Bryant Willoughby

University of Michigan, Dept of Statistics

Stats 510, Instructor: Long Nguyen

Homework 3

Oct 7, 2024, due by 11:59pm Oct 16, 2025

Notes: You may select to solve both problem 4 and problem 5 given below, or only one of them. If you decide the latter, then please toss a coin. If you get a head, do problem 4, otherwise, do problem 5.

1. (Expectations) Do problems 2.14, 2.17, 2.22.
2. (Moments) Do problems 2.30, 2.38, 2.39.
3. Let X be a standard normal variable, i.e., X has pdf $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ for all $x \in (-\infty, +\infty)$. Define a discrete random variable Y by $P(Y = \sqrt{3}) = P(Y = -\sqrt{3}) = 1/6$ and $P(Y = 0) = 2/3$.
   (i) Show that X and Y have the same r-th moment, for each $r = 1, 2, 3, 4, 5$.
   (ii) Find another random variable Z which has the same r-th moments as X and Y for all $r = 1, \ldots, 5$.
4. Let $n \in \mathbb{N}$, define function f_{X_n} as follows: For $x \in \{0, 1, \ldots, n\}$,
$$f_{X_n}(x) = (1/2)\binom{n}{x}(3n)^{-x}(1 - 1/(3n))^{n-x} + (1/2)\binom{n}{x}(6n)^{-x}(1 - 1/(6n))^{n-x}.$$
   and $f_{X_n}(x) = 0$ otherwise.
   (i) Verify that f_{X_n} is a valid pmf for a discrete random variable, which we denote by X_n. Derive the moment generating function for X_n.
   (ii) Show that the sequence of random variables $\{X_n\}$ converges in distribution to a random variable Y. What is the distribution of Y?
5. Let $n \in \mathbb{N}$, define function f_{X_n} as follows: For $x \in \mathbb{R}$,
$$f_{X_n}(x) = (1/2)\frac{1}{\sqrt{2\pi/n}} e^{-\frac{n}{2}x^2} + (1/2)\frac{1}{\sqrt{2\pi/n}} e^{-\frac{n}{2}(x-1)^2}.$$
   (i) Verify that f_{X_n} is a valid pdf for a continuous random variable, which we denote by X_n.
   (ii) Show that the sequence of random variables $\{X_n\}$ converges in distribution to a random variable Y. What is the distribution of Y?

1. (Expectations) Do problems 2.14, 2.17, 2.22.

2.14 (a) Let X be a continuous, nonnegative random variable [$f(x) = 0$ for $x < 0$]. Show that

$$\mathrm{E}X = \int_0^\infty [1 - F_X(x)]\,dx,$$

where $F_X(x)$ is the cdf of X.

(b) Let X be a discrete random variable whose range is the nonnegative integers. Show that

$$\mathrm{E}X = \sum_{k=0}^\infty (1 - F_X(k)),$$

where $F_X(k) = P(X \le k)$. Compare this with part (a).

a) $$\int_0^\infty (1 - F_X(x))\,dx = \int_0^\infty P(X > x)\,dx = \int_0^\infty \int_x^\infty f_X(y)\,dy\,dx$$

$$= \int_0^\infty \left(\int_0^\infty 1(x \le y) f_X(y)\,dy \right) dx$$

$$= \int_0^\infty \left(\int_0^\infty 1(x \le y) f_X(y)\,dx \right) dy$$

$$= \int_0^\infty f_X(y) \left(\int_0^\infty 1(x \le y)\,dx \right) dy$$

$$= \int y f_X(y)\,dy = E(X).$$

b) $$E(X) = \sum_{k=0}^\infty k\,P(X=k)$$

$$= \sum_{k=0}^\infty \sum_{j=1}^k P(X=k), \quad k = \sum_{j=1}^k 1$$

$$= \sum_{k=0}^\infty \sum_{j=1}^\infty 1(j \le k) P(X=k)$$

$$= \sum_{j\ge1} \sum_{k \ge j} P(X=j)$$

$$= \sum_{j \ge 1} P(X \ge j)$$

$$= \sum_{j\ge1} 1 - P(X < j)$$

$$= \sum_{j \ge 1} 1 - P(X \le j-1)$$

$$= \sum_{j\ge1} 1 - F_X(j-1)$$

$$= \sum_{k\ge0} 1 - F_X(k), \quad k = j-1.$$

2.17 A *median* of a distribution is a value m such that $P(X \le m) \ge \frac{1}{2}$ and $P(X \ge m) \ge \frac{1}{2}$. (If X is continuous, m satisfies $\int_{-\infty}^m f(x)\,dx = \int_m^\infty f(x)\,dx = \frac{1}{2}$.) Find the median of the following distributions.

(a) $f(x) = 3x^2, \quad 0 < x < 1$ (b) $f(x) = \dfrac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$

a) $$\int_0^m 3x^2\,dx = x^3 \Big|_0^m = m^3 \overset{\text{set}}{=} \frac{1}{2} \Rightarrow m = \left(\frac{1}{2}\right)^{\frac{1}{3}} = \boxed{0.794}$$

b) $f(x)$ is the cauchy pdf and therefore symmetric about zero, i.e.

$$\int_{-\infty}^0 f(x)\,dx = \int_0^\infty f(x)\,dx \Rightarrow \boxed{m = 0}$$

To show formally:

$$\int_{-\infty}^m f(x)\,dx = \frac{1}{\pi} \int_{-\infty}^m \frac{1}{1+x^2}\,dx = \frac{1}{\pi} \tan^{-1}(x) \Big|_{-\infty}^m = \frac{1}{\pi}\left(\tan^{-1}(m) + \frac{\pi}{2}\right) \overset{\text{set}}{=} \frac{1}{2}$$

$$\Rightarrow \frac{\pi}{2} = \tan^{-1}(m) + \frac{\pi}{2} \Rightarrow \tan^{-1}(m) = 0 \Rightarrow m = 0.$$

2.22 Let X have the pdf

$$f(x) = \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-x^2/\beta^2}, \quad 0 < x < \infty, \quad \beta > 0.$$

(a) Verify that $f(x)$ is a pdf. (b) Find $\mathrm{E}X$ and $\mathrm{Var}\,X$.

a) I will show i) Nonnegativity and ii) Normalization, i.e.

i) For $x \ge 0$ and $\beta > 0$, every factor is nonnegative $\Rightarrow f(x) \ge 0$

ii) Now, I must show $\int_{-\infty}^\infty f(x)\,dx = \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-x^2/\beta^2}\,dx = 1$:

First, compute $A := \int_0^\infty x^2 e^{-x^2/\beta^2}\,dx$ via IBP

let $u = x$ $\quad dv = x e^{-x^2/\beta^2}\,dx$

$du = dx$ $\quad v = \int x e^{-x^2/\beta^2}\,dx$

$w = -x^2/\beta^2$

$dw = -2x/\beta^2\,dx \Rightarrow x\,dx = -\frac{\beta^2}{2}\,dw$

$$= \int e^u\left(-\frac{\beta^2}{2}\,du\right) = -\frac{\beta^2}{2}\int e^u\,du = -\frac{\beta^2}{2} e^{-x^2/\beta^2}$$

$$= uv\Big|_0^\infty - \int_0^\infty v\,du$$

where $\lim_{x\to\infty} x \cdot \left(-\frac{\beta^2}{2} e^{-x^2/\beta^2}\right) = 0$ and $uv|_{x=0} = 0 \Rightarrow uv|_0^\infty = 0$

$$= -\int_0^\infty \left(-\frac{\beta^2}{2} e^{-x^2/\beta^2}\right) dx$$

$$= \frac{\beta^2}{2} \int_0^\infty e^{-x^2/\beta^2}\,dx$$

let $y = x/\beta \Rightarrow x = \beta y \Rightarrow dx = \beta\,dy$

$$= \frac{\beta^2}{2} \int_0^\infty e^{-(\beta y)^2/\beta^2} \beta\,dy = \beta \int_0^\infty e^{-y^2}\,dy$$

$$= \beta \cdot \frac{\sqrt{\pi}}{2} \quad \text{since } \int_{-\infty}^\infty e^{-y^2}\,dy = \sqrt{\pi}$$

Hence,

$$A = \frac{\beta^2}{2} \cdot \beta \frac{\sqrt{\pi}}{2} = \frac{\beta^3\sqrt{\pi}}{4}.$$

Thus,

$$\int_0^\infty f(x)\,dx = \frac{4}{\beta^3\sqrt{\pi}} A = \frac{4}{\beta^3\sqrt{\pi}} \frac{\beta^3\sqrt{\pi}}{4} = 1.$$

b) Find $E(X) = \int_0^\infty x f(x)\,dx$ and $V(X) = E(X^2) - [E(X)]^2$ where $E(X^2) = \int_0^\infty x^2 f(x)\,dx$

compute $B := \int_0^\infty x^3 e^{-x^2/\beta^2}\,dx$ via IBP

$u = x^2$ $\quad dv = x e^{-x^2/\beta^2}\,dx$

$du = 2x\,dx$ $\quad v = \int x e^{-x^2/\beta^2}\,dx$

$= -\frac{\beta^2}{2} e^{-x^2/\beta^2}$ as seen in (a)

$$= uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$= 0 - \int_0^\infty \left(-\frac{\beta^2}{2} e^{-x^2/\beta^2}\right) 2x\,dx$$

$$= \beta^2 \int_0^\infty x e^{-x^2/\beta^2}\,dx \quad \text{via substitution}$$

$w = x^2/\beta^2$

$dw = 2x/\beta^2\,dx \Rightarrow x\,dx = \frac{\beta^2}{2}\,dw$

where $\int_0^\infty x e^{-x^2/\beta^2}\,dx = \frac{\beta^2}{2} \int_0^\infty e^{-w}\,dw = \frac{\beta^2}{2}$ since $[-e^{-w}]_0^\infty = 1$

$$= \beta^2 \cdot \frac{\beta^2}{2} = \frac{\beta^4}{2}.$$

So,

$$\boxed{E(X) = \frac{4}{\beta^3\sqrt{\pi}} B = \frac{4}{\beta^3\sqrt{\pi}} \cdot \frac{\beta^4}{2} = \frac{2\beta}{\sqrt{\pi}}.}$$

Now compute $C := \int_0^\infty x^4 e^{-x^2/\beta^2}\,dx$ via IBP

Let $u = x^3$ $\quad dv = x e^{-x^2/\beta^2}$

$du = 3x^2\,dx$ $\quad v = -\frac{\beta^2}{2} e^{-x^2/\beta^2}$ as seen previously

$$= uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$= \int_0^\infty \left(\frac{\beta^2}{2} e^{-x^2/\beta^2}\right) \cdot 3x^2\,dx$$

$$= \frac{3\beta^2}{2} \int_0^\infty x^2 e^{-x^2/\beta^2}\,dx$$

$$= \frac{3\beta^2}{2} A$$

$$= \frac{3\beta^2}{2} \cdot \frac{\beta^3\sqrt{\pi}}{4} = \frac{3\beta^5\sqrt{\pi}}{8}.$$

Hence,

$$E(X^2) = \frac{4}{\beta^3\sqrt{\pi}} C = \frac{4}{\beta^3\sqrt{\pi}} \cdot \frac{3\beta^5\sqrt{\pi}}{8} = \frac{3}{2}\beta^2$$

Therefore,

$$\boxed{V(X) = E(X^2) - [E(X)]^2 = \frac{3}{2}\beta^2 - \left(\frac{2\beta}{\sqrt{\pi}}\right)^2 = \beta^2\left(\frac{3}{2} - \frac{4}{\pi}\right)}$$

2. (Moments) Do problems 2.30, 2.38, 2.39.

2.30 Find the moment generating function corresponding to

(a) $f(x) = \frac{1}{c}, \quad 0 < x < c.$

(b) $f(x) = \frac{2x}{c^2}, \quad 0 < x < c.$

(c) $f(x) = \frac{1}{2\beta} e^{-|x-\alpha|/\beta}, \quad -\infty < x < \infty, \quad -\infty < \alpha < \infty, \quad \beta > 0.$

(d) $P(X = x) = \binom{r+x-1}{x} p^r (1-p)^x, \quad x = 0, 1, \ldots, \quad 0 < p < 1, r > 0$ an integer.

a) $E(e^{tX}) = \int_0^c e^{tx}\frac{1}{c}\,dx = \frac{1}{ct}e^{tx}\Big|_0^c = \frac{1}{ct}e^{tc} - \frac{1}{ct} = \underline{\frac{1}{ct}(e^{tc}-1)}$

b) $E(e^{tX}) = \int_0^c \frac{2x}{c^2}e^{tx}\,dx = \frac{1}{c^2}\int_0^c 2x e^{tx}\,dx$

↳ IBP: $u = 2x \quad dv = e^{tx}dx$
$du = 2 \quad v = \int e^{tx}dx = \frac{1}{t}e^{tx}dx$

$$= \frac{1}{c^2}\left[\frac{2x}{t}e^{tx}\Big|_0^c - \int_0^c \frac{2}{t}e^{tx}dx\right]$$
$$= \frac{1}{c^2}\left[\frac{2c}{t}e^{tc} - \left(\frac{2}{t^2}e^{tx}\right)\Big|_0^c\right]$$
$$= \frac{1}{c^2}\left[\frac{2c}{t}e^{tc} - \left(\frac{2}{t^2}e^{tc} - \frac{2}{t^2}\right)\right]$$
$$= \frac{2e^{tc}}{ct} - \frac{2e^{tc}}{c^2t^2} + \frac{2}{c^2t^2} = \underline{\frac{2}{c^2t^2}(cte^{tc} - e^{tc} + 1)}$$

c) $E(e^{tX}) = \int_{-\infty}^{\infty}\frac{1}{2\beta}e^{-|x-\alpha|/\beta}dx = \int_{-\infty}^{\alpha}\frac{1}{2\beta}e^{(x-\alpha)/\beta}e^{tx}dx + \int_{\alpha}^{\infty}\frac{1}{2\beta}e^{-(x-\alpha)/\beta}e^{tx}dx$

Since $\left[\int_{-\infty}^{\infty} f(|x-a|)dx = \underbrace{\int_{-\infty}^{a} f(a-x)dx}_{x<a} + \underbrace{\int_{a}^{\infty} f(x-a)dx}_{x>a}\right]$ FS

$$= \int_{-\infty}^{\alpha}\frac{1}{2\beta}e^{x/\beta - \alpha/\beta + tx}dx + \int_{\alpha}^{\infty}\frac{1}{2\beta}e^{-x/\beta + \alpha/\beta + tx}dx$$
$$= \frac{e^{-\alpha/\beta}}{2\beta}e^{x(\frac{1}{\beta}+t)}\frac{1}{(\frac{1}{\beta}+t)}\Big|_{-\infty}^{\alpha} - \frac{e^{\alpha/\beta}}{2\beta}e^{-x(\frac{1}{\beta}-t)}\frac{1}{(\frac{1}{\beta}-t)}\Big|_{\alpha}^{\infty} \quad (1)$$
$$= \left(\frac{e^{-\alpha/\beta+\alpha/\beta+\alpha t}}{2\beta(\frac{1}{\beta}+t)} - 0\right) - \left(0 - \frac{e^{\alpha/\beta-\alpha/\beta+\alpha t}}{2\beta(\frac{1}{\beta}-t)}\right)$$
$$= \frac{e^{\alpha t}}{2\beta(\frac{1}{\beta}+t)} + \frac{e^{\alpha t}}{2\beta(\frac{1}{\beta}-t)}$$
$$= \frac{e^{\alpha t}}{2+2\beta t} + \frac{e^{\alpha t}}{2-2\beta t}$$
$$= \frac{(2-2\beta t)e^{\alpha t} + (2+2\beta t)e^{\alpha t}}{(2+2\beta t)(2-2\beta t)}$$
$$= \frac{2e^{\alpha t} - 2\beta \cancel{t} e^{\alpha t} + 2e^{\alpha t} + 2\beta \cancel{t} e^{\alpha t}}{4 - 4\beta t + 4\beta t - 4\beta^2 t^2}$$
$$\underline{\frac{4e^{\alpha t}}{4-4\beta^2t^2}, \quad -1/\beta < t < 1/\beta}$$

s.t. (1) converges, i.e. the integral can only be evaluated appropriately when $e^{x(\frac{1}{\beta}+t)}\Big|_{x=-\infty} > 0$ iff $\frac{1}{\beta}+t < 0 \Rightarrow t < -\frac{1}{\beta}$ and $e^{-x(\frac{1}{\beta}-t)}\Big|_{x=\infty} > 0$ iff $\frac{1}{\beta} - t < 0 \Rightarrow t > \frac{1}{\beta}$.

d) $E(e^{tX}) = \sum_{x=0}^{\infty}\binom{r+x-1}{x}p^r(1-p)^x e^{tx}$

$$= \sum_{x=0}^{\infty}\binom{r+x-1}{x}p^r((1-p)e^t)^x$$
$$= \sum_{x=0}^{\infty}\binom{r+x-1}{x}p^r((1-p)e^t)^x\frac{(1-(1-p)e^t)^r}{(1-(1-p)e^t)^r}$$
$$= \frac{p^r}{(1-(1-p)e^t)^r}\underbrace{\sum_{x=0}^{\infty}\binom{r+x-1}{x}(1-(1-p)e^t)^r((1-p)e^t)^x}_{1}$$

Since above Negative Binomial $(r, 1-(1-p)e^t)$ sums to 1

$$= \underline{\frac{p^r}{(1-(1-p)e^t)^r}, \quad t < -\log(1-p)}$$

since $0 < (1-(1-p)e^t) < 1$ b/c $1-(1-p)e^t$ is a prob.
$\Rightarrow 0 < 1-(1-p)e^t < 1$
$\Rightarrow (1-p)e^t < 1$ since $1-(1-p)e^t > 0$
FS: $\ln(AB) = \ln(A) + \ln(B)$ $\Rightarrow \log(1-p) + t < \log(1) = 0$
$\Rightarrow t < -\log(1-p)$.

2.38 Let X have the negative binomial distribution with pmf

$$f(x) = \binom{r+x-1}{x}p^r(1-p)^x, \quad x = 0, 1, 2, \ldots,$$

where $0 < p < 1$ and $r > 0$ is an integer.

(a) Calculate the mgf of X.
(b) Define a new random variable by $Y = 2pX$. Show that, as $p \downarrow 0$, the mgf of Y converges to that of a chi squared random variable with $2r$ degrees of freedom by showing that

$$\lim_{p\to 0} M_Y(t) = \left(\frac{1}{1-2t}\right)^r, \quad |t| < \frac{1}{2}.$$

a) $M_X(t) = E(e^{tX}) = \underline{\frac{p^r}{(1-(1-p)e^t)^r}, \quad t < -\log(1-p)}$ (as completed above in 2.30d)

b) Let $Y = 2pX$. Then,

$$M_Y(t) = M_{2pX}(t) = M_X(2pt) = \frac{p^r}{(1-(1-p)e^{2pt})^r}$$

since $\forall a,b \in \mathbb{R}$, $M_{aX+b}(t) = e^{bt}M_X(at)$

Next,

$$\lim_{p\to 0} M_Y(t) = \frac{p^r}{(1-(1-p)e^{2pt})^r} = \lim_{p\to 0}\left(\frac{p}{1-(1-p)e^{2pt}}\right)^r$$
$$= \lim_{p\to 0}\left(\frac{p'}{\frac{d}{dp}[1-(1-p)e^{2pt}]\ (*)}\right)^r$$

by L'hopital's rule: $\underbrace{\lim_{x\to a}\frac{f(x)}{g(x)}}_{\text{indeterminate}} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

where $(*)$: $-((1-p)(2te^{2pt}) + e^{2pt}(-1))$
$= -e^{2pt}(2t(1-p)-1) = e^{2pt}(1-2t(1-p))$

$$= \lim_{p\to 0}\left(\frac{1}{e^{2pt}(1-2t(1-p))}\right)^r$$
$$= \underline{\left(\frac{1}{1-2t}\right)^r, \quad |t| < \frac{1}{2}}. \blacksquare$$

2.39 In each of the following cases calculate the indicated derivatives, justifying all operations.

(a) $\frac{d}{dx}\int_0^x e^{-\lambda t}\,dt$ (b) $\frac{d}{d\lambda}\int_0^\infty e^{-\lambda t}\,dt$

(c) $\frac{d}{dt}\int_t^1 \frac{1}{x^2}\,dx$ (d) $\frac{d}{dt}\int_1^\infty \frac{1}{(x-t)^2}\,dx$

a) verify $\frac{d}{dx}\left[\int_0^x e^{-\lambda t}dt\right] = \frac{d}{dx}\left[\frac{-1}{\lambda}e^{-\lambda t}\Big|_0^x\right] = \frac{d}{dx}\left[\left(-\frac{1}{\lambda}e^{-\lambda x}+\frac{1}{\lambda}\right)\right] = \frac{-1}{\lambda}(-\lambda)e^{-\lambda x} = e^{-\lambda x}$. ∎

b) verify $\frac{d}{d\lambda}\int_0^\infty e^{-\lambda t}dt = \frac{d}{d\lambda}\left[\frac{-1}{\lambda}e^{-\lambda t}\Big|_0^\infty\right] = \frac{d}{d\lambda}\left[0+\frac{1}{\lambda}\right] = \frac{d}{d\lambda}(\lambda^{-1}) = \frac{-1}{\lambda^2}$. ∎

c) verify $\frac{d}{dt}\left[\int_t^1 \frac{1}{x^2}dx\right] = \frac{d}{dt}\left[\frac{-1}{x}\Big|_t^1\right] = \frac{d}{dt}\left[-1+\frac{1}{t}\right] = -\frac{1}{t^2}$. ∎

d) verify $\frac{d}{dt}\int_1^\infty (x-t)^{-2}dx = \frac{d}{dt}\left[-(x-t)^{-1}\Big|_1^\infty\right] = \frac{d}{dt}\left(\frac{1}{1-t}\right) = \frac{1}{(1-t)^2}$.

3. Let X be a standard normal variable, i.e., X has pdf $f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ for all $x \in (-\infty, +\infty)$. Define a discrete random variable Y by $P(Y=\sqrt{3}) = P(Y=-\sqrt{3}) = 1/6$ and $P(Y=0) = 2/3$.

(i) Show that X and Y have the same r-th moment, for each $r = 1,2,3,4,5$.

(ii) Find another random variable Z which has the same r-th moments as X and Y for all $r = 1,\dots,5$.

<u>Recall</u>: $\forall n \in \mathbb{N}$, $E(X^n) = \frac{d^n}{dt^n}M_X(t)\Big|_{t=0}$

i) For $X \sim N(0,1)$,

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(tx-\frac{x^2}{2}\right)dx.$$

Now complete the square in the exponent:

$$tx-\frac{x^2}{2} = \frac{-1}{2}(x^2-2tx) = -\frac{1}{2}((x-t)^2-t^2) = -\frac{(x-t)^2}{2}+\frac{t^2}{2}.$$

So,

$$M_X(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-t)^2}{2}+\frac{t^2}{2}\right)dx = e^{t^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-(x-t)^2/2}dx$$

let $u = x-t$ $du = dx$

$$= e^{t^2/2}\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-u^2/2}du}_{1 \text{ (standard normal)}}$$

Thus, $M_X(t) = e^{t^2/2}$. Now evaluate $M_X^{(1)}(0), M_X^{(2)}(0), \dots, M_X^{(5)}(0)$:

$E(X) := M_X'(0) = \frac{d}{dt}e^{t^2/2}\Big|_{t=0} = te^{t^2/2}\Big|_{t=0} = 0$

$E(X^2) := M_X''(0) = \frac{d}{dt}(te^{t^2/2})\Big|_{t=0} = e^{t^2/2}+t(te^{t^2/2})\Big|_{t=0} = (1+t^2)e^{t^2/2}\Big|_{t=0} = 1.$

$E(X^3) := M_X^{(3)}(0) = \frac{d}{dt}((1+t^2)e^{t^2/2})\Big|_{t=0} = 2te^{t^2/2}+(1+t^2)\cdot(te^{t^2/2})\Big|_{t=0} = (3t+t^3)e^{t^2/3}\Big|_{t=0} = 0$

$E(X^4)$: $M_X^{(4)}(t) = \frac{d}{dt}((3t+t^3)e^{t^2/2}) = (3+3t^2)e^{t^2/2}+(3t+t^3)\cdot(te^{t^2/2}) = (3+6t^2+t^4)e^{t^2/2}$

$\Rightarrow M_X^{(4)}(0) = 3$

$E(X^5) := M_X^{(5)}(0) = 0$ since odd moments vanish by symmetry (as seen above).

For discrete Y given by $P(Y=\sqrt{3}) = P(Y=-\sqrt{3}) = \frac{1}{6}$ and $P(Y=0) = \frac{2}{3}$,

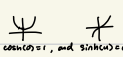
$$M_Y(t) = \sum_y e^{yt}f_Y(y) = \frac{1}{6}e^{\sqrt{3}t}+\frac{2}{3}e^0+\frac{1}{6}e^{-t\sqrt{3}}$$

$$= \frac{1}{6}(e^{\sqrt{3}t}+e^{-\sqrt{3}t})+\frac{2}{3},$$

where $\cosh(u) = \frac{e^u+e^{-u}}{2}$

$$= \frac{1}{3}\cosh(\sqrt{3}t)+\frac{2}{3}$$

of interest,



$\frac{d}{dt}\cosh(u) = \sinh(u)\cdot u'$, $\frac{d}{dt}\sinh(u) = \cosh(u)\cdot u'$, $\cosh(0) = 1$, and $\sinh(u) = 0$

Given $M_Y(t) = \frac{1}{3}\cosh(\sqrt{3}t)+\frac{2}{3}$, find $E(Y), E(Y^2), \dots, E(Y^5)$:

$E(Y)$: $M_Y'(t) = \frac{1}{3}(\sqrt{3})\sinh(\sqrt{3}t)\Big|_{t=0} = \frac{1}{3}(\sqrt{3})\cdot 0 = 0$

$E(Y^2)$: $M_Y''(t) = \frac{1}{3}(\sqrt{3})^2\cosh(\sqrt{3}t) = \cosh(\sqrt{3}t)\Big|_{t=0} = 1$

$E(Y^3)$: $M_Y^{(3)}(t) = \frac{1}{3}(\sqrt{3})^3\sinh(\sqrt{3}t) = \sqrt{3}\sinh(\sqrt{3}t)\Big|_{t=0} = 0$

$E(Y^4)$: $M_Y^{(4)}(t) = \frac{1}{3}(\sqrt{3})^4\cosh(\sqrt{3}t) = 3\cosh(\sqrt{3}t)\Big|_{t=0} = 3$

$E(Y^5) := M_Y^{(5)}(0) = 0$ since odd moments vanish by symmetry

ii) Define $Z := P(Z=1) = P(Z=-1) = \frac{1}{6}$, $P(Z=2) = P(Z=-2) = \frac{1}{12}$, and $P(Z=0) = \frac{1}{2}$.
Then,

$$M_Z(t) = E(e^{tZ}) = \frac{1}{6}(e^t+e^{-t})+\frac{1}{12}(e^{2t}+e^{-2t})+\frac{1}{2}$$

$$= \frac{1}{3}\cosh(t)+\frac{1}{6}\cosh(2t)+\frac{1}{2}$$

Next, we can compute $E(Z), E(Z^2), \dots, E(Z^5)$.
As discussed in (i), the odd moments vanish by symmetry, i.e.

$E(Z) = E(Z^3) = E(Z^5) = 0$, and

$E(Z^2)$: $M_Z''(t) = \frac{1}{3}\cosh(t)+\frac{2}{3}\cosh(2t)\Big|_{t=0} = \frac{1}{3}+\frac{2}{3} = 1$

$E(Z^4)$: $M_Z^{(4)}(t) = \frac{1}{3}\cosh(t)+\frac{8}{3}\cosh(2t)\Big|_{t=0} = \frac{1}{3}+\frac{8}{3} = 3$.

Therefore, Z contains the same r-th moments as X and Y for $r = 1,\dots,5$.

4. Let $n \in \mathbb{N}$, define function f_{X_n} as follows: For $x \in \{0, 1, \ldots, n\}$,

$$f_{X_n}(x) = (1/2)\binom{n}{x}(3n)^{-x}(1 - 1/(3n))^{n-x} + (1/2)\binom{n}{x}(6n)^{-x}(1 - 1/(6n))^{n-x}.$$

and $f_{X_n}(x) = 0$ otherwise.

(i) Verify that f_{X_n} is a valid pmf for a discrete random variable, which we denote by X_n. Derive the moment generating function for X_n.

(ii) Show that the sequence of random variables $\{X_n\}$ converges in distribution to a random variable Y. What is the distribution of Y?

i) Each summand is a binomial pmf: $\text{Bin}(n, \frac{1}{3n})$ and $\text{Bin}(n, \frac{1}{6n})$. Their avg. is a valid pmf.

Recall: For $X \sim \text{Bin}(n,p)$, $M_X(t) = [(1-p)+pe^t]^n$.

Here, the mgf is a mixture of mgf's, i.e.

$$M_{X_n}(t) = \frac{1}{2}\left(1 - \frac{1}{3n} + \frac{e^t}{3n}\right)^n + \frac{1}{2}\left(1 - \frac{1}{6n} + \frac{e^t}{6n}\right)^n = \frac{1}{2}\left(1 + \frac{e^t-1}{3n}\right)^n + \frac{1}{2}\left(1 + \frac{e^t-1}{6n}\right)^n.$$

ii) As $n \to \infty$, $(1 + \frac{a}{n})^n \to e^a$. Hence,

$$M_{X_n}(t) \to \frac{1}{2}\exp\left(\frac{e^t-1}{3}\right) + \frac{1}{2}\exp\left(\frac{e^t-1}{6}\right),$$

the mgf of the mixture $\frac{1}{2}\text{Poi}(\frac{1}{3}) + \frac{1}{2}\text{Poi}(\frac{1}{6})$.

Therefore,

$X_n \xrightarrow{d} Y$ where Y is the Poisson mixture.

5. Let $n \in \mathbb{N}$, define function f_{X_n} as follows: For $x \in \mathbb{R}$,

$$f_{X_n}(x) = (1/2)\frac{1}{\sqrt{2\pi/n}}e^{-\frac{n}{2}x^2} + (1/2)\frac{1}{\sqrt{2\pi/n}}e^{-\frac{n}{2}(x-1)^2}.$$

(i) Verify that f_{X_n} is a valid pdf for a continuous random variable, which we denote by X_n.

(ii) Show that the sequence of random variables $\{X_n\}$ converges in distribution to a random variable Y. What is the distribution of Y?

For $n \in \mathbb{N}$, define

$$f_{X_n}(x) = \frac{1}{2}\frac{1}{\sqrt{2\pi/n}}e^{-\frac{n}{2}x^2} + \frac{1}{2}\frac{1}{\sqrt{2\pi/n}}e^{-\frac{n}{2}(x-1)^2}, \quad x \in \mathbb{R} \qquad (1)$$

where each (equally weighted) term is the density of a normal curve, i.e.

$$\frac{1}{\sqrt{2\pi/n}}e^{-\frac{n}{2}x^2} \sim N(0, \tfrac{1}{n}) \quad \text{and} \qquad (2)$$

$$\frac{1}{\sqrt{2\pi/n}}e^{-\frac{n}{2}(x-1)^2} \sim N(1, \tfrac{1}{n}) \qquad (3)$$

i) verify that f_{X_n} is a valid pdf, i.e.

i) $f_{X_n}(x) \geq 0$ for all $x \in \mathbb{R}$ and ii) $\int_{-\infty}^{\infty} f_{X_n}(x)\,dx = 1$.

Nonnegativy (i) holds since (1) is comprised of two nonnegative Gaussian densities, i.e.

(2) ≥ 0 $\forall x \in \mathbb{R}$ and (3) ≥ 0 $\forall x \in \mathbb{R}$. Thus, $(1) := f_{X_n} \geq 0$ $\forall x \in \mathbb{R}$.

We con also verify (ii), i.e.

$$\int_{-\infty}^{\infty} f_{X_n}(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\underbrace{\cancel{\frac{1}{\sqrt{2\pi/n}}e^{-\frac{n}{2}x^2}}}_{1}\,dx + \frac{1}{2}\int_{-\infty}^{\infty}\underbrace{\cancel{\frac{1}{\sqrt{2\pi/n}}e^{-\frac{n}{2}(x-1)^2}}}_{1} = \frac{1}{2} + \frac{1}{2} = 1. \blacksquare$$

ii) By definition, the MGF of X_n is

$$M_{X_n}(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_{X_n}(x)\,dx$$

$$= \frac{1}{2}\sqrt{\frac{n}{2\pi}}\int_{-\infty}^{\infty} e^{tx}e^{-\frac{n}{2}x^2}\,dx + \frac{1}{2}\sqrt{\frac{n}{2\pi}}\int_{-\infty}^{\infty} e^{tx}e^{-\frac{n}{2}(x-1)^2}\,dx.$$

To compute the first integral

$$I_1 := \sqrt{\frac{n}{2\pi}}\int_{-\infty}^{\infty} e^{tx - \frac{n}{2}x^2}\,dx,$$

complete the square in the exponent, i.e.

$$tx - \frac{n}{2}x^2 = -\frac{n}{2}\left(x^2 - \frac{2t}{n}x\right) = -\frac{n}{2}\left(\left(x - \frac{t}{n}\right)^2 - \frac{t^2}{n^2}\right) = -\frac{n}{2}\left(x - \frac{t}{n}\right)^2 + \frac{t^2}{2n}.$$

Therefore,

$$I_1 = \sqrt{\frac{n}{2\pi}}e^{t^2/2n}\int_{-\infty}^{\infty} e^{-\frac{n}{2}(x - \frac{t}{n})^2}\,dx$$

Let $u = x - \frac{t}{n} \Rightarrow du = dx$

where $\int_{-\infty}^{\infty} e^{-\frac{n}{2}u^2}\,du = \sqrt{\frac{2\pi}{n}}$.

$$\Rightarrow I_1 = \sqrt{\frac{n}{2\pi}}e^{t^2/2n}\sqrt{\frac{2\pi}{n}} = e^{t^2/2n}.$$

Similarly, for the second integral

$$I_2 := \sqrt{\frac{n}{2\pi}}\int_{-\infty}^{\infty} e^{tx - \frac{n}{2}(x-1)^2}\,dx,$$

$$tx - \frac{n}{2}(x-1)^2 = t(x-1) + t - \frac{n}{2}(x-1)^2.$$

So,

$$I_2 = e^t\sqrt{\frac{n}{2\pi}}\int_{-\infty}^{\infty} e^{t(x-1) - \frac{n}{2}(x-1)^2}\,dx = e^t\sqrt{\frac{n}{2\pi}}\int_{-\infty}^{\infty} e^{st - \frac{n}{2}s^2}\,ds, \text{ letting } s = x - 1$$

$$= e^t\sqrt{\frac{n}{2\pi}}e^{t^2/2n}\sqrt{\frac{2\pi}{n}} \text{ by the some above calculation}$$

$$= e^t e^{t^2/2n}.$$

Thus,

$$M_{X_n}(t) = \frac{1}{2}e^{t^2/2n} + \frac{1}{2}e^t e^{t^2/2n} = \frac{1}{2}e^{t^2/2n}(1 + e^t).$$

$$\Rightarrow \lim_{n \to \infty} M_{X_n}(t) = \lim_{n \to \infty}\frac{1}{2}e^{t^2/2n}(1 + e^t) = \frac{1}{2}(1 + e^t).$$

Now, define

$M_Y(t) := \frac{1}{2}(1 + e^t)$ is the mgf of $Y \sim \text{Bernoulli}(1/2)$

since $E(e^{tY}) = \frac{1}{2}e^{0 \cdot t} \cdot \frac{1}{2}e^{1 \cdot t}$.

This implies that

$X_n \xrightarrow{d} Y$ for Bernoulli(1/2) by the mgf convergence thereom.