Homework 2; Due Monday, 01/22/2018

Answer-Only Questions. Credit based solely on the answer.

Question 1. Construct truth tables for the following:

(a) (not A) or (not B)

A	B	not A	not B	(not A) or (not B)
T	F	F	T	T
T	T	F	F	F
F	F	T	T	T
F	T	T	F	F

(b) (A or (not B)) and (B or (not$A$))

A	B	not A	not B	(A or (not B))	(B or (not$A$))	(A or (not B)) and (B or (not$A$))
T	F	F	T	T	F	F
T	T	F	F	T	T	T
F	F	T	T	T	T	T
F	T	T	F	F	T	F

(c) $A \implies$ not(B or C)

A	B	C	B or C	not(B or C)	$A \implies$ not(B or C)
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	F	T	T
F	T	T	T	F	T
F	T	F	T	F	T
F	F	T	T	F	T
F	F	F	F	T	T

Question 2. Consider the following sets:
$A = \{2, 7, \text{cat}, \text{plutonium}, \text{chalk}\}$, $B = \{\text{red}, \text{blue}, 3, 4, 7, \text{chalk}\}$, $C = \{2, 7\}$, and $X = A \cup B \cup C$
For which $x \in X$ are the following conditional statements true? Write you answer as the SET of elements which make the conditional statement true.

(a) $x \in A \implies x \in C$

Solution. {2,7}

(b) $(x \in C \text{ or } x \in A) \implies x \in A$

Solution. {2, 7, cat, plutonium, chalk}

(c) $x \in C \implies x \in A$

Solution. {2,7}

(d) $(x \in A \text{ and } x \in B) \implies x \in C$

Solution. {7, chalk}

Question 3. Consider the following sets:
$A = \{2, 4, 6, 8, 10\}$, $B = \{1, 2, 3, 4, 5\}$, $C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and $D = \{2k \mid k \in \mathbb{N}\}$.
Decide whether the following quantified statements are true.

(a) $\forall x \in A$, $x \in B$

Solution. False

(b) $\exists x \in A$, such that $x \in B$

Solution. True

(c) $\forall x \in A$, $x \in D$

Solution. True

(d) $\forall x \in D$, $x \in C$

Solution. False

(e) $\exists x \in B$, such that $x \in C \backslash A$

Solution. True

(f) $\forall x \in A$, $x \in C \cap D$

Solution. True

(g) $\exists x \in B \cap D$

Solution. True

Question 4. For each of the following statements, rewrite them using quantifiers and state the truth value for each one:

(a) For all x in $\mathbb{Z}$, there exists an integer p that divides x.

Solution. $\forall x \in \mathbb{Z}$, $\exists p \in \mathbb{N}|x$
$p = (-\infty, 0) \cup (0, \infty)$

(b) There is an x in $\mathbb{Q}$ such that $x^2 = 2$.

Solution. $\exists x \in \mathbb{Q}$, $x^2 = 2$
$x = \{-1.414, 1.414\}$

(c) There exists an x in $\mathbb{Z}$ such that $\sqrt{x}$ is an integer.

Solution. $\exists x \in \mathbb{Z}$, $\sqrt{x} \in \mathbb{N}$
$x = \{1, 4, 9, 16, 25, 36, ...\}$

(d) For all x in $\mathbb{Q}$, $\sqrt{x}$ is a rational number.

Solution. $\forall x \in \mathbb{Q}$, $\sqrt{x} \in \mathbb{Q}$
$x = \{a \in \mathbb{Q} : a \geq 0\}$

Full-Justification Questions. Provide a proof for the questions below.

Question 5. Which of the following are statements? Explain. If it is a conditional statement but not a statement , explain why.

(a) Pythagoras was vicious.

Solution. Not a statement. This sentence is based on an opinion and cannot be proven true or false.

(b) The square root of a negative number is a negative number.

Solution. Statement. This statement is false because the square root of a negative number does not exist. However, since it does not depend on a specific variable, it is not a conditional statement. It is simply a false statement.

(c) $4x^3 - 9x^2 + 3x - 25 = 0$

Solution. Conditional Statement. Whether the statement can be proven true or false depends on the variable "x". For some values the statement is true but for other values the statement is false.

(d) Socrates was a very wise man.

Solution. Not a statement. This sentence is based on an opinion and cannot be proven true or false.

(e) There exists a blackboard in every classroom in Wells Hall.

Solution. Statement. Since this statement is either true or false and does not depend on a variable to change it, it is not a conditional statement. The only thing that matters is if evey classroom has a blackboard or not.

(f) Blackboards are superior to whiteboards.

Solution. Not a statement. This sentence is based on an opinion and cannot be proven true or false.

(g) If p is prime, its only positive divisors are 1 and p.

Solution. Statement. This statement is true because it is unambiguous and no other variable can prove it to be false. A prime number is a number that is only divisible by 1 and itself, and that is exactly what this statement is saying.

(h) For all θ such that $0 \leq \theta \leq \pi$, $\sin\theta$ is positive.

Solution. Statement. This is true because all $\sin\theta$ values where $0 \leq \theta \leq \pi$ are in fact positive. It is exactly true and no other variables can change that.

Question 6. Let n be an integer. Prove the following statement: If 6 divides n then n is even.
Hint: The definition of *divides* is in the supplemental document.

Solution. We claim that $n \in \mathbb{N}$. We want to prove that if $6|n$, n is an even number. Assume $n \in \mathbb{N}$ is generic. We can pick n to be multiples of 6 since every multiple of 6 is an integer. Therefore, $x = \{n \in \mathbb{N}, 6n\}$. Since 6 itself is an even number, every value of 6n must also be eve for some $n \in \mathbb{N}$. This is true because adding up the same even number over and over again will result in bigger even numbers. Therefore, if $6|n$, we have proven that n is an even number. QED

Question 7. Prove the following statement: $\forall y > 0,\ \exists x \in \mathbb{R}$ such that $e^{3x} = \dfrac{y}{2}$.

Solution. We want to prove that for every y value greater than 0, there exists an x that is the element of the real numbers set such that $e^{3x} = \dfrac{y}{2}$. Let $x \in \mathbb{R}$ and $y > 0$ be generic. We will choose x to be equal to 0. Thus, $e^{3x} = 1$. We can also choose y to be equal to 2. Therefore, $\frac{y}{2} = 1$ since $\frac{2}{2} = 1$. Thus, $e^{3x} = \dfrac{y}{2}$ because $1 = 1$. Therefore, we have conditionally proven that $\forall y > 0,\ \exists x \in \mathbb{R}$ such that $e^{3x} = \dfrac{y}{2}$. QED

Question 8. Prove the following statement: $\exists n \in \mathbb{Z}$ such that $\forall m \in \mathbb{Z}$, $mn = m$.

Solution. We want to prove that there exists an n that is an element of the integers such that for every m that is an integer, $mn = m$. We claim $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$. Let $n \in \mathbb{Z}$ be generic. We choose n to equal 1. Since $nm = 1$ and $n = 1$, $m * 1 = m$. Considering $1 \in \mathbb{Z}$, we have proven that $\exists n \in \mathbb{Z}$ such that $\forall m \in \mathbb{Z}$, $mn = m$. QED