Homework 7; Due Tuesday, 2/27/2018

Answer-Only Questions. Credit based solely on the answer.

Question 1. For each of the following statements, write the negated statement. Then, answer if the **original statement** is true or false.

(a) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, [x^2 = y^2] \implies [\exists s \in \{1, -1\}, xs = y]$

Solution. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, [x^2 = y^2]$ and $[\forall s \in \{1, -1\}, xs \neq y]$

The original statement is true.

(b) $\forall p \in \mathbb{P}_4 \setminus \mathbb{P}_3, \exists r \in \mathbb{R}, \exists s \in \mathbb{R} \setminus \{r\},\ (p(r) = 0$ and $p(s) = 0)$

Solution. $\exists p \in \mathbb{P}_4 \setminus \mathbb{P}_3, \forall r \in \mathbb{R}, \forall s \in \mathbb{R} \setminus \{r\},\ (p(r) \neq 0$ or $p(s) \neq 0)$

The original statement is false.

(c) Assume A and B are sets. The statement is $[(\forall a \in A, a \notin B) \wedge (\forall b \in B, b \notin A)] \implies [A \cap B = \emptyset]$

Solution. $[(\forall a \in A, a \notin B) \wedge (\forall b \in B, b \notin A)]$ and $[A \cap B \neq \emptyset]$

The original statement is true.

Full-Justification Questions. Provide a proof for the questions below.

Question 2. Prove, using induction, that for all $n \in \mathbb{N}$ the following equality holds.

$$\sum_{i=1}^{n} \frac{2}{(i+1)(i+2)} = \frac{n}{(n+2)}$$

Solution. We will use mathematical induction.
The conditional statement is $P(n) : \sum_{i=1}^{n} \frac{2}{(i+1)(i+2)} = \frac{n}{(n+2)}$
Base Case: We will check to see if $P(1)$ is true.
$\sum_{i=1}^{n} \frac{2}{(i+1)(i+2)} = \frac{2}{(2)(3)} = \frac{2}{6} = \frac{1}{3}$
and $\frac{n}{n+2} = \frac{1}{1+2} = \frac{1}{3}$
Since $\frac{1}{3} = \frac{1}{3}$, $P(1)$ holds.
Inductive Assumption: We will assume $k \in \mathbb{N}$ is fixed and generic.
We will assume $P(k)$ is true.

Thus, $P(k) : \sum_{i=1}^{k} \frac{2}{(i+1)(i+2)} = \frac{k}{(k+2)}$ is true.
Inductive Step: Now, we want to prove that $P(k+1)$ is true.
$P(k+1) : \sum_{i=1}^{k+1} \frac{2}{(i+1)(i+2)} = \frac{k+1}{(k+1+2)}$
$\frac{2}{(K+1+1)(k+1+2)} + \frac{k}{k+2} = \frac{k+1}{k+1+2}$
$\frac{2}{(k+2)(k+3)} + \frac{k}{k+2} = \frac{k+1}{k+3}$
$\frac{2+k(k+3)}{(k+2)(k+3)} = \frac{k+1}{k+3}$
$\frac{k^2+3k+2}{(k+2)(k+3)} = \frac{k+1}{k+3}$
$\frac{(k+2)(k+1)}{(k+2)(k+3)} = \frac{k+1}{k+3}$
$\frac{k+1}{k+3} = \frac{k+1}{k+3}$
Therefore, $P(k+1)$ is true.
Therefore, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$. QED

Question 3. Let a sequence (list) of numbers be defined by $a_1 = 5$, $a_{n+1} = \sqrt{a_n + 30}$. Prove that $a_n < 6$ for all $n \in \mathbb{N}$.

Solution. We will use proof by mathematical induction.
The conditional statement is $P(n) : a_n < 6$
Base Case: Check $P(1)$
$P(1) : a_1 + 5$
Since $5 < 6$, $P(1)$ holds.
Inductive Assumption: Assume $P(k)$ is fixed and true
Thus, $a_k < 6$
Inductive Step: We will now prove that $P(k+1)$ is true.
$a_{k+1} = \sqrt{a_k + 30}$
$a_k < 6$
$a_k + 30 < 6 + 30$
$\sqrt{a_k + 30} < \sqrt{6 + 30}$
$ak + 1 < 6$
We have concluded that $P(k+1)$ is true
Thus, by the principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$

Question 4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = e^x + x$.

(a) Prove that f is strictly increasing. That is to say, prove that for any $a, b \in \mathbb{R}$

$$(a < b) \implies (e^a + a < e^b + b)$$

(Hint: You may assume the fact that e^x is strictly increasing. See the supplementary document for the definition of a strictly increasing function.)

Solution. We want to prove that f is strictly increasing.

We assume that $a < b$

Thus, $e^a < e^b$

Therefore, $e^a + a < e^b + b$

Since we assume e^x is strictly increasing, we can conclude that f is strictly increasing. QED

(b) Prove that f is injective. (Hint: If $x_1 \neq x_2$, then either $x_1 > x_2$ or $x_2 > x_1$. Consider which of the definition of injective or its contrapositive version will be more useful here.)

Solution. We will assume that $x \neq y$

Then, $|x - y| > 0$

Thus, $e^{|x-y|} > e^0$

Since $e^{|x-y|} > 1$, we know that $e^x > e^y$ or $e^y > e^x$.

Thus, $e^x + x > e^y + y$ or $e^y + y > e^x + x$

Thus, $e^x + x \neq e^y + y$

Hence, $f(x) \neq f(y)$

Therefore, f is injective.

Question 5. Determine if the following statement is true or false by considering its contrapositive. Prove the statement if it is true.

If x and y are two integers for which $3(x - y)$ is even, then x and y have the same parity.

Note: We say that integers x and y have the same parity if

$$(x \text{ is even and } y \text{ is even}) \text{ or } (x \text{ is odd and } y \text{ is odd}).$$

Solution. This statement is true.
We want to prove that if x and y are two integers for which $3(x - y)$ is even, then x and y have the same parity.
We will use proof by contrapositive.
The contrapositive statement is that if x and y do not have the same parity, then x and y are two integers for which $3(x - y)$ is odd.
Assume that x and y do not have the same parity.
Thus, we have two possibilities.
Case 1: x is odd and y is even
Thus, $x = 2k + 1$ and $y = 2k$ for some $k \in \mathbb{Z}$
Therefore, $3(2k + 1 - 2k)$ is odd
Thus, $3(1) = 3$ is odd.
Case 1 holds because this is true.
Case 2: x is even and y is odd

Thus, $x = 2k$ and $y = 2k + 1$ for some $k \in \mathbb{Z}$
Therefore, $3(2k - 2k + 1)$ is odd
Thus, $3(1) = 3$ is odd
Case 2 holds because thus is true.
Since we have proven both possible cases to be true, the contrapositive statement is true.
Since the contrapositive statement is equivalent to the original statement, we are done. QED

Question 6. Let $\Phi : \mathbb{P}_3 \to \mathbb{P}_1 \times \mathbb{R} \times \mathbb{R}$ be given by the assignment rule

$$\Phi(p) = (p'', p'(0), p(0)).$$

Recall $\mathbb{P}_1 \times \mathbb{R} \times \mathbb{R} = \{(p, a, b) \mid p \in \mathbb{P}_1 \text{ and } a, b \in \mathbb{R}\}$ is the set containing all triples where the first slot contains a polynomial of degree 1 or less and the second two slots contain real numbers. Here is an example input and output for the function Φ. Let $p \in \mathbb{P}_3$ be given by $p(x) = 2x^3 + x^2 + x - 1$. Then

$$\Phi(p) = (p'', 1, -1) \quad \text{where } p''(x) = 12x + 2.$$

(a) Prove that Φ is a bijection.

Solution. We will first prove that Φ is surjective.

Let $m \in \mathbb{P}_1 \times \mathbb{R} \times \mathbb{R}$ be generic

We have $m = (cx + d, b, a)$

Choose $p = a + bx + cx^2 + dx^3$

Thus, $\Phi(p) = (cx + d, b, a)$ which is equal to m.

Therefore, Φ is surjective.

Now we will prove that Φ is injective.

We have $a, b \in \mathbb{P}_1 \times \mathbb{R} \times \mathbb{R}$

Assume $\Phi(a) = \Phi(b)$

Therefore, $(a_3 + a_4x, a_2, a_1) = (b_3 + b_4x, b_2, b_1)$.

We now have $a_0 + a_1x + a_2x^2 + a_3x^3 = b_0 + b_1x + b_2x^2 + b_3x^3$.

Since the coefficients are the same, $a = b$

Therefore, Φ is injective.

Since Φ is both surjective and injective, we can conclude that it is bijective. QED

(b) Write down the domain, co-domain, and assignment rule for Φ^{-1}.

Solution. $\Phi^{-1} : \mathbb{P}_1 \times \mathbb{R} \times \mathbb{R} \to \mathbb{P}_3$

$\Phi^{-1}(q, c, d) = \frac{ax^3}{6} + \frac{bx^2}{2} + cx + d$

(c) For a generic $(q, c, d) \in \mathbb{P}_1 \times \mathbb{R} \times \mathbb{R}$, confirm that $\Phi(\Phi^{-1}(q, c, d)) = (q, c, d)$.

Solution. Assume $q = ax + b$ for some $a, b \in \mathbb{R}$

$\Phi(\Phi^{-1}(q, c, d)) = ((\frac{ax^3}{6} + \frac{bx^2}{2} + cx + d)'', (\frac{ax^3}{6} + \frac{bx^2}{2} + cx + d)'(0), (\frac{a(0)^3}{6} + \frac{b(0)^2}{2} + c(0) + d))$

$= ((ax + b, (\frac{a(0)^2}{6} + b(0) + c), d))$

$= (q, c, d)$

Confirmed