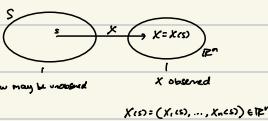


4.6: Multivariate distributions

Def: A random vector denoted $X = (X_1, \dots, X_n)$ is a function that maps elements of a sample space S into \mathbb{R}^n



Now, we may specify probabilities, for $A \subset \mathbb{R}^n$:

$$P(X \in A) := P(S : X(s) \in A)$$

If X takes countably many values, then we say X is a discrete RV, which is associated w/ π .

$$\text{joint probability mass function (pmf)} := f(x) = f(x_1, \dots, x_n) = P(X_1=x_1, \dots, X_n=x_n)$$

$$\text{Hence for } A \subset \mathbb{R}^n, P(X \in A) = \sum_{x \in A} f(x)$$

$$\text{Similarly, pmf: } f(x) = f(x_1, \dots, x_n)$$

$$\text{where } P(X = (x_1, \dots, x_n) \in A) = \int_A f(x_1, \dots, x_n) dx_1 \dots dx_n$$

$$\text{Expectation: } E(g(x)) = \begin{cases} \int_{\mathbb{R}^n} g(x) f(x) dx, & X \text{-cont.} \\ \sum_{x \in \mathbb{R}^n} g(x) \pi(x), & X \text{-discrete} \end{cases}$$

$$\text{marginal dist: } f_{X_1, \dots, X_k | X_{k+1}, \dots, X_n} = \begin{cases} \int_{\mathbb{R}^{n-k}} \dots \int_{\mathbb{R}} f_{X_1, \dots, X_n}(x_1, \dots, x_n) dx_{k+1} \dots dx_n \\ \sum_{x_{k+1}, \dots, x_n} f_{X_1, \dots, X_n}(x_1, \dots, x_n) \end{cases}$$

$$\text{e.g.: } (X_1, \dots, X_n) \sim f_{X_1, \dots, X_n}$$

$$f_{X_1, X_2, Y}(x_1, x_2, y) = \int_{\mathbb{R}^2} f_{X_1, X_2, Y}(x_1, x_2, y_1, y_2) dy_1 dy_2$$

$$\text{conditional dist: } f_{X_1, \dots, X_n | X_1, \dots, X_k}(x_1, \dots, x_n) = \frac{f_{X_1, \dots, X_n}(x_1, \dots, x_n, x_{k+1}, \dots, x_n)}{f_{X_1, \dots, X_k}(x_1, \dots, x_k)}$$

Remarks: - admits factorization: $f_{X_1, \dots, X_n}(x_1, \dots, x_n) = f_{X_1, \dots, X_k}(x_1, \dots, x_k) f_{X_{k+1}, \dots, X_n}(x_{k+1}, \dots, x_n)$

$$\text{chain formula: } f_{X_1, \dots, X_n}(x_1, \dots, x_n) = f_{X_1}(x_1) f_{X_2 | X_1}(x_2 | x_1) f_{X_3 | X_1, X_2}(x_3 | x_1, x_2) \dots f_{X_n | X_1, \dots, X_{n-1}}(x_n | x_1, \dots, x_{n-1})$$

$$\text{e.g.: } f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2 | X_1}(x_2 | x_1) = f_{X_1}(x_1) f_{X_2}(y_1 | x_1) f_{X_2 | X_1}(x_2 | y_1)$$

$$\text{Prop. Suppose } \begin{cases} X_1, \dots, X_n \text{-discrete} \\ Z, T \text{-continuous} \end{cases} \text{ and let } f_{X_1, \dots, X_n, Z, T}(x_1, \dots, x_n, z, t) = f_{X_1}(x_1) f_{X_2 | X_1}(x_2 | x_1) \dots f_{X_n | X_1, \dots, X_{n-1}}(x_n | x_1, \dots, x_{n-1}) f_{Z | X_1, \dots, X_n}(z | x_1, \dots, x_n) f_{T | X_1, \dots, X_n, Z}(t | x_1, \dots, x_n, z)$$

Then, $\forall A, B, C, D \subset \mathbb{R}$,

$$P(X \in A, Y \in B, Z \in C, T \in D) = \sum_{x \in A} \sum_{y \in B} \int_C \int_D f_{X_1, \dots, X_n, Z, T}(x_1, \dots, x_n, z, t) dz dt$$

$$\text{Binomial dist: } \begin{cases} m \text{ independent trials} \\ n \text{ possible outcomes } w_i \\ \text{"cell prob": } (p_1, \dots, p_m) \end{cases} \sum_{i=1}^m p_i^{w_i}$$

Let (X_1, \dots, X_n) : counts of each of n outcomes

$$X_1 + \dots + X_n = m$$

Now, joint pdf for $x_1, \dots, x_n \in \mathbb{R}^n$, $\sum_{i=1}^n x_i = m$

$$f(x_1, \dots, x_n) = \begin{cases} \frac{m!}{x_1! \dots x_n!} p_1^{x_1} \dots p_n^{x_n} & \text{and 0 otherwise} \\ \end{cases}$$

Remarks: By multinomial formula

$$\sum_{x_1+...+x_n=m} (x_1 \dots x_n) p_1^{x_1} \dots p_n^{x_n} = (p_1 + \dots + p_n)^m =$$

Marginal distributions:

$$\begin{aligned} f_{X_1}(x_1) &= \sum_{x_1+...+x_n=m-x_1} \frac{m!}{x_1! \dots x_n!} p_1^{x_1} \dots p_n^{x_n} \\ &= p_1^{x_1} \frac{m!}{(m-x_1)!} \sum_{x_2+...+x_n=m-x_1} p_2^{x_2} \dots p_n^{x_n} \\ &= \frac{m!}{x_1!(m-x_1)!} p_1^{x_1} (p_1 + \dots + p_n)^{m-x_1} \\ &\propto \frac{m!}{x_1!(m-x_1)!} p_1^{x_1} (1-p_1)^{m-x_1} \end{aligned}$$

$\Rightarrow X_1 \sim \text{Binomial}(m, p_1)$.

Conditional distributions

$$\begin{aligned} f_{X_1, \dots, X_m|X_{m+1}=k} &= \frac{f_{X_1, \dots, X_m}(x_1, \dots, x_m)}{f_{X_{m+1}}(k)} \\ &\propto \frac{m!}{x_1! \dots x_m!} p_1^{x_1} \dots p_m^{x_m} \\ &\propto \frac{m!}{x_1! \dots x_m!} (p_1)^{x_1} \dots (p_m)^{x_m} \\ &\propto \frac{(m-x_1)!}{x_1! \dots x_m!} \left(\frac{p_1}{1-p_1}\right)^{x_1} \dots \left(\frac{p_m}{1-p_m}\right)^{x_m} \end{aligned}$$

Hence, $X_1, \dots, X_m | X_{m+1}=k \sim \text{multinomial}(m-k, p)$, cdf prob: $\left(\frac{p_1}{1-p_1}, \dots, \frac{p_m}{1-p_m}\right)$

Notice: - "invariance" property by marginalization and conditioning;

holds for many dist's in the exponential family;

holds for multivariate normal (continuous)

Recall (independence): X and Y independent if $\forall x, y \in \mathbb{R}$, $f_{X,Y}(x,y) = f_X(x)f_Y(y)$

$$\boxed{\text{Def (mutual independence)}: f_{X_1, \dots, X_n}(x_1, \dots, x_n) = \prod_{i=1}^n f_{X_i}(x_i)}$$

- Many properties established for bivariate random vector can be naturally extended

$$\text{Then, } E[g_1(x_1) \dots g_n(x_n)] = E[g_1(x_1)] \dots E[g_n(x_n)]$$

$$\boxed{\text{Thm: If } X_1, \dots, X_n \text{- mutually independent RVs, let } \epsilon = 1, \dots, n \\ \text{then } M_\epsilon(\epsilon) = M_{X_1}(\epsilon) \dots M_{X_n}(\epsilon)}$$

corollary: in particular, if X_1, \dots, X_n iid, then $M_\epsilon(\epsilon) = (M_{X_1}(1))^\epsilon$

Properties

i) If $X_i \sim \text{Gamma}(k_i, \beta)$, X_i - independent.

then $X_1, \dots, X_n \sim \text{Gamma}(k_1 + \dots + k_n, \beta)$

$$\boxed{\text{ii) If } X_i \stackrel{\text{ind}}{\sim} N(\mu_i, \sigma_i^2), \text{ let } a_i, b_i \in \mathbb{R}. \\ \text{then } \mathbf{Z} = \sum_{i=1}^n a_i X_i + b_i \sim N\left(\sum_{i=1}^n a_i \mu_i + b_i, \sum_{i=1}^n a_i^2 \sigma_i^2\right) \\ = \mathbf{a}^T \mathbf{X} + b^T \mathbf{1}}$$

(Lemma: Suppose $(X_1, \dots, X_n) \sim f_{X_1, \dots, X_n}$,

X_1, \dots, X_n are mutually independent r.v.s,

$$\boxed{f_{X_1, \dots, X_n}(x_1, \dots, x_n) = g_1(x_1) \dots g_n(x_n)}$$

for some function g_1, \dots, g_n

Then, if X_1, \dots, X_n are mutually independent,

let $g_i(x_i)$: function of x_i , $i=1, \dots, n$

then $g_1(x_1), \dots, g_n(x_n)$ are also mutually independent