

# Principal Component Analysis

## PCA in a view or coordinate rotation

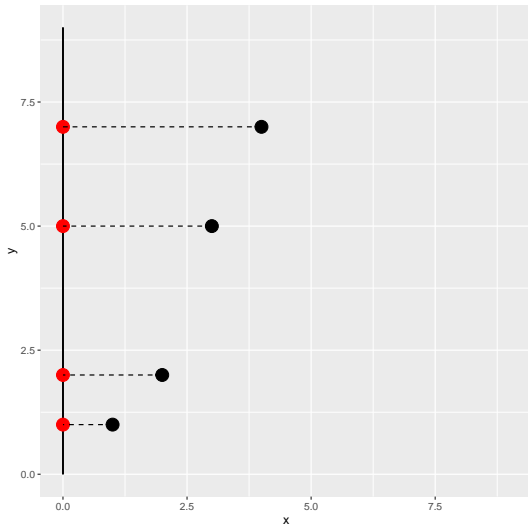
## Variance of the Projection

<hr/>	
x	y
<hr/>	
1	1
2	2
3	5
4	7
<hr/>	

- ▶  $V(x) = 1.67$
- ▶  $V(y) = 7.58$
- ▶ Total variance:  $V(x) + V(y) = 9.25$

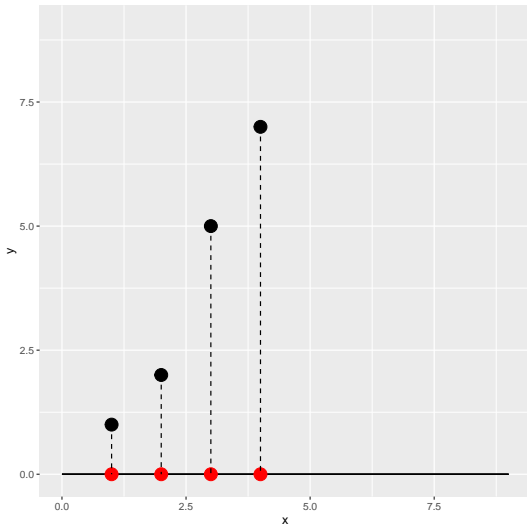
Variance: 7.58

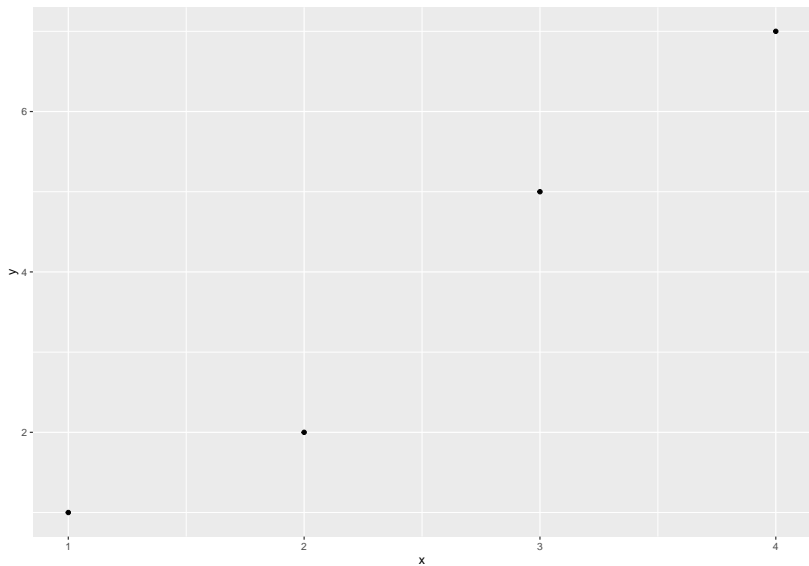
Direction z: [0,1]



Variance: 1.67

Direction  $z$ :  $[1,0]$

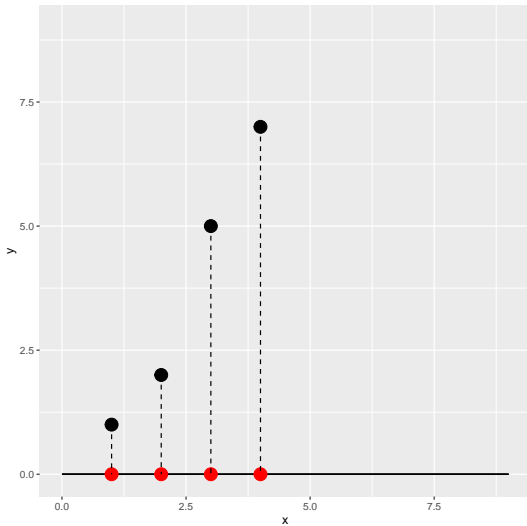




[1] 9.25

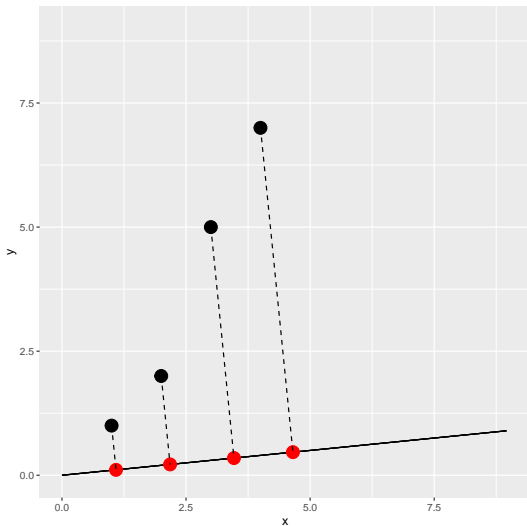
Variance: 1.67

Direction  $z$ :  $[1,0]$

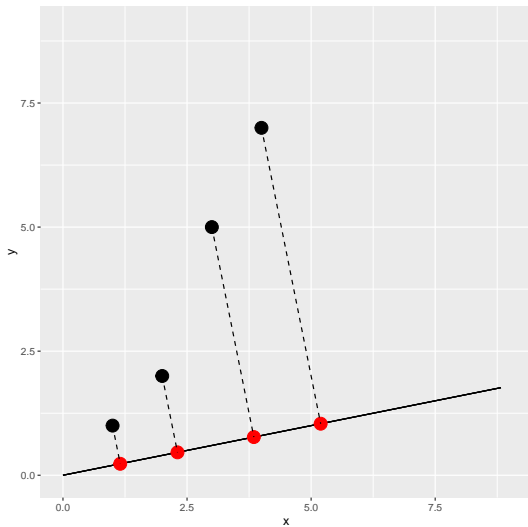




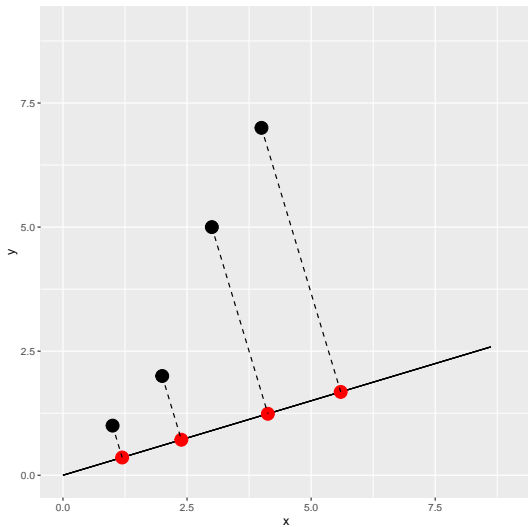
Variance: 2.42  
Direction z: [1,0,1]



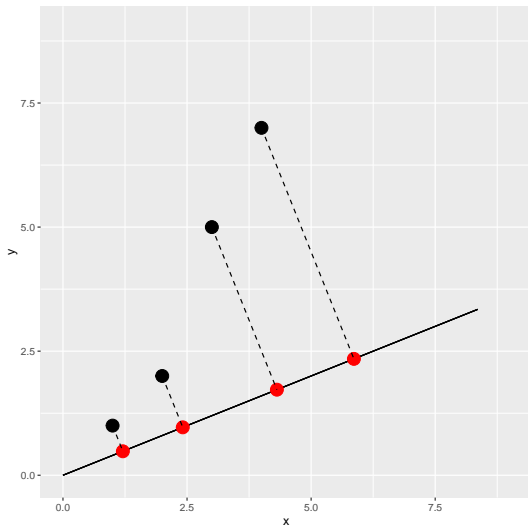
Variance: 3.24  
Direction  $z$ : [1,0.2]



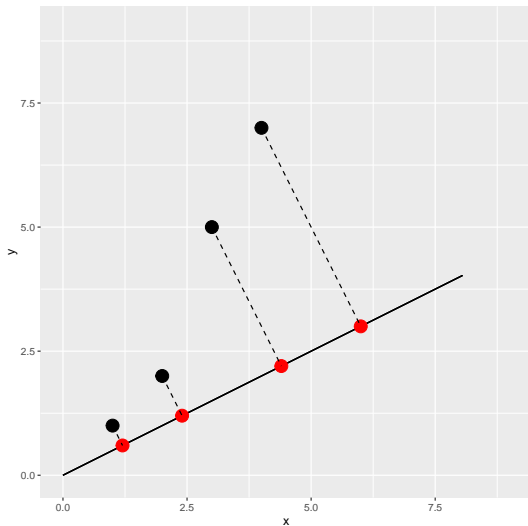
Variance: 4.08  
Direction z: [1,0.3]



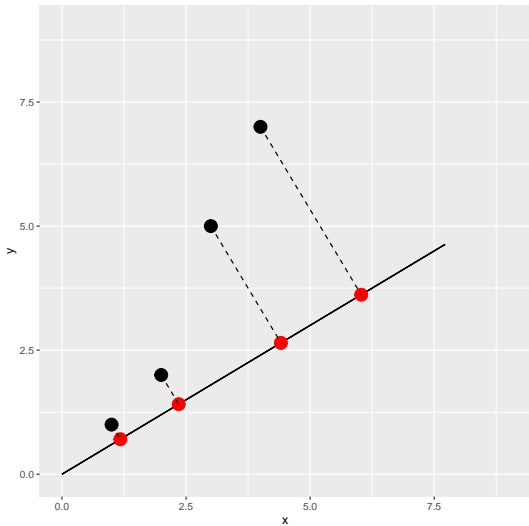
Variance: 4.9  
Direction z: [1,0.4]



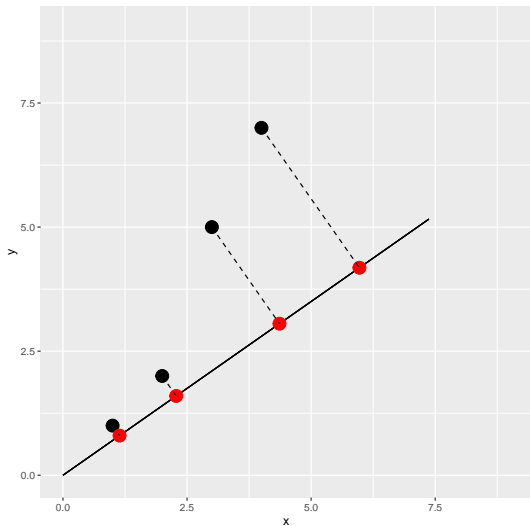
Variance: 5.65  
Direction  $z$ : [1,0.5]



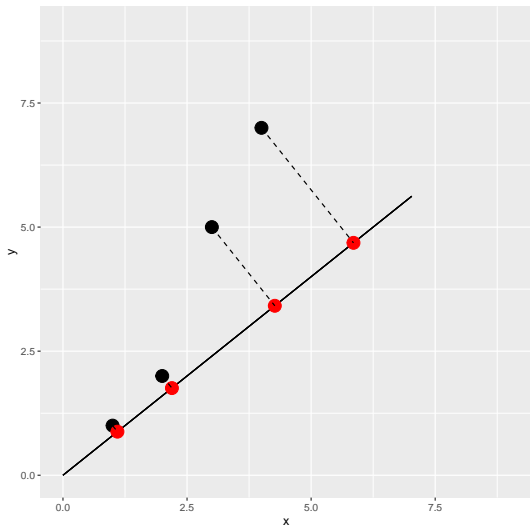
Variance: 6.32  
Direction  $z$ : [1,0.6]



Variance: 6.9  
Direction  $z$ :  $[1, 0.7]$

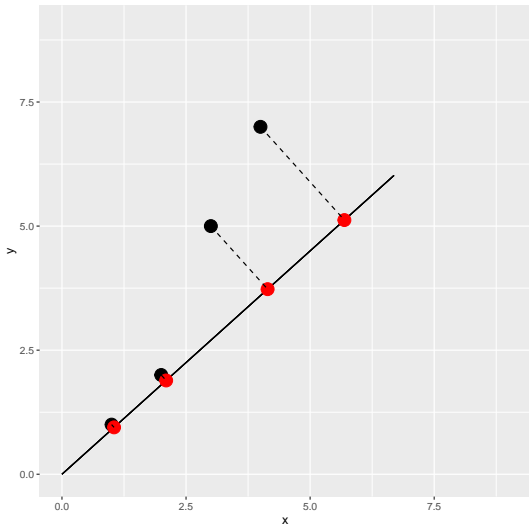


Variance: 7.39  
Direction  $z$ : [1,0.8]



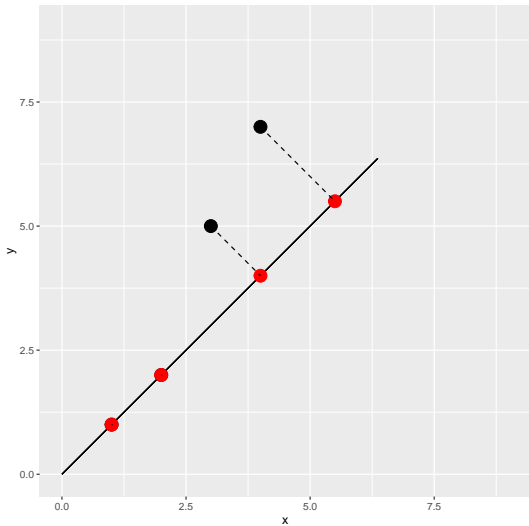


Variance: 7.8  
Direction z: [1,0.9]

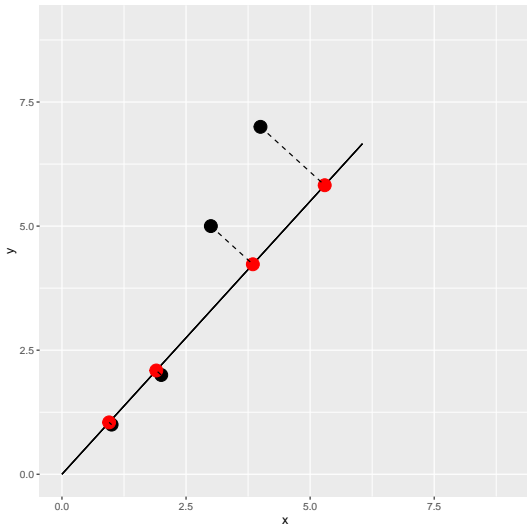


Variance: 8.13

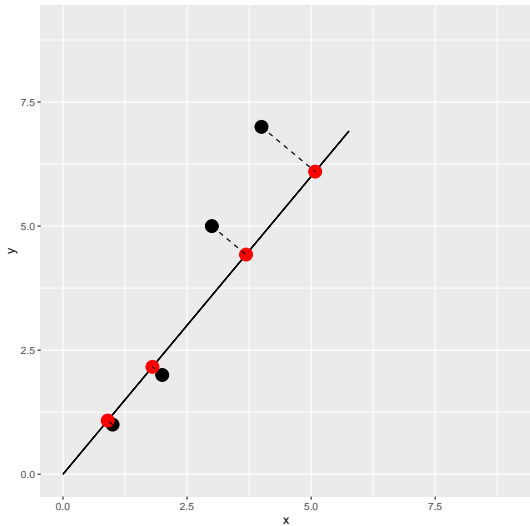
Direction  $z$ : [1, 1]



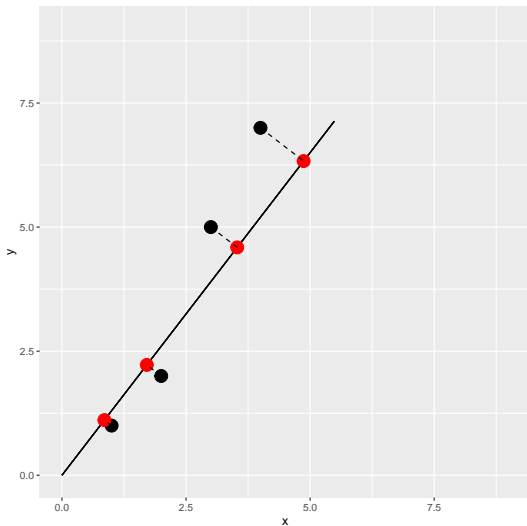
Variance: 8.39  
Direction  $z$ :  $[1, 1, 1]$



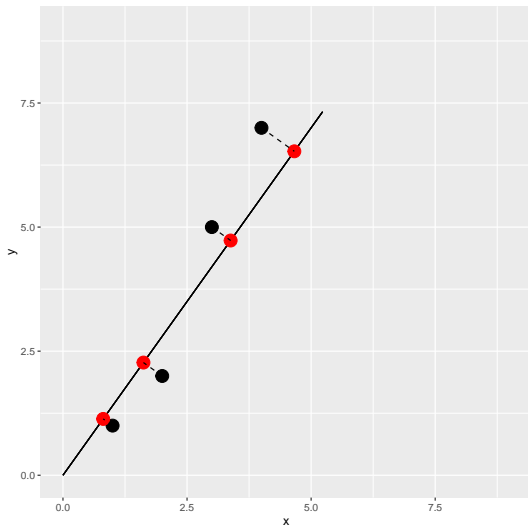
Variance: 8.6  
Direction  $z$ : [1, 1.2]



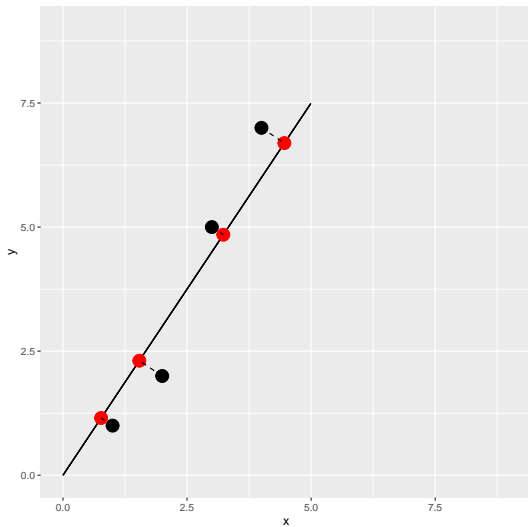
Variance: 8.77  
Direction  $z$ : [1, 1.3]



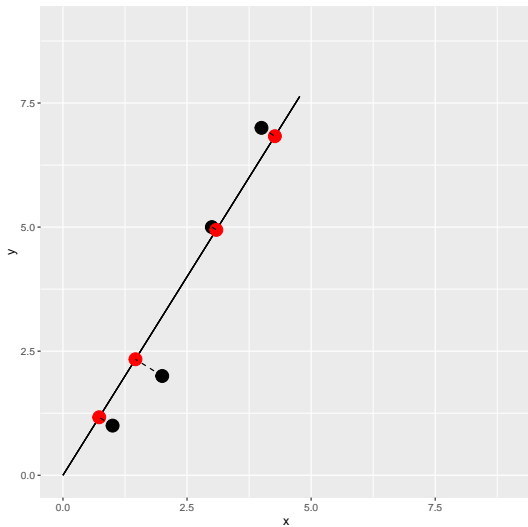
Variance: 8.9  
Direction  $z$ : [1, 1.4]



Variance: 8.99  
Direction  $z$ : [1, 1.5]

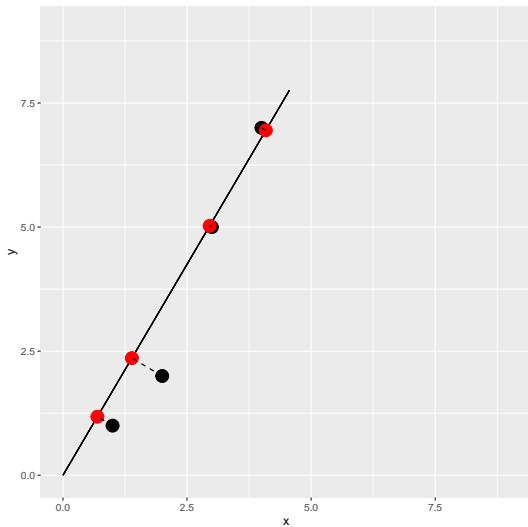


Variance: 9.07  
Direction  $z$ : [1, 1.6]

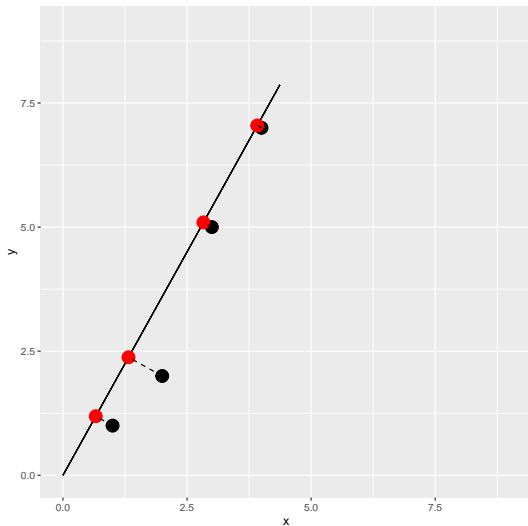




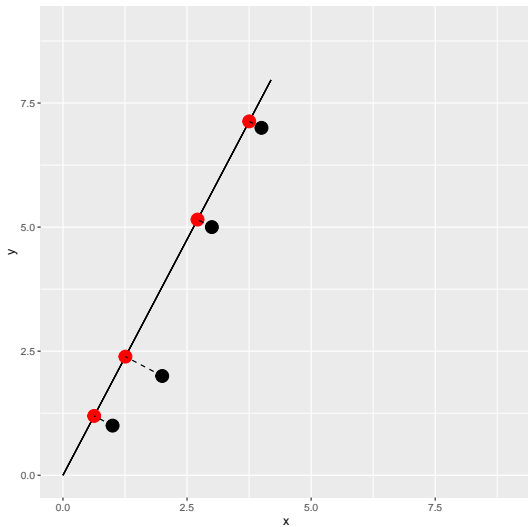
Variance: 9.12  
Direction  $z$ :  $[1, 1.7]$



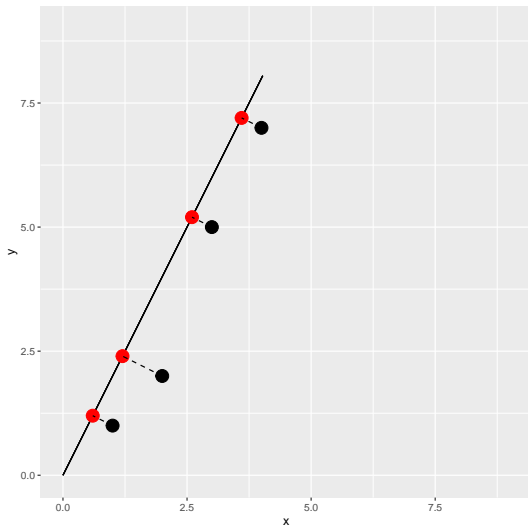
Variance: 9.16  
Direction  $z$ :  $[1, 1.8]$



Variance: 9.18  
Direction  $z$ : [1, 1.9]

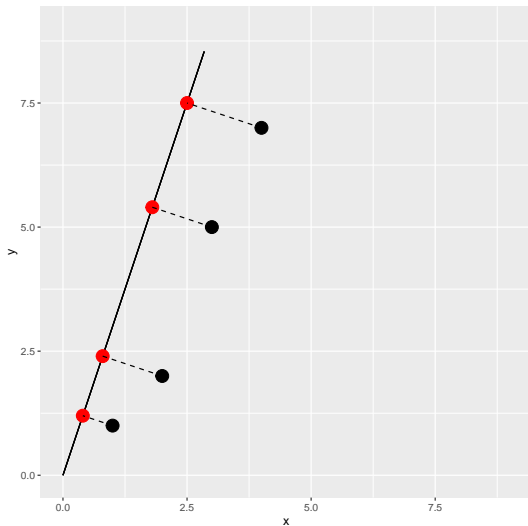


Variance: 9.2  
Direction  $z$ : [1,2]



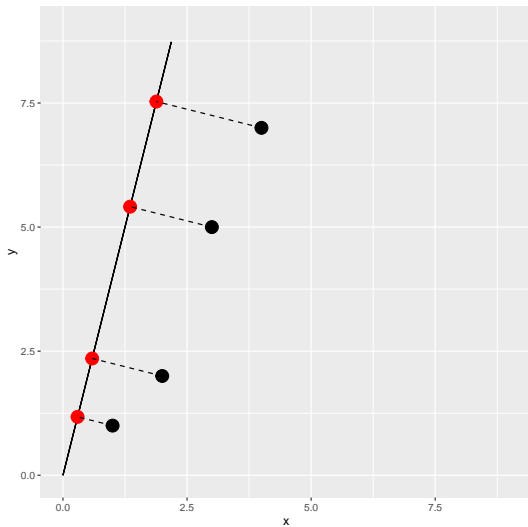
Variance: 9.09

Direction  $z$ : [1,3]



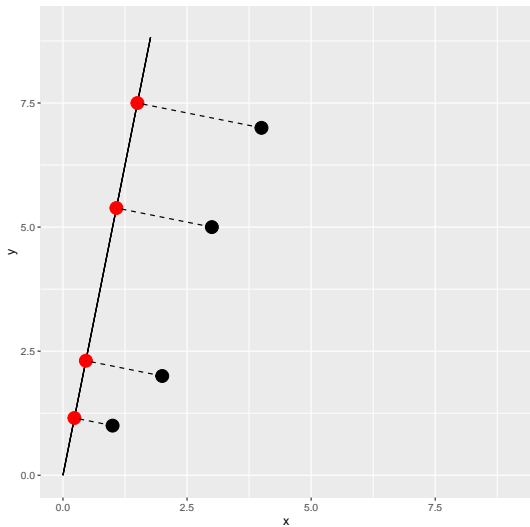
Variance: 8.88

Direction  $z$ : [1,4]



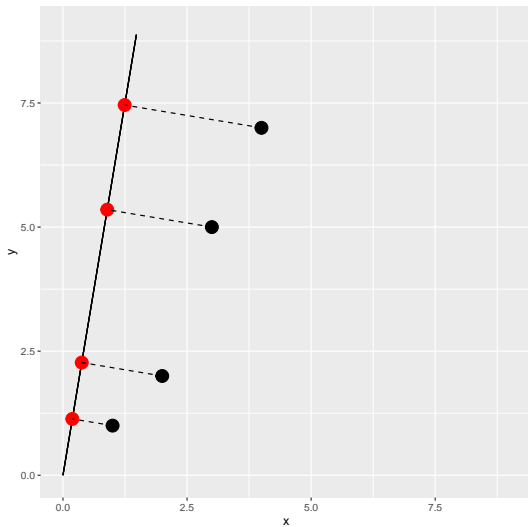
Variance: 8.7

Direction  $z$ : [1,5]



Variance: 8.56

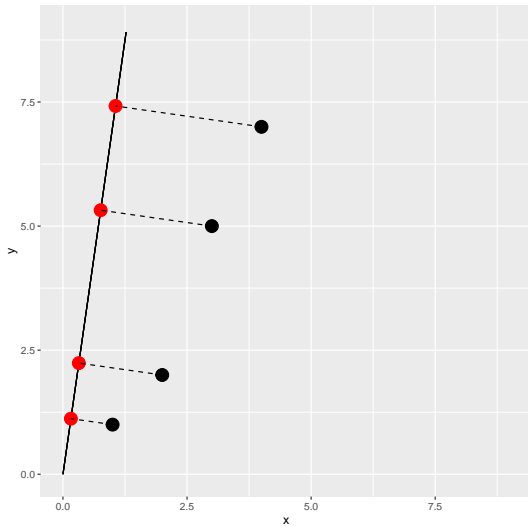
Direction  $z$ : [1,6]





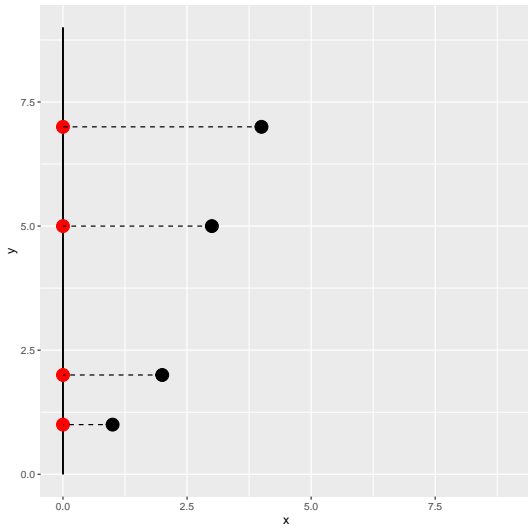
Variance: 8.45

Direction  $z$ : [1,7]



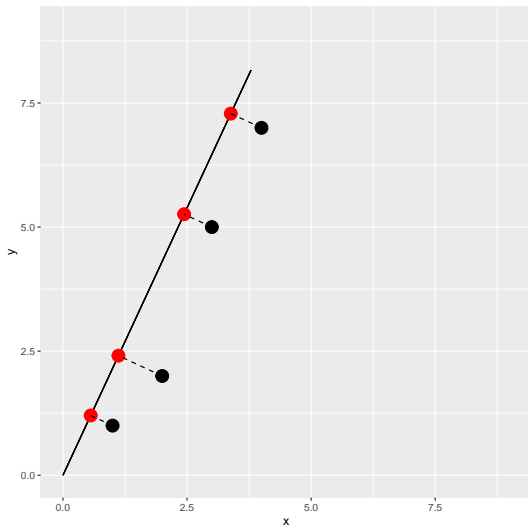
Variance: 7.58

Direction  $z$ :  $[0, 1]$



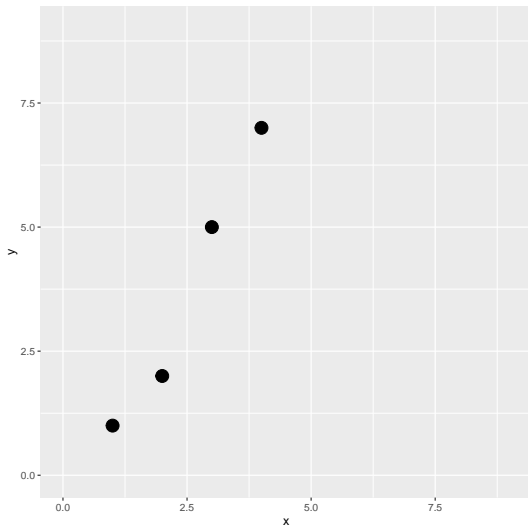
Variance: 9.21

Direction  $z$ : [0.42, 0.91]



Variance: 0.04

Direction  $z$ :  $[-0.91, 0.42]$



## Rotation Matrix or PC Loading

►  $\Phi =$

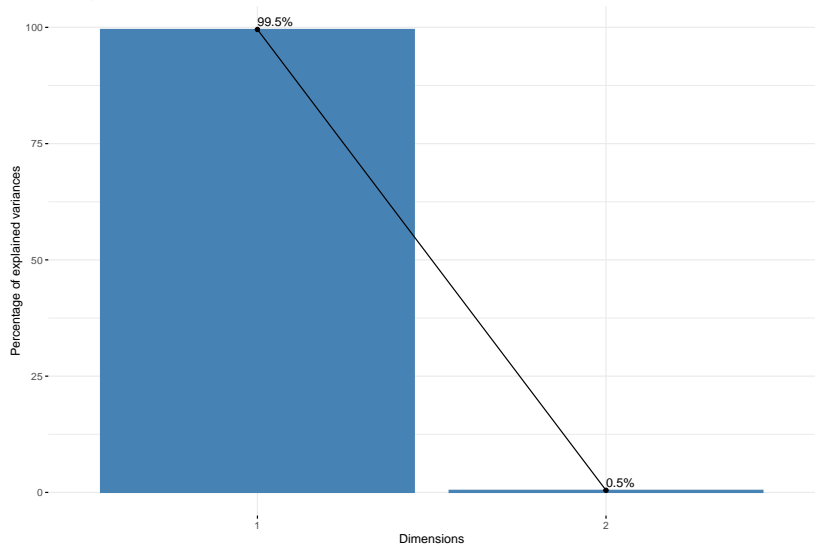
	PC1	PC2
x	0.42	-0.91
y	0.91	0.42

## PC Scores

►  $Z = X \cdot \Phi =$

	PC1	PC2
[1,]	1.33	-0.49
[2,]	2.66	-0.97
[3,]	5.80	-0.62
[4,]	8.03	-0.68

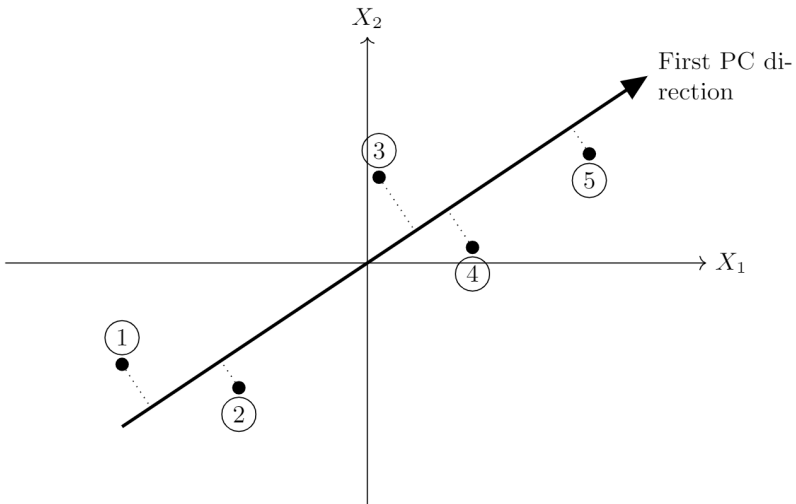
Scree plot

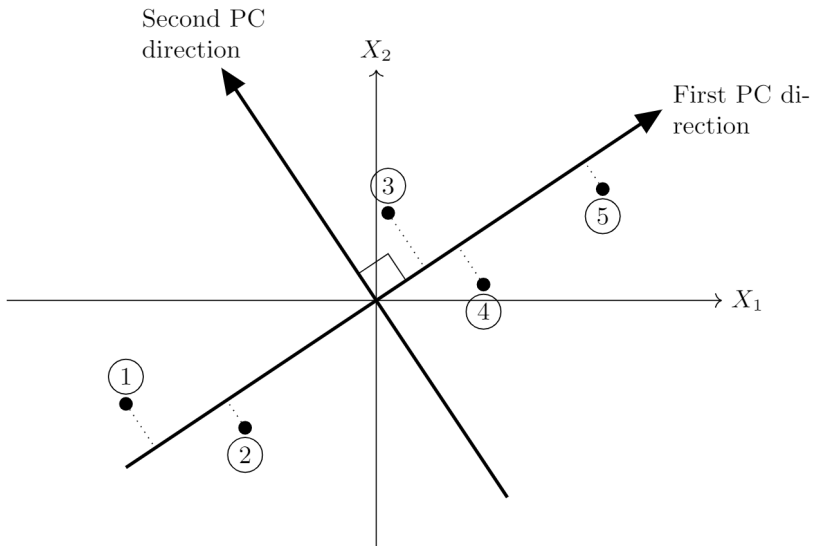


► What direction maximizes the variance?



- ▶ What direction maximizes the variance?
- ▶ The first principal component





# Formula

- ▶ Write down matrix form of the example

$$X \rightarrow X \cdot \phi = Z$$

- ▶  $\phi$  is PC loading
- ▶  $z$  is PC scores

## In general

Original data matrix  
(fat matrix!)

$$\begin{array}{ccccc} \underline{X_1} & \underline{X_2} & \cdots & \cdots & \underline{X_p} \\ \left( \begin{array}{ccccc} x_{11} & x_{12} & \cdots & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & \cdots & x_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & \cdots & x_{np} \end{array} \right) \end{array}$$

$\mathbf{X}$

New data matrix  
(thin matrix!)

$$\begin{array}{ccc} \underline{Z_1} & \cdots & \underline{Z_M} \\ \left( \begin{array}{ccc} z_{11} & \cdots & z_{1M} \\ z_{21} & \cdots & z_{2M} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nM} \end{array} \right) \end{array}$$

$\mathbf{X}\phi_1 \quad \cdots \quad \mathbf{X}\phi_M$

compressed  
 $\longrightarrow$

## Example

	Independent variables	
Observation	$X_1$	$X_2$
1	-2	2
2	2	-2

- The data set consists of only these two observations.
- The first principal component loading for  $X_1$ ,  $\phi_{11}$ , is 0.7071.
- The first principal component loading for  $X_2$ ,  $\phi_{21}$ , is negative.

Calculate the first principal component score for Observation 1.

## PC Loadings

	First PC	Second PC
Murder	0.5359	-0.4182
Assault	0.5832	-0.1880
UrbanPop	0.2782	0.8728
Rape	0.5434	0.1673

# How many PC should we use?

## ► Performance during two sporting events

X100m	Long.jump	Shot.put	High.jump	X400m	X110m.hurdle	Discus
11.04	7.58	14.83	2.07	49.81	14.69	43.75
10.76	7.40	14.26	1.86	49.37	14.05	50.72
11.02	7.23	14.25	1.92	48.93	14.99	40.87
11.34	7.09	15.19	2.10	50.42	15.31	46.26
11.13	7.30	13.48	2.01	48.62	14.17	45.67
10.83	7.31	13.76	2.13	49.91	14.38	44.41



# Scree Plot

