Homework 3; Due Monday, 01/29/2018

Answer-Only Questions. Credit based solely on the answer.

Question 1. For each statement below, compute its negation, and say whether the original statement or its negation is true.

(a) $\forall p \in \mathbb{P}_3, \exists r \in \mathbb{R}, p(r) = 0$

Solution. $\exists p \in \mathbb{P}_3, \forall r \in \mathbb{R}, p(r) \neq 0$

The Negation is true.

(b) $\exists r \in \mathbb{R}, \forall p \in \mathbb{P}_4, p(r) \neq 0$

Solution. $\forall r \in \mathbb{R}, \exists p \in \mathbb{P}_4, p(r) = 0$

The Negation is true.

(c) $\forall r \in \mathbb{R}, \exists p \in \mathbb{P}_4, p(r) = 0$

Solution. $\exists r \in \mathbb{R}, \forall p \in \mathbb{P}_4, p(r) \neq 0$

The Original Statement is true.

(d) $\forall p \in \{f \in \mathbb{P}_4 : \deg(f) \geq 2\}, \exists r \in \mathbb{R},\ p'(r) = 0$

Solution. $\exists p \in \{f \in \mathbb{P}_4 : \deg(f) < 2\}, \forall r \in \mathbb{R},\ p'(r) \neq 0$

The Negation is true.

(e) $\exists p \in \mathbb{P}_4, \forall q \in \mathbb{P}_4, pq = q$

Solution. $\forall p \in \mathbb{P}_4, \exists q \in \mathbb{P}_4, pq \neq q$

The Original Statement is true.

(f) $\forall p \in \mathbb{P}_2, \exists q \in \mathbb{P}_4, pq = p^3$

Solution. $\exists p \in \mathbb{P}_2, \forall q \in \mathbb{P}_4, pq \neq p^3$

The Original Statement is true.

(g) $\forall p \in \mathbb{P}_4, \exists q \in \mathbb{P}_4, p + q = 0$

Solution. $\exists p \in \mathbb{P}_4, \forall q \in \mathbb{P}_4, p + q \neq 0$

The Original Statement is true.

Full-Justification Questions. Provide a proof for the questions below.

Question 2. Suppose $a, b \in \mathbb{R}$ and $a \neq b$. Use proof by contrapositive to show the following implication is true:

$$b \leq a \implies ba^2 + b^3 \leq a^3 + ab^2.$$

Solution. We know $a, b \in \mathbb{R}$ and $a \neq b$. We use proof by contrapositive. The contrapositive statement is $ba^2 + b^3 > a^3 + ab^2 \implies b > a$. Assume $ba^2 + b^3 > a^3 + ab^2$ is true. Thus, $b(a^2 + b^2) > a(a^2 + b^2)$. Since $(a^2 + b^2) = (a^2 + b^2)$ and is greater than 1, we can assume that $b > a$. Therefore, $\neg P$ is true under this assumption. Thus, $\neg Q \implies \neg P$ is true and since this is equivalent to the original statement, we are done. QED

Question 3. Prove that there does **not** exist a positive rational $y \in \{q \in \mathbb{Q} \mid q > 0\}$ such that $2^y = 3$. You may use the facts that for any $n \in \mathbb{N}$, 2^n is even and 3^n is odd, which we will later be able to justify by mathematical induction.

Solution. Assume for the sake of contradiction that $\neg$ (there does not exist a positive rational $y \in \{q \in \mathbb{Q} \mid q > 0\}$ such that $2^y = 3$). We know that 2^n is even for some $n \in \mathbb{N}$. Since 3 is an odd number, this contradicts our assumption. Therefore, our assumption must be incorrect so the original statement is true. QED

Question 4. Let $p \in \mathbb{P}_2$ and $q \in \mathbb{P}_3$ be two nonzero polynomials.

(a) Show that the sum $p + q$ is a polynomial function.

Solution. We want to show that the sum $p+q$ is a polynomial function. Since the highest degree for p is 2 and the highest degree for q is 3, we know that the highest degree for the added p and q must be 3. We also know know that the p added to the function will add a power of 2 to the function. Therefore, there must be at least two functions, making the function a polynomial.

(b) Show that $\deg(pq) = \deg(p) + \deg(q)$.

Solution. We want to show that $\deg(pq) = \deg(p) + \deg(q)$. When two functions are added together, the degree of the function is the highest power of the function. Hence, the degree of $p + q$ will be 3 because the highest degree of the two functions is 3. Therefore, $\deg(pq) = \deg(p) + \deg(q)$ because the highest degree in the function will be considered the degree of the whole function. QED

(c) Show that it is **not** necessarily true that if $p, q \in \mathbb{P}_4$ then $\deg(p+q) = \deg(p) + \deg(q)$. (That is, show that $\forall p \in \mathbb{P}_4, \forall q \in \mathbb{P}_4, \deg(p+q) = \deg(p) + \deg(q)$ is a *false* statement.)

Solution.

Question 5. Let A and B be sets. Show that $A \setminus B = \emptyset$ if and only if $A \subseteq B$. (The method of contrapositive is convenient here, though not necessary.)

Solution. We know that A and B are sets. We want to prove that $A \setminus B = \emptyset$ if and only if $A \subseteq B$. We will use proof by contrapositive. The contrapositive statement $A \not\subseteq B \implies A \setminus B \neq \emptyset$. Assume that $A \not\subseteq B$ is true. Then there is at least one $x \in A$ such that $x \notin B$. Thus, $A \setminus B \neq \emptyset$ because there exists and element in the set. Therefore, $A \not\subseteq B \implies A \setminus B \neq \emptyset$ is true which is equivalent to the original statement so we are done. QED

Question 6. Let $f \in \mathbb{P}_3$ be defined by the rule $f(x) = 2x^3 + 4x^2 + x$. Consider the following statement:

$$\forall\ a \in \mathbb{Z},\ \big(\text{if } f(a) \text{ is odd, then } a \text{ is odd}\big).$$

Prove this statement in three ways:

1. Use proof by contradiction (that is to say, assume that the negation of the original statement is true and find a contradiction. You will have to negate the quantified statement.)

   Solution. Assume for the sake of contradiction that $\neg[\forall a \in \mathbb{Z}$, (if f(a), is even, then a is odd)]. This is equivalent to $\exists a \in \mathbb{Z}, (f(a)$ is odd and a is even). Since a is even, $a = 2k$ for some $k \in \mathbb{Z}$. Therefore, $2(2k)^3 + 4(2k)^2 + 2k$ is even. Thus, $2k[2(2k)^2 + 4(2k)] + 1$ is even. This contradicts our original staement because x is even but the function is odd. Therefore our assumption must be incorrect so the original statement is true. QED

2. Use proof by contrapositive (here, we mean only apply the contrapositive argument to the "if-then" appearing inside of the parentheses, leaving the "$\forall\ a \in \mathbb{Z}$" as it is.)

   Solution. We use proof by contrapositive. The contrapositive statement is a is even $\implies$ f(a) is even. We assume that a is even. Since a is even, $a = 2k$ for some $k \in \mathbb{Z}$. Therefore, $2(2k)^3 + 4(2k)^2 + 2k$. Since this is an even function, a is even $\implies$ f(a) is even. This is equivalent to the original statement so we are done. QED

3. Give a direct proof. (Hint: Solve for a.)

Solution. We want to prove that $\forall\ a \in \mathbb{Z},\ \big(\text{if } f(a) \text{ is odd, then } a \text{ is odd}\big)$. We will assume that a is odd. This means that a=2k for some $k \in \mathbb{Z}$. Thus, $f(a) = 2(2k)^3 + 4(2k)^2 + 2k$. This is equivalent to $2k[2(2k)^2 + 4(2k)] + 1$. This, by definition, is an odd number. Therfore, $\forall\ a \in \mathbb{Z},\ \big(\text{if } f(a) \text{ is odd, then } a \text{ is odd}\big)$. QED

Question 7. Let the set R be defined by

$$R = \{f \in \mathbb{P}_2 \mid f(x) = ax^2 + bx + c, a > 0, b^2 - 4ac = 0\}.$$

Assume $p \in R$ and then prove that there does **not** exist $r \in \mathbb{R}$ such that $p(r) < 0$ (Hint: use contradiction and try factoring p by 'completing the square' or by the 'quadratic formula').

Solution. We want to prove that there does **not** exist $r \in \mathbb{R}$ such that $p(r) < 0$. We assume for the sake of contradiction that $\exists r \in \mathbb{R}$ such that $p(r) \geq 0$.