Homework 4; Due Monday 02/05/2018

Answer-Only Questions. Credit based solely on the answer.

Question 1. Let $a, b \in \mathbb{Z}$. Then we say that a **divides** b if there exists a $k \in \mathbb{Z}$ such that

$$ak = b.$$

We use the notation $a|b$ to mean "a divides b". Similarly, $a \nmid b$ means "a does not divide b".
Determine whether the following statements are true or false, where $a, b, c \in \mathbb{Z}$. If the statement is false, give a counterexample.

(a) $11|132$

(b) If $a|bc$ then $a|b$ or $a|c$.

(c) If $a|b$ and $b|c$, then $a|c$.

(d) If $a|b$ then $a|(-b)$.

(e) If $a|(b-c)$ then $a|b$ and $a|c$.

(f) If $a|b$ and $a|(b+c)$ then $a|c$.

Solution. .
(a) True
(b) False. $6|12$ but $6 \nmid 3$ and $6 \nmid 4$. Also, $4x3 = 12$
(c) True
(d) True
(e) True
(f) True

Question 2. (a) Let the statement, R be defined as

R: "If a natural number n divides every integer, then $n = 1$."

Rewrite the statement, R, in the form

If (statement-1) and (statement-2), then $n = 1$,

where statement-1 expresses n as an element of a set, and statement-2 is a quantified statement involving $n|a$, with $a \in \mathbb{Z}$.

Solution. If $n \in \mathbb{N}$ and $n|a$ $\forall a \in \mathbb{Z}$, then $n = 1$.

(b) Negate the statement, R, from part (a). (hint, this will be much easier to do if you have correctly rewritten R as requested in part (a))

Solution. $n \in \mathbb{N}$ and $n|a$ $\forall a \in \mathbb{Z}$ and $n \neq 1$

(c) Write the contrapositive of the statement, R, from part (a). (hint, this will be much easier to do if you have correctly rewritten R as requested in part (a))

Solution. If $n \neq 1$, then $n \notin \mathbb{N}$ or $n \not| a$ $\forall a \in \mathbb{Z}$

(d) State whether the statement R, from part (a), is true or false.

Solution. True

Question 3. Give an example of each the following.

(a) A function $f : \mathbb{R} \to \mathbb{R} \setminus \{0\}$

Solution. $f(f) = \frac{1}{f}$

(b) A function $g : \mathbb{N} \to \mathbb{Q}$

Solution. $f(g) = \sqrt{g}$

(c) A function $h : \mathbb{P}_3 \to \mathbb{P}_4$

Solution. $f(h) = \int$ h

(d) A function $k : \{a, e, i, o, u, y\} \to \{1, 3, 5, 7\}$

Solution. $f = k^0 + 4$

Question 4. For each of the following assignment rules, fill in the appropriate blank with a choice of domain and/or codomain that makes the triple, (domain, co-domain, f) into valid functions.

(a) $f(x) = \frac{1}{\sqrt{x}}$

Solution. Domain: $\mathbb{N}$ Codomain: $(0, \infty)$

(b) $f(x) = 2^x$

Solution. Domain: $\mathbb{R}$ Codomain: $\mathbb{R}$

Solution. Domain: $[0, 6]$ Codomain: $\{1, 2, 4, 8, 16, 32, 64\}$

Solution. Domain: $(5, 10)$ Codomain: $[1.6094398, 2.302585]$

Full Justification Questions. Provide complete justifications for your responses.

Question 5. Let $p(x) = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$ with $a \neq 0$.

(a) Prove by taking the contrapositive that if $p(x) \neq 0$ for all $x \in \mathbb{R}$ then $p(x)$ cannot be factored. (That is, for all $\alpha, \beta \in \mathbb{R}$ we have $p(x) \neq a(x - \alpha)(x - \beta)$.)

Solution. We want to prove that $\forall \alpha, \beta \in \mathbb{R}$, $p(x) \neq a(x-\alpha)(x-\beta)$. We will use proof by contrapositive. The contrapositive statement is if $p(x) = a(x-\alpha)(x-\beta)$, then $\exists \alpha, \beta \in \mathbb{R}$. Assume $p(x) = a(x-\alpha)(x-\beta)$ is true for some $\alpha, \beta \in \mathbb{R}$. Thus, $p(\beta) = 0$ if $p(x) = 0$ for some $x \in \mathbb{R}$. This means $p(x) \neq 0$ $\forall x \in \mathbb{R}$ and this implies that $p(x)$ cannot be factored. Thus, the contrapositive statement is true and since this is equivalent to the original statement, we are done. QED

(b) Reprove part (a) using contradiction instead of contrapositive. You should see that the proofs are in many ways very similar.

Solution. We want to prove that $\forall \alpha, \beta \in \mathbb{R}$, $p(x) \neq a(x-\alpha)(x-\beta)$. We will use proof by contradiction. We will assume that $p(x) = 0$ for all $x \in \mathbb{R}$. Thus, $p(x) = a(x - \alpha)(x - \beta)$ is true for some $\alpha, \beta \in \mathbb{R}$. This contradicts that $p(x) \neq 0$ $\forall x \in \mathbb{R}$. Therefore, $p(x) = 0$ for all $x \in \mathbb{R}$ cannot be true. Thus, the original statement is true. QED

Question 6. Let $r \in \mathbb{R} \setminus \{1\}$. Prove by induction that

$$r + r^2 + \ldots + r^n = \frac{r - r^{n+1}}{1 - r}$$

holds for all $n \in \mathbb{N}$.

Solution. We want to prove that $\sum_{i=1}^{n} i = \frac{r-r^{n+1}}{1-r}$ $\forall n \in \mathbb{N}$.
$p(n) : \sum_{i=1}^{n} i = \frac{r-r^{n+1}}{1-r}$
Base Case: check $p(1)$.
$sum_{i=1}^{n} i = 1$
and
$= \frac{r-r^{n+1}}{1-r}$
$= \frac{r-r^2}{1-r}$

$= \frac{r(r-1)}{1-r}$
so $1 = 1$
Inductive Step: Fix $k \in \mathbb{N}$.
We want to prove $p(k) \implies p(k+1)$.
Therefore assume $p(k)$ is true.
$p(k) : \sum_{i=1}^{k} i = \frac{r-r^{k+1}}{1-r}$
We want to show $p(k+1) : \sum_{i=1}^{k+1} i = \frac{r-r^{k+2}}{1-r}$
$= \frac{(1-r)(r^k)}{1-r}$
$= (r)(r^k)$
$= r^k + 1$
Hence, $\sum_{i=1}^{k+1} i = r^k + 1$ which is $p(k+1)$
Therefore, by the principal of mathematical induction, $p(n)$ is true for all $n \in \mathbb{N}$. QED

Question 7. Our goal in this problem is to show that

$$\frac{d}{dx}(x^q) = qx^{q-1}$$

for all $q \in \mathbb{Q}$. We will do it in steps. You may use as given (i.e. you do not have to prove) the product, quotient, and chain rules as well as the facts that

$$\frac{d}{dx}(x) = 1 \quad \text{and} \quad \frac{d}{dx}(1) = 0.$$

(a) Use induction to prove that for all $n \in \mathbb{N}$ we have

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Solution. We want to prove that $\sum_{i=1}^{n} i = \dfrac{d}{dx}(x^n) = nx^{n-1}$is true for all $n \in \mathbb{N}$.

$p(n) = \sum_{i=1}^{n} i = nx^{n-1}$.

Base Case: Check $p(1)$

$\sum_{i=1}^{1} i = 1$ and $nx^{n-1} = 1x^{1-1} = 1(1)$

and $1 = 1$ is true.

Inductive step: Fix $k \in \mathbb{N}$.

We want to prove $p(k) \implies p(k+1)$.

Therefore assume $p(k)$ is true.

$p(k) : \sum_{i=1}^{k} i = kx^{k-1}$

We want to show $p(k+1) : \sum_{i=1}^{k+1} i = (k+1)x^k$

We will use the product rule to solve this.

Thus, using the product rule, this is equal to $(n^k - 1)(x^k) + (k^x)$
Hence, $\sum_{i=1}^{k+1} i = (n^k - 1)(x^k) + (k^x)$, which is $p(k+1)$.
Therefore, by the principal of mathematical induction, $p(n)$ is true for all $n \in \mathbb{N}$. QED

(b) Using the result of (a), prove that for all $n \in \mathbb{N}$ we have

$$\frac{d}{dx}\left(x^{-n}\right) = -nx^{-n-1}.$$

(Hint: The quotient rule may be useful here, and you will not need to invoke any induction.)

Solution.

(c) Using the results of part (a) and (b) prove that for all $n \in \mathbb{Z} \setminus \{0\}$

$$\frac{d}{dx}\left(x^{1/n}\right) = \frac{1}{n}x^{\frac{1}{n}-1}$$

(Hint: If $y(x) = x^{1/n}$ then $(y(x))^n = x$. Now use the chain rule to find an expression that contains $\frac{dy}{dx}$ and isolate $\frac{dy}{dx}$ to be by itself on one side of the expression.)

Solution.

(d) Using the above parts, prove that for all $\frac{p}{q} \in \mathbb{Q}$

$$\frac{d}{dx}\left(x^{p/q}\right) = \frac{p}{q}x^{p/q-1}$$

(Hint: Chain rule may be useful.)

Solution.