Homework 9; Due Monday, 03/19/2018

Quick Answer Questions. No work needed. No partial credit available.

Question 1. Rewrite the following sets in a simpler form without the use of indexed unions or intersections.

(a) $\displaystyle\bigcap_{n\in\mathbb{N}} \left\{x \in \mathbb{R} \mid x^2 \geq \frac{n}{2}\right\}$

Solution. $\emptyset$

(b) $\displaystyle\bigcap_{p\in\mathbb{P}_5} \{x \in \mathbb{R} \mid p(x) = 0\}$

Solution. $\emptyset$

(c) $\displaystyle\bigcup_{p\in\mathbb{P}_4} \{x \in \mathbb{R} \mid p(x) = 0\}$

Solution. $\mathbb{R}$

(d) $\displaystyle\bigcap_{n\in\mathbb{N}} \{(a,b) \in \mathbb{R}^2 \mid b = a^n\}$

Solution. $\{(0,0),(1,1)\}$

(e) For $a \in \mathbb{R}$, define the element $p_a \in \mathbb{P}_2$ as $p_a(x) = -x^2 + a^2$. Define the set P as

$$P = \{p_a \in \mathbb{P}_2 \mid a \in \mathbb{N} \cup \{0\}\}$$

Compute

$$A = \bigcup_{p\in P} \{r \in \mathbb{R} \mid p(r) = 0\}.$$

Solution. $\mathbb{Z}$

Full Justification Questions. Provide complete justifications for your responses.

Question 2. Let $S \subseteq \mathbb{N}$ be any subset of the natural numbers with the following two properties:

1. $1 \in S$
2. If $k \in S$, then $(k+1) \in S$.

Prove that $S = \mathbb{N}$. (Hint: Use proof by contradiction. Let $T = \mathbb{N} \setminus S$ and consider the minimum, a.k.a. smallest, element of T. You can read about the properties of the smallest element, or as we say the minimal element, in Theorem 3.6 on page 11 of the supplementary document.)

Note that in proving the above statement, you are essentially proving why the Principle of Mathematical Induction works.

Solution. We will use proof by contradiction
Assume $S \neq \mathbb{N}$
Choose $T = \mathbb{N} - S$
Let $t \subseteq T$
$T \neq 1$ since $1 \in S$
So $t > 2$ implies that $t - 1 \notin T$
Since $(t-1) \in S$, then $(t-1) + 1 \in S$
This is our contradiction.
Thus, our assumption is incorrect, so $S = \mathbb{N}$
We have proven that this is a principle of mathematical induction because P is true if $p(1)$ is true and $p(k) \implies p(k+1)$ is true for all $n \in \mathbb{N}$

Question 3. Let $A = \{p \in \mathbb{P}_3 : p(0) = 5\}$ and define a function $f : A \to \mathbb{P}_3$ by the rule

$$f\big(p\big)(x) = xp'(x) + 27.$$

(a) Prove or disprove that f is injective.

Solution. f is Injective

(b) Prove or disprove that f is surjective.

Solution. f is Surjective
Let $p \in \mathbb{P}_3$ be generic
Choose $p(x) = \int \frac{p-27}{x} dx$
Thus, $f(p)(x) = x(\frac{p-27}{x} + 27 = p$
Thus, f is surjective.

(c) If f is a bijection, compute its inverse function.

Solution. $f^{-1}(p) = \int \frac{p-27}{x} dx$

Question 4. For each $n \in \mathbb{N}$, define an interval

$$A_n = \left(-\frac{4}{n}, 1 - \frac{2}{n}\right].$$

Define the sets

$$U = \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad I = \bigcap_{n=1}^{\infty} A_n$$

(a) Give a simpler expression of U and I, including a proof that your expression is correct.

Solution. U= (-4,1) I= $\emptyset$

(b) Determine if U and I are bounded sets.

Solution. U is bounded.

Let $x \in U$ be generic. Choose $M = 5$.

Thus, for all $x \in U, |x| \leq 5$, which shows that U is bounded.

I is bounded.

Let $x \in I$ be generic. Choose $M = 5$.

Thus, for all $x \in I, |x| \leq 5$, which shows that I is bounded

Question 5. Prove, using the definition of convergent sequences, that the sequence with terms $a_n = \dfrac{n-1}{2n^2}$ for $n \in \mathbb{N}$ is convergent.

Solution. Let $L = 0$
Let $\epsilon > 0$ be generic.
Choose $N = \frac{1}{2\epsilon}$
So $|\frac{n-1}{2n^2} - L| = \frac{n-1}{2n^2} < \frac{n}{2n^2}$
$= \frac{1}{2n} = \epsilon$
Thus, $|\frac{n-1}{2n^2} - L| < \epsilon$
Thus, a_n is convergent.

Question 6. Prove, using the definition of convergent sequences, that $\left\{\dfrac{(-1)^{n+2}}{n+3}\right\}_{n\in\mathbb{N}}$ is a convergent sequence.

Solution. Let L = 0
Let $\epsilon > 0$ be generic.
Choose $N > \frac{1}{\epsilon}$
Then for any $n > N$, $|\frac{(-1)^{n+2}}{n+3} - 0| = \frac{1}{n+3} < \frac{1}{n} < \frac{1}{N} < \epsilon$.
The last inequality is true since $N > \frac{1}{\epsilon}$ and $\frac{1}{2N} < \epsilon$

Question 7. Show that the sequence $\left\{\frac{1+(-1)^n}{2}\right\}_{n\in\mathbb{N}}$ is not convergent.

Solution. Let $L \in \mathbb{R}$ be generic
Choose $\epsilon = 0$
Let $N \in \mathbb{N}$ be generic
Now we will do cases on L.
If L ¿ 0 then choose n=2N+1 where 2n+1≥ N
Then $|\frac{1+(-1)^n}{2} - L| = |0 - L| = |-L| = L > \epsilon = 0$
If $L \leq 0$ then choose n=2N where 2N ≥ N
Then $|\frac{1+(-1)^n}{2} - L| = |1_L| = 1 - (-|L|) = 1 + |L| > 0 = \epsilon$
Thus, the sequence is not convergent.