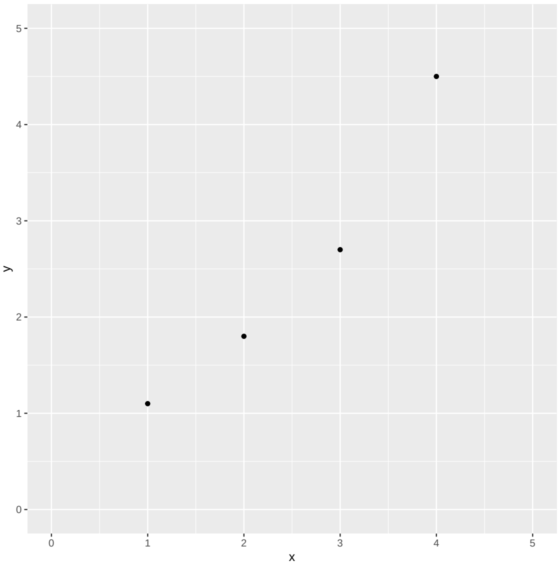
# Simple Linear Model

# 1. Motivation

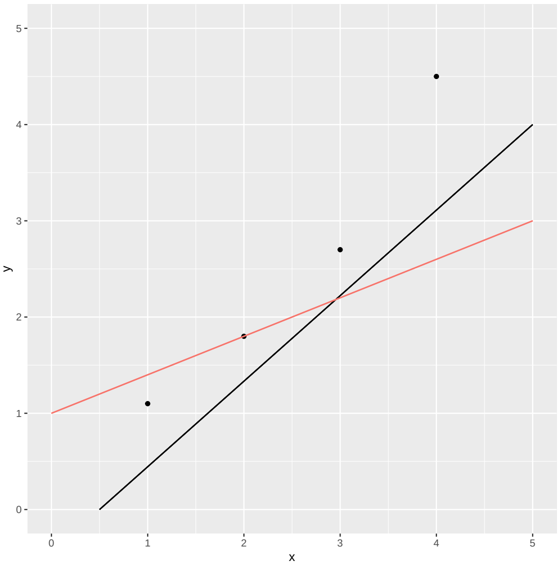
- Given the data

| $x$ | $y$ |
|---|---|
| 1 | 1.1 |
| 2 | 1.8 |
| 3 | 2.7 |
| 4 | 4.5 |

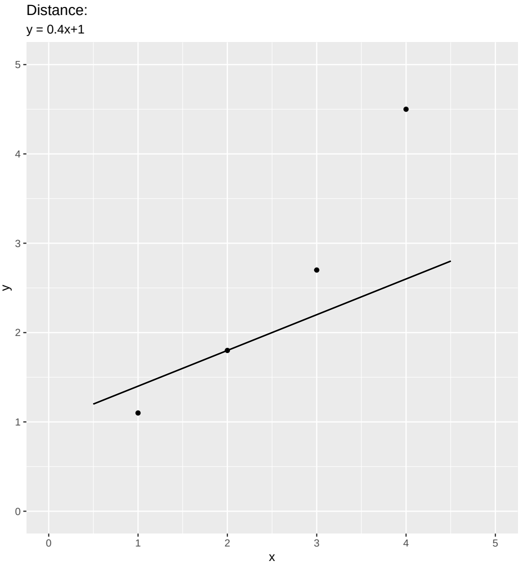
# Scatter plot



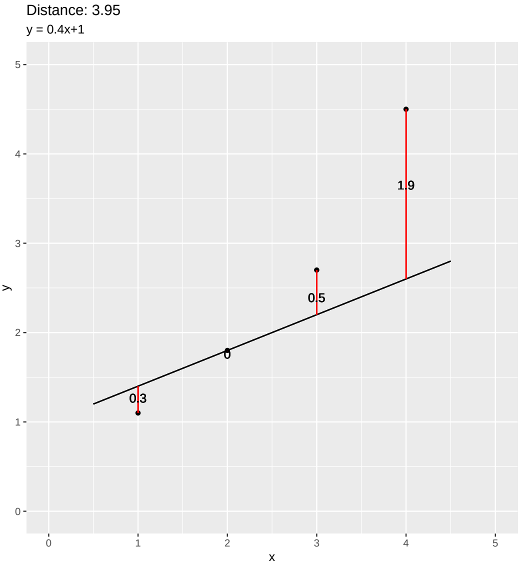
# Which line is closer to the points?



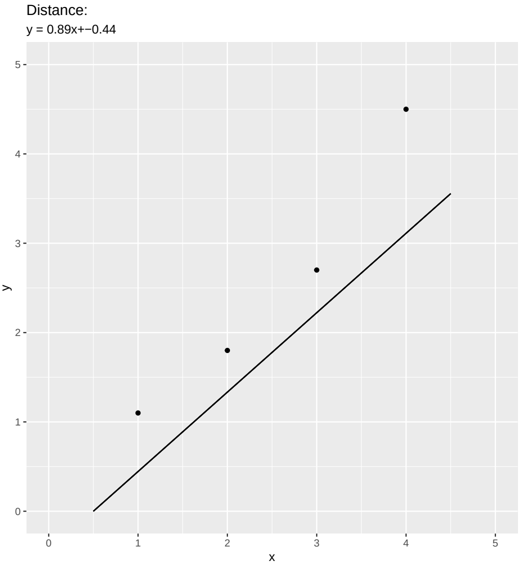
# Squared Distance between a line and points



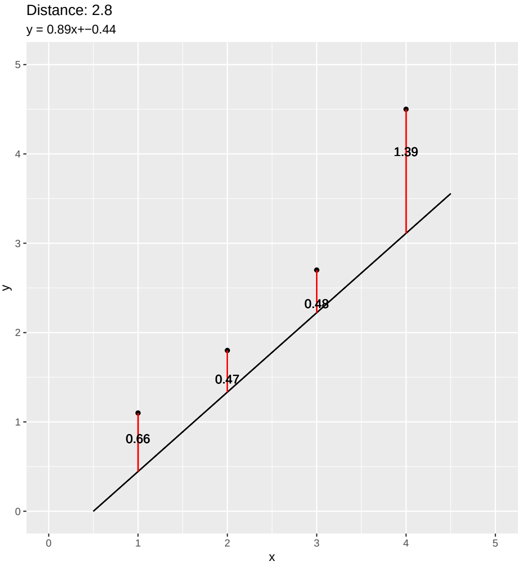
# Squared Distance between a line and points



# Squared Distance between a line and points



# Squared Distance between a line and points



# Linear Model

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

- Model Assumptions
  - (A1) The response variable $y_i$ is a random variable and the predictor $x_i$ is non-random
  - (A2) $\epsilon_i \sim^{iid} N(0, \sigma^2)$

# Parameters Estimation

# The best fitted line

- The least squared methods give us the formula for the closest line or the best fitted line:

$$y = \hat{\beta}_1 x + \hat{\beta}_0$$

- The estimated parameters $\hat{\beta}_0$ and $\hat{\beta}_1$ are

$$\begin{aligned}
\hat{\beta}_1 &= \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \\
&= \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}} = \frac{S_{xy}}{S_{xx}} \\
\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x}
\end{aligned}$$

# Example: Calculate from Data

| $x$ | $y$ |
|---|---|
| 1 | 1.1 |
| 2 | 1.8 |
| 3 | 2.7 |
| 4 | 4.5 |

| | $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|---|
| | 1 | 1.1 | | |
| | 2 | 1.8 | | |
| | 3 | 2.7 | | |
| | 4 | 4.5 | | |
| $\sum$ | | | | |

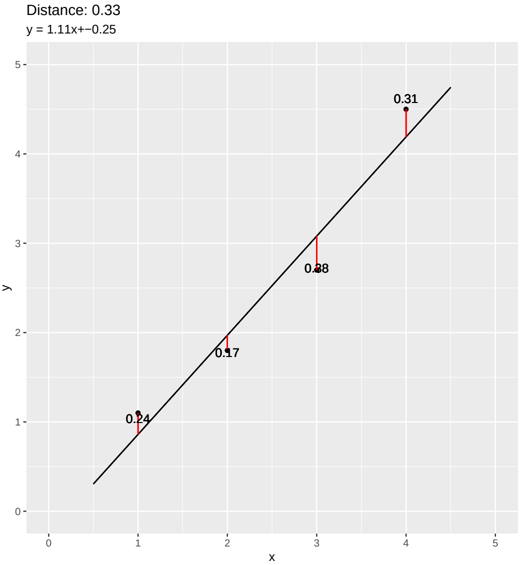
| | $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|---|
| | 1 | 1.1 | | |
| | 2 | 1.8 | | |
| | 3 | 2.7 | | |
| | 4 | 4.5 | | |
| $\sum$ | | | | |

- $\bar{x} = \frac{1+2+3+4}{4} = 2.5$
- $\bar{y} = \frac{1.1+1.8+2.4+4.5}{4} = 2.525$

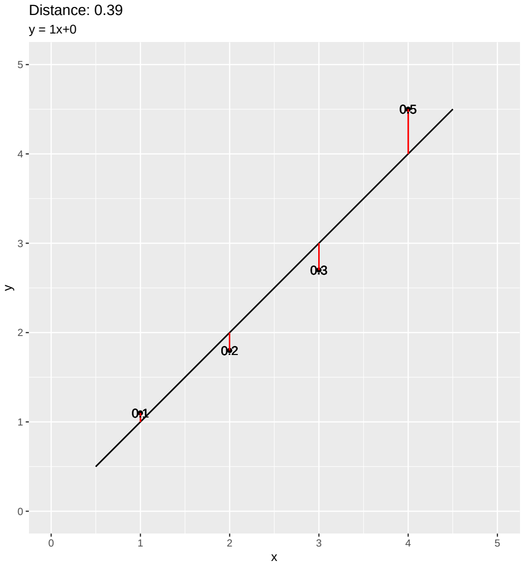
| | $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|---|
| | 1 | 1.1 | 1.1 | 1 |
| | 2 | 1.8 | 3.6 | 4 |
| | 3 | 2.7 | 8.1 | 9 |
| | 4 | 4.5 | 18 | 16 |
| $\sum$ | | | 30.8 | 30 |

- $\hat{\beta}_1 = \frac{\sum xy - n\bar{x}\bar{y}}{\sum x^2 - n\bar{x}^2} = 1.11$
- $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} = -0.25$

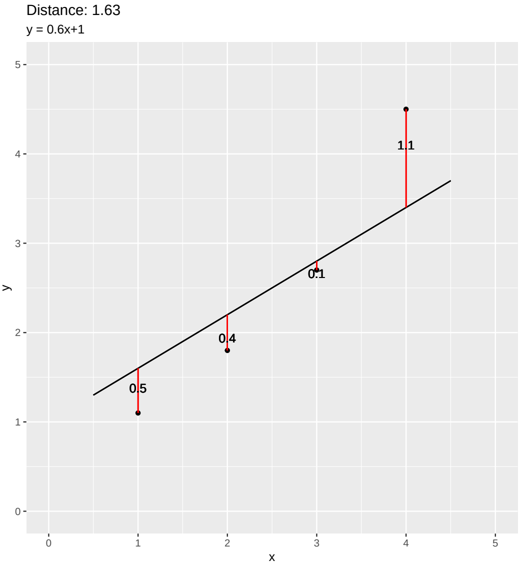
# Best fitted line



# Some other lines



# Some other lines



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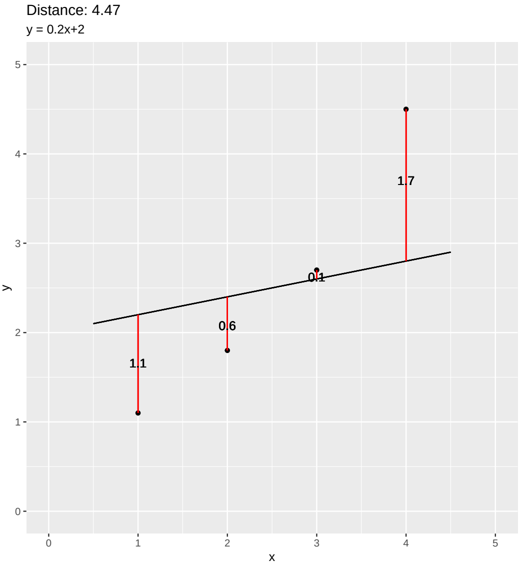

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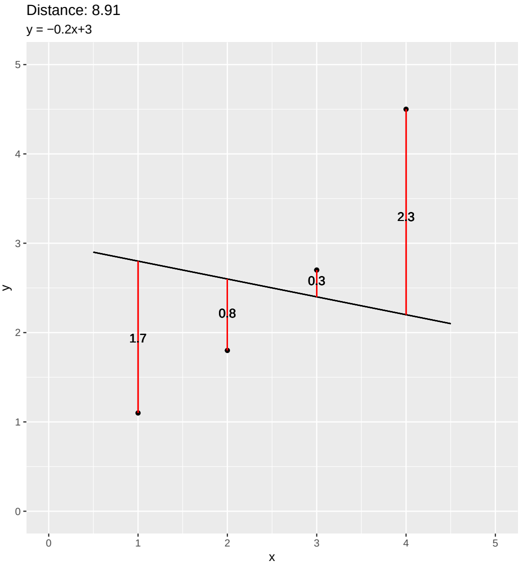

# Example: Calculate from Sumations

The regression model is $y = \beta_0 + \beta_1 x + \epsilon$. There are six observations. The summary statistics are

$$\sum y_i = 42,$$
$$\sum x_i = 21,$$
$$\sum x_i^2 = 91,$$
$$\sum x_i y_i = 187,$$
$$\sum y_i^2 = 390$$

Calculate the least squares estimate of $\beta_1$.

# Example: Calculate from Sumations

The regression model is $y = \beta_0 + \beta_1 x + \epsilon$. There are five observations. The summary statistics are

$$
\begin{aligned}
\sum y_i &= 30, \\
\sum x_i &= 15, \\
\sum (x_i - \bar{x})(y_i - \bar{y}) &= 25, \\
\sum (x_i - \bar{x})^2 &= 10, \\
\sum (y_i - \bar{y})^2 &= 64,
\end{aligned}
$$

Write the equation of the best fitted line using the least squares method.

# Goodness of Fit

# Coefficient of Determination

| | |
|---|---|
| $y_1$ | $\bar{y}$ |
| $y_2$ | $\bar{y}$ |
| $y_3$ | $\bar{y}$ |
| $\vdots$ | |

$\rightarrow$ Residual SS $= \underbrace{(y_1-\bar{y})^2 + (y_2-\bar{y})^2 + \cdots}_{\sum (y_i-\bar{y})^2}$

- Baseline model:

$$y = \beta_0 + \epsilon$$

  - In this model, $y_i$ is estimated by one number, $\bar{y}$
- Linear Model:

$$y = \beta_0 + \beta_1 x + \epsilon$$

  - In this model, $y_i$ is estimated by

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

| | |
|---|---|
| $y_1$ | $\hat{y}$ |
| $y_2$ | $\hat{y}_2$ |
| $y_3$ | $y_3$ |
| $\vdots$ | $\vdots$ |

Residual SS $= (y_1 - \hat{y}_1)^2 + (y_2 - \hat{y}_2)^2 + \cdots$

RSS of the baseline

$$
\begin{aligned}
\sum_{i=1}^{n}(y_i - \bar{y})^2 &= \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 + \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 + \underbrace{2\sum_{i=1}^{n}(y_i - \hat{y}_i)(\hat{y}_i - \bar{y})} \\
&= \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 + \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2.
\end{aligned}
$$

TSS

total SS

RSS = TSS

RSS

Res SS

=

TSS = RSS + Res SS

0

$$\underbrace{\sum_{i=1}^{n}(y_i - \bar{y})^2}_{\text{TSS}} = \underbrace{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}_{\text{RSS}} + \underbrace{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}_{\text{Reg SS}}$$

# Coefficient of Determination

$$R^2 = 1 - \frac{RSS}{TSS} = \frac{RegSS}{TSS}$$

- $R^2$ runs from 0 to 1. The larger $R^2$, the better the model

when: $RSS = 0 \Rightarrow$ perfect fit

$RSS = TSS \Rightarrow$ as good as the baseline model

# Example

You are given the following results from a regression model.

$\hat{y}_i$

| Observation number (i) | $y_i$ | $\hat{f}(x_i)$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 2 | 3 |
| 3 | 3 | 7 |
| 4 | 5 | 9 |
| 5 | 9 | 10 |

RSS

Calculate the sum of squared errors (SSE), the total sum squares (TSS), and the regression sum squares, and the $R^2$ of the model.

Res SS

| Observation number (i) | $y_i$ | $\hat{f}(x_i)$ | $(y_i - \hat{y}_i)^2$ | $(y_i - \bar{y})^2$ |
|---|---|---|---|---|
| 1 | 1 | 1 | $(1-1)^2$ | $(1-4)^2$ |
| 2 | 2 | 3 | $(2-3)^2$ | $(2-4)^2$ |
| 3 | 3 | 7 | $(3-7)^2$ | $(3-4)^2$ |
| 4 | 5 | 9 | $(5-9)^2$ | $(5-4)^2$ |
| 5 | 9 | 10 | $(9-10)^2$ | $(9-4)^2$ |
| $\Sigma$ | | | 34 | 40 |

$$\bar{y} = \frac{1+2+3+5+9}{5} = \frac{20}{5} = 4$$

$RSS = SSE = 34$

$TSS = 40$

$Reg\ SS = 40 - 34 = 6$

$$R^2 = 1 - \frac{RSS}{TSS} = 1 - \frac{34}{40} = \boxed{.15}$$

For a simple linear regression model the total sum of squares (TSS) is 150 and the $R^2$ statistic is 0.7. Calculate the sum of squares of the residuals for this model.

$TSS = 150$

$R^2 = .7$

$RSS = ?$

$$R^2 = 1 - \frac{RSS}{TSS}$$

$$\Rightarrow .7 = 1 - \frac{RSS}{150} \Rightarrow =$$

$$\Rightarrow \frac{RSS}{150} = .3 \Rightarrow RSS = 150 * .3 = \boxed{45}$$

# F-test

- i.i.d model (Baseline Model)

$$H_0 : \quad y = \beta_0 + \epsilon$$

- SLR model

$$H_a : \quad y = \beta_0 + \beta_1 x + \epsilon$$

when $\beta_1 = 0$
$\Rightarrow$ SLR turns to the baseline model.

$$\underbrace{\mathrm{H}_0 : \ \beta_1 = 0}_{\text{i.i.d. model}} \qquad \text{vs.} \qquad \underbrace{\mathrm{H}_a : \ \beta_1 \neq 0}_{\text{SLR model}},$$

$$F = \frac{\text{Reg SS}/1}{\text{RSS}/(n-2)}$$

- The smaller p-value (the larger F-statistics) supports $H_1$
- Small p-value ($\leq .05$): We reject $H_0$. The linear model is a significant improvement over the baseline model.
- Large p-value ($> .05$): Fail to reject $H_0$

# Example

Two actuaries are analyzing car accident claims for a group of $n = 52$ participants. The predictor $x$ is driving experience (years).

Actuary 1 uses the following regression model:

$$Y = \beta + \epsilon$$

Actuary 2 uses the following regression model:

$$Y = \beta_0 + \beta_1 \times x + \epsilon$$

$$F = \frac{\text{Res SS}}{\text{TSS}/(n-2)} = \frac{120}{150/(52-2)} = \frac{120}{3}$$

$$\boxed{F = 40}$$

$$R^2 = \frac{120}{150} = \frac{4}{5} = 80\%$$

The residual sum of squares for the regression of Actuary 2 is 120 and the total sum of squares is 150. = TSS

Calculate the F-statistic to test whether the model of Actuary 2 is a significant improvement over the model of Actuary 1.

# t-test

- We use t-test to test the value of $\beta_1$ and $\beta_0$

$$t(\hat{\beta}_j) = \frac{\hat{\beta}_j - d}{\text{SE}(\hat{\beta}_j)}, \quad j = 0,1,$$

$$\text{SE}(\hat{\beta}_0) = \sqrt{s^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)} = \sqrt{\frac{s^2 \sum_{i=1}^n x_i^2}{nS_{xx}}}$$

and $$\text{SE}(\hat{\beta}_1) = \sqrt{\frac{s^2}{S_{xx}}}.$$

$$s^2 = \frac{\text{RSS}}{n-2} = \frac{\sum_{i=1}^n e_i^2}{n-2},$$

$H_0$ : $\beta_1 = d$

$H_\alpha$ $\to$ $H_\alpha \neq d$

$H_\alpha < d$

$H_\alpha > d$

when $d = 0$

# Example

The results of fitting five observations by the regression model, $y = \beta_0 + \beta_1 x + \epsilon$, are given below.

p - value

$\beta_0 = 0$

| | Estimate | Std. Error | t value | Pr(>\|t\|) |
|---|---|---|---|---|
| Intercept | -1.5 | 0.7416 | -2.023 | 0.13631 |
| x | 2.5 | 0.2236 | 11.180 | 0.00153 |

$\beta_1 = 0$

Determine the test results of the hypothesis $H_0 : \beta_1 = 0$ against $H_\alpha : \beta_1 \neq 0$.

$p = .00153 < .05 \Rightarrow$ Reject $H_0$. The SLR is a significant improvement over the baseline model.