Length and Dot Product

Learning Goals: Students learn about dot products and lengths
- Unit vectors
- Perpendicularity
- Angles
- Schwarz inequality

A very common way to combine two vectors into a single number is as in the following example:

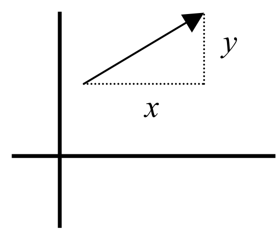
Recall our vector **h** = (50, 30, 60) (and how this really is a column!) of the number of shirts we have on hand. We might also have a vector **p** = ($6, $7, $8) of the price of each kind of shirt. How much is our inventory worth? Obviously we want to multiply and add 50·$6 + 30·$7 + 60·$8 = $990.

This multiply-and-add operation on two vectors is so common that we give it a name—several names in fact:

> **Definition:** the *dot* (or *inner*, or *scalar*) *product* of two vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ is the sum of the products $\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2 + \cdots + v_nw_n$.

Notice that because of commutativity of ordinary multiplication, dot products also commute. And because of distributivity of ordinary multiplication, we have a kind of associativity of dot products: $c(\mathbf{v}\cdot\mathbf{w}) = (c\mathbf{v})\cdot\mathbf{w}$. We also get distributivity of dot products: $\mathbf{v}\cdot(\mathbf{x} + \mathbf{y}) = \mathbf{v}\cdot\mathbf{x} + \mathbf{v}\cdot\mathbf{y}$.

Let's look at the dot product in two dimensions to see what happens with geometry. Then we'll see if it it extends to more dimensions.



First, let's take the dot product of $\mathbf{v} = (x, y)$ with itself: $\mathbf{v}\cdot\mathbf{v} = x^2 + y^2$. From the Pythagorean Theorem this is exactly the square of the distance from the tail of the vector to its tip. We will call this distance the *length* (or *norm*) of the vector **v** and write it $\|\mathbf{v}\|$. Thus we define $\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$.

This would work in three dimensions as well, because we can use the Pythagorean Theorem twice. In fact, we can imagine using it three or more times in even higher dimensions. So in all dimensions we make the definition that the length (or norm) of a vector is the square root of its dot product with itself.

Some of the most important kinds of vectors are those whose length is one (just like one is one of the most important numbers). We call such vectors *unit vectors*.

Consider $(a\mathbf{v})\cdot(a\mathbf{v}) = a^2(\mathbf{v}\cdot\mathbf{v})$. If we now take square roots of both sides, we find that $\|a\mathbf{v}\| = |a|\,\|\mathbf{v}\|$. Don't forget that absolute value! From this, we get the following two ideas:

- Any non-zero vector **v** can be turned into a unit vector by dividing it by its own lengths: we write $\hat{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|$. This is a vector in the same direction as **v**, but having unit length. In fact, we call $\hat{\mathbf{v}}$ the *direction* of **v**.
- Conversely, we can multiply both sides of the equation above by $\|\mathbf{v}\|$, to obtain $\mathbf{v} = \|\mathbf{v}\|\,\hat{\mathbf{v}}$, that is, any vector is equal to its direction times its length.

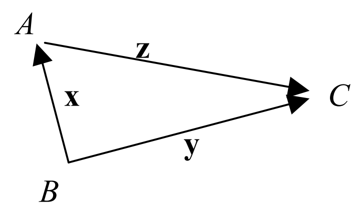
There are special unit vectors that point in the coordinate directions. The vector **i** points in the x-direction, **j** in the y-direction, and **k** in the z-direction. (The names change in more than three dimensions, so we won't run out of letters!)

Now let's explore what happens when a dot product is zero. We'll check first in two-dimensions again. So look at the vectors (3, 1) and (–2, 6). It is easy to check that their dot product is zero: $3\cdot(-2) + 1\cdot 6 = 0$. What geometric connection does this have?

Look at the line segment along (3, 1). It has rise 1 and run 3, so its slope is 1/3. Similarly, the slope along the vector (–2, 6) is –3. The opposite reciprocal. What do we know about opposite reciprocals? They are slopes of *perpendicular* lines. That is, in this particular case, the lines running through these two vectors are perpendicular.

Is this always the case? Not quite. If one of the vectors we have is **0**, it's dot product with everything is, of course, 0. But that doesn't make anything perpendicular. In fact, since **0** doesn't go in any meaningful direction, there is nothing to compare it to, so it is not perpendicular to anything (or it is to everything, who can say?), For this reason we use the word *orthogonal* to indicate that the dot product of two vectors is zero. We reserve *perpendicular* for *nonzero* vectors that really can be perpendicular.

But is it true that the dot product of perpendicular vectors is zero, and vice versa (if the vectors aren't zero)? Yes. Here's the proof:



Consider vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. Let point B be at the tail of **x**, and A be at its head. Move **y** so that it's tail is at B and head at C. Then the vector from A to C is, of course, $\mathbf{z} = \mathbf{y} - \mathbf{x}$. Using the Pythagorean theorem, we have $\|\mathbf{z}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$. Expanding these out using the dot product formula gives $(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2 = x_1^2 + x_2^2 + \cdots + x_n^2 + y_1^2 + y_2^2 + \cdots + y_n^2$. Multiplying out the terms on the left gives $x_1^2 - 2x_1y_1 + y_1^2 + \cdots$. Cancelling all the squares on both sides of the equation yields $-2x_1y_1 - 2x_2y_2 - \cdots - 2x_ny_n = 0$, or $-2\mathbf{x}\cdot\mathbf{y} = 0$ which is what we needed. Note that we can work backward to show that if the dot product is zero, then the vectors are perpendicular.

We can actually do even better than this. Even if angle B is not a right angle, we can apply the Law of Cosines to find that $\|\mathbf{z}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\,\|\mathbf{y}\|\cos(\theta)$ where θ is the measure of angle B. Expanding out the lengths and canceling the squares (and the –2) then gives us the handy formula $\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\|\cos(\theta)$. So dot products can be used to measure the exact angle between two vectors.

We also obtain the important *Schwarz inequality*: $|\mathbf{x}\cdot\mathbf{y}| \le \|\mathbf{x}\|\,\|\mathbf{y}\|$, with equality only when **x** and **y** have the same (or opposite) direction, because $-1 \le \cos(\theta) \le 1$ for any angle.

Reading: 1.2
Problems: 1.2: 1, 2, <u>3</u>, <u>4</u>, 6, <u>8</u>, 11, <u>13</u>, <u>16</u>, 19, 21, 27