Homework 12; Due Monday, 4/9/2018

Answer-Only Questions. Credit based solely on the answer.

Question 1. For each of the following relations, determine whether each one is symmetric, transitive, or reflexive. If the relation is an equivalence relation, give the equivalence classes.

(a) Define $\sim$ on the set $\mathbb{P}_3$ by $p \sim q$ if $p(x) \neq q(x)$ for all $x \in \mathbb{R}$.

Solution. Symmetric

(b) For the set $\mathbb{N}$, define $a \sim b$ if $a|b$ and $b|a$.

Solution. Reflexive, Symmetric, Transitive.

Therefore, it is an equivalence relation.

The Equivalence classes are:

$[a] = \{n \in \mathbb{N} | n - a\}$

(c) For the set $\mathbb{Z}$, define $a \sim b$ if $a|b$ and $b|a$.

Solution. None

(d) On the set $\mathbb{P}_5$, define $p \sim q$ if $p(3) \geq q(3)$.

Solution. Reflexive, Transitive

(e) For a set A, define a relation on its power set $\mathcal{P}(A)$ by $B \sim C$ if $C \subseteq B$.

Solution. Reflexive, Transitive.

(f) For a finite set A, define a relation on its power set $\mathcal{P}(A)$ by $B \sim C$ if $|B| = |C|$.

Solution. Reflexive, Symmetric, Transitive.

Therefore, it is an equivalence relation.

The Equivalence classes are:

$[B] = \{D \in \mathcal{P}(A) | S \leq B\}$

(g) For the set of Michigan State students, say that two students are related if they have taken a class together.

Solution. Reflexive, Transitive

Question 2. (a) Given the assignment $f(n) = \frac{4n+2}{4}$, find an appropriate choice of domain so that the codomain of f is $\mathbb{N}$. What is the range of this function? (Recall the range of a function $f : X \to Y$ is the set $\{y \in Y \mid \exists x \in X, f(x) = y\}$.)

Solution. Domain= $\frac{2k+1}{2}$ for $k \in \mathbb{N} \cup \{0\}$

Range= $\mathbb{N}$

(b) Repeat part (a) for $g(n) = \frac{n}{4} + \frac{n}{8}$.

Solution. Domain= $\frac{8x}{3}$ for $x \in \mathbb{N}$

Range= $\mathbb{N}$

(c) Give an example of a rule defining a function with domain $A = \mathbb{N} \cup \{0\}$ and codomain $B = \{1, 2, ..., 598\}$.

Solution. $f : A \to$ B

where $f(n) = 1$

Full Justification Questions. Provide complete justifications for your responses.

Question 3. Define a relation on

$$L = \{f \in \mathbb{P}_1 \mid f(x) = ax \text{ for some } a \in \mathbb{R}\}$$

by $f \sim g$ if and only if $\left|\int_0^1 f(x)\, dx\right| = \left|\int_0^1 g(x)\, dx\right|$.

(a) Prove that $\sim$ is an equivalence relation on L.

Solution. In order to prove that $\sim$ is an equivalence relation on L, we must prove that it is Reflexive, Symmetric, and Transitive.

(i)Reflexive: Let $f \in L$

Then $|\int_0^1 f(x)dx| = |\int_0^1 f(x)dx|$ which is true.

(ii) Symmetric: Let $f, g \in L$ and assume $f \sim g$

Then $|\int_0^1 g(x)dx| = |\int_0^1 f(x)dx|$

(iii)Transitive: Let $f, g, h \in L$

Assume $f \sim g$ and $g \sim h$

Thus, $|\int_0^1 f(x)dx| = |\int_0^1 fg(x)dx|$ and $|\int_0^1 g(x)dx| = |\int_0^1 h(x)dx|$

Therefore, we know $|\int_0^1 f(x)dx| = |\int_0^1 h(x)dx|$

Since $\sim$ is reflexive symmetric, and transitive, we can conclude that it is an equivalence relation.

(b) Find and give a complete list of all the distinct equivalence classes of $\sim$. Prove that your list is accurate and complete. (Recall, this means you must show that your equivalence classes are pairwise disjoint and that their union is L.)

Question 4. Prove that for all $n \in \mathbb{N}$ and for every polynomial, p, of degree n,

$$\frac{d^{n+1}}{dx^{n+1}}p(x) = 0$$

for all $x \in \mathbb{R}$. Hint: Use induction on the degree of p; you may assume (without proof) the power rule for differentiation.

Solution. We will use proof by Mathematical Induction
Conditional Statement: $P(n) : \frac{d^{n+1}}{dx^{n+1}}p(x) = 0$
Since $p(x) = \sum_{i=0}^{n} a_i x^i$, the conditional statement becomes $P(n) : \frac{d^{n+1}}{dx^{n+1}} \sum_{i=0}^{n} a_i x^i = 0$
Now we check the base case, where n=1
$P(1) : \frac{d^{1+1}}{dx^{1+1}} \sum_{i=0}^{1} a_i x^i$
$= \frac{d^2}{dx^2} a_0 + a_1 x^i = 0$
$0 = 0$ so P(1) checks out
For our inductive assumption, we want to prove that $P(k) \implies P(k = 1)$ so we assume $P(k)$ is true.
$P(k) : \frac{d^{k+1}}{dx^{k+1}} \sum_{i=0}^{k} a_i x^i = 0$ for some $k \in \mathbb{N}$
Thus, $P(k+1) : \frac{d^{k+2}}{dx^{k+2}} \sum_{i=0}^{k+1} a_i x^i = 0$
When expanding the summation, we get $\frac{d^{k+2}}{dx^{k+2}} \sum_{i=0}^{k+1} a_i x^i = \frac{d^{k+2}}{dx^{k+2}}(a_{k+1}x^{k+1}) + \frac{d}{dx}(\frac{d^{k+1}}{dx^{k+1}} \sum_{i=0}^{k} a_i x^i)$
Using the inductive assumption above, this is equal to $\frac{d^{k+2}}{dx^{k+2}}(a_{k+1}x^{k+1}) + \frac{d}{dx}(0) = \frac{d^{k+2}}{dx^{k+2}}(a_{k+1}x^{k+1})$
Thus, $\frac{d^{k+2}}{dx^{k+2}}(a_{k+1}x^{k+1}) = \frac{d^{k+1}}{dx^{k+1}}(\frac{d}{dx}a_{k+1}x^{k+1})$
$= \frac{d^{k+1}}{dx^{k+1}}[(k+1)(a_{k+1}x^k)]$
Now we know that $[(k+1)a_{k+1}x^k]$ is in $P(k)$ since it is in $\mathbb{R}$
So $\frac{d^{k+1}}{dx^{k+1}}[(k+1)a_{k+1}x^k] = 0$, which is $P(k+1)$
Thus, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$

Question 5. Consider the set of functions

$$A = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) = \sin(nx) \text{ for some } n \in \mathbb{Z}\}.$$

Prove or disprove the following two statements:

(a) For all $f \in A$, there exists an $a \in \mathbb{N} \cup \{0\}$ such that $f''(x) + af(x) = 0$ for all $x \in \mathbb{R}$.

Solution. This statement is true.

Let $f \in A$ be generic.

Choose $a = n^2$ for some $n \in \mathbb{Z}$ (Note that $n \in Z$ so $n^2 \in \mathbb{N} \cup \{0\}$)

We want to show that $f''(x) + af(x) = 0$ for all $x \in \mathbb{R}$.

So $-n^2 sin(nx) + asin(nx) = 0$

$\implies -n^2 sin(nx) + n^2 sin(nx) = 0$

Thus, $0 = 0$

Since this is true, we have proven the original statment. QED

(b) There exists $a \in \mathbb{Z}$ such that for all $f \in A$, $f''(x) + af(x) = 0$ for all $x \in \mathbb{R}$.

Solution. This statement is true.

Let $f \in A$ be generic.

Choose $a = n^2$ for some $n \in \mathbb{Z}$

We want to show that $f''(x) + af(x) = 0$ for all $x \in \mathbb{R}$.

So $-n^2 sin(nx) + asin(nx) = 0$

$\implies -n^2 sin(nx) + n^2 sin(nx) = 0$

Thus, $0 = 0$

Since this is true, we have proven the original statment. QED

Question 6. Consider the function $F : \mathbb{P}_2 \to \mathbb{P}_2$ given by

$$F(p)(x) = -x^2 \cdot p\left(\frac{1}{x}\right)$$

1. Show that F is a bijection.

Solution. We will show that F is a bijection by proving it is injective and surjective.

Injective: Let $m, n \in \mathbb{P}_2$ be generic

Assume $F(p)(m) = F(p)(n)$

This means $-m^2(p)(\frac{1}{m}) = -n^2(p)(\frac{1}{n})$

$\implies \frac{-m^2}{m} = \frac{-n^2}{n}$

$\implies -m = -n \implies m = n$

Thus, F is injective.

Now, we will prove that F is surjective.

Let $y \in \mathbb{P}_2$ be generic

Set $x = \frac{-y}{p}$

Then $f(p)(x) = -(\frac{-y}{p})^2(p)(\frac{1}{\frac{-y}{p}})$

$\implies \frac{-y^2}{p^2}\frac{-p^2}{y}$

$\implies \frac{-y^2}{y} = y$

Therefore, F is surjective.

Since F is both injective and surjective, F is bijective. QED

2. Show that $F^{-1} = F$.

Solution. Define $F^{-1} : \mathbb{P}_2 \to \mathbb{P}_2$ by $F^{-1}(p)(y) = (\frac{-y}{p})$ as our inverse function and verify $F[F^{-1}(p)(y)] = F(p)(\frac{-y}{p}) = y$ (as proven above) and $F^{-1}(F(p)(x)) = F^{-1}(-x^2 \cdot p\left(\frac{1}{x}\right)) = x$

Question 7. For each $n \in \mathbb{N}$, $n > 1$, define the interval, I_n, as

$$\left[\frac{1}{n}, 1 + \frac{1}{n}\right].$$

1. Define $X = \bigcup_{n=2}^{\infty} I_n$. Find a simpler expression for X and provide a proof that this simpler expression is equal to X.

Solution. Claim: $X = \bigcup_{n=2}^{\infty} I_n = (0, 1.5]$

First, we will prove $X \subseteq (0, 1.5]$

Let $x \in X$ be generic

Then $x \in I_n$ for some $n \in \mathbb{N}, n > 1$

So $x \in [\frac{1}{n}, 1 + \frac{1}{n}]$

Hence $\frac{1}{n} \leq x \leq 1 + \frac{1}{n}$

Since $n > 1, \frac{1}{n} > 0$

Hence, $0 < \frac{1}{n} \leq x \leq 1 + \frac{1}{n} \leq 1.5$

Thus, $0 < x \leq 1.5$

So $x \in (0, 1.5]$

Hence, $X \subseteq (0, 1.5]$

Next, we prove $(0, 1.5] \subseteq X$

Let $x \in (0, 1.5]$ be generic

Choose $n \leq \frac{1}{x-1}$ by the Archimedean Principle

Then $x - 1 \leq \frac{1}{n}$

So $x \leq 1 + \frac{1}{n}$

Now choose $\frac{1}{x} \leq n$ by the Archimedean Principle.

Then $\frac{1}{n} \leq x$

So $\frac{1}{n} \leq x \leq 1 + \frac{1}{n} \implies x \in F_n \implies x \in X \implies (0, 1.5] \subseteq X$

Since $X \subseteq (0, 1.5]$ and $(0, 1.5] \subseteq X, X = (0, 1.5]$

2. Define $Y = \bigcap_{n=2}^{\infty} I_n$. Find a simpler expression for Y and provide a proof that this simpler expression is equal to Y.

Solution. Claim: $Y = \bigcap_{n=2}^{\infty} I_n = [0.5, 1)$
First we prove $Y \subseteq [0.5, 1)$
Assume $y \in Y$ is generic.
We will use proof by contradiction.
Assume there exists a $y \in Y$ such that $y \notin [0.5, 1)$
If $y \notin [0.5, 1)$, Then $y < 0.5$ or $y \geq 1$
We consider the case $y \geq 1$
By the Archimedean Principle of $\mathbb{R}$, there exists $x_0 \in \mathbb{N}$ such that $x_0 \geq \frac{1}{y-1}$ and since $x_0^2 \geq x_0$, we have $y > 1 + \frac{1}{x_0^2}$
This implies $y \notin [\frac{1}{x_0^2}, 1 + \frac{1}{x_0^2}]$, which contradicts the original assumption that $y \in Y$
Thus, we can conclude that $Y \subseteq [0.5, 1)$
Next we will prove that $[0.5, 1) \subseteq Y$
Let $y \in [0.5, 1)$ be generic
So $0.5 \leq y < 1$
Since $n > 1, \frac{1}{n} \leq 0.5$
So $\frac{1}{n} \leq 0.5 \leq y < 1 < 1 + \frac{1}{n}$
So $y \in F_n$ for every $n \in \mathbb{N}, n > 1$
Therefore, $y \in Y$
Hence, $[0.5, 1) \subseteq Y$
Since $Y \subseteq [0.5, 1)$ and $[0.5, 1) \subseteq Y, Y = [0.5, 1)$