4.2a Conditional distributions

Given a bivariate random vector $(X, Y) \in \mathbb{R}^2$
we are interested in, say

$$P(Y \in B \mid X \in A) \quad , \text{ for } A, B \subset \mathbb{R}.$$

Examples

① $(X, Y) = (\text{height}, \text{weight})$ of a (random) person

$$P(Y > 200 \text{ lb.} \mid X = 6') \quad ?$$

② X = number people out of 100 sampled in a Iowa poll who said they'd vote for John Smith

Y = John Smith wins Iowa on the election day

$$P(Y = \text{yes} \mid X = 53) \quad ?$$

Let (X,Y) be discrete.

$$P(Y \in B \mid X \in A) = \frac{P(X \in A \cap Y \in B)}{P(X \in A)}$$

$$P(X \in A \cap Y \in B) = \sum_{x \in A} \sum_{y \in B} f_{XY}(x,y)$$

$$P(X \in A) = \sum_{x \in A} \underbrace{\sum_{y} f_{XY}(x,y)}_{}$$

$$= \sum_{x \in A} f_X(x)$$

So

$$P(Y \in B \mid X \in A) = \sum_{x \in A} \sum_{y \in B} f_{XY}(x,y) \Big/ \sum_{x \in A} f_X(x)$$

$$= \sum_{y \in B} \frac{\sum_{x \in A} f_{XY}(x,y)}{\sum_{x \in A} f_X(x)}.$$

in particular, let $A = \{x\}$, $B = \{y\}$, then

$$P(Y = y \mid X = x) = \frac{f_{XY}(x,y)}{f_X(x)}$$ $\rightarrow$ conditional pmf

Def. Let (X,Y) be a discrete random vector with joint pmf $f_{XY}(x,y)$

- Then, for any x such that $f_X(x) = P(X=x) > 0$ the conditional pmf of Y given that $X=x$ is a function of y denoted by $f(y|x)$ and

$$f(y|x) = P(Y=y \mid X=x)$$

$$= \frac{f_{XY}(x,y)}{f_X(x)}$$

$$= \frac{f_{XY}(x,y)}{\sum_y f_{XY}(x,y)}.$$

Remark:

- $f(y|x) \geq 0 \quad \forall y$

- $$\sum_y f(y|x) = \sum_y \frac{f_{XY}(x,y)}{\sum_y f_{XY}(x,y)}$$

$$= \frac{\sum_y f_{XY}(x,y)}{\sum_y f_{XY}(x,y)} = 1.$$

- Sometimes, we use notation $f_{Y|X}(y|x)$

- From definition, both the marginal pmf f_X, f_Y and the conditional pmf $f_{Y|X}, f_{X|Y}$ are completely determined by the joint pmf. f_{XY}

Moreover $$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

implies $$\boxed{f_{XY}(x,y) = f_X(x)\, f_{Y|X}(y|x)}$$

Similarly, $$\boxed{f_{XY}(x,y) = f_Y(y)\, f_{X|Y}(x|y)}$$

whenever $f_X(x) > 0$ and $f_Y(y) > 0$

So the marginal pmf and conditional pmf are combined to completely determine the joint pmf

Let's consider now a
continuous bivariate random vector $(X,Y) \in \mathbb{R}^2$
we'll develope a similar notion of
conditional pdf for the continuous case by
mimicking the conditional pmf for the discrete case

$\boxed{\text{Heuristic but incorrect argument}}$

$$P(Y \in B \mid X = x) = \frac{P(Y \in B, X = x)}{P(X = x)}$$

$$P(Y \in B, X = x) = \int_B f_{XY}(x,y)\, dy$$

$$P(X = x) \text{ "="} f_X(x) = \int_{\mathbb{R}} f_{XY}(x,y)\, dy$$

Hence $P(Y \in B \mid X = x) = \dfrac{\int_B f_{XY}(x,y)\, dy}{f_X(x)}$

$$= \int_B \underbrace{\frac{f_{XY}(x,y)}{f_X(x)}}_{\substack{\text{define this integrand to be} \\ \text{the conditional p.d.f.}}} dy \quad (*)$$

"the conditional" p.d.f.

The problem with above argument is than both

$P(X=x)=0$ and

$P(Y\in B,\ X=x) \leq P(X=x) = 0.$

The correct treatment is to use limits:

define

$$\boxed{P(Y\in B \mid X=x) := \lim_{\varepsilon\to 0} P(Y\in B \mid X\in(x-\varepsilon, x+\varepsilon))}$$

provided than such a limit exists and is unique

To assess the RHS and its limit, we need to use the joint pdf:

$$P(Y\in B \mid X\in(x-\varepsilon, x+\varepsilon)) = \frac{P(Y\in B \cap X\in(x-\varepsilon, x+\varepsilon))}{P(X\in(x-\varepsilon, x+\varepsilon))} \quad \text{(by def)}$$

$$= \frac{\int_B \int_{x-\varepsilon}^{x+\varepsilon} f_{XY}(x,y)\, dx\, dy}{\int_{x-\varepsilon}^{x+\varepsilon} f_X(x)\, dx}$$

Under suitably mild conditions (see "Tools")

$$\lim_{\varepsilon \to 0} \int_B \frac{\left(\int_{x-\varepsilon}^{x+\varepsilon} f_{XY}(x,y)\, dx \right)}{\left(\int_{x-\varepsilon}^{x+\varepsilon} f_X(x)\, dx \right)}\, dy$$

(interchanging limit and integral)

$$= \int_B \left(\lim_{\varepsilon \to 0} \frac{\frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f_{XY}(x,y)\, dx}{\frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f_X(x)\, dx} \right) dy$$

$$= \boxed{\int_B \frac{f_{XY}(x,y)}{f_X(x)}\, dy.}$$

(if $f_{XY}(x,y)$ continuous at (x, B))

which is (*) that we aimed for.

We'll take the integrand as the definition for the conditional pdf for the continuous setting.

Def. Let (X,Y) be a bivariate random vector with joint pdf $f_{XY}(x,y)$ (and marginal pdfs $f_X(x)$ and $f_Y(y)$)

Then, for any x such that $f_X(x) > 0$, the **conditional pdf of Y given $X=x$** is

$$f_{Y|X}(y|x) := \frac{f_{XY}(x,y)}{f_X(x)} = \frac{f_{XY}(x,y)}{\int f_{XY}(x,y)\,dy}$$

Remark.

- $f_{Y|X}(y|x) \geq 0 \quad \forall y, \forall x$ (where $f_X(x) > 0$)
- $\int f_{Y|X}(y|x)\,dy = 1$
- The following identities hold

$$f_{XY}(x,y) = f_X(x)\, f_{Y|X}(y|x) \leftarrow \text{when } f_X(x) > 0$$

$$= f_Y(y)\, f_{X|Y}(x|y) \leftarrow \text{when } f_Y(y) > 0$$

Conditional distributions

- Given a bivariate Random vector (X,Y) endowed with a joint pdf / pmf $f_{XY}(x,y)$, we have defined the conditional pdf / pmf

$$f_{Y|X}(y|x)$$

which defines a conditional distribution for Y given that $X=x$, for each $x \in R$.

- By varying x, we actually have a collection of distributions for Random variable Y described by the collection of conditional pdf / pmf on y.

$$\left\{ f_{Y|X}(y|x) \right\}_{x \in \mathbb{R}}.$$

- This is what we mean when we speak of the conditional distribution of a Random variable Y given a Random variable X.

Conditional expectation

Given $(X,Y) \sim f_{XY}$

$g : \mathbb{R} \to \mathbb{R}$ a function

$$\boxed{\begin{aligned} \mathbb{E}[g(Y) \mid x] &:= \mathbb{E}[\, g(Y) \mid X = x \,] \\ &:= \int g(y)\, f_{Y|X}(y|x)\, dy \end{aligned}}$$

$\uparrow$
Σ if Y discrete

Example!

Let $(X,Y) \sim f_{XY}(x,y) = \begin{cases} e^{-y}, & 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$

- $f_{XY}(x,y) \geq 0 \quad \forall x,y \in \mathbb{R}^2$
- $\int_{\mathbb{R}^2} f_{XY}(x,y)\, dx\, dy$

$$= \int_0^\infty \int_0^y e^{-y}\, dx\, dy = \int_0^\infty y e^{-y}\, dy$$

$$= \int_0^\infty (-y)\, de^{-y}$$

$$\overset{ibp}{=} \int_0^\infty e^{-y}\, dy = 1.$$

- So this is a valid joint pdf.

- Marginal computation: $f_X(x) = 0 \quad \forall x \leq 0.$

for $x > 0$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$$

$$= \int_x^\infty f_{XY}(x,y)\, dy$$

$$= \int_x^\infty e^{-y}\, dy$$

$$= -e^{-y}\Big|_x^\infty = e^{-x}.$$

for $y>0$:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= \int_0^y f_{XY}(x,y)\,dx$$

$$= \int_0^y e^{-y}\,dx = ye^{-y}.$$

$$f_Y(y) = 0 \quad \forall\, y \le 0.$$

. Conditional pdf: if $y \geq x > 0$ then

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} = e^{x-y}.$$

$$= 0 \quad \text{if } y < x.$$

if $x \in (0,y)$:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$= \frac{e^{-y}}{ye^{-y}} = \frac{1}{y}.$$

$$f_{X|Y}(x|y) = 0 \quad \text{otherwise}.$$

- Conditional expectations

$$\mathbb{E}[Y \mid X=x] = \int y \, f_{Y|X}(y|x) \, dy$$

$$= \int_x^\infty y \, e^{x-y} \, dy$$

$$= e^x \int_x^\infty y e^{-y} \, dy$$

$$= x+1. \qquad \text{(check!)}$$

$$\mathbb{E}[Y^2 \mid X=x] = \int_x^\infty y^2 e^{x-y} \, dy$$

$$= e^x \int_x^\infty y^2 e^{-y} \, dy$$

$$= x^2 + 2x + 2 \qquad \text{(check!)}$$

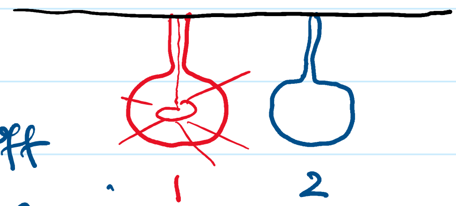
Hence $\text{var}(Y \mid X=x) = x^2+2x+2 - (x+1)^2$

$$= 1.$$

Example 2

Given two light bulbs with lifelengths $X, Z \geq 0$

$X, Z \overset{iid}{\sim} \mathrm{Exp}(\lambda)$



Light bulb 1 is ON, 2 is off.

Light bulb 2 is turned ON as soon as 1 burns out.

Let $Y = X + Z$ be the time bulb 2 burns out

$$P(Y \leq y \mid X = x) = P(Z \leq y - x \mid X = x)$$

$$= P(Z \leq y - x) \quad \text{since } Z \perp X$$

$$= F_Z(y - x) = 1 - e^{-\lambda(y-x)} \quad \text{if } y > x$$

Hence

$$f_{Y|X}(y|x) = \frac{d}{dy}\left(1 - e^{-\lambda(y-x)}\right)$$

$$= \lambda e^{-\lambda(y-x)}$$

in particular, if $\lambda = 1$ then this gives the same Conditional pdf as the previous example

example

With $f_X(x) = \lambda e^{-\lambda x} \mathbb{1}(x>0)$

and $f_{Y|X}(y|x) = \begin{cases} \lambda e^{-\lambda(y-x)} & , y>x \\ 0 & \text{otherwise} \end{cases}$.

The joint pdf for (X,Y) is completely determined via

$$f_{XY}(x,y) = f_X(x)\, f_{Y|X}(y|x)$$

$$= \lambda^2 e^{-\lambda y} \quad \text{if} \quad y>x>0$$

and $f_{XY}(x,y) = 0$ otherwise

Remark: There are two equivalent ways to define a joint distribution for a bivariate random vector (X,Y):

① By specifying joint pdf $f_{XY}(x,y)$ $\forall x,y$

② By specifying pdf for X, $f_X(x)$ and the conditional pdf of Y given X, $f_{Y|X}(y|x)$ and set

$$f_{XY}(x,y) := \begin{cases} f_X(x)\, f_{Y|X}(y|x) & \text{if } f_X(x) > 0 \\ 0 & \text{o.w.} \end{cases}$$

the pdf f_X then becomes the marginal pdf with respect to the joint $f_{X,Y}$, i.e. the identity

$$\int_y f_{XY}(x,y)\, dy = f_X(x)$$

holds.

② is usually "easier" in applications