1.3 Conditional probabilities

Def. if A, B are events in sample space S and $P(B) > 0$

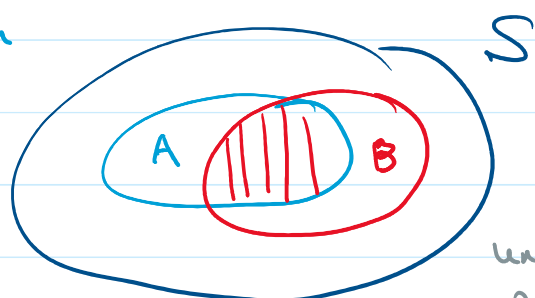
Then the conditional probability of A given B is

$$\boxed{P(A|B) := \frac{P(A \cap B)}{P(B)}}$$

Remark.

- We also say, $P(A \text{ true} \mid B \text{ true}) = \dfrac{P(\text{both } A, B \text{ are true})}{P(B \text{ is true})}$

- Venn diagram



"Renormalizing" under (new) information B.

- Equivalently $P(A \cap B) = P(B)\, P(A|B)$

colloquially, "joint prob = marginal × conditional"

- if $P(A) > 0$ $\quad P(B \cap A) = P(A) \times P(B|A)$

$$= P(B) \cdot P(A|B)$$

Hence

$$\boxed{P(A|B) = \frac{P(B|A)\; P(A)}{P(B)}}$$

BAYES' FORMULA.

EXAMPLE

Four cards are dealt from the top of a well-shuffled deck. Then

$$P(\text{the 4 cards are aces}) = \frac{1}{\#\ \text{4-card hands}} = \frac{1}{\binom{52}{4}}$$

$$= \frac{4 \cdot 3 \cdot 2 \cdot 1}{52 \cdot 51 \cdot 50 \cdot 49}$$

But we may also write

$$P(\text{the 4 cards are aces})$$
$$= P(\text{1st card is ace}) \times P(\text{2nd card is ace} \mid \text{first ace})$$
$$\times P(\text{3rd card is ace} \mid \text{first two are aces})$$
$$\times P(\text{4th card is ace} \mid \text{first three are aces})$$

$$= \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49} \qquad \square$$

CONDITIONAL PROBABILITIES Can "SURPRISE" us

[due to how they are defined and how we use them]

EXAMPLE : "THREE Prisoners"

Three prisoners A, B and C

One of them is chosen at random to be pardoned.

A asks the warden, who is supposed to keep secret:

- which among B and C will be executed?

to which, the warden think for a while, and says:

- B is to be executed.

Warden's thinking: The probability that either A or B or C gets pardon is 1/3.
Between B and C at least one is executed.
So, I gave A no new information on his life.

A's thinking: Great news! Since either C or I gets the pardon, My chance of being alive has gone up to 1/2.

Let A, B, C denote the event A, B, C gets pardon, Resp.
Then $P(A) = P(B) = P(C) = 1/3$.

if Prisoner pardoned is	then, Warden tells A	Prob.
A	$\rightarrow$ B dies	1/2
	$\rightarrow$ C dies	1/2
B	$\rightarrow$ C dies	1
C	$\rightarrow$ B dies	1

Let W denote the event "warden tells A that B dies"

Then warden's Reasoning is Captured by

$$P(A \mid W) = \frac{P(A \cap W)}{P(W)} = \frac{P(A)\,P(W \mid A)}{P(W)}$$

$$= \frac{(1/3) \times (1/2)}{(1/3) \times (1/2) + (1/3) \times 1} = \frac{1/6}{1/2} = \frac{1}{3}$$

A's reasoning Comes from

$A \subset B^c$

$$P(A \mid B^c) = \frac{P(A \cap B^c)}{P(B^c)} = \frac{P(A)}{1 - P(B)} = \frac{1/3}{2/3} = \frac{1}{2}$$

$\boxed{\text{WHO is RIGHT ?}}$

REMARKS

- Suppose that A heard from an unprompted speaker announcing that B is to be executed. Should he be more hopeful?

- Conditional probabilities provide the means to quantify how we may have different assessments of the same event (due to different information we use or have access to)

  what event / information we choose to condition on can affect the answer!

- Conditional probabilities provide the mathematical machinery of Bayesian Statistics, which bases the inference on the posterior distribution, i.e., cond. prob. of quantities of interest conditionally given the observed data.

- Conditional probabilities are obtained usually via the Bayes' formula.