Homework 5; Due Monday, 2/12/2018

Quick Answer Questions. No work needed. No partial credit available.

Question 1. Give an example of each of the following:

(a) An assignment rule with domain $A = \{a, b, 4, 39, 23\}$, and codomain $B = \{\text{cat}, d, e, f, 2, 3\}$ that defines an injective function

Solution. $f : A \to B$

$f(a) = cat$

$f(b) = d$

$f(4) = e$

$f(39) = f$

$f(23) = 2$

(b) A function $f : \mathbb{N} \to \{x \in \mathbb{Z} : x \text{ is even}\}$ that is bijective.

Solution.

$$f(x) = \begin{cases} x & x \text{ is even.} \\ 1 - x & x \text{ is odd.} \end{cases}$$

(c) A codomain S so that the function $h : \mathbb{N} \to S$ defined below is bijective.

$$h(n) = \begin{cases} n & n \text{ is even.} \\ -2n & n \text{ is odd.} \end{cases}$$

Solution. $\{x \in \mathbb{Z} : x \text{ is even } \setminus \{0\}\}$.

Question 2. Write the contrapositive of each of the following statements, and answer true or false to indicate the truth value of the contrapositive:

(a) If $a > 0$ then $\forall b, c \in \mathbb{R}$, $(b > c \Rightarrow ab > ac)$.

Solution. $\exists b, c \in \mathbb{R}$, $(b > c \text{ and } ab \leq ac) \implies a \leq 0$

True

(b) If $\cos(x^2 - 7x + 12) = 1$, then $x = 3$.

Solution. If $x \neq 3$, then $\cos(x^2 - 7x + 12) \neq 1$

False (x can equal 4)

(c) If x is divisible by 6 then x is not prime.

Solution. If x is prime, then x is not divisible by 6.

True

Full Justification Questions. Provide complete justifications and/or proofs for your responses.

Question 3. Prove or disprove the following statements, using contradiction:

(a) If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the sides of a **right** triangle and $\mathbf{c}$ is the hypotenuse then $\mathbf{c} < \mathbf{a} + \mathbf{b}$. (Hint: $\mathbf{a}^2 + \mathbf{b}^2 = \mathbf{c}^2$).

Solution. We want to prove that if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the sides of a **right** triangle and $\mathbf{c}$ is the hypotenuse then $\mathbf{c} < \mathbf{a} + \mathbf{b}$. We assume for the sake of contradiction that a, b, c are the sides of a right triangle and c is the hypotenuse and $c \geq a + b$. When we square both sides we get $c^2 \geq a^2 + 2ab + b^2$. We assume a and b are positive since we cannot have a negative length on the side of a triangle. Thus, $c^2 \geq a^2 + b^2$ is true under this assumption, but this is a contradiction since the Pythagorean Theorem states that $a^2 + b^2 = c^2$. Therefore, $c \geq a + b$ cannot be true, so the original statement is true. QED

(b) $\mathbb{R}^+ = \{x \in \mathbb{R} : \ x > 0\}$, the set of positive real numbers, does not have a minimum element, also called a smallest element. (Note: Look up the definition of the minimum element in the supplementary document after theorem 3.6)

Solution. We want to prove that the set of positive real numbers does not have a minimum element. We assume for the sake of contradiction that the set of positive real numbers does have a minimum element. We will let this minimum element be n. Since $0 < \frac{n}{2} < n$, we know that there is a positive real number $(\frac{n}{2})$ that is less than n. This is a contradiction since n was the minimum element in the set of positive real numbers. Therefore, our original statement is true. QED

Question 4. Determine if the following statements are true or false, and provide a proof of your answer.

(a) The function $g : \mathbb{P}_1 \to \mathbb{P}_2$, $g(p)(x) = \int_0^x p(t)dt$ is injective.

Solution. True

We want to prove that the function $g : \mathbb{P}_1 \to \mathbb{P}_2$, $g(p)(x) = \int_0^x p(t)dt$ is injective.

Let $p, q \in \mathbb{P}_1$ be generic.

We let $\int_0^x p(t)dt = \int_0^x q(t)dt$

Then $p(x) = a_1x + a_0$ for some $a_1, a_0 \in \mathbb{R}$ and $q(x) = b_1x + b_0$ for some $b_1, b_0 \in \mathbb{R}$.

Assume $g(p) = g(q)$.

$\int_0^x p(t) = 0 \implies p(x) = 0$

Hence, $p(t) = q(t)$ and $g : \mathbb{P}_1 \to \mathbb{P}_2$ because $p(x) = 0$.

Thus, we have proven that the function is injective. QED

(b) The function

$$g : \mathbb{R} \to \mathbb{R}$$
$$g(x) = \begin{cases} -x+1 & x > 0 \\ -x^2 & x \leq 0 \end{cases}$$

is injective.

Solution. False (Not injective)

Choose $x = 1$ and $y = 0$.

Then $1 \neq 0$ so $x \neq y$.

But $-(1) + 1 = 0$ and $-(0)^2 = 0$.

So $g(x) = g(y)$.

Therefore, g is not injective. QED

Question 5. Determine if the following functions are surjective. Provide a proof.

(a) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x + 4y, -3x - 12y)$.

Solution. Not Surjective

We want to prove that $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x + 4y, -3x - 12y)$ is not surjective.

We will use proof by contradiction.

Therefore, assume $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x + 4y, -3x - 12y)$ is surjective.

We will choose $(x, y) = (2, 4)$ which is in our codomian $\mathbb{R}^2$.

So $x + 4y = 2$

$x = 2 - 4y$.

We also have $-3x - 12y = 4$

$x = -\frac{4}{3} - 4y$

Set both x values equal to each other.

Thus, $2 - 4y = -\frac{4}{3} - 4y$.

But $2 \neq -\frac{4}{3}$.

This is a contradiction, so our original statement is true.

The function is not surjective. QED

(b)

$$g : \mathbb{N} \to \mathbb{Z}$$
$$g(n) = \begin{cases} \frac{n+1}{2} & n \text{ is odd.} \\ -\frac{n}{2} & n \text{ is even.} \end{cases}$$

Solution. Not surjective

Choose $y = 0$ which is in our codomain $\mathbb{Z}$.

Then for $n \in \mathbb{N}$ generic in our domain, $g(n) = -\frac{n}{2} = 0$ or $g(n) = \frac{n+1}{2} = 0$

In either case, n is equal to 0 or -1, neither of which is in the domain of natural numbers.

Therefore, $g(n)$ is not surjective.

Question 6. Define the Fibonacci sequence by:

$$a_0 = 0, a_1 = 1, a_2 = 1$$
$$a_n = a_{n-1} + a_{n-2} \quad n \geq 2$$

Show, using induction:

$$a_1^2 + a_2^2 + ... + a_n^2 = a_n a_{n+1}$$

Solution. We will use mathematical induction.
The conditional statement is $P(n) =a_1^2 + a_2^2 + ... + a_n^2 = a_n a_{n+1}$.
Base Case: $P(1) : a_1^2 = a_1 \times a_0$.
$a_1^2 = 1$
$a_1 \times a_2 = 1$
Inductive Step: Fix $k \in \mathbb{N}$.
We want to prove $p(k) \implies p(k+1)$.
Assume $P(k)$ is true for some generic $k \in \mathbb{N}$
Thus, $a_1^2 + ... + a_k^2 = a_k \times a_{k+1}$ is true.
Now we prove that $P(k+1)$ is true.
$= a_1^2 + ... + a_k^2 + a_{k+1}^2 = a_k \times a_{k+1} + a_{k+1}^2$.
$= a_{k+1}(a_k + a_{k+1}) = a_{K+2}$
Thus $P(k+1)$ is true.
So $\forall n \in \mathbb{N}$, $P(n)$ is true by mathematical induction. QED

Question 7. Define a function, F, as

$$F : \mathbb{P}_2 \to \mathbb{R}^3$$

via the rule,

$$\text{if } p(x) = a_0 + a_1x + a_2x^2,$$
$$\text{then } F(p) = (a_0, a_0 + a_2, a_0 + 3a_1 + 4a_2).$$

The function F given above is bijective, which you can assume without proof. Find (and give) the assignment rule for F^{-1} and specify its domain and codomain. Confirm that for a generic $p \in \mathbb{P}_2$ with $p(x) = a_0 + a_1x + a_2x^2$, that you have $[F^{-1}(F(p))](x) = a_0 + a_1x + a_2x^2$.

Solution. We will define $F^{-1} : \mathbb{R}^3 \to \mathbb{P}_2$ as $F^{-1}(u, v, w) = u + ux - \frac{-4vx}{3} + \frac{wx}{3} + vx^2 - ux^2$
Let $p \in \mathbb{P}_2$ be generic
$p(x) = -a_0 + a_1x + a_2x^2$
Thus, $[F^{-1}(F(p))](x) = F^{-1}(a_0, a_0 + a_2, a_0 + 3a_1 + 4a_2)](x)$
$= a_0 + a_0x + \frac{4(a_0+a_2)x}{3} + \frac{x(a_0+3a_1+4a_2)}{3} + x^2(a_0 + a_2 + a_0))$
$= a_0 + a_1x + a_2x^2$