2.4 Tools

INTERCHANGE INTEGRAL AND DIFFERENTIAL

Thm (Leibniz's Rule) if

- $f(x,\theta)$, $a(\theta)$, $b(\theta)$ are differentiable with respect to θ
- $\frac{\partial f}{\partial \theta}(x,\theta)$ is continuous on $x \in (x_0, x_1)$ ← can be weakened

Then for $a(\theta), b(\theta) \in (x_0, x_1)$:

$$\frac{d}{d\theta} \int_{a(\theta)}^{b(\theta)} f(x,\theta)\, dx = f(b(\theta),\theta) \frac{d}{d\theta} b(\theta) - f(a(\theta),\theta) \frac{d}{d\theta} a(\theta) + \int_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial \theta} f(x,\theta)\, dx$$

REMARK

- if $a(\theta) = a$, $b(\theta) = b$ then

$$\frac{d}{d\theta} \int_a^b f(x,\theta)\, dx = \int_a^b \frac{\partial f}{\partial \theta}(x,\theta)\, dx$$

Proof

$$\text{LHS} = \lim_{\delta \to 0} \frac{1}{\delta} \left(\int_{a(\theta+\delta)}^{b(\theta+\delta)} f(x,\theta+\delta)\, dx - \int_{a(\theta)}^{b(\theta)} f(x,\theta)\, dx \right)$$

$$= \lim_{\delta \to 0} \frac{1}{\delta} \left[\int_{a(\theta)}^{b(\theta)} \left(f(x,\theta+\delta) - f(x,\theta) \right) dx + \int_{a(\theta+\delta)}^{a(\theta)} f(x,\theta+\delta)\, dx + \int_{b(\theta)}^{b(\theta+\delta)} f(x,\theta+\delta)\, dx \right]$$

$$+ \int_{a(\theta+\delta)}^{a(\theta)} f(x,\theta+\delta)\,dx + \int_{b(\theta)} f(x,\theta+\delta)\,dx \Big]$$

$$=: A + B + C$$

fixed, finite boundaries

where

$$A = \lim_{\delta\to 0} \frac{1}{\delta} \int_{a(\theta)}^{b(\theta)} f(x,\theta+\delta) - f(x,\theta)\ dx$$

$$= \int_{a(\theta)}^{b(\theta)} \lim_{\delta\to 0} \frac{1}{\delta} \big(f(x,\theta+\delta) - f(x,\theta) \big)\ dx$$

$$= \int_{a(\theta)}^{b(\theta)} \frac{\partial f}{\partial \theta}(x,\theta)\ d\theta$$

(Since $\frac{\partial f}{\partial \theta}$ is cont. in x and next theorem)

$$B = \lim_{\delta\to 0} -\frac{1}{\delta} \int_{a(\theta)}^{a(\theta+\delta)} f(x,\theta+\delta)\ dx$$

$$= \lim_{\delta\to 0} - \frac{a(\theta+\delta)-a(\theta)}{\delta} \times \frac{1}{a(\theta+\delta)-a(\theta)} \int_{a(\theta)}^{a(\theta+\delta)} f(x,\theta+\delta)\ dx$$

$$= - \frac{d}{d\theta} a(\theta) \times f(a(\theta),\theta)$$

($a(\theta)$ is diff)

Likewise

$$C = \frac{d}{d\theta} b(\theta)\ f(b(\theta),\theta) .$$

□

To strengthen the above Result for infinite domain of the integral, we need a new tool

THM (interchanging limit and integral)

Suppose

- $h(x,y)$ is continuous at $y=y_0$ for each fixed x
- There is a function $g(x)$ such that

envelope condition
$$\begin{cases} |h(x,y)| \leq g(x) & \forall x,y \\ \int_{-\infty}^{\infty} g(x)\,dx < \infty. \end{cases}$$

Then
$$\lim_{y\to y_0} \int_{-\infty}^{\infty} h(x,y)\,dx = \int_{-\infty}^{\infty} \lim_{y\to y_0} h(x,y)\,dx$$

Remark

- g is called the dominating OR envelope function of f.
- known as (a version / consequence of) Lebesgue's Dominated Convergence Theorem (D.C.T.)

INTERCHANGE Differential and integral

THM. Suppose

- For each x, $f(x,\theta)$ is differentiable WRT θ, at $\theta \in \Theta$
- For each $\theta \in \Theta$, there is a function $g(\cdot,\theta)$ and $\delta_0 > 0$ such that

envelope condition
$$\begin{cases} \left| \frac{\partial}{\partial\theta} f(x,\theta)\Big|_{\theta=\theta'} \right| \leq g(x,\theta') \quad \forall\, \theta' \in (\theta-\delta_0, \theta+\delta_0) \\ \int_{-\infty}^{\infty} g(x,\theta)\, dx < \infty \quad \text{for each } \theta \in \Theta. \end{cases}$$

Then
$$\frac{d}{d\theta} \int_{-\infty}^{\infty} f(x,\theta)\, dx = \int_{-\infty}^{\infty} \frac{d}{d\theta} f(x,\theta)\, dx.$$
holds for each $\theta \in \Theta$.

Remark.

- Leibniz's Rule holds under this weaker envelope condition for $f(x,\theta)$.
- if we need only to differentiate at $\theta = \theta_0$ Then it is sufficient that the envelope condition be satisfied for a neighborhood of θ_0 i.e. $\forall\, \theta \in (\theta_0 - \delta_0, \theta_0 + \delta_0)$ for some $\delta_0 > 0$, AND only a dominating function $g(x,\theta_0)$ (for fixed $\theta = \theta_0$) is required.
- This THM is a direct consequence of Lebesgue's Dominated convergence Theorem.

Example (Exponential)

$$X \sim \mathrm{Exp}(\lambda)$$

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda} \qquad x \in [0, +\infty)$$

Moments: $$\mathbb{E}X^n = \int_0^\infty \frac{1}{\lambda} x^n e^{-x/\lambda}\, dx \quad , n = 1, 2, \ldots$$

Wish to differentiate

$$\frac{d}{d\lambda} \mathbb{E}X^n = \frac{d}{d\lambda} \int_0^\infty$$

$$= \int_0^\infty \frac{d}{d\lambda} \qquad (*) \text{ need to justify this}$$

$$= \int_0^\infty x^n \left(-\frac{1}{\lambda^2} e^{-x/\lambda} + \frac{1}{\lambda} e^{-x/\lambda} \frac{x}{\lambda^2} \right) dx$$

$$= \int_0^\infty x^n e^{-x/\lambda} \frac{1}{\lambda^2} \left(\frac{x}{\lambda} - 1 \right) dx$$

$$= \int_0^\infty x^{n+1} e^{-x/\lambda} \frac{1}{\lambda^3} - x^n e^{-x/\lambda} \frac{1}{\lambda^2}\, dx$$

$$= \frac{1}{\lambda^2} \mathbb{E}X^{n+1} - \frac{1}{\lambda} \mathbb{E}X^n$$

This gives a Recursion Relation

$$\boxed{\mathbb{E}X^{n+1} = \lambda \mathbb{E}X^n + \lambda^2 \frac{d}{d\lambda} \mathbb{E}X^n.}$$

Similar identities hold for a broad family of distributions

Similar identities hold for a broad family of distributions

justifying the interchange of $\int$ and $\frac{d}{d\lambda}$

the derivative of the integrand is

$$\frac{d}{d\lambda} f(x,\lambda) = x^n e^{-x/\lambda} \frac{1}{\lambda^2} \left(\frac{x}{\lambda} - 1\right)$$

Need to find a dominating function, $g(\cdot,\lambda)$ s.t

① $\left| \frac{d}{d\theta} f(x,\theta)\Big|_{\theta=\theta'} \right| < g(x,\lambda) \quad \forall \theta' \in (\lambda-\delta_0, \lambda+\delta_0)$ for some $\delta_0 > 0$

② and g integrable wrt x.

$$\left| \frac{d}{d\theta} f(x,\theta) \right| = x^n e^{-x/\theta} \frac{1}{\theta^2} \left| \frac{x}{\theta} - 1 \right|$$

$$\leq x^n e^{-x/\theta} \frac{1}{\theta^2} \left(\frac{x}{\theta} + 1\right), \quad \text{since } x > 0$$

$$\leq x^n e^{-x/(\theta+\delta_0)} \frac{1}{(\theta-\delta_0)^2} \left(\frac{x}{\theta-\delta_0} + 1\right) =: g(x,\theta)$$

The last inequality holds $\forall x > 0$, $\forall \theta > \delta_0$.

Thus ① holds for the chosen $g(x,\lambda)$.

To verify ②

$$\int g(x,\lambda)\,dx = \int x^{n+1} e^{-x/\lambda+\delta_0} \frac{1}{(\lambda-\delta_0)^3} + \int x^n e^{-x/\lambda+\delta_0} \frac{1}{(\lambda-\delta_0)^2}\,dx$$

$=$ multiple of $(n+1)$-th moment of an exponential RV

$+$ multiple of n-th moment of another Exp. RV

+ multiple of n-th moment of another exp. RV $< +\infty$. $\square$

Example (Gaussian)

$$X \sim N(0,1)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R}$$

MGF $\quad M_X(t) = \mathbb{E}\, e^{tX} = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$

"Generating Moments" via

$$\frac{d}{dt} M_X(t) = \frac{d}{dt} \mathbb{E}\, e^{tX}$$

$$(*) \quad = \mathbb{E} \frac{d}{dt} e^{tX} = \mathbb{E}\, X e^{tX}.$$

justifying $(*)$ by finding dominating function for

$$\left| \frac{d}{dt} e^{tx} e^{-x^2/2} \right|$$

$$= \left| x e^{tx} e^{-x^2/2} \right| = |x|\, e^{tx} e^{-x^2/2}$$

Let $0 < \delta_0 < t$:

if $x \geq 0$: $\quad |x| e^{tx} e^{-x^2/2} \leq x\, e^{(t+\delta_0)x} e^{-x^2/2}$

if $x < 0$: $\quad |x| e^{tx} e^{-x^2/2} \leq (-x)\, e^{(t-\delta_0)x} e^{-x^2/2}$.

Take the RHS as $g(x,t)$ for $x \geq 0$ and $x < 0$.

Then $\left| \frac{d}{dt} e^{tx} e^{-x^2/2} \Big|_{t=t'} \right| \leq g(x,t) \quad \forall\, t' \in (t-\delta_0, t+\delta_0)$

The domination holds.

Then, one can check that $g(\cdot, t)$ is integrable $\square$

INTERCHANGING SUM AND DIFFRENTIAL

Suppose $\sum_{x=0}^{\infty} h(x,\theta)$ exists (i.e. converges pointwise) $\forall \theta \in (a,b)$

Moreover, assume

1. $\frac{\partial}{\partial \theta} h(x,\theta)$ is continuous in θ for each x
2. $\sum_{x=0}^{\infty} \frac{\partial}{\partial \theta} h(x,\theta)$ converges uniformly for all θ in a closed subinterval of (a,b)

Then

$$\boxed{\frac{d}{d\theta} \sum_{x=0}^{\infty} h(x,\theta) = \sum_{x=0}^{\infty} \frac{\partial}{\partial \theta} h(x,\theta)}$$

Remark.
This is a consequence of Lebesgue's D.C.T.

Example $X \sim \text{Geometric}(\theta)$, $0<\theta<1$

$$P(X=x) = \theta(1-\theta)^x \quad , \quad x = 0,1,2,\ldots$$

this is a valid pmf since

$$\sum_{x=0}^{\infty} \theta(1-\theta)^x = 1$$

(*) Differentiating both sides wrt θ, assuming we can interchange $\sum$ and $\frac{d}{d\theta}$:

$$\sum_{x=0}^{\infty} (1-\theta)^x - \theta x(1-\theta)^{x-1} = 0$$

$$\Rightarrow \underbrace{\sum_{x=0}^{\infty} (1-\theta)^x}_{\frac{1}{1-(1-\theta)}} = \sum_{x=0}^{\infty} \theta x(1-\theta)^{x-1}$$

$$= \frac{1}{1-\theta} \sum_{x=0}^{\infty} \theta x(1-\theta)^x$$

$$\frac{1}{\theta} = \frac{1}{1-\theta} EX$$

Hence $EX = \dfrac{\theta}{1-\theta}$.

justifying (*)

- $h(x,\theta) = \theta(1-\theta)^x$

$\forall \theta \in (0,1) \quad \sum_{x=0}^{k} h(x,\theta) = \theta \sum_{x=0}^{k}(1-\theta)^x = \theta \frac{1-(1-\theta)^{k+1}}{1-(1-\theta)} \to 1$ as $k \to \infty$.

- $\frac{\partial}{\partial\theta} h(x,\theta) = (1-\theta)^x - \theta x (1-\theta)^{x-1}$

continuous in θ

So h is continuously diff. in θ.

- "Uniform convergence" of $\sum_{x=0}^{k} \frac{\partial}{\partial\theta} h(x,\theta)$

Pointwise Convergence: for each $\theta \in (0,1)$

$$S_k(\theta) := \sum_{x=0}^{k} \frac{\partial}{\partial\theta} h(x,\theta) = \sum_{x=0}^{k} (1-\theta)^x - \theta x (1-\theta)^{x-1}$$

$$= \frac{1-(1-\theta)^{k+1}}{\theta} + \theta \sum_{x=0}^{k} \frac{\partial}{\partial\theta} (1-\theta)^x$$

$$= \frac{1-(1-\theta)^{k+1}}{\theta} + \theta \frac{d}{d\theta} \sum_{x=0}^{k} (1-\theta)^x$$

$$= \frac{1-(1-\theta)^{k+1}}{\theta} + \theta \frac{d}{d\theta} \frac{1-(1-\theta)^{k+1}}{\theta}$$

$$= \frac{1-(1-\theta)^{k+1}}{\theta} + \theta \frac{(k+1)(1-\theta)^k \theta - (1-(1-\theta)^{k+1})}{\theta^2}$$

$$= \frac{1-(1-\theta)}{\theta} + \theta \frac{\quad}{\theta^2}$$

$$= (k+1)(1-\theta)^k$$

So $S_k(\theta) \longrightarrow 0$ as $k \to \infty$.

Uniform Convergence: Take any $[c,d] \in (0,1)$

check:

$$\sup_{\theta \in [c,d]} |S_k(\theta) - 0| = \sup_{\theta \in [c,d]} (k+1)(1-\theta)^k$$

$$\longrightarrow 0 \text{ as } k \to \infty$$

$\square$

INTERCHANGING INTEGRAL AND SUM

THM

Suppose $\sum_{x=0}^{\infty} h(x,\theta)$ exists (i.e. converges pointwise) $\forall \theta \in [a,b]$

Moreover, assume

1. $h(x,\theta)$ is continuous in θ for each fixed x
2. $\sum_{x=0}^{\infty} h(x,\theta)$ converges uniformly on $[a,b]$

Then

$$\boxed{\int_a^b \sum_{x=0}^{\infty} h(x,\theta)\, d\theta = \sum_{x=0}^{\infty} \int_a^b h(x,\theta)\, d\theta .}$$

Remark.

This is also a consequence of Lebesgue's D.C.T.