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Stats 510, Instructor: Long Nguyen

Homework 5

Nov 17, due by 11:59pm Nov 25, 2025

1. Let f_{XY} denote the pmf or pdf of a bivariate vector (X, Y). f_X denotes the marginal pmf/pdf of X and $f_{Y|X}$ stands for the conditional pmf/pdf of Y given X.

   (i) Let A and B be subsets of X and Y's domains, respectively, such that $P(X \in A) > 0$. Produce an example in the discrete setting to demonstrate that

   $$P(Y \in B|X \in A) \neq \sum_{x \in A} \sum_{y \in B} f_{Y|X}(y|x).$$

   It suffices if you simply take X and Y to be binary random variables in your example.

   (ii) Derive a correct expression of $P(Y \in B|X \in A)$ in terms of *only* $f_{Y|X}$ and f_X (do this for the discrete setting, and then proceed to the continuous setting).

   (iii) Let $B = \{y\}$, a singleton. Derive $P(Y \in B|X \in A)$ in terms of *only* $f_{X|Y}$ and f_Y (do this for the discrete setting, and then proceed to the continuous setting).

2. Do problems 4.5, 4.9, 4.19, 4.23.

3. Do problems 4.32, 4.34, 4.42.

1. Let f_{XY} denote the pmf or pdf of a bivariate vector (X, Y). f_X denotes the marginal pmf/pdf of X and $f_{Y|X}$ stands for the conditional pmf/pdf of Y given X.

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i) Let $X, Y \in \{0,1\}$ with the following joint pmf:

	Y=0	Y=1
X=0	0.1	0.2
X=1	0.3	0.4

Note that this is a valid pdf:

$f_{XY}(x,y) \geq 0 \ \forall x, y$ and it sums to 1.

Let $A = \{0,1\}$, $B = \{1\}$.

Then $P(Y\in B \mid X \in A) = P(Y=1) = f_{XY}(0,1) + f_{XY}(1,1) = 0.2 + 0.4 = 0.6$ (*)

But $\sum_{x\in A}\sum_{y\in B} f_{Y|X}(y|x) = f_{Y|X}(1|0) + f_{Y|X}(1|1)$

$= \frac{0.2}{0.1+0.2} + \frac{0.4}{0.3+0.4}$

$= \frac{26}{21} \neq$ (*) and an invalid probability.

ii) $P(Y\in B \mid X\in A) = \dfrac{P(X\in A, Y\in B)}{P(X\in A)}$

where $P(X\in A, Y\in B) = \sum_{x\in A}\sum_{y\in B} f_{XY}(x,y)$

and $P(X\in A) = \sum_{x\in A}\sum_{y} f_{XY}(x,y) = \sum_{x\in A} f_X(x)$

So,

$$P(Y\in B\mid X\in A) = \frac{\sum_{x\in A}\sum_{y\in B} f_{XY}(x,y)}{\sum_{x\in A} f_X(x)}$$

where $f_{XY}(x,y) = f_{Y|X}(y|x) f_X(x)$

Thus,

$$P(Y\in B\mid X\in A) = \frac{\sum_{x\in A}\sum_{y\in B} f_{Y|X}(y|x) f_X(x)}{\sum_{x\in A} f_X(x)}$$

$$\boxed{P(Y\in B\mid X\in A) = \sum_{x\in A}\left(\frac{f_X(x)}{\sum_{x'\in A} f_X(x')} \cdot \sum_{y\in B} f_{Y|X}(y|x)\right)}$$

i.e. the weighted average of conditional prob.'s.

For the continuous case,

replacing PMF's w/ PDF's and sums with integrals

$$P(Y\in B\mid X\in A) = \frac{\int_A\int_B f_{XY}(x,y)\,dy\,dx}{\int_A f_X(x)\,dx}$$

$$\boxed{P(Y\in B\mid X\in A) = \frac{\int_A f_X(x)\int_B f_{Y|X}(y|x)\,dy\,dx}{\int_A f_X(x)\,dx}}$$

iii) Letting $B = \{y\}$,

$$P(Y=y\mid X\in A) = \frac{P(Y=y, X\in A)}{P(X\in A)}$$

where $f_{XY}(x,y) = f_{X|Y}(x|y) f_Y(y)$

$$= \frac{\sum_{x\in A} P(X=x\mid Y=y)P(Y=y)}{\sum_{x\in A} P(X\in A)}$$

and $f_X(x) = \sum_{y'} f_{X|Y}(x|y') f_Y(y')$

$$\boxed{P(Y=y\mid X\in B) = \frac{f_Y(y)\sum_{x\in A} f_{X|Y}(x|y)}{\sum_{x\in A}\sum_{y'} f_{X|Y}(x|y') f_Y(y')}}$$

Now assume X, Y - continuous.

To interpret $P(Y=\{y\}\mid X\in A)$, we treat this as a conditional density:

$$f_{Y|X\in A}(y) = \frac{f_Y(y)\int_{x\in A} f_{X|Y}(x|y)\,dx}{\int_{x\in A} f_X(x)\,dx}$$

$$\Rightarrow \boxed{f_{Y|X\in A}(y) = \frac{f_Y(y)\int_A f_{X|Y}(x|y)\,dx}{\int_A\int f_{X|Y}(x|y') f_Y(y)'\,dy'\,dx}}$$

4.5 (a) Find $P(X > \sqrt{Y})$ if X and Y are jointly distributed with pdf

$$f(x, y) = x + y, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1.$$

(b) Find $P(X^2 < Y < X)$ if X and Y are jointly distributed with pdf

$$f(x, y) = 2x, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1.$$

a) $P(X>\sqrt{Y}) = \int_0^1\int_{\sqrt{y}}^1 (x+y)\,dx\,dy = \int_0^1 \left(\frac{x^2}{2} + xy \Big|_{x=\sqrt{y}}^{x=1}\right) dy$

$= \int_0^1 (\frac{1}{2}+y) - (\frac{y}{2} + y^{3/2})\,dy$

$= \frac{y}{2} + \frac{y^2}{2} - \frac{y^2}{4} - \frac{2}{5}y^{5/2}\Big|_0^1$

$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} - \frac{2}{5}$

$= \frac{10}{20} + \frac{10}{20} - \frac{5}{20} - \frac{8}{20}$

$\boxed{= \frac{7}{20}.}$

b) $X^2 < Y < X \Rightarrow \begin{cases} X^2 < Y \Rightarrow X < \sqrt{Y} \\ Y < X \Rightarrow X > Y \end{cases}$

$P(X^2<Y<X) = \int_0^1\int_y^{\sqrt{y}} 2x\,dx\,dy$

$= \int_0^1 (x^2 \Big|_y^{\sqrt{y}})\,dy = \int_0^1 y - y^2\,dy$

$= \frac{y^2}{2} - \frac{y^3}{3}\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \boxed{\frac{1}{6}.}$

4.9 Prove that if the joint cdf of X and Y satisfies

$$F_{X,Y}(x, y) = F_X(x)F_Y(y),$$

then for any pair of intervals (a, b), and (c, d),

$$P(a \leq X \leq b, c \leq Y \leq d) = P(a \leq X \leq b)P(c \leq Y \leq d).$$

$P(a\leq X\leq b, c\leq Y\leq d) = P(X\leq b, c\leq Y\leq d) - P(X\leq a, c\leq Y\leq d)$

$= P(X\leq b, Y\leq d) - P(X\leq b, Y\leq c) - P(X\leq a, Y\leq d) - P(X\leq a, Y\leq c)$

$= F(b,d) - F(b,c) - F(a,d) - F(a,c)$

$= F_X(b)F_Y(d) - F_X(b)F_Y(c) - F_X(a)F_Y(d) - F_X(a)F_Y(c)$

$= P[X\leq b][P(Y\leq d) - P(Y\leq c)] - P(X\leq a)[P(Y\leq d) - P(Y\leq c)]$

$= P(X\leq b)[P(c\leq Y\leq d)] - P(X\leq a)[P(c\leq Y\leq d)]$

$= P(a\leq X\leq b)P(c\leq Y\leq d)$

4.19 (a) Let X_1 and X_2 be independent $\mathrm{n}(0, 1)$ random variables. Find the pdf of $(X_1 - X_2)^2/2$.

(b) If $X_i, i = 1, 2$, are independent $\mathrm{gamma}(\alpha_i, 1)$ random variables, find the marginal distributions of $X_1/(X_1 + X_2)$ and $X_2/(X_1 + X_2)$.

a) Let $X_1, X_2 \sim N(0,1)$ and independent.

Since both X_1 and X_2 are standard normals and independent:

$X_1 - X_2 \sim N(0, \mathrm{Var}(X_1 - X_2)) = N(0, 1+1) = N(0,2)$

Now standardize, letting $Z = \frac{X_1 - X_2}{\sqrt{2}} \Rightarrow Z \sim N(0,1)$

Then

$Y = \frac{(X_1-X_2)^2}{2} = \left(\frac{X_1-X_2}{\sqrt{2}}\right)^2 = Z^2 \sim \chi^2_1.$

So, its pdf is $f(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2}$, $y>0$

b) Let $X_1 \sim \mathrm{Gamma}(\alpha_1, 1)$, $X_2 \sim \mathrm{Gamma}(\alpha_2, 1)$, $X_1 \perp\!\!\!\perp X_2$.

Notice that $\frac{X_2}{X_1+X_2} = 1 - \frac{X_1}{X_1+X_2}$

i.e. both X_1 and X_2 complements of each other on (0,1).

So, define the transformation:

$$\begin{cases} Y_1 = \frac{X_1}{X_1+X_2} \in (0,1) \\ Y_2 = X_1 + X_2 \in (0, \infty) \end{cases}$$

This change-of-variables maps the random pair (X_1, X_2) to (Y_1, Y_2) with the following inverse transformation:

$Y_2 = X_1 + X_2 \Rightarrow X_1 + X_2 = Y_2$

$Y_1 = \frac{X_1}{X_1+X_2} \Rightarrow X_1 = Y_1 Y_2$

$X_2 = Y_2 - X_1 = Y_2 - Y_1Y_2 = Y_2(1-Y_1)$

$$\begin{cases} X_1 = Y_1Y_2 \\ X_2 = (1-Y_1)Y_2 \end{cases}$$

Find the Jacobian determinant:

$$J = \begin{vmatrix} \partial x_1/\partial y_1 & \partial x_1/\partial y_2 \\ \partial x_2/\partial y_1 & \partial x_2/\partial y_2 \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{vmatrix} = y_2(1-y_1) + y_1y_2 = y_2$$

Since $X_1 \perp X_2$, the joint PDF is

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{\Gamma(\alpha_1)} x_1^{\alpha_1-1}e^{-x_1} \cdot \frac{1}{\Gamma(\alpha_2)} x_2^{\alpha_2-1}e^{-x_2}$$

Now, substituting $x_1 = y_1y_2$, $x_2 = (1-y_1)y_2$:

$$f_{X_1,X_2}(y_1y_2, (1-y_1)y_2) = \frac{(y_1y_2)^{\alpha_1-1}e^{-y_1y_2}}{\Gamma(\alpha_1)} \cdot \frac{[(1-y_1)y_2]^{\alpha_2-1}e^{-(1-y_1)y_2}}{\Gamma(\alpha_2)}.$$

Computing the joint PDF:

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(y_1y_2, (1-y_1)y_2)\cdot y_2$$

$$= \frac{y_1^{\alpha_1-1}(1-y_1)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \cdot y_2^{\alpha_1+\alpha_2-1}e^{-y_2}$$

$$= \underbrace{\left[\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y_1^{\alpha_1-1}(1-y_1)^{\alpha_2-1}\right]}\cdot\underbrace{\left[\frac{1}{\Gamma(\alpha_1+\alpha_2)} y_2^{\alpha_1+\alpha_2-1}e^{-y_2}\right]}$$

$\Rightarrow Y_1 \sim \mathrm{Beta}(\alpha_1, \alpha_2)$, $Y_2 \sim \mathrm{Gamma}(\alpha_1+\alpha_2, 1)$

Thus, $\boxed{\frac{X_1}{X_1+X_2} \sim \mathrm{Beta}(\alpha_1, \alpha_2)}$

It can be shown that $1 - \frac{X_1}{X_1+X_2} = \boxed{\frac{X_2}{X_1+X_2} \sim \mathrm{Beta}(\alpha_2, \alpha_1)}$:

PDF of $X \sim \mathrm{Beta}(\alpha_1, \alpha_2)$ is

$$f_X(x) = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha_1-1}(1-x)^{\alpha_2-1}, \quad 0<x<1$$

Define $Y = 1 - X \Rightarrow Y^{-1} = X = 1 - Y$, $\left|\frac{dx}{dy}\right| = 1$.

Then $f_Y(y) = f_X(1-y)\cdot 1 = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}(1-y)^{\alpha_1-1}y^{\alpha_2-1} \sim \mathrm{Beta}(\alpha_2, \alpha_1).$ ∎

4.23 For X and Y as in Example 4.3.3, find the distribution of XY by making the transformations given in (a) and (b) and integrating out V.

(a) $U = XY,\ V = Y$
(b) $U = XY,\ V = X/Y$

Let $X \sim \text{Beta}(\alpha, \beta)$, $Y \sim \text{Beta}(\alpha+\beta, \gamma)$, $X \perp\!\!\!\perp Y$.

The joint pdf of (X, Y) is

$$f_{XY}(x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} y^{\alpha+\beta-1}(1-y)^{\gamma-1} \quad \Bigg| \begin{matrix} 0<x<1 \\ 0<y<1 \end{matrix} \qquad (*)$$

a) Find the distribution of $U=XY$ by transforming to variables (U,V), computing the joint density $f_{U,V}(u,v)$, and integrating out V to find marginal dist. of U.

Define transformation $:= \begin{cases} U = XY \\ V = Y \end{cases} \Rightarrow$ inverse transformation $:= \begin{cases} X = U/V \\ Y = V \end{cases}$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

Now substitute $x = (u/v)$ and $y = v$ into $(*)$ and compute

$$f_{U,V}(u,v) = f_X\left(\tfrac{u}{v}\right) \cdot f_Y(v) \cdot \left|\tfrac{1}{v}\right| \quad \text{since } X \perp\!\!\!\perp Y$$

$$= \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{u}{v}\right)^{\alpha-1}\left(1-\frac{u}{v}\right)^{\beta-1}\right] \cdot \left[\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} v^{\alpha+\beta-1}(1-v)^{\gamma-1}\right] \cdot \frac{1}{v}$$

where $\begin{cases} \left(\frac{u}{v}\right)^{\alpha-1}\left(1-\frac{u}{v}\right)^{\beta-1} \cdot \frac{1}{v} = u^{\alpha-1} v^{\beta-1} \\ \left(1-\frac{u}{v}\right)^{\beta-1} = \left(\frac{v-u}{v}\right)^{\beta-1} = (v-u)^{\beta-1} v^{-(\beta-1)} \end{cases}$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(v-u)^{\beta-1}(1-v)^{\gamma-1}, \quad 0<u<v<1 : \begin{cases} \text{constrained by } x \in (0,1),\ \frac{u}{v} \in (0,1) \Rightarrow 0<u<v \\ \text{constrained by } y \in (0,1),\ 0<v<1 \end{cases}$$

Next, $f_U(u) = \int_u^1 f_{UV}(u,v)\,dv = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1} \int_u^1 (v-u)^{\beta-1}(1-v)^{\gamma-1}\,dv$ via substitution:

letting $y = \frac{v-u}{1-u} \Rightarrow v = y(1-u)+u \Rightarrow dv = (1-u)\,dy$

when $\begin{matrix} v=u \Rightarrow y=0 \\ v=1 \Rightarrow y=1 \end{matrix}$ $\Bigg| \begin{matrix} v-u = y(1-u) \\ 1-v = 1-y+yu+u \\ = (1-y)(1-u) \end{matrix}$

Rewriting the integral:

$$\int_0^1 [y(1-u)]^{\beta-1}[(1-y)(1-u)]^{\gamma-1}(1-u)\,dy$$

$$= (1-u)^{\beta-1+\gamma-1+1}\int_0^1 y^{\beta-1}(1-y)^{\gamma-1}\,dy$$

$$= (1-u)^{\beta+\gamma-1} \cdot B(\beta,\gamma) = (1-u)^{\beta+\gamma-1} \cdot \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$$

$$\Rightarrow f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1}$$

$$\Rightarrow \boxed{U = XY \sim \text{Beta}(\alpha, \beta+\gamma)}$$

b) Find the distribution of $U=XY$ by transforming to variables (U,V), computing the joint density $f_{U,V}(u,v)$, and integrating out $V = X/Y$ to find marginal dist. of U.

From the transformation $:= \begin{cases} U = XY \\ V = X/Y \end{cases}$

we can find the inverse transformation $:= \begin{cases} X = \sqrt{UV} \\ Y = \sqrt{U/V} \end{cases}$ since: $\Bigg| \begin{matrix} UV = XY \cdot \frac{X}{Y} = X^2 \Rightarrow X = \sqrt{UV} \\ V = \frac{X}{Y} \Rightarrow Y = \frac{X}{V} = \frac{\sqrt{UV}}{V} = \sqrt{\frac{U}{V}} \end{matrix}$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{1}{2}\sqrt{v/u} & \frac{1}{2}\sqrt{u/v} \\ \frac{1}{2}\sqrt{1/uv} & -\frac{1}{2}\sqrt{u/v^3} \end{vmatrix} = \frac{-1}{4v} - \frac{1}{4v} = \frac{-1}{2v} \Rightarrow |J| = \frac{1}{2v}.$$

$$f_{U,V}(u,v) = f_{X,Y}(\sqrt{uv}, \sqrt{u/v}) \cdot \frac{1}{2v}$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\left[\sqrt{uv}^{\,\alpha-1}(1-\sqrt{uv})^{\beta-1}\sqrt{\tfrac{u}{v}}^{\,\alpha+\beta-1}\left(1-\sqrt{\tfrac{u}{v}}\right)^{\gamma-1}\right]\frac{1}{2v}, \quad 0<u<v<1/u<1$$

The set $\{0<x<1, 0<y<1\}$ is mapped onto the set $\{0<u<v<\frac{1}{u}, 0<u<1\}$:

constrained by $x \in (0,1)$, $x = \sqrt{uv} \in (0,1) \Rightarrow 0<uv<1 \Rightarrow v<1/u$

constrained by $y \in (0,1)$, $y = \sqrt{u/v} \in (0,1) \Rightarrow 0<u/v<1 \Rightarrow u<v$

so, for $u \in (0,1)$, $u<v<1/u$

Then

$$f_U(u) = \int_u^{1/u} f_{U,V}(u,v)\,dv = \underbrace{\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1}}_{A} \int_u^{1/u}\left(\frac{1-\sqrt{uv}}{1-u}\right)^{\beta-1}\left(\frac{1-\sqrt{u/v}}{1-u}\right)^{\gamma-1}\frac{(\sqrt{u/v})^{\beta}}{2v(1-u)}\,dv \quad \text{via substitution:}$$

letting $z = \frac{\sqrt{u/v}-u}{1-u} \Rightarrow \sqrt{u/v} = (1-u)z + u \Rightarrow v = \frac{u}{[(1-u)z+u]^2}$

$\Rightarrow dz = \frac{-\sqrt{u/v}}{2(1-u)v}\,dv$ and $\Bigg| \begin{matrix} v=u \rightarrow z=1 \\ v=1/u \rightarrow z=0 \end{matrix}$

$$= A\int z^{\beta-1}(1-z)^{\gamma-1}\,dz, \text{ i.e. the kernel of Beta}(\beta,\gamma)$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1}\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$$

$$\Rightarrow f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1}$$

$\Rightarrow \boxed{U = XY \sim \text{Beta}(\alpha, \beta+\gamma)}$ (as seen in (a))

4.32 (a) For the hierarchical model

$$Y|\Lambda \sim \text{Poisson}(\Lambda) \quad \text{and} \quad \Lambda \sim \text{gamma}(\alpha, \beta)$$

find the marginal distribution, mean, and variance of Y. Show that the marginal distribution of Y is a negative binomial if α is an integer.

(b) Show that the three-stage model

$$Y|N \sim \text{binomial}(N, p), \quad N|\Lambda \sim \text{Poisson}(\Lambda), \quad \text{and} \quad \Lambda \sim \text{gamma}(\alpha, \beta)$$

leads to the same marginal (unconditional) distribution of Y.

a) Suppose $Y|\Lambda \sim \text{Poisson}(\Lambda)$
$\Lambda \sim \text{Gamma}(\alpha, \beta)$.

For $y = 0, 1, 2, \dots$ the marginal dist. for Y is:

$$f_Y(y) = \int_0^\infty f_{Y|\Lambda}(y|\lambda) f_\Lambda(\lambda)\,d\lambda$$

$$= \int_0^\infty \frac{\lambda^y e^{-\lambda}}{y!} \cdot \frac{\lambda^{\alpha-1}e^{-\lambda/\beta}}{\Gamma(\alpha)\beta^\alpha}\,d\lambda$$

$$= \frac{1}{y!\Gamma(\alpha)\beta^\alpha}\underbrace{\int_0^\infty \lambda^{y+\alpha-1}\exp\left(-\lambda\left(1+\tfrac{1}{\beta}\right)\right)d\lambda}$$

kernel of the Gamma dist: $\int_0^\infty \lambda^{c-1}e^{-d\lambda} = \frac{\Gamma(c)}{d^c}$

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where $\Bigg| \begin{matrix} c = y+\alpha \\ d = 1+\frac{1}{\beta} = \frac{\beta+1}{\beta} \end{matrix}$

$$= \frac{1}{y!\Gamma(\alpha)\beta^\alpha} \cdot \frac{\Gamma(y+\alpha)}{\left(\frac{\beta+1}{\beta}\right)^{y+\alpha}}$$

$$= \frac{\Gamma(y+\alpha)}{y!\Gamma(\alpha)}\left(\frac{\beta}{\beta+1}\right)^y\left(\frac{1}{\beta+1}\right)^\alpha$$

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$$= \binom{y+\alpha-1}{y}\left(\frac{\beta}{\beta+1}\right)^y\left(\frac{1}{\beta+1}\right)^\alpha.$$

If α is a positive integer, then

$$\boxed{Y \sim \text{NegBin}\left(\alpha, p = \tfrac{1}{1+\beta}\right)}$$

Using the law of total expectation,

$$\boxed{E(Y) = E(E(Y|\Lambda)) = E(\Lambda) = \alpha\beta}$$

Similarly, using the law of total variance,

$$\text{var}(Y) = E[\text{var}(Y|\Lambda)] + \text{var}[E(Y|\Lambda)]$$
$$= E(\Lambda) + \text{var}(\Lambda)$$
$$= \alpha\beta + \alpha\beta^2$$
$$\boxed{\text{var}(Y) = \alpha\beta(1+\beta)}$$

b) Suppose $Y|N \sim \text{Binomial}(N, p)$
$N|\Lambda \sim \text{Poisson}(\Lambda)$
$\Lambda \sim \text{Gamma}(\alpha, \beta)$

Show that this 3-stage model leads to the same marginal (unconditional) dist. of Y

For $y = 0, 1, \dots$

$$P(Y=y|\Lambda=\lambda) = \sum_{n=y}^\infty P(Y=y|N=n, \Lambda=\lambda)P(N=n|\Lambda=\lambda)$$

where $\begin{cases} P(Y=y|N=n,\Lambda=\lambda) = \binom{n}{y}p^y(1-p)^{n-y} \\ P(N=n|\Lambda=\lambda) = \frac{e^{-\lambda}\lambda^n}{n!} \end{cases}$

$$= \sum_{n=y}^\infty \binom{n}{y}p^y(1-p)^{n-y} \cdot \frac{e^{-\lambda}\lambda^n}{n!}$$

where $\binom{n}{y} = \frac{n!}{y!(n-y)!}$

$$= \sum_{n=y}^\infty \frac{1}{y!(n-y)!}p^y(1-p)^{n-y} \cdot e^{-\lambda}\lambda^n$$

via change-of-var: letting $m = n-y$
$\Rightarrow n = m+y$

$$= \frac{p^y e^{-\lambda}\lambda^y}{y!}\sum_{m=0}^\infty \frac{[(1-p)\lambda]^m}{m!}$$

recall (power series expansion): $e^a = \sum_{m=0}^\infty \frac{a^m}{m!}$

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where sum is Taylor expansion of $e^{(1-p)\lambda}$

$$= \frac{p^y e^{-\lambda}\lambda^y}{y!} \cdot e^{(1-p)\lambda}$$

$$= \frac{(p\lambda)^y}{y!}e^{-p\lambda}$$

i.e. the PMF of the Poisson dist. w/ mean $p\lambda$:

$$Y|\Lambda = \lambda \sim \text{Poisson}(p\lambda)$$

Now integrate over the Gamma prior on $\Lambda \sim \text{Gamma}(\alpha, \beta)$:

$$f_Y(y) = \int_0^\infty P(Y=y|\lambda) \cdot f_\Lambda(\lambda)\,d\lambda$$

where $\begin{cases} f_\Lambda(\lambda) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\lambda^{\alpha-1}e^{-\lambda/\beta} \\ P(Y=y|\lambda) = \frac{(p\lambda)^y}{y!}e^{-p\lambda} \end{cases}$

$$= \frac{p^y}{y!\Gamma(\alpha)\beta^\alpha}\int_0^\infty \underbrace{\lambda^{y+\alpha-1}e^{-\lambda(p+\frac{1}{\beta})}}$$

kernel of Gamma dist

$$= \Gamma(y+\alpha) \cdot \left(\frac{\beta}{1+p\beta}\right)^{y+\alpha}$$

$$= \frac{1}{y!\Gamma(\alpha)(p\beta)^\alpha} \cdot \Gamma(y+\alpha) \cdot \left(\frac{p\beta}{1+p\beta}\right)^{y+\alpha}$$

where, again if α is a positive integer, then

$$\boxed{Y \sim \text{NegBinom}\left(\alpha, \tfrac{1}{1+p\beta}\right)}$$

4.34 (a) For the hierarchy in Example 4.4.6, show that the marginal distribution of X is given by the *beta-binomial distribution,*

$$P(X=x)=\binom{n}{x}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(x+\alpha)\Gamma(n-x+\beta)}{\Gamma(\alpha+\beta+n)}.$$

(b) A variation on the hierarchical model in part (a) is

$$X|P \sim \text{negative binomial}(r,P) \quad \text{and} \quad P\sim \text{beta}(\alpha,\beta).$$

Find the marginal pmf of X and its mean and variance. (This distribution is the *beta-Pascal.*)

a) Suppose

$X|P \sim \text{Binomial}(P)$, $i=1,\dots,n$

$P\sim \text{Beta}(\alpha,\beta)$

$EX = E[E(X|P)] = E(nP) = n\frac{\alpha}{\alpha+\beta}$

Show that the marginal dist. follows a beta-binomial distribution:

$$P(X=x)=\binom{n}{x}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(x+\alpha)\Gamma(n-x+\beta)}{\Gamma(\alpha+\beta+n)}$$

Start w/ the conditional PMF:

$$P(X=x|P=p)=\binom{n}{x}p^x(1-p)^{n-x}.$$

we also know for $P\sim\text{Beta}(\alpha,\beta)$:

$$f_P(p)=\frac{1}{B(\alpha,\beta)}p^{\alpha-1}(1-p)^{\beta-1}=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}$$

Now

$$P(X=x)=\binom{n}{x}\int_0^1 p^x(1-p)^{n-x}\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}dp$$

$$=\binom{n}{x}\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\underbrace{\int_0^1 p^{x+\alpha-1}(1-p)^{n-x+\beta-1}dp}$$

FS Beta Function: $=\int_0^1 p^{a-1}(1-p)^{b-1}dp=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

where here $a=x+\alpha$, $b=n-x+\beta$

$$=\binom{n}{x}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(x+\alpha)\Gamma(n-x+\beta)}{\Gamma(n+\alpha+\beta)}.$$

b) Suppose $X|P\sim \text{NegBin}(r,P)$, $P\sim\text{Beta}(\alpha,\beta)$

Start with

$$f_X(x)=\int_0^1 f_{X|P}(x|p)f_P(p)\,dp$$

where $\begin{cases} f_{X|P}(x|p)=\binom{r+x-1}{x}p^r(1-p)^x \\ \text{Beta}:=f_P(p)=\frac{1}{B(\alpha,\beta)}p^{\alpha-1}(1-p)^{\beta-1}\end{cases}$

$$=\int_0^1\binom{r+x-1}{x}p^r(1-p)^x\cdot\frac{1}{B(\alpha,\beta)}p^{\alpha-1}(1-p)^{\beta-1}dp$$

$$=\binom{r+x-1}{x}\cdot\frac{1}{B(\alpha,\beta)}\int_0^1 p^{r+\alpha-1}(1-p)^{x+\beta-1}dp$$

where $\int_0^1 p^{a-1}(1-p)^{b-1}dp=B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

$$=\binom{r+x-1}{x}\cdot\frac{1}{B(\alpha,\beta)}\cdot B(r+\alpha,x+\beta)$$

$$\boxed{f_X(x)=\underbrace{\binom{r+x-1}{x}\cdot\frac{\Gamma(r+\alpha)\Gamma(x+\beta)}{\Gamma(r+\alpha+x)}\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}, \quad x=0,1,\dots}_{\text{Beta-Pascal Dist.}}}$$

Now use the law of iterated expectation and variance:

$$\boxed{EX=E[E(X|P)]=E\left[r\cdot\frac{1-P}{P}\right]=r\cdot E\left[\frac{1-P}{P}\right]=\frac{r\beta}{\alpha-1}}$$

Since $E\left[\frac{1-P}{P}\right]=\int_0^1\left(\frac{1-p}{p}\right)\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}dp$

$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 p^{\alpha-1-1}(1-p)^{\beta+1-1}dp$$

$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha-1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta)}=\frac{\beta}{\alpha-1}$$

Similarly,

$$\boxed{\begin{aligned}\text{var}X&=E[\text{var}(X|P)]+\text{var}[E(X|P)]\\&=E\left[\frac{r(1-P)}{P^2}\right]+\text{var}\left(\frac{r(1-P)}{P}\right)\\&=r\frac{(\beta+1)(\alpha+\beta)}{\alpha(\alpha-1)}+r^2\frac{\beta(\alpha+\beta-1)}{(\alpha-1)^2(\alpha-2)}\end{aligned}}$$

Since $E\left[\frac{1-P}{P^2}\right]=\int_0^1\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-2-1}(1-p)^{\beta+1-1}dp=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha-2)\Gamma(\beta+1)}{\Gamma(\alpha+\beta-1)}=\frac{(\beta+1)(\alpha+\beta)}{\alpha(\alpha-1)}$

and $\text{var}\left(\frac{1-P}{P}\right)=E\left[\left(\frac{1-P}{P}\right)^2\right]-\left[E\left(\frac{1-P}{P}\right)\right]^2=\frac{\beta(\beta+1)}{(\alpha-2)(\alpha-1)}-\left(\frac{\beta}{\alpha-1}\right)^2=\frac{\beta(\alpha+\beta-1)}{(\alpha-1)^2(\alpha-2)}$

where $E\left[\left(\frac{1-P}{P}\right)^2\right]=\int_0^1\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha-2)\Gamma(\beta+2)}{\Gamma(\alpha-2+\beta+2)}=\frac{\beta(\beta+1)}{(\alpha-2)(\alpha-1)}$

4.42 Let X and Y be independent random variables with means μ_X,μ_Y and variances σ_X^2, σ_Y^2. Find an expression for the correlation of XY and Y in terms of these means and variances.

we want $\rho_{XY,Y}=\frac{\text{cov}(XY,Y)}{\sigma_{XY}\sigma_Y}$.

First compute the covariance:

$$\begin{aligned}\text{cov}(XY,Y)&=E[XY\cdot Y]-E[XY]\cdot E[Y]=E(XY^2)-EX\,EY\,EY, \quad X\perp Y\\&=EX\,EY^2-EX\,EY\,EY, \quad X\perp Y\\&=\mu_X E(Y^2)-\mu_X\mu_Y^2\\&=\mu_X[E(Y^2)-\mu_Y^2]\\&=\mu_X\sigma_Y^2\end{aligned}$$

Next, compute

$$\text{var}(XY)=E[(XY)^2]-(E(XY))^2$$

where $E[(XY)^2]=E(X^2)E(Y^2)=(\sigma_X^2+\mu_X^2)(\sigma_Y^2+\mu_Y^2)$

and $[E(XY)]^2=(\mu_X\mu_Y)^2$

$$\Rightarrow \text{var}(XY)=(\sigma_X^2+\mu_X^2)(\sigma_Y^2+\mu_Y^2)-\mu_X^2\mu_Y^2$$
$$=\sigma_X^2\sigma_Y^2+\mu_X^2\sigma_Y^2+\sigma_X^2\mu_Y^2$$

Therefore,

$$\boxed{\rho_{XY,Y}=\frac{\text{cov}(XY,Y)}{\sqrt{\text{var}(XY)}\cdot\sigma_Y}=\frac{\mu_X\sigma_Y^2}{\sigma_Y\cdot\sqrt{\sigma_X^2\sigma_Y^2+\mu_X^2\sigma_Y^2+\sigma_X^2\mu_Y^2}}}$$