University of Michigan, Dept of Statistics

Stats 510, Instructor: Long Nguyen

Exam 1
October 10, 2024

Solve the following problems.

1. (15pt) Given three events A, B and C, which are subsets of a sample space $\mathcal{S}$. Suppose that $P(A) = 1/2$, $P(B) = 1/4$ and $P(C) = 1/8$. Assume further that A and B are pairwise independent, A and C are pairwise independent, while B and C are disjoint.

   (i) (2pt) Show that A and B and C cannot be mutually independent.

   (ii) (8pt) Find $P(A \cup B \cup C)$, $P(A \cup C|A \cap B)$, $P(A \cap B|A \cup C)$, and $P(B \cup C|A^c)$.

   (iii) (5pt) Let N be the count of the number of events among the A, B and C that are true. Show that N is a random variable (i.e., by writing N precisely as a function of $s \in \mathcal{S}$). Find the pmf of N.

2. (5pt) Suppose that 11% of men and 9% of women are left-handed. A person is chosen at random and that person is right-handed. What is the probability that that person is a woman? (Assume that there are the same number of men and women).

3. (20pt) Let function f be defined as follows:

$$f(x) = \begin{cases} x & \text{if } x \in (0,1], \\ 2-x & \text{if } x \in (1,2), \\ 0 & \text{otherwise.} \end{cases}$$

   (i) (4pt) Show that f is a valid pdf for a random variable X.

   (ii) (6pt) What is the cdf F_X? Find $\mathbb{E}X$ and $\text{Var}(X)$.

   (iii) (6pt) Let $Y = (1/\lambda)X - 1$ for $\lambda > 0$. Find the cdf of Y, $\mathbb{E}Y$ and $\text{Var}(Y)$.

   (iv) (4pt) Let $Y_n = (1/n^2)X - 1$. What is the limit of the sequence of random variables Y_n as $n \to \infty$? Justify your answer.

4. (10pt) Let function f be given as follows, for $x \in (-\infty, +\infty)$,

$$f(x) = \frac{1}{4}e^{-|x|} + \frac{e^{-x}}{2(1+e^{-x})^2}.$$

   (i) (3pt) Show that f is a valid pdf for a random variable X.

   (ii) (3pt) Show that the moment generating function exists for a neighborhood of 0 (that is, $M_X(t) < \infty$ for $t \in (-t_0, t_0)$ for some $t_0 > 0$).

   (iii) (4pt) For $t \in (-t_0, t_0)$ from part (ii), find the integral expression for $\frac{d}{dt}M_X(t)$, and evaluate it at $t = 0$. Justify your operations.