3.4 Location and scale families

Fact. if $f(x)$ is a valid pdf on $\mathbb{R}$
then for any $\mu \in \mathbb{R}$, $\sigma > 0$
the function

$$g(x|\mu,\sigma) := \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right), \quad x \in \mathbb{R}$$

is a valid pdf on $\mathbb{R}$.

Proof

- clearly, $g \geq 0$ because $f \geq 0$.
- check the integral:

$$\int_{\mathbb{R}} g(x|\mu,\sigma)\, dx$$

$$= \int_{\mathbb{R}} \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) dx$$

Let $(x-\mu)/\sigma = y$
$\Rightarrow x = \mu + \sigma y$
$\Rightarrow dx = \sigma\, dy$

$$= \int \frac{1}{\sigma} f(y)\, \sigma\, dy$$

$$= \int f(y)\, dy = 1.$$

Definition Let $f(x)$ be any pdf (on $\mathbb{R}$).

The family of pdf $\left\{ g(x|\mu,\sigma) := \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right),\ \mu \in \mathbb{R}, \sigma > 0 \right\}$

is called the a location - scale family of distributions

$\begin{cases} \mu : & \text{location parameter} \\ \sigma : & \text{scale parameter} \end{cases}$

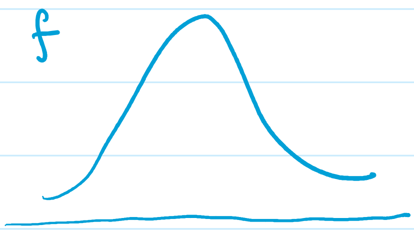
Special cases:

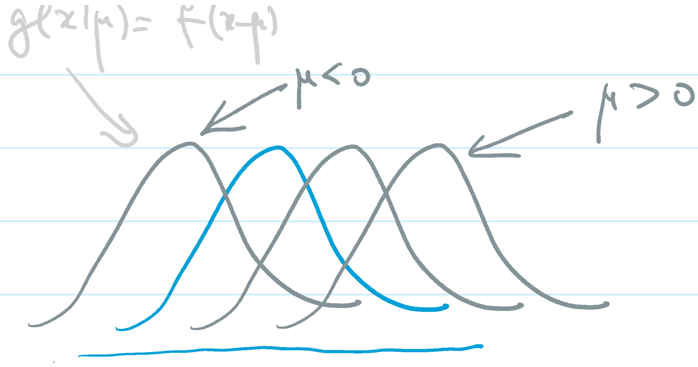
1. $\{ g(x|\mu) := f(x-\mu) \mid \mu \in \mathbb{R} \}$ is a location family

2. $\{ g(x|\sigma) := \frac{1}{\sigma} g\left(\frac{x}{\sigma}\right) \mid \sigma > 0 \}$ is a scale family

$g(x|\mu) = f(x-\mu)$

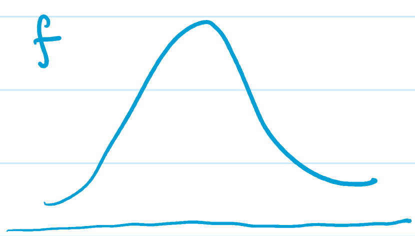
$\mu < 0$ $\mu > 0$

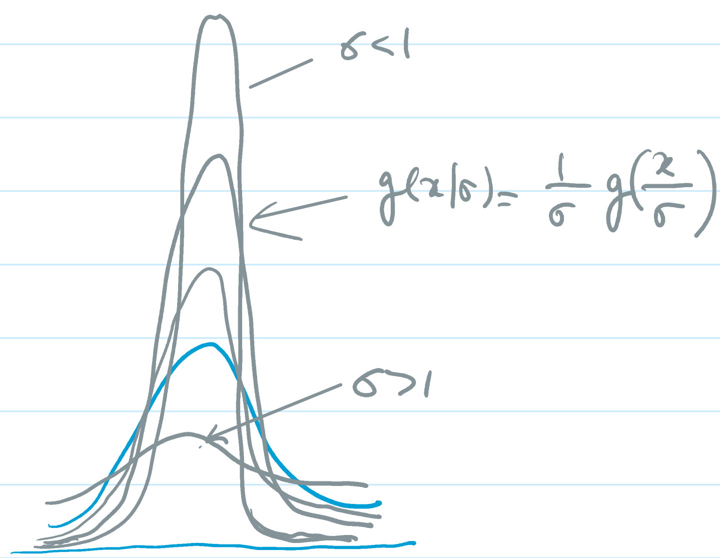




$$\boxed{\text{if } X \sim f \quad \text{then} \quad X+\mu \sim g(x) = f(x-\mu)}$$

$$\boxed{\text{if } X \sim f \quad \text{then} \quad \sigma X \sim g(x) = \frac{1}{\sigma} f\left(\frac{x}{\sigma}\right)}$$





$$\boxed{\text{if } X \sim f \quad \text{then} \quad \sigma X+\mu \sim g(x) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)}$$

① Let $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ then we obtain l-s family

$$\left\{ g(x|\mu,\sigma) = \frac{1}{\sigma} f\left(\frac{1}{\sigma}(x-\mu)\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \,\middle|\, \begin{matrix}\mu\in\mathbb{R}\\ \sigma>0\end{matrix} \right\}$$

② Let $f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}$, then we obtain

$$\left\{ g(x|\mu,\sigma) = \frac{1}{\Gamma(\alpha)} \frac{1}{\sigma} \left(\frac{x-\mu}{\sigma}\right)^{\alpha-1} e^{-\frac{x-\mu}{\sigma}} \,\middle|\, \begin{matrix}\mu\in\mathbb{R}\\ \sigma>0\end{matrix} \right\}$$

Not a gamma family

$$\left\{ g(x|\sigma) = \frac{1}{\Gamma(\alpha)} \frac{1}{\sigma} \left(\frac{x}{\sigma}\right)^{\alpha-1} e^{-\frac{x}{\sigma}} \,\middle|\, \sigma\in\mathbb{R}_+ \right\}$$

$$\equiv \left\{ \text{Gamma}(\alpha,\sigma) \,\middle|\, \sigma\in\mathbb{R}_+ \right\}$$

③ Let $f(x) = e^{-\lambda} \lambda^x / x!$, $x = 0, 1, 2, \ldots$

define $g(x|\sigma) = \frac{1}{\sigma} e^{-\lambda} \frac{\lambda^{x/\sigma}}{(x/\sigma)!}$ for $x = \sigma y$, $y \in \mathbb{N}$, $\sigma > 0$.

then g is a valid pmf supported by multiples of σ.

Not a Poisson family but inherits many properties of Poisson dist.

Theorem Suppose Y is a R.V. with pdf $f(z)$ and $\mathbb{E}Y$ and $\operatorname{var} Z$ exists

if X is a R.V. with pdf $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$, then

$$\boxed{\begin{cases} \mathbb{E}X = \sigma \mathbb{E}Y + \mu \\ \operatorname{var} X = \sigma^2 \operatorname{var} Y. \end{cases}}$$