1.1 Sets and probabilities

Probability theory can be viewed as Calculus of Random Variables

to be defined

Random variables are our devices for capturing random phenomena or outcomes of experiments

natural

engineered

Definition

Sample space S is the space of all possible outcomes of a particular experiment.

Examples

Sample space S can be concrete or complicated or downright abstract.

- Coin tossing : $S = \{H, T\}$ (head/tail)
- SAT scores : $S = \{200, \dots, 800\}$ → finite
- Human heights : $S = (0, 7)$ ft → continuous
- Text Messages produced by ChatGPT : $S = ?$ → finite? → infinite? → countable?

$\boxed{\text{Def.}}$ A collection of possible outcomes is called an event, i.e., any subset of a sample space S

We'll be speaking of "probability of an event" $\equiv$ formally, probability of sets (subsets of S)

Most events of interest can be described by the usual operations on sets

- Union $A \cup B$
- intersection $A \cap B$
- complementation A^c

{ commutative, associative, distributive, de Morgan's

Set Theory Basics

outcomes for us

- Set S represents a collection of elements
- A subset A of S is a collection of elements in S
  A is also a set
- Given any two sets A and B "in"

$$\boxed{A \subset B} \text{ if and only if } \boxed{x \in A \Rightarrow x \in B}$$

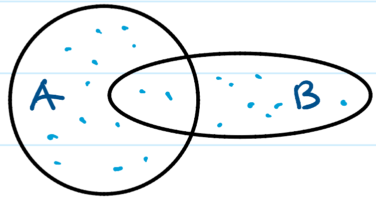
$$A = B \quad \text{iff} \quad A \subset B \text{ and } B \subset A$$

- A set can be empty and denoted by $\boxed{\phi}$.

Operations of sets

Union :

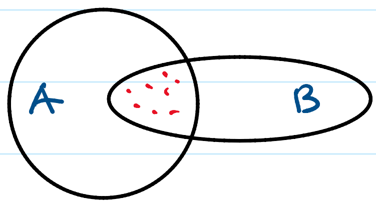
Venn diagram



$$A \cup B := \{ x \mid x \in A \text{ OR } x \in B \}$$

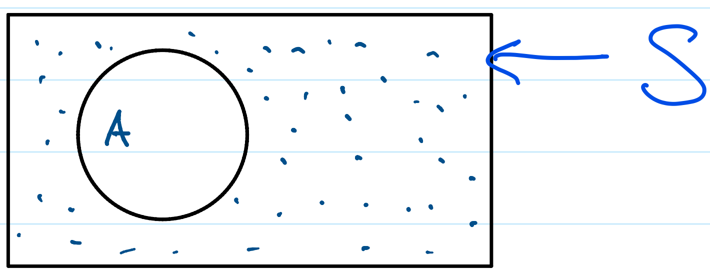
"defined as"

intersection :



$$A \cap B := \{ x \mid x \in A \text{ and } x \in B \}$$

Complementation :



$$A^c := \{ x \mid x \notin A \}$$

$\boxed{\text{THEOREM}}$

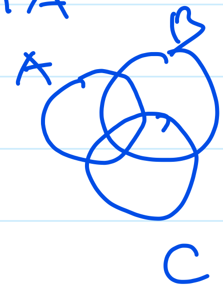
Laws on Operations of Sets

Commutativeness: $A \cup B = B \cup A \quad \forall A, B \subset S$

$$A \cap B = A \cap B = B \cap A$$

Associativity: $A \cup B \cup C =: A \cup (B \cup C) = (A \cup B) \cup C$

$$A \cap B \cap C =: A \cap (B \cap C) = (A \cap B) \cap C$$



Distributive law: $\rightarrow$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

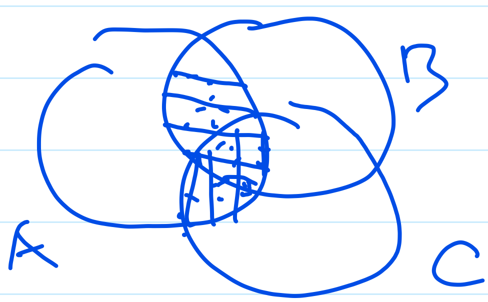
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

De Morgan's law

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c.$$

informal proof: Venn diagram



$\boxed{\text{Example}}$ Select a card at Random ← Experiment

$S = \{C, D, H, S\}$

an event can be

$A = \{C, D\}$ or $B = \{H\}$

or $C = \phi$, etc.

$\boxed{\text{Example}}$ S = all possible heights of a human

an event is a subset of real numbers in $(0, 7)$ ft

or $(0, 213.36)$ cm

Want to put probability numbers on (large) number of events

- easy if S finite
- more challenging is S infinite

Sigma Algebra

Def. A collection of subsets of S is called a sigma-algebra (or Borel field), denoted by $\mathcal{B}$, if:

(i) $\phi \in \mathcal{B}$

(ii) $A^c \in \mathcal{B}$ whenever $A \in \mathcal{B}$

(iii) if $A_1, A_2, \ldots \in \mathcal{B}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$

$\mathcal{B}$ closed under complementation and countable unions

Remarks

$$A, B \in \mathcal{B} \overset{(iii)}{\Rightarrow} A \cup B \in \mathcal{B} \overset{(ii)}{\Rightarrow} (A \cup B)^c \in \mathcal{B} \Rightarrow A^c \cap B^c \in \mathcal{B},$$

- $S \in \mathcal{B}$
- $\mathcal{B}$ is also closed under countable intersections, by DeMorgan's
- The countable operations allow us to capture sufficiently large collection of events of interest.

$$A, B \in \mathcal{B} \Rightarrow A^c \in \mathcal{B} \; ; \; B^c \in \mathcal{B}$$

$$\overset{(iii)}{\Rightarrow} \underbrace{A^c \cup B^c}_{} \in \mathcal{B}$$

$$\overset{\text{deMorgan's}}{\Rightarrow} (A \cap B)^c \in \mathcal{B}$$

$$\overset{(ii)}{\Rightarrow} A \cap B \in \mathcal{B}$$

- given $A_1, A_2, \ldots \subset S$, what do I mean by $A_1 \cup A_2 \cup \ldots = \bigcup_{i=1}^{\infty} A_i$

EXAMPLES

1. $S = \{1, 2, \ldots, n\}$ (a finite set)

Take $\mathcal{B}$ to be a sigma-algebra contains every element of S

Then $\mathcal{B} =$ a collection of all subsets of S

$|\mathcal{B}| = 2^n$

2. $S = \mathbb{Z}$ (all integers)

$\mathcal{B}_1 := \{ \phi, S,$ subset of all odd numbers, subset of all even numbers $\}$

$\mathcal{B} :=$ collection of all subsets of $\mathbb{Z}$

$\mathcal{B}_1$ has 4 members, but $\mathcal{B}$ is infinite and uncountable

3. Let $S = (-\infty, \infty) = \mathbb{R}$, the Real line.

Let $\mathcal{B} :=$ sigma algebra containing all sets of forms
$[a,b]$, $(a,b]$, (a,b), $[a,b)$, $a,b \in \mathbb{R}$

How can you imagine $\mathcal{B}$?

Let $\mathcal{B}' :=$ sigma algebra containing all sets of form
$[a,b)$ for $a, b \in \mathbb{Q}$ (Rationals)

It is incredible that

$$\mathcal{B} = \mathcal{B}' := \text{Borel sigma algebra of } \mathbb{R}$$