week 9

3.2: continuous Distributions

- continuous distributions put prob. mass on cont. spaces

Formally, a continuous RV is one whose cdf is a continuous function

Def (Uniform Dist.): Let $X \sim \text{Uniform}([a,b])$, $a<b$.

$$f(x|a,b) = \begin{cases} \frac{1}{b-a} & , x \in [a,b] \\ 0 & , \text{otherwise} \end{cases}$$

Remarks: $E(X) = \frac{b+a}{2}$, $V(X) = \frac{(b-a)^2}{12}$

Recall: $\Gamma(\alpha) := \int_0^\infty t^{\alpha-1} e^{-t} dt$, $\alpha > 0$

change-of-var $t = x/\beta$, $\beta > 0$:

$$\Gamma(\alpha) = \int_0^\infty (x/\beta)^{\alpha-1} e^{-x/\beta} (1/\beta) dx$$

$$= \beta^{-\alpha} \int_0^\infty x^{\alpha-1} e^{-x/\beta} dx$$

$$\Rightarrow \Gamma(\alpha)\beta^\alpha = \int_0^\infty x^{\alpha-1} e^{-x/\beta} dx$$

Def (Gamma Dist.): Let $X \sim \text{Gamma}(\alpha, \beta)$ for $\alpha > 0$: shape, $\beta > 0$: scale

$$\boxed{f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, x > 0}$$

Remark: sometimes we use $\delta = 1/\beta$ instead as rate parameter

More on Gamma Function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

$$\Rightarrow \Gamma(\alpha+1) = \int_0^\infty t^\alpha e^{-t} dt$$

$$= -\int_0^\infty t^\alpha de^{-t} \quad \text{via IBP}$$

$$= t^\alpha e^{-t} \Big|_\infty^0 + \int_0^\infty e^{-t} dt^\alpha$$

$$= 0 + \alpha \underbrace{\int_0^\infty e^{-t} t^{\alpha-1} dt}_{\Gamma(\alpha)}$$

so, $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

$$\Gamma(1) = \int_0^\infty e^{-t} dt = e^{-t} \Big|_\infty^0 = 1$$

$\Gamma(2) = 1\Gamma(1) = 1$

$\Gamma(3) = 2\Gamma(2) = 2!$

...

$\Gamma(n) = (n-1)\Gamma(n-1) = \cdots = (n-1)!$

Back to the Gamma dist.:

$$E(X) = \int_0^\infty x f(x|\alpha,\beta) dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x \cdot x^{\alpha-1} e^{-x/\beta} dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \Gamma(\alpha+1)\beta^{\alpha+1}$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \alpha\Gamma(\alpha)\beta^{\alpha+1} = \alpha\beta.$$

Similarly,

$$E(X^2) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \Gamma(\alpha+2)\beta^{\alpha+2}$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} (\alpha+1)\alpha\Gamma(\alpha)\beta^{\alpha+2} = (\alpha+1)\alpha\beta^2$$

$$\Rightarrow V(X) = E(X^2) - E^2(X)$$

$$= (\alpha+1)\alpha\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$$

note: $X \sim Gamma(\alpha, \beta)$. Then,

$$\frac{X}{\beta} \sim Gamma(\alpha, 1)$$

MGF

$$M_X(t) = E(e^{tX}) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty e^{tx} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty \exp\left(\frac{-x-\beta t}{\beta}\right) x^{\alpha-1} dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha}\Gamma(\alpha)\left(\frac{\beta}{1-\beta t}\right)^\alpha, \text{ provided } 1-\beta t > 0$$

$$= (1-\beta t)^{-\alpha}, \; t < 1/\beta$$

consider cdf for $Gamma(\alpha, \beta)$, $\alpha \in \mathbb{N}$, $\forall x > 0$

$$P(X \le x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^x t^{\alpha-1} e^{-t/\beta} dt \quad \text{via IBP}$$

$$= \frac{1}{(\alpha-1)!\beta^\alpha}\int_0^x t^{\alpha-1}(-\beta) de^{-t/\beta}$$

$$= \frac{1}{(\alpha-1)!\beta^\alpha}\left(t^{\alpha-1}\beta e^{-t/\beta}\Big|_x^0 + \beta\int_0^x e^{-t/\beta} dt^{\alpha-1}\right)$$

$$= -\frac{1}{(\alpha-1)!}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-x/\beta} + \frac{1}{(\alpha-2)!\beta^{\alpha-1}}\int_0^x e^{-t/\beta} t^{\alpha-2} dt$$

$$= -P(Y = \alpha-1) + cdf_{Gamma(\alpha-1, \beta)}(x)$$

$Y \sim Poisson(x/\beta)$

recall: $p(y) = e^{-\lambda}\frac{\lambda^y}{y!}$

$$= -P(Y=\alpha-1) - P(Y=\alpha-2) + cdf_{Gamma(\alpha-2,\beta)}(x)$$

$= \cdots$

$$= -P(Y=\alpha-1) - \cdots - P(Y=1) + cdf_{Gamma(1,\beta)}(x) \quad (*)$$

where $cdf_{Gamma(1,\beta)}(x) = \frac{1}{\beta}\int_0^x e^{-t/\beta} dt = -e^{-t/\beta}\Big|_0^x = 1 - e^{-x/\beta}$

so, $(*) = -P(Y=\alpha-1) - \cdots - P(Y=1) + 1 - e^{-x/\beta}$

$= -P(Y=\alpha-1) - \cdots - P(Y=1) - P(Y=0) + 1$

$= P(Y \ge \alpha)$

Therefore, we have shown

$$P(X \le x) = P(Y \ge \alpha) \quad \text{for} \begin{cases} X \sim Gamma(\alpha, \beta) \\ Y \sim Poisson(\frac{x}{\beta}) \end{cases}$$

$$\Leftrightarrow \int_0^x pdf_{Gam}(t)\, dt = \sum_{y=\alpha}^\infty pmf_{Poi}(y)$$

Special cases of Gamma Dist (α, β)

1) Let $\begin{cases} \alpha = p/2, \; p\text{-integer} \\ \beta = 2 \end{cases}$

$$f(x|p) = \frac{1}{\Gamma(\frac{p}{2})2^{p/2}} x^{p/2-1} e^{-x/2}, \; x > 0$$

$$\sim \chi^2_p \sim \underbrace{\sum_{i=1}^p z_i^2 \text{ where } z_i \sim N(0,1)}_{\text{see later}}$$

2) Let $\alpha = 1$, $\beta > 0$, noting that $\Gamma(1) = 1$

$$f(x|\beta) = \frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right), \; x > 0$$

$\sim$ Exponential (β).

⋮

Memoryless Property (like Geom. Dist)

⋮

3) Let $X \sim Exp(\beta)$, $\beta > 0$

Let $Y = X^{1/\gamma}$, $\gamma > 0$

Then $P(Y \le y) = P(X \le y^\gamma)$

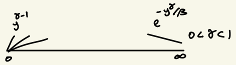
$$= 1 - e^{-y^\gamma/\beta}$$

$\Rightarrow$ Y has pdf

$$f_Y(y) = \frac{d}{dy}P(Y \le y) = \frac{\gamma}{\beta} y^{\gamma-1} e^{-y^\gamma/\beta}, \; 0 < y < \infty$$

$\sim Weibull(\gamma, \beta)$

and useful for modelling extreme/rare events:



3.2b: Continuous Dist. (cont.)

Def (Normal Dist):= $X \sim \text{Normal}(\mu, \sigma^2)$, $\mu \in \mathbb{R}$, $\sigma > 0$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right), \; x \in \mathbb{R}}$$

Remark: Parameters μ: mean, σ^2: variance

Prop (identities):

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx = 1 = \int f(x)dx \qquad (1)$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int x e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx = \mu = \int x f(x) dx \qquad (2)$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int x^2 e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx = \mu^2 + \sigma^2 = E^2(X) + V(X) \qquad (3)$$

$\forall \mu \in \mathbb{R}, \sigma > 0$.

Fact: If $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0,1)$.

We say $Z = \frac{X-\mu}{\sigma}$ is a standard normal r.v.

Proof. $E(Z) = \frac{1}{\sigma}(E(X) - \mu) = \frac{1}{\sigma}(\mu - \mu) = 0$

$V(Z) = V\left(\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right) = \frac{1}{\sigma^2}V(X) = 1$

$$P(Z \le z) = P\left(\frac{X-\mu}{\sigma} \le z\right) = P(X \le \mu + \sigma z)$$

$$= \int_{-\infty}^{\mu+\sigma z} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

letting $x = \mu + \sigma y$

$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}\sigma^2 y^2\right) \sigma \, dy$$

$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) dy$$

the pdf of y is defined as

$$f(z) = \frac{d}{dz} P(Z \le z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \quad Z \sim N(0,1)$$

Proof (1). Need to show $\frac{1}{\sqrt{2\pi\sigma^2}} \int e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx = 1$

By change-of-var, letting $z = \frac{1}{\sigma}(x-\mu)$,

it suffices to prove (1) via

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$$

The integrand is symmetric, so we claim

$$\int_0^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{\frac{\pi}{2}}$$

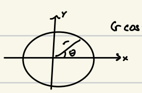
$$\Leftrightarrow \left(\int_0^{\infty} e^{-\frac{1}{2}x^2} dx\right)^2 = \frac{\pi}{2}$$

$$\Leftrightarrow \left(\int_0^{\infty} e^{-\frac{1}{2}x^2} dx\right)\left(\int_0^{\infty} e^{-\frac{1}{2}y^2} dy\right) = \frac{\pi}{2}$$

$$\Leftrightarrow \int_0^{\infty}\int_0^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dx\, dy = \frac{\pi}{2}$$

Another change-of-variables, using polar coords:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \text{for } r \ge 0, \theta \in [0, \pi/2)$$



$(r\cos\theta, r\sin\theta) = (x,y)$

$dx = dr\cos\theta - r\sin\theta\, d\theta$
$dy = dr\sin\theta + r\cos\theta\, d\theta$

$$\frac{dx\,dy}{dr\,d\theta} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \quad \text{(determinant } ad-bc\text{)}$$

Hence $dx\,dy = r\,dr\,d\theta$

So,

$$\int_0^{\pi/2}\int_0^{\infty} e^{-\frac{1}{2}r^2} r\, dr\, d\theta$$

$$= \int_0^{\pi/2} -e^{-\frac{1}{2}r^2}\Big|_0^{\infty} d\theta$$

$$= \int_0^{\pi/2} d\theta = \pi/2.$$

$$\vdots$$

finish