3.2a Continuous distributions

Continuous distributions put probability mass on cont. spaces

Formally, a continuous random variable is one whose cdf is a continuous function.

Uniform distribution

$X \sim \text{Uniform}([a,b])$, $a<b$

if its pdf takes the form

$$f(x|a,b) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}.$$

Check: f is a valid pdf

$\mathbb{E}X = (b+a)/2$

$\text{var}\, X = (b-a)^2/12$.

Gamma distribution

Recall the Gamma function $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt \quad , \alpha > 0$

- Change of variable $t = x/\beta \quad , \quad \beta > 0$:

$$\Gamma(\alpha) = \int_0^\infty (x/\beta)^{\alpha-1} e^{-x/\beta} (1/\beta)\, dx$$

$$= \beta^{-\alpha} \int_0^\infty x^{\alpha-1} e^{-x/\beta}\, dx$$

$$\Rightarrow \quad \Gamma(\alpha)\beta^{\alpha} = \int_0^\infty x^{\alpha-1} e^{-x/\beta}\, dx$$

So the function $f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} \quad , x > 0$

is a valid pdf (that we call Gamma pdf)

Say $X \sim \text{Gamma}(\alpha, \beta) \quad , \begin{cases} \alpha > 0 & \text{shape} \\ \beta > 0 & \text{scale} \end{cases}$

Sometimes we use $b = 1/\beta$ as rate parameter

More on Gamma function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$$

$$\Rightarrow \Gamma(\alpha+1) = \int_0^\infty t^{\alpha} e^{-t}\, dt$$

$$= -\int_0^\infty t^{\alpha}\, d e^{-t}$$

integration by parts

$$= \underbrace{t^{\alpha} e^{-t} \Big|_\infty^0}_{} + \int_0^\infty e^{-t}\, d t^{\alpha}$$

$$= 0 + \alpha \underbrace{\int_0^\infty e^{-t} t^{\alpha-1}\, dt}_{\Gamma(\alpha)}$$

So $\Gamma(\alpha+1) = \alpha \Gamma(\alpha)$

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = -e^{-t}\Big|_0^\infty = 1$$

$$\Rightarrow \Gamma(2) = 1\,\Gamma(1) = 1$$

$$\Gamma(3) = 2\,\Gamma(2) = 2!$$

$$\ldots$$

$$\Gamma(n) = (n-1)\Gamma(n-1) = \ldots = (n-1)!$$

Back to Gamma distribution

$$\mathbb{E}X = \int_0^\infty x f(x|\alpha,\beta)\,dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x \cdot x^{\alpha-1} e^{-x/\beta}\,dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \Gamma(\alpha+1)\beta^{\alpha+1}$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \alpha\Gamma(\alpha)\beta^{\alpha+1} = \alpha\beta$$

Similarly,

$$\mathbb{E}X^2 = \frac{1}{\Gamma(\alpha)\beta^\alpha} \Gamma(\alpha+2)\beta^{\alpha+2}$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} (\alpha+1)\alpha\Gamma(\alpha)\beta^{\alpha+2} = (\alpha+1)\alpha\beta^2$$

$$\mathrm{var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2$$

$$= (\alpha+1)\alpha\beta^2 - \alpha^2\beta^2 = \alpha\beta^2.$$

Moment Generating function

$$M_X(t) = \mathbb{E}\, e^{tX}$$

$$e^{-x\left(\frac{1}{\beta} - t\right)}$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty e^{tx} x^{\alpha-1} e^{-x/\beta}\, dx$$

$$\frac{1-\beta t}{\beta}$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \Gamma(\alpha) \left(\frac{\beta}{1-\beta t}\right)^\alpha, \quad \text{provided } 1-\beta t > 0$$

$$= (1-\beta t)^{-\alpha}, \quad \text{if } t < 1/\beta.$$

Consider the cdf for Gamma(α, β), $\alpha \in \underline{\underline{\mathbb{N}}}$

$\forall x > 0$

$$P(X \leq x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^x t^{\alpha-1} e^{-t/\beta}\, dt$$

$$= \frac{1}{(\alpha-1)!\,\beta^\alpha} \int_0^x t^{\alpha-1} (-\beta)\, d e^{-t/\beta}$$

i.b.p.

$$= \frac{1}{(\alpha-1)!\,\beta^\alpha} \left(t^{\alpha-1} \beta e^{-t/\beta} \Big|_x^0 + \beta \int_0^x e^{-t/\beta}\, dt^{\alpha-1} \right)$$

$$= \underbrace{-\frac{1}{(\alpha-1)!} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-x/\beta}} + \underbrace{\frac{1}{(\alpha-2)!\,\beta^{\alpha-1}} \int_0^x e^{-t/\beta} t^{\alpha-2}\, dt}$$

$$= -P(Y = \alpha-1) + \text{cdf}_{\text{Gamma}(\alpha-1, \beta)}(x)$$

$Y \sim$ Poisson(x/β)

keep going

$$= -P(Y=\alpha-1) - P(Y=\alpha-2) + \text{cdf}_{\text{Gamma}(\alpha-2, \beta)}(x)$$

$$= \ldots$$

$$= -P(Y=\alpha-1) - \ldots - P(Y=1) + \text{cdf}_{\text{Gamma}(1, \beta)}(x)$$

$(*)$

$$\mathrm{cdf}_{\mathrm{Gamma}(1,\beta)}(x) = \frac{1}{\beta}\int_0^x e^{-t/\beta}\,dt$$

$$= -e^{-t/\beta}\Big|_0^x = 1 - e^{-x/\beta}$$

So $(*)$

$$= -P(Y=\alpha-1) - \dots - P(Y=1) + 1 - \underbrace{e^{-x/\beta}}$$

$$= -P(Y=\alpha-1) - \dots - P(Y=1) - P(Y=0) + 1$$

$$= P(Y \geq \alpha)$$

we have shown the interesting and surprising connection

$$P(X \leq x) = P(Y \geq \alpha)$$

where $X \sim \mathrm{Gamma}(\alpha, \beta)$
$Y \sim \mathrm{Poisson}(x/\beta)$

Special cases of Gamma distribution (α, β)

① Let $\begin{cases} \alpha = p/2, & p \text{ integer} \\ \beta = 2 \end{cases}$

$$f(x|p) = \frac{1}{\Gamma(p/2)\, 2^{p/2}} x^{p/2-1} e^{-x/2}, \quad x > 0$$

is the chi-square pdf with p degrees of freedom (which is also, as we'll learn later, the pdf of the sum of square of p independent standard normal (Gaussian) variables)

② Let $\alpha = 1$, $\beta > 0$ noting that $\Gamma(1) = 1$

$$f(x|\beta) = \frac{1}{\beta} e^{-x/\beta}, \quad x > 0$$

which is the exponential pdf.

Exponential distribution (like the geometric dist) has the memoryless property

For $s > t \geq 0$

$$\boxed{P(X > s \mid X > t) = P(X > s - t)}$$

e.g. X modelling time of occurance.

indeed $$P(X>s \mid X>t) = \frac{P(X>s \cap X>t)}{P(X>t)}$$

$$= \frac{P(X>s)}{P(X>t)} = \frac{e^{-s/\beta}}{e^{-t/\beta}}$$

$$= e^{-(s-t)/\beta}$$

$$= P(X > s-t)$$

③ if Let $X \sim \text{Exp}(\beta)$, $\beta > 0$

$Y = X^{1/\gamma}$, $\gamma > 0$

Then $P(Y \leq y) = P(X \leq y^{\gamma})$

$= 1 - e^{-y^{\gamma}/\beta}$

$\Rightarrow$ Y has pdf

$$f_Y(y) = \frac{d}{dy} P(Y \leq y)$$

$$= \frac{\gamma}{\beta} y^{\gamma - 1} e^{-y^{\gamma}/\beta}, \quad 0 < y < \infty$$

This defines the pdf of Weibull (γ, β)

very useful for modelling extreme / rare events.