2.2 Expectation

Def. The expectation of a Random variable $g(X)$ is

$$\boxed{\mathbb{E}\, g(X) := \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x)\, dx, & \text{if } X \text{ continuous} \\ \sum_{x \in \mathcal{X}} g(x) \underbrace{f_X(x)}_{P(X=x)}, & \text{if } X \text{ discrete} \end{cases} \quad (*)}$$

Remark

- Also known as "expected value", "average" of a RV or of the probability distribution of the RV ($g(X)$)

- What is the expectation of X?
  if $X \in \mathbb{R}$ then by letting $g(x) := x$

$$\mathbb{E} X = \begin{cases} \int x f_X(x)\, dx & \text{if } X \text{ cont} \\ \sum_{x \in \mathcal{X}} x f_X(x) & \text{if } X \text{ dis.} \end{cases} \quad (**)$$

- if the domain $\mathcal{X}$ is not a subset of $\mathbb{R}$ (Euclidean space) then $(**)$ may be invalid but a notation of $\mathbb{E}X$ may still be defined via the expectations of $\{\mathbb{E}\, g(X)\}$ in $(*)$.

Examples.

1. $X \sim \text{Exp}(\lambda)$ Exponential distribution

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda}, \quad x \in \mathcal{X} = [0, +\infty)$$

$$\lambda > 0$$

Check

$$\int_{\mathcal{X}} f_X(x)\, dx = \int_0^\infty \frac{1}{\lambda} e^{-x/\lambda}\, dx = e^{-x/\lambda}\Big|_{-\infty}^{\infty} = 1.$$

$$\begin{aligned}
\mathbb{E}X &= \int_0^\infty x \frac{1}{\lambda} e^{-x/\lambda}\, dx \\
&= -\int_0^\infty x \, d\left(e^{-x/\lambda}\right) \\
&= -x e^{-x/\lambda}\Big|_0^\infty + \int_0^\infty e^{-x/\lambda}\, dx \quad \text{integration by part} \\
&= 0 + \lambda e^{-x/\lambda}\Big|_\infty^0 \\
&= \lambda.
\end{aligned}$$

Examples.

2. $X \sim \text{Binomial}(n,p)$ Binomial distribution

$$f_X(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x \in \mathcal{X} = \{0,\dots,n\}$$
$$n \in \mathbb{N}, \quad p \in (0,1).$$

Check

$$\sum_{x \in \mathcal{X}} f_X(x) = \sum_{x=0} \binom{n}{x} p^x (1-p)^{1-x}$$

"binomial identity"

$$= (p + (1-p))^n$$
$$= 1.$$

$$\mathbb{E}X = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

see below

$$= \sum_{x=1}^{n} n \binom{n-1}{x-1} p^x (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{n-x}$$

$y = x-1$

$$= np \sum_{y=0}^{n-1} \binom{n-1}{y} p^y (1-p)^{n-1-y}$$

$$= np$$

Examples.

3. $X \sim$ Cauchy

if

$$f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \quad x \in \mathcal{X} = \mathbb{R}$$

Check

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{1}{1+\tan^2 y}\,d\tan y$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{1}{1+\frac{\sin^2 y}{\cos^2 y}}\frac{1}{\cos^2 y}\,dy$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\,dy = 1.$$

$$E|X| = \int_{-\infty}^{\infty}\frac{1}{\pi}\frac{|x|}{1+x^2}\,dx$$

$$= \int_{0}^{\infty}\frac{1}{\pi}\frac{|x|}{1+x^2}\,dx + \int_{-\infty}^{0}\frac{1}{\pi}\frac{|x|}{1+x^2}\,dx$$

$$= 2\int_{0}^{\infty}\frac{1}{\pi}\frac{x}{1+x^2}\,dx = \lim_{M\to\infty}\frac{1}{\pi}\int_0^M \frac{2x}{1+x^2}\,dx$$

for any $M > 0$

$$\int_0^M \frac{2x}{1+x^2}\,dx = \log(1+x^2)\Big|_0^M = \log(1+M^2) \to \infty \text{ as } M\to\infty.$$

Hence $\mathbb{E}|X| = \infty$

and $\mathbb{E}X$ is undefined (does not exist!)

NOTE: Expectation is associated with a distribution

4. Let X be a Random variable with dist P_X

Let $X_1, \dots, X_n$ are n mutually independent Random variables that have identical distributions as X

i.e. $X_1 \dots X_n$ is an n-iid sample of P_X

The empirical distribution of $\{X_1 \dots X_n\}$ is a a probability distribution, denoted by P_n, such that if $Y \sim P_n$ then

$Y \in \mathcal{Y} = \{X_1, \dots, X_n\}$ and

$P(Y = X_i) = \frac{1}{n}$ for $i = 1, \dots, n$

Thus, Y is discrete (regardless of X), and

$$\mathbb{E}Y = \sum_{y \in \mathcal{Y}} P(Y = y) \times y$$

$$= \frac{1}{n}(X_1 + \dots + X_n)$$

Colloquially, this $\mathbb{E}Y$ is also known as the "average" of the (data) sample $X_1, \dots, X_n$.

"LINEARITY of Expectation"

THM. Let X and Y be two Real-valued R.V.'s for which the expectations exist. Let $a, b, c \in \mathbb{R}$.

(i) $\mathbb{E}(aX + bY + c) = a\mathbb{E}X + b\mathbb{E}Y + c.$

(ii) if $X \geq 0$ almost surely, i.e. $P(X \geq 0) = 1$ then $\mathbb{E}X \geq 0$.

(iii) if $X \geq Y$ almost surely, i.e. $P(X \geq Y) = 1$ then $\mathbb{E}X \geq \mathbb{E}Y$.

(iv) if $P(X \in [a,b]) = 1$ then $\mathbb{E}X \in [a,b]$.

Example.

Recall $X \sim \text{Binomial}(n, p)$

X can be viewed as the number of heads obtained by tossing a coin n times indep. where p = prob. of getting a head.

Let $Y_i = \begin{cases} 1 & \text{if the coin turns at the i-th toss.} \\ 0 & \text{otherwise} \end{cases}$

Then $\mathbb{E}Y_i = 1 \times p + 0 \times (1-p) = p.$

$$X = Y_1 + Y_2 + \dots + Y_n$$

By linearity of expectation

$\mathbb{E}X = \mathbb{E}Y_1 + \dots + \mathbb{E}Y_n = np.$

Proof.

(i) This is not a complete proof, but it conveys the main idea.

Assume X and Y may be expressed as functions of a random variable Z, i.e.,

$$\begin{cases} X = g_1(Z) \\ Y = g_2(Z) \end{cases} \quad (*)$$

Suppose Z is a continuous RV with f_Z.

Then

$$\begin{aligned}
& \mathbb{E}\, aX + bY + c \\
&= \mathbb{E}\, a g_1(Z) + b g_2(Z) + c \\
&= \int \left(a g_1(z) + b g_2(z) + c \right) f_Z(z)\, dz \\
&= a \underbrace{\int g_1(z) f_Z(z)\, dz}_{\mathbb{E} g_1(Z)} + b \underbrace{\int g_2(z) f_Z(z)\, dz}_{\mathbb{E} g_2(Z)} + c \underbrace{\int f_Z(z)\, dz}_{1} \\
&= a \mathbb{E} g_1(Z) + b \mathbb{E} g_2(Z) + c \\
&= a \mathbb{E} X + b \mathbb{E} Y + c.
\end{aligned}$$

linearity of integration

The proof proceeds similarly if Z is discrete.

[The proof is not complete because we assumed $(*)$] $\square$

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Proof of (ii) for the case X is a continuous RV.

$P(X \geq 0) = 1 \Rightarrow P(X<0) = 0$, so $f_X(x) = 0 \ \forall x < 0.$

why?

Hence,

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_{0}^{\infty} x f_X(x)\,dx \geq 0 \quad \text{, since the integrand} \geq 0.$$

(iii) and (iv) is a direct consequence of (i) and (ii).