Homework 10; Due Monday, 3/26/2018

Quick Answer Questions. No work needed. No partial credit available.

Question 1. Find all of the $x \in \mathbb{Z}$ such that $x \equiv 2 \mod 3$ and $x \equiv 3 \mod 7$. As a hint, you should consider solutions mod $3 \times 7 = 21$.

Solution. $\{x \in \mathbb{Z} : x \equiv 17 \mod 21\}$

Question 2. Consider the rule f given by $f(p) = [\frac{d}{dx}p]^2$ for functions $p : \mathbb{R} \to \mathbb{R}$. Note that this rule is not a function, as the domain and codomain of f have not yet been specified.

(a) Choose a domain and codomain, both contained in $\mathbb{P}_2$, for the rule f that makes f into a surjective, but not injective, function.

Solution. $f : \{\mathbb{Z}\} \to \mathbb{Z}$

(b) Choose a domain and codomain, both contained in $\mathbb{P}_2$, for the rule f that makes f into an injective, but not surjective, function.

Solution. $f : \{1, 3, 5, 7\} \to \{0\}$

(c) Choose a domain and codomain, both contained in $\mathbb{P}_2$, for the rule f that makes f into an bijective function.

Solution. $f : \{1\} \to \{0\}$

Question 3. (a) Give an example of a sequence which converges to to 5.

Solution. $a_n = (5 + \frac{1}{n})$

(b) Give an example of a sequence which does not converge and $a_n < a_{n-1}$ for all n.

Solution. $a_n = -n$

(c) Give an example of a sequence of irrational numbers which converges to a rational number.

Solution. $a_n = \frac{\sqrt{5}}{n}$

Full Justification Questions. Provide complete justifications for your responses.

Question 4. Using the division lemma, prove that there are no natural numbers n for which $n^2 \equiv 2 \mod 5$. (Hint: Use cases on the remainder of n when divided by 5.)

Solution. We want to prove that if $n^2 \equiv 2 \mod 5$, then $n \notin \mathbb{N}$
Assume that $n^2 \equiv 2 \mod 5$ is true
Thus, by the division lemma (which we can use since every natural number is an integer), n must fall into one of the following 5 cases:
$n \equiv 0 \mod 5, n \equiv 1 \mod 5, n \equiv 2 \mod 5, n \equiv 3 \mod 5, n \equiv 4 \mod 5$.
In order for our original statement to be true, one of these cases must imply that $n^2 \equiv 2 \mod 5$
We will check each case.
Case 1: If r=0, then $n \equiv 0 \mod 5$
So $n = 5x$ for some $x \in \mathbb{N}$
Thus, $n^2 = (5x)^2 = 25x^2 = 5(5x^2)$
So n^2 is of the form $0 \mod 5$
Case 2: If r=1, then $n \equiv 1 \mod 5$
So $n = 5x + 1$ for some $x \in \mathbb{N}$
Thus, $n^2 = (5x + 1)^2 = 25x^2 + 10x + 1 = 5(5x^2 + 2x) + 1$
So n^2 is of the form $1 \mod 5$
Case 3: If r=2, then $n \equiv 2 \mod 5$
So $n = 5x + 2$ for some $x \in \mathbb{N}$
Thus, $n^2 = (5x + 2)^2 = 25x^2 + 20x + 4 = 5(5x^2 + 4x) + 4$
So n^2 is of the form $4 \mod 5$
Case 4: If r=3, then $n \equiv 3 \mod 5$
So $n = 5x + 3$ for some $x \in \mathbb{N}$
Thus, $n^2 = (5x + 3)^2 = 25x^2 + 30x + 9 = 25x^2 + 30x + 5 + 4 = 5(5x^2 + 6x + 1) + 4$
So n^2 is of the form $4 \mod 5$
Case 5: If r=4, then $n \equiv 4 \mod 5$
So $n = 5x + 4$ for some $x \in \mathbb{N}$
Thus, $n^2 = (5x + 4)^2 = 25x^2 + 40x + 16 = 25x + 40x + 15 + 15 = 5(5x^2 + 8x + 3) + 1$
So n^2 is of the form $1 \mod 5$
Since none of the possible cases result in n^2 being of the form $2 \mod 5$, we have proven that $n \notin \mathbb{N}$ because the division lemma only uses integers and since we do not have an integer that works, we can assume that there is not a natural number that works as well. QED

Question 5. Let $n > 1$ be a fixed natural number.

(a) Prove the following: if x is of the form $x = nk + 1$ for some natural number k then $x^2 \equiv 1 \mod n$.

Solution. We let $n > 1$ be a fixed natural number.

We want to prove that if $x = nk + 1$ for some $k \in \mathbb{N}$ then $x^2 \equiv 1 \mod n$

Since $x = nk + 1$, $x^2 = (nk + 1)^2 = nk^2 + 2nk + 1 = n([n]k^2 + 2k) + 1$

Thus, $x^2 = n(d) + 1$ for some $d = (n)k^2 + 2k$

Therefore, x^2 is of the form $1 \mod n$ when $x = nk + 1$ for some $k \in \mathbb{N}$. QED

(b) Find a $y \in \mathbb{N}$ and an $n \in \mathbb{N}$ such that $y \not\equiv 1 \mod n$ and $y^2 \equiv 1 \mod n$.

Solution. Choose $n = 5$ and choose $y \equiv 4 \mod n$
Thus, when plugging in n, we have $y \equiv 4 \mod 5$
Therefore, $y = 5x + 4$ for some $x \in \mathbb{N}$
Hence, $y^2 = (5x+4)^2 = 25x^2 + 40x + 16 = 25x^2 + 40x + 15 + 1 = 5(5x^2 + 8x + 3) + 1$
So $y^2 = 5(r) + 1$for some $r = 5x^2 + 8x + 3$
Thus, $y^2 \equiv 1 \mod n$ for $n = 5$ when $y \equiv 4 \mod n$ which is not equal to $1 \mod n$

Question 6. Fix three natural numbers a, b, c and suppose that $g = \gcd(a, b) = \gcd(a, c)$. Prove that $\gcd(b, c) \geq g$, and give an example where $\gcd(b, c) > g$.

Solution. We want to prove that if $g = gcd(a, b) = gcd(a, c)$ then $gcd(b, c) \geq gcd(a, c)$ for some $a, b, c \in \mathbb{N}$
By the definition of gcd, we let $a = kg, b = lg, c = mg$ for some $k, l, m \in \mathbb{N}$
Thus, we can see that b and c can be factored by at least g, so $gcd(b, c) \geq g$. QED
Example: Choose $a = 2, b = 4, c = 8$
Thus, $gcd(a, b) = gcd(a, c) = 2$ and $gcd(b, c) = 4$ and $4 > 2$

Question 7. Define a sequence by

$$a_n = 5 + \frac{\sin n}{2n}.$$

State whether the sequence converges or diverges. If the sequence converges determine what limit it converges to and prove it. If the sequence does not converge, prove it.

Solution. The sequence converges.
The limit the sequence converges to is 5.
Let $L = 5$ and let $\epsilon > 0$ be generic.
Choose $N = \lceil \frac{1}{2\epsilon} \rceil + 1$
Then for any $n \geq N$, $|5 + \frac{sin(n)}{2n} - 5| = |\frac{sin(n)}{2n}|$
$= \frac{1}{2n} \leq \frac{1}{2N} = \frac{1}{2(\lceil \frac{1}{2\epsilon} \rceil + 1)} = \frac{1}{2\frac{1}{2\epsilon}} = \epsilon$
So $|5 + \frac{sin(n)}{n} - 5| < \epsilon$
Thus, we have proven that this sequence converges to 5. QED

Question 8. Prove that the following sequence does not converge

$$a_n = \frac{2n}{5} - \left\lfloor \frac{2n}{5} \right\rfloor.$$

We note here that the notation $\lfloor x \rfloor$ is the integer floor function for the real variable, x. In particular,

$$\lfloor x \rfloor = \max\{z \;:\; z \in \mathbb{Z}, \text{ and } z \leq x\}.$$

What this means in a practical sense is that if x is **positive** and has a decimal expansion, $\lfloor x \rfloor$ will be the unique integer obtained by taking the decimal expansion of x and removing all terms to the right of the decimal.
(Hint, write out at least the first 16 terms in the sequence... you should see a pattern!)

Solution. We will prove that the sequence $a_n = \frac{2n}{5} - \left\lfloor \frac{2n}{5} \right\rfloor$. does not converge.
Let $L \in \mathbb{R}$ be generic.
Then choose $\epsilon = \frac{1}{5}$ and let $N \in \mathbb{N}$ be generic.
We do cases on L.
If $L > \frac{2}{5}$, then choose $n = 5N$ which has $5N \geq N$
Then $|\frac{2n}{5} - \left\lfloor \frac{2n}{5} \right\rfloor - L| = |\frac{10N}{5} - \left\lfloor \frac{10N}{5} \right\rfloor - L| = |0 - L| = |-L| = L > \frac{2}{5} > \frac{1}{5} = \epsilon$
If $L \leq \frac{2}{5}$ choose $n = 5N - 3$ which has $5N - 3 > N$
Then, $|\frac{2n}{5} - \left\lfloor \frac{2n}{5} \right\rfloor - L| = |\frac{10N-6}{5} - \left\lfloor \frac{10N-6}{5} \right\rfloor - L|$
$= |\frac{4}{5} - L| = \frac{4}{5} - L > \frac{2}{5} > \frac{1}{5} = \epsilon$
Therefore, the sequence does not converge. QED