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Stats 510, Instructor: Long Nguyen

Homework 4

Nov 3, 2025, due by 11:59pm Nov 11, 2025

1. (Discrete distributions) Do problems 3.6, 3.13, 3.15.
2. (Continuous distributions) Do problems 3.20, 3.23, 3.25.
3. (Exponential families/ Inequalities) Do problems 3.28, 3.46.

1) Discrete Distributions

3.6 A large number of insects are expected to be attracted to a certain variety of rose plant. A commercial insecticide is advertised as being 99% effective. Suppose 2,000 insects infest a rose garden where the insecticide has been applied, and let X = number of surviving insects.

(a) What probability distribution might provide a reasonable model for this experiment?

(b) Write down, but do not evaluate, an expression for the probability that fewer than 100 insects survive, using the model in part (a).

(c) Evaluate an approximation to the probability in part (b).

a) According to the setup, X = # surviving insects

We can reasonably assume that $\underline{X \sim \text{Binomial}(n=2000, p=0.01)}$ since: $\begin{cases} \cdot \text{ Fixed number of trials } (n=2000) \\ \cdot \text{ Each trial has same prob. } p \\ \cdot \text{ Each trial independent} \\ \cdot \text{ Binary outcomes (survive/not)} \end{cases}$

where p: Prob(survival).

$\Rightarrow \underline{P(X=x|n,p) = \binom{n}{x} p^x (1-p)^{n-x}}$

b) $P(X<100) = \sum_{x=0}^{99} \binom{2000}{x} (0.01)^x (0.99)^{2000-x}$

c) Since n is large and p is small, a normal approximation to the binomial is reasonable:

Let $X \sim \text{Binomial}(n,p)$ s.t. $E(X)=np$, $V(X)=np(1-p)$.

If $\begin{cases} np \to \infty \\ np(1-p) \to \infty \end{cases}$ then $X \approx \text{Normal}(np, np(1-p))$

First compute mean and variance:

$\mu = np = 2000(0.01) = 20$

$\sigma^2 = np(1-p) = 2000(0.01)(0.99) = 19.8 \Rightarrow \sigma = \sqrt{19.8}$

Now, using the normal approximation: $X \sim \text{Bin}(2000, 0.01) \approx \text{Normal}(20, \sigma^2 = 19.8)$,

$P(X<100) \approx P(Y<100)$, $Y \sim \text{Normal}(20, 19.8)$

$\approx P(Y<99.5)$, continuity correction

$= P\left(Z < \frac{99.5-20}{\sqrt{19.8}}\right) \approx 1$, i.e. P(fewer than 100 insects survive) is almost certain.

Alternatively, can approximate Binomial(2000, 0.01) by Poisson(λ),

letting $np \to \lambda$, i.e. $2000(0.01) = 20 \to \lambda$ as $n \to \infty$

(where $n=200$ is reasonably large).

Thus, $P(X<100) \approx P(\text{Poisson}(20) < 100)$

$= 1 - P(\text{Poisson}(20) \geq 100)$

$\approx 1 - 0$

≈ 1 (same approximating result as when using Normal approx.)

3.13 A *truncated* discrete distribution is one in which a particular class cannot be observed and is eliminated from the sample space. In particular, if X has range $0, 1, 2, \ldots$ and the 0 class cannot be observed (as is usually the case), the 0-*truncated* random variable X_T has pmf

$$P(X_T = x) = \frac{P(X=x)}{P(X>0)}, \quad x = 1, 2, \ldots.$$

Find the pmf, mean, and variance of the 0-truncated random variable starting from

(a) $X \sim \text{Poisson}(\lambda)$.

(b) $X \sim$ negative binomial(r, p), as in (3.2.10).

For any X with support $0, 1, \ldots$, we have the mean and variance of the 0-truncated X_T:

$$E(X^T) = \sum_{x=1}^{\infty} x P(X_T = x) = \sum_{x=1}^{\infty} x \frac{P(X=x)}{P(X>0)}$$

$$= \frac{1}{P(X>0)} \sum_{x=1}^{\infty} x P(X=x)$$

$$= \frac{1}{P(X>0)} \sum_{x=0}^{\infty} x P(X=x) = \frac{E(X)}{P(X>0)}$$

Similarly, $E(X_T^2) = \frac{E(X^2)}{P(X>0)}$.

Thus, $V(X_T) = E(X_T^2) - E^2(X_T) = \frac{E(X^2)}{P(X>0)} - \left(\frac{E(X)}{P(X>0)}\right)^2$

a. For Poisson(λ), $P(X>0) = 1 - P(X=0) = 1 - \frac{e^{-\lambda}\lambda^0}{0!} = 1 - e^{-\lambda}$.

Therefore, $P(X_T = x) = \frac{e^{-\lambda}\lambda^x}{x!(1-e^{-\lambda})}$, $x = 1, 2, \ldots$

Thus, $E(X_T) = \lambda/(1-e^{-\lambda})$ and

$V(X_T) = \frac{\lambda^2+\lambda}{1-e^{-\lambda}} - \left(\frac{\lambda}{1-e^{-\lambda}}\right)^2$

b. For Negative Binomial(r,p), $P(X>0) = 1 - P(X=0)$

$= 1 - \binom{r-1}{0} p^r (1-p)^0 = 1 - p^r$.

Then,

$P(X_T = x) = \frac{\binom{r+x-1}{x} p^r (1-p)^x}{1-p^r}$, $x = 1, 2, \ldots$

$E(X_T) = \frac{r(1-p)}{p(1-p^r)}$

$V(X_T) = \frac{r(1-p) + r^2(1-p)^2}{p^2(1-p^r)} - \left[\frac{r(1-p)}{p(1-p^r)}\right]^2$

3.15 In Section 3.2 it was claimed that the Poisson(λ) distribution is the limit of the negative binomial(r,p) distribution as $r \to \infty$, $p \to 1$, and $r(1-p) \to \lambda$. Show that under these conditions the mgf of the negative binomial converges to that of the Poisson.

The mgf for the negative binomial is $M(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r$. Let $\begin{cases} q = 1-p \Rightarrow p = 1-q \\ rq \to \lambda \text{ as } r \to \infty, q \to 0 \end{cases}$

where $\frac{1-q}{1-qe^t} = \left[1 + \frac{(1-q)-(1-qe^t)}{1-qe^t}\right] = \left[1 + \frac{q(e^t-1)}{1-qe^t}\right] = \left[1 + \frac{1}{r}\left(\frac{r(1-p)(e^t-1)}{1-(1-p)e^t}\right)\right]$ where $\frac{r(1-p)(e^t-1)}{1-(1-p)e^t} \longrightarrow \frac{\lambda(e^t-1)}{1} = \lambda(e^t-1)$ as $r \to \infty$, $p \to 1$ and $r(p-1) \to \lambda$

and $\underline{\lim_{r \to \infty} \left[1 + \frac{1}{r}\left(\frac{r(1-p)(e^t-1)}{1-(1-p)e^t}\right)\right]^r = \exp(\lambda(e^t-1))\text{, i.e. the mgf of Poisson}(\lambda).}$

Recall: $\lim_{r \to \infty} \left(1 + \frac{a_r}{r}\right)^r = e^{\lim a_r}$ if $a_r \to a$

2) continuous Distributions

3.20 Let the random variable X have the pdf

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}, \quad 0 < x < \infty.$$

(a) Find the mean and variance of X. (This distribution is sometimes called a *folded normal.*)
(b) If X has the folded normal distribution, find the transformation $g(X) = Y$ and values of α and β so that $Y \sim \text{gamma}(\alpha, \beta)$.

a) $E(X) = \int_0^\infty x\, f(x)\, dx = \int_0^\infty x \cdot \frac{2}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \frac{2}{\sqrt{2\pi}} \int_0^\infty x e^{-x^2/2}\, dx$ via substitution:

let $u = \frac{x^2}{2}$ $du = x\,dx$

$x = 0 : u = 0$
$x = \infty : u = \infty$

$$= \frac{2}{\sqrt{2\pi}} \left[\int_0^\infty e^{-u}\, du = -e^{-u} \Big|_0^\infty = 1 \right]$$

$$\Rightarrow \underline{E(X) = \frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}}$$

Similarly, $E(X^2) = \int_0^\infty x^2 f(x)\, dx = \frac{2}{\sqrt{2\pi}} \underbrace{\int_0^\infty x^2 e^{-x^2/2}\, dx}_{\sqrt{\pi/2}} = \frac{2}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} = 1$

Thus, $\underline{V(X) = 1 - \frac{2}{\pi}}$

b) Suppose $Z \sim N(0,1)$ and let

$X = |Z| \Rightarrow X \sim$ Folded Normal $(0,1)$

$\Rightarrow \underline{Y = X^2 = |Z|^2 = Z^2 \sim \chi^2_1 = \text{Gamma}(\alpha = \tfrac{1}{2}, \beta = 2).}$

3.23 The *Pareto distribution*, with parameters α and β, has pdf

$$f(x) = \frac{\beta\alpha^\beta}{x^{\beta+1}}, \quad \alpha < x < \infty, \quad \alpha > 0, \quad \beta > 0.$$

(a) Verify that $f(x)$ is a pdf.
(b) Derive the mean and variance of this distribution.
(c) Prove that the variance does not exist if $\beta \leq 2$.

a. $f(x) \geq 0 \ \forall x \in \mathcal{X}$, i.e. nonnegativity holds.

$\int_\alpha^\infty x^{-\beta-1} dx = \frac{-1}{\beta} x^{-\beta} \Big|_\alpha^\infty = \frac{1}{\beta\alpha^\beta}$ and thus $f(x)$ integrates to 1.

b. $E(X^n) = \int_\alpha^\infty x^n f(x)\, dx = \int_\alpha^\infty x^n \cdot \frac{\beta\alpha^\beta}{x^{\beta+1}}\, dx = \beta\alpha^\beta \int_\alpha^\infty x^{n-\beta-1}\, dx$,

evaluating $\int_\alpha^\infty x^k dx$ where $k = n - \beta - 1$ $\begin{cases} \text{integral converges if } k \leq -1 \\ \text{integral diverges if } k \geq -1 \end{cases}$

since $\int_\alpha^\infty x^k dx = \frac{x^{k+1}}{k+1} \Big|_\alpha^\infty$; to converge as $x \to \infty$, we need $x^{k+1} \to 0$,
i.e. $k+1 < \infty \Rightarrow k < -1 \Rightarrow n - \beta - 1 < -1 \Rightarrow n < \beta$

$$= \beta\alpha^\beta \left[\frac{x^{n-\beta}}{n-\beta} \Big|_{x=\alpha}^{x=\infty} \right]$$

where $n < \beta \Rightarrow$ convergent case

$$= \beta\alpha^\beta \left(0 - \frac{\alpha^{n-\beta}}{n-\beta} \right)$$

$$= \beta\alpha^\beta \cdot \frac{\alpha^{n-\beta}}{\beta - n} = \frac{\beta\alpha^n}{\beta - n}$$

Therefore,

$\underline{E(X) = \frac{\beta\alpha}{\beta - 1}}$, exists if $\beta > 1$

$E(X^2) = \frac{\beta\alpha^2}{\beta - 2}$, exists if $\beta > 2$

$\Rightarrow \underline{V(X) = E(X^2) - E^2(X) = \frac{\beta\alpha^2}{\beta - 2} - \left(\frac{\beta\alpha}{\beta - 1} \right)^2}$; c. If $\beta \leq 2$, the integral of the second moment is infinite.
variance expression is only finite when both moments exist, i.e $\beta > 2$.

3.25 Suppose the random variable T is the length of life of an object (possibly the lifetime of an electrical component or of a subject given a particular treatment). The *hazard function* $h_T(t)$ associated with the random variable T is defined by

$$h_T(t) = \lim_{\delta \to 0} \frac{P(t \leq T < t + \delta | T \geq t)}{\delta}.$$

Thus, we can interpret $h_T(t)$ as the rate of change of the probability that the object survives a little past time t, given that the object survives to time t. Show that if T is a continuous random variable, then

$$h_T(t) = \frac{f_T(t)}{1 - F_T(t)} = -\frac{d}{dt} \log\left(1 - F_T(t)\right).$$

If T is continuous, then

$$P(t \leq T \leq t+\delta | T \geq t) = \frac{P(t \leq T \leq t+\delta, t \leq T)}{P(t \leq T)}$$

$$= \frac{P(t \leq T \leq t+\delta)}{P(t \leq T)}$$

$$= \frac{F_T(t+\delta) - F_T(t)}{1 - F_T(t)}$$

Therefore, from the definition of the derivative,

$$h_T(t) = \lim_{\delta \to 0} \frac{F_T(t+\delta) - F_T(t)}{\delta} = \frac{F_T'(t)}{1 - F_T(t)} = \frac{f_T(t)}{1 - F_T(t)}$$

Also, $\frac{-d}{dt}[\log(1 - F_T(t))] = \frac{-1}{1 - F_T(t)}(-f_T(t)) = h_T(t)$.

3.28 Show that each of the following families is an exponential family.

(a) normal family with either parameter μ or σ known

(b) gamma family with either parameter α or β known or both unknown

(c) beta family with either parameter α or β known or both unknown

(d) Poisson family

(e) negative binomial family with r known, $0 < p < 1$

a) (i) μ known: $f(x|\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right)$,

$h(x)=1$, $c(\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} I_{(0,\infty)}(\sigma^2)$, $w_1(\sigma^2) = \frac{-1}{2\sigma^2}$, $t_1(x) = (x-\mu)^2$

(ii) σ^2 known: $f(x|\mu) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(\frac{-x^2}{2\sigma^2}\right)\exp\left(\frac{-\mu^2}{2\sigma^2}\right)\exp\left(\mu\frac{x}{\sigma^2}\right)$

$h(x) = \exp\left(\frac{-x^2}{2\sigma^2}\right)$, $c(\mu) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(\frac{-\mu^2}{2\sigma^2}\right)$, $w_1(\mu)=\mu$, $t_1(x) = \frac{x}{\sigma^2}$

b) (i) α known: $f(x|\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}$

$h(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}$, $x>0$ $c(\beta) = 1/\beta^\alpha$ $w_1(\beta) = \frac{1}{\beta}$, $t_1(x) = -x$

(ii) β known: $f(x|\alpha) = e^{-x/\beta}\frac{1}{\Gamma(\alpha)\beta^\alpha}\exp((\alpha-1)\log x)$

$h(x) = e^{-x/\beta}$, $x>0$ $c(\alpha) = \frac{1}{\Gamma(\alpha)\beta^\alpha}$, $w_1(\alpha)=\alpha-1$, $t_1(x) = \log x$

(iii) α, β unknown: $f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\exp\left((\alpha-1)\log x - \frac{x}{\beta}\right)$

$h(x) = I_{\{x>0\}}(x)$ $c(\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}$, $w_1(\alpha) = \alpha-1$ $t_1(x) = \log x$

$w_2(\beta) = -1/\beta$ $t_2(x) = x$

c) (i) α known: $f(x|\beta) = \frac{1}{B(\alpha,\beta)}x^{\alpha-1}\exp((\beta-1)\log(1-x))$

$h(x) = x^{\alpha-1}I_{[0,1]}(x)$, $c(\beta) = \frac{1}{B(\alpha,\beta)}$, $w_1(\beta) = \beta-1$ $t_1(x) = \log(1-x)$

(ii) β known: $f(x|\alpha) = \frac{1}{B(\alpha,\beta)}(1-x)^{\beta-1}\exp((\alpha-1)\log x)$

$h(x) = (1-x)^{\beta-1}I_{[0,1]}(x)$ $c(\alpha) = \frac{1}{B(\alpha,\beta)}$ $w_1(\alpha) = \alpha-1$ $t_1(x) = \log x$

(iii) α, β unknown: $f(x|\alpha,\beta) = \frac{1}{B(\alpha,\beta)}\exp((\alpha-1)\log x + (\beta-1)\log(1-x))$

$h(x) = I_{[0,1]}(x)$, $c(\alpha,\beta) = \frac{1}{B(\alpha,\beta)}$, $w_1(\alpha) = \alpha-1$ $t_1(x) = \log x$

$w_2(\beta) = \beta-1$ $t_2(x) = \log(1-x)$

d) $f(x|\theta) = \frac{e^{-\theta}}{x!}\cdot\theta^x = \frac{e^{-\theta}}{x!}\cdot\exp(x\log\theta)$

$h(x) = \frac{1}{x!}I_{\{0,1,2,\dots\}}(x)$ $c(\theta) = e^{-\theta}$, $w_1(\theta) = \log\theta$, $t_1(x) = x$

e) r known: $f(x|p) = \binom{x-1}{r-1}p^r(1-p)^{r-x}$, $0<p<1$

$= \binom{x-1}{r-1}\left(\frac{p}{1-p}\right)^r\exp(\log(1-p)+x)$

$h(x) = \binom{x-1}{r-1}I_{\{r,r+1,\dots\}}$, $c(p) = \left(\frac{p}{1-p}\right)^r$, $w_1(p) = \log(1-p)$, $t_1(x) = x$

3.46 Calculate $P(|X - \mu_X| \geq k\sigma_X)$ for $X \sim \text{uniform}(0,1)$ and $X \sim \text{exponential}(\lambda)$, and compare your answers to the bound from Chebychev's Inequality.

For $X \sim$ uniform$(0,1)$, $\mu = 1/2$ and $\sigma^2 = 1/12$; thus,

$$P(|X-\mu| > k\sigma) = P\left(X \le \tfrac{1}{2} - k\sigma\right) + P\left(X > \tfrac{1}{2} + k\sigma\right)$$
$$= \left(\tfrac{1}{2} - \tfrac{k}{\sqrt{12}}\right) + \left(1 - \tfrac{1}{2} - \tfrac{k}{\sqrt{12}}\right) = 1 - \tfrac{2k}{\sqrt{12}}$$

where if $\frac{1}{2} - k\sigma < 0$ or $\frac{1}{2} + k\sigma > 1$,
then part or all of the prob. mass lies outside $(0,1)$

Hence, when $k \geq \sqrt{3}$ $\begin{cases} \frac{1}{2} - k\sigma \le 0 \\ \frac{1}{2} + k\sigma \ge 1 \end{cases}$ and the entire prob. region lies w/in $(0,1)$.

$$= \begin{cases} 1 - \frac{2k}{\sqrt{12}}, & k < \sqrt{3} \\ 0, & k > \sqrt{3} \end{cases}$$

Similarly, for $X \sim$ exponential(λ), $\mu = \lambda$, $\sigma^2 = \lambda^2$; then,

$$P(|X-\lambda| \geq k\lambda) = P(X \le \lambda - k\lambda) + P(X \geq \lambda + k\lambda)$$
$$= \underbrace{P(X \le \lambda(1-k))}_{\text{left-tail (1)}} + \underbrace{P(X \geq \lambda(1+k))}_{\text{right-tail (2)}}$$

(1): = Exp. Dist. has support for $x \geq 0$, so

if $k>1$, then $\lambda(1-k) < 0$, outside of the support $\Rightarrow P(X \le \lambda(1-k)) = 0$

$k \le 1$, then $\lambda(1-k) \geq 0$, and

$F(x) = 1 - e^{-x/\lambda} \Rightarrow P(X \le \lambda(1-x)) = 1 - e^{-(1-k)}$

(2): = the right tail is always valid:

$P(X \geq \lambda(1+k)) = 1 - (1 - e^{-(1+k)}) = e^{-(1+k)}$

$$\Rightarrow P(|X-\mu| > k\sigma) = \begin{cases} 1 + e^{-(k+1)} - e^{-k-1}, & k \le 1 \\ e^{-(k+1)}, & k \geq 1 \end{cases}$$

Chebychev's inequality gives the bound: $P(|X-\mu| > k\sigma) \le 1/k^2$; it can be shown that:

Comparison of probabilities

k	u(0, 1) exact	exp(λ) exact	Chebychev
.1	.942	.926	100
.5	.711	.617	4
1	.423	.135	1
1.5	.134	.0821	.44
$\sqrt{3}$	0	0.0651	.33
2	0	0.0498	.25
4	0	0.00674	.0625
10	0	0.0000167	.01

we see that Chebychev's inequality is quite conservative.