1.1 Sets and probabilities (cont)

Def. Given a sample space S,
a sigma algebra $\mathcal{B}$ associated with S,

a probability function (aka distribution / measure)
is a function P on $\mathcal{B}$ that satisfies:

1. $P(A) \geq 0 \quad \forall A \in \mathcal{B}$
2. $P(S) = 1$.
3. if $A_1, A_2, \dots \in \mathcal{B}$ are pairwise disjoint
   then
   $$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Remark

- A and B are disjoint iff $A \cap B = \phi$.
- The three properties a.k.a. Kolmogorov's Axioms of Prob.
- Property 3 is known as axiom of Countable additivity.

EXAMPLES

- Tossing a fair coin: $S = \{H, T\}$

$$P(\{H\}) := P(\{T\}) := 1/2$$

- To allow possibly unfair coin: $\begin{cases} P(\{H\}) = q \\ P(\{T\}) = 1-q \end{cases}$

for some $q \in [0,1]$

- Tossing 2 coins in a row

$$S = \{HH, HT, TH, TT\}$$

Take

$P(HH) = q_1$, $P(HT) = q_2$, $P(TH) = q_3$, $P(TT) = q_4$

where $q_1 + q_2 + q_3 + q_4 = 1$

Probabilities on Countable Set are easy to define

$\boxed{\text{Thm}}$ Let $S = \{s_1, \dots, s_n\}$

$\mathcal{B}$ any sigma-algebra of subsets of S.

Let $p_1, \dots, p_n \in [0,1]$ s.t. $\sum_{i=1}^{n} p_i = 1$.

Define, for any $A \in \mathcal{B}$

$$\boxed{P(A) := \sum_{i:\, s_i \in A} p_i}$$

Then, P is a valid probability function on $\mathcal{B}$.

$\boxed{\text{Thm}}$ Let $S = \{s_1, s_2, \dots\}$ countably infinite

$\mathcal{B}$ any sigma-algebra of subsets of S.

Let $p_1, p_2, \dots \in [0,1]$ s.t. $\sum_{i=1}^{\infty} p_i = 1$.

Define, for any $A \in \mathcal{B}$

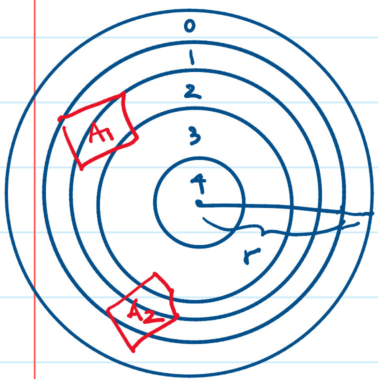
$$\boxed{P(A) := \sum_{i:\, s_i \in A} p_i}$$

Then, P is a valid probability function on $\mathcal{B}$

Proof. verify easily that P satisfies all axioms of prob.

ASSIGNING PROBABILITIES TO UNCOUNTABLE SET IS NON-TRIVIAL

EXAMPLE : A DART THROWING EXPERIMENT



ensuring $P(A_1 \cup A_2) = P(A_1) + P(A_2)$?

- $S = \{$ dart positions on the circle board of Radius $r\}$

S is uncountable
valid P must satisfy countable additivity, among other axioms

- Perhaps we can attempt to define an "uniform" probability distribution.

Consider a simpler sample space

$$S = \{\text{scoring positions } 0, 1, 2, 3, 4\}$$

we may define

$$P(\{i\}) := \frac{\text{area on Ring } i}{\text{area of dart board}}$$

$$= \frac{\pi\left(1-\frac{i}{5}\right)^2 r^2 - \pi\left(1-\frac{i+1}{5}\right)^2 r^2}{\pi r^2}$$

$$= \left(1-\frac{i}{5}\right)^2 - \left(1-\frac{i+1}{5}\right)^2.$$

GENERAL APPROACH

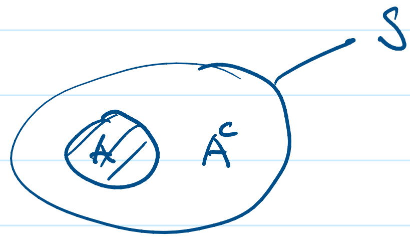
- Placing probabilities on (countably) many "simple" subsets
- using axioms of Prob. AOP to derive probabilities of all Remaining subsets of the sigma algebra $\mathcal{B}$.

See TOOLS

Tools A Prob function P on $\mathcal{B}$ must satisfy the following facts:

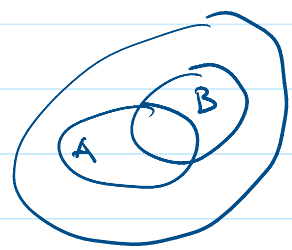
Thm

$$\boxed{\begin{array}{l} \forall A \in \mathcal{B} \\ 0 \le P(A) \le 1 \\ P(A^c) = 1 - P(A) \end{array}}$$



Thm

$$\boxed{\begin{array}{ll} & \forall A, B \in \mathcal{B} \\ 1. & P(B \cap A^c) = P(B) - P(A \cap B) \\ 2. & P(A \cup B) = P(A) + P(B) - P(A \cap B) \\ 3. & \text{if } A \subset B \text{ then } P(A) \le P(B) \end{array}}$$



Proof. By exploiting the three A.O.P.

Cor. Bonferoni's inequality

$$\boxed{P(A \cap B) \ge P(A) + P(B) - 1}$$

Remark. Role of intersections

MORE POWERFUL TOOL : COUNTABLE $\rightarrow$ UNCOUNTABLE

i.e., using probabilities on a countable collection of subsets to determine prob. on an uncountable collection

THM

1. $P(A) = \sum_{i=1} P(A \cap C_i)$ for any partition $(C_1, C_2, \ldots)$ of S

2. $P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i) \quad \forall A_1, A_2, \ldots \in \mathcal{B}.$

Remark

1. A partition $(C_1, C_2, \ldots)$ of S means $C_1, C_2, \ldots$ are pairwise disjoint, and $\bigcup_{i=1}^{\infty} C_i = S$.

2. Examples : Assigning probabilities to the Borel sigma-algebra of subsets in $\mathbb{R}$?