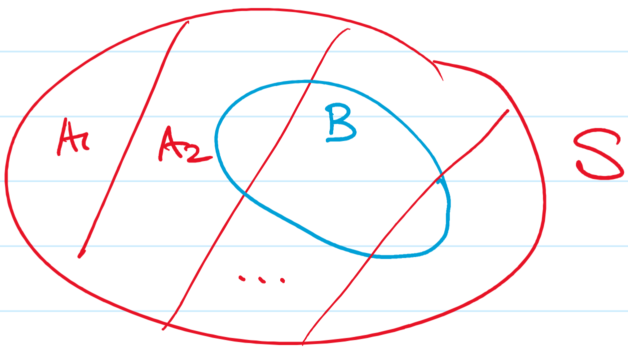
1.3 Conditional probabilities (cont)

Bayes formula : given events $\begin{cases} A, B \in \mathcal{B} \\ P(B) > 0 \end{cases}$

$$\boxed{P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}}$$

A variation



Let $A_1, A_2, \ldots$ be a partition of S

Then
$$P(B) = \sum_i P(B \cap A_i)$$
$$= \sum_i P(B|A_i)\,P(A_i)$$

So,

$$\boxed{P(A_i|B) = \frac{P(B|A_i)\,P(A_i)}{\sum_i P(B|A_i)\,P(A_i)}}$$

up to a multiplying const

Equivalently, $\forall i \quad P(A_i|B) \propto P(B|A_i)\,P(A_i)$

INDEPENDENCE

- We want to be able to convey that an event B has no effect on A, by insisting that

$$\boxed{P(A|B) = P(A)}$$

note the symmetry

Then $P(B|A) = \frac{\cancel{P(A|B)}\, P(B)}{\cancel{P(A)}} = P(B)$

provided $P(A)\, P(B) > 0$

Moreover

$$P(A \cap B) = P(A|B)\, P(B)$$
$$= P(A)\, P(B)$$

$\boxed{\text{Def.}}$ Two events A and B are statistically independent if

$$\boxed{P(A \cap B) = P(A)\, P(B)}$$

Remark. This def does not require $P(A)$ or $P(B) > 0$

A & B independent does not mean $A \cap B = \phi$

EXAMPLES

① Dice Rolling: $S = \{1, \dots, 6\}$
What is the prob of throwing At least one 6 in 4 Rolls?

Assume: the outcome of the Rolls are indep.

$$P(\text{at least one 6 in 4 Rolls}) = 1 - P(\text{no 6 in 4 Rolls})$$
$$= 1 - \prod_{i=1}^{4} P(\text{no 6 in Roll } i)$$
$$= 1 - (5/6)^4 = 0.518$$

independent

Thm if $A \perp\!\!\!\perp B$ then
(i) $A \perp\!\!\!\perp B^c$
(ii) $A^c \perp\!\!\!\perp B$
(iii) $A^c \perp\!\!\!\perp B^c$

MUTUAL INDEPENDENCE OF MULTIPLE EVENTS

$\boxed{\text{Def.}}$ A collection of events $A_1, \dots, A_n$ are **mutually independent**, if for any subcollection $A_{i_1}, \dots, A_{i_k}$, we have

$$\boxed{P(A_{i_1} \cap \dots \cap A_{i_k}) = \prod_{j=1}^{k} P(A_{i_j})}$$

Remark.

- Pairwise independence does <u>not</u> imply mutual indep.
  i.e. given 3 events A, B, C s.t
  $$\left.\begin{array}{l} P(A)P(B) = P(A \cap B) \\ P(A)P(C) = P(A \cap C) \\ P(B)P(C) = P(B \cap C) \end{array}\right. \not\Rightarrow P(A \cap B \cap C) = P(A)P(B)P(C)$$

- Conversely, in general
  $$P(A \cap B \cap C) = P(A)P(B)P(C) \not\Rightarrow P(A \cap B) = P(A)P(B)$$

EXAMPLES

① $S = \left\{ \begin{array}{lll} aaa & bbb & ccc \\ abc & bca & cba \\ acb & bac & cab \end{array} \right\}$

and put prob 1/9 on each element of S.

Let $A_i = \{$ ith place in the triple is occupied by $a \}$

Then $P(A_1) = P(A_2) = P(A_3) = 1/3$ (by symmetry)

$$P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_2 \cap A_3) =$$
$$= P(A_1 \cap A_2 \cap A_3) = P(\{aaa\}) = 1/9$$

So A_1, A_2, A_3 are pairwise independent, but $P(A_1 \cap A_2 \cap A_3) \neq P(A_1) P(A_2) P(A_3)$

(2) Three coin tosses, each of which produces H or T

$$S = \{ HHH, HHT, HTH, HTT, THH, THT, TTH, TTT \}$$

Let's put prob. 1/8 on each element of S, and define

A_i to be the event the i^{th} toss is H, for $i=1,2,3$.

Then $P(A_1) = P(\{HHH, HHT, HTH, HTT\}) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$
$= 1/2$

$P(A_1 \cap A_2) = (\{HHH, HHT\}) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

$P(A_1 \cap A_2 \cap A_3) = P(\{HHH\}) = \frac{1}{8}$.

Likewise $P(A_i) = 1/2 \quad \forall i = 1,2,3$

$P(A_i \cap A_j) = 1/4 \quad \forall i,j \in \{1,2,3\}$

So
$$\begin{cases} P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3) \\ P(A_i \cap A_j) = P(A_i)P(A_j) \quad \forall i \neq j \end{cases}$$

Hence A_1, A_2, A_3 are mutually independent and the coin tosses they represent are "fair"

$P(A_i) = 1/2$

Question: is there another probability distribution on S s.t. the coin tosses are fair and mutually independent?