PPHA 41501 – GAME THEORY

University of Chicago, Harris School of Public Policy

Autumn Quarter 2021, Monday/Wednesday 8:30-9:50am (Central Time)

Syllabus: <http://home.uchicago.edu/~rmyerson/teaching/ppha41501.pdf>

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Teaching Assistants: Kisoo Kim and Jose Pascual Moreno

These slides: <http://home.uchicago.edu/~rmyerson/teaching/ppha41501slides.pdf>
Extensive notes: <http://home.uchicago.edu/~rmyerson/teaching/ppha41501nts.pdf>

Course grades will be based on homework assignments, midterm exam, and final exam. Homework is graded on basis of effort only. You may discuss problems with each other, but you must turn in your own work. Do not copy and paste answers from others' work. Assignments at <https://home.uchicago.edu/~rmyerson/teaching/ppha41501hwk.pdf>

Assignment 1 should be prepared for discussion in class on Sept 29, not to be handed in.

Instructor's office hours: Thursday 8:30-10:00 online, 11:00-noon hybrid.

A **model of decisions under uncertainty** is characterized by:
a set of alternative choices C, a set of possible states of the world S,
a utility function $u:C\times S \to \mathbb{R}$, and a probability distribution p in $\Delta(S)$.
Suppose that C and S are nonempty finite sets. We use the notation:
$\Delta(S) = \{q\in\mathbb{R}^S \mid q(s)\geq 0\ \forall s\in S,\ \sum_{\theta\in S} q(\theta) = 1\}$. ($\forall s\in S$, $[s]\in\Delta(S)$ with $[s](s)=1$.)
The <u>expected utility hypothesis</u> says that an optimal decision should
maximize expected utility $Eu(c) = Eu(c|p) = \sum_{\theta\in S} p(\theta)u(c,\theta)$ over all c in C,
for some utility function u that is appropriate for the decision maker.

<u>**Fact:**</u> Given utility function $u:C\times S\to\mathbb{R}$ and some choice $d\in C$, the set of probability distributions that make d optimal is a closed convex (possibly empty) subset of $\Delta(S)$.
This set (of probabilities that make d optimal) is empty if and only if there exists some randomized strategy σ in $\Delta(C)$ such that $u(d,s) < \sum_{c\in C} \sigma(c)u(c,s)\ \ \forall s\in S$.
When these inequalities hold, we say that d is <u>strongly dominated</u> by σ.

Here we use the same $\Delta(\bullet)$ notation as above, so
$\Delta(C) = \{\sigma\in\mathbb{R}^C \mid \sigma(c)\geq 0\ \forall c\in C,\ \sum_{e\in C} \sigma(e) = 1\}$.

Example 1. Consider an example with choices C = {T,M,B}, state S = {L,R}, and u(c,s):

	L	R
T	7	2
M	2	7
B	5	6

Let the parameter r denote the probability of state R.

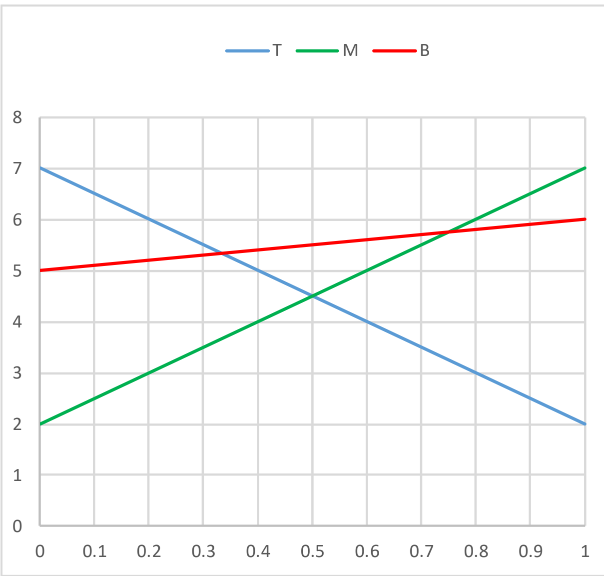
$Eu(T|r) = (1-r)7+r2 = 7-5r$,

$Eu(M|r) = (1-r)2+r7 = 2+5r$,

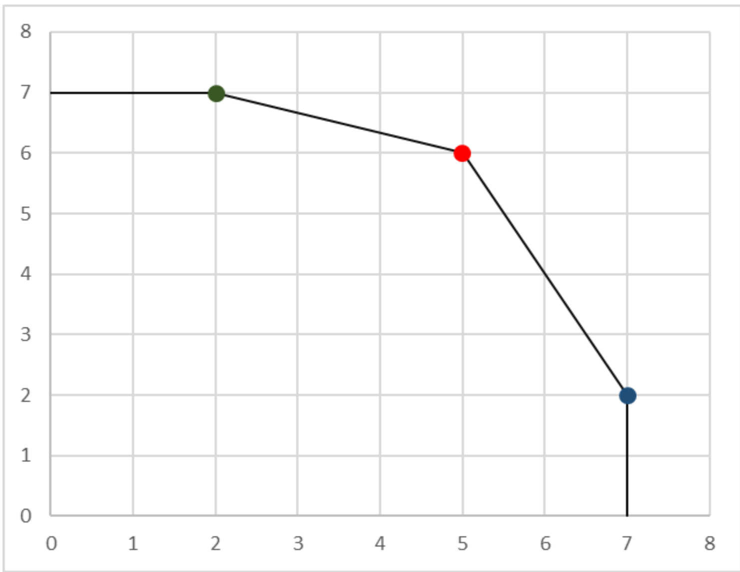
$Eu(T|r) = (1-r)5+r6 = 5+r$.

Then B is optimal when $5(1-r)+6r \geq 2(1-r)+7r$ and $5(1-r)+6r \geq 7(1-r)+2r$, which are satisfied when $3/4 = (5-2)/((5-2)+(7-6)) \geq r \geq (7-5)/((7-5)+(6-2)) = 1/3$. T is optimal when $r \leq 1/3$. M is optimal when $r \geq 3/4$.

(r, Eu(c|r))



(u(c,L), u(c,R))

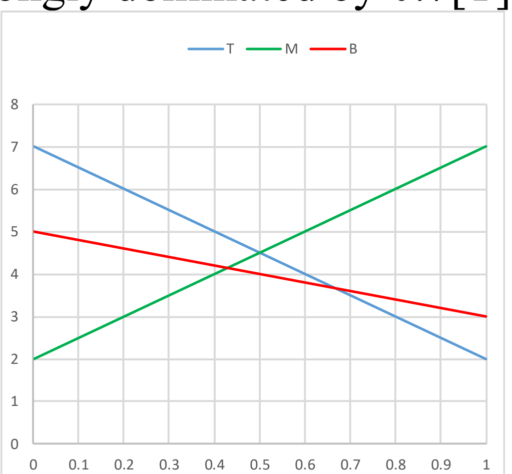


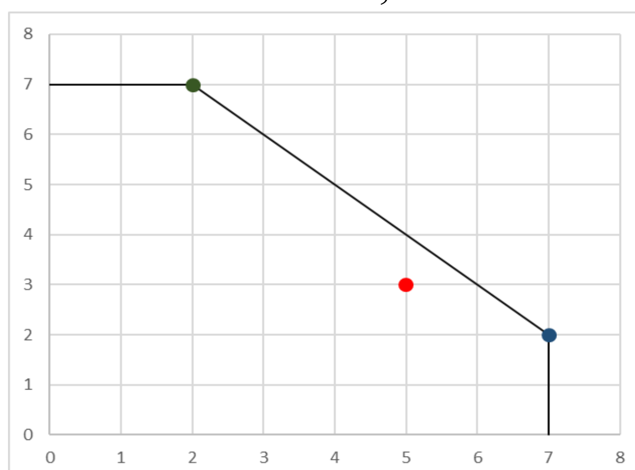
Example 2: As above, C = {T,M,B}, S = {L,R}, and u is same except u(B,R) = 3.

u(c,s):   P(R) = r.

	L	R
T	7	2
M	2	7
B	5	3

B would be optimal when $5(1-r)+3r \geq 7(1-r)+2r$ & $5(1-r)+3r \geq 2(1-r)+7r$,
which are satisfied when $r \geq 2/3$ & $3/7 \geq r$, which is impossible! So B cannot be optimal.
T is optimal when $r \leq 1/2$. M is optimal when $r \geq 1/2$.
Now consider a randomized strategy that chooses T with some probability σ(T) and chooses M with probability σ(M) = 1−σ(T). This strategy is σ(T)[T]+(1−σ(T))[M].
B would be strongly dominated by this randomized strategy σ if
$5 < \sigma(T)7 + (1-\sigma(T))2$ (B worse than σ in state L), and
$3 < \sigma(T)2 + (1-\sigma(T))7$ (B worse than σ in state R).
These are satisfied when $3/5 < \sigma(T) < 4/5$. For example, σ(T) = 0.7 works.
B is strongly dominated by 0.7[T]+0.3[M] ($5 < 0.7\times7+0.3\times2 = 5.5$, $3 < 0.7\times2+0.3\times7 = 3.5$).

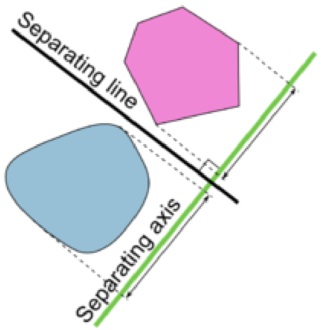
(r, Eu(c|r))


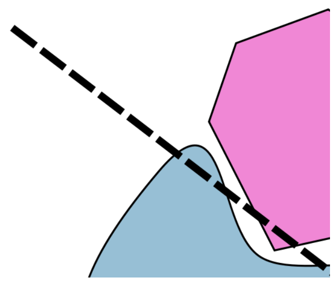
(u(c,L), u(c,R))


Separating Hyperplane Theorem (MWG M.G.2):

Suppose X is a closed convex subset of $\mathbb{R}^N$, and w is a vector in $\mathbb{R}^N$.
Then exactly one of the following two statements is true *(but not both)*:

Either (1) $w \in X$, or (2) there exists a vector $y \in \mathbb{R}^N$ such that $y'w > \max_{x \in X} y'x$.
(Here $y'w = y_1x_1 + \ldots + y_Nx_N = \sum_{i \in \{1,\ldots,N\}} y_ix_i$, with $y = (y_1,\ldots,y_N)$ and $x = (x_1,\ldots,x_N)$.)





Supporting Hyperplane Theorem (MWG M.G.3):

Suppose X is a convex subset of $\mathbb{R}^N$, and w is a vector in $\mathbb{R}^N$. Then exactly one of these two statements is true *(but not both)*: Either (1) w is in the interior of X (relative to $\mathbb{R}^N$), or (2) $\exists y \in \mathbb{R}^N$ such that $y \neq \mathbf{0}$ and $y'w \geq \max_{x \in X} y'x$. (Here $\mathbf{0} = (0,\ldots,0)$.)

<u>**Fact**</u> If X is convex and compact (closed & bounded), then $\max_{x \in X} y'x$ is finite, and this maximum must be achieved at some extreme point in X. (MWG p 946.)

<u>**Fact**</u> For any nonempty finite set C and any $v \in \mathbb{R}^C$, $\max_{\sigma \in \Delta(C)} \sum_{c \in C} \sigma(c)v_c = \max_{c \in C} v_c$, and $\operatorname{argmax}_{\sigma \in \Delta(C)} \sum_{c \in C} \sigma(c)v_c = \{\sigma \in \Delta(C) | \{d | \sigma(d) > 0\} \subseteq \operatorname{argmax}_{c \in C} v_c\}$.

Strong domination Theorem. Given nonempty finite sets C={choices} & S={states}, utility function $u:C\times S\rightarrow\mathbb{R}$, and some $d\in C$, exactly of these two statements is true: either
(1) $\exists\sigma\in\Delta(C)$ such that $u(d,s) < \sum_{c\in C}\sigma(c)u(c,s)\ \ \forall s\in S$, *[d dominated with randomization]*
or (2) $\exists p\in\Delta(S)$ s.t. $\sum_{s\in S} p(s)u(d,s) = \max_{c\in C}\sum_{s\in S} p(s)u(c,s)$. *[d optimal for some beliefs]*

<u>Proof</u>. Let $X = \{x\in\mathbb{R}^S \mid \exists\sigma\in\Delta(C) \text{ s.t. } x_s \leq \sum_{c\in C}\sigma(c)u(c,s)\ \forall s\}$. $X\subseteq\mathbb{R}^S$ is convex.
(1) here is equivalent to: (1′) the vector $u(d) = (u(d,s))_{s\in S}$ is in the interior of X in $\mathbb{R}^S$.
By the Supporting Hyperplane Thm, (1′) is false iff
(2′) $\exists p\in\mathbb{R}^S$ such that $p\neq\mathbf{0}$ and $\sum_{s\in S} p(s)u(d,s) \geq \max_{x\in X}\sum_{s\in S} p(s)x_s$.
We must have $p(s)\geq 0$ for all s, because x in X can have x_s approaching $-\infty$.
So $\sum_{s\in S} p(s) > 0$, from $p\geq\mathbf{0}$ and $p\neq\mathbf{0}$. Dividing by this sum, we can make $\sum_{s\in S} p(s) = 1$.
Furthermore, the maximum of the linear function $p'x$ over $x\in X$ must be achieved at one of the extreme points in X, which are vectors $(u(c,s))_{s\in S}$ for the various $c\in C$:
$\max_{x\in X}\sum_{s\in S} p(s)x_s = \max_{\sigma\in\Delta(C)}\sum_{s\in S} p(s)\sum_{c\in C}\sigma(c)\,u(c,s) = \max_{c\in C}\sum_{s\in S} p(s)u(c,s)$.
So (2′) is equivalent to condition (2) in the theorem here.

Expected Utility Theorem. Let N be a finite set of ***prizes***, and consider a finite sequence of pairs of lotteries $p(i)\in\Delta(N)$ and $q(i)\in\Delta(N)$, for $i\in M=\{1,\ldots,m\}$. Essentially, M is a finite set of ***lottery-comparisons*** of the form "p(i) preferred to q(i)." (Here $p(i) = (p_j(i))_{j\in N}$, $q(i) = (q_j(i))_{j\in N}$. So in the i'th comparison, the probability of getting prize j is $p_j(i)$ in the more-preferred lottery, $q_j(i)$ in the less-preferred lottery.)
Then exactly one of these two statements is true *(not both)*: Either
(1) $\exists\sigma\in\Delta(M)$ such that $\sum_{i\in M}\sigma(i)p_j(i) = \sum_{i\in M}\sigma(i)q_j(i)\ \forall j\in N$, [*substitution axiom is violated*]
or (2) $\exists u\in\mathbb{R}^N$ such that $\sum_{j\in N} p_j(i)u_j > \sum_{j\in N} q_j(i)u_j\ \ \forall i\in M$. [*preferences satisfy utility theory*]

<u>Proof.</u> Let $X = \{\sum_{i\in M}\sigma(i)(q(i)-p(i)) \mid \sigma\in\Delta(M)\}$. X is a closed convex subset of $\mathbb{R}^N$.
Condition (1) here is equivalent to: (1′) the N-vector $\mathbf{0}$ is in X.
By the Separating Hyperplane Thm, (1′) is false iff
(2′) $\exists u\in\mathbb{R}^N$ such that $0 = u'\mathbf{0} > \max_{x\in X} u'x$.
The extreme points of X are vectors $(q(i)-p(i)) = (q_j(i)-p_j(i))_{j\in N}$.
The linear function $u'x = \sum_{j\in N} x_j u_j$ must achieves its maximum over $x\in X$ at some extreme point, so $\max_{x\in X} u'x = \max_{i\in M}\sum_{j\in N} u_j\,(q_j(i)-p_j(i))$.
So (2′) is equivalent to (2) in the theorem here.

<u>**Fact**</u>. Suppose the utility-representation condition (2) is satisfied by $u = (u_j)_{j\in N}$.
Then (2) is also satisfied by $\hat{u}$ if there exists $A>0$ and B such that $\hat{u}_j = Au_j + B\ \forall j\in N$.

Linear Duality Theorem (Farkas's lemma, theorem of the alternatives)

Given any m×n matrix $A = (a_{ij})_{i\in\{1,\ldots,m\},j\in\{1,\ldots n\}}$ and any vector $b = (b_i)_{i\in\{1,\ldots,m\}}$ in $\mathbb{R}^m$, exactly one of the following two conditions is true: Either

(1) $\exists x \in \mathbb{R}^n$ such that $\sum_{j\in\{1,\ldots,n\}} a_{ij}\, x_j \geq b_i\ \forall i\in\{1,\ldots,m\}$;

or (2) $\exists y \in \mathbb{R}^m$ s.t. $y_i \geq 0\ \forall i$, $\sum_{i\in\{1,\ldots,m\}} y_i\, a_{ij} = 0\ \forall j\in\{1,\ldots,n\}$, $\sum_{i\in\{1,\ldots,m\}} y_i\, b_i > 0$.

Sketch of Proof: (1) & (2) cannot both be true, or else we could get $0 > y'(Ax-b) \geq 0$.

If (1) is false then b is not in the closed convex set $\{Ax-z \mid x\in\mathbb{R}^n,\ z\in\mathbb{R}_+{}^m\}$, which by the Separating Hyperplane Theorem implies $\exists y$ such that

$y'b > \max \{y'(Ax-z) \mid x\in\mathbb{R}^n,\ z\in\mathbb{R}_+{}^m\}$. This vector y must satisfy condition (2). So (1) & (2) cannot both be false.

Weak domination Theorem. Given nonempty finite sets C={choices} & S={states}, utility function $u{:}C\times S\rightarrow\mathbb{R}$, and some $d\in C$, exactly of these two statements is true:
(1) $\exists p\in\Delta(S)$ such that $p(s) > 0\ \forall s$, and $\sum_{s\in S} p(s)u(d,s) = \max_{c\in C} \sum_{s\in S} p(s)u(c,s)$;
(2) $\exists \sigma\in\Delta(C)$ s.t. $u(d,s) \leq \sum_{c\in C} \sigma(c)u(c,s)\ \forall s\in S$, with strict < for some s in S;

Proof on page 4 of notes uses Linear Duality Thm. ...With $\varepsilon>0$, consider:

(1′) $\exists p\in\mathbb{R}^S$ such that $\sum_{s\in S} (u(d,s) - u(c,s))\, p(s) \geq 0\ \ \forall c\in C$, $\qquad \sigma(c)$
and $p(s) \geq \varepsilon\ \ \forall s\in S$. $\qquad \delta(s)$

(2′) $\exists(\sigma,\delta) \in \mathbb{R}_+{}^C \times \mathbb{R}_+{}^S$ s.t. $\sum_{c\in C} \sigma(c)(u(d,s) - u(c,s)) + \delta(s) = 0\ \ \forall s\in S$, $\qquad p(s)$

In **game theory** we assume that players are rational and intelligent.
Here rational means that each player acts to maximize his own expected utility.
Here intelligent means that the players know everything that we know about their situation when we analyze it game-theoretically. Intelligence implies that game model that we analyze must be common knowledge among the players, that is, all players know (that all players know)^k the model, $\forall k=\{0,1,2,...\}$.

A **strategic-form game** is characterized by $(N, (C_i)_{i\in N}, (u_i)_{i\in N})$ where
$N = \{1,2,...,n\}$ is the set of players, and, for each player i:
C_i is the set of alternative actions or (pure) strategies that are feasible for i in the game,
and $u_i:C_1\times C_2\times ...\times C_n \to \mathbb{R}$ is player i's utility function in the game.
We generally assume that each player i independently chooses an action in C_i.
If $c = (c_1,c_2,...,c_n)$ is the combination (or profile) of actions chosen by the players then each player i will get the expected utility payoff $u_i(c_1,c_2,...,c_n)$.
We let $C = C_1\times C_2\times ...\times C_n = \times_{i\in N} C_i$ denote the set of all combinations or profiles of actions that the players could choose.
Let C_{-i} denote the set of all profiles of actions that can be chosen by players other than i.
When $c\in C$ is a profile of actions for the players, c_i denotes the action of each player i, c_{-i} denotes the profile of actions for players other than i where they act as in c, and $(c_{-i};d_i)$ denotes the profile of actions in which i's action is changed to d_i but all others choose the same action as in c. So $c = (c_{-i};c_i)$.

A <u>randomized strategy</u> (or mixed strategy) for player i is a probability distribution over C_i, so $\Delta(C_i)$ denotes the set of all randomized strategies for i. (<u>pure</u>=nonrandomized.)

An action d_i for player i is <u>strongly dominated</u> by a randomized strategy $\sigma_i \in \Delta(C_i)$ if

$u_i(c_{-i};d_i) < \sum_{c_i \in C_i} \sigma_i(c_i)\, u_i(c_{-i};c_i) \quad \forall c_{-i} \in C_{-i}$.

An action d_i for player i is <u>weakly dominated</u> by a randomized strategy $\sigma_i \in \Delta(C_i)$ if

$u_i(c_{-i};d_i) \leq \sum_{c_i \in C_i} \sigma_i(c_i)\, u_i(c_{-i};c_i) \quad \forall c_{-i} \in C_{-i}$, with strict inequality (<) for at least one c_{-i}.

The set of player i's <u>best responses</u> to any profile of opponents' actions c_{-i} is

$\beta_i(c_{-i}) = \operatorname{argmax}_{d_i \in C_i} u_i(c_{-i};d_i) = \{d_i \in C_i \mid u_i(c_{-i};d_i) = \max_{c_i \in C_i} u_i(c_{-i};c_i)\}$.

Similarly, if i's beliefs about the other players' actions can be described by a probability distribution μ in $\Delta(C_{-i})$, then the set of player i's best responses to the beliefs μ is

$\beta_i(\mu) = \operatorname{argmax}_{d_i \in C_i} \sum_{c_{-i} \in C_{-i}} \mu(c_{-i})\, u_i(c_{-i};d_i)$.

<u>Fact.</u> If we iteratively eliminate strongly dominated actions for all players until no strongly dominated actions remain, then we get a reduced game in which each remaining action for each player is a best response to some beliefs about the other players' actions. These remaining actions are <u>rationalizable</u>.

If each player j independently uses strategy σ_j in $\Delta(C_j)$, then player i's expected payoff is

$$u_i(\sigma) = u_i(\sigma_{-i};\sigma_i) = u_i(\sigma_1,\sigma_2,\ldots,\sigma_n) = \sum_{c\in C} \left(\prod_{j\in N} \sigma_j(c_j)\right) u_i(c)$$
$$= \Sigma_{c_i\in C_i} \sigma_i(c_i) \, \Sigma_{c_{-i}\in C_{-i}} \left(\prod_{j\in N-i} \sigma_j(c_j)\right) u_i(c_{-i};c_i) = \Sigma_{c_i\in C_i} \sigma_i(c_i) \, u_i(\sigma_{-i};[c_i]).$$

Here $[c_i]\in\Delta(C_i)$ with $[c_i](c_i) = 1$, $[c_i](d_i) = 0$ if $d_i\neq c_i$.

Notice $\Sigma_{c_i\in C_i} \sigma_i(c_i)\big(u_i(\sigma_{-i};[c_i]) - u_i(\sigma)\big) = 0$.

<u>Fact</u> $\sigma_i \in \operatorname{argmax}_{\tau_i\in\Delta(C_i)} u_i(\sigma_{-i};\tau_i)$ if & only if $\{c_i\in C_i | \sigma_i(c_i)>0\} \subseteq \operatorname{argmax}_{d_i\in C_i} u_i(\sigma_{-i};[d_i])$.

The set $\{c_i | \sigma_i(c_i)>0\}$ of actions with positive probability under σ_i is the <u>support</u> of σ_i.

A <u>Nash equilibrium</u> is a profile of actions or randomized strategies such that each player is using a best response to the others. That is $\sigma = (\sigma_1,\ldots,\sigma_n)$ is a Nash equilibrium in randomized strategies iff $\sigma_i \in \operatorname{argmax}_{\tau_i\in\Delta(C_i)} u_i(\sigma_{-i};\tau_i)$ for every player i in N.

<u>Nash's Theorem.</u> Any finite strategic-form game has at least one Nash equilibrium in the set of all randomized strategies ($\times_{i\in N} \Delta(C_i)$).

<u>John Nash's 1950 proof:</u> Apply the Brouwer fixed point theorem to the function $f:\times_{i\in N} \Delta(C_i) \rightarrow \times_{i\in N} \Delta(C_i)$ such that $f(\sigma) = \tau$ iff $\forall i\in N, \forall c_i\in C_i$:

$\tau_i(c_i) = \big(\sigma_i(c_i)+\max\{0,u_i(\sigma_{-i};[c_i])-u_i(\sigma)\}\big) / \sum_{d_i\in C_i} \big(\sigma_i(d_i)+\max\{0,u_i(\sigma_{-i};[d_i])-u_i(\sigma)\}\big)$.

Brouwer Fixed Point Theorem: If X is a nonempty compact convex subset of some finite-dimensional $\mathbb{R}^m$ and $f:X\rightarrow X$ is continuous then $\exists x^*$ such that $f(x^*) = x^*$.

Some basic 2×2 games. N={1,2}, each player i has two possible actions, tables show how players' payoffs (u_1,u_2) depend on their actions. (1 chooses row, 2 chooses column)

1 Cain/Abel game	1: \ 2:	ℓ	k
	L	3,7	0,4
	K	4,0	2,2

2 Prisoners' dilemma	1: \ 2:	f_2	g_2
(friendly)	f_1	5,5	0,6
(aggressive)	g_1	6,0	1,1

3 Stag hunt	1: \ 2:	s_2	h_2
(go for stag)	s_1	5,5	0,4
(go for hares)	h_1	4,0	2,2

4 Battle of sexes	1: \ 2:	f_2	s_2
(football)	f_1	2,1	0,0
(shopping)	s_1	0,0	1,2

5 SymmetricBoS		1: \ 2:	b_2 (q)	a_2 (1-q)	with $\sigma_2(b_2)=q$
(bold)	(p)	b_1	0,0	2,1	$u_1([b_1],\sigma_2) = 0q+2(1-q)$,
(accommodating)	(1-p)	a_1	1,2	0,0	$u_1([a_1],\sigma_2) = 1q+0(1-q)$.

6 OnMinimax	1: \ 2:	L	R
	T	0, 0	0,–1
	B	1, 0	–1, 3

Nonrandom equilibria: 1 (K,k). 2 (g_1,g_2). 3 (s_1,s_2), (h_1,h_2). 4 (f_1,f_2), (s_1,s_2). 5 (b_1,a_2), (a_1,b_2). 6 none.

OnMinimax:

1: \ 2:		L (q)	R (1-q)
(p)	T	0, 0	0,−1
(1-p)	B	1, 0	−1, 3

With $\sigma_2(L)=q$,
$u_1([T],\sigma_2) = 0q+0(1\text{-}q) = 0$,
$u_1([B],\sigma_2) = 1q+(-1)(1-q) = 2q-1$.

With $\sigma_1(T)=p$, $u_2(\sigma_1,[L]) = 0p+(1-p)0 = 0$, $u_2(\sigma_1,[R]) = -1p + 3(1-p) = 3-4p$.
1's best responses: $q<1/2\to[T]$ $(p=1)$, $q>1/2\to[B]$ $(p=0)$, $q=1/2\to\{\text{all } 0\le p\le 1\}$.
2's best responses: $p<3/4\to[R]$ $(q=0)$, $p>3/4\to[L]$ $(q=1)$, $p=3/4\to\{\text{all } 0\le q\le 1\}$.
So unique eqm has $p=3/4$ & $q=1/2$, is $(\sigma_1,\sigma_2) = (3/4[T]+1/4[B], 1/2[L]+1/2[R])$.
Expected payoffs in equilibrium?
$u_1(\sigma_1,\sigma_2) = (3/4)(1/2)0+(3/4)(1/2)0 + (1/4)(1/2)1+(1/4)(1/2)(-1) = 0$
$\quad = u_1([T],\sigma_2) = (1/2)0+(1/2)0 = 0 = (1/2)1 + (1/2)(-1) = u_1([B],\sigma_2)$.
$u_2(\sigma_1,\sigma_2) = u_2(\sigma_1,[L]) = (3/4)0+(1/4)0 = 0 = (3/4)(-1) + (1/4)3 = u_2(\sigma_1,[R])$.

For *symmetricBoS*, we have equilibria with expected payoffs (u_1,u_2) as follows:
(b_1,a_2) with (2,1), (a_1,b_2) with (1,2), $(2/3[b_1]+1/3[a_1], 2/3[b_2]+1/3[a_2])$ with (2/3, 2/3).
Any symmetric game must have a symmetric equilibrium, but the symmetric equilibrium of this game is Pareto-inferior to both non-symmetric equilibria (2>1>2/3)!

The StagHunt has a third randomized equilibrium where each does $2/3[s_i]+1/3[h_i]$.

Example of a game on $C_1 \times C_2 = \mathbb{R}_+ \times \mathbb{R}_+$: almost-symmetric joint-project (E & A Kalai).

$c_1 \geq 0$, $c_2 \geq 0$, $u_1(c_1,c_2) = 25(c_1+c_2)-(c_1+c_2)^2-c_1$, $u_2(c_1,c_2) = 25.2(c_1+c_2)-(c_1+c_2)^2-c_2$.

1st-order conditions for $c_1=\beta_1(c_2)>0$: $0 = \partial u_1/\partial c_1 = 25-2(c_1+c_2)-1 \Rightarrow \beta_1(c_2)=12-c_2$.

1st-order conditions for $c_2=\beta_2(c_1)>0$: $0 = \partial u_2/\partial c_2 = 25.2-2(c_1+c_2)-1 \Rightarrow \beta_2(c_1)=12.1-c_1$.

With "$c_1 \geq 0$" constraint, the boundary condition for an optimum at $c_1=\beta_1(c_2)=0$ would be $0 \geq \partial u_1/\partial c_1 = 25-2(0+c_2)-1 \Rightarrow c_2 \geq 12$.

So we get $\beta_1(c_2) = \max\{12-c_2, 0\}$. Similarly, $\beta_2(c_1) = \max\{12.1-c_1, 0\}$.

The equilibrium is $(c_1,c_2) = (0,12.1)$. It yields payoffs $u_1 = 156.09$, $u_2 = 146.41$.

2×2×...×2 n-person example: reporting a crime. Given $n \geq 2$, $v > c > 0$. (Say $v=100$, $c=1$)
C_i={<u>call</u>, <u>don't</u>}, $u_i(a_1,\ldots a_n) = v-c$ if a_i=<u>call</u>, $u_i = v$ if a_i=<u>don't</u> but some a_j=call, else $u_i=0$.

Let $p = \sigma_i(\text{call})$ in symmetric equilibrium ($0 \leq p \leq 1$).

$u_i(a_{-i};a_i)$	a_i: \ a_{-i}:	<u>all others don't</u> ($(1-p)^{n-1}$)	<u>some others call</u> ($1-(1-p)^{n-1}$)
(p)	<u>call</u>	v−c	v−c
(1−p)	<u>don't</u>	0	v

Can there be a symmetric equilibrium with p=0? No! ($0 < v-c$).
Can there be a symmetric equilibrium with p=1? No (with $n \geq 2$)! ($v-c < v$)

So $0 < p < 1$, support ={call, don't}, and thus $u_i(\sigma_{-i};[\text{call}]) = u_i(\sigma_{-i};[\text{don't}]) = \ \forall i$,
and so $v-c = (1-p)^{n-1}\, 0 + (1-(1-p)^{n-1})v$.
Therefore $(1-p)^{n-1} = c/v$, $p = 1-(c/v)^{1/(n-1)}$.

Notice $p \to 0$ as $n \to \infty$. $\forall n>1$: P(no others call) $= (1-p)^{n-1} = c/v$,
P(no calls) $= (1-p)^n = (c/v)^{n/(n-1)} \to c/v$ as $n \to \infty$ (it increases in n).
(With n=1, get p=1, P(no calls) = 0.)

E(#calls)=np. As $n \to \infty$, #calls→Poisson, mean $np \to \lambda = -\text{LN}(c/v)$, so that $e^{-\lambda} = c/v$.

Schelling's (1960) focal point effect: In a game with multiple equilibria, any salient cultural or environmental factor that focuses people's attention on one equilibrium can generate expectations that people will behave as this equilibrium predicts, so that it becomes rational for everyone to fulfill this prediction, as a self-fulfilling prophecy. (Thomas Schelling, "Bargaining, communication, and limited war," *Journal of Conflict Resolution* 1:19-36 (1957) <https://www.jstor.org/stable/172548>)

Consider an island where every day different matched pairs play the following **rival-claimants** game in various places on the island:

	2 claims	2 defers
1 claims	-1, -1	**9, 0**
1 defers	**0, 9**	0, 0

Nash equilibria:

- (1 claims, 2 defers) with payoffs $(u_1,u_2) = (9,0)$,
- (1 defers, 2 claims) with payoffs $(u_1,u_2) = (0,9)$,
- each claims with independent probability 9/10, with each $Eu_i = 0 = 0.9\times-1 + 0.1\times9$.

Focal equilibrium: They may play the top equilibrium (9,0) when player 1 is recognized as owner here. (Justice, arbitration, divination; transfer of ownership by handshake.)

Social equilibria: anarchy; traditional ownership, legislation of ownership principles; focal arbitration by a recognized leader (duly elected, with limited authority).
(See sections 5 & 6 in "Learning from Schelling's Strategy of Conflict," *Journal of Economic Literature* 47:1109-1125 (2009) <https://www.jstor.org/stable/40651534>)

Computing randomized Nash equilibria. We describe here a procedure for finding Nash equilibria, from section 3.3 of Myerson (1991).

We are given some game, including a given set of players N and, for each i in N, a set of feasible actions C_i for player i and a payoff function $u_i:C_1\times\ldots\times C_n\to\mathbb{R}$ for player i.

The <u>support</u> of a randomized equilibrium is, for each player, the set of actions that have positive probability of being chosen in this equilibrium.

To find a Nash equilibrium, we can apply the following 5-step method:

(1) Guess a support for all players. That is, for each player i, let S_i be a nonempty subset of C_i, and guess that S_i is the set of actions that player i uses with positive probability.

(2) Consider the smaller game where the action set for each player i is reduced to S_i, and try to find an equilibrium where all of these actions get positive probability.

To do this, we need to solve a system of equations for some unknown quantities.

The unknowns: For each player i in N and each action s_i in i's support S_i, let $\sigma_i(s_i)$ denote i's probability of choosing s_i, and let w_i denote player i's expected payoff in the equilibrium. ($\sigma_i(a_i)=0$ if $a_i\notin S_i$.)

The equations: For each player i, the sum of these probabilities $\sigma_i(s_i)$ must equal 1.

$\forall i\in N$ and $\forall s_i \in S_i$, player i's expected payoff when he chooses s_i but all other players randomize independently according to their σ_j probabilities must be equal to w_i.

Let $u_i(\sigma_{-i},[a_i])$ denote player i's expected payoff when he chooses action a_i and all other players are expected to randomize independently according to their σ_j probabilities.

The equations can be written: $\sum_{si\in Si} \sigma_i(s_i) = 1 \;\; \forall i\in N$; & $u_i(\sigma_{-i},[s_i]) = w_i \;\; \forall i\in N \;\; \forall s_i\in S_i$.

Notice that we have as many equations as unknowns (w_i, $\sigma_i(s_i)$).

(3) If the equations in step 2 have no solution, then we guessed the wrong support, and so we must return to step 1 and guess a new support.
Assuming that we have a solution from step (2), continue to (4) and (5)
(4) The solution from (2) would be nonsense if any of the "probabilities" were negative. That is, for every player i in N and every action s_i in i's support S_i, we need $\sigma_i(s_i) \geq 0$.
If these nonnegativity conditions are not satisfied by a solution, then we have not found an equilibrium with the guessed support. So we return to step 1, guess a new support.
If we have a solution that satisfies all these nonnegativity conditions, then it is an equilibrium of the reduced game where each player i can only choose actions in S_i.
(5) A solution from (2) that satisfies the condition in (4) would still not be an equilibrium of the original game, if any player would prefer an action outside the guessed support. Recall $u_i(\sigma_{-i},[s_i]) = w_i \ \forall s_i \in S_i$. So we must ask, for each player i and each action a_i that is in C_i but is not in the guessed support S_i, could i do better than w_i by choosing a_i when all other players act according to their σ_j probabilities? That is, for every action a_i that is in C_i but is not in S_i (so $\sigma_i(a_i)=0$), we need $u_i(\sigma_{-i},[a_i]) \leq w_i$.
If our solution satisfies all these inequalities then it is an equilibrium of the given game. But if any of these inequalities is violated (some $u_i(\sigma_{-i},[a_i]) > w_i$), then we have not found an equilibrium with the guessed support, and so we must return to step 1 and guess a new support. (There are only finitely many possible supports to consider.)

Thus, an equilibrium $\sigma = (\sigma_i(a_i))_{a_i \in C_i, i \in N}$ with payoffs $w = (w_i)_{i \in N}$ must satisfy:
$\sum_{a_i \in C_i} \sigma_i(a_i)=1 \ \forall i \in N$; and $\sigma_i(a_i) \geq 0$ & $u_i(\sigma_{-i},[a_i]) \leq w_i$ with at least one equality $\forall a_i \ \forall i$ (<u>complementary slackness</u>). The support for i is $S_i=\{s_i \in C_i | \ \sigma_i(s_i)>0, \text{ so } u_i(\sigma_{-i},[a_i])=w_i\}$.

Example. Find all Nash equilibria (pure and mixed) of the following 2×3 game:

	Player 2		
Player 1	L	M	R
T	7, 2	2, 7	3, 6
B	2, 7	7, 2	4, 5

There are $(2^2-1)\times(2^3-1) = 3\times 7 = 21$ possible supports.

But it is easy to see that this game has no pure-strategy equilibria.
(2's best response to T is M, but T is not 1's best response to M;
and 2's best response to B is L, but B is not 1's best response to L).
This eliminates the six cases where each player's support is just one action.
Furthermore, when either player is restricted to just one action, the other player always has a unique best response, so there are no equilibria where only one player randomizes.
That is, both players must have at least two actions in the support of any equilibrium.

Thus, we must search for equilibria where the support of randomized strategy is {T,B}, and the support of 2's randomized strategy is {L,M,R} or {M,R} or {L,M} or {L,R}.

		Player 2		
	Player 1	**L (q)**	**M (1–q–r)**	**R (r)**
(p)	**T**	**7, 2**	**2, 7**	**3, 6**
(1–p)	**B**	**2, 7**	**7, 2**	**4, 5**

Guess support is {T,B} for 1 and {L,M,R} for 2?

Write 1's strategy as $\sigma_1 = p[T]+(1-p)[B]$, 2's strategy as $\sigma_2 = q[L]+(1-q-r)[M]+r[R]$, that is $p = \sigma_1(T)$, $1-p = \sigma_1(B)$, $q = \sigma_2(L)$, $r = \sigma_2(R)$, $1-q-r = \sigma_2(M)$.

Player 1 randomizing over {T,B} requires $w_1 = u_1(T,\sigma_2) = u_1(B,\sigma_2)$,

and so $\mathbf{w_1 = 7q+2(1-q-r)+3r = 2q+7(1-q-r)+4r.}$

Player 2 randomizing over {L,M,R} requires $w_2 = u_2(\sigma_1,L) = u_2(\sigma_1,M) = u_2(\sigma_1,R)$,

and so $\mathbf{w_2 = 2p+7(1-p) = 7p+2(1-p) = 6p+5(1-p).}$

We have three equations for three unknowns (p,q,r), but they have no solution (as the two indifference equations for player 2 imply both p=1/2 and p = 3/4, which is impossible).

Thus, there is no equilibrium with this support.

Player 1		L (0)	M (1−r)	R (r)
(p)	T	7, 2	**2, 7**	**3, 6**
(1−p)	B	2, 7	**7, 2**	**4, 5**

Player 2 (columns L, M, R)

<u>Guess support is {T,B} for 1 and {M,R} for 2?</u>

We write 1's strategy as p[T]+(1−p)[B] , 2's strategy as (1−r)[M]+r[R]. (q=0)

Player 1 randomizing over {T,B} requires $w_1 = u_1(T,\sigma_2) = u_1(B,\sigma_2)$,

so $\mathbf{w_1 = 2(1-r)+3r = 7(1-r)+4r}$.

Player 2 randomizing over {M,R} requires $w_2 = u_2(\sigma_1,M) = u_2(\sigma_1,R)$,

so $\mathbf{w_2 = 7p+2(1-p) = 6p+5(1-p)}$.

The solution for these equations is $p = 3/4$ and $r = 5/4$, with $w_1 = 13/4$, $w_2 = 23/4$.

But this would yield $\sigma_2(M) = 1-r = -1/4 < 0$, and so no equilibrium has this support.

(Notice: if player 2 never chose L then T would be dominated by B for player 1.)

		Player 2		
	Player 1	**L (q)**	**M (1−q)**	R (0)
(p)	**T**	**7, 2**	**2, 7**	3, 6
(1−p)	**B**	**2, 7**	**7, 2**	4, 5

Guess support is {T,B} for 1 and {L,M} for 2?

We write 1's strategy as p[T]+(1−p)[B], 2's strategy as q[L]+(1−q)[M]. (r=0)

Player 1 randomizing over {T,B} requires $w_1 = u_1(T,\sigma_2) = u_1(B,\sigma_2)$,

so $\mathbf{w_1 = 7q+2(1-q) = 2q+7(1-q)}$.

Player 2 randomizing over {L,M} requires $w_2 = u_2(\sigma_1,L) = u_2(\sigma_1,M)$,

so $\mathbf{w_2 = 2p+7(1-p) = 7p+2(1-p)}$.

The solution for these equations is $p = 1/2$ and $q = 1/2$, with $w_1 = 9/2$, $w_2=9/2$.

This solution yields nonnegative probabilities for all actions.

We must check that player 2 would not prefer deviating outside her support to R. But

$u_2(\sigma_1,R) = 6p+5(1-p) = 6\times1/2+5\times1/2 = 11/2 > w_2 = u_2(\sigma_1,L) = 2\times1/2+7\times1/2 = 9/2$.

So there is no equilibrium with this support.

	Player 2		
Player 1	**L** **(q)**	M (0)	**R** **(1–q)**
(p) **T**	**7, 2**	2, 7	**3, 6**
(1–p) **B**	**2, 7**	7, 2	**4, 5**

Guess support is {T,B} for 1 and {L,R} for 2? (Nash says some support must work!)

We write 1's strategy as p[T]+(1–p)[B] , 2's strategy as q[L]+(1−q)[R]. (r=1−q)

Player 1 randomizing over {T,B} requires $w_1 = u_1(T,\sigma_2) = u_1(B,\sigma_2)$,

so $\mathbf{w_1 = 7q+3(1-q) = 2q+4(1-q)}$.

Player 2 randomizing over {L,R} requires $w_2 = u_2(\sigma_1,L) = u_2(\sigma_1,R)$,

so $\mathbf{w_2 = 2p+7(1-p) = 6p+5(1-p)}$.

The solution for these equations is $p = 1/3$ and $q = 1/6$, with $w_1 = 11/3$, $w_2 = 16/3$.

This solution yields nonnegative probabilities for all actions.

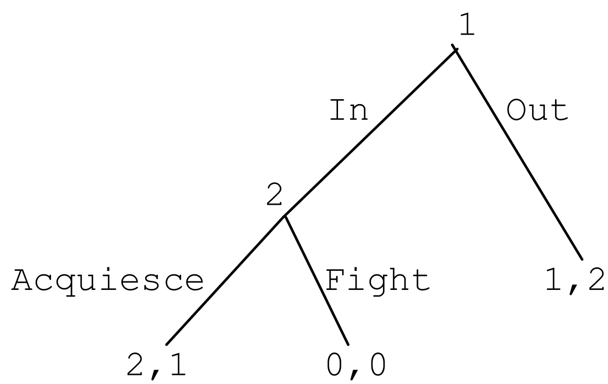
We also need to check that player 2 would not prefer deviating outside her support to M;

$u_2(\sigma_1,M) = 7p+2(1-p) = 7\times1/3+2\times2/3 = 11/3 < w_2 = u_2(\sigma_1,L) = 2\times1/3+7\times2/3 = 16/3$.

Thus, the equilibrium with this support is ((1/3)[T]+(2/3)[B], (1/6)[L]+(5/6)[R]).

Introduction to extensive games with perfect information.
An extensive game with perfect information is a tree diagram with:
nodes, branches, a root node, terminal nodes, decision nodes, player label, terminal payoffs, chance nodes, chance probabilities.

Analytical concepts: the path of play, a strategy for a player, the normal representation in strategic form, subgame, subgame-perfect equilibrium.
A (pure) ***strategy*** for a player here is a complete plan that specifies a feasible move for the player at each of the player's decision nodes in the game tree.

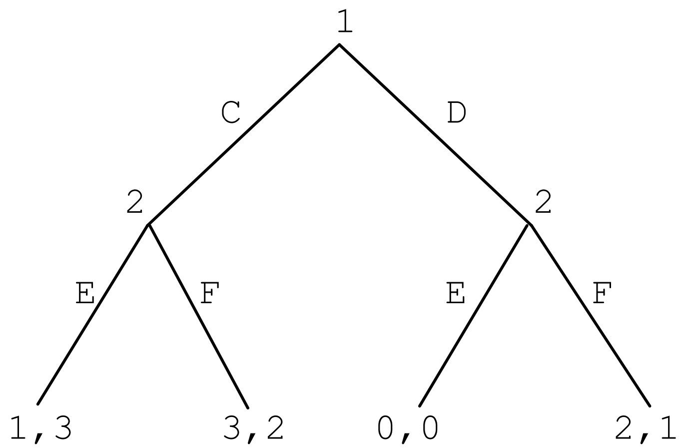


Ex1

Normal representation in strategic form:

Player 1: \ Player 2:	A	F
I	2, 1	0, 0
O	1, 2	1, 2

Subgame-perfect equilibrium (I,A);
other Nash equilibriums: (O, q[F]+(1−q)[A]) for $q \geq 1/2$.



Ex2.

Normal representation in strategic form:

Player 1: \ Player 2:	$E_C E_D$	$E_C F_D$	$F_C E_D$	$F_C F_D$
C	1, 3	1, 3	3, 2	3, 2
D	0, 0	2, 1	0, 0	2, 1

Subgame-perfect eqm $(D, E_C F_D)$.

Other Nash eqms: $(C, p[E_C E_D]+(1-p)[E_C F_D])$ for $p \geq 1/2$, $(D, p[E_C F_D]+(1-p)[F_C F_D])$ for $1>p\geq 1/2$.

Fact: The normal representation in strategic form of an extensive game with perfect information must have at least one equilibrium in pure strategies.
A pure-strategy subgame-perfect equilibrium can be found by backward induction, starting with the analysis of subgames of shortest length.
In strategic form, Nash eqms that do not correspond to subgame-perfect eqms can be eliminated by iterative elimination of weakly dominated or equivalent strategies.
(Player i's pure strategies a_i and b_i are *equivalent* for i iff $u_i(c_{-i};a_i) = u_i(c_{-i};b_i)\ \forall c_{-i}$.)
(Such broad iterative elimination can actually eliminate everything except one sgp eqm.)

The Holdup Problem

Player 1 can invest to improve an asset which he may later sell player 2.
First player 1 chooses an amount $e \geq 0$ (effort) to invest on improving the asset.
Given e, the asset is worth $v_1(e) = e^{0.5}$ to player 1, but is worth $v_2(e) = 2e^{0.5}$ to player 2.
We consider two versions of this game, which differ in how they bargain over the price.

Buyer-offer game First player 1 chooses $e \geq 0$ to invest in the asset.
Player 2 observes this investment e.
Then player 2 chooses a price $p \geq 0$ at which 2 offers to buy the asset.
Player 1 observes this offer, and then can choose to accept or reject it.
Final payoffs are: $u_1(e, p, \text{accept}) = p - e$, $u_2(e, p, \text{accept}) = v_2(e) - p$,
$u_1(e, p, \text{reject}) = v_1(e) - e$, $u_2(e, p, \text{reject}) = 0$.

There is a unique subgame-perfect equilibrium.
At the last stage, player **1 rejects if $p<v_1(e)$, accepts if $p>v_1(e)$**.
So player 2's optimal offer, given e, must be to **offer $p = v_1(e)$, which player 1 accepts**.
(Note: Player 1 is actually indifferent between accepting and rejecting, but there would be no optimal offer for 2 if player 1 rejected in this case of indifference!)
So player 1 expects that his payoff from e will be $v_1(e) - e = e^{0.5} - e$, maximized by **$e=0.25$**
So the equilibrium outcome is: 1 chooses $e = 0.25$, 2 offers $p = 0.25^{0.5} = 0.5$,
and payoffs are $u_1 = 0.5 - 0.25 = 0.25$, $u_2 = 2 \times 0.25^{0.5} - 0.5 = 1 - 0.5 = 0.5$.

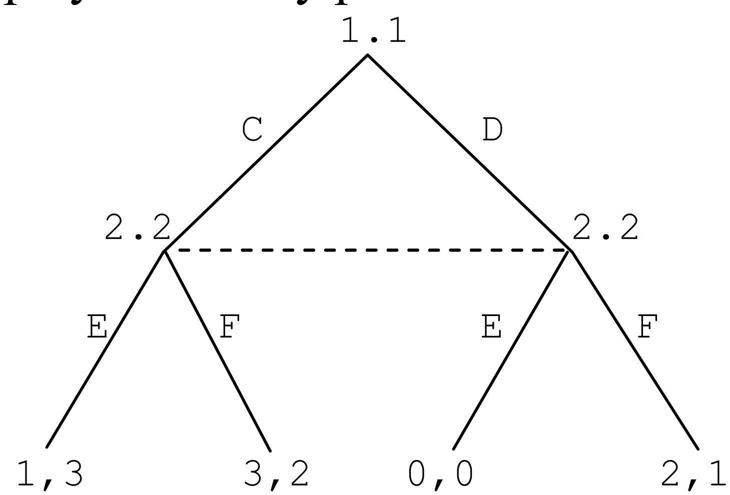
<u>Seller-offer game.</u> First player 1 chooses his investment $e \geq 0$.
Then player 1 chooses the price $p \geq 0$ at which he offers to sell the asset.
Player 2 observes e & p, then 2 can choose to accept or reject 1's offer.
Payoffs are still $u_1(e, p, \text{accept}) = p - e$, $u_2(e, p, \text{accept}) = v_2(e) - p$,
$u_1(e, p, \text{reject}) = v_1(e) - e$, $u_2(e, p, \text{reject}) = 0$.

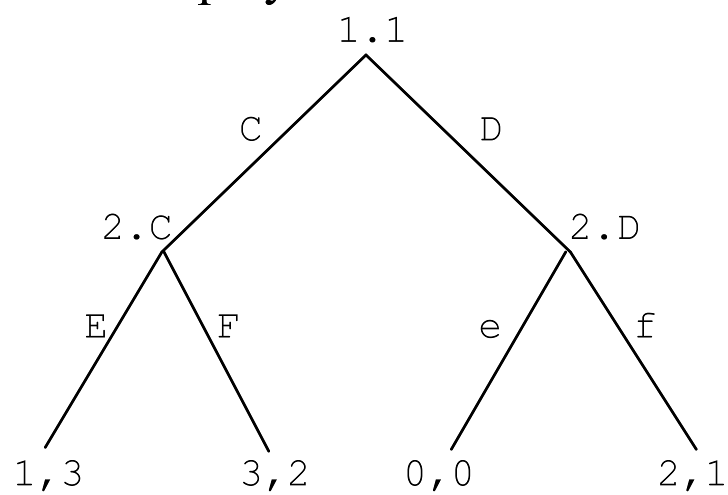
In the unique subgame-perfect equilibrium:
player **2 rejects if $p > v_2(e)$, accepts if $p \leq v_2(e)$**
so given e, player **1 offers $p = v_2(e)$**, which 2 accepts.
So player 1 chooses $e = 1$ to maximize $2e^{0.5} - e$. Thus the equilibrium outcome is:
1 chooses $e = 1$ and offers $p = 2\times 1^{0.5} = 2$, payoffs are $u_1 = 2 - 1 = 1$, $u_2 = 2\times 1^{0.5} - 2 = 0$.
This seller-offer game also has many other Nash equilibria that are not subgame perfect.

The equilibrium sum of payoffs u_1+u_2 is greater in the seller-offer game.
For an efficient outcome, the person who made the first-period investment should have more control in the process of bargaining over the price.
If they were about to play the buyer-offer game, the buyer would be willing to sell her right to set the price for any payment more than 0.5, and the seller would be willing to pay up to 0.75 for the right to set the price.

Both games have many other Nash equilibria that are not subgame-perfect. Consider any $(\hat{e},\hat{p})$ such that $v_2(\hat{e}) \geq \hat{p} \geq \hat{e} + \max_e (v_1(e) - e) = \hat{e}+0.25$ (such as $\hat{e}=1$, $\hat{p}=1.625$), so that each does better than he could alone. With either player offering the price, there is a Nash equilibrium in which 1 invests this $\hat{e}$, and then this price $\hat{p}$ is offered and accepted, but rejection would follow any other investment $e \neq \hat{e}$ or any other price-offer $p \neq \hat{p}$. These Nash equilibria violate sequential rationality, however, as threats to reject prices between $v_1(e)$ and $v_2(e)$ would not be credible.

Introduction to general finite extensive games (with imperfect information).
See sections 2.1-2.2 in my book.
Games in extensive form are shown by tree diagrams with: *nodes, branches, root node, terminal nodes, decision nodes, player labels, terminal payoffs, chance nodes, chance probabilities,* information labels, move labels.
I write "PlayerLabel.InformationLabel" at decision nodes.
An **information set** is a set of decision nodes with the same player & information labels.
A (pure) ***strategy*** for a player is a complete plan that specifies a feasible move for the player at every possible information set of the player.





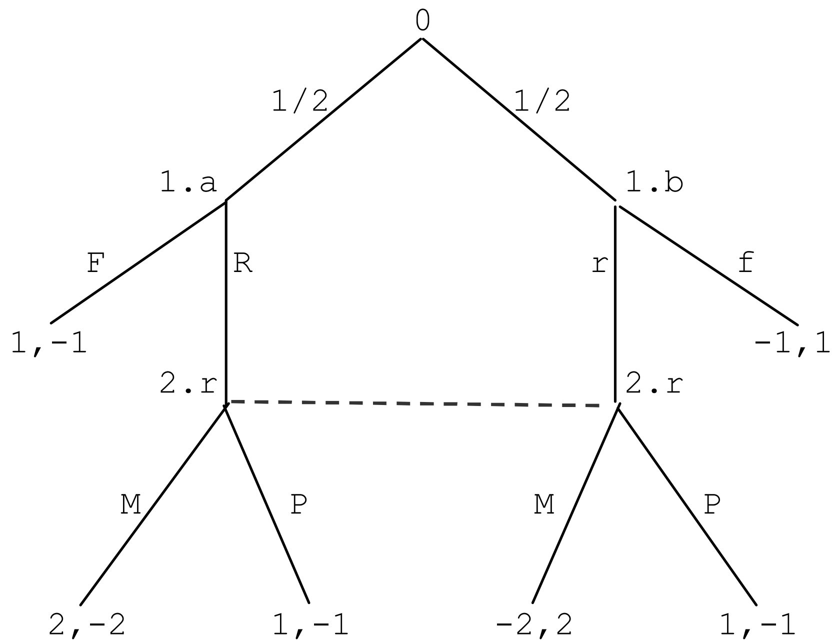
	E	F
C	1,3	3,2
D	0, 0	2,1

Eqm (C,E) $\rightarrow$ (1,3).

	Ee	Ef	Fe	Ff
C	1, 3	1, 3	3, 2	3, 2
D	0, 0	2, 1	0, 0	2, 1

Sgp-eqm (D,Ef) $\rightarrow$ (2,1).

Example: a simple card game (from Figure 2.2 in my book):



Normal representation in strategic form:

1.a/1.b: \ 2:	M	P
Rr	0, 0	1,-1
Rf	.5,-.5	0, 0
Fr	-.5, .5	1,-1
Ff	0, 0	0, 0

Notice that 1's mixed strategies 0.5[Rr]+0.5[Ff] and 0.5[Rf]+0.5[Fr] are both equivalent to the same behavioral strategy (0.5[R]+0.5[F], 0.5[r]+0.5[f]).

The mixed-strategy equilibrium $\tau = ((1/3)[Rr]+(2/3)[Rf], (2/3)[M]+(1/3)[P])$ is equivalent to behavioral strategy $\sigma = ([R], (1/3)[r]+(2/3)[f]; (2/3)[M]+(1/3)[P])$, yields expected payoffs $Eu_1 = 1/3$, $Eu_2 = -1/3$.

The sequential equilibrium consists of the behavioral strategy σ and belief probabilities $\mu_{2.r}(1.a,R) = (1/2)(1)/((1/2)(1)+(1/2)(1/3)) = 3/4$, $\mu_{2.r}(1.b,r) = 1-3/4 = 1/4$.

Analysis of sequential eqm of the card game; finding it in the extensive form.

Unknowns: $\sigma_1(F|1.a) = t$, $\sigma_1(r|1.b) = p$, $\sigma_2(M|2.r) = q$, $\mu_2(1.a,R|2.r) = \alpha$.

(By weak dominance, must exist an eqm with **t=0**; no equilibrium with t>0 unless q=0.)

We need Bayes consistency (**BC**) & sequential rationality (**SR**) at each information set.

Let's try to analyze these conditions working backwards through the tree.

BC@2.r: $\alpha = (1/2)(1-t)\,/\,[(1/2)(1-t)+(1/2)p] = (1-t)/(p+1-t) = 1/(1+p)$ with $t=0$.

SR@2.r: $Eu_2(M|2.r) = \alpha(-2)+(1-\alpha)2$, $Eu_2(P|2.r) = -1$; so prefer M if $\alpha<3/4$, P if $\alpha>3/4$.

First show $0<q<1$ at 2.r: No eqm with [F] & [f] (1's SR would require $q=0$ & $q\geq 2/3$).

?[P] $\Rightarrow$ [r] (SR@1.b), $p=1 \Rightarrow \alpha\leq 1/2$ (BC@2.r), $\Rightarrow$ [M] (SR@2.r) ... No!

?[M] $\Rightarrow$ [f] (SR@1.b), $p=0 \Rightarrow \alpha=1$ (BC@2.r), $\Rightarrow$ [P] (SR@2.r) ... No!

So $0<q<1$, and so **t=0** (SR@1.a). With support {M,P} at 2.r, we must have (by SR@2.r):

$Eu_2(M|2.r) = Eu_2(P|2.r) \Rightarrow \alpha(-2)+(1-\alpha)2 = \alpha(-1)+(1-\alpha)(-1)$, so $\boldsymbol{\alpha = 3/4}$,

But then solving $\alpha=1/(1+p)$ (from BC@2.r) yields $\mathbf{p = 1/3}$.

Then $0<p<1$ & SR@1.b yield $Eu_1(r|1.b) = Eu_1(f|1.b) \Rightarrow q(-2)+(1-q)(1) = -1$,

and so $\mathbf{q = 2/3}$.

Move probabilities, belief probabilities and sequential equilibria

Suppose that we are given some extensive game with imperfect information.

Given a randomized strategy for player i, at any information set s of player i that could occur with positive probability when i plays this strategy, we can compute a probability distribution over the set of possible actions $\{d_i\}$ for player i at this information set.

These probabilities $\sigma_i(d_i|s)$ are called <u>move probabilities</u> (or <u>action probabilities</u>).

That is, the move-probability for any move d_i at any information state s of player i denotes the probability that player i will choose move d_i if information set s occurs.

A <u>behavioral strategy</u> σ_i for player i is a vector that specifies a move-probability distribution for each of player i's information sets.

A <u>behavioral-strategy profile</u> σ is a vector that specifies a behavioral strategy σ_i for each player i, and so it must specify an move probability $\sigma_i(d_i|s)$ for every possible move d_i at every possible information set s of every player i in the game.

Let S_i denote the set of information sets for each player i, and let $D_{i.s}$ denote the set of moves for each player i at each information set s in S_i. Then the <u>mixed strategies</u> are $\tau \in \times_{i\in N} \Delta(\times_{s\in Si} D_{i.s})$, but the <u>behavioral strategies</u> are $\sigma \in \times_{i\in N} \times_{s\in Si} \Delta(D_{i.s})$.

The chance probabilities on all branches that follow chance nodes are given parameters of the extensive game. We assume here that the chance probabilities are all positive.

Given σ, a profile of behavioral strategies for all players, the <u>prior probability</u> $P(x|\sigma)$ of any node x in the tree is the multiplicative product of all chance-probabilities and move-probabilities on the path that leads to this node from the root node.

For any finite set Z, let $\Delta^0(Z)$ denote the full-support probability distributions on Z.

$$\Delta^0(Z) = \{\pi \in \Delta(Z) \mid \pi(z)>0\ \forall z \in Z\}.$$

A full-support behavioral strategy profile assigns strictly positive probability ($\sigma_i(d_i|s)>0$) to every possible move d_i at every information set s of every player i, so that every node x in the tree has positive probability.

The set of full-support behavioral strategy profiles is $\times_{i\in N} \times_{s\in S_i} \Delta^0(D_{i.s})$.

When player i moves at his information set s, the belief probability that player i should assign to any node x in this information set should be, by Bayes's formula,

$$\mu_i(x|s) = P(x|\sigma)/\textstyle\sum_{y\in s} P(y|\sigma).$$

That is, the belief probability $\mu_i(x|s)$ should equal the prior probability of x divided by the sum of prior probabilities of all nodes in the information set s, whenever this formula is well-defined (not 0/0).

A belief system μ is a vector that specifies a belief-probability distribution $\mu_i(\bullet|s)$ over the nodes of each information set s of each player i in the game.

Bayes's formula yields one belief system for any full-support behavioral strategy profile. But for strategies that do not have full support, Bayes's formula may leave some belief probabilities undefined, at information sets where all nodes have prior probability 0.

A beliefs system μ is consistent with a behavioral strategy profile σ iff there exists a sequence of full-support behavioral strategies $\tilde{\sigma}^k$ that converge to σ and yield Bayesian beliefs $\tilde{\mu}^k$ that converge to μ as $k\to\infty$. (All $\tilde{\sigma}^k_i(d_i|s)\to\sigma_i(d_i|s)$ & $\tilde{\mu}^k_i(x|s)\to\mu_i(x|s)$.)

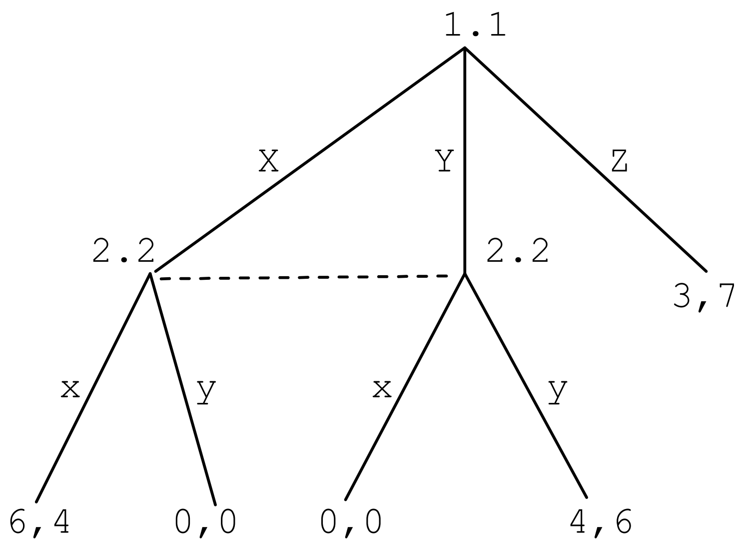
A behavioral-strategy profile σ is sequentially rational given a beliefs system μ iff, at each information set t_i of each player i, $\sigma_i(\bullet|s)$ assigns positive move-probabilities only to moves that maximize i's expected payoff at s, given i's beliefs $\mu_i(\bullet|s)$ about the current node in the information set t_i and given what the behavioral-strategy profile σ specifies about players' behavior after this information set.

A sequential equilibrium is a pair (σ,μ), where σ is a behavioral strategy profile and μ is a belief system, such that σ is sequentially rational given the beliefs system μ, and the beliefs system μ is consistent with the behavioral-strategy profile σ.

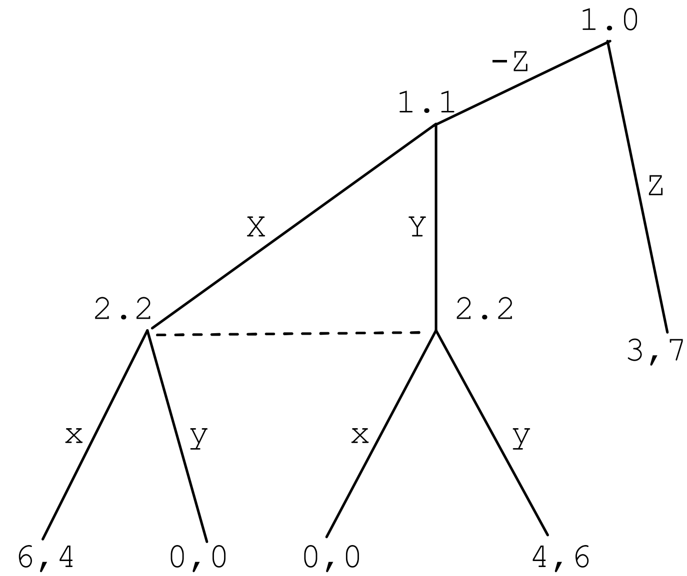
A game has perfect information if every information set consists of just one node.
A game with perfect information can have only one possible beliefs system, which trivially assigns belief probability 1 to every decision node.
For a game with perfect information, a behavioral strategy profile σ is a subgame-perfect equilibrium if it would form a sequential equilibrium together with this (trivial) beliefs system μ.

Given a strategic form game $(N, (C_i)_{i\in N}, (u_i)_{i\in N})$ and any $\varepsilon>0$, a strategy profile σ in $\times_{i\in N} \Delta(C_i)$ is an ε-perfect equilibrium iff, for every $i\in N$ and $c_i\in C_i$, $\sigma_i(c_i)>0$ but if $u_i(\sigma_{-i};[c_i]) < \max_{d_i\in C_i} u_i(\sigma_{-i};[d_i])$ then $\sigma_i(c_i) < \varepsilon$.
A strategy profile σ is a perfect equilibrium iff there is a sequence $\{\sigma^k,\varepsilon^k\}$ such that each σ^k is an ε^k-perfect equilibrium, $\lim_{k\to\infty} \varepsilon^k = 0$, and $\lim_{k\to\infty} \sigma^k = \sigma$.

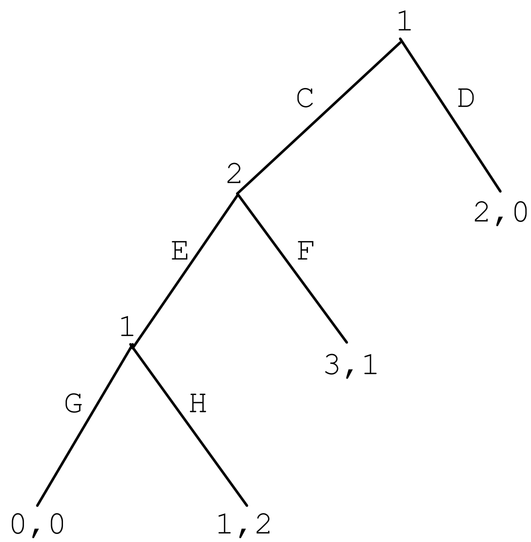
"Find a sequential eqm with outcome (3,7) for these examples." Let $\alpha=\mu_2(X|2.2)$.
With $q=\sigma_2(x|2.2)$, the condition $\sigma_1(Z|1.1)=1$ requires $3\geq 6q$ and $3\geq 4(1-q)$, so $1/2\geq q\geq 1/4$,
but then $0<q<1$ requires $4\alpha+0(1-\alpha)=0\alpha+6(1-\alpha)$, $\alpha=0.6$.



$\sigma_1(Z|1.1)=1$, $1/2\geq\sigma_2(x|2.2)\geq 1/4$, $\alpha=0.6$.



$\sigma_1(Z|1.0)=1$, $\alpha=0.6$, $\sigma_1(X|1.1)=0.6$, $\sigma_2(x|2.2)=0.4$.



Normal represent'n

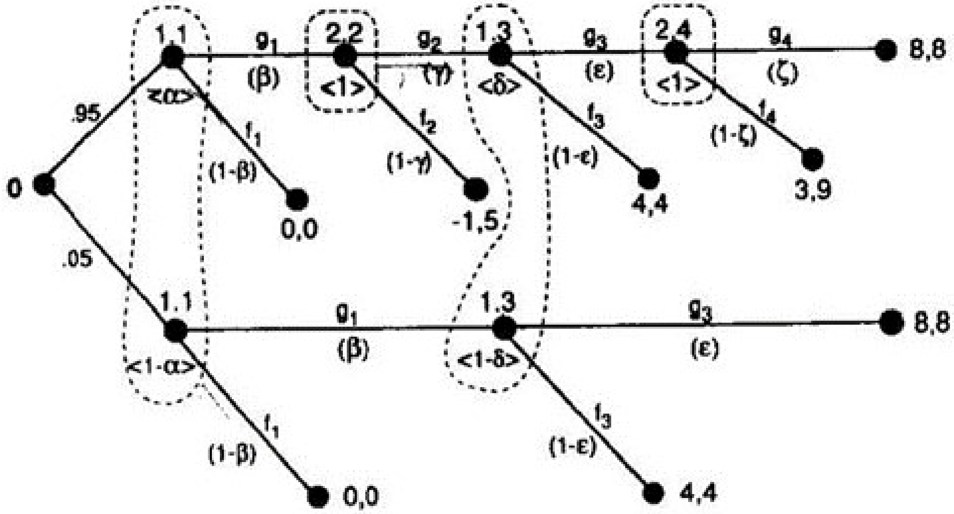
1:\2:	E	F
CG	0,0	3,1
CH	1,2	3,1
DG	2,0	2,0
DH	2,0	2,0

eqms: (DH,E),
also (DG,E), (CG,F)...

Agent-normal or multi-agent representation

1:	3: \ 2:	E	F
C	G	0,0,0	3,1,3
	H	1,2,1	3,1,3
D	G	2,0,2	2,0,2
	H	2,0,2	2,0,2

Computing seq'l eqm in perturbed centipede game from my book section 4.5



	$f_2\bullet$	g_2f_4	g_2g_4
$f_1\bullet$	0, 0	0, 0	0, 0
g_1f_3	–1, 5	4, 4	4, 4
g_1g_3	–1, 5	3, 9	8, 8

	$f_2\bullet$	g_2f_4	g_2g_4
$f_1\bullet$	0, 0	0, 0	0, 0
g_1f_3	–0.75, 4.95	4, 4	4, 4
g_1g_3	–0.55, 5.15	3.25, 8.95	8, 8

$\zeta=\sigma_2(g_4|2.4)$, $\varepsilon=\sigma_1(g_3|1.3)$, $\delta=\mu_1(\text{up}|1.3)$, $\gamma=\sigma_2(g_2|2.2)$, $\beta=\sigma_1(g_1|1.1)$, $\alpha=\mu_1(up|1.1)=0.95$.

By SR@2.4, know $\boldsymbol{\zeta=0}$. So at 1.3, 1 prefers f_3 over g_3 if $4 > 3\delta+8(1-\delta)$, $\delta>4/5$.

By BC@1.3, $\delta = 0.95\beta\gamma/(0.95\beta\gamma+0.05\beta) = 19\gamma/(19\gamma+1)$, $\delta/(1-\delta)=19\gamma$.

At 1.3, guess $\varepsilon=1$? $\Rightarrow$ $\gamma=1$ for SR@2.2 (9>5) $\Rightarrow$ $\delta=0.95 >4/5$ $\Rightarrow$ $\varepsilon=0$, No!

At 1.3, guess $\varepsilon=0$? $\Rightarrow$ $\gamma=0$ for SR@2.2 (5>4) $\Rightarrow$ $\delta=0 <4/5$ $\Rightarrow$ $\varepsilon=1$, No!

So $0<\varepsilon<1$ at 1.3 $\Rightarrow$ $4 = 3\delta+8(1-\delta)$ & $\boldsymbol{\delta=4/5}$ for SR@1.3 $\Rightarrow$ $\boldsymbol{\gamma = 4/19}$ for BC@1.3.

At 2.2, $0<\gamma<1$ $\Rightarrow$ $5=9\varepsilon+4(1-\varepsilon)$ & $\boldsymbol{\varepsilon=1/5}$ for SR@2.2.

So at 1.1 get $Eu_1(g_1f_3) = 0.95((4)4/19+(-1)15/19)+0.05(4) = 1/4 > 0 = Eu_1(f_1)$

$\Rightarrow$ $\boldsymbol{\beta=1}$ for SR@1.1.

Introduction to repeated games Players 1 & 2 meet on $\tau+1$ days, numbered $0,1,2,\ldots,\tau$.
On each day, each player i must choose to be generous (g_i) or selfish (f_i).
On each day k, they get payoffs (u_{1k},u_{2k}) that depend on their actions (c_{1k},c_{2k}) as follows:

Player 1: \ Player 2:	g_2	f_2	
g_1	3, 3	0, 5	(Prisoners' dilemma)
f_1	5, 0	2, 2	

except on the last day τ their payoffs will be:

Player 1: \ Player 2:	g_2	f_2	
g_1	5, 5	0, 4	(Trust game)
f_1	4, 0	2, 2	

On each day, each player knows what both players did on all previous days.
Each player wants to maximize his expected discounted sum of payoffs

$V_i = \sum_{k=0}^{\tau} \delta^k u_{ik}$ for some given *discount factor* δ between 0 and 1.

Basic Fact: $w+w\delta+w\delta^2+\ldots+w\delta^{s-1} = w(1-\delta^s)/(1-\delta)$.

If the first payoff matrix (the prisoners' dilemma) were played once, (f_1,f_2) would be the unique equilibrium, yielding the Pareto-dominated payoff allocation (2,2).
Furthermore, both players $i\in\{1,2\}$ using the strategy ***f_i-always*** would be a subgame-perfect equilibrium of the multi-period game
But in such multi-period games, opportunities to respond later can enlarge the set of equilibria.

Consider the strategy for each player i:

choose g_i until f_1 or f_2 is chosen, but thereafter choose f_i.

Claim: If $\delta \geq 2/3$ then both players using this strategy is a sgp-equilibrium.

The discounted value of *(f_1,f_2)-always* is $\mathbf{F(\tau)} = 2(1-\delta^{\tau+1})/(1-\delta) = 2+\delta F(\tau-1)$, $F(0)=2$.

Discounted value of *(g_1,g_2)-always* is $\mathbf{G(\tau)} = 3(1-\delta^{\tau})/(1-\delta)+5\delta^{\tau} = 3+\delta G(\tau-1)$, $G(0)=5$.

Lemma: If $1>\delta\geq 2/3$ then $G(\tau)-F(\tau) \geq 3$ for all τ.

(*Proof of Lemma by induction:* $G(0)-F(0) = 5-2 = 3$, and then for any $\tau\geq 1$ we get inductively $G(\tau)-F(\tau) = 3-2 + \delta(G(\tau-1)-F(\tau-1)) \geq 1 + (2/3)(3) = 3$.)

The Claim is true for $\tau=0$, because (g_1,g_2) is an equilibrium of the trust game.

Now we argue by induction in $\tau\geq 1$, assuming that the Claim is true for $\tau-1$.

(In any subgame after the first period, both use either the above strategy or *f_i-always*.)

For $\tau\geq 1$, assuming that the above strategies above will be played after the first day, the players' overall payoffs will depend on their first-period choices as follows:

Player 1: \ Player 2:	g_2	f_2
g_1	$3+\delta G(\tau-1)$, $3+\delta G(\tau-1)$	$0+\delta F(\tau-1)$, $5+\delta F(\tau-1)$
f_1	$5+\delta F(\tau-1)$, $0+\delta F(\tau-1)$	$2+\delta F(\tau-1)$, $2+\delta F(\tau-1)$

The equilibrium now requires $3+\delta G(\tau-1) \geq 5+\delta F(\tau-1)$, which is satisfied because $3+\delta G(\tau-1) \geq 3+\delta(3+F(\tau-1)) \geq 5+\delta F(\tau-1)$ when $\delta\geq 2/3$.

Infinitely Repeated games

Infinitely repeated games can be used as simple models of long-term relationships.
The game will be played at an infinite sequence of time periods numbered 1,2,3,...
Suppose that the set of players is {1,2}. In each period k, each player i must choose an action c_{ik} in some set C_i. In period k, each player i's payoff u_{ik} will depend on both players' actions according to some utility function $u_i{:}C_1 \times C_2 \rightarrow \mathbb{R}$; that is, $u_{ik} = u_i(c_{1k}, c_{2k})$.
We assume here that the actions at each period are publicly observable, so each player's action in each period may depend on the history of past actions by both players.
Given any <u>discount factor</u> δ such that $0 \leq \delta < 1$, the δ-<u>discounted sum</u> of player i's payoffs is
$V(u_{i1},u_{i2},u_{i3},...) = u_{i1} + \delta u_{i2} + \delta^2 u_{i3} + ... + \delta^{k-1} u_{ik} + ...$
For a constant payoff x each period, the δ-discounted sum would be $x/(1-\delta)$.
The objective of each player i in the repeated game is to maximize the expected discounted sum of his payoffs, with respect to some discount factor δ, where $0 \leq \delta < 1$.
<u>Fact. (Recursion formula)</u> $V(u_{i1},u_{i2},u_{i3},...) = u_{i1} + \delta V(u_{i2},u_{i3},u_{i4},...)$.
We may describe equilibria of repeated games in terms of a various <u>social states</u>.
At each period of the game, the players expectations about each others' behavior will be determined by one of these social states, which characterizes their relationship.
This state may be called the <u>state of play</u> in the game at this period. (These social states are a characteristic of the equilibrium, not of the game.)

To describe an equilibrium or scenario in terms of social states, we must specify:

(1) Social states We must list the set of social states in this equilibrium. (States may denoted by numbers or named for the kinds of relationships that they represent.)

(2) State-dependent strategies. For each state θ, we must specify (possibly randomized) actions $(\tilde{s}_1(\theta),\tilde{s}_2(\theta))$ that are the players' predicted behavior each period when state of play is θ.

(3) Transitions. For each social state θ, we must specify how the state of play in the next period might change depending on player's actions. We may let $\Theta(a_1,a_2;\theta)$ denote the state of play in the next period after a period when the state of play was θ and the players chose actions (a_1,a_2) (possibly deviating from the prediction $(\tilde{s}_1(\theta),\tilde{s}_2(\theta))$).

(4) Initial state. We must specify which social state is initial state of play in the first period of the game. Here we will generally let state "0" denote this initial state.

Given a scenario as in (1)-(3) above and a discount factor δ, let $V_i(\theta)$ denote the expected δ-discounted sum of player i's payoffs in this scenario when (ignoring (4)) the state of play begins in state θ. These numbers $V_i(\theta)$ can be computed from the recursion equations:

$V_i(\theta) = E[u_i(\tilde{s}_1(\theta),\tilde{s}_2(\theta))) + \delta\ V_i(\Theta(\tilde{s}_1(\theta),\tilde{s}_2(\theta);\theta))]\ \ \forall\theta.$

Fact. A scenario as in (1)-(3) above is a subgame-perfect equilibrium if, for every player i and every state θ, player i could not expect to gain by unilaterally deviating from the prediction $\tilde{s}_i(\theta)$ in a period when the state of play is θ. That is, we have an equilibrium if, for every state θ,

$V_1(\theta) \geq E[u_i(c_1,\tilde{s}_2(\theta))) + \delta\ V_i(\Theta(c_1,\tilde{s}_2(\theta);\theta))]$, for all c_1 in C_1,

$V_2(\theta) \geq E[u_i(\tilde{s}_1(\theta),c_2)) + \delta\ V_i(\Theta(\tilde{s}_1(\theta),c_2;\theta))]$, for all c_2 in C_2.

(This is the *one-deviation condition*: if nobody could ever gain in any state by a one-round deviation, then longer strategic deviations are also not profitable.)

Folk Theorem: When $\delta \approx 1$, subgame-perfect equilibria can achieve almost any feasible payoff allocation that gives each player more than he could guarantee himself when others act punitively.

Example 1 (notes p14). Consider a repeated game where, in each period, the players play a "Prisoners' dilemma" game where must decide whether to "cooperate" or "defect":

	c_2	d_2
c_1	5, 5	0, 6
d_1	6, 0	1, 1

Each player wants to maximize his or her δ-discounted sum of payoffs, for some $0 \leq \delta < 1$.

We first consider a version of the "grim trigger" equilibrium where the states are $\{0, 1\}$.
(State 0 represents "trust" or "friendship"; state 1 represents "distrust".)
The predicted behavior in state 0 is (c_1,c_2). The predicted behavior in state 1 is (d_1,d_2).
In any period when the state of play is 0, if the players' action profile is (d_1,c_2) or (c_1,d_2) then the state of play next period will switch to state 1, otherwise it will remain state 0. When the state of play is 1, the future state of play always remains state 1.

$$\boxed{0\text{: } (c_1,c_2)} \xrightarrow{(d_1,c_2) \text{ or } (c_1,d_2)} \boxed{1\text{: } (d_1,d_2)}$$

The expected discounted values $V_i(\theta)$ for each player i in each states θ satisfy:

$V_1(0) = u_1(c_1,c_2) + \delta V_1(0)$, $V_1(1) = u_1(d_1,d_2) + \delta V_1(1)$,

$V_2(0) = u_2(c_1,c_2) + \delta V_2(0)$, $V_2(1) = u_2(d_1,d_2) + \delta V_2(1)$.

So $V_1(0) = 5 + \delta V_1(0)$, $V_1(1) = 1 + \delta V_1(1)$, and so $\mathbf{V_1(0) = 5/(1-\delta)}$, $\mathbf{V_1(1) = 1/(1-\delta)}$.
Similarly, $\mathbf{V_2(0) = 5/(1-\delta)}$, $\mathbf{V_2(1) = 1/(1-\delta)}$.

For this scenario to be an equilibrium, we need:

$V_1(0) \geq u_1(d_1,c_2) + \delta V_1(1)$, $V_1(1) \geq u_1(c_1,d_2) + \delta V_1(1)$,

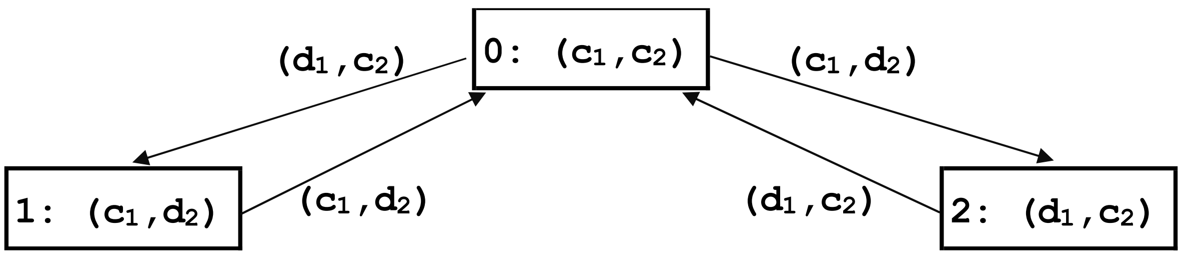
$V_2(0) \geq u_2(c_1,d_2) + \delta V_2(1)$, $V_2(1) \geq u_2(d_1,c_2) + \delta V_2(1)$.

So we need: $5/(1-\delta) \geq 6 + \delta 1/(1-\delta)$ & $1/(1-\delta) \geq 0 + \delta 1/(1-\delta)$.
These are (with $\delta<1$): $5 \geq 6(1-\delta)+\delta 1$ & $1 \geq 0(1-\delta)+\delta 1$, satisfied when $\mathbf{1>\delta\geq 1/5}$.

For the same δ-discounted repeated game:

	c_2	d_2
c_1	5, 5	0, 6
d_1	6, 0	1, 1

Let's consider now another (more forgiving) equilibrium:
The states are {0, 1, 2}. (State 0 is "friendship"; state 1 is "punishing 1"; state 2 is "punishing 2".)



The expected discounted values $V_1(\theta)$ for player 1 in each state θ satisfy the equations:
$V_1(0) = u_1(c_1,c_2) + \delta V_1(0)$, $V_1(1) = u_1(c_1,d_2) + \delta V_1(0)$, $V_1(2) = u_1(d_1,c_2) + \delta V_1(0)$.
Thus $V_1(0) = 5 + \delta V_1(0)$, and so $\mathbf{V_1(0) = 5/(1-\delta)}$;
$V_1(1) = 0 + \delta 5/(1-\delta)$, so $\mathbf{V_1(1) = 5\delta/(1-\delta)}$; and $\mathbf{V_1(2) = 6 + \delta 5/(1-\delta) = (6-\delta)/(1-\delta)}$.
Similarly, $\mathbf{V_2(0) = 5/(1-\delta)}$, $\mathbf{V_2(1) = (6-\delta)/(1-\delta)}$, $\mathbf{V_2(2) = 5\delta/(1-\delta)}$.

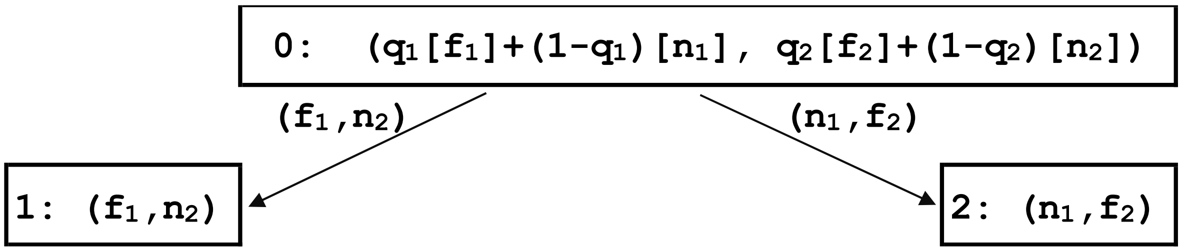
To have a subgame-perfect equilibrium, we need:
$V_1(0) \geq u_1(d_1,c_2) + \delta V_1(1)$, $V_1(1) \geq u_1(d_1,d_2) + \delta V_1(1)$, $V_1(2) \geq u_1(c_1,c_2) + \delta V_1(2)$,
and similar conditions for player 2. These inequalities (for both players) become:
$5/(1-\delta) \geq 6 + \delta 5\delta/(1-\delta)$, $5\delta/(1-\delta) \geq 1 + \delta 5\delta/(1-\delta)$, $(6-\delta)/(1-\delta) \geq 5+\delta(6-\delta)/(1-\delta)$.
With $\delta<1$, these inequalities are equivalent to: $5(1-\delta^2)/(1-\delta) \geq 6$, $5\delta \geq 1$, $6-\delta \geq 5$.
With $(1-\delta^2) = (1-\delta)(1+\delta)$ (and $\delta<1$), the first inequality further simplifies to $5(1+\delta) \geq 6$.
These conditions for a subgame-perfect equilibrium are all satisfied when $\mathbf{1 > \delta \geq 1/5}$.

Example 2 (notes p15). Consider a repeated game where players 1 and 2 play the game below infinitely often. In each round, each player i must decide to fight (f_i) or not (n_i).

	f_2	n_2
f_1	$-1, -1$	$9, 0$
n_1	$0, 9$	$0, 0$

Each player i wants to maximize his δ_i-discounted sum of payoffs, for some $0 \leq \delta_i < 1$.

States: Consider three social states, numbered 0,1,2. The initial state in period 1 is state 0.
Interpretation: state1 = "1 is owner", state2 = "2 is owner", state0 = "fighting for ownership".



Suppose $0<q_i<1\ \forall i$.

The recursion equations for $V_i(\theta)$ in state 1 & state 2 are:
$V_i(1) = u_i(f_1,n_2) + \delta_i V_i(1)$, for i=1,2, and so $\mathbf{V_1(1) = 9/(1-\delta_1)}$ and $\mathbf{V_2(1) = 0/(1-\delta_2) = 0}$;
$V_i(2) = u_i(n_1,f_2) + \delta_i V_i(2)$, for i=1,2, and so $\mathbf{V_1(2) = 0}$ and $\mathbf{V_2(2) = 9/(1-\delta_2)}$.
Equilibrium conditions in state 1: $9/(1-\delta_1) = V_1(1) \geq u_1(n_1,n_2) + \delta_1 V_1(1) = 0 + \delta_1 9/(1-\delta_1)$,
$0 = V_2(1) \geq u_2(f_1, f_2) + \delta_2 V_2(1) = -1 + \delta_2 0$; both are true when $\mathbf{0 \leq \delta_1 < 1}$. State 2 is similar.
In state 0, for player 1 to be willing to randomize between f_1 and n_1, he must expect the same discounted value $V_1(0)$ from choosing f_1 or n_1 this period, and so we must have
$V_1(0) = q_2\mathbf{(}u_1(f_1,f_2) + \delta_1 V_1(0)\mathbf{)} + (1-q_2)\mathbf{(}u_1(f_1,n_2)) + \delta_1 V_1(1)\mathbf{)}$, and
$V_1(0) = q_2\mathbf{(}u_1(n_1,f_2) + \delta_1 V_1(2)\mathbf{)} + (1-q_2)\mathbf{(}u_1(n_1,n_2)) + \delta_1 V_1(0)\mathbf{)}$.
Latter is $V_1(0) = q_2 0 + q_2\delta_1 0 + (1-q_2)0 + (1-q_2)\delta_1 V_1(0)$, which implies $\mathbf{V_1(0) = 0}$.
Then $V_1(0) = q_2(-1) + q_2\delta_1 V_1(0) + (1-q_2)9 + (1-q_2)\delta_1 9/(1-\delta_1)$ implies $\mathbf{q_2 = 9/(10-\delta_1)}$.
For 2 to choose f_2 or n_2 in θ=0 yields similar equations and $\mathbf{V_2(0) = 0,\ q_1 = 9/(10-\delta_2)}$.

A first Bayesian game (notes p17). {Players} = {1,2}, each player i's action is generous or hostile (g_i or h_i). Player 1's type may be contented or envious (1c or 1e), and player 2 thinks these are equally likely. Payoffs (u_1,u_2) depend on both actions and 1's type as follows:

If 1's type is 1c: $p(1c) = 0.5$

	g_2	h_2
g_1	7,7	0,4
h_1	4,0	4,4

If 1's type is 1e: $p(1e) = 0.5$

	g_2	h_2
g_1	3,7	0,4
h_1	4,0	4,4

A common mistake: *Some students try to analyze the game where $u_1(g_1,g_2)$ is the expected utility $0.5(7)+0.5(3) = 5$. So they consider a 2×2 payoff matrix that differs from the (1e) case in that the payoff 3 is replaced by 5, and then they find an "equilibrium" at (g_1,g_2) (as 5>4 for 1, 7>4 for 2). Such analysis is nonsense. This "equilibrium" predicts that each player is sure to act generously. But player 2 knows that g_1 would be dominated by h_1 for player 1 when his type is 1e, and so player 2 must understand that there is at least a probability 0.5 of player 1 acting hostile (h_1).*

A <u>strategy</u> for a player is a complete plan that specifies a feasible action for the player in every possible contingency that the player could find. If the game started before 1 learned his type, 1's ability to condition his action on his type would give him 4 strategies {g_cg_e, g_ch_e, h_cg_e, h_ch_e}. Thus, the <u>normal representation</u> in strategic form of this Bayesian game is:

	g_2	h_2
g_cg_e	5, 7	0, 4
g_ch_e	5.5, 3.5	2, 4
h_cg_e	3.5, 3.5	2, 4
h_ch_e	4, 0	4, 4

...Unique eqm is (h_ch_e, h_2)!

A Bayesian game is defined by a set of players N; a set of actions C_i, a set of types T_i, and a utility function $u_i:(\times_{j\in N} C_j)\times(\times_{j\in T} T_j)\to\mathbb{R}$, for each player i in N; and a probability distribution $p\in\Delta(\times_{j\in N} T_j)$.

The Bayesian game $(N, (C_i, T_i, u_i)_{i\in N}, p)$ is assumed to be common knowledge among the players in the game, but each player i also privately knows his own actual type $\tilde{t}_i\in T_i$, which is a random variable in the model.

Thus, in a Bayesian game, a player's **type** is a random variable that summarizes the player's private information at the start of the game (everything that the player knows beyond what is common knowledge among the players).

We may say that a Bayesian game has private values if each player's payoff depends only on his own type, not on others' types.

If the players' types are independent then $p(t) = \prod_{i\in N} p_i(t_i)$, where p_i is the marginal probability distribution over player i's types (which anyone else would believe about i).

Mixed strategies for i are in $\Delta(C_i^{T_i})$, distributions over functions from T_i to C_i.

But nobody cares about correlations among plans of i's different types; so we can instead analyze behavioral strategies in $\Delta(C_i)^{T_i}$, functions from T_i to distributions over C_i.

A behavioral strategy σ_i specifies conditional probabilities $\sigma_i(c_i|t_i) = \text{Prob}(\text{i does } c_i| \tilde{t}_i=t_i)$, for each action $c_i\in C_i$ and each type $t_i\in T_i$.

Increasing differences and increasing strategies in Bayesian games (notes p18)
We may consider Bayesian games where each player i first learns his type $\tilde{t}_i$, and then each player i chooses his action a_i. We assume here that each player i's type is drawn from some probability distribution p_i, independently of all other players' types, and so in a two-person game the joint distribution of the players' types can be written $p((t_i)_{i\in N}) = p_1(t_1)\,p_2(t_2)$.
Payoffs of each player i may depend on all types and actions according to some $u_i(c_1,c_2,\tilde{t}_1,\tilde{t}_2)$.

Suppose that types and actions are ordered as numbers ($c_i \in C_i \subseteq \mathbb{R}$, $\tilde{t}_i \in T_i \subseteq \mathbb{R}$).

A function $f:\mathbb{R}\to\mathbb{R}$ is (weakly) increasing iff, for all x and $\hat{x}$, $\hat{x} \geq x$ implies $f(\hat{x}) \geq f(x)$.

A function $f:\mathbb{R}\to\mathbb{R}$ is strictly increasing iff, for all x and $\hat{x}$, $\hat{x} > x$ implies $f(\hat{x}) > f(x)$.
Player 1's payoffs satisfy (weakly or strictly) increasing differences if, for every pair of actions c_1 and d_1 such that $c_1>d_1$, the difference $\mathbf{u_1(c_1, c_2, t_1, t_2)-u_1(d_1, c_2, t_1, t_2)}$ is a (weakly or strictly) increasing function of 1's type t_1, no matter what 2's action c_2 and type t_2 may be.
If u_1 is differentiable then the condition for increasing differences is $\partial^2 u_1/\partial c_1 \partial t_1 \geq 0$.

Fact. If 1's payoffs satisfy weakly increasing differences then, for any strategy of player 2, player 1 has some best-response $s_1:T_1\to C_1$ that is weakly increasing ($r_1 \geq t_1 \Rightarrow s_1(r_1) \geq s_1(t_1)$).

Fact. When 1's payoffs have strictly increasing differences then all player 1's best-response strategies must be weakly increasing: if $r_1 > t_1$ and, against some strategy σ_2 for player 2, action c_1 is optimal for type t_1 and action d_1 is optimal for type r_1, then $d_1 \geq c_1$. So in any equilibrium, if type t_1 would choose c_1 with positive probability, and type $r_1 > t_1$ would choose d_1 with positive probability, then $d_1 \geq c_1$.
(By optimality, $Eu_1(c_1,\sigma_2,t_1,\tilde{t}_2) - Eu_1(d_1,\sigma_2,t_1,\tilde{t}_2) \geq 0$ and $0 \geq Eu_1(c_1,\sigma_2,r_1,\tilde{t}_2) - Eu_1(d_1,\sigma_2,r_1,\tilde{t}_2)$, but this would contradict strictly increasing differences if we had $c_1 > d_1$ with $r_1 > t_1$.)

Example (notes p19). Player 1's types are $\{0, .1, .2, .3\}$, each has probability $p_1(t_1) = 1/4$. Player 2 has no private information. 1's actions are $\{T,B\}$, 2's actions are $\{L,R\}$.

Given 1's type t_1, the payoffs are:

		L (q)	R (1–q)
($t_1>\theta$)	T	t_1, 0	t_1, −1
($t_1<\theta$)	B	1, 0	−1, 3

$u_1(T,L,t_1)-u_1(B,L,t_1) = t_1-1$, $u_1(T,R,t_1)-u_1(B,R,t_1) = t_1+1$ => ***increasing differences*** with "T>B"

If player 2 uses $\sigma_2 = q[L]+(1-q)[R]$ then player 1 that would be willing to choose T iff 1's type t_1 satisfies $qt_1+(1-q)t_1 = U_1(T,\sigma_2,t_1) \geq U_1(B,\sigma_2,t_1) = q(1)+(1-q)(-1)$.

With this q and this u_1 function, player 1 would be indifferent between T and B only if 1's type were θ such that $q\theta+(1-q)\theta = q(1)+(1-q)(-1)$. This ***cutoff type*** is $\theta = 2q-1$, with $\mathbf{q = (\theta+1)/2}$.

All types $t_1>\theta$ prefer to do T, and all $t_1<\theta$ prefer to do B against $\sigma_2 = q[L]+(1-q)[R]$.

So although player 1 has $2^4 = 16$ pure strategies ($2^{16}-1$ possible supports) in this Bayesian game, we only need to consider *cutoff strategies* with 9 possible supports:

- ($\theta>.3$) every type would choose [B], so 2 thinks the probability of T is P(T)=0;
- ($\theta=.3$) {0,.1,.2} would choose [B], but .3 might randomize, so 2 thinks $0 \leq P(T) \leq 1/4$;
- ($.2<\theta<.3$) {0,.1,.2} would choose [B], but .3 would choose [T], so 2 thinks P(T) = 1/4;
- ($\theta=.2$) {0,.1} choose [B], .2 could randomize, .3 chooses [T], so $1/4 \leq P(T) \leq 1/2$;
- ($.1<\theta<.2$) {0,.1} choose [B], {.2,.3} choose [T], so P(T) = 1/2;
- ($\theta=.1$) 0 choose [B], .1 could randomize, {.2,.3} choose [T], so $1/2 \leq P(T) \leq 3/4$;
- ($0<\theta<.1$) 0 would choose [B], {.1,.2,.3} would choose [T], so P(T) = 3/4;
- ($\theta=0$) 0 could randomize, {.1, .2,.3} would choose [T], so 2 thinks $3/4 \leq P(T) \leq 1$;
- ($\theta<0$) every type would choose [T], and so 2 thinks P(T) = 1.

There is no equilibrium without 2 randomizing. So we must have $EU_2(L) = EU_2(R)$, that is, $P(T)(0) + (1-P(T))(0) = P(T)(-1) + (1-P(T))(3)$.

So P(T) = 3/4, which corresponds to $0<\theta<.1$. Thus we get $1/2 \leq q = (\theta+1)/2 \leq 11/20$, and $P(T) = \sum_{t1} p_1(t_1)\sigma_1(T|t_1) = 0.25(\sigma_1(T|0)+\sigma_1(T|.1)+\sigma_1(T|.2)+\sigma_1(T|.3)) = 0.25(0+1+1+1)$.

Second variant of the Bayesian game with payoff matrix

		L (q)	R (1−q)
($t_1 > \theta$)	T	t_1, 0	t_1, −1
($t_1 < \theta$)	B	1, 0	−1, 3

Now suppose 1's type is in {0, .1, .2, .3, .4}, each with $p_1(t_1)=1/5$. When $\sigma_2 = q[L]+(1-q)[R]$, we get $u_1(T,\sigma_2|t_1) = (t_1)q+(t_1)(1-q) = t_1$ and $u_1(B,\sigma 2|t_1) = (1)q+(-1)(1-q) = 2q-1$.
Among 1's types, at most one cutoff type θ such that $\theta=2q-1$ can be willing to randomize, and all higher types $t_1>\theta$ must prefer T, and all lower types $t_1<\theta$ must prefer B.
To make player 2 willing to randomize, player 1 must use a strategy such that P(T) = 3/4.
So we need $3/4 = \sum_{t1} p_1(t_1)\sigma_i(T|t_1) = (1/5)\mathbf{(}\sigma_1(T|0) + \sigma_1(T|.1) + \sigma_1(T|.2) + \sigma_1(T|.3) + \sigma_1(T|.4)\mathbf{)}$
To do this with an increasing cutoff strategy, the cutoff must be at $\boldsymbol{\theta = 0.1}$, which gives us
$3/4 = (1/5)(0 + \sigma_1(T|.1) + 1 + 1 + 1)$, and so $\boldsymbol{\sigma_1(T|.1) = (3/4 - 3/5)/(1/5) = 0.75}$.
To make type θ=.1 willing to randomize, we need $0.1= \theta = 2q-1$, so $\mathbf{q = (.1+1)/2 = 11/20}$.

Third variant: Suppose now that player 1's type t_1 is drawn from a Uniform distribution on the interval from 0 to 1, and ε is a given parameter between 0 and 1 (say, ε=0.1).

		L (q)	R (1–q)
($t_1>\theta$)	T	εt_1, 0	εt_1, −1
($t_1<\theta$)	B	1, 0	−1, 3

Player 1's payoffs still satisfy increasing differences with "T>B".

So 1 uses some cutoff strategy: T if $t_1>\theta$, B if $t_1<\theta$, and so 2 thinks $P(T) = P(t_1>\theta) = 1-\theta$.

For player 2 to use $0<q<1$, we need $0 = (1-\theta)(-1)+(\theta)(3)$, and so $\boldsymbol{\theta=1/4}$.

For player 1 to use a θ-cutoff strategy, we need $\varepsilon\theta = (q)(1)+(1-q)(-1)$, and so $\boldsymbol{q=(1+\varepsilon/4)/2}$.

Facts about Uniform distributions.

Suppose that random variable $\tilde{X}$ is drawn from a Uniform distribution on the interval from A to B, given A < B. Then $E(\tilde{X}) = (A+B)/2$, and $\forall\theta\in[A,B]$:

$F(\theta) = P(\tilde{X}\leq\theta) = P(\tilde{X}<\theta) = (\theta-A)/(B-A)$, $f(\theta) = F'(\theta) = 1/(B-A)$, $1-F(\theta) = (B-\theta)/(B-A)$,

$E(\tilde{X}|\,\tilde{X}\leq\theta) = E(\tilde{X}|\,\tilde{X}<\theta) = (A+\theta)/2$, and $E(\tilde{X}|\,\tilde{X}\geq\theta) = E(\tilde{X}|\,\tilde{X}>\theta) = (\theta+B)/2$.

Fourth variant: Suppose 1's type t_1 is drawn from a Uniform distribution on interval from 0 to 1, 2's type t_2 is drawn independently from a Uniform distribution on interval from 0 to 1, and payoffs depend on the types as follows, for some given number ε between 0 and 1:

		L $(t_2>\theta_2)$	R $(t_2<\theta_2)$
$(t_1>\theta_1)$	T	$\varepsilon t_1, \varepsilon t_2$	$\varepsilon t_1, -1$
$(t_1<\theta_1)$	B	$1, \varepsilon t_2$	$-1, 3$

1's payoffs have increasing diff's with "T>B," 2's payoffs have increasing diff's with "L>R".
So we should look for an equilibrium where each uses a cutoff strategy of the form
- player 1 does T if $t_1 > \theta_1$, player 1 does B if $t_1 < \theta_1$,
- player 2 does L if $t_2 > \theta_2$, player 2 does R if $t_2 < \theta_2$,

for some pair of cutoffs θ_1 and θ_2. It is easy to check that neither player's action can be certain to the other, and so these cutoffs θ_1 and θ_2 must be strictly between 0 and 1.
With t_1 Uniform on 0 to 1, the probability of player 1 doing T $(t_1>\theta_1)$ is $1-\theta_1$.
Similarly, the probability of player 2 doing L $(t_2>\theta_2)$ is $1-\theta_2$.
The cutoff types must be indifferent between the two actions. So we have the equations
$\varepsilon\theta_1 = (1)(1-\theta_2) + (-1)\theta_2$, $\varepsilon\theta_2 = (-1)(1-\theta_1) + (3)\theta_1$.
The unique solution to these equations is $\boldsymbol{\theta_1 = (2+\varepsilon)/(8+\varepsilon^2)}$, $\boldsymbol{\theta_2 = (4-\varepsilon)/(8+\varepsilon^2)}$.
Unless a player's type exactly equals the cutoff (which has zero probability), he is not indifferent between his two actions, and he uses the action yielding a higher expected payoff given his type.
As $\varepsilon\to 0$, these equilibria approach the randomized strategies (0.75[T]+0.25[B], 0.5[L]+0.5[R]).
(*Recall "on minimax" example on slide 13 above.*)

In this way, randomized equilibria can become pure-strategy equilibria in Bayesian games where each player has minor private information that determines his optimal action in equilibrium.
This is called purification of randomized equilibria by Bayesian games (Harsanyi, *IJGT*, 1973.)

Akerlof's lemon example, with a given parameter M (say M=1000).
{players}={1,2}, 1's type $\tilde{t}_1$ is Uniform on [0,M]. Player 1 owns a "lemon", worth $\tilde{t}_1$ to him, which would be worth $\tilde{V}=1.5\tilde{t}_1$ to player 2, but only 1 knows these values.

Buyer-offer game: Player 2 offers price $r=c_2\geq 0$, then player 1 says Yes or No;
$u_1(Y,r,t_1) = r-t_1$ & $u_2(Y,r,t_1) = 1.5t_1-r$, $u_1(N,r,t_1) = 0 = u_2(N,r,t_1)$.
Sequential rationality => 1 says $\sigma_1(r,t_1)=Y$ if $r>t_1$, $\sigma_1(r,t_1)=N$ if $r<t_1$.
Against this strategy, any offer $r\leq M$ for 2 yields
$Eu_2(\sigma_1(r,\tilde{t}_1),r,\tilde{t}_1) = \int_0^r (1.5t_1-r)\, dt_1/M = (1.5r^2/2 - r^2)/M = (r/M)(1.5r/2 - r)$
$= P(\tilde{t}_1\leq r)(E(1.5\tilde{t}_1|\tilde{t}_1\leq r) - r) = -0.25r^2/M < 0$ if $c_2>0$.
So in equilibrium, player 2 should bid r=0, no trade!
With $\tilde{V}=1.5\tilde{t}_1$, we get $E(\tilde{V}) = 1.5E(\tilde{t}_1) = 1.5((M+0)/2) = 0.75M$, but
$E(\tilde{V}|\tilde{t}_1\leq r) = 1.5E(\tilde{t}_1|\tilde{t}_1<r) = 1.5((r+0)/2) = 0.75r < E(\tilde{V})$ when $r<M$ (*winner's curse*).

Seller-offer game: Knowing $\tilde{t}_1$, player 1 offers price $\tilde{r} =\sigma_1(\tilde{t}_1)\geq 0$, then player 2 chooses Y or N.
(Player 2's rational response to an offer r depends on 2's beliefs about $\tilde{t}_1$ given $\sigma_1(\tilde{t}_1)=r$.)
Let $\sigma_2(r)\geq 0$ denote the probability that 2 says Yes to price r.
For an equilibrium (σ_1,σ_2), let $p(t_1) = \sigma_2(\sigma_1(t_1)) = P(\text{trade}|\tilde{t}_1=t_1) \geq 0$, $U_1(t_1) = p(t_1)(\sigma_1(t_1)-t_1)$.
$U_1(t_1) = p(t_1)(\sigma_1(t_1)-t_1) \geq p(s_1)(\sigma_1(s_1)-t_1)$, $p(t_1)(\sigma_1(t_1)-s_1) \leq p(s_1)(\sigma_1(s_1)-s_1) = U_1(s_1)$.
These *informational incentive constraints* imply $p(t_1)(s_1-t_1) \geq U_1(t_1)-U_1(s_1) \geq p(s_1)(s_1-t_1)$.
So $p(t_1)$ is weakly decreasing in t_1, and $U_1(t_1) = U_1(M) + \int_{t_1}^M p(s_1)ds_1$ (*information rents*).
Player 2's expected payoff is $U_2 = \int_0^M (1.5t_1-\sigma_1(t_1))p(t_1)dt_1 / M = \int_0^M (0.5t_1p(t_1)-U_1(t_1))dt_1/M$.
With no-trade options, $0 \leq U_1(M)+U_2 = \int_0^M 0.5t_1p(t_1)dt_1/M - \int_0^M\int_{t_1}^M p(s_1)ds_1\, dt_1/M =$
$= \int_0^M 0.5t_1p(t_1)dt_1/M - \int_0^M\int_0^{s_1} dt_1\, p(s_1)ds_1/M = \int_0^M (0.5t_1-t_1)p(t_1)dt_1/M = \int_0^M (-0.5t_1)p(t_1)dt_1/M$
But this implies that $p(t_1)=0$ for all $t_1>0$. Even when the seller offers, they cannot trade!

Admitting public random events into a War of Attrition model
(From <http://home.uchicago.edu/~rmyerson/research/ww1_review.pdf>)
Two nations 1 & 2 are contending for shares of peacetime benefits flows worth 1.
Nation j demands $\mathbf{w_j}$, is offered $\mathbf{v_j} = 1-w_{-j}$ by the other nation -j.
Until an offer is accepted, each j pays conflict cost $\mathbf{c_j}$. $1 > w_j > v_j > 0 > -c_j \ \ \forall j \in \{1,2\}$.
Expected future flows of benefits & costs are discounted at rate $\mathbf{r}>0$.

If $\mathbf{q_j}$ is the probability density of -j accepting j's offer in near future, then nation j is willing to concede now (not wait short dt) only if $\mathbf{(c_j+v_j)dt \geq (w_j/r - v_j/r)q_j dt}$, that is, $\mathbf{q_j \leq \lambda_j}$ where $\boldsymbol{\lambda_j = (c_j+v_j)r/(w_j-v_j)}$ is the ***critical success rate*** for j.
Conflict eqm: each nation j's concession time is an independent exponential random variable, mean $\boldsymbol{\mu_j} = 1/\lambda_{-j}$. $\mathbf{EU_j = v_j/r}$, expected conflict costs cancel win-gains.
Bizarre implication: higher c_j => higher λ_j => lower μ_{-j} => j more likely to win!

Other equilibria: immediate concession expected from -j (always), j never concedes.
The conflict eqm is Pareto-inferior to randomizing equally over 1 or 2 conceding.

This assumed uncertainty only from other's decisions. But battles are random too!
Let q_j denote the probability density of ***decisive victory*** for j (=> -j conceding focal).
Fighting until a decisive victory is equilibrium if each $q_j \geq \lambda_j$.
Then can get higher c_j => higher λ_j => (switch to eqm of j conceding if now $\lambda_j > q_j$).

Why don't nations moderate their demands to end conflict sooner?
Model: Each j has small δ_j rate of becoming ***exhausted type*** that must concede soon.
An unanticipated concession could be taken as indicating an exhausted type.

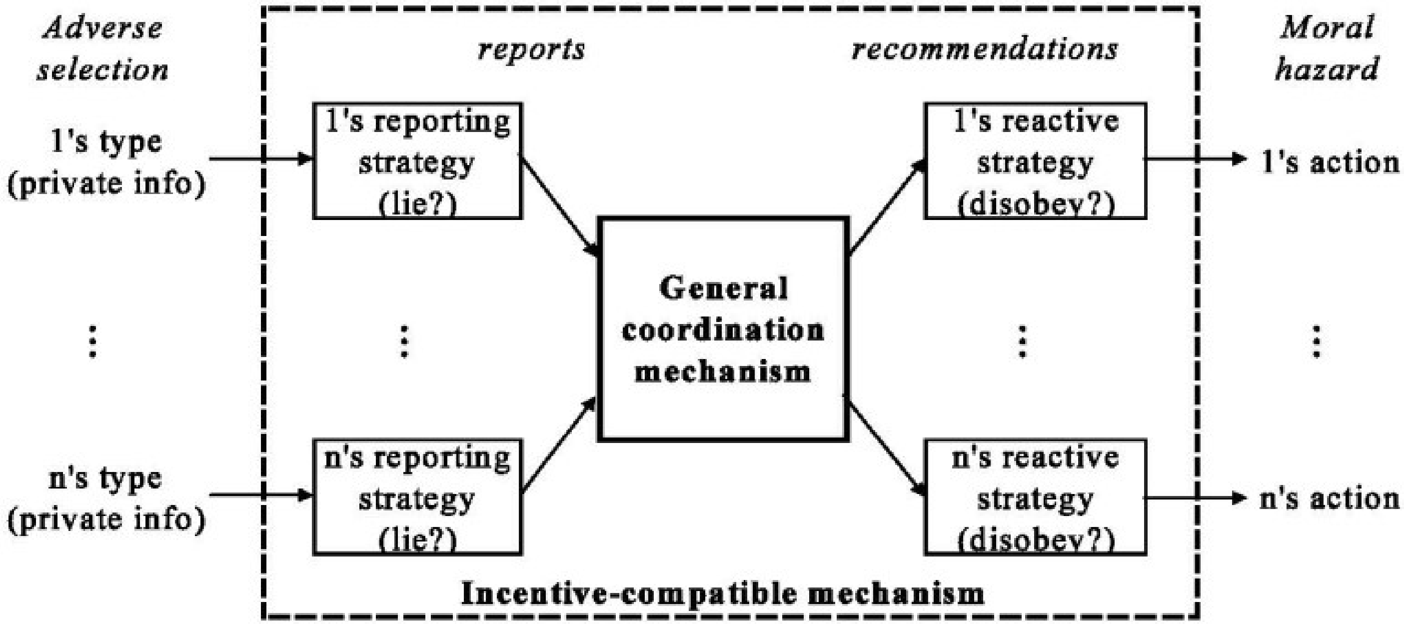


Figure 1. The revelation principle.

For strategic games where players have no private information, these are **correlated equilibra**.
Example from section 6.1 of Myerson 1991: $C_1=\{x_1,y_1\}$, $C_2=\{x_2,y_2\}$.

(u_1,u_2):	x_2	y_2
x_1	5,1	0,0
y_1	4,4	1,5

Nash equilibria in $\Delta(C_1)\times\Delta(C_2)$: $([x_1,],[x_2])$, $([y_1],[y_2])$, $(0.5[x_1]+0.5[y_1],\ 0.5[x_2]+0.5[y_2])$.

Correlated equilibria are joint strategy distributions $\mu\in\Delta(C_1\times C_2)$ such that

$\sum_{c2} \mu(c_1,c_2)(u_1(c_1,c_2)-u_1(d_1,c_2)) \geq 0,\ \ \forall c_1\in C_1,\ \forall d_1\in C_1$;

$\sum_{c1} \mu(c_1,c_2)(u_1(c_1,c_2)-u_1(c_1,d_2)) \geq 0,\ \ \forall c_2\in C_2,\ \forall d_2\in C_2$.

Here these include $0.5[(x_1,x_2)]+0.5[(y_1,y_2)]$, $(1/3)[(x_1,x_2)]+(1/3)[(y_1,x_2)]+(1/3)[(y_1,y_2)]$.

Cheap talk and mediation in a simple sender-receiver game (notes p 23-26)
First player 1 learns his type t_1 in $T_1=\{1a,1b\}$, each with equal probability, then 1 can send some payoff-irrelevant message to player 2, who then chooses and action c_2 in $C_2=\{x,y,z\}$.
Payoffs (u_1,u_2) depend on types and action as follows:

	c_2=x	c_2=y	c_2=z	
t_1=1a	2, 3	0, 2	–1, 0	(p(1a) = 0.5)
t_1=1b	1, 0	2, 2	0, 3	(p(1b) = 0.5)

With cheap talk, there is always a babbling equilibrium where 1's message is independent of his type and 2's action is her ex-ante optimal action y independently of 1's message.

In ***direct communication without noise*** here, there cannot be any equilibrium in which 1 encourages 2 to choose x with positive probability. Notice first that no belief about t_1 could ever make 2 willing to randomize between x and z. If there were any message that 1 could announce that would make 2 willing to choose x (or randomize between x and y), then type 1a would always want to announce such a message (to maximize the probability of x), but any other message that has positive probability must convince 2 that 1's type is 1b and so would cause 2 to choose z; but this in turn would imply that even type 1b should want to send the same message as type 1a.

But more communication is possible with noise. Suppose that player 1 has a carrier pigeon which, if sent, would reach player 2 with probability 0.4. There is an equilibrium in which 1 sends the carrier pigeon (with a note saying "I am type 1a, please do x") if t_1=1a but not if t_1=1b.
If the pigeon does not arrive, player 2's posterior belief about the probability of type 1b is $0.5\times1/(0.5\times1+0.5\times(1-0.4)) = 5/8$, and so player 2 still prefers to choose y, not z.
So noise can help player 1 here to send messages that credibly guide 2's action.

The general characterization of what can be achieved with noise and mediation involves some linear *incentive constraints* that are often easier to analyze than Nash equilibria.

A coordination mechanism or mediation plan is a function $\mu{:}T_1 \rightarrow \Delta(C_2)$, where
$\mu(c_2|t_1)$ = P(mediator recommends "c_2" to 2 if 1 has confidentially reported "t_1").
The coordination mechanism must satisfy the probability constraints:
[1] $\sum_{d2 \in C2} \mu(d_2|t_1) = 1$ and $\mu(c_2|t_1) \geq 0,\ \forall c_2 \in C_2,\ \forall t_1 \in T_1$.

Consider an equilibrium in which player 1 reports honestly and 2 obeys the recommendation.
The expected payoff to type t_1 of player 1 is $U_1(\mu|t_1) = \sum_{d2 \in C2} \mu(d_2|t_1)u_1(c_2,t_1)$.
But player 1's type t_1 could dishonestly report type s_1 and get $\hat{U}_1(\mu,s_1|t_1) = \sum_{d2 \in C2} \mu(d_2|s_1)u_1(c_2,t_1)$.
So to make honesty an equilibrium strategy for 1, we need
[2] $U_1(\mu|t_1) \geq \hat{U}_1(\mu,s_1|t_1),\ \forall t_1 \in T_1,\ \forall s_1 \in T_1$; and

Player 2's expected payoff is $U_2(\mu) = \sum_{t1 \in T1} \sum_{c2 \in C2} p(t_1)\mu(c_2|t_1)u_2(c_2,t_1)$.
But to make obedience an equilibrium strategy for 2, we need
[3] $\sum_{t1 \in T1} p(t_1)\mu(c_2|t_1)(u_2(d_2,t_1) - u_2(c_2,t_1)) \leq 0,\ \forall c_2 \in C_2,\ \forall d_2 \in C_2$.

We say that μ is incentive compatible iff μ satisfies constraints [2] and [3]. Here [2] are informational incentive constraints, and [3] are strategic (or moral-hazard) incentive constraints.
The *revelation principle* tells us that any equilibrium of any cheap-talk communication system can be simulated by an incentive-compatible coordination mechanism.

For our example with payoffs

	c_2=x	c_2=y	c_2=z	
t_1=1a	2, 3	0, 2	–1, 0	(p(1a) = 0.5)
t_1=1b	1, 0	2, 2	0, 3	(p(1b) = 0.5)

the incentive-compatible mechanism that maximizes $U_2(\mu)$ is

	c_2=x	c_2=y	c_2=z
t_1=1a	2/3	1/3	0
t_1=1b	0	2/3	1/3

Summary Overview:

Decision theory: strong and weak *domination by randomized strategies*, domination theorem. *Expected utility* thm: consistent decisions maximize expected utility.

Finite strategic-form games: iterative elimination of dominated strategies in games (rationalizability), best-response functions, finding equilibria in 2x2 and 2x3 games (support of equilibrium, complementary slackness conditions), symmetric equilibriums of larger games (2x2x...x2, 3x3), finding an equilibrium with a support such that...
Games where players choose numbers subject to bounds: find best-response functions & pure equilibriums (using $\partial u_i/\partial a_i$, 1st-order and boundary conditions).
Extensive-form games: (with perfect information, or more generally with information sets): strategies, normal representation in strategic form, mixed strategies, behavioral strategies, subgame-perfect equilibrium, sequential equilibrium (move probabilities, prior probabilities of nodes, belief probabilities at information sets, consistency of beliefs, sequential rationality of strategies, beliefs at 0-probability information sets).
Handling discontinuities in subgame-perfect equilibriums of infinite games with perfect information.
Repeated games: maximizing δ-discounted sum of payoffs; social states, state-dependent strategies, state transition rule, recursive formula for state-dependent values, one-deviation conditions for subgame-perfect equilibriums.
Bayesian games: Privately known types, payoff functions with increasing differences, increasing strategies and cutoff strategies, winner's curse effect.