Homework 6; Due Monday, 2/19/2018

Answer-Only Questions. Credit based solely on the answer.

Question 1. Determine whether the following functions are injective, surjective, both, or neither.

(a) $f : \mathbb{N} \to \mathbb{R}$ with the assignment rule $f(x) = x^7$.

Solution. Injective

(b) $g : \mathbb{R}^2 \to \mathbb{R}$ with the assignment rule $g(x, y) = x^4|x - y|$.

Solution. Neither

(c) $H : \mathbb{P}_1 \to \mathbb{P}_3$ with the assignment rule $H(p) = p^3$.

Solution. Injective

(d) $f : \mathbb{Z} \to \mathbb{Z}$ with the assignment rule $f(n) = \begin{cases} n+4 & \text{if } n < 0 \\ n+2 & \text{if } n \geq 0 \end{cases}$.

Solution. Surjective

(e) $H : \mathbb{P}_2 \to \mathbb{R}$ with the assignment rule $H(p) = -4p'(3) - 10$.

Solution. Surjective

Question 2. Negate the following statements. Then decide which is true the statement or the negation.

(a) $\forall q \in \{p \in \mathbb{Q} \ : \ p \neq 0\}, \ \exists x \in \mathbb{R}$ such that $x\sqrt{|q|} \in \mathbb{N}$.

Solution. $\exists q \in \{p \in \mathbb{Q} : p \neq 0\}, \forall x \in \mathbb{R}$ such that $x\sqrt{|q|} \notin \mathbb{N}$

The Statement is true.

(b) $\forall p \in \mathbb{P}_5, \ \exists (a, b) \in \{(x, y) \in \mathbb{R}^2 : x < y\}$, such that $\int_a^b p(s)\ ds = 0$.

Solution. $\exists p \in \mathbb{P}_5, \forall (a, b) \in \{(x, y) \in \mathbb{R}^2 : x < y\}$, such that $\int_a^b p(s)\ ds \neq 0$

The Negation is true

(c) $\forall (a, b) \in \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}, \ \exists\ \theta \in [0, 2\pi]$ such that $\sin\theta\cos\theta = ab$.

Solution. $\exists (a,b) \in \{(x,y) \in \mathbb{R}^2 \ : \ x^2+y^2=1\}, \ \forall \ \theta \in [0,2\pi]$, such that $\sin\theta\cos\theta \neq ab$
The Statement is true

Full-Justification Questions. Provide a proof for the questions below.

Question 3. Let n be a natural number. Prove that if n does not divide 72, then n does not divide 12.

Solution. We want to prove that if $n \nmid 72$, then $n \nmid 12$.
We will use proof by contrapositive.
The contrapositive statement is $n|12 \implies n|72$.
Assume $n|12$ is true for some $n \in \mathbb{N}$.
We know that if $A|B$ and $B|C \implies A|C$.
Therefore, $n|12$ and $12|72 \implies n|72$.
Since 12 does, in fact, divide 72, then we can conclude that $n|12 \implies n|72$.
Since this is equivalent to the original statement, we are done. QED

Question 4. We say that two integers, m and n, have the same ***parity*** if both m and n are odd or both m and n are even. Use the method of proof by contradiction to prove the following statement:

Suppose $m,n \in \mathbb{Z}$. If $(m+1)^2-(n+1)^2$ is odd, then m and n do not have the same parity.

Solution. We want to prove that if $(m+1)^2-(n+1)^2$ is odd, then m and n do not have the same parity.
We will use proof by contradiction.
We will assume that $(m+1)^2-(n+1)^2$ is even.
Therefore, $(m+1)^2-(n+1)^2 = 2k$ for some $k \in \mathbb{Z}$.
$(m+1)^2 = sk + (n+1)^2$
$m+1 = \sqrt{2k}+n+1$
$m = \sqrt{2k}+n$
Since m was originally an integer, we have our contradiction.
Thus, if $(m+1)^2-(n+1)^2$ is odd, then m and n do not have the same parity is true. QED

Question 5. For each of the parts (b), (c), and (d) of Question 1, give complete proofs for your answers.

Solution. (b) First we will prove that the function $g : \mathbb{R}^2 \to \mathbb{R}$ with the assignment rule $g(x,y) = x^4|x-y|$ is not surjective.
Choose $g(x,y) = -1$ which is in our codomain $\mathbb{R}$.
Then for $x \in \mathbb{R}^2$ generic in our domain, $g(x,y) = x^4|x-y| \geq 0$ since $|x-y| \geq 0$ and $x^4 \geq 0$.
So, in particular, $g(x,y) \neq -1$.
Therefore, g is not surjective.
Now we will prove that the function is not injective.
Choose $m = (0,1)$ and $n = (0,2)$.
Then $(0,1) \neq (0,2)$ so $m \neq n$.
But $0^4|0-1| = 0(1) = 0$ and $0^4|0-2| = 0(2) = 0$.
So $g(m) = g(n)$.
Therefore, g is not injective. QED

Solution. (c) We want to prove that $H : \mathbb{P}_1 \to \mathbb{P}_3$ with the assignment rule $H(p) = p^3$ is injective.
First, we will prove that it is not surjective.
Choose $y = x^2$ which is in our codomain $\mathbb{P}_3$.
Then for $p \in \mathbb{P}_1$ generic in our domain,
$H(p) = p^3$ which will never be equal to x^2.
So $H(p) \neq p^2$
Therefore, H is not surjective.
Now, we will prove that the function is injective.
Let $m, n \in \mathbb{P}_1$ be generic.
Assume $H(m) = H(n)$.
So $m^3 = n^3$.
Therefore $m = n$.
Therefore, H is injective.QED

Solution. (d)First, we will prove that the function is surjective.
Let $y \in \mathbb{Z}$ be generic.
Set $n = y - 4$.
Then $f(n) = f(y-4) = y - 4 + 4 = y$
Now set $n = y - 2$
Then $f(n) = f(y-2) = y - 2 + 2 = y$.
Therefore, $f(n)$ is surjective.
Now, we will prove that the function is not injective.
Choose $x = -1$ and $y = 1$
Then $-1 \neq 1$ so $x \neq y$.
However, $-1 + 4 = 3$ and $1 + 2 = 3$.
So $f(x) = f(y)$.
Therefore, g is not injective. QED

Question 6. Let $F : \mathbb{P}_2 \to \mathbb{P}_1$ with the assignment rule $F(p) = \frac{1}{2}p' + 5$.

(a) Prove that F is surjective.

Solution. Let $y \in \mathbb{P}_2$ be generic.

Set $p = y^2 - 10y$

Then $F(p) = F(y^2 - 10y)$

$= \frac{1}{2}(y^2 - 10y)' + 5$

$= \frac{1}{2}(2y - 10) + 5$

$= (y - 5) + 5 = y$

Therefore, F is surjective. QED

(b) Prove that F is not injective.

Solution. Choose $x = 0$ and $y = 1$.

Then $0 \neq 1$ so $x \neq y$.

However, $\frac{1}{2}(0)' + 5 = 0 + 5 = 5$

and $\frac{1}{2}(1)' + 5 = 0 + 5 = 5$

Therfore, F is not injective. QED

Question 7. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be with the assignment rule $f(x, y) = (7y, x + 8)$.

(a) Prove that this function is bijective.

Solution. First, we will prove that the function is injective.

Suppose $(x_1, y_1), (x_2, y_2) \in \mathbb{R} \times \mathbb{R}$ and that $f(x_1, y_1) = f(x_2, y_2)$. Then:

$f(x_1, y_1) = f(x_2, y_2)$

$(7y_1, x_1 + 8) = (7y_2, x_2 + 8)$.

$x_1 + 8 = x_2 + 8$

$x_1 = x_2$

$7y_1 = 7y_2$

$y_1 = y_2$

Therefore, $(x_1, y_1) = (x_2, y_2)$ so f is injective.

Now, we will prove that the function is surjective.

Assume $u, v \in \mathbb{R} \times \mathbb{R}$ is generic.

Scratch work: $f(x, y) = (u, v)$

$(7y, x + 8) = (u, v)$

$x + 8 = v$ so $x = v - 8$

$7y = u$ so $y = \frac{u}{7}$ (end scratch work)

We will choose $x = v - 8$ and $y = \frac{u}{7}$.

We note that this makes $(x, y) \in \mathbb{R} \times \mathbb{R}$ because $(u, v) \in \mathbb{R} \times \mathbb{R}$. Now, we will plug this into our function.

$f(x, y) = f(v - 8, \frac{u}{7}) = (7\frac{u}{7}, v - 8 + 8)$

$= (u, v)$

Since (u, v) was generic, we conclude that f is surjective.

Since f was both surjective and injective, we can conclude that f is bijective. QED

(b) Compute the assignment rule for the inverse function, which we call f^{-1}

Solution. Scratch work: $f(x, y) = (u, v)$
$(7y, x + 8) = (u, v)$
$x + 8 = v$ so $x = v - 8$
$7y = u$ so $y = \frac{u}{7}$ (end scratch work)
We will let $x = v - 8$ and $y = \frac{u}{7}$.
We can now conclude that $f^{-1} = (x - 8, \frac{y}{7})$

(c) Confirm that for a generic $(x, y) \in \mathbb{R} \times \mathbb{R}$ that we have $f^{-1}(f(x, y)) = (x, y)$.

Solution. We want to show that $f^{-1}(f(x, y)) = (x, y)$ for some generic $(x, y) \in \mathbb{R} \times \mathbb{R}$.
We let $f^{-1} = (7y, x + 8)$
$(= x + 8 - 8, \frac{7y}{7})$
$= (x, y)$.
Therfore, $f^{-1}(f(x, y)) = (x, y)$.

Question 8. Use mathematical induction to prove that for every $n \in \mathbb{N}$, $3 \mid n^3 - n + 9$.

Solution. We will use mathematical induction.
The conditional statement is $P(n) : 3 \mid n^3 - n + 9$
Base Case: For $n = 1, 1^3 - 1 + 9 = 9$
9 is divisible by 3, so $P(1)$ holds.
Inductive Hypothesis: Assume $k \in \mathbb{N}$ is fixed and generic and that $P(k)$ is true.
Therefore we assume $3 \mid k^3 - k + 9$ is true. and for some $x \in \mathbb{Z}$, $k^3 - k + 9 = 3x$.
Inductive Step: We will show that $P(k + 1)$ is also true.
We calculate that $(k + 1)^3 - (k + 1) + 9 = k^3 + 3k^2 + 3k + 1 - k - 1 + 9$
$= (k^3 - k + 9) + (3k^2 + 3k)$
By the induction assumption, this is equal to $3x + (3k^2 + 3k)$
$= 3(x + k^2 + k)$.
Since k and x are integers, $x + k^2 + k$ is also an integer.
Thus, this calculation shows that $(k + 1)^3 - (k + 1) + 9$ is divisible by 3.
We have proven that $P(k+1)$ is true when $P(k)$ is true and so by mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. QED