

### 3.1 Discrete distributions

A Random variable  $X$  is discrete if the range of  $X$  is countable.

**Uniform**

$$X \sim \text{Uniform}(1, N) \quad \text{if} \\ P(X=n) = \frac{1}{N} \quad n=1, \dots, N$$

We write :  $P(X=n | N) = \frac{1}{N}$

$\uparrow$   
parameter

Easy to verify that

$$E X = \frac{1}{N} (1 + \dots + N) = \frac{1}{N} \frac{N(N+1)}{2} = \frac{N+1}{2}$$

$$E X^2 = \frac{1}{N} (1^2 + \dots + N^2) = \frac{1}{N} \frac{N(N+1)(2N+1)}{6} = \frac{(N+1)(2N+1)}{6}$$

$$\text{var } X = E X^2 - (E X)^2 = \frac{(N+1)(2N+1)}{6} - \frac{(N+1)^2}{4} \\ = \frac{(N+1)(N-1)}{12}$$

## Hypergeometric distributions

Experiment:  $N$  balls :  $M$  red,  $N-M$  green

Pick  $K$  balls uniformly at Random  
(without replacement)

Let  $X$  = number of reds,

$$P(X = x \mid N, M, K) = \frac{\binom{M}{x} \binom{N-M}{K-x}}{\binom{N}{K}}, x=0, \dots, K$$

Notes:

• The counting argument implies  $\sum_{x=0}^K \binom{M}{x} \binom{N-M}{K-x} = \binom{N}{K}$

$$\bullet E[X] = \sum_{x=0}^K x \underbrace{\binom{M}{x} \binom{N-M}{K-x}}_{\sim} \frac{1}{\binom{N}{K}}$$

$$= \sum_{x=1}^K M \binom{M-1}{x-1} \binom{N-M}{K-x} \left( \frac{N}{K} \binom{N-1}{K-1} \right)^{-1}$$

$$= \frac{KM}{N} \sum_{x=1}^K \binom{M-1}{x-1} \binom{N-M}{K-x} / \binom{N-1}{K-1}$$

$$\stackrel{y=x-1}{=} \frac{KM}{N} \sum_{y=0}^{K-1} \binom{M-1}{y} \binom{(N-1)-(M-1)}{K-1-y} / \binom{N-1}{K-1}$$

$$= KM \sum_{y=0}^{K-1} P(Y=y \mid N-1, M-1, K-1)$$

$$= \frac{KM}{N} \sum_{y=0}^{k-1} P(Y=y \mid N-1, M-1, k-1) \\ = KM/N.$$

$$\cdot \text{Var } X = \frac{KM}{N} \frac{(N-M)(N-K)}{N(N-1)} .$$

Example .  $N = 25$  items

$M = \#$  defective items

We sample  $K$  items from the lot , say  $k=10$  ,  
and find that none is defective .  
What can we say about  $M$  ?

From above  $X \mid N, M, K \sim \text{Hypergeometric}(N, M, K)$

So

$$P(X=0 \mid N=25, M=6, K=10) = \frac{\binom{M}{0} \binom{N-M}{K-0}}{\binom{N}{K}} = \frac{\binom{6}{0} \binom{19}{10}}{\binom{25}{10}} = 0.028$$

$$\Rightarrow P(X=0 \mid N=25, K=10, M \geq 6) \leq 0.028 .$$

"if  $M \geq 6$  then the observed event " $X=0$ " is  
highly unlikely"

## Bernoulli

$X \sim \text{Bernoulli}(p)$ ,  $0 \leq p \leq 1$

if  $\begin{cases} P(X=1|p) = p \\ P(X=0|p) = 1-p \end{cases}$

$$\mathbb{E} X = 1 \cdot p + 0 \cdot (1-p) = p$$

$$\mathbb{E} X^2 = \mathbb{E} X = p$$

$$\text{var } X = \mathbb{E} X^2 - (\mathbb{E} X)^2 = p - p^2 = p(1-p)$$

## Binomial

Experiment : Perform  $n$  independent and identically distributed (i.i.d.) Bernoulli trials (o).

Let  $Y$  be the number of 1's ( $1 = \text{success}$ ) we call

$$Y \sim \text{Binomial}(n, p)$$

$$P(Y=y | n, p) = \binom{n}{y} p^y (1-p)^{n-y}.$$

$$\left\{ \begin{array}{l} \mathbb{E} Y = np \\ \text{var } Y = np(1-p) \\ M_x(t) = (pe^t + 1-p)^n \end{array} \right.$$

from last chapter.

## Poisson

$X \sim \text{Poisson}(\lambda)$ ,  $\lambda > 0$

if

$$P(X=x | \lambda) = e^{-\lambda} \frac{\lambda^x}{x!}, x=0, 1, 2, \dots$$

Note:  $e^\lambda = 1 + \lambda + \frac{\lambda^2}{2!} + \dots + \frac{\lambda^x}{x!} + \dots$

Poisson distribution arises in modelling the following experiments:

- the number of buses arriving within a window of time
- the number of stars observed in a  $\square$  of sky map
- the number of incidents found in a segment of a network ...

Easy Computation

$$\begin{aligned} E[X] &= \sum_{x=0}^{\infty} x e^{-\lambda} \frac{\lambda^x}{x!} \\ &= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} \\ &= e^{-\lambda} \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \\ &= \lambda. \end{aligned}$$

Similar computation gives  $\text{Var}X = \lambda$ ,  
Last chapter:  $M_X(t) = e^{\lambda(e^t - 1)}$ .

**Question.** "A call operator handles on average  $\frac{5 \text{ calls}}{3 \text{ min}}$  what is the probability that there'll be **no** calls in the next minute ?".

**Solution.**

- Let  $X$  be the number of calls in the next minute.

- Assume  $X \sim \text{Poisson}(\lambda)$

Then  $E(X) = \lambda = 5/3$ .

$$P(X=0 | \lambda=5/3) = e^{-5/3} \frac{\left(\frac{5}{3}\right)^0}{0!} = e^{-5/3} = .189.$$

Recall

Poisson approximation of binomial distributions

if  $X_n \sim \text{Binomial}(n, p_n)$

and  $n p_n \rightarrow \lambda$  as  $n \rightarrow \infty$

then  $X_n \xrightarrow{d} Y$  where  $Y \sim \text{Poisson}(\lambda)$ .

- Question:**
- A typesetter on average makes one error per 500 words typeset.
  - A typical page has 300 words. What is the probability of  $\leq 2$  errors in a five-page essay?

### Solution

- Let  $X$  be the number of error in five pages
- Assume

$$X \sim \text{Binomial}(n, p)$$

where  $n = 300 \cdot 5 = 1500$

$$p = 1/500.$$

Then

$$P(X \leq 2 | n, p) = \sum_{x=0}^2 \binom{1500}{x} \left(\frac{1}{500}\right)^x \left(\frac{499}{500}\right)^{1500-x}$$

$$= 0.4230$$

- Use Poisson approximation:  $X \stackrel{d}{\approx} Y$
- $Y \sim \text{Poisson}(\lambda)$ ,  $\lambda = np = 3$ .

Then

$$P(X \leq 2 | \lambda) \approx P(Y \leq 2 | \lambda)$$

$$= e^{-3} \left(1 + 3 + \frac{3^2}{2!}\right) = 0.4232.$$

## Negative Binomial

Experiment: Count number of independent Bernoulli ( $p$ ) trials until obtaining  $r$  successes (1's).

$X \sim \text{Neg Binomial}(r, p)$ ,  $0 \leq p \leq 1$

$$\text{if } P(X=x | p, r) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, x=r, r+1, \dots$$

Let  $Y = X - r$  (# of failures) then

$$P(Y=y | p, r) = P(X=y+r | p, r) = \binom{y+r-1}{r-1} p^r (1-p)^y$$

$y = 0, 1, 2, \dots$

Since

$$\begin{aligned} \binom{y+r-1}{r-1} &= \binom{y+r-1}{y} = \frac{(y+r-1) \dots (r+1)r}{y!} \\ &= (-1)^y \frac{(-r)(-r-1) \dots (-r-y+1)}{y!} \\ &=: (-1)^y \binom{-r}{y} \end{aligned}$$

$$P(Y=y | p, r) = (-1)^y \binom{-r}{y} p^r (1-p)^y$$

this gives the name "negative binomial"

## Additional facts

$$EY = \sum_{y=0}^{\infty} y \binom{y+r-1}{r-1} p^r (1-p)^y$$

$$= r(1-p)/p \quad (\text{check!})$$

$$\text{var}Y = r(1-p)/p^2 \quad (\text{check!})$$

$$\text{set } \mu = r(1-p)/p \Rightarrow \frac{1-p}{p} = \frac{\mu}{r} \Rightarrow \frac{1}{p} = \frac{\mu+r}{r}$$

$$\text{Hence } \text{var}Y = \frac{EY}{p} = \frac{\mu(\mu+r)}{r}$$

$$\boxed{\text{var}Y = \frac{1}{r} \mu^2 + \mu}$$

Quadratic Relation

## Poisson Approximation

if  $r \rightarrow \infty$ ,  $p \rightarrow 1$  such that  $r(1-p) \rightarrow \lambda$

then  $\begin{cases} EY \rightarrow \lambda \\ \text{var}Y \rightarrow \lambda \end{cases}$

Moreover  $Y \xrightarrow{d} \text{Poisson}(\lambda)$  (exercise!)

Geometric

$X \sim \text{Geometric}(p)$

$$\text{if } P(X=x | p) = p(1-p)^{x-1}, \quad x=1, 2, \dots$$

This is special case of Neg Binomial ( $p, r=1$ )

$$\text{So } \left\{ \begin{array}{l} E(X) = (1-p)/p \\ \text{var}(X) = (1-p)/p^2. \end{array} \right.$$

$$\text{Note: } P(X > x) = \sum_{k=x+1}^{\infty} p(1-p)^{k-1} = (1-p)^x.$$

$$\text{So if } s > t \\ P(X > s | X > t) = \frac{P(X > t) \cap (X > s)}{P(X > t)}$$

$$= \frac{P(X > s)}{P(X > t)} = (1-p)^{s-t}$$

$$P(X > s | X > t) = P(X > s - t)$$

This is called the "Memoryless" property