Advanced Microeconomics for Policy Analysis I

Fall 2021

Problem Set 6

Production

Problem 1. Find and draw the smallest production set Y in R^2 that satisfies

1. Possibility of inaction.
2. Free disposal.
3. Closed.
4. No free lunch.
5. Irreversibility.
6. Constant returns to scale.
7. Convexity.

Hint: follow the list in the given order.

Given the set you found, would you say these assumptions are sufficient to obtain an interesting theory of production?

Problem 2. Consider a production set $Y \subset \mathbb{R}^n$. Let good 1 be output. All other $n-1$ goods are inputs. Let $-z$ denote such vector. Show that Y is convex if, and only if, the production function $f(z)$ is concave.

Problem 3. Consider a profit-maximizing firm. The firm uses $n-1$ inputs, z, to produce a single output, q, according to a production function $f(z)$. Let $y = (q, -z)$. A generic price vector is denoted by $p \in \mathbb{R}^n$ and it is assumed to satisfy $p \gg 0$. We partition this vector into two components: $p = (p_q, w)$, where $p_q > 0$ is the price of output and $w \in R^{n-1}_{++}$ is the vector price of inputs. The firm takes all prices as given. The maximization problem of the firm is therefore

$$\max_{y} \; py \qquad \text{(PMP)}$$
$$\text{s.t.: } q \leq f(z)$$

Let $\pi(p)$ denote the value function of the problem (i.e., maximized profits).

a. Argue that, at the optimum (if one exists), the constraint of the problem must bind.

b. Show that $\pi(q)$ is a convex function of p.

   Hint: Use a revealed preference argument. The steps will be very similar to the ones employed in proving that the expenditure function $M(p, \overline{U})$ is concave in p.

c. Using a revealed preference argument, show that the law of supply always holds, that is: if $p_q' > p_q$, then $q^{*\prime} \geq q^*$.

   Show also that the quantity demanded of an input is non-increasing in its price.

d. Assuming that the the solution to the maximization problem, $y^*(p)$, is differentiable at p (which implies it is a singleton at p), use the Envelope Theorem to prove the "derivative" version of the result in part (c): $\partial q^*/\partial p_q \geq 0$, and $\partial z_j^*/\partial w_j \leq 0$.

e. Can you show that the cross-price effects are symmetric? That is, $\partial y_j^*/\partial p_k = \partial y_k^*/\partial p_j$ for all $k, j = \{1, \ldots, n\}$.

   Notice that, in order to write these derivatives in the most general way, we are denoting output q by y_1 and its price p_q by p_1. The analogous remark applies to the case of inputs and its prices.

f. Consider the cost-minimization problem of our firm:

$$\min_{z \geq 0} \; wz \qquad \text{(CMP)}$$
$$\text{s.t.: } \; q \leq f(z)$$

   where q is now a parameter of the problem. The firm is only choosing inputs z.

   Show that cost minimization is a *necessary* condition for profit-maximization. That is, if $y^* = (q^*, -z^*)$ solves (PMP), z^* solves (CMP) for $q = q^*$.

   Hint: The proof of this result is similar to the one of the Duality Lemma (Lemma 8.1) in the lecture notes.

g. Let $c(q, w)$ denote the value function of the problem (CMP). We call this object the **cost function** of the firm. Show that the profit-maximization problem (PMP) is *equivalent* to

$$\max_{q \geq 0} pq - c(q, w) \qquad \text{(PMP')}$$

   That is, show that,

   a. If $y^* = (q^*, -z^*)$ solves (PMP), then q^* solves (PMP').
   b. If q^* solves (PMP'), then $(q^*, -z^*)$ solves (PMP), where z^* is the solution to (CMP) for $q = q^*$.

   Hint: Once again, base your proof on a revealed preference argument (plus the result from part (f), at some points).

Problem 4. Solve the profit-maximization problem of a firm that produces a single output according to a technology $f(z)$, with $z \in \mathbb{R}^2_+$, where

a. $f(z) = \sqrt{z_1 + z_2}$;

b. $f(z) = \sqrt{\min\{z_1, z_2\}}$;

c. $f(z) = (z_1^\rho + z_2^\rho)^{1/\rho}$, with $\rho < 1$.

Denote by $w_i > 0$ the price of input $i = 1, 2$, and by $p_q > 0$ the price of output.

In each case, find the profit function $\pi(p)$ (i.e., maximized profits).

Verify that, whenever the solution exists and it is a singleton, $\partial\pi(p)/\partial p_i = y_i^*(p)$, for all i.

Important: be aware of Theorem 5.1 and Corollary 5.1. So, before doing anything, you should check whether the production function exhibits nondecreasing or constant returns to scale. If Theorem 5.1 and/or Corollary 5.1 apply, solve the cost minimization problem (CMP) (instead of the profit maximization problem, PMP) to find the demand for inputs and the cost function. Then solve (PMP').