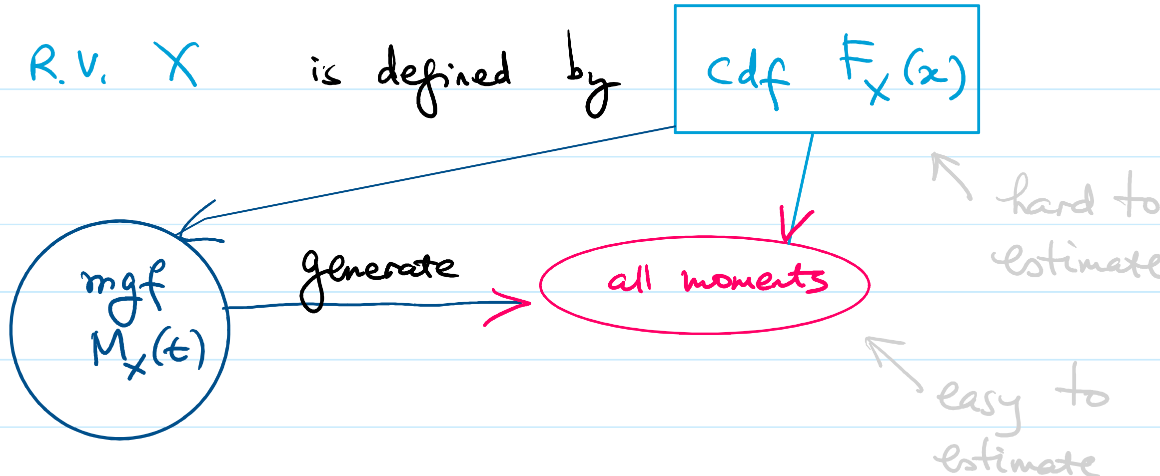
2.3 Moments (cont)



1/ Does the mgf uniquely determine the cdf ?

2/ Does the set of moments $\{\mathbb{E}X^n\}$ uniquely determine the cdf ?

THM.

Let $F_X(x)$ and $F_Y(y)$ be two cdf's all of whose moments exist.

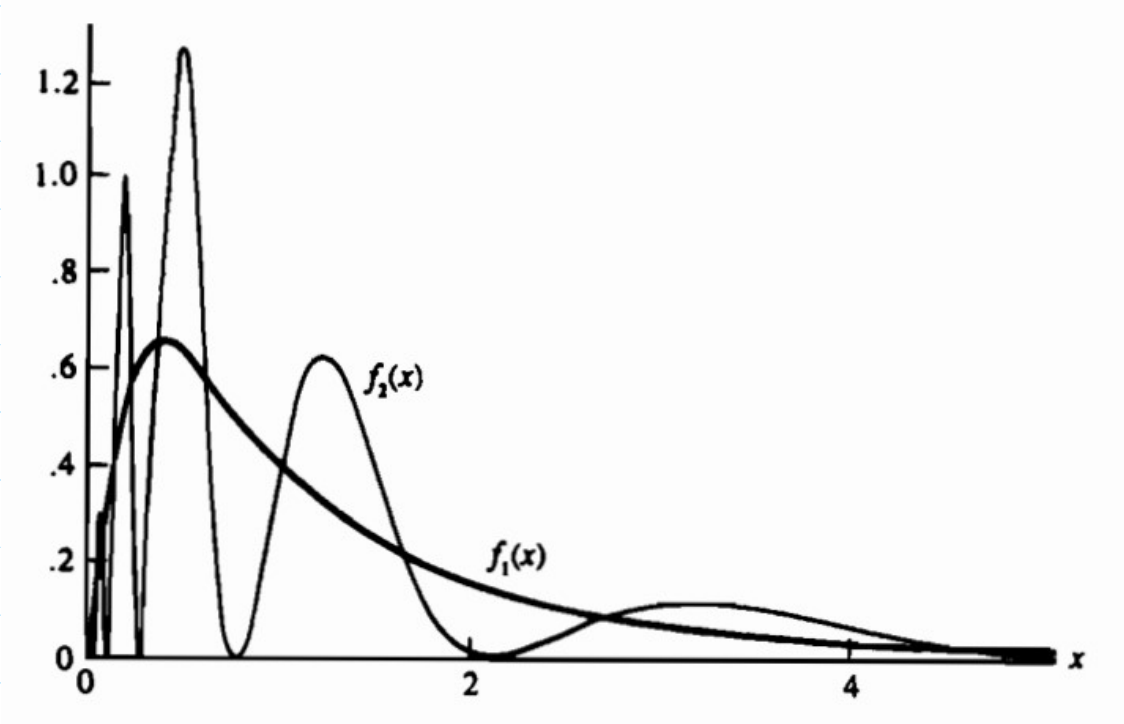
(i) if X and Y have bounded support, then

$$F_X(u) = F_Y(u) \quad \forall u \iff \mathbb{E}X^r = \mathbb{E}Y^r \quad \forall r = 0, 1, 2, \ldots$$

(ii) if $M_X(t)$ and $M_Y(t)$ exist, and $M_X(t) = M_Y(t)$ for all t in some neighborhood of 0, then $F_X(u) = F_Y(u)$ $\forall u$.

what happen if X and Y have unbounded Support?

$X \sim f_1$, $Y \sim f_2$



$$f_1(x) = \frac{1}{\sqrt{2\pi}\, x} e^{-\frac{1}{2}(\log x)^2}, \quad x \in [0, +\infty)$$

$$f_2(x) = f_1(x) \cdot \left(1 + \sin\left(2\pi \log x\right)\right)$$

Then it can be checked that

$$\mathbb{E}X^r = \mathbb{E}Y^r = e^{r^2/2}, \quad r = 0, 1, 2, \ldots$$

The moments do not capture all information about the distribution in the unbounded support scenarios.

Why do we care?

One answer lies in the need to approximate Random variables

Thm (Convergence of MGFs leads to Convergence of cdf)

- Suppose $X_1, X_2, \ldots$ is a sequence of Random Variables, each with mgf $M_{X_i}(t)$.
- Suppose $\lim_{i \to \infty} M_{X_i}(t) \longrightarrow M_X(t)$

for all t in a neighborhood of 0, and $M_X(t)$ a mgf.

THEN $X_i \longrightarrow X$ in distribution,

where X is a Random Variable with mgf $M_X(t)$

i.e. $F_{X_i}(x) \longrightarrow F_X(x)$ at all points x where the cdf F_X is continuous

Proof idea:

- $$M_X(t) = \mathbb{E}\, e^{tX} = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx$$

  is (in fact) the Laplace transform of the pdf $f_X(x)$

- From the theory of Laplace transform

  $$f_X(x) \xrightarrow{\mathcal{L}} M_X(t)$$

  is one-to-one, provided $M_X(t)$ is defined for a neighborhood of 0.

  Hence, f_X is uniquely determined by function M_X

- From here, one can establish a degree of regularity for the inverse map of the Laplace transform.

Example (Poisson approximation)

- Let $X \sim \text{Binomial}(n,p)$

  we know $\begin{cases} \mathbb{E}X = np \\ \operatorname{var}X = np(1-p) \end{cases}$

It is known that as n gets large, X "behaves" like a Poisson variable

- $Y \sim \text{Poisson}(\lambda)$ if

$$f_Y(y) = e^{-\lambda}\frac{\lambda^y}{y!} \quad , y = 0, 1, 2, \dots$$

$$\mathbb{E}Y = \operatorname{var}Y = \lambda.$$

$$M_Y(t) = e^{\lambda(e^t - 1)}.$$

- we will establish that

$$P(X \leq x) \approx P(Y \leq x) \quad \text{as } n \uparrow \infty.$$

$$\forall x \in \mathbb{N}$$

More precisely we also need $p \downarrow 0$

s.t. $np = \text{const} \equiv \lambda.$

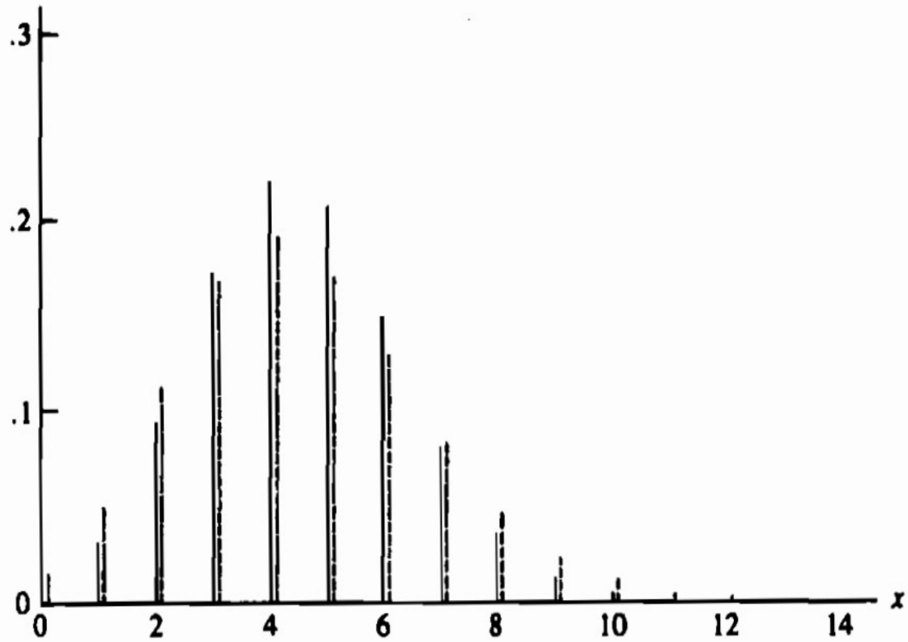


Figure 2.3.3. *Poisson (dotted line) approximation to the binomial (solid line)*, $n = 15$, $p = .3$

• Suppose $\begin{cases} np = \lambda = \text{const} \\ n \uparrow \infty, \quad p \downarrow 0. \end{cases}$

$$M_X(t) = (pe^t + 1 - p)^n$$

$$= (1 + p(e^t - 1))^n$$

$$= \Big(1 + \frac{1}{n}(e^t - 1)\underbrace{np}_{\lambda}\Big)^n$$

$$= \Big(1 + \frac{1}{n}(e^t - 1)\lambda\Big)^{\frac{n}{(e^t-1)\lambda}(e^t-1)\lambda}$$

$$\longrightarrow e^{(e^t - 1)\lambda} \quad \text{as } n \to \infty.$$

Hence $M_X(t) \longrightarrow M_Y(t) \quad \forall t$

By the Convergence theorem of mgf,

if $X_n \sim$ Binomial (n, p), $\lambda = np$. Then

$$F_{X_n}(x) \longrightarrow F_Y(x) \quad \text{as } n \to \infty \text{ for } x \in \mathbb{R}$$

where F_Y is continuous.

For Poisson, F_Y is a step function with Continuity at $y \notin \mathbb{N}$.

for $y \in \mathbb{N}$, $F_Y(y) = F_Y(y + \frac{1}{2})$ Continuous at $y + \frac{1}{2}$.

So, $P(X_n \le x) \longrightarrow P(Y \le x)$ as $n \to \infty$

$\forall x \in N$ (in fact, here $\forall x \in \mathbb{R}$)

Remark Still holds if $X_n \sim$ Binomial (n, p_n) where

$n \to \infty$, $p_n \to 0$ s.t. $np_n \to \lambda$.

Thm. $\forall a, b \in \mathbb{R}$

$$M_{aX+b}(t) = e^{bt} M_X(at)$$

Proof.

$$\begin{aligned} M_{aX+b}(t) &= \mathbb{E}\, e^{t(aX+b)} \\ &= \mathbb{E}\, e^{taX} e^{bt} \\ &= e^{bt}\, \mathbb{E}\, e^{taX} \\ &= e^{bt} M_X(at) \end{aligned}$$

$\square$.