Homework 8; Monday, 3/12/2018

Answer-Only Questions. Credit based solely on the answer.

Question 1. Compute the indexed unions and intersections below. For example, you should give a simpler description of the set, X, as presented by the solution given in part (a).

(a) $X = \bigcap_{k=3}^{7} \{k-2, k-1, k, k+1, k+2, k+3, k+4\}$

Solution. $X = \{5, 6, 7\}$

(b) $X = \bigcap_{k=1}^{5} [k, k+5]$

Solution. $X = [5, 6]$

(c) $X = \bigcup_{n\in\mathbb{N}} [n-3, n-2)$

Solution. $X = [-2, \infty)$

(d) $X = \bigcup_{k=2}^{51} \{n \in \mathbb{Z} \mid -2k \leq n \leq k\}$

Solution. $X = [-102, 51]$

Question 2. Say whether each set is bounded or not. If the set in question is in fact bounded, give a suitable choice for bound M (here, M is from definition 7.2 of bounded set from the supplementary document). For example, you should proceed as indicated by the answer given for part (a).

(a)

$$A = \bigcup_{j=-7}^{-5} \{j-1, j, j+1, j+2, j+3, j+4, j+5\}$$

Solution. A is bounded. For example, we can choose $M = 10$.

(b)

$$X = \bigcup_{n\in\mathbb{N}} \{r \in \mathbb{R} \mid 9r^2 + (-1)^n n^2 = 0\}.$$

Solution. X is unbounded.

(c)

$$Y = \bigcap_{n=1}^{9} \Big\{ x \in \mathbb{R} \mid \sin\Big(\frac{x}{n}\Big) = 0 \Big\}.$$

Solution. Y is bounded. Choose $M = 1$

(d)

$$W = \bigcap_{n \in \mathbb{N}} \Big\{ x \in \mathbb{R} \mid \frac{e^x}{2^n} \geq 1 \Big\}.$$

Solution. W is unbounded.

Question 3. Say whether the following quantified statements are true or false. If a statement is false, give a counterexample to show this is the case.

(a) $\forall p \in \mathbb{P}_3, \exists q \in \mathbb{P}_3, \deg(pq) = 2.$

Solution. False. Let $\mathbb{P} = n^3 + 1$

Thus, $q(x) \notin \mathbb{P}_3$

(b) $\forall p \in \mathbb{P}_3, \forall q \in \mathbb{P}_3, p - q \in \mathbb{P}_3.$

Solution. True

(c) $\exists p \in \mathbb{P}_3, \forall q \in \mathbb{P}_3, \frac{q}{p} \in \mathbb{P}_3.$

Solution. True

(d) $\forall p \in \{f \in \mathbb{P}_2 \mid \deg(f) = 2\}, \exists q \in \{g \in \mathbb{P}_2 \mid \deg(g) \geq 1\}, \exists r \in \{h \in \mathbb{P}_2 \mid \deg(h) \geq 1\}, p = qr.$

Solution. True

(e) $\forall p \in \mathbb{P}_2, \forall \alpha \in \mathbb{R}, \Big(\big(p(\alpha) = 0\big) \implies \big(\exists \beta \in \mathbb{R} \backslash \{\alpha\}, p(\beta) = 0\big) \Big).$

Solution. False. Let $p = x^2 - 2$. Thus, $p(\beta) \neq 0$

Full Justification Questions. Provide complete justifications for your responses.

Question 4. Define the sets

$$E = \bigcup_{n \in \mathbb{Z}} \left\{ x \in \mathbb{R} : |x - 2n| < \frac{1}{2} \right\}$$

and

$$F = \bigcup_{m \in \mathbb{Z}} \left[2m - \frac{3}{4}, 2m + 1 \right).$$

Prove that $E \subseteq F$.

Solution. Let $e \in E$ be generic.
We know that $2n - \frac{1}{2} < e < 2n + \frac{1}{2}$ for some $n \in Z$
Thus, $2n - \frac{3}{4} < 2n - \frac{1}{2}$ and $2n + \frac{1}{2} < 2n + 1$
Hence, $2n - \frac{3}{4} < 2n - \frac{1}{2} < e < 2n + \frac{1}{2} < 2n + 1$
Therefore, we know that $e \in E$ and $e \in F$ which implies that $E \subseteq F$

Question 5. Prove, using contradiction, that the equation $x^3 + x^2 = 1$ has no rational solutions.
Hint: You should assume that any rational solution $\frac{p}{q}$ to the above equation has been simplified, i.e., that p and q have no common factor. Substitute $\frac{p}{q}$ into $x^3 + x^2 = 1$ and consider the following four (exhaustive) cases, and obtain a contradiction in each:
Case 1: p even, q even (this case is impossible by the first sentence of this hint).
Case 2: p even, q odd (get an even/odd contradiction, e.g., p^3 even, and p^2q even, so $p^3 + p^2q$ even, but q^3 is odd).
Case 3: p odd, q even (even/odd contradiction).
Case 4: p odd, q odd (even/odd contradiction).

Question 6. Let $q \in \mathbb{P}_2$ be the polynomial defined by the rule $q(x) = 3x + 2$ for every $x \in \mathbb{R}$. Define the function

$$\Phi : \mathbb{P}_2 \longrightarrow \mathbb{P}_2,$$

where for any $p \in \mathbb{P}_2$ (domain of Φ), $\Phi(p) \in \mathbb{P}_2$ (codomain of Φ) is the polynomial defined by the rule

$$\big[\Phi(p)\big](x) = (q \circ p)(x) \quad \forall x \in \mathbb{R}.$$

(a) Compute $\Phi(p), \Phi(r)$, and $\Phi(f)$, where $p, r, f \in \mathbb{P}_2$ are the polynomials defined by the rules

$$p(x) = -1, \;\; r(x) = x + \pi, \quad \text{and } f(x) = x^2 - x + 5 \quad \forall x \in \mathbb{R}.$$

In each case, write your answer in the form $ax^2 + bx + c$ for appropriate $a, b, c \in \mathbb{R}$.

Solution. $\Phi(p) = -1$

$\Phi(r) = 3x + 3\pi + 2$

$\Phi(f) = 3x^2 - 3x + 17$

(b) Show that Φ is injective.

Solution. Let $m, n \in \mathbb{P}_2$ be generic.

Assume $\Phi(m) = \Phi(n)$

Since $f(m) = 3m + 2$ and $f(n) = 3n + 2$, $3m + 2 = 3n + 2$

Thus, $n = m$

Therefore, Φ is injective.

(c) Show that Φ is surjective.

Solution. Let $x \in \mathbb{P}_2$ be generic.

Choose $p = \frac{-2}{3} + \frac{y}{3}$

Then, $\Phi(p) = 3(\frac{-2}{3} + \frac{y}{3}) + 2$

$= -2 + y + 2 = y$

So Φ is surjective.

(d) By parts (a) and (b), Φ is a bijection. Find a formula for Φ^{-1}, and verify that indeed Φ^{-1} is the inverse of Φ.

Solution. $[\Phi^{-1}(p)](x) = \frac{x-2}{3}$
$\Phi^{-1}[\Phi(p)](x) = \Phi^{-1}(3p(x) + 2) = \frac{(3p(x)+2-2}{3}$
$= p(x)$

Question 7. Let $A = \{x \in \mathbb{N} \mid \sqrt{x} \in \mathbb{N}\}$. Prove that A is not bounded.

Solution. Let $k \in A$ be generic
We know by the Archimedian Principle that $\exists N \in \mathbb{N}$ such that $N > M$
When we let $x_0 = \sqrt{N}$, $x_0 \in B$ and $|x_0| > M$
Therefore, since there will always be a larger natural number, A is unbounded.

Question 8. (a) Show that any set $A \subseteq \mathbb{R}$ with $|A| = 2$ is bounded.

Solution. Let $x_1, x_2 \in \mathbb{R}$ be genric for some $x_1 > x_2$.

Let $X \in Y$ be generic as well.

We know that $\exists X \in Y > x_2$ by the AP.

Hence, $|x_2| < X$ which means A is bounded.

(b) Show that any set $B \subseteq \mathbb{R}$ with $|B| = 3$ is bounded.

Solution. Let $x_1, x_2, x_3 \in \mathbb{R}$ be generic for some $x_3 > x_2 > x_1$

Let $X \in Y$ be generic as well.

We know that $\exists X \in Y > x_3$ by the AP.

Hence, $|x_3| < X$ which means B is bounded.

(c) Show using induction that $\forall\ n \in \mathbb{N}$, if $A \subseteq \mathbb{R}$ and $|A| = n$ then A is a bounded set.

Solution. We will use proof by induction

The conditional statement is $\forall\ n \in \mathbb{N}$, if $A \subseteq \mathbb{R}$ and $|A| = n$ then A is a bounded set.

Base case: check n=1

n=1 shows that A is a bounded set

Inductive Assumption: assume P(k) is true

We want to priove that P(k) $\implies$ P(k+1)

Therefore, by the principle of mathematical induction, P(n) is true for all $n \in \mathbb{N}$