Vectors and Linear Combinations

Learning Goal: Students become familiar with vectors and linear combinations

- Scalar multiples, addition, and combinations
- Algebra and geometry
- Columns and rows

Vectors and operations

Many times we are faced with "quantities" that are not represented by single numbers, but by collections of numbers. Such collections are called *vectors*.

For instance, let's say you sell t-shirts. In your store, you have a bunch of small, medium, and large t-shirts, and manufacturer shipments to the store might contain various amounts of all three sizes. The amount you have on hand might be represented by an inventory vector: $\begin{bmatrix} S \\ M \\ L \end{bmatrix}$. For example, $\begin{bmatrix} 50 \\ 30 \\ 60 \end{bmatrix}$ would mean that you have 50 small shirts, 30 mediums, and 60 large on hand. We might represent this vector by a single letter (notation: **bold** in typed materials, $\overrightarrow{\text{arrowed}}$ if hand-written), say **h** (for on-**h**and), We might also receive a shipment **s** of new t-shirts, $\mathbf{s} = \begin{bmatrix} 30 \\ 50 \\ 40 \end{bmatrix}$, from the manufacturer. Now how many do we have?

The simple answer is to simply *add* the two vectors: $\mathbf{h} + \mathbf{s} = \begin{bmatrix} 50 \\ 30 \\ 60 \end{bmatrix} + \begin{bmatrix} 30 \\ 50 \\ 40 \end{bmatrix} = \begin{bmatrix} 80 \\ 80 \\ 100 \end{bmatrix}$, and this will tell us the new amounts of each kind of shirt we have on hand.

Some vocabulary: the individual numbers inside the vector are called *components*. They are indexed by their position within the vector, and written in plain type with a subscript. So, for instance, $s_3 = 60$, while $h_1 = 50$. Apparently, to add two vectors, you simply add the corresponding components. This remains true no matter how many components your vectors have—you can add any two vectors having the same number of components by adding the corresponding components. The number of components a vector has is called its *dimension*. You can't add vectors of differing dimensions because if there are more components of one vector than the other, there won't be anything to add to the highest numbered components of the higher dimensional vector.

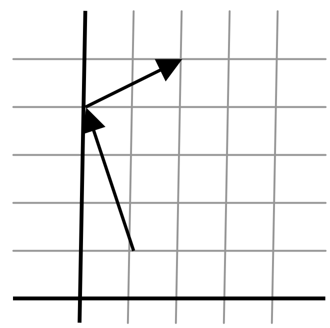
We can also multiply a vector by a number. For instance, if you receive three shipments of t-shirts, the result is that you received $3\mathbf{s} = 3\begin{bmatrix} 30 \\ 50 \\ 40 \end{bmatrix} = \begin{bmatrix} 3 \cdot 30 \\ 3 \cdot 50 \\ 3 \cdot 40 \end{bmatrix}$ new t-shirts. In other words, we can multiply a vector by a number (vocabulary: numbers are called *scalars*, and this process is called *scalar multiplication*) by simply multiplying each components of the vector by the scalar.

Addition and scalar multiplication possess many of the properties of addition and multiplication you might expect: commutativity, associativity, distributivity. If you multiply a vector **v** by the number 1, that is $1\cdot\mathbf{v}$, you get **v** back. There is a special vector, **0**, that acts as an additive identity. Actually, there's a **0** for each dimension, and all of the components of **0** are 0. Adding **v** to $-1\cdot\mathbf{v} = -\mathbf{v}$ gives $\mathbf{v} - \mathbf{v} = \mathbf{0}$.

The most common thing to do in linear algebra is to use both operations. Given vectors **v** and **w**, and scalars a and b, we take $a\mathbf{v} + b\mathbf{w}$. This is called a *linear combination* of the vectors **v** and **w**. By choosing a and b wisely, we can find $\mathbf{v} + \mathbf{w}$ ($a = b = 1$), $\mathbf{0}$ ($a = b = 0$), $\mathbf{v} - \mathbf{w}$ ($a = 1$, $b = -1$), and multiples of **v** (a anything, $b = 0$). Again, notice that because of the additions taking place, both vectors must be the same dimension. We often look at the entire collection of *all possible* linear combinations of two (or more!) vectors.

<u>Geometry of vectors</u>

For algebra purposes, all we need are the numeric components of vectors (and we can get abstract enough that we don't even need that). But it really helps in understanding to have a visual representation of vectors. The best such is to think of a vector as a displacement. For instance, the vector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is a displacement of -1 in the x-direction, and 3 in the y-direction. If you start from the origin, you will end up at the point (-3, 1). If you start at the point (5, 5) you end up at the point (4, 8). That is an important concept—it doesn't matter where you start, just the amount and direction in which you moved.



This helps us give geometric meaning to the basic operation. Addition is moving one way, then another. If you move by $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and then by $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, your total displacement is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, which is their sum. In the picture, we start at (1, 1), move by **v** to (0, 4), and then by **w** to (2, 5), which does have the indicated displacement from the original point (1, 1).

Geometrically speaking, a scalar multiple $a\mathbf{v}$ is just moving a times as far in the same direction as **v**. If a is negative, we move in the opposite direction.

Think about taking all possible linear combinations of **v** and **w** shown above. Start by thinking of all the possibilities where b is zero. So we are taking all scalar multiples of **v**, This gives us a whole line, for wherever we start we can go as far in the direction of **v** as we like, and as far in the opposite direction too because our multiple can be negative.

If we then add all possible multiples of **w**, we end up with an entire plane. What do you think happens if we take all possible combinations of one, two, or three vectors that each have three dimensions?

One last item. You have seen that I've always written vectors as columns of numbers. If we attach our vectors to the origin (a natural place to start) in the geometric picture, then a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ corresponds to the point (x, y), and from now on we will use this as a space saver. But beware, for there is also the "row vector" [x y], and this is an entirely different object! For most purposes, it doesn't matter whether we distinguish row vectors and column vectors, but let's not take any chances for now. Almost all vectors you deal with regularly are technically column vectors. Use either the vertical notation or the ordered n-tuple notation unless you know for sure that you are dealing with a row vector, and then use the brace-with-no-commas notation. Row vectors are commonly used for things that "aren't quite real" like angular momentum or magnetic field.

Reading: 1.1
Problems: 1.1: 1 – 6, 8, 9, 12 – 14, 15, 16, 18, 20 – 22, 25 (note this is asking for a new set of vectors **u**, **v**, and **w**, *not* the ones shown in the diagram), 28