1.4 Random variables (cont)

Def. The probability mass function (pmf) of a discrete Random variable X is

$$f_X(x) = P_X(X=x) \quad \forall x.$$

Example (last page)

$X \sim$ Geometric (p) , $0<p<1$

has

$$f_X(x) := P(X=x) := \begin{cases} (1-p)^{x-1} p & , x=1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

However, this concept is NOT useful for Continuous Random variables.

Indeed, if X is continuous R.V. then $F_X(x)$ is a continuous function of x.

Then

$$
\begin{aligned}
P(X=x) &= \lim_{\varepsilon \downarrow 0} P\big(X \in (x-\varepsilon, x+\varepsilon]\big) \\
&= \lim_{\varepsilon \downarrow 0} P(X \le x+\varepsilon) - P(X \le x-\varepsilon) \\
&= \lim_{\varepsilon \downarrow 0} F_X(x+\varepsilon) - F_X(x-\varepsilon) \\
&= F_X(x) - F_X(x) \\
&= 0 .
\end{aligned}
$$

More useful is the c.d.f F_X and, for continuous Random variables, the notion of p.d.f.

Def. the probability density function (pdf), namely, $f_X(x)$, of a continuous Random variable X is a function that satisfies

$$\boxed{F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt .}$$

for (almost) all x.

Remarks

(absolutely continuous)

① if F_X is differentiable, then the pdf always exist,

(almost always)

$$f_X(x) = \frac{d}{dx} F_X(x).$$

② F_X, or f_X contain all information there is about the distribution of Random variable X.

We say

$$X \sim F_X \quad \text{or} \quad X \sim f_X \quad (\text{Equivalently})$$

③ $\boxed{\text{Theorem}}$ A function $f_X(x)$ is a pdf (or pmf) of a Random variable X

if and only if

(i) $f_X(x) \geq 0 \quad \forall x$

(ii)

$$\sum_x f_X(x) = 1 \quad (\text{for pmf})$$

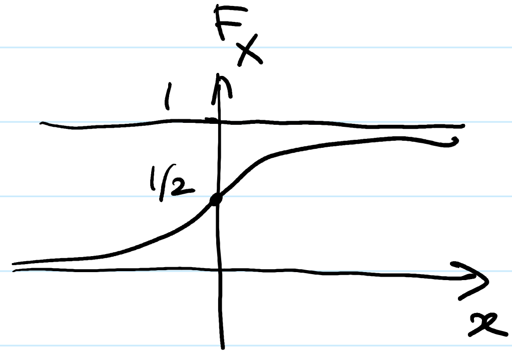
or

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \quad (\text{for pdf})$$

Example

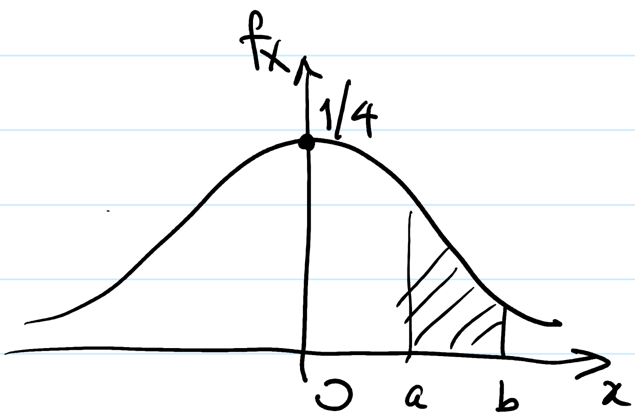
Recall the logistic cdf:

$$F_X(x) = \frac{1}{1+e^{-x}}$$



which gives

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{e^{-x}}{(1+e^{-x})^2}$$



Under which

$$P(X \in (a,b)) = \int_a^b \frac{e^{-x}}{(1+e^{-x})^2}\, dx.$$