Homework 1; Due Thu, 01/18/2018

Answer-Only Questions. Credit based solely on the answer.

Question 1. Answer the following with true or false.

(a) $\sqrt[3]{3} < \sqrt[2]{2}$

Solution. False

(b) $a^2 + 2a + 1 \geq 0$

Solution. True

(c) $(a + b)^4 = a^4 + b^4$

Solution. False

(d) $(a - b)^2 - b^2 = a(a - 2b)$

Solution. False

(e) $(a + b)^2 \geq a^2 + b^2$

Solution. True

(f) If $a > 0$, $b > 0$ then $(a + b)^2 > a^2 + b^2$

Solution. True

(g) If $c \neq 0$ then $a > b \Rightarrow ac > bc$

Solution. False

(h) $(a^3b^5c^2)^{\frac{1}{2}}a^{-2}c^5\frac{1}{b^3} = \frac{c^6}{\sqrt{ab}}$

Solution. False

Question 2. For each of the following choices of A, B below, answer the following questions:

(i) $|A| = ?$

(ii) $|B| = ?$

(iii) true or false $A \subset B$

(iv) true or false $B \subseteq A$

(v) true or false $A = B$

(a) $A = \{-5, -5, 5, 5\}$, $B = \{-5, 5\}$

Solution. i=2 ii=2 iii=False iv=True v=True

(b) $A = \{a, b, \{c, d\}\}$, $B = \{c, d\}$

Solution. i=3 ii=2 iii=False iv=False v=False

(c) $A = \mathbb{N}$, $B = \mathbb{Z}$

Solution. i= infinite elements ii= infinite elements iii= True iv= False v=False

(d) $A = \{\mathbb{Z}, \{\emptyset\}\}$, $B = \emptyset$

Solution. i=2 ii=0 iii=False iv=True v=False

(e) $A = \{\mathbb{N}\}$, $B = \mathbb{N} \cup \{\mathbb{N}\}$

Solution. i= 1 ii= infinite elements iii= False iv= True v= True

(f) $A = \{\emptyset, \{\emptyset\}\} \cup \emptyset$, $B = \{\mathbb{Q}, \emptyset\} \cup \{\{\emptyset\}\}$

Solution. i=2 ii= infinite elements iii= True iv= False v= False

(g) $A = \mathbb{R} \cap \mathbb{Q}$, $B = \mathbb{N} \cup \mathbb{Z}$

Solution. i= infinite elements ii= infinite elements iii= False iv= True v= False

Question 3. Determine whether the following statements are true or false.

(a) If $n \in \mathbb{N}$, then $n^3 > 0$.

Solution. True

(b) If $k \in \mathbb{Z}$ then $k < n$ for some $n \in \mathbb{Z}$

Solution. False

(c) When $a > 0$ and $b < 0$ then $ba > 0$.

Solution. False

(d) For $x \in \mathbb{R}$, $\sin^2(x) + \cos^2(x) = 1$.

Solution. True

(e) If $a, b, c \in \mathbb{R}$ then $a^2 + b^2 + c^2 \geq ab + bc + ca$ (Hint: multiply both sides by 2)

Solution. True

Full Justification Questions. Provide a full justifications for your responses.

Question 4. Decide when the following conditional statements are true.

(a) Let the domain be given as $\mathbb{N}$, and for $n \in \mathbb{N}$, the conditional is $P(n) : n^2 > 9$.

Solution. $n \geq 4$.

n is an element in the natural numbers. Whenever $n \geq 4$, the result is greater than 9. If n is equal to 3, 2, or 1, the result is 9 or less.

(b) Let the domain be given as $\mathbb{R}$, and for $x \in \mathbb{R}$, the conditional is $Q(x) : x^2 - 3x + 1 = -1$.

Solution. $x \in A$ where $A = \{1, 2\}$

x can be any real number. We Solve for x using the quadratic formula.

(c) Let the domain be given as $\mathbb{R}$, and for $\theta \in \mathbb{R}$, the conditional is $R(\theta)$: "the triangle with angles $\hat{a} = \theta, \hat{b} = \theta + 10^0, \hat{c} = 180^0 - \hat{a} - \hat{b}$ is an acute triangle". (Note, you have to consider both for which θ the conditional is true as well as for which θ the conditional even makes sense.)

Solution. $\theta = 50^0$

θ can be any real number. Since we are dealing with an acute triangle, each angle must be less than 90^0. Using logic, 50^0 is the only degree that fits the given conditions. Each angle is less than 90^0 and the three angles (50^0, 60^0, 70^0) add up to 180^0.

Question 5. Given the set $A = \{x : \cos(x) = 0\}$ prove the following:

(a) If $B = \{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, -\frac{\pi}{2}, -\frac{3\pi}{2}\}$ then $B \subset A$.

Solution. We claim that $A = \{x : \cos(x) = 0\}$ and $B = \{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, -\frac{\pi}{2}, -\frac{3\pi}{2}\}$ First, we will prove that $B \subset A$. Let $x \in A$ and $x \in B$ be generic. This means that $cos(x) = 0$ and $x \in B$. Therefore, every element in B is an element in A and we can conclude that $B \subset A$. We will now prove that $B \neq A$. Let $n \in A$ be generic. Assume that $n = \frac{7\pi}{2}$. Hence, $n \notin B$, but $n \in A$. Therefore, $B \neq A$. QED

(b) If $D = \{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$ then $D = A$. (Recall, you must show two separate statements: $A \subseteq D$ and $D \subseteq A$.)

Solution. We claim that $A = \{x : \cos(x) = 0\}$ and $D = \{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$. We will prove that $D = A$. We will first prove that $A \subseteq D$. Let $n \in A$ be generic. This means that $n \in \{x : \cos(x) = 0\}$. Since $D = \frac{\pi}{2} + k\pi$ for some $k \in \mathbb{Z}$, $n \in D$. Therefore, $A \subseteq D$. We will now prove that $D \subseteq A$. Let $x \in D$ be generic. This means that $x \in \{\frac{\pi}{2} + k\pi\}$ for some $k \in \mathbb{Z}$. Since $A = cos(x)$ for some x, $x \in A$. Therefore, $D \subseteq A$. Therefore, $D = A$. QED

(c) If $C = \{x : \sin(x) = 1\}$ then $C \subset A$.

Solution. We claim that $A = \{x : \cos(x) = 0\}$ and $C = \{x : \sin(x) = 1\}$. We will prove that $C \subset A$. Let $n \in C$ and $n \in A$ be generic. Assume $n = \frac{\pi}{2}$. Since $cos(\frac{\pi}{2}) = 0$ and $sin(\frac{\pi}{2}) = 1$, there exists an element that is a part of C and a part of A. Therefore, $C \in A$. QED

(d) If $E = \{x : \cot(x) = 1\}$ then $E \cap A = \emptyset$.

Solution. We claim that $A = \{x : \cos(x) = 0\}$ and $E = \{x : \cot(x) = 1\}$. We will prove that $E \cap A = \emptyset$. Let $n \in E \cap A$ be generic. There does not exist an element that fits set A and set E. Since there is nothing in the set $E \cap A$, $E \cap A = \emptyset$. QED

Question 6. Let the sets A, B, C, be given as:

$$A = \{x \in \mathbb{R} : \ x^3 - 4x^2 + 4x < 0\}$$
$$B = \{z \in \mathbb{Z} \ : \ z = \frac{v}{3}, \text{ for some } v \in \mathbb{Z}\}$$
$$C = \{\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}\}.$$

Prove that $A \cap B = C$.
(Recall, you must show two separate statements: $A \cap B \subseteq C$ and $C \subseteq A \cap B$. Hint, you may benefit from factoring the polynomials that appear in the sets.)

Solution. You should get to know the "align" environment! It is a really cool presentation tool. Suppose you want to display multiple lines of an equation, and organize where they line up. You can use "align∗" (it has a "∗" to tell LaTeXto suppress the numbering of the lines in the equation). Let's make these lines line up (using the "&" symbol, and the "\\" says to start a new line), just before the first curly bracket of each set:

$$\begin{aligned} X =&\{ \text{ blah blah hey blah } \} \\ Y =&\{ \text{ blah hey blah hey } \}. \end{aligned}$$

Question 7. Assume that A and B are sets such that $A \cap B = B$.

(a) Prove that this assumed information shows that $B \subseteq A$.

Solution. We assume that A and B are sets such that $A \cap B = B$. We will prove that $B \subseteq A$. Assume $B \subseteq A$ is generic. Let $n \in A \cap B$. This means that $n \in A$ and $n \in B$. Therefore $B \subseteq A \cap B$, which means $B = A \cap B$. Since $A \cap B$ and $n \in A \cap B$ and $n \in A$, $B \subseteq A$. QED

(b) Given an example of A and B, that are both non-empty, for which it is true that $A \cap B = B$.

Solution. $A = \{1, 2, 3, 4\}$ $B = \{1, 2, 3\}$
This is a valid example because the only elements that occur in both A and B are 1, 2, and 3. These are also the only elements in B. Because of this, $A \cap B = B$.