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Stats 510, Instructor: Long Nguyen

Homework 1

Issued September 3, 2025, due by 11:59pm September 10, 2025

1. Do problems 1.1, 1.2.

   (Hint: In problem 1.2, recall the set difference operation: $A \setminus B := \{x|x \in A \text{ and } x \notin B\}$.)

2. Approximately one-third of all human twins are identical (one-egg), and two-thirds are fraternal (two-egg) twins. Identical twins are necessarily the same sex, with male and female being equally likely. Among fraternal twins, approximately one-fourth are both female, on-fourth are both male, and half are one male and one female. Finally, among all U.S. births, approximately 1 in 90 is a twin birth. Define the following events: $A = \{\text{a U.S. birth results in twin females}\}$, $B = \{\text{a U.S. birth results in identical twins}\}$, and $C = \{\text{a U.S. birth results in twins}\}$.

   (i) State, in words, the events $A \cap B \cap C$, $B \setminus A$, $A \setminus B$, $A \cup B$, and $C \setminus (A \cap B)$.

   (ii) Find the probabilities of all such events in (i).

3. Let the sample space S be the real line. Suppose that a sigma algebra $\mathcal{B}$ contains all half-closed intervals of the form $(-\infty, a]$ where a is a rational number. (Note: $(-\infty, a] = \{x|x \leq a\}$). Show that the following sets are elements of $\mathcal{B}$

   (i) all singleton sets $\{a\}$ where a is a rational number.

   (ii) all singleton sets $\{a\}$ where a is a real number.

   (iii) all intervals of the form $(a, b]$, where a and b are real numbers.

   (iv) Give an example of an element of $\mathcal{B}$ that is neither empty set, nor S, nor any of the forms mentioned above.

   (Hint: This is done by verifying that the set in question can be obtained from known elements of $\mathcal{B}$ via countably many set operations. Use the fact that any real number can be constructed as the limit of a sequence of rational numbers).

4. Let $A_1 \subset A_2 \ldots \subset A_n \subset \ldots S$ be an increasing sequence of subsets in a sigma algebra $\mathcal{B}$ associated with a sample space S. The limit of this sequence of subsets is defined as

   $$A := \lim A_n := \cup_{i=1}^{\infty} A_i := \{x|x \in A_i \text{ for some } i < \infty\}.$$

   Let P be a probability function on $\mathcal{B}$. Use the axioms of probability to show that

   (i) The sequence of $P(A_n)$ increases to a finite limit.

   (ii) In fact, that limit is equal to $P(A)$.

5. (Counting) Do problems 1.20, 1.21.

1. Do problems 1.1, 1.2.

(Hint: In problem 1.2, recall the set difference operation: $A \setminus B := \{x | x \in A \text{ and } x \notin B\}$.)

1.1) For each of the following experiments, describe the sample space

a. Toss a coin four times: Each toss has two possible outcomes: H or T.

$\boxed{S = \{HHHH, HHHT, HHTH, HTHH, THHH, HHTT, \dots, TTTT\}.}$

S is a set of all sequences of length 4; there are $2^4 = 16$ total number of outcomes

b. Count the number of insect-damaged leaves on a plant:

S consists of all possible nonnegative integers, i.e.

$\boxed{S = \{0, 1, 2, 3, \dots, N\}}$ where N: total number of leaves on the plant

$\Leftrightarrow \boxed{S = \{0, 1, 2, 3, \dots\}}$ if N not specified

c. Measure the lifetime (in hours) of a particular brand of light bulb

Lifetime is a positive real-valued measurement: $\boxed{S = (0, \infty)}$

d. Record the weights of 10-day old rats

If only one rat is measured, the sample space is $\boxed{S = (0, \infty)}$

Suppose a group of n rats is measured. Then, $\underline{S = (0, \infty)^n}$, the n-dimensional positive real space.

e. Observe the proportion of defectives in a shipment of electronic components

If the shipment has N components, $\boxed{S = \{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\}} \Leftrightarrow \boxed{S = [0, 1]}$ if N is very large or unspecified.

1.2) Verify the following identities; Recall: $A \setminus B := \{x | x \in A \text{ and } x \notin B\}$

a. $A \setminus B = A \setminus (A \cap B) = A \cap B^c$

$x \in A \setminus (A \cap B) \Leftrightarrow x \in A$ and $(x \notin A \cap B)$

$\Leftrightarrow x \in A$ and $(x \in (A \cap B)^c)$

$\Leftrightarrow x \in A$ and $(x \in A^c$ or $x \in B^c)$, Demorgan's

$\Leftrightarrow (x \in A$ and $x \in A^c)$ or $(x \in A$ and $x \in B^c)$, Distributive

$\Leftrightarrow x \in \emptyset$ or $(x \in A$ and $x \in B^c)$

$\Leftrightarrow x \in A$ and $x \in B^c$

$\Leftrightarrow x \in A \cap B^c$

$\Leftrightarrow x \in A$ and $x \notin B$

$(\Rightarrow) x \in A \setminus B.$ ∎

$A \setminus B = A \setminus (A \cap B) = A \cap B^c$

Proof. $A \setminus B = \{x : x \in A, x \notin B\}$

Also, $A \setminus (A \cap B) = \{x : x \in A, x \notin (A \cap B)\}$

If $x \in A$ and $x \notin (A \cap B)$, then necessarily $x \notin B$. Hence,

$A \setminus (A \cap B) = \{x : x \in A, x \notin B\} = A \setminus B.$

Moreover,

$A \cap B^c = \{x : x \in A, x \in B^c\} = \{x : x \in A, x \notin B\}.$

Thus, $A \setminus B = A \setminus (A \cap B) = A \cap B^c.$ ∎

b. $B = (B \cap A) \cup (B \cap A^c)$

$x \in (B \cap A) \cup (B \cap A^c) \Leftrightarrow (x \in B$ and $x \in A)$ or $(x \in B$ and $x \notin A)$

$\Leftrightarrow x \in B$ and $(x \in A$ or $x \in A^c)$, Distributive

$\Leftrightarrow x \in B$ and $x \in \emptyset$

$\Leftrightarrow x \in B.$ ∎

Proof. $B = (B \cap A) \cup (B \cap A^c)$

Let $x \in B$. Then either $x \in A$ or $x \in A^c$.

- If $x \in A$, then $x \in B \cap A$
- If $x \in A^c$, then $x \in B \cap A^c$

Thus, $x \in (B \cap A) \cup (B \cap A^c)$.

Conversely, if $x \in (B \cap A) \cup (B \cap A^c)$, then clearly $x \in B$.

Therefore, $B = (B \cap A) \cup (B \cap A^c).$ ∎

c. $B \setminus A = B \cap A^c$

$x \in B \setminus A \Leftrightarrow x \in B$ and $x \notin A$

$\Leftrightarrow x \in B \cap A^c.$ ∎

Proof. $B \setminus A = B \cap A^c$.

By definition, $B \setminus A = \{x : x \in B, x \notin A\}$.

On the other hand,

$B \cap A^c = \{x : x \in B, x \in A^c\} = \{x : x \in B, x \notin A\}$

Hence,

$B \setminus A = B \cap A^c.$ ∎

d. $A \cup B = A \cup (B \cap A^c)$

$x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

$\Leftrightarrow x \in A$ or $(x \in B$ and $x \notin A)$

$\Leftrightarrow A \cup (B \cap A^c).$ ∎

Proof. $A \cup B = A \cup (B \cap A^c)$

First, let $x \in A \cup (B \cap A^c)$

- If $x \in A$, then $x \in A \cup B$
- If $x \in B \cap A^c$, then $x \in B$, so $x \in A \cup B$

Thus,

$A \cup (B \cap A^c) \subseteq A \cup B.$

Conversely, let $x \in A \cup B$.

- If $x \in A$, then clearly $x \in A \cup (B \cap A^c)$
- If $x \in B$ and $x \notin A$, then $x \in B \cap A^c$, hence $x \in A \cup (B \cap A^c)$.

Therefore,

$A \cup B \subseteq A \cup (B \cap A^c)$

Combining both inclusions,

$A \cup B = A \cup (B \cap A^c).$ ∎

2. Approximately one-third of all human twins are identical (one-egg), and two-thirds are fraternal (two-egg) twins. Identical twins are necessarily the same sex, with male and female being equally likely. Among fraternal twins, approximately one-fourth are both female, on-fourth are both male, and half are one male and one female. Finally, among all U.S. births, approximately 1 in 90 is a twin birth. Define the following events: $A = \{$a U.S. birth results in twin females$\}$, $B = \{$a U.S. birth results in identical twins$\}$, and $C = \{$a U.S. birth results in twins$\}$.

(i) State, in words, the events $A \cap B \cap C$, $B \setminus A$, $A \setminus B$, $A \cup B$, and $C \setminus (A \cap B)$.

(ii) Find the probabilities of all such events in (i).

Let $A = \{$a U.S. birth results in twin females$\}$

$B = \{$a U.S. birth results in identical twins$\}$

$C = \{$a U.S. birth results in twins$\}$

i) $A \cap B \cap C$: births that are identical twin females ($A, B \subseteq C$, thus, just $A \cap B$)

$B \setminus A$: births that are identical twins but not twin females, i.e. identical twin males

$A \setminus B$: births that are twin females but not identical, i.e. fraternal twin females

$A \cup B$: births that are twin females or identical twins (including their overlap)

$C \setminus A \cap B$: births that are twins but not identical twin females (all twin births except identical FF)

ii) $P(A \cap B \cap C) = P(A)P(B)P(C) = \frac{1}{90}\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{540}$

$P(B \setminus A) = P(C) \cdot P(B \setminus A | C)$

$= \frac{1}{90} P(\text{identical}) P(\text{MM} | \text{identical}) = \frac{1}{90}\left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{540}$

$P(A \setminus B) = P(C) \cdot P(A \setminus B | C)$

$= \frac{1}{90} P(\text{fraternal}) P(\text{FF} | \text{fraternal}) = \frac{1}{90}\left(\frac{2}{3}\right)\left(\frac{1}{4}\right) = \frac{1}{540}$

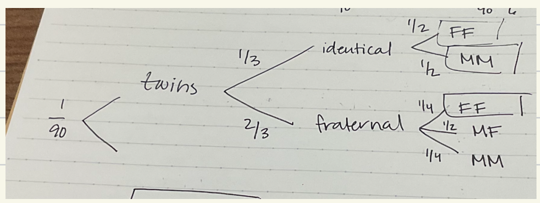
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{90}\left[\frac{1}{3}\frac{1}{2} + \frac{2}{3}\frac{1}{4}\right] + \left[\frac{1}{90} \cdot \frac{1}{3}\right] - \left[\frac{1}{90} \cdot \frac{1}{3} \cdot \frac{1}{2}\right]$

$= \frac{1}{90}\left(\frac{1}{3}\right) + \frac{1}{90}\left(\frac{1}{3}\right) - \frac{1}{540}$

$= \frac{1}{180}$

$P(C \setminus (A \cap B)) = P(C) - P(A \cap B) = \frac{1}{90} - \left(\frac{1}{540}\right) = \frac{1}{108}$



3. Let the sample space S be the real line. Suppose that a sigma algebra $\mathcal{B}$ contains all half-closed intervals of the form $(-\infty, a]$ where a is a rational number. (Note: $(-\infty, a] = \{x | x \leq a\}$). Show that the following sets are elements of $\mathcal{B}$

(i) all singleton sets $\{a\}$ where a is a rational number.

(ii) all singleton sets $\{a\}$ where a is a real number.

(iii) all intervals of the form $(a, b]$, where a and b are real numbers.

(iv) Give an example of an element of $\mathcal{B}$ that is neither empty set, nor S, nor any of the forms mentioned above.

(Hint: This is done by verifying that the set in question can be obtained from known elements of $\mathcal{B}$ via countably many set operations. Use the fact that any real number can be constructed as the limit of a sequence of rational numbers).

Solution: Let $S = \mathbb{R}$ and let $\mathcal{B}$ be the σ-algebra on S that contains every half-closed ray $(-\infty, q]$ with $q \in \mathbb{Q}$.

For any real $b \in \mathbb{R}$, choose a decreasing sequence of rationals $q_n \downarrow b$. Then

$$(-\infty, b] = \bigcap_{n=1}^{\infty} (-\infty, q_n] \in \mathcal{B},$$

since $\mathcal{B}$ is closed under countable intersections. Hence all rays $(-\infty, b]$ with real b are in $\mathcal{B}$.

Also, for any real a,

$$(-\infty, a) = \bigcup_{q < a, q \in \mathbb{Q}} (-\infty, q] \in \mathcal{B},$$

since $\mathbb{Q}$ is countable and $\mathcal{B}$ is closed under countable unions.

recall: $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c \Rightarrow$ above solution w.r.t. $\mathbb{R}$ applies for $\mathbb{Q}$.

i) For rational a,

$\{a\} = (-\infty, a] \setminus (-\infty, a) = (-\infty, a] \cap ((-\infty, a))^c \in \mathcal{B}.$

ii) The same identity works for any real a:

$\{a\} = (-\infty, a] \setminus (-\infty, a) = (-\infty, a] \cap ((-\infty, a))^c \in \mathcal{B}$

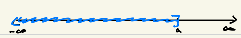
iii) For $a, b \in \mathbb{R}$

$(a, b] = (-\infty, b] \setminus (-\infty, a] = (-\infty, b] \cap ((-\infty, a])^c \in \mathcal{B}.$

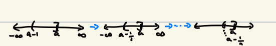
iv) The set of rationals $\mathbb{Q}$ belongs to $\mathcal{B}$ b/c it is a countable union of singletons: $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\} \in \mathcal{B}$.

It is neither $\emptyset$ nor S, not a singleton, and not an interval of the form $(a, b]$ or a ray $(-\infty, a]$. Hence $\mathbb{Q}$ is a valid example,

Let $S = \mathbb{R}$ w/ a $\mathcal{B}$ that contains all intervals of the form $(-\infty, a]$ for $a \in \mathbb{Q}$ (rational).



$\Leftrightarrow \{x | x \leq a, a \in \mathbb{Q}\}.$



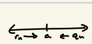
i) Let $a \in \mathbb{Q}$. I will show that $\{a\} \in \mathcal{B}$

For $n \in \mathbb{N}$, $a - \frac{1}{n} \in \mathbb{Q}$.

Then, $(-\infty, a) = \lim_{n \to \infty} \bigcup_{i=1}^{n} (-\infty, a - \frac{1}{n}) \in \mathcal{B}.$

Since $(-\infty, a] \in \mathcal{B}$,

$\{a\} = (-\infty, a] \setminus (-\infty, a) \in \mathcal{B}.$ ∎



ii) Let $a \in \mathbb{R}$, $n \in \mathbb{N}$

Pick rationals $q_n \downarrow a$ (decreasing sequence of rationals converging to a from above, i.e. $q_n > a \ \forall n$.)

Then,

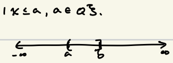
$(-\infty, a] = \lim_{n \to \infty} \bigcap_{i=1}^{n} (-\infty, q_n] \in \mathcal{B}.$ (*)

Pick rationals $r_n \uparrow a$ (increasing sequence of rationals converging to a from below, i.e. $r_n < a \ \forall n$)

Then,

$(-\infty, a) = \lim_{n \to \infty} \bigcup_{i=1}^{n} (-\infty, r_n] \in \mathcal{B}.$

Thus, $\{a\} = (-\infty, a] \setminus (-\infty, a) \in \mathcal{B}$ ∎



iii) Let $a, b \in \mathbb{R}$ where $a < b$.

From (ii), we know that $(-\infty, a] \in \mathcal{B}$ via (*).

Similarly, $(-\infty, b] \in \mathcal{B}$.

Therefore, $(a, b] = (-\infty, b] \setminus (-\infty, a] \in \mathcal{B}.$ ∎

iv) Let $a \in \mathbb{R}$.

From (ii), $\{a\} \in \mathcal{B}$.

Thus, $\underline{\{a\}^c = \mathbb{R} \setminus \{a\} \in \mathcal{B}}$

as $\mathcal{B}$ is closed under complementation. ∎

4. Let $A_1 \subset A_2 \ldots \subset A_n \subset \ldots S$ be an increasing sequence of subsets in a sigma algebra $\mathcal{B}$ associated with a sample space S. The limit of this sequence of subsets is defined as

$$A := \lim_{n\to\infty} A_n := \cup_{i=1}^{\infty} A_i := \{x | x \in A_i \text{ for some } i < \infty\}.$$

Let P be a probability function on $\mathcal{B}$. Use the axioms of probability to show that

(i) The sequence of $P(A_n)$ increases to a finite limit.

(ii) In fact, that limit is equal to $P(A)$.

Solution: Let $(A_n)_{n\geq 1}$ be an increasing sequence of sets:

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots \subset S,$$

and let $A := \lim_{n\to\infty} A_n = \bigcup_{i=1}^{\infty} A_i$.

Define the disjoint sets $B_1 := A_1$, $B_n := A_n \setminus A_{n-1}$ $(n \geq 2)$.

Then, the sets $B_1, B_2, \ldots$ are pairwise disjoint, and we have

$$A_n = \bigcup_{i=1}^{n} B_i, \quad A = \bigcup_{i=1}^{\infty} B_i$$

i) By finite additivity on disjoint unions,

$$P(A_{n+1}) = P(A_n \cup (A_{n+1} \setminus A_n)) = P(A_n) + P(A_{n+1} \setminus A_n) \geq P(A_n).$$

Thus $\{P(A_n)\}$ is nondecreasing. Moreover, since $A_n \subseteq S$,

$$P(A_n) \leq P(S) = 1.$$

Hence, $(P(A_n))$ is a nondecreasing sequence bounded above by 1, so it converges to a finite limit. ∎

ii) By countable additivity on disjoint unions,

$$P(A_n) = \sum_{i=1}^{n} P(B_i), \quad P(A) = \sum_{i=1}^{\infty} P(B_i).$$

Therefore,

$$\lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i) = \sum_{i=1}^{\infty} P(B_i) = P(A). ∎$$

I refer to the axioms of probability as AOP(1, 2, or 3).

i) $A_n \subset A_{n+1} \iff x \in A_n \Rightarrow x \in A_{n+1}$

$$\Rightarrow P(A_n) \leq P(A_{n+1}).$$

By AOP(1),

$$P(A_n) \geq 0$$

and AOP(2) asserts

$$P(A_n) \leq P(S) = 1.$$

Thus, the sequence of $P(A_n)$ converges to a finite limit, i.e.

$$\lim_{n\to\infty} P(A_n) = L \text{ for some } L \in [0,1]. ∎$$

ii) Define $B_1 := A_1$, $B_n := A_i \setminus A_{i-1}$, $i \geq 2$.

The sets B_i $(i=1,2,\ldots)$ are disjoint and

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i = A.$$

By AOP(3),

$$P(A) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i).$$

$$\Rightarrow \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i) = \sum_{i=1}^{\infty} P(B_i) = P(A). ∎$$

5. (Counting) Do problems 1.20, 1.21.

1.2) Let E_i: event that day i receives no call, $i = 1, \ldots, 7$. We want

$$P(\text{at least one each day}) = 1 - P\left(\bigcup_{i=1}^{7} E_i\right).$$

By the inclusion-exclusion principle,

$$P\left(\bigcup_{i=1}^{7} E_i\right) = \sum_{k=1}^{7} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq 7} P(E_{i_1} \cap \cdots \cap E_{i_k}).$$

If k specific days get no calls, then all 12 calls must fall among the remaining $7-k$ days. Since each call independently chooses a day uniformly,

$$P(E_{i_1} \cap \cdots \cap E_{i_k}) = \left(\frac{7-k}{7}\right)^{12}.$$

There are $\binom{7}{k}$ ways to choose which k days are empty. Hence,

$$P\left(\bigcup_{i=1}^{7} E_i\right) = \sum_{k=1}^{7} (-1)^{k+1} \binom{7}{k} \left(\frac{7-k}{7}\right)^{12}.$$

Therefore,

$$P(\text{at least one call each day}) = \sum_{k=0}^{7} (-1)^{k} \binom{7}{k} \left(\frac{7-k}{7}\right)^{12}.$$

Equivalently,

$$P = \frac{1}{7^{12}} \sum_{k=0}^{7} (-1)^k \binom{7}{k} (7-k)^{12}.$$

Computing this sum gives

$$P = \frac{3{,}162{,}075{,}840}{13{,}841{,}287{,}201} \approx 0.2285.$$

1.20) My telephone rings 12 times each week, the calls being randomly distributed among the 7 days. What is the prob. that I get at least one call each day? (A: 0.2285)

We are modeling 12 telephone calls that arrive during the week where

- Each call is a distinct event
- $\forall$ call, decide which of 7 days it happens

$\Rightarrow$ An outcome is a sequence of 12 choices s.t. each choice is one of 7 days, i.e. $(d_1, d_2, \ldots, d_{12})$, $d_i \in \{1, 2, \ldots, 7\}$

By the Fundamental Thm. of Counting (FToC) and supported by the ordered, with replacement scheme,

$$\underline{|\Omega| = 7^{12}} \qquad \underset{\text{choice 1}}{\underline{1\text{-}7}}\ \underset{\text{choice 2}}{\underline{1\text{-}7}} \cdots \underset{\text{choice 12}}{\underline{1\text{-}7}}$$

Let A_j = set of assignments that miss day j. We want assignments in the following assignments: $(A_1 \cup \cdots \cup A_7)^c$.

Then, by inclusion-exclusion principle:

$$\#\{\text{no empty days}\} = \sum_{i=0}^{7} (-1)^i \binom{7}{i} \left(\frac{7-i}{7}\right)^{12}$$

$$= 1 - \sum_{i=1}^{7} (-1)^{i+1} \binom{7}{i} \left(\frac{7-i}{7}\right)^{12} \quad (\star)$$

where
- $(-1)^i \iff$ alternating signs; corrects overcounting, e.g. subtracting assignments that misses one day double-counts assignments that miss two particular days, etc. So, inclusion-exclusion gives the exact correction; Generalizes: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$,
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$
can prove via induction; instead taking this principle to be true.
- $\binom{7}{i}$: choose i days to exclude for 7 choose i choices. Then, every call must go to one of the remaining $(7-i)$ days $\Rightarrow \left(\frac{7-i}{7}\right)^{12}$ assignments.

$$\text{So, } P\left(\begin{matrix}\{\text{at least one} \\ \text{call a day}\}\end{matrix}\right) = \frac{\star}{|\Omega|} \underset{\text{technology}}{= \cdots\cdots =} \frac{3162075840}{13841287201} \approx \boxed{0.2285}. ∎$$

We enumerate all integer partitions of 12 into 7 positive nonzero integers, then count the number of ways to arrange the calls under the given partition. For example, for the partition $(1, 1, 1, 1, 2, 3, 3)$, we have $\binom{12}{3}$ ways of choosing the first three calls, $\binom{12-3}{3}$ ways of choosing the second three calls, and $\binom{7}{2}$ ways of placing these subsets of calls on a day of the week. Then we have $\binom{9-3}{2}$ ways of choosing the next two calls, and $\binom{7-1}{1}$ ways of chosing the day of the week to place those calls on. Then the rest of the calls must be assigned, one each to the remaining days, so that is 4 options of call for the first day, 3 for the second and so on, giving 4! ways to assign the remaining calls. We use a similar logic and get the following counts for the 7 partitions:

$$\text{Partition } (1,1,1,1,1,1,6): \binom{12}{6}\binom{7}{1} \cdot 6! = 4,656,960$$

$$\text{Partition } (1,1,1,1,1,2,5): \binom{12}{5}\binom{7}{1} \cdot \binom{7}{2}\binom{6}{1} \cdot 5! = 83,825,280$$

$$\text{Partition } (1,1,1,1,1,3,4): \binom{12}{4}\binom{7}{1} \cdot \binom{8}{3}\binom{6}{1} \cdot 5! = 139,708,800$$

$$\text{Partition } (1,1,1,1,2,2,4): \binom{12}{4}\binom{7}{1} \cdot \binom{8}{2}\binom{6}{2}\binom{6}{2} \cdot 4! = 523,908,000$$

$$\text{Partition } (1,1,1,1,2,3,3): \binom{12}{3}\binom{9}{3}\binom{7}{2} \cdot \binom{6}{2}\binom{5}{1} \cdot 4! = 698,544,000$$

$$\text{Partition } (1,1,1,2,2,2,3): \binom{12}{3}\binom{7}{1} \cdot \binom{9}{2}\binom{7}{2}\binom{5}{2}\binom{6}{3} \cdot 3! = 1,397,088,000$$

$$\text{Partition } (1,1,2,2,2,2,2): \binom{12}{2}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{7}{5} \cdot 2! = 314,344,800$$

and the total of the above counts is 3,162,075,840, which multiplied by $\frac{1}{7^{12}}$ is approximately .2285 as desired.

1.21) We have a closet containing n pairs of shoes, i.e. $2n$ individual shoes in total. Suppose $2r$ shoes are chosen at random, with $2r<n$.

The total number of ways to choose $2r$ shoes from $2n$ is $\binom{2r}{2n}$.

To avoid having a matching pair, we must choose at most one shoe from each pair.

- First choose which $2r$ pairs will contribute one shoe each: $\binom{n}{2r}$.
- From each of the chosen pairs, pick exactly one shoe (left/right): 2^{2r}

Thus, the number of favorable outcomes is

$$\binom{n}{2r}2^{2r}$$

Therefore, the required prob. is

$$P=\frac{\binom{n}{2r}2^{2r}}{\binom{2n}{2r}}. \blacksquare$$

1.21) A closet contains n pairs of shoes. If $2r$ shoes are chosen at random ($2r<n$), what is the prob. that there will be no matching pair in the sample? $\left(A: \frac{\binom{n}{2r}2^{2r}}{\binom{2n}{2r}}\right)$

We want $P(\{\text{no matching pairs in sample}\})$

$=P(\{\text{all chosen shoes from different pairs}\})$

We are choosing $2r$ shoes from $2n$ shoes, i.e. $\binom{2n}{2r}$.

Now consider how many ways to choose $2r$ shoes s.t. no full pair is chosen, i.e.

1a) choose which pairs will contribute shoes, i.e. $\binom{n}{2r}$

1b) choose one shoe from each selected pair

$\forall$ of the $2r$ chosen pairs, there are 2 choices, i.e. 2^{2r}

Thus, $\boxed{P(\{\text{no matching pair}\})=\frac{\binom{n}{2r}2^{2r}}{\binom{2n}{2r}}}$