# 4.3 Bivariate transformation

- Let $(X,Y)$ be a bivariate Random vector

  $g : \mathbb{R}^2 \to \mathbb{R}^2$

  $g(x,y) := (g_1(x,y), g_2(x,y)) \in \mathbb{R}^2$

Then $(U,V) := g(X,Y)$ is a Random bivariate vector

$\forall A \in \mathbb{R}^2 \quad P((U,V) \in A) = P((X,Y) \in g^{-1}(A))$

Discrete Case

- if $(X,Y)$ is discrete, then So is $(U,V)$

$$f_{UV}(u,v) = \sum_{\substack{x,y: \\ g_1(x,y)=u \\ g_2(x,y)=v}} f_{XY}(x,y) \qquad (*)$$

Example

Let $X \sim \text{Poisson}(\theta)$

$Y \sim \text{Poisson}(\lambda)$

$X \perp\!\!\!\perp Y$

First way

We know
$$M_X(t) = \mathbb{E}\, e^{tX}$$
$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\theta} \lambda^x}{x!}$$
$$= \sum_{x=0}^{\infty} e^{-\theta} \frac{(e^t)^x}{x!}$$
$$= e^{\theta(e^t - 1)}$$

Hence
$$M_{X+Y}(t) = M_X(t)\, M_Y(t) \quad (\text{since } X \perp\!\!\!\perp Y)$$
$$= e^{\lambda(e^t-1)}\, e^{\theta(e^t-1)}$$
$$= e^{(\lambda+\theta)(e^t-1)},$$

which is the MGF for $\text{Poisson}(\lambda+\theta)$.

Second way

- Write $\begin{pmatrix} u \\ v \end{pmatrix} = g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) := \begin{pmatrix} x+y \\ y \end{pmatrix}$

Then, for $u \geqslant v$

- $$f_{uv}(u,v) = \sum_{\substack{x,y \\ y=v \\ x+y=u}} f_{xy}(x,y)$$

$$= f_{xy}(u-v, v)$$

$$= f_x(u-v)\, f_y(v) \qquad \text{Since } X \perp\!\!\!\perp Y$$

$$= e^{-\theta} \frac{\theta^{u-v}}{(u-v)!} \quad e^{-\lambda} \frac{\lambda^v}{v!}$$

- Obtain $f_U(u)$ by marginalization:

$$f_U(u) = \sum_v f_{UV}(u,v) \quad , \text{ Since } f_{UV}(u,v)=0 \text{ if } v>u$$

$$= e^{-(\theta+\lambda)} \sum_{v=0}^{u} \theta^{u-v} \lambda^v \frac{1}{(u-v)!\, v!}$$

$$= e^{-(\theta+\lambda)} \frac{1}{u!} \underbrace{\sum_{v=0}^{u} \theta^{u-v} \lambda^v \binom{u}{v}}$$

$$= e^{-(\theta+\lambda)} \frac{1}{u!} (\theta+\lambda)^u \quad \text{(binomial formula)}$$

Hence $U = X+Y \sim \text{Poisson}(\theta+\lambda)$.

Continuous case

# Change of variable formula

Let $(X,Y)$ be a continuous bivariate vector

$(X,Y) \sim f_{XY}$

Let $g: \mathbb{R}^2 \to \mathbb{R}^2$ be a one-to-one mapping

i.e. $g^{-1}(\{u,v\})$ has at most one element.

Define $(U,V) = g(X,Y)$, what is $f_{UV}(u,v)$?

Let $A = (\{(x,y)\}, f_{XY}(x,y) > 0)$
support of $f_{XY}$

and
$$B = g(A) := \{ g(x,y) \mid (x,y) \in A \}$$

Write $\begin{pmatrix} u \\ v \end{pmatrix} := g(x,y) := \begin{pmatrix} g_1(x,y) \\ g_2(x,y) \end{pmatrix}$

which has the inverse function

$$\begin{pmatrix} x \\ y \end{pmatrix} = h(u,v) := g^{-1}(u,v) := \begin{pmatrix} h_1(u,v) \\ h_2(u,v) \end{pmatrix}$$

Then the pdf $f_{UV}(u,v)$ is given by the "change of variable" formula

$$\boxed{f_{UV}(u,v) = f_{XY}\left(h_1(u,v),\ h_2(u,v)\right) \, |J|}$$

where $J$ is the determinant of the Jacobian matrix

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{vmatrix}$$

$$= \frac{\partial h_1}{\partial u} \frac{\partial h_2}{\partial v} - \frac{\partial h_1}{\partial v} \frac{\partial h_2}{\partial u} .$$

Example

Let $X \sim N(0,1)$

$Y \sim N(0,1)$ , $X \perp\!\!\!\perp Y$

First way , via MGF

we already proved from Sec 4.2:

$X+Y \sim N(0,2)$

$X-Y \sim N(0,2)$

Second way , via Change of variable formula

Let $\begin{cases} U = X+Y \\ V = X-Y \end{cases}$

the mapping $\begin{cases} u = g_1(x,y) := x+y \\ v = g_2(x,y) := x-y \end{cases}$

is one-to-one , and admits the inverse:

$$\begin{cases} x = h_1(u,v) = \frac{1}{2}(u+v) \\ y = h_2(u,v) = \frac{1}{2}(u-v) \end{cases}.$$

The Jacobian of the transformation $h$:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$\left| \frac{\partial y}{\partial u} \quad \frac{\partial y}{\partial v} \right| \qquad \left| \frac{1}{2} \quad -\frac{1}{2} \right|$$

Hence

$$f_{UV}(u,v) = f_{XY}\left(\tfrac{1}{2}(u+v), \tfrac{1}{2}(u-v)\right) \cdot \frac{1}{2}$$

$X \perp\!\!\!\perp Y$

$$= f_X\left(\tfrac{1}{2}(u+v)\right) f_Y\left(\tfrac{1}{2}(u-v)\right) \cdot \frac{1}{2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{1}{2}(u+v)\right)^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{1}{2}(u-v)\right)^2} \cdot \frac{1}{2}$$

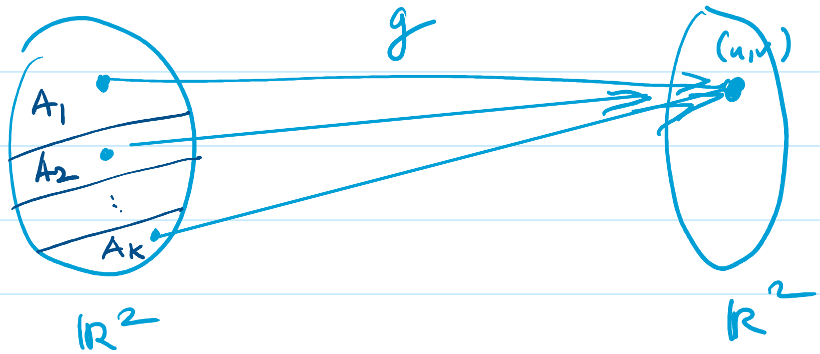
$$= \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{4}u^2} \cdot \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{4}v^2}$$

$f_{UV}(u,v)$ factorizes, so $U \perp\!\!\!\perp V$ and

$U \sim N(0, 2)$

$V \sim N(0, 2)$.

When the transformation $(x,y) \mapsto g(x,y) \in \mathbb{R}^2$ is many-to-one




if we may partition the support of $f_{XY}$

$$A = \{ (x,y) \mid f_{XY}(x,y) > 0 \}$$

into disjoint subsets

$$A = A_1 \cup \dots \cup A_K$$

such that $g$ is one-to-one from each $A_i$ to $g(A_i)$, $i=1,\dots,K$.

Let $h_i = (h_{i1}, h_{i2})$ be the inverse function of $g$ restricted to domains $A_i \to g(A_i)$ and $J_i$ be the corresponding Jacobian

Then,

$$\boxed{f_{UV}(u,v) = \sum_{i=1}^{K} f_{XY}(h_{i1}(u,v), h_{i2}(u,v)) \, |J_i|}$$

Example.

if $X, Y \sim N(0,1)$, $X \perp\!\!\!\perp Y$

what is the distribution of $\begin{cases} U = X/Y \\ V = |Y| \end{cases}$ ?

Note $(x,y)$ and $(-x,-y)$ map to the same $(u,v)$

so partition

$$A = \mathbb{R}^2 = \underbrace{\mathbb{R}\times\mathbb{R}_+}_{A_1} \cup \underbrace{\mathbb{R}\times\mathbb{R}_-}_{A_2} \cup \underbrace{\mathbb{R}\times\{0\}}_{A_3}$$

if $(x,y) \in A_3$ $f_{XY}(x,y) = f_X(x) f_Y(y)$ but $(u,v)$ is not defined. Not of concern,

since $P((X,Y) \in A_3) = 0$.

$\boxed{\text{For } A_1}$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} uv \\ v \end{pmatrix}$

$$J_1 = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

$\boxed{\text{For } A_2}$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -uv \\ -v \end{pmatrix} \Rightarrow J_2 = \begin{vmatrix} -v & -u \\ 0 & -1 \end{vmatrix} = v.$

Hence

$$f_{UV}(u,v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(uv)^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} |v| + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-uv)^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-v)^2} |v|$$

$$f_{UV}(u,v) = \frac{1}{\sqrt{2\pi}} e \quad \frac{1}{\sqrt{2\pi}} e \quad |v| + \frac{1}{\sqrt{2\pi}} e^{-} \quad \frac{1}{\sqrt{2\pi}} e \quad |v|$$

$$= \frac{1}{\pi} e^{-\frac{1}{2}v^2(u^2+1)} v \quad , \text{ for } v > 0.$$

$f_{UV}(u,v) = 0 \quad \forall v < 0.$ Thus,

$$f_U(u) = \int_0^\infty \frac{1}{\pi} e^{-\frac{1}{2}v^2(u^2+1)} v \, dv$$

$$= \frac{1}{\pi(u^2+1)} e^{-\frac{1}{2}v^2(u^2+1)} \Big|_{\infty}^{0}$$

$$= \frac{1}{\pi(u^2+1)} \quad \text{for all } u \in \mathbb{R}$$

$\Rightarrow U = X/Y \sim$ Cauchy (1).