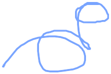
# Generalized Linear Models



# Example 1

- We would like to predict $admit$ given $gre$ and $gpa$

response

predictors

| admit | gre | gpa |
|---|---|---|
| 0 | 380 | 3.61 |
| 1 | 660 | 3.67 |
| 1 | 800 | 4.00 |
| 1 | 640 | 3.19 |
| 0 | 520 | 2.93 |
| 1 | 760 | 3.00 |
| 1 | 560 | 2.98 |
| 0 | 400 | 3.08 |
| 1 | 540 | 3.39 |
| 0 | 700 | 3.92 |

- Can we use linear model here?

$$admit = \beta_0 + \beta_1 \cdot gre + \beta_2 \cdot gpa$$

- Multiple Linear Regression cannot handle binary/categorical response

- Linear Model

$$admit = \beta_0 + \beta_1 \cdot gre + \beta_2 \cdot gpa$$

- Logistic Regression

$$P(admit = 1) = \frac{1}{1 + exp(-\beta_0 - \beta_1 \cdot gre - \beta_2 \cdot gpa)}$$

- When fit the logistic regression on the data we obtain:

$$P(admit = 1) = \frac{1}{1 + exp(-0.73 - 0.02 \cdot gre + 3.57 \cdot gpa)}$$

600

4.0

GRE = 600

GPA = 4.0

- For example, with a student having 380 GRE and 3.61 gpa, the model will predict

$$P(admit = 1) = \frac{1}{1 + exp(-0.73 - 0.02 \cdot 600 + 3.57 \cdot 4.0)} = 0.01$$

- This means that the chance of the student being admitted is 0.01 or the student will not be admitted by the model prediction.

# Logistic Regression

$E(Y) = 0 \cdot P(Y=0) + 1 \cdot P(Y=1)$

$= \boxed{P(Y=1)}$

- Suppose the response $y$ can only takes two values 0 and 1. The logistic regression models the probability of $y = 1$ as follows.

$$E(Y) = \pi = P(y = 1) = \frac{1}{1 + exp(-\beta_0 - \beta_1 x_1 - \beta_2 x_2)}$$

or, equivalently

$= g_1(\pi)$

$$\ln\left(\frac{\pi}{1-\pi}\right) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

$g_2(\pi) = \pi = E(Y) = \beta_0 + \beta_1 X_1 + \beta_2 X_2$ (MLR)

# Generalized Linear Model

- The GLM models $\mu = E(y)$ as follows.

$$g(\mu) = \beta_0 + \beta_1 x_1 + ... + \beta_p x_p = x'\beta$$

  where $y$ is assumed to follow an exponential distribution family.
- Exponential distribution family includes all the basic distribution such as normal distribution, binomial distribution, Poisson distribution...
- $g(\mu)$ is called the canonical link function
- For logistic regression, the link function is a logit function

$$g(x) = \ln\left(\frac{x}{1-x}\right)$$

# Some GLMs

$$g(\mu) = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p = x'\beta$$

| Distribution | Canonical Link Function | Mathematical Form |
|---|---|---|
| Normal | Identity | $g(\mu) = \mu$ |
| Binomial | Logit | $g(\pi) = \ln[\pi/(1-\pi)]$ |
| Poisson | Natural log | $g(\mu) = \ln \mu$ |
| Gamma | Inverse | $g(\mu) = 1/\mu$ |
| Inverse Gaussian | Squared inverse | $g(\mu) = 1/\mu^2$ |

# Example 2

A statistician uses logistic regression to model a probability of success of a random variable. The estimated parameters for the intercepts and two predictors are $\hat{\beta}_0 = 0.02$, $\hat{\beta}_1 = -0.4$, and $\hat{\beta}_2 = 0.3$. Calculate the predicted probability of success at $x_1 = 1$ and $x_2 = 1$.

$$\ell\left(\frac{\pi}{1-\pi}\right) = .02 - .4x_1 + .3x_2$$

$$\pi = \frac{1}{1 + e^{-.02 + .4x_1 - .3x_2}} = \boxed{.48}$$

# Example 3

$\pi = \dfrac{1}{1+e^{-\beta_0-\beta_1 x_1}}$

$\ln\left(\dfrac{\pi}{1-\pi}\right) = \beta_0+\beta_1 x_1$

A statistician uses logistic regression to model a probability of success of a random variable. You are given

- There is one predictors and an intercept in the model
- The estimates of success at $x = 4$ and $x = 6$ are 0.8 and 0.9, respectively.

Calculate $\hat{\beta}_1$ the estimated slope of the model.

$x=4, \Rightarrow \pi = .8 \Rightarrow$ $\ln\left(\dfrac{.8}{.2}\right) = \beta_0+\beta_1\cdot 4$

$x=6 \Rightarrow \pi = .9 \Rightarrow$ $\ln\left(\dfrac{.9}{.1}\right) = \beta_0+\beta_1\cdot 6$

$\ln 4 = \beta_0+\beta_1 4$

$\ln 9 = \beta_0+\beta_1\cdot 6$

$\ln 4-\ln 9 = -2\beta_1$

$\Rightarrow \beta_1 = \dfrac{\ln 4-\ln 9}{-2} = \boxed{.405}$

## Example 4

You are given the following for a fitted GLM. Calculate the modeled probability of an Urban driver having an accident.

| | | | |
|---|---|---|---|
| Response variable | Occurrence of Accidents | | |
| Response distribution | Binomial → ① | | |
| Link | Logit → ② | | |

| Parameter | df | $\hat{\beta}$ | se |
|---|---|---|---|
| $x_1$ Intercept | 1 | −2.358 $\hat{\beta}_0$ | 0.048 |
| Area | 2 | | |
| Suburban | 0 | 0.000 | |
| Urban | 1 | 0.905 $\hat{\beta}_1$ | 0.062 |
| Rural | 1 | −1.129 | 0.151 |

$$\ln\left(\frac{\pi}{1-\pi}\right) = \hat{\beta}_0 + \hat{\beta}_1 \cdot 1$$

(1)

$$\pi = \frac{1}{1 + e^{-\hat{\beta}_0 - \hat{\beta}_1}}$$

(2)

$$= \frac{1}{1 + e^{2.358 - .905}} = \boxed{.1895}$$

# Odd of an event

$$\ln\left(\boxed{\frac{\pi}{1-\pi}}\right) = \beta_0 + \beta_1 x_1 + \cdots$$

ln(odd of -1 = 1)

- The odds of an event A is the ratio of the probability that A occurs to the probability that A does not occur.
- The odd of tossing an coin and see Tail is 1:1
- The odd of rolling a die and see number 6 is $\frac{1/6}{5/6} =$ 1:5

- Logistic regression in terms of Odd

$$\ln(\text{Odd of Success}) = \ln\left(\frac{\pi}{1-\pi}\right) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

# Possion Regression and Other link functions

- Poisson Regression are used when the response are count variables, for example: the number of claims of a customer...
- The response is assume to follow a Poisson distribution and the link function used is a log link function, $ln$.

$$\underset{\underset{g(\mu)}{\parallel}}{\ln(\mu)} = \underbrace{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n}$$

# Example 5

You are given the following when fitted a GLM model. Calculate the predicted $Y$ value for a female with age of 22.

| | | |
|---|---|---|
| Response variable | $Y$ | |
| Response distribution | Poisson | |
| Link | log | |
| AIC | 221.254 | |
| **Parameter** | $\hat{\beta}$ | s.e.$(\hat{\beta})$ |
| Intercept | 5.421 | 0.228 |
| $x_1$ Gender | | |
| Male | 0.000 | 0.000 |
| Female | −0.557 | 0.217 |
| $x_2$ Age | 0.107 | 0.002 |

$\log(Y) = 5.421 - .557 + .107 * 22$

$= 7.218$

$\Rightarrow Y = e^{7.218}$

$= 1363.759$

# Example 6

You are given the following GLM output. Calculate the predicted premium for an insured in Risk Group 2 with Vehicle Symbol 2.

| | | | | |
|---|---|---|---|---|
| | Response variable | Pure Premium | | |
| | Response distribution | Gamma | | |
| | Link | log | | |
| | **Parameter** | **df** | $\hat{\beta}$ | |
| | Intercept | 1 | 4.78 | $\hat{\beta}_0$ |
| $+1$ | Risk Group | 2 | | |
| | Group 1 | 0 | 0.00 | |
| | Group 2 | 1 | −0.20 | $\hat{\beta}_1$ |
| $+2$ | Group 3 | 1 | −0.35 | |
| | Vehicle Symbol | 1 | | |
| | Symbol 1 | 0 | 0.00 | |
| | Symbol 2 | 1 | 0.42 | $\hat{\beta}_2$ |

$\log(y) = 4.78 - .2 + .42$

$= 5$

$\Rightarrow y = e^{5}$

$= 148.41$

# Example 7

You are given the following output of an GLM. Calculate the probability of a policy with 5 years of tenure that experienced at a 10% prior rate increase and has 100,000 in amount of insurance will retain into the next policy term.

| | | | |
|---|---|---|---|
| Response variable | | retention | |
| Response distribution | | binomial | |
| Link | | square root | |
| Pseudo $R^2$ | | 0.6521 | |
| **Parameter** | **df** | $\hat{\beta}$ | |
| Intercept | 1 | 0.6102 | $\hat{\beta}_0$ |
| $x_1$ Tenure | | | |
| $<5$ years | 0 | 0.0000 | |
| $\geq 5$ years | 1 | 0.1320 | $\hat{\beta}_1$ |
| $x_2$ Prior Rate Change | | | |
| $<0\%$ | 1 | 0.0160 | |
| [0%,10%] | 0 | 0.0000 | $\hat{\beta}_2$ |
| $>10\%$ | 1 | −0.0920 | |
| $x_3$ Amount of Insurance (000's) | 1 | 0.0015 | $\hat{\beta}_3$ |

$$\sqrt{y} = .6102 + .1320 + 0 + .0015(100)$$

$$\sqrt{y} = .8922$$

$$y = .8922^2 = \boxed{.796}$$