Advanced Microeconomics for Policy Analysis I

Fall 2021

Problem Set 5

Envelope Theorem and Shepard's Lemma

Problem 1. Define the function f by $f(x,r) = x^{1/2} - rx$, where $x \geq 0$. On a graph with r on the horizontal axis, sketch the function for several values of x. (E.g. sketch the functions $(1/2)^{1/2} - r/2$, $1^{1/2} - r$, and $2^{1/2} - 2r$.) Sketch, in addition, the value function f^*, where $f^*(r)$ is the maximal value of $f(x,r)$ for each given value of r.

Problem 2. The output of a good is $x^a y^{1-a}$, where $x \geq 0$ and $y \geq 0$ are the amounts of two inputs and $a \in (0,1)$ is a parameter. A government-controlled firm is directed to maximize output subject to meeting the constraint $2x + y = b$, with $b > 0$.

a. Solve the firm's problem.

b. Use the envelope theorem to find how the maximal output changes as the parameter b varies.

c. Given your results in part (a), compute the value function of the problem. Verify that its derivative with respect to b is equal to the expression you found in part (b).

Problem 3. A firm's output depends on the amount $x \geq 0$ of labor and labor productivity, a, according to the function $f(x,a)$. f is increasing in a, for any $x \geq 0$. The price of the firm's output is $p > 0$ and the price of labor is $w > 0$. Determine the sign of the derivative of the firm's maximal profit with respect to a.

Duality in consumer theory

Problem 4. Suppose that $u(\cdot)$ is a continuous utility function representing a locally non-satiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^n_+$. Assume all prices are positive, so $p \gg 0$.

(i) Show that, for any p, the compensated demand $x^c(p,U)$ satisfies homogeneity of degree zero in p: $x^c(\alpha p, U) = x^c(p,U)$ for any p, U, and $\alpha > 0$.

(ii) Based on (i), show that $D_p x^c(p,U) \cdot p = 0$, where $D_p x^c(p,U)$ is the gradient with respect to vector p.

(iii) Based on (ii) and the compensated law of demand, show that every good ℓ has at least one substitute.

Problem 5. Consider a consumer with preferences defined on $\mathbb{R}^3_+$ and represented by the utility function $u(\boldsymbol{x}) = (x_1 - a_1)^\alpha (x_2 - a_2)^\beta (x_3 - a_3)^\gamma$, where all parameters are strictly positive. Assume also that $I > p_1 a_1 + p_2 a_2 + p_3 a_3$.

a. Using the fact that any strictly monotonic transformation of u represents the same preference relation, show that there is no loss of generality in assuming that $\alpha + \beta + \gamma = 1$.

   From now on assume that $\alpha + \beta + \gamma = 1$.

b. Find the Marshallian demand functions, and the indirect utility function.

   Trick: rename the variables, let $\tilde{x}_i := x_i - a_i$ for all $i = 1, 2, 3$, and rename income too, $\tilde{I} := I - p_1 a_1 - p_2 a_2 - p_3 a_3 > 0$. Rewrite the problem in terms of these new variables and parameters. Now your maximization problem looks more standard, doesn't it? Solve this new problem, and at the end remember that you want expressions for the optimal values of the original variables, x_i, in terms of the original parameters.

c. Given your results in part (b), verify that points (1) and (3) in Theorem 4.1: the demand functions are homogeneous of degree 0 in p and I, and $px^*(p, I) = I$.

d. Verify that the indirect utility function is homogeneous of degree 0 in p and I, strictly increasing in I, and decreasing in p_i for all i.

e. Verify Roy's Identity: compute $\partial V / \partial p_i$ and $\partial V / \partial I$, and show that the negative of their ratio is equal to x_i^*, for all $i = 1, 2, 3$.

f. Find the compensated demand functions, and the expenditure function, for some utility level $\overline{U} > 0$.

   Suggestion: for solving the expenditure minimization problem, apply the same trick as in part (b).

g. Evaluate the compensated demand functions at $\overline{U}$ equal to the value function you found in part (b). Is Lemma 8.1(1) verified?

   Evaluate the Marshallian demand functions you found in (b) at I equal to the value function you found in part (f). Is Lemma 8.1(2) verified?

h. Verify Shepard's Lemma: Show that the derivative of the expenditure function with respect to each price is equal to the corresponding compensated demand function.

i. Verify the Slutsky equation: for each good i do one verification for $k = i$ (own-price) and another one for $k \neq i$ (cross-price).

j. Verify that the own-price effect, $\partial x_i^c / \partial p_i$, is negative, and the cross-price effects, $\partial x_i^c / \partial p_k$, are symmetric: $\partial x_i^c / \partial p_k = \partial x_k^c / \partial p_i$ for all i, k.