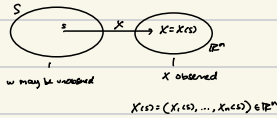


Week 14

4.6: Multivariate Distributions

DEF: A random vector defined $X = (X_1, \dots, X_n)$ is a function that maps elements of a sample space S into \mathbb{R}^n



Now, we may specify probabilities, for $A \in \mathbb{R}^n$:

$$P(X \in A) = P(\{s \in S : X(s) \in A\})$$

If X takes countably many values, then we say X is a discrete RV, which is associated w/ a

joint probability mass function (pmf): $f_X(x) = P(X_1 = x_1, \dots, X_n = x_n)$

Hence for $A \in \mathbb{R}^n$, $P(X \in A) = \sum_{x \in A} P(X=x) = \sum_{x \in A} f_X(x)$

Similarly, pdf: $f_X(x) = f_X(x_1, \dots, x_n)$

where $P(X = (X_1, \dots, X_n) \in A) = \int_A f_X(x) dx = \int_{x_1=-\infty}^{\infty} \dots \int_{x_n=-\infty}^{\infty} f_X(x_1, \dots, x_n) dx_1 \dots dx_n$

expectation: $Eg(X) = \begin{cases} \sum_{x \in \mathbb{R}^n} g(x) f_X(x) & X \text{ discrete} \\ \int_{\mathbb{R}^n} g(x) f_X(x) dx & X \text{ continuous} \end{cases}$

marginal dist: $f_{X_1, \dots, X_k}(x_1, \dots, x_k) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f_{X_1, \dots, X_n}(x_1, \dots, x_n) dx_{k+1} \dots dx_n$

eg: $(X_1, \dots, X_n) \sim f_X(x_1, \dots, x_n)$

$f_{X_1, X_2}(x_1, x_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X_1, X_2, X_3}(x_1, x_2, x_3) dx_3 dx_4$

conditional dist: $f_{X_1, \dots, X_n}(x_1, \dots, x_n | X_{k+1}, \dots, X_n = (x_{k+1}, \dots, x_n)) = \frac{f_{X_1, \dots, X_n}(x_1, \dots, x_n, x_{k+1}, \dots, x_n)}{f_{X_{k+1}, \dots, X_n}(x_{k+1}, \dots, x_n)}$

REMARKS: admits factorization: $f_{X_1, \dots, X_n}(x_1, \dots, x_n) = f_{X_1, \dots, X_k}(x_1, \dots, x_k) f_{X_{k+1}, \dots, X_n}(x_{k+1}, \dots, x_n)$

chain formula: $f_{X_1, \dots, X_n}(x_1, \dots, x_n) = f_{X_1}(x_1) f_{X_2|X_1}(x_2|x_1) f_{X_3|X_1, X_2}(x_3|x_1, x_2) \dots f_{X_n|X_1, \dots, X_{n-1}}(x_n|x_1, \dots, x_{n-1})$

eg: $f_{X_1, X_2, X_3}(x_1, x_2, x_3) = f_{X_1}(x_1) f_{X_2|X_1}(x_2|x_1) f_{X_3|X_1, X_2}(x_3|x_1, x_2)$

PROP. SUPPORT: $\begin{cases} X_1, Y \text{ discrete} \\ Z, T \text{ continuous} \end{cases}$ and let $f_{X_1, X_2, X_3}(x_1, x_2, x_3) = \underbrace{f_{X_1}(x_1)}_{\text{pmf}} \underbrace{f_{X_2|X_1}(x_2|x_1)}_{\text{conditional pdf}} = \underbrace{f_{X_1}(x_1)}_{\text{pdf}} \underbrace{f_{X_2|X_1}(x_2|x_1)}_{\text{conditional pdf}}$

Then, $\forall A, B, Z \in \mathbb{R}$,

$$P(X \in A, Y \in B, Z \in C, T \in D) = \sum_{x \in A} \sum_{y \in B} \int_D \int_C f_{X_1, X_2, X_3}(x_1, x_2, x_3) dx_3 dx_4$$

EX: multinomial dist: $\begin{cases} n \text{ independent trials} \\ n \text{ possible outcomes w/} \\ \text{"cell probs"} (p_1, \dots, p_n), \sum_{i=1}^n p_i = 1 \end{cases}$

(let X_1, \dots, X_n : counts of each of n outcomes
 $X_1 + \dots + X_n = n$)

Now, joint pmf for $x_1, \dots, x_n \in \mathbb{N}^n$, $\sum_{i=1}^n x_i = m$

$$f(x_1, \dots, x_n) = \binom{m}{x_1, \dots, x_n} p_1^{x_1} \dots p_n^{x_n} \\ = \frac{m!}{x_1! \dots x_n!} p_1^{x_1} \dots p_n^{x_n} \quad \text{and 0 otherwise}$$

Remarks: By multinomial formula

$$\sum_{\substack{x_1, \dots, x_n \in \mathbb{N}^n \\ x_1 + \dots + x_n = m}} \binom{m}{x_1, \dots, x_n} p_1^{x_1} \dots p_n^{x_n} = (p_1 + \dots + p_n)^m = 1$$

Marginal distributions:

$$f_{X_n}(x_n) = \sum_{x_1 + \dots + x_{n-1} = m - x_n} \frac{m!}{x_1! \dots x_{n-1}! x_n!} p_1^{x_1} \dots p_{n-1}^{x_{n-1}} p_n^{x_n} \\ = p_n^{x_n} \frac{m!}{x_n! (m-x_n)!} \sum_{x_1 + \dots + x_{n-1} = m - x_n} \frac{(m-x_n)!}{x_1! \dots x_{n-1}!} p_1^{x_1} \dots p_{n-1}^{x_{n-1}} \\ = \frac{m!}{x_n! (m-x_n)!} p_n^{x_n} (p_1 + \dots + p_{n-1})^{m-x_n} \\ = \frac{m!}{x_n! (m-x_n)!} p_n^{x_n} (1-p_n)^{m-x_n} \\ \Rightarrow X_n \sim \text{Binomial}(m, p_n).$$

Conditional distributions

$$f_{X_1, \dots, X_{n-1}}(x_1, \dots, x_{n-1} | x_n = 1) = \frac{f_{X_1, \dots, X_n}(x_1, \dots, x_{n-1}, x_n)}{f_{X_n}(x_n)} \\ = \frac{\frac{m!}{x_1! \dots x_{n-1}! x_n!} p_1^{x_1} \dots p_{n-1}^{x_{n-1}} p_n^{x_n}}{\frac{m!}{x_n! (m-x_n)!} p_n^{x_n} (1-p_n)^{m-x_n}} \\ = \frac{(m-x_n)!}{x_1! \dots x_{n-1}!} \left(\frac{p_1}{1-p_n} \right)^{x_1} \dots \left(\frac{p_{n-1}}{1-p_n} \right)^{x_{n-1}}$$

Hence $X_1, \dots, X_{n-1} | X_n = x_n \sim \text{Multinomial}(m-x_n, \text{vec prob: } (\frac{p_1}{1-p_n}, \dots, \frac{p_{n-1}}{1-p_n}))$

Notice: "invariance" property by marginalization and conditioning;

holds for many dist's in the exponential family;

holds for multivariate normal (continuous)

Recall (independence): X and Y independent if $\forall x, y \in \mathbb{R}$, $f_{XY}(x, y) = f_X(x) f_Y(y)$

$$\text{Def (Mutual Independence)} := \boxed{f(x_1, \dots, x_n) = \prod_{i=1}^n f_{X_i}(x_i)}$$

Many properties established for bivariate random vector can be naturally extended

$$\text{Thm: } \boxed{E[g_1(X_1) \dots g_n(X_n)] = E[g_1(X_1)] \dots E[g_n(X_n)]}$$

$$\text{Thm: If } X_1, \dots, X_n \text{ mutually independent RV's, let } \mathcal{F} = \mathcal{F}_1, \dots, \mathcal{F}_n \\ \text{then } \boxed{M_{\mathcal{F}}(t) = M_{\mathcal{F}_1}(t_1) \dots M_{\mathcal{F}_n}(t_n)}$$

conclusion: in particular, if $X_1, \dots, X_n \sim \text{i.i.d.}$, then $M_{\mathcal{F}}(t) = (M_{\mathcal{F}_1}(t_1))^n$

Properties

$$1) \text{ If } X_i \sim \text{Gamma}(k_i, \beta), X_i \text{ independent.} \\ \text{then } X_1, \dots, X_n \sim \text{Gamma}(k_1 + \dots + k_n, \beta)$$

$$2) \text{ If } X_i \stackrel{\text{ind}}{\sim} N(\mu_i, \sigma_i^2), \text{ let } a_i, b_i \in \mathbb{R}. \\ \text{then } Z := \sum_{i=1}^n (a_i X_i + b_i) \sim N\left(\sum_{i=1}^n (a_i \mu_i + b_i), \sum_{i=1}^n a_i^2 \sigma_i^2\right) \\ = a^T X + b^T \mathbf{1}$$

Lemma: Suppose $(X_1, \dots, X_n) \sim f_{X_1, \dots, X_n}$.

X_1, \dots, X_n are mutually independent r.v.'s.

$$\boxed{f_{X_1, \dots, X_n}(x_1, \dots, x_n) = g_1(x_1) \dots g_n(x_n)}$$

for some functions g_1, \dots, g_n

Thm: If X_1, \dots, X_n are mutually independent,

(let $g(x_i)$ be function of $x_i, i=1, \dots, n$)

then $g_1(X_1), \dots, g_n(X_n)$ are also mutually independent