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Stats 510, Instructor: Long Nguyen

Homework 2

Issued September 23, 2025, due by 11:59pm September 30, 2025

1. Do problems 1.38, 1.47, 1.53, 2.2, 2.4.
2. Let X, Y, Z be real-valued continuous random variables with the pdf, respectively, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $f_Y(y) = \frac{1}{\sqrt{8\pi}} e^{-y^2/8}$, and $f_Z(z) = \frac{1}{\sqrt{8\pi}} e^{-(z-1)^2/8}$.
   (i) Show directly that the random variables X, $Y/2$, $(Z-1)/2$ and $(1-Z)/2$ all have identical distributions. (Hint: Examine the corresponding cdf's.).
   (ii) Show that $P(X > 0) = 1/2$ and in fact, $P(X > 0) = P(Y \geq 0) = P(Z \leq 1)$.
   (iii) Let U be a chi squared random variable with 1 degree of freedom. Show that $P(U \leq 1) < P(X \leq 1)$.
3. This question asks you to prove a theorem in the lecture notes (Theorem 2 in "Probability integral transform" of Section 2.1). Let X be a real-valued random variable with cdf $F_X(x)$. Recall that the inverse function for the (right-continuous) F_X can be defined as follows, for $0 < y < 1$,

More general than the simple case as seen in lecture (week 5)

$$F_X^{-1}(y) := \inf\{x : F_X(x) \geq y\}.$$

   Moreover, in the above definition if a set is empty then its infimum is defined to be $+\infty$ as a convention. We also define $F_X^{-1}(0) := -\infty$. Let U be a uniform random variable in $(0,1)$, and $Z := F_X^{-1}(U)$. Show that Z has the same distribution as that of X in the following two scenarios:
   (i) X is a discrete random variable taking values in a finite set $\mathcal{X} = \{a_1, \ldots, a_k\} \subset \mathbb{R}$, for some $k \in \mathbb{N}$.
   (ii) X is a continuous random variable.

1. Do problems 1.38, 1.47, 1.53, 2.2, 2.4.

1.38) Prove each of the following statements (Assume that any conditioning event has positive probability)

a) Please Grade this for correctness:

$P(A) = P(A \cap B) + P(A \cap B^c)$

but, $(A \cap B^c) \subset B^c$ and $P(B^c) = 1 - P(B) = 0$.

So, $P(A \cap B^c) = 0$ and $P(A) = P(A \cap B)$.

Thus,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{1} = P(A). \blacksquare$$

Please Grade this for Accuracy & comment if/where incorrect:

If $P(B) = 1$, then $P(A|B) = P(A)$ for any A:

P. $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

$= \frac{P(B)P(A)}{P(B)}$, since $P(B) = 1 \Rightarrow P(B|A) = P(B) = 1$

$= P(A). \blacksquare$

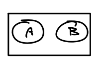
b. If $A \subset B$, then $P(B|A) = 1$ and $P(A|B) = P(A)/P(B)$

$A \subset B$ implies $A \cap B = A$. Thus,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

and also,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}. \blacksquare$$



c. If A and B are mutually exclusive, then

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}$$

P. Let A, B - mutually exclusive events, i.e. $A \cap B = \emptyset$ s.t

$P(A \cap (A \cup B)) = P(A)$ and $P(A \cup B) = P(A) + P(B)$

Then,

$$P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A \cup B)} = \frac{P(A)}{P(A) + P(B)}. \blacksquare$$

d. $P(A \cap B \cap C) = P(A|B \cap C) P(B|C) P(C)$

P. $P(A \cap B \cap C) = P((A \cap B) \cap C)$

$= P(A \cap B | C) P(C)$

$= \frac{P((A \cap B) \cap C)}{P(C)} P(C)$

$= \frac{P(A \cap (B \cap C))}{P(C)} P(C)$

$= \frac{P(A|B \cap C) P(B \cap C)}{P(C)} P(C).$

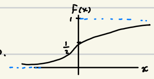
$= \frac{P(A|B \cap C) \frac{P(B|C)}{P(C)}}{P(C)} P(C) = P(A|B \cap C) P(B|C) P(C). \blacksquare$

1.47) Prove that the following functions are cdfs: <u>Note</u>: all of the functions are continuous, hence right continuous. Thus, only need to check the limit, and that they are nondecreasing

I will use the following theorem to show that each subsequent function is a valid cdf:

> <u>Thm.</u> The function $F_X(x)$ is a cdf i.f.f.
>
> (i) $\lim_{x\to-\infty} F(x)=0$ and $\lim_{x\to\infty} F(x)=1$
>
> (ii) $F(x)$ is a nondecreasing function of x
>
> (iii) $F(x)$ is right-continuous, i.e. $\lim_{x\downarrow x_0} F(x)=F(x_0)$

a. $F(x)=\frac{1}{2}+\frac{1}{\pi}\tan^{-1}(x),\ x\in(-\infty,\infty)$

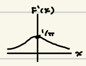


i) $\lim_{x\to-\infty} F(x)=\lim_{x\to-\infty}\frac{1}{2}+\frac{1}{\pi}\tan^{-1}(x)=\frac{1}{2}+\frac{1}{\pi}\left(\frac{-\pi}{2}\right)=0.$

$\lim_{x\to\infty} F(x)=\lim_{x\to\infty}\frac{1}{2}+\frac{1}{\pi}\tan^{-1}(x)=\frac{1}{2}+\frac{1}{\pi}\left(\frac{\pi}{2}\right)=1.$

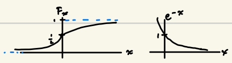
ii) $F'(x)=\frac{d}{dx}\left[\frac{1}{2}+\frac{1}{\pi}\tan^{-1}(x)\right]=\underbrace{\frac{1}{\pi}\frac{1}{1+x^2}}_{>0}$ $\Rightarrow$ increasing

$\Rightarrow$ nondecreasing



iii) $F(x)$ is continuous and therefore right-continuous, i.e.

$\lim_{\varepsilon\downarrow 0} F_X(x+\varepsilon)=F_X(x).$ ∎



b. $F(x)=(1+e^{-x})^{-1},\ x\in(-\infty,\infty)$

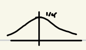
i) $\lim_{x\to-\infty} F(x)=\lim_{x\to-\infty}\frac{1}{1+e^{-x}}=0$ since $\lim_{x\to-\infty}e^{-x}=\infty$

$\lim_{x\to\infty} F(x)=1$ since $\lim_{x\to\infty}e^{-x}=0$

ii) $F'(x)=\frac{d}{dx}\left[(1+e^{-x})^{-1}\right]=\frac{e^{-x}}{(1+e^{-x})^2}>0$

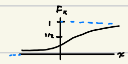
$\Rightarrow$ increasing

$\Rightarrow$ nondecreasing



iii) $F(x)$ is (left+right) continuous and $\lim_{\varepsilon\downarrow 0}F(x+\varepsilon)=F_X(x).$ ∎

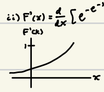
c. $F(x)=e^{-e^{-x}},\ x\in(-\infty,\infty)$



i) $\lim_{x\to-\infty} F(x)=\lim_{x\to-\infty}e^{-(e^{-x})}=\lim_{x\to-\infty}\frac{1}{e^{e^{-x}}}=0$ since $\lim_{x\to-\infty}e^{-x}=\infty$

$\lim_{x\to\infty}F(x)=\lim_{x\to\infty}\frac{1}{e^{e^{-x}}}=1$ since $\lim_{x\to\infty}e^{-x}=0$

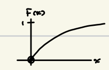
ii) $F'(x)=\frac{d}{dx}\left[e^{-e^{-x}}\right]=\underbrace{e^{-e^{-x}-x}}_{>0}$ since $\begin{cases} g(f(x))=e^{-e^{-x}} \\ g(x)=e^{x} \\ f(x)=-e^{-x} \\ \frac{d}{dx}g(f(x))=g'(f(x))f'(x) \\ \quad =e^{-e^{-x}}e^{-x}\end{cases}$



iii) $F(x)$ is continuous and therefore right-continuous, i.e.

$\lim_{\varepsilon\downarrow 0}F_X(x+\varepsilon)=F_X(x).$ ∎

d. $F(x)=1-e^{-x},\ x\in(0,\infty)$



i) $\lim_{x\to-\infty}F(x)=0$ since $F_X(x)=0$ for $x\le 0$.

$\lim_{x\to\infty}F(x)=\lim_{x\to\infty}1-e^{-x}=1$ since $\lim_{x\to\infty}e^{-x}=0.$

ii) $F'(x)=\frac{d}{dx}\left[1-e^{-x}\right]=-\frac{d}{dx}e^{-x}=e^{-x}>0$

$\Rightarrow$ increasing

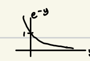
$\Rightarrow$ nondecreasing

iii) $F(x)$ is continuous and therefore right-continuous, i.e.

$\lim_{\varepsilon\downarrow 0}F_X(x+\varepsilon)=F_X(x).$ ∎

e. For some $\varepsilon\in(0,1)$: $F_Y(y)=\begin{cases}\frac{1-\varepsilon}{1+e^{-y}}, & y<0,\\ \varepsilon+\frac{(1-\varepsilon)}{1+e^{-y}}, & y\ge 0.\end{cases}$

<u>recall</u>:



i) $\lim_{y\to-\infty}F(y)=\lim_{y\to-\infty}\frac{1-\varepsilon}{1+e^{-y}}=0$ since $\lim_{y\to-\infty}e^{-y}=\infty$ and $1-\varepsilon\in(0,1)$

$\lim_{y\to\infty}F(y)=\lim_{y\to\infty}\varepsilon+\frac{(1-\varepsilon)}{1+e^{-y}}=\varepsilon+(1-\varepsilon)=1$ since $\lim_{y\to\infty}e^{-y}=0.$

ii) when $y<0$, $F'(y)=(1-\varepsilon)\frac{d}{dy}(1+e^{-y})^{-1}=\underbrace{\underbrace{(1-\varepsilon)}_{>0}\frac{e^{-y}}{\underbrace{(1+e^{-y})^2}_{>0}}}_{>0}$

when $y\ge 0$, $F'(y)=(1-\varepsilon)\frac{e^{-y}}{(1+e^{-y})^2}>0,$

and $\lim_{y\to 0^+}F(y)=\lim_{y\to 0^+}\left[\varepsilon+\frac{(1-\varepsilon)}{1+e^{-y}}\right]=\varepsilon+\frac{(1-\varepsilon)}{2}>\frac{1-\varepsilon}{2}=\lim_{y\to 0^-}\left[\frac{1-\varepsilon}{1+e^{-y}}\right]=\lim_{y\to 0^-}F(y).$

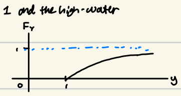
Therefore, $F(y)$ is nondecreasing.

iii) For $y<0$, $\lim_{\varepsilon\downarrow 0}F_Y(y+\varepsilon)=F_Y(y)$

$y\ge 0$, $\lim_{\varepsilon\downarrow 0}F_Y(y+\varepsilon)=F_Y(y),$

i.e. right-continuity holds for the piecewise function $F(y)$. ∎

1.53) A certain river floods every year. Suppose that the low-water mark is set at 1 and the high-water mark Y has distribution function $F_Y(y)=P(Y\le y)=1-\frac{1}{y^2},\ 1\le y<\infty$



a. Verify that $F_Y(y)$ is a cdf

i) $\lim_{y\to-\infty}F(y)=0$ since $F_Y(y)=0$ for $y<1$

$\lim_{y\to\infty}F(y)=1$ since $\lim_{y\to\infty}\frac{1}{y^2}=0.$

ii) For $y\le 1$, $F_Y(y)=0$ is constant. For $y>1$, $\frac{d}{dy}F(y)=\frac{2}{y^3}$ so F_Y is increasing and therefore $\forall\ y$, F_Y is nondecreasing.

iii) F_Y is continuous and hence right continuous.

b. Find the pdf of Y, i.e. $f_Y(y)$

$f_Y(y)=F'(y)=\frac{d}{dy}\left[1-\frac{1}{y^2}\right]=\frac{d}{dy}(-y^{-2})=(-)-2y^{-3}=\frac{2}{y^3},\quad 1\le y<\infty$

so, $f_Y(y)=\begin{cases}\frac{2}{y^3}, & 1\le y<\infty\\ 0, & \text{otherwise}\end{cases}$

c. If the low-water mark is reset at 0 and we use a unit of msmt that is 1/10 of that given previously, the high-water mark becomes $Z=10(Y-1)$.

Find $F_Z(z)=P(Z\le z)$

$=P(10(Y-1)\le z)$

$=P(Y-1\le \frac{z}{10})$

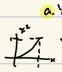
$=P(Y\le\frac{z}{10}+1)$

$=1-\frac{1}{(\frac{z}{10}+1)^2}.$

Thus, $F_Z(z)=\begin{cases}0 & \text{if } z\le 0\\ 1-\frac{1}{(\frac{z}{10}+1)^2} & \text{if } z>0\end{cases}$ since $\begin{cases}y=1\Rightarrow z=10(1-1)=0\\ y=\infty\Rightarrow z=10(\infty)=\infty\end{cases}$

2.2) In each of the following find the pdf of Y: $\boxed{\underline{\text{change-of-var (formula)}}:=\ f_Y(y)=f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right|}$

a. $Y=X^2$ and $f_X(x)=1,\ 0<x<1$



$Y=g(X)$ where $g(x)=x^2$ for $\begin{cases}\mathcal{X}=(0,1)\\ \mathcal{Y}=(0,1)\end{cases}$

$F_Y(y)=P(Y\le y),\quad y\in(0,1)$

$=P(X^2\le y)$

$=P(|X|\le\sqrt{y}),\ x\in(0,1)$

$=P(X\le\sqrt{y})$

$=F_X(\sqrt{y})$

Given $f_X(x)=1,\ x\in(0,1)$

then $F_X(x)=\int_0^x f_X(t)\,dt=\int_0^x dt=t\Big|_0^x=x$

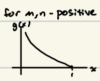
$\Rightarrow F_X(\sqrt{y})=\sqrt{y}$

$=\sqrt{y}$

$f_Y(y)=\frac{d}{dy}F_Y(y)=\frac{d}{dy}y^{\frac{1}{2}}=\frac{1}{2}y^{-\frac{1}{2}}=\boxed{\underbrace{\frac{1}{2\sqrt{y}},\ y\in(0,1).}_{f_Y(y)}}$

b. $Y=-\log(X)$ and $f_X(x)=\frac{(n+m+1)!}{n!m!}x^n(1-x)^m,\ 0<x<1$ for m,n - positive integers

$Y=g(X)$ where $g(x)=-\log(x)$ for $\begin{cases}\mathcal{X}=(0,1)\\ \mathcal{Y}=(0,\infty)\end{cases}$



$y=-\log(x)\Rightarrow\log(x)=-y\Rightarrow x=e^{-y}=g^{-1}(y)\ \forall\ y>0.$

$\frac{d}{dy}[g^{-1}(y)]=\frac{d}{dy}(e^{-y})=-e^{-y}$

and $g\downarrow$ on $\mathcal{X}=(0,1)$.

so,

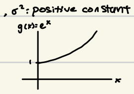
$f_Y(y)=f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right|$

$=\frac{(n+m+1)!}{n!m!}(e^{-y})^n(1-e^{-y})^m e^{-y}$

$=\boxed{\underbrace{\frac{(n+m+1)!}{n!m!}e^{-y(n+1)}(1-e^{-y})^m,\ y\in(0,\infty)}_{f_Y(y)}}$

c. $Y=e^X$ and $f_X(x)=\frac{1}{\sigma^2}x\exp\left(\frac{-(x/\sigma)^2}{2}\right),\ 0<x<\infty,\ \sigma^2$: positive constant

$Y=g(X)$ where $g(x)=e^x$ for $\begin{cases}\mathcal{X}=(0,\infty)\\ \mathcal{Y}=(1,\infty)\end{cases}$



$y=e^x\Rightarrow x=\log(y)=g^{-1}(y)\ \forall\ y>1$

$\frac{d}{dy}[g^{-1}(y)]=\frac{d}{dy}[\log(y)]=\frac{1}{y}$

and $g\uparrow$ on $\mathcal{X}=(0,\infty)$.

so,

$f_Y(y)=f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right|$

$=\frac{1}{\sigma^2}\log(y)\exp\left\{\frac{-(\log(y)/\sigma)^2}{2}\right\}\cdot\frac{1}{y}$

$=\frac{1}{\sigma^2}\frac{\log(y)}{y}\exp\left\{\frac{-1}{2}\left(\frac{\log(y)}{\sigma}\right)^2\right\},\ y\in(1,\infty)$

2.4) Let λ- fixed, positive constant and $f_X(x) = \begin{cases} \frac{1}{2}\lambda e^{-\lambda x}, & x \geq 0 \\ \frac{1}{2}\lambda e^{\lambda x}, & x < 0 \end{cases}$

a. verify that $f(x)$ is a pdf

$f(x)$ is a pdf since it is positive and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} \frac{\lambda}{2} e^{\lambda x}\,dx + \int_0^{\infty} \frac{\lambda}{2} e^{-\lambda x}\,dx = \frac{\lambda}{2}\left[\frac{1}{\lambda}e^{\lambda x}\right]\Big|_{-\infty}^{0} + \frac{\lambda}{2}\left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^{\infty} = \frac{1}{2} + \frac{1}{2} = 1$$

b. If X is a RV w/ pdf given by $f(x)$, Find $P(X<t)$ for all t. Evaluate all integrals.

Let X be a RV with density $f(x)$.

$$P(X<t) = \begin{cases} \int_{-\infty}^{t} \frac{1}{2}\lambda e^{\lambda x}\,dx & , t<0 \\ \int_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda x}\,dx + \int_0^t \frac{1}{2}\lambda e^{-\lambda x}\,dx, & t \geq 0 \end{cases}$$

where $\int_{-\infty}^{t} \frac{1}{2}\lambda e^{\lambda x}\,dx = \frac{1}{2}e^{\lambda x}\Big|_{-\infty}^{t} = \frac{1}{2}e^{\lambda t}$

and $\int_0^t \frac{1}{2}\lambda e^{-\lambda x}\,dx = -\frac{1}{2}e^{-\lambda x}\Big|_0^t = \frac{-1}{2}e^{-\lambda t} + \frac{1}{2}$.

Therefore,

$$P(X<t) = \begin{cases} \frac{1}{2}e^{\lambda t} & , t<0 \\ 1-\frac{1}{2}e^{-\lambda t} & , t \geq 0 \end{cases}$$

c. Find $P(|X|<t)$ for all t. Evaluate all integrals

$$P(|X|<t) = P(-t<X<t) = \int_{-t}^{0} \frac{1}{2}\lambda e^{\lambda x}\,dx + \int_0^t \frac{1}{2}\lambda e^{-\lambda x}\,dx = -\int_0^{-t} \frac{1}{2}\lambda e^{\lambda x}\,dx + \int_0^t \frac{1}{2}\lambda e^{-\lambda x}\,dx$$

$$= \frac{\lambda}{2}\left[\frac{1}{\lambda}e^{\lambda x}\right]\Big|_{-t}^{0} - \frac{1}{2}e^{-\lambda t} + \frac{1}{2}$$

$$= \frac{1}{2}[1-e^{-\lambda t}] + \frac{1}{2}[-e^{-\lambda t}+1]$$

$$= 1-e^{-\lambda t}.$$

2. Let X, Y, Z be real-valued continuous random variables with the pdf, respectively, $f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, $f_Y(y) = \frac{1}{\sqrt{8\pi}}e^{-y^2/8}$, and $f_Z(z) = \frac{1}{\sqrt{8\pi}}e^{-(z-1)^2/8}$.

(i) Show directly that the random variables X, $Y/2$, $(Z-1)/2$ and $(1-Z)/2$ all have identical distributions. (Hint: Examine the corresponding cdf's.).

(ii) Show that $P(X>0) = 1/2$ and in fact, $P(X>0) = P(Y \geq 0) = P(Z \leq 1)$.

(iii) Let U be a chi squared random variable with 1 degree of freedom. Show that $P(U \leq 1) < P(X \leq 1)$.

a. Let X, Y, Z - continuous RV's w/ the following pdf's:

$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) \sim N(0,1)$,

$f_Y(y) = \frac{1}{\sqrt{8\pi}}e^{-y^2/8} = \frac{1}{\sqrt{8\pi}}\exp\left(-\frac{y^2}{8}\right) \sim N(0,4)$,

$f_Z(z) = \frac{1}{\sqrt{8\pi}}e^{-(z-1)^2/8} = \frac{1}{\sqrt{8\pi}}\exp\left(-\frac{(z-1)^2}{8}\right) \sim N(1,4)$

Since recall: Normal(m, σ^2) has the following pdf: $\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-m)^2}{2\sigma^2}\right), \sigma>0$.

Let $X \sim N(0,1)$ with previously defined $f_X(x)$.

I will show that $X \stackrel{d}{=} \frac{Y}{2} \iff 2X \stackrel{d}{=} Y$.

Given $f_X(x)$, $X = \frac{Y}{2} \Rightarrow Y = g(x) = 2X$ for $\begin{cases} x \in \mathbb{R} \\ y \in \mathbb{R} \end{cases}$

Then, $y = 2x \Rightarrow x = \frac{y}{2} = g^{-1}(y)$,

$\frac{d}{dy}[g^{-1}(y)] = \frac{1}{2}y' = \frac{1}{2}$,

and $g(x) = 2x \uparrow$.

So,

$f_Y(y) = f_X(g^{-1}(y)) \left|\frac{d}{dy}g^{-1}(y)\right|$

$= \frac{1}{\sqrt{2\pi}}\exp\left(\frac{(-y/2)^2}{2}\right)\frac{1}{2}$

$= \frac{1}{\sqrt{8\pi}}\exp\left(\frac{-y^2}{8}\right)$.

Similarly, to show $X \stackrel{d}{=} \left(\frac{Z-1}{2}\right) \iff Z \stackrel{d}{=} 2X+1$,

$Z = g_1(X) = 2X+1$ for $\begin{cases} x \in \mathbb{R} \\ z \in \mathbb{R} \end{cases}$.

Next,

$z = 2x+1 \Rightarrow x = \frac{z-1}{2} = g_1^{-1}(z)$

$\frac{d}{dz}[g_1^{-1}(z)] = \frac{1}{2}\frac{d}{dy}(z-1) = \frac{1}{2}$

and $g_1(x) = 2x+1 \uparrow$.

Thus,

$f(z) = f_X(g_1^{-1}(z)) \left|\frac{d}{dz}g_1^{-1}(z)\right|$

$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\left(\frac{z-1}{2}\right)^2}{2}\right)\frac{1}{2}$

$= \frac{1}{\sqrt{8\pi}}\exp\left(-\frac{(z-1)^2}{8}\right)$.

The same process follows to show $X \stackrel{d}{=} \frac{(1-Z)}{2} \iff Z \stackrel{d}{=} 1-2X$:

$g_2(x) = z = 1-2x$ for $x \in \mathbb{R}, z \in \mathbb{R}$

where

$z = 1-2x \Rightarrow x = \frac{1-z}{2} = g_2^{-1}(z)$, $\frac{d}{dz}[g_2^{-1}(z)] = \frac{-1}{2}$, and $g_2 \downarrow$

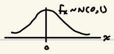
So,

$f(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\left(\frac{1-z}{2}\right)^2}{2}\right)\frac{1}{2} = \frac{1}{\sqrt{8\pi}}\exp\left(\frac{-(z-1)^2}{2}\right)$.

Consequently, I have shown that

$X \stackrel{d}{=} \frac{Y}{2} \stackrel{d}{=} \frac{(Z-1)}{2} \stackrel{d}{=} \frac{(1-Z)}{2} \sim f_X(x)$.

b. We know $X \sim N(0,1)$:



So, $P(X>0) = \int_0^{\infty} f_X(x)\,dx = \frac{1}{2} = \int_{-\infty}^{0} f_X(x)\,dx$ by symmetry of $f_X(x) \sim N(0,1)$.

Similarly, for $Y \sim N(0,4)$ and $Z \sim N(1,4)$,

$P(X>0) = P(Y \geq 0) = P(Z \leq 1) = \frac{1}{2}$ by symmetry of the continuous normal pdf and hence equal prob. on either side of the mean.

iii) Let $U \sim \chi_1^2$. I will show that $P(U \leq 1) < P(X \leq 1)$.

We know that $\chi_1^2 = X^2$ for $X \sim N(0,1)$

So, $P(U \leq 1) = P(\chi_1^2 \leq 1)$

$= P(X^2 \leq 1)$

$= P(|X| \leq 1) = P(-1 \leq X \leq 1) = \Phi(1) - \Phi(-1)$

Because $\Phi(-1) > 0$, $\Phi(1) - \Phi(-1) < \Phi(1) = P(X \leq 1)$

3. This question asks you to prove a theorem in the lecture notes (Theorem 2 in "Probability integral transform" of Section 2.1). Let X be a real-valued random variable with cdf $F_X(x)$. Recall that the inverse function for the (right-continuous) F_X can be defined as follows, for $0 < y < 1$,

$$F_X^{-1}(y) := \inf\{x : F_X(x) \geq y\}.$$

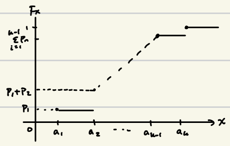
Moreover, in the above definition if a set is empty then its infimum is defined to be $+\infty$ as a convention. We also define $F_X^{-1}(0) := -\infty$. Let U be a uniform random variable in $(0, 1)$, and $Z := F_X^{-1}(U)$. Show that Z has the same distribution as that of X in the following two scenarios:

(i) X is a discrete random variable taking values in a finite set $\mathcal{X} = \{a_1, \ldots, a_k\} \subset \mathbb{R}$, for some $k \in \mathbb{N}$.

(ii) X is a continuous random variable.

i) Suppose $P(X=a_n) = P_n \ \forall n \in \{1, \ldots, k\}$

$$F_X^{-1}(u) = \begin{cases} -\infty, & u = 0 \\ a_1, & 0 < u \leq P_1 \\ a_2, & P_1 < u \leq P_1 + P_2 \\ \vdots & \\ a_{k-1}, & \sum_{i=1}^{k-2} P_n < u \leq \sum_{i=1}^{k-1} P_n \\ a_k, & \sum_{i=1}^{k-1} P_n < u \leq 1 \end{cases} \iff$$



$$= \begin{cases} -\infty, & u = 0 \\ a_n, & \sum_{i=1}^{n-1} P_n < u \leq \sum_{i=1}^{n} P_n \ \forall n \in \{1, \ldots, k\} \end{cases}$$

so, $P(F_X^{-1}(U) = a_n) = P\left(\sum_{i=1}^{n-1} P_n < U \leq \sum_{i=1}^{n} P_n\right)$

$= \sum_{i=1}^{n} P_n - \sum_{i=1}^{n-1} P_n$ since $U \sim \text{unif}(0,1)$

$= P_n$

$= P(X = a_n)$.

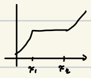
therefore, we have shown that

$Z := F_X^{-1}(U) \stackrel{d}{=} X.$ ∎

Scratch Work (Don't Grade)

to show $u \leq F(z)$ in the following form :

(where monotonicity doesn't hold)



if $u < F(z)$, then $F^{-1}(u) \leq z$

if $u = F(z)$ s.t. $x_1 \leq z \leq x_2$

if $z < x_1$, then $F^{-1}(u) < z$

if $z > x_2$, then $F^{-1}(u) > z$ and

if $x_1 < z < x_2$, then $F^{-1}(u) =$

ii) Let X - continuous RV

$P(Z \leq z) = P(F_X^{-1}(U) \leq z)$ (#)

$= P(F_X(F_X^{-1}(U)) \leq F_X(z))$

$= P(U \leq F_X(z))$ (★)

where (#) ⇒ (★) since F_X is monotonic

and (★) ⇒ (#): Given $u \leq F(z)$, need to show that $\inf\{x : F(x) \geq u\} \leq z$

since $u \leq F(z)$, $z \in \{x : F(x) \geq u\}$

so $\inf\{x : F(x) \geq u\} \leq z$.

$= F_X(z)$

$= P(X \leq z)$