week 10

3-2b: continuous Distributions

Showed that $\sqrt{\pi} = \sqrt{2}\int_0^\infty e^{-\frac{1}{2}x^2}dx = \frac{1}{\sqrt{2}}\int_{-\infty}^\infty e^{-\frac{1}{2}x^2}dx$] Prove/show

Also obtain $\Gamma(1/2) = \sqrt{\pi}$

<u>Recall</u>: Let $X \sim N(0,1)$. Then $Y := X^2 \sim \chi^2_{df=1}$

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2}$$
$$= \frac{1}{\Gamma(\frac{1}{2})\, 2^{1/2}} y^{-1/2} e^{-y/2} \quad \text{since } \Gamma(1/2) = \sqrt{\pi}$$

which is pdf for $Gamma(\alpha = \frac{1}{2}, \beta = 2)$

<u>Recall</u> (Poisson Approx.): $X \sim Binomial(n,p)$. Then $\begin{cases} E(X) = np \\ V(X) = np(1-p) \end{cases}$

Let $\begin{cases} n \to \infty, p \to 0 \\ np \to \lambda \end{cases}$, Then $X \xrightarrow{d} Poisson(\lambda)$.

<u>Normal Approx.</u> when $\begin{cases} np \to \infty \\ np(1-p) \to \infty \end{cases}$

then $\underline{X \approx Normal(np, np(1-p))}$

<u>Remark</u>: We will later show via CLT that

$$\frac{1}{\sqrt{n}}(X - np) \xrightarrow{d} Normal(0, p_*(1-p_*)) \quad \text{if } \begin{cases} n \to \infty \\ p \to p_* \end{cases}$$

$$\Rightarrow \frac{1}{\sqrt{n}}(X - np) \approx Normal(0, p(1-p))$$
$$X - np \approx Normal(0, np(1-p))$$
$$X \approx Normal(np, np(1-p))$$

<u>Ex</u>: Let $X \sim Binomial(n=25, p=0.6)$

Then $E(X) = np = 15$
$V(X) = np(1-p) = 2.45^2$ * Poisson approx. not appropriate; why?

Now use Normal approx:

$P(X \le 13) \approx P(Y \le 13)$, $Y \sim N(15, 2.45)$

$= P(Z \le \frac{13-15}{2.45})$, $Z \sim N(0,1)$

$= P(Z \le -0.82) = 0.206$

vs. direct computation, i.e.

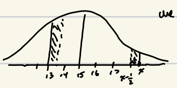
$P(X \le 13) = \sum_{x=0}^{13} \binom{25}{x} 0.6^x (0.4)^{25-x} = 0.267$

Better yet, use the <u>continuity correction</u>:

$P(X \le 13)$, $x \in \mathbb{N}$

why $\approx P(Y \le 13.5)$, $Y \in \mathbb{R}$

$= P(Z \le \frac{13.5-15}{2.45}) = 0.271$



we use $\begin{cases} P(X \le x) \approx P(Y \le x + 1/2) \\ P(X \ge x) \approx P(Y \ge x - 1/2) \end{cases}$

. without this correction, we tend to underestimate prob.'s at the tails

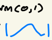
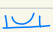
3-2c: continuous Distributions

<u>Def (Beta Dist.)</u>: = Let $X \sim Beta(\alpha, \beta)$, $\alpha, \beta > 0$

$$\boxed{f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \text{ for } x \in (0,1)}$$

<u>Remarks</u>: <u>Beta Function</u>: $= B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}$

<u>Fact</u>: $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \Rightarrow Beta(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

<u>Beta pdfs for different (α, β)</u>:
- $\alpha = \beta = 1 \Rightarrow Beta(1,1) \equiv Uniform(0,1)$
- $\alpha > 1, \beta > 1 \Rightarrow$ Beta is unimodal 
- $\alpha < 1, \beta < 1 \Rightarrow$ Beta is 
- $\alpha < \beta$  or 
- $\alpha > \beta$  or 

(n-th) moments: $E(X^n) = \frac{1}{B(\alpha,\beta)} \int_0^1 x^n x^{\alpha-1}(1-x)^{\beta-1}\,dx$

$$= \frac{1}{B(\alpha,\beta)} B(\alpha+n, \beta)$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+n)\Gamma(\beta)}{\Gamma(\alpha+\beta+n)}$$

$$= \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(\alpha+\beta+n)}$$

$n=1$: $E(X) = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(\alpha+\beta+1)}$

where $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$

$$= \frac{\Gamma(\alpha+\beta)\,\alpha\,\Gamma(\alpha)}{\Gamma(\alpha)(\alpha+\beta)\Gamma(\alpha+\beta)} = \frac{\alpha}{\alpha+\beta}$$

Similarly, $E(X^2) = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+2)}{\Gamma(\alpha)\Gamma(\alpha+\beta+2)} = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$

So, $V(X) = E(X^2) - E^2(X) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2}$

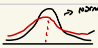
$$= \frac{\alpha}{(\alpha+\beta)} \frac{(\alpha+1)(\alpha+\beta) - \alpha(\alpha+\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Def (Cauchy Dist): Let $X \sim \text{Cauchy}(\theta)$, $\theta \in \mathbb{R}$.

$$\boxed{f(x) = \frac{1}{\pi}\frac{1}{1+(x-\theta)^2},\ x \in \mathbb{R}}$$

Remarks: · Cauchy pdf also bell-shaped:



· θ is the median (location) of X

Recall: $\begin{cases} E|X| = \infty \\ E(X) \text{ DNE} \end{cases}$

Prop. Let $X, Y \overset{iid}{\sim} N(0,1)$. Then $\frac{X}{Y} \sim \text{Cauchy}(0)$

Def (Log-Normal Dist): If $Y \sim N(\mu, \sigma^2)$, then $X = e^Y \sim \text{LogNormal}$, i.e.
$Y = \log(X) \sim N(\mu, \sigma^2)$ s.t.

$$\boxed{f(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{x} \exp\left(\frac{-1}{2\sigma^2}(\log(x) - \mu)^2\right)}$$

3-3: Exponential Families

Def: A family of pdf or pmf is called an exponential family if it has the form

$$\boxed{f(x|\theta) = c(\theta)\,h(x) \exp\left\{ \sum_{i=1}^{k} w_i(\theta)\,t_i(x) \right\}},\ x \in \mathcal{X}$$

$$= c(\theta)\,h(x) \exp\langle w(\theta), t(x)\rangle \quad (*)$$

Remarks: where $w(\theta) = (w_1(\theta), \dots, w_k(\theta))$
$t(x) = (t_1(x), \dots, t_k(x))$
↳ vector of sufficient statistics (coming later)

Here,
$h(x) \geq 0$
$t(x) = (t_1(x), \dots, t_k(x))$ depends only on x (data)
$w(\theta) = (w_1(\theta), \dots, w_k(\theta))$ depends only on θ (parameters)

Clearly, $c(\theta)$ is the reciprocal of the normalizing constant, i.e.

$$\int f(x|\theta) = 1 \Rightarrow \boxed{\frac{1}{c(\theta)} = \int_{x \in \mathcal{X}} h(x)\, e^{\langle w(\theta), t(x)\rangle}\,dx} \text{ (or sum if discrete)}$$

Ex (Binomial): Let $X \sim \text{Binomial}(n, p)$.

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x},\ x = 0, 1, \dots, n \quad \theta = p$$

$$= (1-p)^n \binom{n}{x}\left(\frac{p}{1-p}\right)^x$$

where $e^{x \ln\left(\frac{a}{b}\right)} = \left(\frac{a}{b}\right)^x$

$$= \underbrace{(1-p)^n}_{c(\theta)} \underbrace{\binom{n}{x}}_{h(x)} \exp\left\{ \underbrace{x}_{t_1(x)} \underbrace{\log \frac{p}{1-p}}_{w_1(\theta)} \right\} \quad k=1$$

__Ex (Normal)__: Let $X \sim N(\mu, \sigma^2)$.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right), \quad x \in \mathbb{R}$$

$$\theta = (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(x^2 - 2x\mu + \mu^2)\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\mu^2/2\sigma^2} \exp\left\{\underbrace{\frac{-1}{2\sigma^2}}_{w_1(\theta)} \underbrace{x^2}_{t_1(x)} + \underbrace{\frac{\mu}{\sigma^2}}_{w_2(\theta)} \underbrace{x}_{t_2(x)}\right\}$$

__Remarks__: other examples of dist.'s which are in the exponential family:

Poisson, Geometric, NegBinom, Gamma, Exponential, LogNormal

Among dist.'s not in this (exp.) family are:

Cauchy, Mixtures

__Thm.__ If X is a RV with pdf/pmf in the exponential family of form (*) then,

$$(1) := E\left[\sum_{i=1}^{k} \frac{\partial w_i(\theta)}{\partial \theta_j} t_i(X)\right] = \frac{\partial}{\partial \theta_j} \log(1/c(\theta)), \quad j = 1, \dots, k$$

$$(2) := \text{Var}\left(\sum_{i=1}^{k} \frac{\partial w_i(\theta)}{\partial \theta_j} t_i(X)\right) = \frac{\partial^2}{\partial \theta_j^2} \log(1/c(\theta)) - E\left(\sum_{j=1}^{k} \frac{\partial^2 w_i(\theta)}{\partial \theta_j^2} t_i(X)\right)$$

*what!

__Remark__: In words, differentiating the logarithm of the normalizing constant w.r.t. a parameter results in suitable expectations of sufficient statistics

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__Corollary (Simplified)__: Suppose

$$w(\theta) = \theta = (\theta_1, \theta_2, \dots, \theta_k) \in \mathbb{R}^k$$
$$t(x) = (t_1(x), t_2(x), \dots, t_k(x)) \in \mathbb{R}^k$$

then $f(x|\theta) = c(\theta) h(x) \exp\langle \theta, t(x) \rangle$ and

$$\boxed{\begin{aligned} (1) &:= E(t_j(X)) = \frac{\partial}{\partial \theta_j} \log(1/c(\theta)) \\ (2) &:= \text{Var}(t_j(X)) = \frac{\partial^2}{\partial \theta_j^2} \log(1/c(\theta)) \end{aligned}}$$

__Proof.__ proceeds simply via calculus

__Ex (Binomial)__: $f_X(x|p) = \underbrace{(1-p)^n}_{c(\theta)} \underbrace{\binom{n}{x}}_{h(x)} \exp\left\{\underbrace{x}_{t_1(x)} \underbrace{\log \frac{p}{1-p}}_{w_1(\theta)}\right\}$, $k = 1$

$$c(\theta) = (1-p)^n, \quad \theta = p$$

$$\frac{\partial}{\partial \theta} \log(1/c(\theta)) = \frac{-\partial}{\partial p} n \log(1-p) = \frac{n}{1-p}$$

$$\frac{\partial}{\partial \theta} w_1(\theta) = \frac{\partial}{\partial p}(\log p - \log(1-p)) \quad ?$$

$$= \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}$$

Applying the thm:

$$\frac{\partial}{\partial \theta} \log(1/c(\theta)) = E \frac{\partial}{\partial \theta} w_1(\theta) t_1(X)$$

$$\frac{n}{1-p} = E \frac{1}{p(1-p)} x$$

$$n = E \frac{x}{p}$$

$$\Rightarrow E(X) = np$$

__Ex (Normal)__: Given $f(x) = \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-\mu^2}{2\sigma^2}\right)}_{c(\theta)} \exp\left(\underbrace{\frac{-1}{2\sigma^2}}_{w_1(\theta)} \underbrace{x^2}_{t_1(x)} + \underbrace{\frac{\mu}{\sigma^2}}_{w_2(\theta)} \underbrace{x}_{t_2(x)}\right)$

$$w(\theta) = \left(\frac{-1}{2\sigma^2}, \frac{\mu}{\sigma^2}\right)$$

$$\frac{\partial}{\partial \mu} \log(1/c(\theta)) = \frac{\partial}{\partial \mu}\left(\log\left(\sqrt{2\pi\sigma^2} + \frac{\mu^2}{2\sigma^2}\right)\right)$$

$$= \frac{\mu}{\sigma^2}$$

$$\frac{\partial}{\partial \mu} w_2(\theta) = \frac{\partial}{\partial \mu}\left(\frac{\mu}{\sigma^2}\right) = \frac{1}{\sigma^2}$$

Applying the thm.

$$\frac{\partial}{\partial \mu} \log\left(\frac{1}{c(\theta)}\right) = E\left(\frac{\partial}{\partial \mu} w_2(\theta)\right) t_2(X) + \left(\underbrace{\frac{\partial}{\partial \mu} \cancel{w_1}(\theta)}_{0}\right) t_1(X)$$

$$\frac{m}{\sigma^2} = \frac{1}{\sigma^2}E(X)$$

$$\Rightarrow E(X) = m.$$

To obtain $E(t_2(X)) = EX^2$:

$$\frac{\partial}{\partial\sigma}\log\left(\frac{1}{c(\theta)}\right) = \frac{\partial}{\partial\sigma}\left(\log\sqrt{2\pi\sigma^2} + \frac{m^2}{2\sigma^2}\right)$$
$$= \frac{1}{\sigma} - m^2\frac{1}{\sigma^3}$$

$$\frac{\partial}{\partial\sigma}w_1(\theta) = \frac{\partial}{\partial\sigma}\left(-\frac{1}{2\sigma^2}\right) = \frac{1}{\sigma^3}$$

$$\frac{\partial}{\partial\sigma}w_2(\theta) = \frac{\partial}{\partial\sigma}\left(\frac{m}{\sigma^2}\right) = -2m\frac{1}{\sigma^3}$$

Applying thm:

$$\frac{\partial}{\partial\sigma}\log\left(\frac{1}{c(\theta)}\right) = E\left(\frac{\partial}{\partial\sigma}w_1(\theta)\right)t_1(X) + \left(\frac{\partial}{\partial\sigma}w_2(\theta)\right)t_2(X)$$

$$\frac{1}{\sigma} - m^2\frac{1}{\sigma^3} = \frac{1}{\sigma^3}EX^2 - \frac{2m}{\sigma^3}E(X)$$

$$\sigma^2 - m^2 = EX^2 - 2m^2$$

Hence $EX^2 = \sigma^2 + m^2$

$$\Rightarrow V(X) = E(X^2) - E^2(X)$$
$$= EX^2 - m^2$$
$$= \sigma^2.$$

3.4: Location and Scale Families

Fact: If $f(x)$ is a valid pdf on $\mathbb{R}$,
then $\forall\, m \in \mathbb{R}, \sigma > 0$,
the function
$$g(x|m,\sigma) := \frac{1}{\sigma}f\left(\frac{x-m}{\sigma}\right), x \in \mathbb{R}$$
is a valid pdf on $\mathbb{R}$

Proof. $g \geq 0$ b/c $f \geq 0$.
Check integral: $\int_{\mathbb{R}} g(x|m,\sigma)dx = \int_{\mathbb{R}} \frac{1}{\sigma}f\left(\frac{x-m}{\sigma}\right)dx$

letting $y = \frac{x-m}{\sigma} \Rightarrow x = m + \sigma y$
$\Rightarrow dx = \sigma dy$

$$= \int_{\mathbb{R}} \frac{1}{\sigma}f(y)\sigma dy = \int_{\mathbb{R}} f(y)dy = 1. \blacksquare$$

Def: Let $f(x)$ be any pdf (on $\mathbb{R}$).
The family of pdf $\left\{g(x|m,\sigma) := \frac{1}{\sigma}f\left(\frac{x-m}{\sigma}\right) \,\middle|\, \begin{matrix} m \in \mathbb{R} \\ \sigma > 0\end{matrix}\right\}$
is called a location-scale family of distributions for $\begin{cases} m: \text{location param.} \\ \sigma: \text{scale param.}\end{cases}$

Special cases: 1) $\{g(x|m) := f(x-m) \mid m \in \mathbb{R}\}$ is location family
2) $\{g(x|\sigma) := \frac{1}{\sigma}g\left(\frac{x}{\sigma}\right) \mid \sigma > 0\}$ is scale family

Prop. If $X \sim f$, then $X + m \sim g(x) = f(x-m)$

If $X \sim f$, then $\sigma X \sim g(x) = \frac{1}{\sigma}f\left(\frac{x}{\sigma}\right)$

If $X \sim f$, then $\sigma X + m \sim g(x) = \frac{1}{\sigma}f\left(\frac{x-m}{\sigma}\right)$

Ex(1): Let $f(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)$. Then, the l-s family is:

$$\left\{g(x|m,\sigma) = \frac{1}{\sigma}f\left(\frac{1}{\sigma}(x-m)\right) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right) \,\middle|\, \begin{matrix} m \in \mathbb{R} \\ \sigma > 0\end{matrix}\right\}$$

Ex(2): Let $f(x) = \frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}$; then

$$\left\{g(x|m,\sigma) = \frac{1}{\Gamma(\alpha)}\frac{1}{\sigma}\left(\frac{x-m}{\sigma}\right)^{\alpha-1}e^{-\frac{x-m}{\sigma}} \,\middle|\, \begin{matrix} m \in \mathbb{R} \\ \sigma > 0\end{matrix}\right\}$$

NOT a Gamma family (incorrect support)

However,
$$\left\{g(x|\sigma) = \frac{1}{\Gamma(\alpha)}\frac{1}{\sigma}\left(\frac{x}{\sigma}\right)^{\alpha-1}e^{-x/\sigma} \,\middle|\, \sigma \in \mathbb{R}_+\right\}$$
$$= \{\text{Gamma}(\alpha,\sigma) \mid \sigma \in \mathbb{R}_+\}$$

Ex(3): Let $f(x) = e^{-\lambda}\frac{\lambda^x}{x!}$, $x = 0,1,\dots$ ($X \sim \text{Poisson}(\lambda)$)

Define $g(x\mid\sigma) = \frac{1}{\sigma}e^{-\lambda}\frac{\lambda^{x/\sigma}}{(x/\sigma)!}$ for $\underbrace{x=\sigma y}_{\text{?}}, y\in\mathbb{N}, \sigma>0$

Then, g is a valid pmf supported by multiples of σ.

note: not a Poisson family but inherits many properties of Poisson dist.

> **Thm.** Suppose Y is a RV w/ pdf $f(y)$ and $E(Y), V(Y)$ exist.
> If X is a RV w/ pdf $\frac{1}{\sigma}f\left(\frac{x-\mu}{\sigma}\right)$, then $\boxed{\begin{cases} E(X) = \sigma E(Y) + \mu \\ V(X) = \sigma^2 \text{Var}(Y)\end{cases}}$

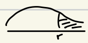
Proof (sketch): If $Y \sim f$, let $Z := \mu + \sigma Y \sim \frac{1}{\sigma}f\left(\frac{z-\mu}{\sigma}\right)$ and $Z \overset{d}{=} X$.

$$\Rightarrow \begin{cases} EX = EZ = \mu + \sigma EY \\ VX = VZ = \sigma^2 \text{Var}(Y)\end{cases}$$

3.5: Inequalities and Identities

- when we can't calculate prob's, it is important to estimate them by inequalities to obtain bounds

> **Thm (Chebychev's Inequality):** Let X be a RV, $g(x) \ge 0\ \forall x$
> Then, $\boxed{P(g(X)\ge r) \le \frac{E(g(X))}{r}\quad \forall r>0}$



$$\Leftrightarrow Eg(X) \ge r\cdot P(g(X)\ge r)$$

Proof. $Eg(X) = \int_{-\infty}^{\infty} g(x)f_X(x)\,dx = \int_{g(x)\ge r} + \underbrace{\int_{g(x)<r}}_{\ge 0 \text{ since } g(x)\ge 0}$

$$\ge \int_{x:g(x)\ge r} g(x)f(x)\,dx$$

where $g(x)\ge r$

$$\ge \int_{x:g(x)\ge r} r f(x)\,dx$$

$$= rP(g(X)\ge r)$$

and divide both sides by r to conclude. ∎

Ex: Let $g(x) = \frac{(x-\mu)^2}{\sigma^2}$ where $\begin{cases}\mu = E(X)\\ \sigma^2 = V(X)\end{cases}$

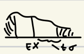
Then, $P\left(\frac{(X-\mu)^2}{\sigma^2}\ge r\right) \le \frac{1}{r}E\frac{(X-\mu)^2}{\sigma^2}$

$= \frac{1}{r}$?

write $r = t^2$, $t>0$ to obtain

$\underline{P(|X-\mu|\ge t\sigma)\le \frac{1}{t^2}}$

$\underline{P(|X-\mu|< t\sigma)\ge 1-\frac{1}{t^2}}$



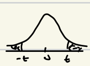
Take $t=2$: $P(|X-\mu|\ge 2\sigma)\le 1/4 = 25\%$

$t=3$: $P(|X-\mu|\ge 3\sigma)\le 1/9 = 11.1\%$

$t=4$: $P(|X-\mu|\ge 4\sigma)\le 1/16 = 6.25\%$

...

> **Thm.** (tighter inequality for normal tails)
> If $Z\sim N(0,1)$, then for $t>0$: $\boxed{P(|Z|\ge t)\le\sqrt{\frac{2}{\pi}}\frac{e^{-t^2/2}}{t}}$



Remark: If $t=2$, $P(|Z|\ge 2)\le\sqrt{2/\pi}\,\frac{e^{-2}}{2} = 0.054 \ll 1/4$ (from above Chebyshev)

> **Corollary:** In general, $\underbrace{\sqrt{\frac{2}{\pi}}\frac{e^{-t^2/2}}{t}}_{\text{exponentially small}} \ll \underbrace{\frac{1}{t^2}}_{\text{polynomially true}}$

Proof. $P(|Z|\ge t) = 2P(Z\ge t)$ by symmetry of f_Z

$$= 2\int_t^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\,dz$$

$$\le 2\int_t^{\infty}\frac{1}{\sqrt{2\pi}}\frac{z}{t}e^{-\frac{1}{2}z^2}\,dz$$

$$= \frac{2}{\sqrt{2\pi}}\frac{1}{t}e^{-\frac{1}{2}z^2}\Big|_{\infty}^{t}$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{t}e^{-\frac{1}{2}t^2}$$ ∎