Advanced Microeconomics for Policy Analysis I

Fall 2021

Problem Set 7

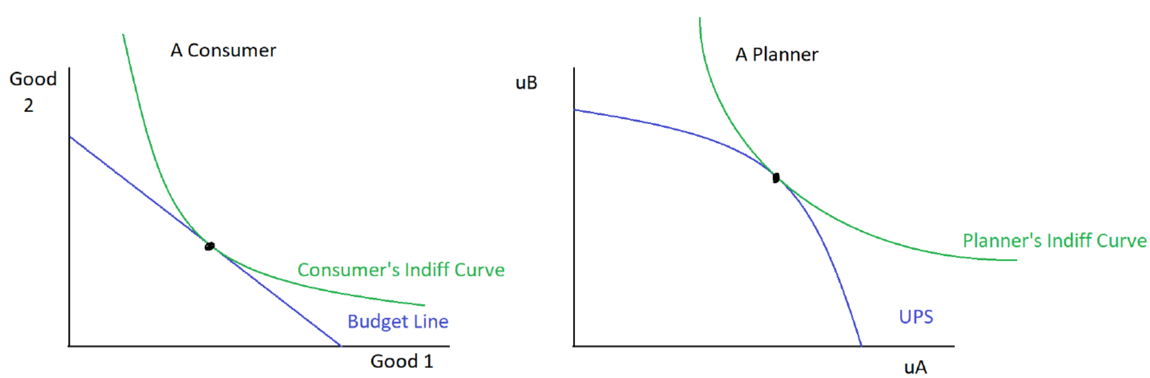
Pareto Optimal allocations, Walrasian Equilibria, and First and Second Welfare Theorems

This problem set is rather long, but I want you to really understand Pareto Optimal (PO) allocations, their connection to weighted utilitarian Social Welfare Functionals (SWFs) and with Walrasian Equilibria (WE). So here it is. Let me first explain what we are going to do.

In **Problem 1** we will understand and exploit Theorem 6.2. This theorem says that, under some conditions, you can definitely find any PO allocation by means of solving a planner's problem, whose preferences are represented by a weighted utilitarian SWF, i.e., $W(u(a)) = \sum_h \alpha^h u^h(a)$. For *a particular*, given vector of weights $\{\alpha^h\}$, you will find *a particular* PO allocation. By considering *all possible* vectors $\{\alpha^h\}$, you will find *all possible* PO allocations. You will work with, and find the set of PO allocations for different economies. Once you find this set, you will be asked to draw this set in Edgeworth Boxes. And you will be also asked to draw the corresponding utility possibility frontiers. This last step anticipates Problem 2.

In **Problem 2** we will study another way of finding PO allocations. This other method will be much more directly related to finding the utility possibility frontier of a given economy. And once we have the utility possibility frontier of an economy, you can analyze what *other* planners/policymakers/policy analysts, with other preferences (i.e., other SWFs) will choose. You will have to find this other planners' most preferred PO allocations.

To solve the planner's problem, we treat the planner as any other consumer we have analyzed before: choosing the bundle that maximizes preferences subject to a constraint. But what's the planner's constraint? Look at the figure below.



On the left panel we have a standard consumer, choosing amounts of goods 1 and 2 to maximize utility given a budget constraint. On the right panel we have an example of a planner, who wants to maximize her utility (the SWF) by choosing utility levels for each individual in the economy (there are only 2 individuals in this economy). And the constraint is the utility possibility set (UPS), which indicates all possible combinations of utility levels in this economy. The points on the frontier of the UPS correspond to Pareto Optimal allocations. A solution to the planner's problem in this case is a pair of utility levels (u_A, u_B).

In **Problem 3** you will have to find WE of particular economies. And we also want to work with the connections between WE and PO allocations stated by the First and Second Welfare Theorems. This leads to Problem 4.

In **Problem 4** you will have to take the WE that you found in Problem 3, and verify that its allocation is indeed PO. This is what the First Welfare Theorem guarantees. Finally, you will verify the Second Welfare Theorem: you will have to take one of the PO allocations you found in Problem 1, and verify that it can be decentralized, or implemented as a WE by correctly choosing lump sum transfers that adjust the endowments of the individuals.

Happy solving.

Problem 1. On PO allocations and Theorem 6.2.

(a) Find the set of Pareto Optimal allocations for the following economies, each of them populated by 2 individuals, A and B, who have preferences over 2 commodities, represented by utility functions $u_A(x, y)$ and $u_B(x, y)$, respectively, where x is the quantity consumed of good 1 and y is the quantity consumed of good 2. The social endowment is $\omega = (\omega_x, \omega_y) = (1, 1)$.

(i) $u_A(x, y) = x^\beta y^{1-\beta}$ and $u_B(x, y) = x^\gamma y^{1-\gamma}$, $\beta \in (0, 1)$, $\gamma \in (0, 1)$, $\beta \neq \gamma$; Define $\theta := \gamma(1-\beta) / (1-\gamma)\beta$ and use this object, otherwise the solutions will be a mess.

(ii) $u_A(x, y) = \min\{x, y\}$ and $u_B(x, y) = x^\gamma y^{1-\gamma}$, $\gamma \in (0, 1)$;

(iii) $u_A(x, y) = \min\{x, y\}$ and $u_B(x, y) = x + y$;

(iv) $u_A(x, y) = \min\{x, y\}$ and $u_B(x, y) = \max\{x, y\}$ (Harder).

Read all the following hints before solving the problems:

- Hint 1: first read Theorem 6.2 and make sure you try to understand what it says. To find PO allocations, solve the planner's problem, for *some* $\alpha \in [0, 1]$:

$$\begin{aligned} &\max_{x_A, x_B, y_A, y_B \geq 0} \alpha u_A(x_A, y_A) + (1-\alpha) u_B(x_B, y_B) \\ s.t. \quad : \quad & x_A + x_B \leq \omega_x; \; y_A + y_B \leq \omega_y. \end{aligned}$$

  Then, the entire set of PO allocations is obtained by considering all values of $\alpha \in [0, 1]$. Notice that, in economy (iv), concavity is not satisfied for agent B. Don't worry.

- Hint 2: for economy (i), I suggest you to follow these steps: (1) argue that the resource constraints will bind at the optimum; (2) substitute $y_B = \omega_y - y_A$ and $x_B = \omega_x - x_A$ into the objective function; (3) solve for optimal x_A and y_A. When doing so, consider only the cases $\alpha \in (0, 1)$ and don't worry about corner solutions (they can and will happen, but do not consider them when solving). Then, when you find the solutions x_A^* and y_A^*, analyze for which values of α these solutions are indeed interior (i.e., $x_A^*(\alpha) \in (0, \omega_x)$ and $y_A^*(\alpha) \in (0, \omega_y)$).

- Hint 3: you can use Edgeworth's Box to have an idea of what's the set of PO allocations: (1) Draw the box, the length of the horizontal axes is ω_x and the length of the vertical ones is ω_y; (2) Choose any one of the two agents (say A); (3) Choose a utility level for agent A (i.e., pick an indifference curve for A); (4) Evaluate how much you can increase *the other agent*'s, B's, utility while keeping A at the utility level you chose in step (3); (5) The allocation that allows you to give B the highest possible utility while keeping A at the utility level you chose in (3) is a PO allocation. Then, you can find the entire set of PO allocations by considering all possible and feasible utility levels of A. (Of course you cannot consider them all, but doing this exercise a few times will give you a good idea of what's the set of PO allocations that you should find.) I strongly recommend you to follow this procedure especially with economy (iv).

(b) Use the Edgeworth Box to find the set of PO allocations for the case $u_A(x, y) = \min\{x, y\}$ and $u_B(x, y) = \min\{ax, y\}$ with $a > 1$. No need to do any math, but indicate precisely what's the set of PO allocations.

(c) In economy (i), consider the following allocation: $(x_A, y_A, x_B, y_B) = (a, a, b, b)$, with $a, b \in [0, 1]$ and $a + b = 1$. Is the allocation PO? Explain briefly.

Hint: just look at the MRSs.

(d) In an Edgeworth Box (with y on the y-axis and x on the x-axis), draw the set of PO allocations you found in economy (i), for the following configurations of parameters:

(i) $\beta = 0.9$, $\gamma = 0.1$

(ii) $\beta = 0.75$, $\gamma = 0.25$

(iii) $\beta = 0.51$, $\gamma = 0.49$

Notice that this will be a scatter plot, where you draw $(x_A^*(\alpha), y_A^*(\alpha)) \in [0, 1]^2$ for the different values of α. Use Excel or something like that.

(e) Take again economy (i) from part (a). Maximized utilities as a function of α are

$$u_A(\alpha) = (x_A^*(\alpha))^{\beta} (y_A^*(\alpha))^{1-\beta}$$

and

$$u_B(\alpha) = (x_B^*(\alpha))^{\gamma} (y_B^*(\alpha))^{1-\gamma}.$$

Draw a scatter plot with $u_B(\alpha)$ on the y-axis and $u_A(\alpha)$ on the x-axis for the different values of α, for the following configurations of parameters:

(i) $\beta = 0.9$, $\gamma = 0.1$

(ii) $\beta = 0.75$, $\gamma = 0.25$

(iii) $\beta = 0.51$, $\gamma = 0.49$

What you are drawing are the utility possibility frontiers of these economies.

Problem 2. Finding the utility possibility frontier directly, and finding solutions to different planners' problems.

Consider an economy populated by 2 individuals, A and B, who have preferences over 2 commodities, represented by utility functions $u_A(x,y)$ and $u_B(x,y)$, respectively, where x is the quantity consumed of good 1 and y is the quantity consumed of good 2. The social endowment is $\omega = (\omega_x, \omega_y)$. We are going to draw the utility possibility frontier. That is, we will create a two-dimensional graph, where $\overline{u}_B$, the utility *level* of agent B, is on the y-axis, and $\overline{u}_A$, the utility *level* of agent A, is on the x-axis. How do we do this? Well, what we want to find is the maximum $\overline{u}_B$ for every possible $\overline{u}_A$. So the problem we want to solve is

$$\begin{aligned}
\overline{u}_B(\overline{u}_A) &= \max_{x_A,x_B,y_A,y_B\geq 0} u_B(x_B,y_B) \\
s.t. &: u_A(x_A,y_A) = \overline{u}_A \\
x_A + x_B &\leq \omega_x \\
y_A + y_B &\leq \omega_y
\end{aligned}$$

$\overline{u}_B(\overline{u}_A)$ is the value function of this problem: it is the maximum level of u_B given that we give agent A a utility level of $\overline{u}_A$. So this is the function that we want to graph. (I have notationally omitted the dependence of $\overline{u}_B$ on ω_x and ω_y, which are also parameters of the problem and are therefore also arguments of the value function.) The solutions (x_A, x_B, y_A, y_B) to the above problem are PO allocations, so this is another way of finding PO allocations.

What are the values that this parameter $\overline{u}_A$ can take? If $u_A(x_A,y_A)$ is increasing, the lowest possible value of u_A is reached at $x_A = y_A = 0$, and the highest possible value is reached at $x_A = \omega_x$ and $y_A = \omega_y$. So $\overline{u}_A \in [u_A(0,0), u_A(\omega_x,\omega_y)]$.

Consider $u_A(x,y) = x^\beta y^{1-\beta}$ and $u_B(x,y) = x^\beta y^{1-\beta}$, with $\beta \in (0,1)$.

(a) Solve the above problem and find the PO allocations and the value function $\overline{u}_B$.

(b) Draw the function $\overline{u}_B(\overline{u}_A)$ for

(i) $\omega_x = \omega_y = 1$, and $\beta = \frac{1}{2}$;

(ii) $\omega_x = 2$, $\omega_y = 1$ and $\beta = \frac{1}{2}$;

(iii) $\omega_x = 2$, $\omega_y = 1$ and $\beta = \frac{3}{4}$;

(iv) $\omega_x = 2$, $\omega_y = 1$ and $\beta = \frac{1}{4}$.

(c) Briefly explain the differences in the graphs you found in (b): compare (ii) with (i); and (iii) with (ii); and (iv) with (ii).

(d) Suppose a planner/policymaker in this economy has preferences over allocations $a = (x_A, y_A, x_B, y_B)$ represented by the SWF

$$W(u(a)) = \alpha u_A(x_A, y_A) + (1-\alpha) u_B(x_B, y_B).$$

For a given $\alpha \in [0,1]$, this planner can choose the allocation she wants to implement by looking at the utility possibility frontier that you found before and choosing some point $(\overline{u}_A, \overline{u}_B)$ on this frontier. So the planner's problem is to choose the best (u_A, u_B) on the frontier:

$$\begin{aligned} &\max_{u_A, u_B} \alpha u_A + (1-\alpha) u_B \\ s.t. \quad : \quad & u_B = \overline{u}_B(u_A), \end{aligned}$$

where $\overline{u}_B(u_A)$ is the function that you found before in part (a).

Take your graph of $\overline{u}_B(u_A)$ for the case $\omega_x = \omega_y = 1$ and $\beta = \frac{1}{2}$. Indicate, *graphically*, what's the planner's solution for each possible value of $\alpha \in [0,1]$.

(e) Suppose another planner in this economy has preferences represented by the SWF

$$W(u) = \min\{u_A, u_B\}.$$

Take your graph of $\overline{u}_B(u_A)$ for the case $\omega_x = \omega_y = 1$ and $\beta = \frac{1}{2}$. Indicate, *graphically*, what's the planner's solution.

(f) Suppose another planner in this economy has preferences represented by the SWF

$$W(u) = u_A^{\alpha} u_B^{1-\alpha}, \ \alpha \in (0,1).$$

Take your graph of $\overline{u}_B(u_A)$ for the case $\omega_x = \omega_y = 1$ and $\beta = \frac{1}{2}$. Indicate, *graphically*, what's the planner's solution. (The graph is enough, but I won't stop you if you want to find the solution analytically. It is not hard.)

Problem 3. Walrasian Equilibria.

Consider the economy of Problem 1, part (a)(i).

(a) Find the Walrasian Equilibria (prices and allocations) when endowments are $\omega = \left(\omega_x^A, \omega_y^A, \omega_x^B, \omega_y^B\right)$.

(b) How does the equilibrium price vary with ω_x^A? And with ω_y^A? Interpret.

(c) How does the equilibrium price vary with β? Interpret.

Problem 4. First and Second Welfare Theorems: the connection between PO allocations and WE.

Consider the same economy as in Problem 3. Take $\beta = \frac{3}{4}$, $\gamma = \frac{1}{4}$, $\omega_x^A = \omega_x^B = \omega_y^A = \omega_y^B = \frac{1}{2}$.

(a) Compute the values of the equilibrium prices and the equilibrium allocations.

(b) Verify the First Welfare Theorem: Show that there exists an $\alpha \in [0, 1]$ such that the PO allocation from Problem 1, part (a)(i), coincides with the WE allocation you just found in part (a).

(c) Now take any PO allocation that you found in Problem 1, part (a)(i), for any α. Let's call it $a^* (\alpha) = (x_A^* (\alpha), y_A^* (\alpha), x_B^* (\alpha), y_B^* (\alpha))$.

Verify the Second Welfare Theorem: Is there a price p^* and a set of transfers (t_A, t_B) such that $(p^*, a^* (\alpha))$ is a WE of this economy when endowments are $\omega_x^{A\prime} = \omega_x^A + t_x^A$, $\omega_x^{B\prime} = \omega_x^B + t_x^B$, $\omega_y^{A\prime} = \omega_y^A + t_y^A$, $\omega_y^{B\prime} = \omega_y^B + t_y^B$? What is the value of $t_x^A + t_x^B$ and $t_y^A + t_y^B$? What do these values mean?